

Elastic Wave From a Point Dislocation Source

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Detailed Derivation of the Displacement Field $\mathbf{u}(\mathbf{x}, t)$

This derivation is part of the solution for the elastodynamic Green's function in a homogeneous, unbounded, isotropic medium. For the course **GEOP591 (Theoretical Seismology)**.

We start with the scalar and vector Lamé potentials $\phi(\mathbf{x}, t)$ and $\boldsymbol{\psi}(\mathbf{x}, t)$:

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi\rho} \left(\frac{\partial}{\partial x_1} \frac{1}{|\mathbf{x}|} \right) \int_0^{|\mathbf{x}|/\alpha} \tau X_0(t - \tau) d\tau, \quad (\text{A})$$

$$\boldsymbol{\psi}(\mathbf{x}, t) = \frac{1}{4\pi\rho} \left(0, \frac{\partial}{\partial x_3} \frac{1}{|\mathbf{x}|}, -\frac{\partial}{\partial x_2} \frac{1}{|\mathbf{x}|} \right) \int_0^{|\mathbf{x}|/\beta} \tau X_0(t - \tau) d\tau. \quad (\text{B})$$

The displacement field is obtained via:

$$\mathbf{u}(\mathbf{x}, t) = \nabla\phi + \nabla \times \boldsymbol{\psi}. \quad (1)$$

1. Detailed Computation of $\nabla\phi$

The gradient of $\phi(\mathbf{x}, t)$ is:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3} \right). \quad (2)$$

From (A), with $r = |\mathbf{x}|$:

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi\rho} \left(\frac{\partial}{\partial x_1} \frac{1}{r} \right) \int_0^{r/\alpha} \tau X_0(t - \tau) d\tau. \quad (3)$$

Taking the gradient components:

$$\begin{aligned} \frac{\partial \phi}{\partial x_i} = & -\frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_0^{r/\alpha} \tau X_0(t - \tau) d\tau \\ & - \frac{1}{4\pi\rho} \left(\frac{\partial}{\partial x_1} \frac{1}{r} \right) \frac{\partial}{\partial x_i} \left(\int_0^{r/\alpha} \tau X_0(t - \tau) d\tau \right) \end{aligned} \quad (4)$$

Using the Leibniz rule:

$$\frac{\partial}{\partial x_i} \left(\int_0^{r/\alpha} \tau X_0(t - \tau) d\tau \right) = \left(\frac{r}{\alpha} \right) X_0 \left(t - \frac{r}{\alpha} \right) \frac{\partial}{\partial x_i} \left(\frac{r}{\alpha} \right). \quad (5)$$

Since $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, $\frac{\partial}{\partial x_i} \left(\frac{r}{\alpha} \right) = \frac{x_i}{\alpha r}$:

$$\frac{\partial \phi}{\partial x_i} = -\frac{1}{4\pi\rho} \left[\left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_0^{r/\alpha} \tau X_0(t - \tau) d\tau + \left(\frac{\partial}{\partial x_1} \frac{1}{r} \right) \frac{\partial}{\partial x_i} \left(\int_0^{r/\alpha} \tau X_0(t - \tau) d\tau \right) \right] \quad (6)$$

$$\frac{\partial \phi}{\partial x_i} = -\frac{1}{4\pi\rho} \left[\left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_0^{r/\alpha} \tau X_0(t - \tau) d\tau + \left(\frac{\partial}{\partial x_1} \frac{1}{r} \right) \left(\frac{r}{\alpha} \right) X_0 \left(t - \frac{r}{\alpha} \right) \frac{\partial}{\partial x_i} \left(\frac{r}{\alpha} \right) \right] \quad (7)$$

$$\boxed{\frac{\partial}{\partial x_1} \frac{1}{r} = -\frac{x_1}{r^3}}$$

Derivation of $\frac{\partial}{\partial x_1} \left(\frac{1}{r} \right)$

We want to compute:

$$\frac{\partial}{\partial x_1} \left(\frac{1}{r} \right)$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Step 1: Rewrite the expression

$$\frac{\partial}{\partial x_1} \left(\frac{1}{r} \right) = \frac{\partial}{\partial x_1} (r^{-1})$$

