Green's Function for the 2D SH Wave Equation with a Line Source

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1 Derivation of the 2D Green's Function for the Wave Equation

The 2D SH wave equation in stress-displacement formulation is given as:

1. Equation of motion:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} + f_y$$

2. Constitutive relations:

$$\sigma_{xy} = \mu \frac{\partial u_y}{\partial x}$$

$$\sigma_{yz} = \mu \frac{\partial u_y}{\partial z}$$

Where:

- u_y is the displacement in the y direction
- ρ is the density
- μ is the shear modulus
- $f_y = A\delta(x)\delta(z)\delta(t)$ is the source term (a line source along y-axis)
- $f_y = (0, A\delta(x)\delta(z)\delta(t), 0)$

A is a constant having the dimensions of impulse per unit length. Only the y component of displacement is excited by this source.

2 Substitute Constitutive Relations into Equation of Motion

First, let's substitute the stress components into the equation of motion:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \frac{\partial u_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial u_y}{\partial z}\right) + A\delta(x)\delta(z)\delta(t)$$

Assuming μ is constant, this simplifies to:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \mu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + A\delta(x)\delta(z)\delta(t)$$

Divide both sides by ρ:

$$\frac{\partial^2 u_y}{\partial t^2} = \frac{\mu}{\rho} \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + \frac{A}{\rho}\delta(x)\delta(z)\delta(t)$$

Let $\beta = \sqrt{\mu/\rho}$ (shear wave velocity), and $\tilde{A} = A/\rho$:

$$\boxed{\frac{\partial^2 u_y}{\partial t^2} = \beta^2 \nabla^2 u_y + \tilde{A}\delta(x)\delta(z)\delta(t)}$$

We seek to find the causal Green's function $u_y(x, z, t)$ for this equation, i.e. the response to $A\delta(x)\delta(z)\delta(t)$.

We define the Green's function $G(\mathbf{x}, z, t)$ as:

$$u_y(x, z, t) = G(x, z, t)$$

$$\frac{\partial^2 G}{\partial t^2} - \beta^2 \nabla^2 G = \tilde{A}\delta(x)\delta(z)\delta(t)$$

3 Spatial Fourier Transform

Apply the 2D spatial Fourier transform in x, z. Let

$$\tilde{G}(k_x, k_z, t) = \iint_{-\infty}^{\infty} G(x, z, t) e^{-i(k_x x + k_z z)}\, dx\, dz,$$

with inverse

$$G(x, z, t) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{G}(k_x, k_z, t) e^{i(k_x x + k_z z)}\, \mathrm{d}k_x\, \mathrm{d}k_z$$

Under this transform, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} = -k^2$ with $k^2 = k_x^2 + k_z^2$, and $\delta(x)\delta(z) = 1$.

The wave equation in **k**-space becomes

$$\boxed{\frac{\partial^2 \tilde{G}}{\partial t^2} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho}\delta(t).}$$

4 Solution to the ODE in Time

The given equation is a second-order partial differential equation (PDE) for $\tilde{G}(t)$ with respect to time t, and it can be treated as an ordinary differential equation (ODE) in t since there are no other independent variables present. The equation is:

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho}\delta(t).$$

This is an inhomogeneous ODE due to the Dirac delta function $\delta(t)$. To solve it, we proceed as follows:

1. Solve the Homogeneous Equation

First, consider the homogeneous version (without the delta function):

$$\frac{d^2 \tilde{G}}{dt^2} + \beta^2 k^2 \tilde{G} = 0.$$

The characteristic equation is:

$$r^2 + \beta^2 k^2 = 0 \implies r = \pm i\beta k.$$

Thus, the general solution to the homogeneous equation is:

$$\tilde{G}_h(t) = C_1 \cos(\beta k t) + C_2 \sin(\beta k t),$$

where C_1 and C_2 are constants.

