2010年、交替方向乘子法（ADMM）求解图像复原ROF-TV模型

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目 录 CONTENTS

1、符号说明

2. Basic notation. Without loss of generality, we represent a grayscale image as an $N \times N$ matrix. The Euclidean space $\mathbb{R}^{N\times N}$ is denoted as V. The discrete gradient operator is a mapping $\nabla : V \to Q$, where $Q = V \times V$. For $u \in V$, ∇u is given by

$$(\nabla u)_{i,j} = ((\mathring{D}_x^+ u)_{i,j}, (\mathring{D}_y^+ u)_{i,j}),$$

with

$$(\mathring{D}_x^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j}, & 1 \leq j \leq N-1, \\ u_{i,1} - u_{i,N}, & j = N, \end{cases}$$
$$(\mathring{D}_y^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j}, & 1 \leq i \leq N-1, \\ u_{1,j} - u_{N,j}, & i = N, \end{cases}$$

We denote the usual inner product and Euclidean norm of V as $(\cdot,\cdot)_V$ and $\|\cdot\|_V$, respectively. We also equip the space Q with inner product $(\cdot,\cdot)_Q$ and norm $\|\cdot\|_Q$, which are defined as follows. For $p = (p^1, p^2) \in Q$ and $q = (q^1, q^2) \in Q$,

$$(p, q)_Q = (p^1, q^1)_V + (p^2, q^2)_V$$

and

$$\|p\|_Q = \sqrt{(p, p)_Q}.$$

In addition, we mention that at each pixel (i, j),

$$|p_{i,j}| = |(p^1_{i,j}, p^2_{i,j})| = \sqrt{(p^1_{i,j})^2 + (p^2_{i,j})^2},$$

2、图像复原ROF-TV模型

2.1、图像复原问题

$$f = Ku + n, \tag{3.1}$$

图像恢复的目标是从f恢复u，由于问题通常是病态的，我们不能直接从(3.1)解出u。

应考虑解的正则化，其中最基本也是最成功的图像正则化模型是ROF模型：

2.2、ROF-TV模型

$$\min_{u \in V} \left\{ F_{\text{rof}}(u) = R_{\text{rof}}(\nabla u) + \frac{\alpha}{2} \|Ku - f\|_V^2 \right\}, \tag{3.2}$$

where

$$R_{\text{rof}}(\nabla u) = \text{TV}(u) = \sum_{1 \le i,j \le N} |(\nabla u)_{i,j}| \tag{3.3}$$

3、ADMM求解TV-L2

3.1、变量分离

求解ROF恢复模型(3.2)的困难在于TV项的不可微性，为此，对于∇u引入一个辅助变量p∈Q，我们可以将不可微项和2-范数平方项的计算分离。令$\mathrm{p}=\nabla u$，模型(3.2)等价于

$$\min_{u\in V,p\in Q}\left\{G_{\mathrm{rof}}(u,p)=R_{\mathrm{rof}}(p)+\frac{\alpha}{2}\|Ku-f\|_V^2\right\} \quad \text{s.t.} \quad p=\nabla u, \tag{3.5}$$

3.2、ADMM求解

对凸优化问题（3.5），ADMM求解步骤：

（1）、构建增广拉格朗日函数

$$\mathcal{L}_{rof}(v,q;\lambda)=R_{rof}(q)+\frac{\alpha}{2}\|Kv-f\|_V^2+(\lambda,q-\nabla v)_Q+\frac{\gamma}{2}\|q-\nabla v\|_Q^2$$

3、ADMM求解TV-L2

3.2、ADMM求解

（2）、迭代计算

$$\mathcal{L}_{rof}(v,q;\lambda) = R_{rof}(q) + \frac{\alpha}{2}\|Kv-f\|_V^2 + (\lambda, q-\nabla v)_Q + \frac{\gamma}{2}\|q-\nabla v\|_Q^2$$

固定拉格朗日乘子 λ^k.