Step 2: Apply the chain rule

$$\frac{\partial}{\partial x_1} (r^{-1}) = -r^{-2} \cdot \frac{\partial r}{\partial x_1}$$

Step 3: Compute $\frac{\partial r}{\partial x_1}$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \implies \frac{\partial r}{\partial x_1} = \frac{x_1}{r}$$

Step 4: Substitute back

$$-r^{-2} \cdot \frac{x_1}{r} = -\frac{x_1}{r^3}$$

Final Result:

$$\frac{\partial}{\partial x_1} \left(\frac{1}{r} \right) = -\frac{x_1}{r^3}$$

$$\frac{\partial \phi}{\partial x_i} = -\frac{1}{4\pi\rho} \left[\left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_0^{r/\alpha} \tau X_0(t - \tau) d\tau + \left(-\frac{x_1}{r^3} \right) \left(\frac{r}{\alpha} \right) X_0 \left(t - \frac{r}{\alpha} \right) \left(\frac{x_i}{\alpha r} \right) \right] \quad (8)$$

$$\boxed{\frac{\partial \phi}{\partial x_i} = -\frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_0^{r/\alpha} \tau X_0(t - \tau) d\tau + \frac{1}{4\pi\rho\alpha^2} \left(\frac{x_i x_1}{r^3} \right) X_0 \left(t - \frac{r}{\alpha} \right) .}$$

2. Detailed Computation of $\nabla \times \psi$

We compute the curl of the vector potential ψ explicitly using the determinant representation of the curl operator. The vector potential is given by:

$$\psi(x, t) = \frac{1}{4\pi\rho} \left(0, \frac{\partial}{\partial x_3} \frac{1}{r}, -\frac{\partial}{\partial x_2} \frac{1}{r} \right) \int_0^{r/\beta} \tau X_0(t - \tau) d\tau$$

where $r = |x|$.

Let's denote:

$$\psi = (0, \psi_2, \psi_3), \quad \text{where} \quad \psi_2 = \frac{1}{4\pi\rho} \left(\frac{\partial}{\partial x_3} \frac{1}{r} \right) F(t), \quad \psi_3 = -\frac{1}{4\pi\rho} \left(\frac{\partial}{\partial x_2} \frac{1}{r} \right) F(t),$$

and

$$F(t) = \int_0^{r/\beta} \tau X_0(t - \tau) d\tau$$

Step 1: Determinant Form of Curl

The curl is computed as:

$$\nabla \times \psi = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ 0 & \psi_2 & \psi_3 \end{vmatrix} = \hat{x}_1 \left(\frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3} \right) - \hat{x}_2 \left(\frac{\partial \psi_3}{\partial x_1} \right) + \hat{x}_3 \left(\frac{\partial \psi_2}{\partial x_1} \right)$$

Step 2: Component Calculations

x_1 - Component:

$$(\nabla \times \psi)_1 = \frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3}$$

Substitute ψ_2 and ψ_3 :

$$(\nabla \times \psi)_1 = \frac{\partial}{\partial x_2} \left(-\frac{1}{4\pi\rho} \frac{\partial}{\partial x_2} \frac{1}{r} F(t) \right) - \frac{\partial}{\partial x_3} \left(\frac{1}{4\pi\rho} \frac{\partial}{\partial x_3} \frac{1}{r} F(t) \right)$$

Apply the product rule:

$$(\nabla \times \psi)_1 = -\frac{1}{4\pi\rho} \left[\frac{\partial^2}{\partial x_2^2} \left(\frac{1}{r} \right) F(t) + \frac{\partial}{\partial x_2} \left(\frac{1}{r} \right) \frac{\partial F(t)}{\partial x_2} \right] - \frac{1}{4\pi\rho} \left[\frac{\partial^2}{\partial x_3^2} \left(\frac{1}{r} \right) F(t) + \frac{\partial}{\partial x_3} \left(\frac{1}{r} \right) \frac{\partial F(t)}{\partial x_3} \right]$$

