Gauss-Hermite Quadrature for Mutual Information

Drew Mitchell

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1 Probability Distribution Definitions

Calculate mutual information using numerical integration for a complex signal model $\mathbf{z}(\boldsymbol{\mu}, \mathbf{k})$.

$$\begin{aligned}
\mathbf{z}(\boldsymbol{\mu}, \mathbf{k}) &= \int_\Omega M(\mathbf{x})\, e^{-s(\boldsymbol{\mu},\mathbf{x})} e^{-2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} + \boldsymbol{\nu} \\
&= \mathcal{G}(\boldsymbol{\mu}, \mathbf{k}) + \boldsymbol{\nu} \\
s(\boldsymbol{\mu}, \mathbf{x}) &= \frac{T_E}{T_2^*(\mathbf{x})} + \mathrm{i}\left[2\pi\gamma\alpha B_0 T_E \Delta u(\boldsymbol{\mu}, \mathbf{x}) + T_E \Delta\omega_0(\mathbf{x})\right] \\
\boldsymbol{\nu} &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\nu), \quad \boldsymbol{\Sigma}_\nu = \begin{bmatrix} \sigma_{\nu,r}^2 & 0 \\ 0 & \sigma_{\nu,i}^2 \end{bmatrix}
\end{aligned} \tag{1}$$

Therefore, the probability distribution for $p(\mathbf{z} \mid \boldsymbol{\mu})$ is

$$p(\mathbf{z} \mid \boldsymbol{\mu}) = \mathcal{N}(\mathcal{G}(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu) \tag{2}$$

The tissue properties can be described by the following piecewise functions.

$$\begin{aligned}
\boldsymbol{\mu}(\mathbf{x}) &= \sum_{n=1}^{N} \boldsymbol{\mu}_n U(\mathbf{x} - \Omega_n) \\
&\bigcup_{n=1}^{N} \Omega_n = \Omega, \quad \Omega_n \cap \Omega_m = \varnothing \\
U(\mathbf{x} - \Omega_n) &= \begin{cases} 1, & x \in \Omega_n \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{3}$$

Assume normal distribution for the model parameter, optical attenuation coefficient $\boldsymbol{\mu}$.

$$p(\boldsymbol{\mu}) = \mathcal{N}(\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) \tag{4}$$

$$\mathbf{m}_\mu = \begin{bmatrix} m_{\mu_1} \\ \vdots \\ m_{\mu_N} \end{bmatrix} \tag{5a}$$

$$\boldsymbol{\Sigma}_\mu = \begin{bmatrix} \sigma_{\mu_1} & 0 & \cdots & 0 & 0 \\ 0 & \sigma_{\mu_2} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \sigma_{\mu_{N-1}} & 0 \\ 0 & 0 & \cdots & 0 & \sigma_{\mu_N} \end{bmatrix} \tag{5b}$$

Also of note, because the real and imaginary components of $\mathbf{z}$ are assumed independent, the covariance matrix $\boldsymbol{\Sigma}_z$ is diagonal, and the following simplification results.

$$p(\mathbf{z} \mid \boldsymbol{\mu}) = \mathcal{N}_{\mathbf{z}}(\mathcal{G}(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu) = \mathcal{N}_{z_r}(\mathcal{G}_r(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu)\,\mathcal{N}_{z_i}(\mathcal{G}_i(\boldsymbol{\mu}), \boldsymbol{\Sigma}_\nu) = p(z_r \mid \boldsymbol{\mu})\,p(z_i \mid \boldsymbol{\mu}) \tag{6}$$

Similarly, the various tissue types are independent, and the covariance matrix $\boldsymbol{\Sigma}_\mu$ is also diagonal.

$$p(\boldsymbol{\mu}) = \mathcal{N}_{\boldsymbol{\mu}}(\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) = \prod_{i=1}^{N} \mathcal{N}_{\mu_i}(m_{\mu_i}, \sigma_{\mu_i}) = \prod_{i=1}^{N} p(\mu_i) \tag{7}$$

2 Problem Statement

The most approachable way to solve mutual information is to begin with the difference of entropies definition.

$$\begin{aligned} I(\boldsymbol{\mu}; \mathbf{z}) &= H(\mathbf{z}) - H(\mathbf{z} \mid \boldsymbol{\mu}) \\ &= \frac{1}{2}\ln\left((2\pi e)^2 \cdot |\boldsymbol{\Sigma}_z|\right) - \frac{1}{2}\ln\left((2\pi e)^2 \cdot |\boldsymbol{\Sigma}_\nu|\right) \end{aligned} \tag{8}$$

