

Analytical Solution to 2D SH ViscoElastic Wave Equation

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To describe a viscoelastic medium, we modify the stress-strain relation because the conservation of momentum is independent of the material behavior. The viscoelastic stress-strain relation can be described by generalizing the purely elastic case by introducing frequency-dependent complex moduli (or quality factor, Q) or time-domain convolution integrals described by the Boltzmann superposition and causality principle:

$$\sigma(t) = \int_{-\infty}^t \Psi(t-t') \dot{\epsilon}(t') dt'$$

$\Psi(t)$ is the relaxation function.

1 Governing Wave Equation

2D SH Wave:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} + f_y \quad (1)$$

$$\sigma_{yx} = \mu \frac{\partial u_y}{\partial x} \quad (2)$$

$$\sigma_{yz} = \mu \frac{\partial u_y}{\partial z}$$

1D SH Wave:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + f_y,$$

- ρ : Mass density (kg/m³)

- $u_y(x, t)$: Transverse displacement field (m)
- σ_{yx} : Shear stress component (Pa)
- f_y : External force density (N/m³)
- x : Spatial coordinate along propagation direction
- t : Time coordinate

ViscoElastic Wave Equation

Kevin-Voigt Model:

which adds a velocity-dependent damping term to the stress-strain relation. The modified stress components become:

$$\sigma_{yx} = \mu \frac{\partial u_y}{\partial x} + \eta \frac{\partial}{\partial t} \left(\frac{\partial u_y}{\partial x} \right), \quad (3)$$

$$\sigma_{yz} = \mu \frac{\partial u_y}{\partial z} + \eta \frac{\partial}{\partial t} \left(\frac{\partial u_y}{\partial z} \right), \quad (4)$$

where η is the viscosity coefficient. Substituting these into the equation of motion:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \mu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) + \eta \frac{\partial}{\partial t} \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) + A \delta(x) \delta(z) \delta(t). \quad (5)$$

Let $\beta = \sqrt{\mu/\rho}$ and introduce the damping parameter $\alpha = \eta/(2\rho)$.

The equation becomes:

$$\boxed{\frac{\partial^2 u_y}{\partial t^2} = \beta^2 \nabla^2 u_y + 2\alpha \frac{\partial}{\partial t} \nabla^2 u_y + \frac{A}{\rho} \delta(x) \delta(z) \delta(t)}, \quad (6)$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

we seek to find the causal Green's function $u_y(x, z, t)$ for the wave equation in (6) i.e the response to the source $A \delta(x) \delta(z) \delta(t)$.

We define the Green's function $G(\mathbf{x}, \mathbf{z}, t)$ as the solution to:

$$\frac{\partial^2 G}{\partial t^2} - \beta^2 \nabla^2 G - 2\alpha \frac{\partial}{\partial t} \nabla^2 G = \frac{A}{\rho} \delta(x) \delta(z) \delta(t). \quad (7)$$

Spatial Fourier Transform

Apply the 2D spatial Fourier transform in x, z . Let

$$\tilde{G}(k_x, k_z, t) = \iint_{-\infty}^{\infty} G(x, z, t) e^{-i(k_x x + k_z z)} dx dz,$$

with inverse

$$G(x, z, t) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{G}(k_x, k_z, t) e^{i(k_x x + k_z z)} dk_x dk_z$$

Under this transform, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} = -k^2$ with $k^2 = k_x^2 + k_z^2$, and $\delta(x)\delta(z) = 1$.

The wave equation in \mathbf{k} -space becomes

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + \beta^2 k^2 \tilde{G} - 2\alpha \frac{\partial}{\partial t} (-k^2) \tilde{G} = \frac{A}{\rho} \delta(t),$$

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + 2\alpha k^2 \frac{\partial}{\partial t} \tilde{G} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho} \delta(t),$$

This is a **damped harmonic oscillator equation** with a Dirac delta forcing.

Solve the Transformed Equation

Homogeneous Solution (for $RHS = 0$)

The equation becomes:

$$\ddot{\tilde{G}} + 2\alpha k^2 \dot{\tilde{G}} + \beta^2 k^2 \tilde{G} = 0.$$

Assume solutions of the form $\tilde{G} \propto e^{st}$, leading to the characteristic equation:

$$s^2 + 2\alpha k^2 s + \beta^2 k^2 = 0.$$

Characteristic Roots and Damping Cases

Solving for the roots r of the characteristic equation:

$$s = -\alpha k^2 \pm \sqrt{\alpha^2 k^4 - \beta^2 k^2}$$

Three Damping Cases:

- **Overdamped:** $\alpha^2 k^2 > \beta^2$ (exponential decay, no oscillations)
- **Critically damped:** $\alpha^2 k^2 = \beta^2$ (fastest decay without oscillation)
- **Underdamped:** $\alpha^2 k^2 < \beta^2$ (oscillatory decay)

Underdamped Solution (Most Common in Wave Physics)

Define the natural frequency and damping coefficient:

$$\omega_0^2 = \beta^2 k^2, \quad \gamma = \alpha k^2$$

The roots become:

$$s = -\gamma \pm i\omega_d, \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2}$$

The homogeneous solution is:

$$\tilde{G}_h(t) = e^{-\gamma t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$$

where ω_d is the damped natural frequency.