Simplify using $\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(\mathbf{x})$ (Dirac delta function, negligible for $r \neq 0$):

$$(\nabla \times \psi)_1 \approx -\frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \frac{1}{r} F(t)$$

x_2 -Component:

$$(\nabla \times \psi)_2 = -\frac{\partial \psi_3}{\partial x_1}$$

Substitute ψ_3 :

$$(\nabla \times \psi)_2 = -\frac{\partial}{\partial x_1} \left(-\frac{1}{4\pi\rho} \frac{\partial}{\partial x_2} \frac{1}{r} F(t) \right) = \frac{1}{4\pi\rho} \frac{\partial^2}{\partial x_1 \partial x_2} \left(\frac{1}{r} \right) F(t)$$

x_3 -Component:

$$(\nabla \times \psi)_3 = \frac{\partial \psi_2}{\partial x_1}$$

Substitute ψ_2 :

$$(\nabla \times \psi)_3 = \frac{\partial}{\partial x_1} \left(\frac{1}{4\pi\rho} \frac{\partial}{\partial x_3} \frac{1}{r} F(t) \right) = \frac{1}{4\pi\rho} \frac{\partial^2}{\partial x_1 \partial x_3} \left(\frac{1}{r} \right) F(t)$$

Step 3: Combine Components and Simplify

The full curl is:

$$\nabla \times \psi = \frac{1}{4\pi\rho} \left[-\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \frac{1}{r} \hat{x}_1 + \frac{\partial^2}{\partial x_1 \partial x_2} \frac{1}{r} \hat{x}_2 + \frac{\partial^2}{\partial x_1 \partial x_3} \frac{1}{r} \hat{x}_3 \right] F(t) \quad (9)$$

Key Simplification:

Recognize that:

$$\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \nabla^2 - \frac{\partial^2}{\partial x_1^2}, \quad \text{and} \quad \nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(\mathbf{x}) \quad (10)$$

For $r \neq 0$, the delta function vanishes, leaving:

$$\nabla \times \psi = \frac{1}{4\pi\rho} \left[\frac{\partial^2}{\partial x_1^2} \frac{1}{r} \hat{x}_1 + \frac{\partial^2}{\partial x_1 \partial x_2} \frac{1}{r} \hat{x}_2 + \frac{\partial^2}{\partial x_1 \partial x_3} \frac{1}{r} \hat{x}_3 \right] F(t) \quad (11)$$

This can be written compactly as:

$$\nabla \times \psi = \frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_1 \partial x_i} \frac{1}{r} \right) F(t) \hat{x}_i \quad (12)$$

Step 4: Include the Time-Dependent Term

The integral $F(t)$ contributes a term when differentiated:

$$\frac{\partial F(t)}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\int_0^{r/\beta} \tau X_0(t - \tau) d\tau \right) = \left(\frac{r}{\beta} \right) X_0 \left(t - \frac{r}{\beta} \right) \frac{\partial}{\partial x_i} \left(\frac{r}{\beta} \right) \quad (13)$$

Since $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, we get:

$$\frac{\partial F(t)}{\partial x_i} = \frac{x_i}{\beta^2} X_0 \left(t - \frac{r}{\beta} \right) \quad (14)$$

Thus, the dominant term in $\nabla \times \psi$ at $r \neq 0$ is:

$$\nabla \times \psi \approx \frac{1}{4\pi\rho\beta^2} \left(\delta_{i1} - \frac{x_i x_1}{r^2} \right) X_0 \left(t - \frac{r}{\beta} \right) \hat{x}_i \quad (15)$$

Final Result for $\nabla \times \psi$

$$\boxed{\nabla \times \psi = \frac{1}{4\pi\rho\beta^2} \left(\delta_{i1} - \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_1} \right) X_0 \left(t - \frac{r}{\beta} \right) \hat{x}_i}$$

Combine $\nabla\phi$ and $\nabla \times \psi$

The total displacement field is:

$$u_i(x, t) = \frac{\partial\phi}{\partial x_i} + (\nabla \times \psi)_i$$

Substitute the expressions for $\frac{\partial\phi}{\partial x_i}$ and $(\nabla \times \psi)_i$:

$$\begin{aligned} u_i(x, t) = & -\frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_0^{r/\alpha} \tau X_0(t - \tau) d\tau \\ & + \frac{1}{4\pi\rho\alpha^2} \left(\frac{x_i x_1}{r^3} \right) X_0 \left(t - \frac{r}{\alpha} \right) \\ & + \frac{1}{4\pi\rho\beta^2} \left(\delta_{i1} - \frac{x_i x_1}{r^2} \right) X_0 \left(t - \frac{r}{\beta} \right) \end{aligned}$$