2. Find a Particular Solution for the Inhomogeneous Equation

The inhomogeneous term is $\frac{A}{\rho}\delta(t)$. To handle the delta function, integrate the original equation over a small interval around $t = 0$ (from $t = -\epsilon$ to $t = +\epsilon$):

$$\int_{-\epsilon}^{\epsilon} \frac{d^2\tilde{G}}{dt^2}\, dt + \beta^2 k^2 \int_{-\epsilon}^{\epsilon} \tilde{G}\, dt = \frac{A}{\rho} \int_{-\epsilon}^{\epsilon} \delta(t)\, dt.$$

The first integral gives the jump in the first derivative:

$$\left. \frac{d\tilde{G}}{dt} \right|_{-\epsilon}^{+\epsilon} + \beta^2 k^2 \cdot 0 = \frac{A}{\rho},$$

since $\tilde{G}$ is continuous at $t = 0$ (so its integral over an infinitesimal interval vanishes), and $\int \delta(t)\, dt = 1$. Thus:

$$\left. \frac{d\tilde{G}}{dt} \right|_{0^+} - \left. \frac{d\tilde{G}}{dt} \right|_{0^-} = \frac{A}{\rho}.$$

Assuming $\tilde{G}$ is initially at rest for $t < 0$ (i.e., $\tilde{G}(t) = 0$ and $\frac{d\tilde{G}}{dt} = 0$ for $t < 0$), we have:

$$\left. \frac{d\tilde{G}}{dt} \right|_{0^+} = \frac{A}{\rho}.$$

3. Apply Initial Conditions

For $t > 0$, the solution is the homogeneous solution:

$$\tilde{G}(t) = C_1 \cos(\beta k t) + C_2 \sin(\beta k t).$$

Apply the initial conditions at $t = 0^+$:

- Continuity of $\tilde{G}$ at $t = 0$ implies $\tilde{G}(0^+) = 0$, so:

$$C_1 = 0.$$

- The jump condition for the derivative gives:

$$\left.\frac{d\tilde{G}}{dt}\right|_{0^+} = \beta k C_2 = \frac{A}{\rho} \implies C_2 = \frac{A}{\rho\beta k}.$$

Thus, the solution for $t > 0$ is:

$$\tilde{G}(t) = \frac{A}{\rho\beta k}\sin(\beta k t).$$

For $t < 0$, $\tilde{G}(t) = 0$.

4. Final Solution

The solution can be written compactly using the Heaviside step function $H(t)$:

$$\tilde{G}(t) = \frac{A}{\rho\beta k}\sin(\beta k t)\, H(t).$$

Verification

Differentiate $\tilde{G}(t)$:

$$\frac{d\tilde{G}}{dt} = \frac{A}{\rho}\cos(\beta k t)\, H(t) + \frac{A}{\rho\beta k}\sin(\beta k t)\delta(t) = \frac{A}{\rho}\cos(\beta k t)\, H(t),$$

since $\sin(\beta k t)\delta(t) = 0$. Differentiate again:

$$\frac{d^2\tilde{G}}{dt^2} = -\frac{A\beta k}{\rho}\sin(\beta k t)\, H(t) + \frac{A}{\rho}\cos(\beta k t)\delta(t).$$

Substitute into the original equation:

$$-\frac{A\beta k}{\rho}\sin(\beta k t)\, H(t) + \frac{A}{\rho}\delta(t) + \beta^2 k^2\left(\frac{A}{\rho\beta k}\sin(\beta k t)\, H(t)\right) = \frac{A}{\rho}\delta(t).$$

The terms involving $\sin(\beta k t)$ cancel, leaving:

$$\frac{A}{\rho}\delta(t) = \frac{A}{\rho}\delta(t),$$

which confirms the solution is correct.

Final Answer

The solution to the ODE is:

$$\boxed{\tilde{G}(t) = \frac{A}{\rho\beta k}\sin(\beta k t)\, H(t).}$$

$\omega = \beta k$, so

$$\boxed{\tilde{G}(t) = \frac{A}{\rho\omega}\sin(\omega t)\, H(t).}$$

5 Inverse Transform and Bessel Integral

We now invert to real space.

