Assignment for

Computer Science Theory for the Information Age

Day 2

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Exercise 1. How does the volume of a sphere of radius two behave as the dimension of the space increases? What if the radius was large than two but a constant independent of d? What function of d would the radius need to be for a sphere of radius r to have approximately constant volume as the dimension increases?

Answer.

take[1]

$$V(r,d) = \frac{2}{d}\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} r^d \tag{1}$$

for the sphere of radius 2, we have

$$V(2,d) = \frac{2}{d}\frac{2^d \pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{2}$$

see into the radio of

$$\frac{V(2,d)}{V(2,d-2)} = \frac{4\pi}{\frac{d}{2}-1} \tag{3}$$

we can conclude that

Proposition. *The volume of sphere of radius 2 draws bigger, than smaller, and finally goes to infinitesimal as the dimension of the space increases.*

When the radius $r \neq 2$ but remains constant indepent from d, see into the radio

$$\frac{V(r,d)}{V(r,d-2)} = \frac{r^2\pi}{\frac{d}{2}-1} \tag{4}$$

the conclusion goes exactly the same

Proposition. *The volume of sphere of constant radius draws bigger, than smaller, and finally goes to infinitesimal as the dimension of the space increases.*

Note. To be more serious and accurate, there may be (r,d) with $(V(r,d-1) < V(r,d) \cap V(r,$ $d) > V(r,d+1))$ for $V(r,d)$ is not continuous since $\Gamma\left(\frac{d}{2}\right)$ is not the same for odd and even d, howerver the number of such (r,d) is countable and rather small.

1. *Computer Science Theory for the Information Age*, p10, Lemma 2.1

Exercise 2. Consider the upper hemisphere of a unit-radius sphere in d-dimensions. What is the height of the maximum volume cylinder that can be placed entirely inside the hemisphere? As you increase the height of the cylinder, you need to reduce the cylinder's radius so that it will lie entirely within the hemisphere.

Answer.

Denote h as the height of the cylinder, then the volume of the cylinder is

$$V(h) = h\,(1-h^2)^{\frac{d-1}{2}}\,V(d-1) = V(d-1)\,\sqrt{h^2\,(1-h^2)^{d-1}} \tag{5}$$

and when

$$h^2 = \frac{1-h^2}{d-1} \tag{6}$$

that is

$$h = \frac{1}{\sqrt{d}} \tag{7}$$

the volume $V(h)$ gets its maximum.