Assuming $\sigma^2_{\nu,r} = \sigma^2_{\nu,i}$, then

$$|\boldsymbol{\Sigma}_\nu| = \sigma^2_\nu \sigma^2_\nu - 0 = \sigma^4_\nu \tag{9}$$

Calculation of $\boldsymbol{\Sigma}_z$ is less straightforward.

$$\begin{aligned} \boldsymbol{\Sigma}_z &= \begin{bmatrix} \sigma_{rr} & \sigma_{ri} \\ \sigma_{ir} & \sigma_{ii} \end{bmatrix} \\ \sigma_{rr} &= \mathrm{E}\left[(z_r - \mathrm{E}[z_r])^2\right] \\ \sigma_{ri} &= \mathrm{E}\left[(z_r - \mathrm{E}[z_r])(z_i - \mathrm{E}[z_i])\right] \\ \sigma_{ir} &= \mathrm{E}\left[(z_i - \mathrm{E}[z_i])(z_r - \mathrm{E}[z_r])\right] \\ \sigma_{ii} &= \mathrm{E}\left[(z_i - \mathrm{E}[z_i])^2\right] \end{aligned} \tag{10}$$

The values of σ_{rr}, σ_{ri}, σ_{ir}, and σ_{ii} all need to be calculated numerically.

Alternatively, $\int p(z \mid \mu)p(\mu)$ can probably be calculated analytically using [Bromiley 2003]. The product of three normal distributions is a scaled normal distribution, so after converting N_{z_i} and N_{z_r} to distributions in terms of μ for the integration, the normal distribution integrates to 1, and the scaling factor comes out as a function of z_r and z_i, $S_{fgh}(z_r, z_i)$, for the remaining two integrals. This leaves potentially very difficult numerical integrations at this step.

3 Gauss-Hermite Quadrature

Gaussian quadrature:

$$\int_{-\infty}^{\infty} \exp^{-x^2} f(x)\,dx \approx \sum_{i=1}^{N} \omega_i f(x_i) \tag{11}$$
$$\omega_i = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 \left[H_{n-1}(x_i)\right]^2}$$

where n is the number of sample points used, $H_n(x)$ is the physicists' Hermite polynomial, x_i are the roots of the Hermite polynomial, and ω_i are the associated Gauss-Hermite weights.

Substitution for normal distributions using Gauss-Hermite quadrature:

$$\mathrm{E}\left[h(y)\right] = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) h(y)\,dy \tag{12}$$

h is some function of y, and random variable Y is normally distributed.

$$x = \frac{y-\mu}{\sqrt{2}\sigma} \Leftrightarrow y = \sqrt{2}\sigma x + \mu \tag{13}$$

$$\begin{aligned} \mathrm{E}\left[h(y)\right] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \exp\left(-x^2\right) h\left(\sqrt{2}\sigma x + \mu\right) dx \\ &\approx \frac{q}{\sqrt{\pi}} \sum_{i=1}^{N} \omega_i h\left(\sqrt{2}\sigma x_i + \mu\right) \end{aligned} \tag{14}$$

4 Mutual Information Calculation

$$I(\boldsymbol{\mu}; \mathbf{z}) = H(\mathbf{z}) - H(\mathbf{z} \mid \boldsymbol{\mu}) = \frac{1}{2}\ln\left((2\pi e)^2 \cdot |\boldsymbol{\Sigma}_z|\right) - \frac{1}{2}\ln\left((2\pi e)^2 \cdot |\boldsymbol{\Sigma}_\nu|\right) \tag{15}$$

$$|\boldsymbol{\Sigma}_\nu| = \sigma_\nu^2 \sigma_\nu^2 - 0 = \sigma_\nu^4 \tag{16}$$

$$\mathbf{\Sigma}_z = \begin{bmatrix} \sigma_{rr} & \sigma_{ri} \\ \sigma_{ir} & \sigma_{ii} \end{bmatrix} \tag{17}$$

$$\begin{aligned} \sigma_{rr} &= \mathrm{E}\left[\left(z_r - \mathrm{E}\left[z_r \right] \right)^2 \right] \\ \sigma_{ri} &= \mathrm{E}\left[\left(z_r - \mathrm{E}\left[z_r \right] \right) \left(z_i - \mathrm{E}\left[z_i \right] \right) \right] \\ \sigma_{ir} &= \mathrm{E}\left[\left(z_i - \mathrm{E}\left[z_i \right] \right) \left(z_r - \mathrm{E}\left[z_r \right] \right) \right] \\ \sigma_{ii} &= \mathrm{E}\left[\left(z_i - \mathrm{E}\left[z_i \right] \right)^2 \right] \end{aligned}$$