任给定 q^k，更新 v.

$$v^{k+1} = \arg\min_{v\in V}\left\{\frac{\alpha}{2}\|Kv-f\|_V^2 - (\lambda^k, \nabla v)_Q + \frac{\gamma}{2}\|q^k - \nabla v\|_Q^2\right\}.$$

任给定 v^k，更新 q

$$q^{k+1} = \arg\min_{q\in Q}\left\{R_{rof}(q) + (\lambda^k, q)_Q + \frac{\gamma}{2}\|q-\nabla v^k\|_Q^2\right\}.$$

更新拉格朗日乘子

$$\lambda^{k+1} = \lambda^k + \gamma(q^{k+1} - \nabla v^{k+1}).$$

3、ADMM求解TV-L2

3.2、ADMM求解

$$\mathscr{L}_{\text{rof}}(v,q;\mu) = R_{\text{rof}}(q) + \frac{\alpha}{2}\|Kv - f\|_V^2 + (\mu, q - \nabla v)_Q + \frac{r}{2}\|q - \nabla v\|_Q^2, \tag{4.1}$$

where $\mu \in Q$ is the Lagrange multiplier and r is a positive constant. For the augmented Lagrangian method for (3.5), we consider the following saddle-point problem:

$$\begin{array}{ll} \text{Find} & (u,p;\lambda) \in V \times Q \times Q \\ \text{s.t.} & \mathscr{L}_{\text{rof}}(u,p;\mu) \le \mathscr{L}_{\text{rof}}(u,p;\lambda) \le \mathscr{L}_{\text{rof}}(v,q;\lambda) \ \forall (v,q;\mu) \in V \times Q \times Q. \end{array} \tag{4.2}$$

Algorithm 4.1. Augmented Lagrangian method for the ROF model.

1. *Initialization:* $\lambda^0 = 0$;
2. *For* $k = 0, 1, 2, \ldots$: compute (u^k, p^k) as an (approximate) minimizer of the augmented Lagrangian functional with the Lagrange multiplier λ^k, i.e.,

$$(u^k, p^k) \approx \arg \min_{(v,q)\in V\times Q} \mathscr{L}_{\text{rof}}(v,q;\lambda^k), \tag{4.7}$$

where $\mathscr{L}_{\text{rof}}(v,q;\lambda^k)$ is defined in (4.1); update

$$\lambda^{k+1} = \lambda^k + r(p^k - \nabla u^k). \tag{4.8}$$

3、ADMM求解TV-L2

3.3、子问题求解

We separate (4.7) into the following two subproblems:

(4.9)
$$\min_{v \in V} \frac{\alpha}{2}\|Kv - f\|_V^2 - (\lambda^k, \nabla v)_Q + \frac{r}{2}\|q - \nabla v\|_Q^2$$

for a given q, and

(4.10)
$$\min_{q \in Q} R_{\mathrm{rof}}(q) + (\lambda^k, q)_Q + \frac{r}{2}\|q - \nabla v\|_Q^2$$

for a given v.

3、ADMM求解TV-L2

3.3、子问题求解

对v子问题（4.9），可用FFT求解

$$\mathcal{F}(v) = \frac{\alpha \mathcal{F}(K^*)\mathcal{F}(f) + \mathcal{F}(\nabla^*)[\mathcal{F}(\lambda^k) + \gamma \mathcal{F}(q)]}{\alpha \mathcal{F}(K^*)\mathcal{F}(K) + \gamma \mathcal{F}(\nabla^* \nabla)}$$

$$v = \mathcal{F}^{-1}[\quad\quad\quad]$$

若令 $\nabla^* = -\mathrm{div}$，$\nabla^*\nabla = -\Delta$，有：

$$v = \mathcal{F}^{-1}\left(\frac{\alpha\mathcal{F}(K^*)\mathcal{F}(f) - \mathcal{F}(\mathring{D}_x^-)\mathcal{F}((\lambda^1)^k + rq^1) - \mathcal{F}(\mathring{D}_y^-)\mathcal{F}((\lambda^2)^k + rq^2)}{\alpha\mathcal{F}(K^*)\mathcal{F}(K) - r\mathcal{F}(\triangle)}\right), \tag{4.11}$$

where $\lambda^k = ((\lambda^1)^k, (\lambda^2)^k)$ and $q = (q^1, q^2)$; Fourier transforms of operators such as $K, \mathring{D}_x^-, \mathring{D}_y^-$,

