Write $\{x_n\} \underset{c}{\rightarrow} \alpha$ if $\lim_{n\to\infty}(x_1 + \cdots + x_n)/n = \alpha$. A function $f(x)$ is said to be *C. c.*, (Cesàro continuous) at $x=\alpha$ if $\{x_n\} \underset{c}{\rightarrow} \alpha$ implies $\{f(x_n)\} \underset{c}{\rightarrow} f(\alpha)$. Show that if $f(x)$ is of the form $Ax+B$ then it is *C. c.* at every value of x, and that if $f(x)$ is *C. c.* at even a single value $x=\alpha$, then $f(x)$ is of the form $Ax+B$.

4217. *Proposed by P. A. Piza, San Juan, P. R.*

Let a and b be positive integers whose sum is the square c^2 and whose difference is the cube d^3. Given that each of c^2, d^3, and a is a four digit integer and that the sum of the four digits contained in each is equal to b, find a and b.

4218. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $T \equiv ABCD$, if a point L with the normal coordinates (x, y, z, t) is such that its associates $(-x, y, z, t)$, $(x, -y, z, t)$, $(x, y, -z, t)$, $(x, y, z, -t)$ are on the circumsphere, it coincides with the point whose distances to the planes of the faces BCD, CDA, DAB, ABC are proportional to the radii of the circumcircles of these faces (second Lemoine point for T), and conversely.

4219. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron $ABCD$ with the altitudes AA', BB', CC', DD', let H' be the inverse of the orthocenter H with respect to the circumsphere, which the lines $H'A$, $H'B$, $H'C$, $H'D$ meet again in A_1, B_1, C_1, D_1. Show that the tetrahedrons $A_1B_1C_1D_1$ and $A'B'C'D'$ are similar and that the volume of the first is 27 times that of the second.

SOLUTIONS

Bound for a Finite Sum

4155 [1945, 220]. *Proposed by G. W. Wishard, Norwood, Ohio*

Find a formula for the sum of the series

$$1 + 2^2 + 3^3 + \cdots + n^n.$$

If such a formula is impossible find superior and inferior limits for the sum.

Solution by F. Underwood, University College, Nottingham. Let S_n denote the sum of the given series whose νth term is denoted by u_ν, then a first (crude) upper limit for S_n is given as follows,

$$S_{n-1} < (n-1)u_{n-1} = (n-1)^n u_n; \qquad S_n = S_{n-1} + u_n < 2u_n.$$

The first few cases show the crudity of this limit; thus $u_4 = 256$, $S_4 = 288$; $u_5 = 3125$, $S_5 = 3413$; and so a much closer inequality can be expected. Now

$$\frac{u_{n-1}}{u_n} = \frac{(n-1)^{n-1}}{n^n} = \frac{1}{n-1}\left(1 - \frac{1}{n}\right)^n = \frac{v_n}{n-1}, \qquad v_n = \left(1 - \frac{1}{n}\right)^n,$$

$$v_n - v_{n-1} = \left(1 - \frac{1}{n}\right)^n - \left(1 - \frac{1}{n-1}\right)^{n-1}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{n-1} - \left(1-\frac{1}{n-1}\right)^{n-1},$$

$$= \left(1-\frac{1}{n}\right)\left(1+\frac{1}{n-1}\right)^{-(n-1)} - \left(1-\frac{1}{n-1}\right)^{n-1}$$

$$= \left(1+\frac{1}{n-1}\right)^{-(n-1)}\left\{\left(1-\frac{1}{n}\right) - \left(1-\frac{1}{(n-1)^2}\right)^{n-1}\right\}.$$

For $n>1$

$$\left(1-\frac{1}{(n-1)^2}\right)^{n-1} < \left(1-\frac{1}{(n-1)^2}\right) < 1-\frac{1}{n}.$$