$$p\left(\mathbf{z}\right) = \int p\left(\mathbf{z} \mid \boldsymbol{\mu}\right) p\left(\boldsymbol{\mu}\right) d\boldsymbol{\mu} \tag{18}$$

$$p\left(\mathbf{z} \mid \boldsymbol{\mu}\right) = \mathcal{N}_{\mathbf{z}}\left(\mathcal{G}\left(\boldsymbol{\mu}\right), \mathbf{\Sigma}_\nu\right) = \mathcal{N}_{z_r}\left(\mathcal{G}_r\left(\boldsymbol{\mu}\right), \sigma_\nu\right) \mathcal{N}_{z_i}\left(\mathcal{G}_i\left(\boldsymbol{\mu}\right), \sigma_\nu\right) = p\left(z_r \mid \boldsymbol{\mu}\right) p\left(z_i \mid \boldsymbol{\mu}\right) \tag{19}$$

$$p\left(\boldsymbol{\mu}\right) = \mathcal{N}_{\boldsymbol{\mu}}\left(\mathbf{m}_\mu, \mathbf{\Sigma}_\mu\right) = \prod_{i=1}^{N} \mathcal{N}_{\mu_i}\left(m_{\mu_i}, \sigma_{\mu_i}\right) = \prod_{i=1}^{N} p\left(\mu_i\right) \tag{20}$$

The signal measurement probability distribution $p\left(\mathbf{z}\right)$ can be approximated using Gauss-Hermite quadrature. The calculation is performed separately for the real and imaginary components, $p\left(z_r\right)$ and $p\left(z_i\right)$, because z_r and z_i are assumed independent.

$$\begin{aligned} p\left(z_r\right) &= \int_{-\infty}^{\infty} p\left(z_r \mid \boldsymbol{\mu}\right) \prod_{n=1}^{N} p\left(\mu_n\right) d\boldsymbol{\mu} = \int_{-\infty}^{\infty} p\left(z_r \mid \boldsymbol{\mu}\right) \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_{\mu_n}} \exp\left(-\frac{\left(\mu_n - m_{\mu_n}\right)^2}{2\sigma_{\mu_n}^2} \right) d\boldsymbol{\mu} \\ &= \int_{-\infty}^{\infty} p\left(z_r \mid \boldsymbol{\mu}\right) \prod_{n=1}^{N} \frac{1}{\sqrt{\pi}} \exp\left(-x_n^2\right) d\mathbf{x} \approx \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p p\left(z_r \mid \boldsymbol{\mu}_p\right) \end{aligned} \tag{21}$$

Q=number of quadrature points; N=number of tissue types

$$\begin{aligned} & p\left(z_r \mid \boldsymbol{\mu}\right) = p\left(z_r \mid \mu_1, \ldots, \mu_N\right) \\ & p\left(z_r \mid \boldsymbol{\mu}_p\right) = p\left(z_r \mid \mu_{1,q}, \ldots, \mu_{N,q}\right) \\ & d\boldsymbol{\mu} = d\mu_1 \cdots d\mu_N \\ & \mu_{i,q} = \sqrt{2}\sigma_{\mu_i} x_{i,q} + m_{\mu_i} \\ & d\mu_{i,q} = \sqrt{2}\sigma_{\mu_i} dx_{i,q} \end{aligned}$$

An identical calculation for the imaginary component results in

$$p\left(z_i\right) \approx \pi^{-N/2} \sum_{i=1}^{N} \sum_{q=1}^{Q} \omega_{i,q} p\left(z_i \mid \boldsymbol{\mu}_p\right) \tag{22}$$

$$\begin{aligned} p\left(z_i \mid \boldsymbol{\mu}_p\right) &= p\left(z_i \mid \mu_{1,q}, \ldots, \mu_{N,q}\right) \\ \mu_{i,q} &= \sqrt{2}\sigma_{\mu_i} x_{i,q} + m_{\mu_i} \end{aligned}$$

Calculate expectation values for real and imaginary components by definition and approximating $p(\mathbf{z})$ with Gauss-Hermite quadrature as shown above.

$$
\begin{aligned}
\mathrm{E}[z_r] &= \int_{-\infty}^{\infty} z_r p(z_r)\, dz_r \approx \int_{-\infty}^{\infty} z_r \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p p\left(z_r \mid \boldsymbol{\mu}_p\right) dz_r \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \int_{-\infty}^{\infty} z_r p\left(z_r \mid \boldsymbol{\mu}_p\right) dz_r = \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \mathrm{E}\left[z_r \mid \boldsymbol{\mu}_p\right] = \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \mathcal{G}_r\left(\boldsymbol{\mu}_p\right)
\end{aligned}
\tag{23}
$$