Particular Solution (Delta Forcing)

Now include the **impulse forcing term** $\delta(t)$. This introduces a discontinuity in $\frac{d\tilde{G}}{dt}$ at $t = 0$.

The total solution is:

$$\tilde{G}(t) = \tilde{G}_h(t), \quad t > 0$$

Initial Conditions

- $\tilde{G}(0) = 0$ (continuity of displacement)
- Jump in derivative from delta function: Integrate the ODE around $t = 0$:

$$\int_{-\epsilon}^{+\epsilon} \left(\frac{d^2 \tilde{G}}{dt^2} + 2\alpha k^2 \frac{d\tilde{G}}{dt} + \beta^2 k^2 \tilde{G} \right) dt = \int_{-\epsilon}^{+\epsilon} \frac{A}{\rho} \delta(t) dt$$

Since $\tilde{G}(t)$ and $\frac{d\tilde{G}}{dt}$ are finite at $t = 0$, only $\frac{d^2 \tilde{G}}{dt^2}$ contributes:

$$\left. \frac{d\tilde{G}}{dt} \right|_{0^+} - \left. \frac{d\tilde{G}}{dt} \right|_{0^-} = \frac{A}{\rho}$$

Assuming zero initial velocity before impulse: $\frac{d\tilde{G}}{dt}(0^-) = 0$, so:

$$\left. \frac{d\tilde{G}}{dt} \right|_{0^+} = \frac{A}{\rho}$$

Final Solution

We apply the initial conditions:

- $\tilde{G}(0) = 0$
- $\frac{d\tilde{G}}{dt}(0^+) = \frac{A}{\rho}$

Recall the general solution:

$$\tilde{G}(t) = e^{-\gamma t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$$

- Apply $\tilde{G}(0) = 0 \Rightarrow C_1 = 0$
- Compute the derivative:

$$\frac{d\tilde{G}}{dt} = e^{-\gamma t} C_2 (-\gamma \sin(\omega_d t) + \omega_d \cos(\omega_d t))$$

- Evaluate at $t = 0$:

$$\frac{d\tilde{G}}{dt}(0) = C_2 \omega_d = \frac{A}{\rho} \Rightarrow C_2 = \frac{A}{\rho \omega_d}$$

Final Solution in Wavenumber Domain

$$\tilde{G}(k, t) = \frac{A}{\rho\omega_d} e^{-\gamma t} \sin(\omega_d t), \quad t \geq 0,$$

where:

- $\gamma = \alpha k^2$ (damping coefficient)
- $\omega_d = \sqrt{\beta^2 k^2 - \alpha^2 k^4}$ (damped frequency)

The solution can be written compactly using the Heaviside step function $H(t)$:

$$\tilde{G}(k_x, k_z, t) = \frac{A}{\rho\omega_d} e^{-\gamma t} \sin(\omega_d t) H(t).$$

$$\tilde{G}(t) = \frac{A}{\rho\sqrt{\beta^2 k^2 - \alpha^2 k^4}} e^{-\alpha k^2 t} \sin\left(\sqrt{\beta^2 k^2 - \alpha^2 k^4} t\right) H(t)$$

Key Observations

1. **Damping Effect:** The term $e^{-\alpha k^2 t}$ shows **wavenumber-dependent damping** (higher k modes decay faster).
2. **Dispersion:** Frequency ω_d depends on k , indicating dispersive waves.
3. **Singularity at $t = 0$:** Matches the delta source behavior.

Inverse Transform and Bessel Integral

Green's Function Solution

The wavenumber-domain solution is:

$$\tilde{G}(k_x, k_z, t) = \frac{A}{\rho\sqrt{\beta^2 k^2 - \alpha^2 k^4}} e^{-\alpha k^2 t} \sin\left(t\sqrt{\beta^2 k^2 - \alpha^2 k^4}\right) H(t)$$

where $k^2 = k_x^2 + k_z^2$.

Inverse Fourier Transform

To obtain the space-time solution, we compute the 2D inverse Fourier transform:

$$G(x, z, t) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{G}(k_x, k_z, t) e^{i(k_x x + k_z z)} dk_x dk_z$$

Step 1: Use Radial Symmetry

Let $r = \sqrt{x^2 + z^2}$. Since $\tilde{G}(k_x, k_z, t)$ depends on $k = \sqrt{k_x^2 + k_z^2}$, we switch to polar coordinates in k -space:

Let:

- $k_x = k \cos \theta$
- $k_z = k \sin \theta$
- $dk_x dk_z = k dk d\theta$

Also:

$$k_x x + k_z z = kr \cos(\theta - \phi)$$

where $\phi = \tan^{-1}(z/x)$.