3、ADMM求解TV-L2

3.3、子问题求解

对q子问题（4.10）

For (4.10), we actually have the following closed form solution [8, 40, 39]:

$$q_{i,j} = \begin{cases} \left(1 - \frac{1}{r}\frac{1}{|w_{i,j}|}\right) w_{i,j}, & |w_{i,j}| > \frac{1}{r}, \\ 0, & |w_{i,j}| \leq \frac{1}{r}, \end{cases} \tag{4.12}$$

where

$$w = \nabla v - \frac{\lambda^k}{r}. \tag{4.13}$$

4、ADMM求解TV-L1

4.1、图像复原ROF模型

$$\min_{u\in V}\{E(u)=R(\nabla u)+F(Ku)\}, \tag{2}$$

where

$$R(\nabla u)=\mathrm{TV}(u)=\sum_{1\le i,j\le N}|(\nabla u)_{i,j}|, \tag{3}$$

不同噪声的保真项形式:

1. Gaussian noise:
$$F(Ku)=\frac{\alpha}{2}\|Ku-f\|^2,$$
2. Impulsive noise:
$$F(Ku)=\alpha\|Ku-f\|_{L^1},$$
3. Poisson noise (assuming $f_{i,j}>0, \forall i,j$, as in [36]):
$$F(Ku)=\begin{cases}\alpha\sum\limits_{1\le i,j\le N}((Ku)_{i,j}-f_{i,j}\log(Ku)_{i,j}), & (Ku)_{i,j}>0,\forall\ i,j\\ +\infty, & \text{otherwise}\end{cases},$$

4、ADMM求解TV-L1

$$\min_{u\in V}\{F_{rof}(u)=R_{rof}(\nabla u)+\alpha\|Ku-f\|_1\} \quad (1).$$

若只引入辅助变量 $p=\nabla u$.

$$\min_{u\in V, p\in Q}\{G_{rof}(u,p)=R_{rof}(p)+\alpha\|Ku-f\|_1\} \quad (2).$$

$$s.t.\quad p=\nabla u.$$

其 ALF:

$$\mathcal{L}_{rof}(u,p;\lambda)=R_{rof}(p)+\alpha\|Ku-f\|_1+(\lambda,p-\nabla u)+\frac{r}{2}\|p-\nabla u\|_Q^2 \quad (3)$$

u-子问题

$$\min_{u\in V}\{\underbrace{\alpha\|Ku-f\|_1}_{\text{不可微}}-(\lambda,\nabla u)_Q+\frac{r}{2}\|p-\nabla u\|_Q^2\}$$

4、ADMM求解TV-L1

$$\min_{u\in V}\{F_{rof}(u)=R_{rof}(\nabla u)+\alpha\|Ku-f\|_1\} \qquad (1)$$

引入辅助变量 $p=\nabla u\in Q$, $z=Ku-f\in V$.

转化为优化问题:

$$\min_{u\in V,\, p\in Q,\, z\in V}\{G_{rof}(u,p,z)=R_{rof}(p)+\alpha\|z\|_1\} \qquad (2)$$

$$\text{s.t.} \quad p=\nabla u, \quad z=Ku-f$$

(2)的 ALF:

$$\mathcal{L}_{rof}(u,p,z;\lambda_p,\lambda_z)=R_{rof}(p)+\alpha\|z\|_1+(\lambda_p,p-\nabla u)+(\lambda_z,z-(Ku-f))+\frac{r_p}{2}\|p-\nabla u\|^2+\frac{r_z}{2}\|z-(Ku-f)\|^2 \qquad (3)$$

4、ADMM求解TV-L1

$$\mathcal{L}_{rof}(u,p,z;\lambda_p,\lambda_z)=R_{rof}(p)+\alpha\|z\|_1+(\lambda_p,p-\nabla u)$$
$$+(\lambda_z,z-(Ku-f))+\frac{r_p}{2}\|p-\nabla u\|^2+\frac{r_z}{2}\|z-(Ku-f)\|^2 \quad (3)$$

(3)转化为如下3个子问题.

u-子问题：给定 p,z.

$$\min_{u\in V}\{(\lambda_p^k,-\nabla u)+(\lambda_z^k,-(Ku-f))+\frac{r_p}{2}\|p-\nabla u\|^2\} \quad (4).$$

p-子问题：给定 u,z

$$\min_{p\in Q}\{R_{rof}(p)+(\lambda_p^k,p)+\frac{r_p}{2}\|p-\nabla u\|^2\} \quad (5)$$

z-子问题：给定 u,p

$$\min_{z\in V}\{\alpha\|z\|_1+(\lambda_z^k,z)+\frac{r_z}{2}\|z-(Ku-f)\|^2\} \quad (6)$$