Similarly for the imaginary component,

$$
\mathrm{E}[z_i] \approx \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \mathcal{G}_i\left(\boldsymbol{\mu}_p\right) \tag{24}
$$

The variance can be calculated as follows:

$$
\begin{aligned}
\mathrm{E}\left[\left(z_r - \mathrm{E}[z_r]\right)^2\right] &= \mathrm{E}\left[z_r^2\right] - \left(\mathrm{E}[z_r]\right)^2 = \int_{-\infty}^{\infty} z_r^2 p(z_r)\, dz_r - \left(\mathrm{E}[z_r]\right)^2 \\
&\approx \int_{-\infty}^{\infty} z_r^2 \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p p\left(z_r \mid \boldsymbol{\mu}_p\right) dz_r - \left(\mathrm{E}[z_r]\right)^2 \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \int_{-\infty}^{\infty} z_r^2 p\left(z_r \mid \boldsymbol{\mu}_p\right) dz_r - \left(\mathrm{E}[z_r]\right)^2 = \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \\
&\quad \cdot \int_{-\infty}^{\infty} z_r^2 \frac{1}{\sqrt{2\pi}\sigma_\nu} \exp\left(-\frac{\left(z_r - \mathcal{G}_r\left(\boldsymbol{\mu}_p\right)\right)^2}{2\sigma_\nu^2}\right) dz_r - \left(\mathrm{E}[z_r]\right)^2 \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \int_{-\infty}^{\infty} \left(x^2\sigma_\nu^2 + 2x\sigma_\nu \mathcal{G}_r\left(\boldsymbol{\mu}_p\right) + \mathcal{G}_r^2\left(\boldsymbol{\mu}_p\right)\right) \\
&\quad \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx - \left(\mathrm{E}[z_r]\right)^2 \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \left(\sigma_\nu^2 + \mathcal{G}_r^2\left(\boldsymbol{\mu}_p\right)\right) - \left(\mathrm{E}[z_r]\right)^2 \\
&= \pi^{-N/2}\sigma_\nu^2 + \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \mathcal{G}_r^2\left(\boldsymbol{\mu}_p\right) - \left(\mathrm{E}[z_r]\right)^2
\end{aligned}
\tag{25}
$$

Again, an identical calculation for the imaginary component yields

$$\mathrm{E}\left[\left(z_i - \mathrm{E}\left[z_i\right]\right)^2\right] = \pi^{-N/2}\sigma_\nu^2 + \pi^{-N/2}\sum_{p=1}^{Q^N}\omega_p\mathcal{G}_i^2\left(\boldsymbol{\mu}_p\right) - \left(\mathrm{E}\left[z_i\right]\right)^2 \tag{26}$$

The off-diagonal elements of Σ_z are equal.

$$\mathrm{E}\left[\left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right)\right] = \mathrm{E}\left[\left(z_i - \mathrm{E}\left[z_i\right]\right)\left(z_r - \mathrm{E}\left[z_r\right]\right)\right] \tag{27}$$