Using polar coordinates in $\mathbf{k}$:
$k_x = k\cos\phi$,
$k_z = k\sin\phi$,
$d^2k = dk_x dk_z = k\, dk\, d\phi$, and
$\mathbf{k}\cdot\mathbf{r} = kr\cos(\phi-\theta)$ where $r = \sqrt{x^2+z^2}$.
$\mathbf{x} = (r\cos\theta, r\sin\theta)$
$\mathbf{k} = (k\cos\phi, k\sin\phi)$
$\mathrm{d}k_x\,\mathrm{d}k_z = k\,\mathrm{d}k\,\mathrm{d}\phi$

The inverse transform gives

$$G(x,z,t) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\tilde{G}(k_x,k_z,t)e^{i(k_x x + k_z z)}\,\mathrm{d}k_x\,\mathrm{d}k_z = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\tilde{G}(k_x,k_z,t)e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}k_x\,\mathrm{d}k_z$$

$$G(x,z,t) = \frac{A}{(2\pi)^2\rho}H(t)\int_0^{2\pi}\int_0^{\infty}\frac{\sin(\omega t)}{\omega}e^{ikr\cos(\phi-\theta)}\,k\,dk\,d\phi.$$

$$G(x,z,t) = \frac{A}{(2\pi)^2\rho} H(t) \int_0^{2\pi} \int_0^{\infty} \frac{\sin(\beta k t)}{\beta k} e^{ikr\cos(\phi-\theta)}\, k\, dk\, d\phi.$$

Evaluate the Angular Integral

Simplify the angular integral using the identity:

$$\int_0^{2\pi} e^{ikr\cos\phi} d\phi = 2\pi J_0(kr)$$

This yields

$$G(x,z,t) = \frac{A}{2\pi\rho\beta} H(t) \int_0^{\infty} J_0(kr)\sin(\beta k t)\, dk.$$

where J_0 is the Bessel function of the first kind.

Evaluate the Radial Integral

The remaining radial integral is a standard Bessel–sine integral. This integral can be evaluated using tables of Hankel transforms or contour integration. The radial integral evaluate to:

$$\int_0^{\infty} \sin(\beta k t) J_0(kr)\, \mathrm{d}k = \begin{cases} \frac{1}{\sqrt{\beta^2 t^2 - r^2}} & \text{for } \beta t > r \\ 0 & \text{otherwise } (\beta t < r) \end{cases}$$

Thus the integral vanishes unless $\beta t > r$; for $\beta t > r$ it equals $1/\sqrt{(ct)^2 - r^2}$. Including the Heaviside step for causality, one obtains:

$$\int_0^{\infty} J_0(kr)\sin(\beta k t)\, dk = \frac{H(\beta t - r)}{\sqrt{(\beta t)^2 - r^2}}.$$

Substituting back gives the space-time Green's function.

6 Final Green's Function (Time Domain)

Combining all results:

$$G(x,z,t) = \frac{H(t)}{2\pi\beta} \frac{H(\beta t - r)}{\sqrt{\beta^2 t^2 - r^2}} \tag{1}$$

By the Heaviside function, it follows that:

- For $t > 0$ and $r < \beta t$, the function is finite and decays as r approaches βt.
- For $t \leq 0$ or $r \geq \beta t$, the function is zero due to the Heaviside step functions.

Final Simplified Form:

$$G(x, z, t) = \begin{cases} \frac{1}{2\pi\beta\sqrt{\beta^2 t^2 - r^2}} & \text{if } t > 0 \text{ and } r < \beta t, \\ 0 & \text{otherwise.} \end{cases}$$

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By combining the Heaviside Functions in equation 1:

- $H(t)H(\beta t - r)$ is equivalent to $H(t - r/\beta)$ because:
  - For $t > 0$ and $r < \beta t, t > r/\beta$.
  - Thus, $H(t - r/\beta)$ captures both conditions.

So the final simplified form of equation 1 is:

$$G(x, z, t) = \frac{1}{2\pi\beta^2} \frac{H(t - r/\beta)}{\sqrt{t^2 - \frac{r^2}{\beta^2}}}.$$

This result agrees with the well-known 2D wave Green's function (for $\beta = 1$)

$$\boxed{G(x, z, t) = \frac{H(t - r)}{2\pi\sqrt{t^2 - r^2}}}$$

Replacing the factor A/ρ. The factor $1/(2\pi\rho\beta)$ ensures the correct amplitude for the given source strength A.

Thus in summary, the time-domain Green's function is:

$$\boxed{u_y(x, z, t) = G(x, z, t) = \frac{A}{2\pi\rho\beta} \frac{H(\beta t - r)}{\sqrt{(\beta t)^2 - r^2}}}$$

which is valid for $r = \sqrt{x^2 + z^2}$. This is the **retarded (causal) solution of the 2D SH wave equation for an impulse line source**.