The integral term can be split into two parts to highlight the contributions from P-waves (α) and S-waves (β):

$$\int_0^{r/\alpha} \tau X_0(t - \tau) d\tau = \int_0^{r/\beta} \tau X_0(t - \tau) d\tau - \int_{r/\alpha}^{r/\beta} \tau X_0(t - \tau) d\tau$$

Thus, $u_i(x, t)$ becomes:

$$\begin{aligned} u_i(x, t) = & -\frac{1}{4\pi\rho} \frac{\partial^2}{\partial x_i \partial x_1} \left(\frac{1}{r} \right) \int_0^{r/\alpha} \tau X_0(t - \tau) d\tau \\ & + \frac{1}{4\pi\rho\alpha^2} \frac{x_i x_1}{r^3} X_0 \left(t - \frac{r}{\alpha} \right) \\ & + \frac{1}{4\pi\rho} \frac{\partial^2}{\partial x_i \partial x_1} \left(\frac{1}{r} \right) \int_0^{r/\beta} \tau X_0(t - \tau) d\tau \\ & + \frac{1}{4\pi\rho\beta^2} \left(\delta_{i1} - \frac{x_i x_1}{r^2} \right) X_0 \left(t - \frac{r}{\beta} \right) \end{aligned}$$

Note on terms:

1. First line: Near-field P-wave contribution (potential term)
2. Second line: Far-field P-wave radiation pattern
3. Third line: Near-field S-wave contribution (curl term)

4. Fourth line: Far-field S-wave radiation pattern

- Corrected integral upper limit from τ/α to r/α for dimensional consistency

Combined Integral Form of Displacement Field

The displacement field can be expressed more compactly by combining the integral terms:

$$\begin{aligned} u_i(x, t) = & \frac{1}{4\pi\rho} \frac{\partial^2}{\partial x_i \partial x_1} \left(\frac{1}{r} \right) \left(\int_0^{r/\beta} - \int_0^{r/\alpha} \right) \tau X_0(t - \tau) d\tau \\ & + \frac{1}{4\pi\rho\alpha^2 r} \left(\frac{x_i x_1}{r^2} \right) X_0 \left(t - \frac{r}{\alpha} \right) \\ & + \frac{1}{4\pi\rho\beta^2 r} \left(\delta_{i1} - \frac{x_i x_1}{r^2} \right) X_0 \left(t - \frac{r}{\beta} \right) \end{aligned}$$

The total displacement field is:

$$\begin{aligned} u_i(\mathbf{x}, t) = & \frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_{r/\alpha}^{r/\beta} \tau X_0(t - \tau) d\tau \\ & + \frac{1}{4\pi\rho\alpha^2 r} \left(\frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_1} \right) X_0 \left(t - \frac{r}{\alpha} \right) \\ & + \frac{1}{4\pi\rho\beta^2 r} \left(\delta_{i1} - \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_1} \right) X_0 \left(t - \frac{r}{\beta} \right). \end{aligned}$$

The total displacement can also be written as:

$$\begin{aligned} u_i(\mathbf{x}, t) = & \frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_{r/\alpha}^{r/\beta} \tau X_0(t - \tau) d\tau \\ & + \frac{1}{4\pi\rho\alpha^2 r} \left(\frac{x_i x_1}{r^2} \right) X_0 \left(t - \frac{r}{\alpha} \right) \\ & + \frac{1}{4\pi\rho\beta^2 r} \left(\delta_{i1} - \frac{x_i x_1}{r^2} \right) X_0 \left(t - \frac{r}{\beta} \right) \end{aligned}$$

where:

- α is the P-wave velocity
- β is the S-wave velocity
- $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the radial distance
- $X_0(t)$ is the source time function

- δ_{i1} is the Kronecker delta