Advanced Appliations of Synthetic MR and MAGiC

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1 Physics Model

The steady-state magnetization M_{TD} at a specific delay time T_D can be found as a function of flip angle θ, repetition time T_R, excitation pulse α, and relaxation time T_1:

$$M_{TD} = M_0 \frac{1 - (1 - \cos(\theta))e^{-\frac{T_D}{T_1}} - \cos(\theta)e^{-\frac{T_R}{T_1}}}{1 - \cos(\theta)e^{-\frac{T_R}{T_1}} \cos(\alpha)}$$

where, M_0 is the unsatuarated magnetization.

2 Mathematical Framework

The underlying philosophy and assumptions within our approach is that the physics models are 1st order accurate or within 70-80% of the needed accuracy and the error is adequate within the assumed Gaussian noise. Gaussian distributions provide analytical representations of the random variables of interest (ie T1, T2) within the Bayesian setting and provide a crux for understanding. In particular, we say that a random variable η belongs to a multi-variate normal distribution of mean $\mu \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n \times n}$

$$\eta \sim \mathcal{N}(\mu, \Sigma) \Rightarrow p(\eta) = \frac{1}{2\ \pi\ \det \Sigma} \exp\left(-\frac{1}{2}\|\mu - \eta\|^2_\Sigma\right)$$

1. Our data acquistion model, $\mathcal{G}(\vec{k}, \theta) : \mathbb{R}^a \times \mathbb{R}^m \to \mathbb{R}^n$, maps deterministic acquisition parameters, $\vec{k} \in \mathbb{R}^a$, and uncertain parameters, $\theta \in \mathbb{R}^m$ to observables, $\vec{z} \in \mathbb{R}^n$ (or $\vec{z} \in \mathbb{C}^n$). Explicitly, we will assume that the measurement models are corrupted by zero mean white noise noise of a **known** covariance matrix, $\Sigma_z \in \mathbb{R}^{n \times n}$

$$\begin{aligned} \vec{z} &= \mathcal{G}(\vec{k}; \theta) + \eta \qquad \eta \sim \mathcal{N}(0, \Sigma_z) \\ \vec{k} &= (\text{TE, TR, etc}) \\ \theta &= (\text{T1, T2, etc}) \end{aligned} \tag{1}$$

   η may be interpreted as the measurement noise or the acquisition noise in the sensor model. For a deterministic measurement model $\mathcal{G}$, the conditional probablity distribution has an explicit analytical form and may be written as a **known** Gaussian distribution.

$$p(\vec{z}|\theta) = \mathcal{N}(\mathcal{G}(\vec{k}; \theta), \Sigma_z)$$

2. Additional **known** information is the prior probability distributions for the model parameters, $p(\theta)$. For simplicity, assume that Prior parameters are Gaussian distributed of **known** mean, $\hat{\theta}$ and covariance, Σ_θ

$$\theta \sim \mathcal{N}(\hat{\theta}, \Sigma_\theta)$$

3. Bayes theorem is fundamental to the approach. The probability of the measurements $p(z)$ must be interprited in terms of the known information. The probability of the measurements may be derived from the marginalization of the joint probability and has the interpretation as the projection of the joint probability onto the measurement axis.

$$p(z) = \int_\theta p(\theta, z)\ d\theta = \int_\theta p(z|\theta)\ p(\theta)\ d\theta$$

4. The concept of informational entropy [Madankan et al., 2015], $H(Z)$, provides a mathematically rigorous framework to look for measurement acquisition parameters, $\vec{k}$, with the high information content of the reconstruction. Given a probability space $(\Omega, \mathcal{F}, p)$ (probability maps from the sigma-algebra of possible events $p : \mathcal{F} \to [0, 1]$ sigma-algebra, $\mathcal{F}$, defined on set of 'outcomes' Ω [Durrett, 2010]), we will define information of an event as proportional to the inverse probability.

$$\text{information} \equiv \frac{1}{p(z)}$$

Intuitively, when a low probability event occurs this provides high information. The informational entropy is an *average* of the information content for a sigma algebra of events $\mathcal{F}$

$$H(Z) = \int_Z p(z) \ln \frac{1}{p(z)} \, dz \qquad p(z) = \int_\theta p(z|\theta) \; p(\theta) \; d\theta$$

Hence this entropy measure is an average of the information content for a given set of events, $\mathcal{F}$, and is proportional to the variance or uncertainty in which the set of events occur. This agrees with thermodynamic entropy; if the information containing events are completely spread out such as in a uniform distribution, the entropy is maxmized. The entropy is zero for a probability distribution in which only one event occurs. Zero information is gained when the same event always occurs ($0 \ln \frac{1}{0} = 0$). Intuitively, we want to find acquisition parameters, $\vec{k}$, for which the measurements are most uncertain

$$\max_k H(Z) \quad \Leftrightarrow \quad \min_k \int_Z dz \underbrace{\int_\theta d\theta \; p(z|\theta) \; p(\theta)}_{p(z)} \underbrace{ln \left(\int_\theta d\theta \; p(z|\theta) \; p(\theta) \right)}_{ln \; p(z)}$$

Alternatively we may consider this entropy maximization problem as a sensitivity analysis for the variance of the measurement Z, ie . $\max_k H(Z) \approx \max_k \text{Var}(Z)$

$$\bar{Z} = \mathbb{E}[Z] = \int_Z dz \; z \underbrace{\int_\theta d\theta \; p(z|\theta) \; p(\theta)}_{p(z)}$$

$$\mathbb{E}[(Z - \bar{Z})^2] = \int_Z dz \; (z - \bar{Z})^2 \underbrace{\int_\theta d\theta \; p(z|\theta) \; p(\theta)}_{p(z)}$$

$$\propto \int_Z dz \; (z - \bar{z})^2 \int_\theta d\theta \exp\left(-\frac{1}{2} \| z - \mathcal{G}(\vec{k}, \theta) \|^2_{\Sigma_z} \right) \exp\left(-\frac{1}{2} \| \theta - \hat{\theta} \|^2_{\Sigma_\theta} \right)$$

Probilistic integrals may be computed from uncertainty quantification techniques [Fahrenholtz et al., 2013].

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