So:

$$G(x, z, t) = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} \tilde{G}(k_x, k_z, t) e^{ikr \cos(\theta - \phi)} k d\theta dk$$

Simplify the angular integral using the Bessel function. Use the identity:

$$\int_0^{2\pi} e^{ikr \cos(\theta - \phi)} d\theta = 2\pi J_0(kr)$$

Thus:

$$G(x, z, t) = \frac{1}{2\pi} \int_0^\infty \tilde{G}(k_x, k_z, t) k J_0(kr) dk$$

Step 2: Plug in Expression for $\tilde{G}(k, t)$

Substitute the expression for $G(k, t)$:

$$G(x, z, t) = \frac{A}{2\pi\rho} \int_0^\infty H(t) \frac{k J_0(kr)}{\sqrt{\beta^2 k^2 - \alpha^2 k^4}} e^{-\alpha k^2 t} \sin\left(t\sqrt{\beta^2 k^2 - \alpha^2 k^4}\right) dk \quad (8)$$

$$G(x, z, t) = \frac{AH(t)}{2\pi\rho} \int_0^\infty \frac{e^{-\alpha k^2 t} \sin\left(t\sqrt{\beta^2 k^2 - \alpha^2 k^4}\right)}{\sqrt{\beta^2 - \alpha^2 k^2}} J_0(kr) dk. \quad (9)$$

where:

$$\alpha = \frac{\eta}{2\rho}$$

This is the exact solution for the Green's function in the time-space domain for a damped 2D SH wave equation.

Small Damping Approximation: $\alpha \rightarrow 0$ (Weakly Damped $\alpha \ll \beta$)

When damping is very small, the solution to the underdamped system (equation 9) becomes nearly that of a pure 2D SH wave equation. This happens when the viscosity, η , approaches zero, such that α also approaches zero.
For small α , then:

$$\sqrt{\beta^2 - \alpha^2 k^2} \approx \beta$$

$$\sin\left(t\sqrt{\beta^2 k^2 - \alpha^2 k^4}\right) \approx \sin(\beta kt)$$

The exponential becomes:

$$e^{-\alpha k^2 t} \approx 1$$

This leads to:

$$G(x, z, t) \approx \frac{AH(t)}{2\pi\rho} \int_0^\infty \frac{\sin(\beta kt)}{\beta k} \cdot k J_0(kR) dk = \frac{AH(t)}{2\pi\rho\beta} \int_0^\infty \sin(\beta kt) J_0(kr) dk$$

This is the pure 2D SH Wave Solution.

Critically Damped Solution of Second-Order ODE with Delta Source

We start from the non-homogeneous ODE:

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + 2\alpha k^2 \frac{\partial \tilde{G}}{\partial t} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho} \delta(t)$$

Let:

$$\gamma = \alpha k^2, \quad \omega_0^2 = \beta^2 k^2$$

Taking the Laplace transform with zero initial conditions:

$$s^2 \tilde{G}(s) + 2\gamma s \tilde{G}(s) + \omega_0^2 \tilde{G}(s) = \frac{A}{\rho}$$

$$\tilde{G}(s) = \frac{A/\rho}{s^2 + 2\gamma s + \omega_0^2}$$

—
Critically damped case: $\gamma^2 = \omega_0^2 \Rightarrow \alpha^2 k^4 = \beta^2 k^2$

This implies the characteristic equation has a repeated real root:

$$s^2 + 2\gamma s + \gamma^2 = (s + \gamma)^2$$

So:

$$\tilde{G}(s) = \frac{A/\rho}{(s + \gamma)^2}$$

Using the Laplace transform identity:

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s + a)^2} \right\} = t e^{-at}$$

We get:

$$\tilde{G}(t) = \frac{A}{\rho} \cdot t e^{-\gamma t} \cdot H(t)$$

Substituting back $\gamma = \alpha k^2$:

$$\boxed{\tilde{G}(t) = \frac{A}{\rho} \cdot t e^{-\alpha k^2 t} \cdot H(t)}$$

This solution exhibits the behaviour of a critically damped system: no oscillation and the fastest return to equilibrium without overshooting.