$$
\begin{aligned}
\mathrm{E}\left[\left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right)\right] &= \int_{-\infty}^{\infty} \left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right) p\left(\mathbf{z}\right) d\mathbf{z} \\
&= \int_{-\infty}^{\infty} \left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right) \int_{-\infty}^{\infty} p\left(\mathbf{z} \mid \boldsymbol{\mu}\right) p\left(\boldsymbol{\mu}\right) d\boldsymbol{\mu} d\mathbf{z} \\
&\approx \int_{-\infty}^{\infty} \left(z_r - \mathrm{E}\left[z_r\right]\right)\left(z_i - \mathrm{E}\left[z_i\right]\right) \cdot \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p p\left(\mathbf{z} \mid \boldsymbol{\mu}_p\right) d\mathbf{z} \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \int_{-\infty}^{\infty} \left(z_r - \mathrm{E}\left[z_r\right]\right) p\left(z_r \mid \boldsymbol{\mu}_p\right) dz_r \\
&\quad \cdot \int_{-\infty}^{\infty} \left(z_i - \mathrm{E}\left[z_i\right]\right) p\left(z_i \mid \boldsymbol{\mu}_p\right) dz_i \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \\
&\quad \cdot \int_{-\infty}^{\infty} \left(z_r - \mathrm{E}\left[z_r\right]\right) \frac{1}{\sqrt{2\pi}\sigma_\nu} \exp\left(-\frac{\left(z_r - \mathcal{G}_r\left(\boldsymbol{\mu}_p\right)\right)^2}{2\sigma_\nu^2}\right) dz_r \\
&\quad \cdot \int_{-\infty}^{\infty} \left(z_i - \mathrm{E}\left[z_i\right]\right) \frac{1}{\sqrt{2\pi}\sigma_\nu} \exp\left(-\frac{\left(z_i - \mathcal{G}_i\left(\boldsymbol{\mu}_p\right)\right)^2}{2\sigma_\nu^2}\right) dz_i \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \\
&\quad \cdot \int_{-\infty}^{\infty} \left(\sigma_\nu x + \mathcal{G}_r\left(\boldsymbol{\mu}_p\right) - \mathrm{E}\left[z_r\right]\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\
&\quad \cdot \int_{-\infty}^{\infty} \left(\sigma_\nu y + \mathcal{G}_i\left(\boldsymbol{\mu}_p\right) - \mathrm{E}\left[z_i\right]\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \left(\left(\mathcal{G}_r\left(\boldsymbol{\mu}_p\right) - \mathrm{E}\left[z_r\right]\right)\left(\mathcal{G}_i\left(\boldsymbol{\mu}_p\right) - \mathrm{E}\left[z_i\right]\right)\right) \\
&= \pi^{-N/2} \sum_{p=1}^{Q^N} \omega_p \mathcal{G}_r\left(\boldsymbol{\mu}_p\right) \mathcal{G}_i\left(\boldsymbol{\mu}_p\right) - \mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right]
\end{aligned}
\tag{28}
$$

Thus, the numerical approximation to the covariance matrix $\boldsymbol{\Sigma}_z$ is then

$$\mathbf{\Sigma}_z = \left[\begin{array}{cc} \sigma_{rr} & \sigma_{ri} \\ \sigma_{ir} & \sigma_{ii} \end{array} \right] \tag{29}$$

$$\sigma_{rr} = \pi^{-N/2}\sigma_\nu^2 + \pi^{-N/2}\sum_{p=1}^{Q^N}\omega_p\mathcal{G}_r^2\left(\boldsymbol{\mu}_p\right) - \left(\mathrm{E}\left[z_r\right]\right)^2$$

$$\sigma_{ii} = \pi^{-N/2}\sigma_\nu^2 + \pi^{-N/2}\sum_{p=1}^{Q^N}\omega_p\mathcal{G}_i^2\left(\boldsymbol{\mu}_p\right) - \left(\mathrm{E}\left[z_i\right]\right)^2$$

$$\begin{aligned}\sigma_{ri} &= \sigma_{ir} \\ &= \pi^{-N/2}\sum_{p=1}^{Q^N}\omega_p\mathcal{G}_r\left(\boldsymbol{\mu}_p\right)\mathcal{G}_i\left(\boldsymbol{\mu}_p\right) - \mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right]\end{aligned}$$

The determinant of the covariance matrix is easily calculated.

$$\begin{aligned}\left|\mathbf{\Sigma}_z\right| = \Bigg(&\pi^{N/2}\sigma_\nu^2 \\ &+\pi^{-N/2}\sum_{p=1}^{Q^N}\omega_p\mathcal{G}_r^2\left(\boldsymbol{\mu}_p\right) - \left(\mathrm{E}\left[z_r\right]\right)^2\Bigg)\cdot\left(\pi^{N/2}\sigma_\nu^2 + \pi^{-N/2}\sum_{p=1}^{Q^N}\omega_p\mathcal{G}_i^2\left(\boldsymbol{\mu}_p\right) - \left(\mathrm{E}\left[z_i\right]\right)^2\right) \\ &-\left(\pi^{-N/2}\sum_{p=1}^{Q^N}\omega_p\mathcal{G}_r\left(\boldsymbol{\mu}_p\right)\mathcal{G}_i\left(\boldsymbol{\mu}_p\right) - \mathrm{E}\left[z_r\right]\mathrm{E}\left[z_i\right]\right)^2\end{aligned} \tag{30}$$

The mutual information calculation is straightforward once the value of $\left|\mathbf{\Sigma}_z\right|$ is known.

$$I\left(\boldsymbol{\mu};\mathbf{z}\right) = H\left(\mathbf{z}\right) - H\left(\mathbf{z}\mid\boldsymbol{\mu}\right) = \frac{1}{2}\ln\left(\left(2\pi e\right)^2\cdot\left|\mathbf{\Sigma}_z\right|\right) - \frac{1}{2}\ln\left(\left(2\pi e\right)^2\cdot\left|\mathbf{\Sigma}_\nu\right|\right) \tag{31}$$