7 Key Points

The solution satisfies:

- $G = 0$ for $t < r/\beta$ (causality). This means the solution is zero before the wave arrives.
- Integrable singularity at the wavefront $t = r/\beta$
- Wavefront propagates at speed β
- Amplitude decays as $1/\sqrt{t}$ for large t
- The solution is cylindrically symmetric (depends only on r)

8 Frequency-Domain (Hankel) Form

Equivalently, one can work in the frequency domain by taking the Fourier transform in time. Writing

$$u_y(x, z, t) = \int_{-\infty}^{\infty} \tilde{u}_y(x, z, \omega) e^{-i\omega t}\, d\omega,$$

the equation becomes the 2D Helmholtz equation

$$\nabla^2 \tilde{u}_y + \frac{\omega^2}{\beta^2} \tilde{u}_y = -\frac{A}{\rho\beta^2} \delta(x)\delta(z).$$

The fundamental solution of $(\nabla^2 + k^2)\tilde{G} = -\delta$ in two dimensions is well known to be proportional to the Hankel function $H_0^{(1)}$. In fact one finds

$$\tilde{u}_y(r, \omega) = \frac{A}{\rho\beta^2} \tilde{G}(r, \omega), \quad \tilde{G}(r, \omega) = \frac{i}{4} H_0^{(1)}\left(\frac{\omega r}{\beta}\right).$$

Hence

$$\boxed{\tilde{u}_y(r, \omega) = \frac{iA}{4\rho\beta^2} H_0^{(1)}\left(\frac{\omega r}{\beta}\right),}$$

which represents an outgoing cylindrical wave. This matches the known 2D Helmholtz Green's function

$$G(k; r) = \frac{i}{4} H_0^{(1)}(kr)$$

(with $k = \omega/\beta$ and including the $A/(\rho\beta^2)$ prefactor).

Where:

- $H_0^{(1)}$ is the **Hankel function of the first kind**, order zero

- $k = \omega/\beta$ is the wavenumber

- $r = \sqrt{x^2 + z^2}$ is the radial distance from the source

This Hankel function describes **outgoing cylindrical waves**—perfect for 2D problems.

One may verify that the inverse Fourier transform of this $\tilde{u}_2$ in ω recovers the time-domain result above.

Asymptotic Forms of $H_0^{(1)}(kr)$. $k = \frac{\omega}{\beta}$

1. Near-field (Small argument: $kr \ll 1$)

As $kr \to 0$, the Hankel function has the asymptotic expansion:

$$H_0^{(1)}(kr) \approx \frac{2i}{\pi}\left(\ln\left(\frac{kr}{2}\right) + \gamma - i\frac{\pi}{2}\right), \quad \gamma \text{ is Euler–Mascheroni constant}$$

Thus:

$$\tilde{u}_y^{\text{near}}(r,\omega) \approx \frac{iA}{4\rho\beta^2} \cdot \frac{2i}{\pi}\left(\ln\left(\frac{kr}{2}\right) + \gamma - i\frac{\pi}{2}\right)$$

Simplifying:

$$\tilde{u}_y^{\text{near}}(r,\omega) \approx -\frac{A}{2\pi\rho\beta^2}\left(\ln\left(\frac{kr}{2}\right) + \gamma - i\frac{\pi}{2}\right)$$

- **Interpretation**:
  - Logarithmic divergence as $r \to 0$
  - This is a complex-valued, non-radiating field (reactive field)
  - Dominated by evanescent energy and stored energy near the source

2. Far-field (Large argument: $kr \gg 1$)

As $kr \to \infty$, the Hankel function behaves like:

$$H_0^{(1)}(kr) \approx \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)}$$

So:

$$\tilde{u}_y^{\text{far}}(r,\omega) \approx \frac{iA}{4\rho\beta^2}\sqrt{\frac{2}{\pi k r}}e^{i(kr-\pi/4)}$$

- **Interpretation**:
  - Describes an outgoing cylindrical wave
  - Amplitude decays as $r^{-1/2}$, slower than 3D r^{-1}
  - The phase varies linearly with kr, indicating radiating energy