Hence $v_n - v_{n-1} > 0$ for $n>1$. Also $v_2 = 1/4$ and $\lim_{n\to\infty} v_n = 1/e$. Hence

$$\frac{1}{4} = v_2 < v_3 < v_4 < \cdots < v_n < \frac{1}{e},$$

$$S_{n-1} < 2u_{n-1} < \frac{2u_n}{e(n-1)},$$

$$S_n = S_{n-1} + u_n < u_n\left[1+\frac{2}{e(n-1)}\right], \quad \text{for} \quad n>1.$$

Again

$$u_{n-1} > \frac{u_n}{4(n-1)}, \quad \text{for} \quad n>2,$$

$$S_n = S_{n-1} + u_n > u_{n-1} + u_n > u_n\left[1+\frac{1}{4(n-1)}\right].$$

Hence, for $n>2$,

$$n^n\left[1+\frac{1}{4(n-1)}\right] < S_n < n^n\left[1+\frac{2}{e(n-1)}\right].$$

Editorial Note. The proposer gave the inequalities $n^n < S_n < 2n^n$. He also gave a table for the first ten values of S_n: it will suffice here to give the following extracts:

$n =$	6,	7,	8,	9
$S_n =$	50069,	873612,	17650828,	405071317.

The table suggested to him $n^n + n^{(n-1)}/2$ *might* be taken as a superior limit. A third, incomplete, solution uses the inequalities

$$\frac{n!e^n}{\sqrt{2\pi n}} \frac{12n-1}{12n} < n^n < \frac{n!e^n}{\sqrt{2\pi n}},$$

which are modifications of inequalities in Serret's *Algebra*, vol. 2, p. 249, §391.

Perfect Squares

4157 [1945, 220]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the base such that a number of eight digits of the form *ababcdcd* can be the square of a number of four digits *mnmn*, where the numbers of two digits *ab* and *cd*, or *ab* and *mn*, or *cd* and *mn* are consecutive.

Solution by E. P. Starke, Rutgers University. Put B for the base and A, C' M for the two-digit numbers *ab*, *cd*, *mn*, respectively. Then the hypothesis gives

$$A(B^6 + B^4) + C(B^2 + 1) = M^2(B^2 + 1)^2,$$

or

$$A(B^2 - 1) + (A + C)/(B^2 + 1) = M^2.$$

The integer $(A+C)/(B^2+1)$ must have the value unity, so that

(1) $$A(B^2 - 1) = M^2 - 1,$$

(2) $$A + C = B^2 + 1.$$

(a) Suppose A and M are consecutive. Then $M=A+1$. ($M=A-1$ is inconsistent with $M<B^2$ and (1).) Hence (1) becomes $M=B^2-2$, and at once $A=B^2-3$, $C=04$. Thus there are squares of the proposed form for every base $B>4$. *e.g.* when $B=10$, $9898^2=97970404$.

(b) Suppose $A=C\pm 1$, reducing (2) to $2A\pm 1=B^2+1$. We may eliminate A between this equation and (1) with the result

$$2M^2 = B^4 + B^2 \quad \text{or} \quad 2M^2 = B^4 - B^2 + 2.$$

The first of these is impossible since by (2) B is even, and hence M^2 must contain 2 as a factor to an odd power. For the second, solutions are possible: the simplest is $B=4$, $M=11$. Thus, in the system of base 4, $2323^2=20202121$.

(c) Suppose $M=C\pm 1$. Using equations (1), (2) and eliminating A and C, we may put the results in the form

$$(5B^2 - 3)^2 + 16 = 5(2M + B^2 - 1)^2 \quad \text{or} \quad (5B^2 + 1)^2 - 16 = 5(2M + B^2 - 1)^2.$$

For the first there seems to be no simple integral solution, but the second yields $B=7$, $M=31$. Thus in the system of base 7 we have $4343^2=26264242$.

Solved also by the proposer.

Editorial Note. The proposer gave three examples, the first two of which are above: base 4, $(2323)^2$; base 7, $(4343)^2$ base 7, $(6565^2)=64640404$.

No derivation was given.