4、ADMM求解TV-L1

u-子问题，由FFT可求得

$$\mathcal{F}(u)=\frac{\mathcal{F}(K^*)[\mathcal{F}(\lambda_z^k)+\gamma_z\mathcal{F}(z)+\gamma_z\mathcal{F}(f)]+\mathcal{F}(\nabla^*)[\mathcal{F}(\lambda_p^k+\gamma_p\mathcal{F}(p)]}{\gamma_z\mathcal{F}(K^*)\mathcal{F}(K)+\gamma_p\mathcal{F}(\nabla^*\nabla)}.$$

p-子问题有解

$$p_{i,j}=\max\left(0,1-\frac{1}{\gamma_p|w_{i,j}|}\right)w_{i,j}$$

$$w=\nabla u-\frac{\lambda_p^k}{\gamma_p}\in Q.$$

z-子问题有解.

$$z_{i,j}=\max\left(0,1-\frac{\alpha}{\gamma_z|H_{i,j}|}\right)H_{i,j}.$$

$$H=Ku-f-\frac{\lambda_z^k}{\gamma_z}\in V.$$

4、ADMM求解TV-L1

4.1.2. *Solving the* $p-$*sub problem* (12). Similarly to [7, 56, 54, 57], (12) has the following closed form solution

$$p_{i,j} = \max(0, 1 - \frac{1}{r_p|\mathbf{w}_{i,j}|})\mathbf{w}_{i,j}, \quad \forall\ i, j \tag{15}$$

where

$$\mathbf{w} = \nabla u - \frac{\lambda_p^k}{r_p} \in Q. \tag{16}$$

Here we would like to provide a geometric interpretation of the formulae (15), which is different from the view point of sub-differential [56]. According to the definition of $R(p)$ and $\|\cdot\|_Q$, we rewrite the problem (12) as

$$\min_{p\in Q}\{\sum_{1\le i,j\le N}|p_{i,j}| + \frac{r_p}{2}\sum_{1\le i,j\le N}|p_{i,j} - (\nabla u - \frac{\lambda_p^k}{r_p})_{i,j}|^2 + \text{Constant}\}.$$

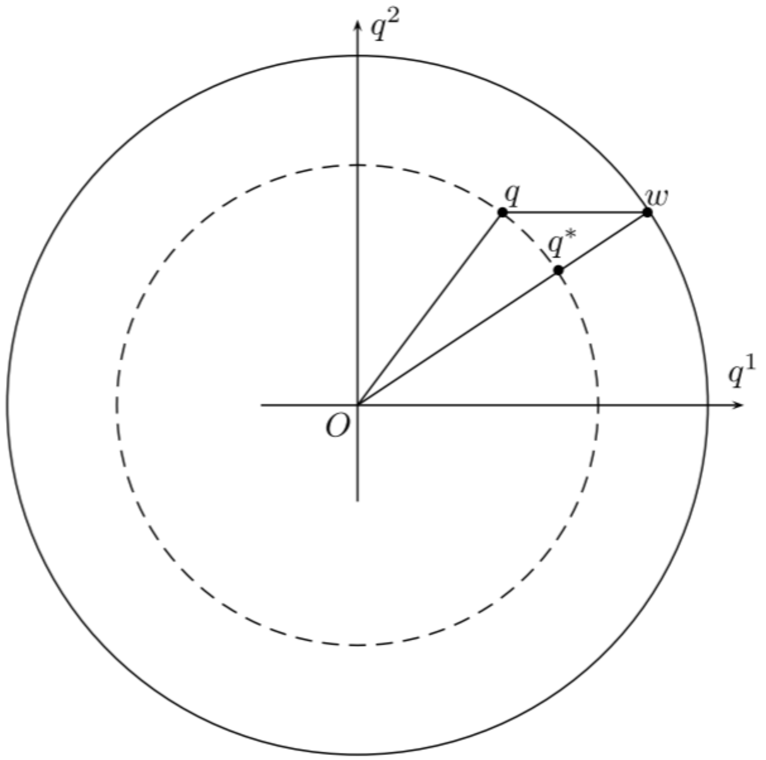
As one can see, the above problem is decomposable and at each pixel (i, j), the problem takes the form as follows

$$\min_{q\in\mathbb{R}^2}\{|q| + \frac{r_p}{2}|q - w|^2\}, \tag{17}$$

where $w \in \mathbb{R}^2$; see Fig. 1.