Overdamped Solution of Second-Order ODE with Delta Source

We are given the non-homogeneous ODE:

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + 2\alpha k^2 \frac{\partial \tilde{G}}{\partial t} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho} \delta(t)$$

Let:

$$\gamma = \alpha k^2, \quad \omega_0^2 = \beta^2 k^2$$

Take the Laplace transform of both sides (assuming zero initial conditions):

$$s^2 \tilde{G}(s) + 2\gamma s \tilde{G}(s) + \omega_0^2 \tilde{G}(s) = \frac{A}{\rho}$$

$$\tilde{G}(s) = \frac{A/\rho}{s^2 + 2\gamma s + \omega_0^2}$$

—
Overdamped case: $\gamma^2 > \omega_0^2 \Rightarrow \alpha^2 k^4 > \beta^2 k^2$

Define:

$$\lambda = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{\alpha^2 k^4 - \beta^2 k^2}$$

Then the characteristic roots are:

$$s_1 = -\gamma + \lambda, \quad s_2 = -\gamma - \lambda$$

So the Laplace domain solution becomes:

$$\tilde{G}(s) = \frac{A/\rho}{(s - s_1)(s - s_2)} = \frac{A}{2\lambda\rho} \left(\frac{1}{s - s_1} - \frac{1}{s - s_2} \right)$$

Take the inverse Laplace transform:

$$\tilde{G}(t) = \frac{A}{2\lambda\rho} (e^{s_1 t} - e^{s_2 t}) H(t)$$

Substitute s_1 and s_2 explicitly:

$$\tilde{G}(t) = \frac{A}{2\rho\sqrt{\alpha^2 k^4 - \beta^2 k^2}} \left[e^{(-\alpha k^2 + \sqrt{\alpha^2 k^4 - \beta^2 k^2})t} - e^{(-\alpha k^2 - \sqrt{\alpha^2 k^4 - \beta^2 k^2})t} \right] H(t)$$

This solution is real and decaying (no oscillations), characteristic of overdamped systems.

Underdamped Solution of Second-Order ODE with Delta Source

Consider the non-homogeneous ODE:

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + 2\alpha k^2 \frac{\partial \tilde{G}}{\partial t} + \beta^2 k^2 \tilde{G} = \frac{A}{\rho} \delta(t)$$

Define:

$$\gamma = \alpha k^2, \quad \omega_0^2 = \beta^2 k^2$$

Step-by-step Solution Using Laplace Transform

We apply the Laplace transform $\mathcal{L}\{\cdot\}$ with respect to time t , using the standard transforms:

- $\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s^2 \tilde{f}(s) - sf(0) - f'(0),$
- $\mathcal{L}\left\{\frac{df}{dt}\right\} = s \tilde{f}(s) - f(0),$
- $\mathcal{L}\{\delta(t)\} = 1.$

Assume zero initial conditions (i.e., $\tilde{G}(0) = 0, \tilde{G}'(0) = 0$). Taking the Laplace transform of both sides:

$$\mathcal{L}\left\{\frac{\partial^2 \tilde{G}}{\partial t^2}\right\} + 2\alpha k^2 \mathcal{L}\left\{\frac{\partial \tilde{G}}{\partial t}\right\} + \beta^2 k^2 \mathcal{L}\{\tilde{G}\} = \frac{A}{\rho}$$

Laplace transforming both sides (assuming zero initial conditions) gives:

$$s^2 \tilde{G}(s) + 2\gamma s \tilde{G}(s) + \omega_0^2 \tilde{G}(s) = \frac{A}{\rho} \Rightarrow \tilde{G}(s) = \frac{A/\rho}{s^2 + 2\gamma s + \omega_0^2}$$

Underdamped case: $\gamma^2 < \omega_0^2 \Rightarrow \alpha^2 k^4 < \beta^2 k^2$

Let:

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\beta^2 k^2 - \alpha^2 k^4}$$

Then the denominator becomes:

$$s^2 + 2\gamma s + \omega_0^2 = (s + \gamma)^2 + \omega_d^2$$

Underdamped Solution of Second-Order ODE with Delta Source – Contd

So:

$$\tilde{G}(s) = \frac{A/\rho}{(s + \gamma)^2 + \omega_d^2}$$

Using the Laplace inverse identity:

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s + a)^2 + \omega^2} \right\} = \frac{1}{\omega} e^{-at} \sin(\omega t)$$

We obtain:

$$\tilde{G}(t) = \frac{A}{\rho \omega_d} e^{-\gamma t} \sin(\omega_d t) \cdot H(t)$$

Substitute $\gamma = \alpha k^2$, $\omega_d = \sqrt{\beta^2 k^2 - \alpha^2 k^4}$:

$$\tilde{G}(t) = \frac{A}{\rho \sqrt{\beta^2 k^2 - \alpha^2 k^4}} \cdot e^{-\alpha k^2 t} \sin \left(\sqrt{\beta^2 k^2 - \alpha^2 k^4} t \right) H(t)$$

This solution captures the oscillatory decay of the underdamped system — a sine wave modulated by an exponential decay.