5.3. **T1 Calculations.** Applying STMCB processing to our general signal model, Eqn (4), we know that the complex amplitude, $C_j \in \mathbb{C}$, depends on (1) acquisition parameters - repetition time, TR, and flip angle, θ_N, as well as (2) tissue dependent properties - spin lattice relaxation, $T1_j$, proton density M_{0_j}, and initial phase offset, ϕ_j.

$$C_j = \frac{M_{0_j} \sin(\gamma\theta_N)\left(1 - e^{-TR/T1_j}\right)}{\left(1 - \cos(\gamma\theta_N) e^{-TR/T1_j}\right)} e^{-i\phi_j} \in \mathbb{C}$$

Multiple methods may be used to extract T1 information from the complex signal amplitude.

- The T1 may be found as the slope of the linearized signal

$$\frac{|C_j|}{\sin(\gamma\theta_N)} = e^{-TR/T1_j} \frac{|C_j|}{\tan(\gamma\theta_N)} + M_{0_j}\left(1 - e^{-TR/T1_j}\right)$$

  Given two flip angles, $\theta_N = \theta_1,\ \theta_2$, a B1 correction map γ, and the TR

$$e^{-TR/T1_j} = \frac{\frac{|C_j(\theta_2)|}{\sin(\gamma\theta_2)} - \frac{|C_j(\theta_1)|}{\sin(\gamma\theta_1)}}{\frac{|C_j(\theta_2)|}{\tan(\gamma\theta_2)} - \frac{|C_j(\theta_1)|}{\tan(\gamma\theta_1)}} \equiv m_j \in \mathbb{R} \quad \Rightarrow \quad T1_j = \frac{-TR}{\ln(m_j)}$$

- A low flip angle may be obtained prior to the scan.

$$\cos(\beta) \approx 1 \qquad \Rightarrow \qquad |C_j(\beta)| = \frac{M_{0_j} \sin(\beta)\left(1 - e^{-TR/T1_j}\right)}{\left(1 - \cos(\beta) e^{-TR/T1_j}\right)} \approx M_{0_j} \sin(\beta)$$

  The ratio of this low flip angle value to values acquired under normal conditions may be used to eliminate the M_{0_j} dependence.

$$\rho \equiv \frac{|C_j(\theta)|}{|C_j(\beta)|} = \frac{\sin(\theta)\left(1 - e^{-TR/T1_j}\right)}{\sin(\beta)\left(1 - \cos(\theta) e^{-TR/T1_j}\right)}$$

  Given β, θ, and TR, solve the nonlinear equation for T1.
  - Assuming a linear temperature dependence of the T1(u) [Rieke and Butts Pauly, 2008], the ratio of the amplitudes, ρ, is implicitly a temperature dependent measurement corrupted by noise $\nu \sim \mathcal{N}(0, \sigma)$

$$\rho(u) = \frac{\sin(\theta)\left(1 - e^{-TR/T1(u)}\right)}{\sin(\beta)\left(1 - \cos(\theta) e^{-TR/T1(u)}\right)} + \nu \qquad T1(u) = T1(u_{\text{ref}}) + m(u - u_{\text{ref}}) \qquad T1(u_{\text{ref}}) \sim \mathcal{U}(T1_{lb}, T1_{ub})$$

  In order to detect a temperature dependent signal change, the variance of the signal ratio must be greater than the noise variance.

$$\frac{\delta\rho(u)}{\delta u}\delta u > \sigma \tag{22}$$

TODO

- validate T1 values for both the double FA method and the ratio method against T1 values obtained from gold standard IR
- Verify temperature induced T1 measurements are viable. ie. when does equation (22) hold ? @madankan, is this condition satified in some sense under uncertainty in the reference T1 value ? $T1(u_{\text{ref}}) \sim \mathcal{U}(T1_{lb}, T1_{ub})$

5.4. **Root finding.** To find the roots of $N(z)$ and $D(z)$ the Companion matrix method can be used. It can be verified by direct computation that the so called Companion matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -c_{n-1} \end{pmatrix} \tag{23}$$

has the characteristic polynomial $p(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1} + t^n$. Thus the eigenvalues of (23) are the roots of $p(t)$. To find the eigenvalues the QR algorithm can be used. The algorithm performs iterations of the form.

$$A_{k+1} = R_k Q_k,$$

where Q_k is an orthogonal matrix and R_k an upper triangular matrix, such that $A_k = Q_k R_k$, i.e. the QR decomposition of A_k, and $A_0 = A$. It can be shown, that A_k has the same eigenvalues as A and that it converges to a triangular matrix, the *Schur form*. Thus, the eigenvalues of A and be read off the diagonal of A_k after convergence. Usually A is transformed into a upper Hessenberg matrix to reduce the costs of the QR decomposition during each iteration. However in our case, where $p(t)$ is of very small degree direct computation should be sufficient. One method for performing QR decomposition is Gram-Schmidt orthonormalization. This is a process for orthonormalizing a set of vectors in an inner product space, and