4、ADMM求解TV-L1

首先，可以验证(和想象)潜在的最小值应该位于实心圆的内部。通过构造对称点，我们可以进一步证明，潜在的最小值应该位于与w相同的象限。
因此，只需要考虑那些位于实心圆内部和第一象限的点，
例如:对于这样一个点q，我们画一个虚线圆，以O为圆心，|q|为半径。假设这个圆在q*处与线段Ow相交。由三角不等式得到



$$|q| + |q - w| \geq |w| = |q^*| + |q^* - w|.$$

Since $|q| = |q^*|$, we obtain

$$|q - w| \geq |q^* - w|,$$

indicating

$$|q| + \frac{r_p}{2}|q - w|^2 \geq |q^*| + \frac{r_p}{2}|q^* - w|^2.$$

This means the solution of the problem (17) will locate on the line segment Ow. Denoting $q = \beta w$ with $0 \leq \beta \leq 1$, we hence simplify (17) to be the following 1-dimensional problem

$$\min_{0 \leq \beta \leq 1} \{\beta |w| + \frac{r_p}{2}(\beta - 1)^2 |w|^2\}. \tag{18}$$

The above (18) can be solved exactly, with a closed form solution as

$$\beta^* = \max(0, 1 - \frac{1}{r_p |w|}).$$

5、实验结果

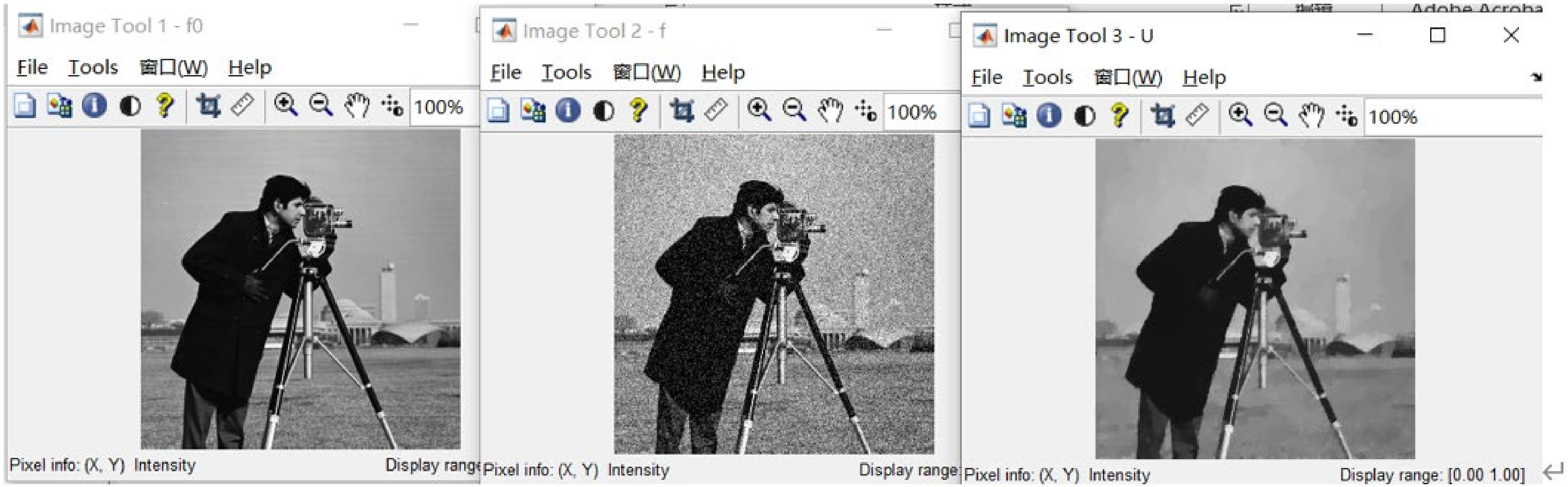
（1）、复原高斯噪声（TV-L2）

```
f = imnoise(f0,'gaussian',0.01);
```

```
factPSNR =

   20.3166

The alpha is 10, r_P is 5, SNR1 is 14.2815, PSNR1 is 26.5168,
```



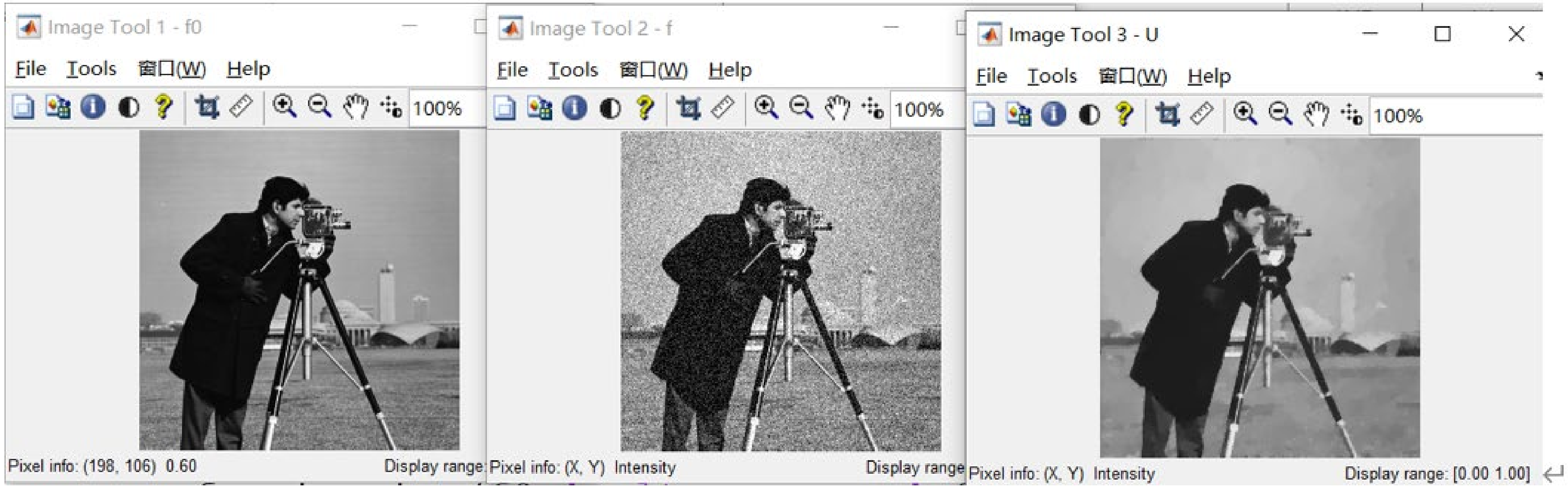
5、实验结果

```
f = imnoise(f0,'gaussian',0.05);
```

```
factPSNR =

   19.2126

The alpha is 10, r_P is 5, SNR1 is 11.1238, PSNR1 is 23.3591,
```



5、实验结果

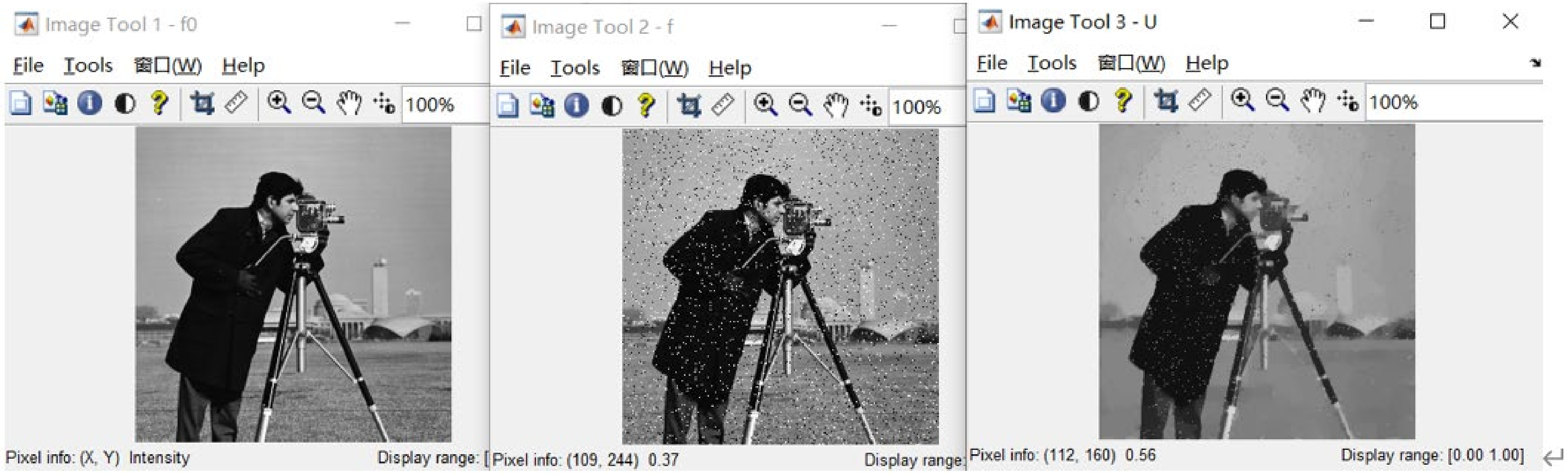
（2）、复原脉冲噪声（椒盐）（TV-L1）

```
f = imnoise(f0,'salt & pepper',0.05);
```

factPSNR =

    18.0849

The alpha is 7, r_P is 10, SNR1 is 11.3517, PSNR1 is 23.5869,



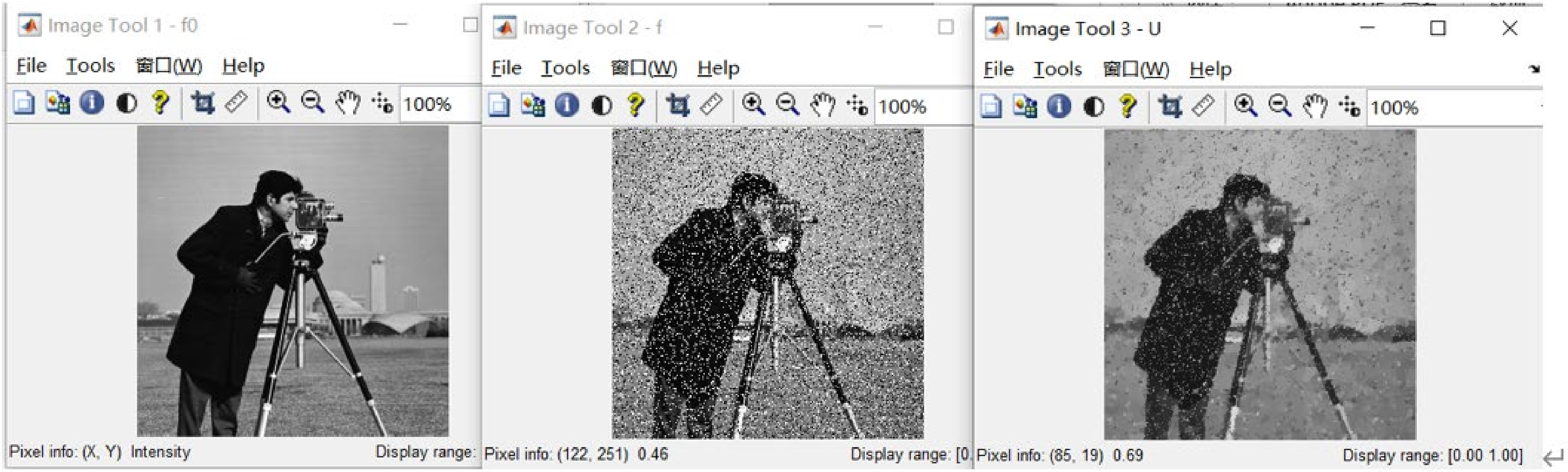
5、实验结果

```
f = imnoise(f0,'salt & pepper',0.2);

factPSNR =

    12.0611

The alpha is 7, r_P is 10, SNR1 is 7.3963, PSNR1 is 19.6316,
```



感 谢 老 师 的 悉 心 指 导

T H A N K　Y O U !

汇 报 人 | X X X

指 导 老 师 | XXX