Assignment for

Computer Science Theory for the Information Age

Day 5

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Exercise 1. Let $x_i, 1 \leqslant i \leqslant n$, be a set of indicator variables with identical probability distributions. Let $x = \sum_{i=1}^{n} x_i$ and suppose $E(x) \to \infty$. Show that if the x_i are statistically independent the $\mathrm{Prob}(x=0) \to 0$.

Proof.

$$\begin{aligned} \sigma^2(x) &= E(x^2) - E^2(x) \\ &= n\,E(x_1^2) + 2\,C_n^2 E^2(x_1) - n^2\,E^2(x_1) \\ &= n(E(x_1^2) - E^2(x_1)) \\ &= n\,\sigma^2(x_i) \\ &= \frac{n}{4} \end{aligned} \tag{1}$$

use the Chabyshev's inequality, we have

$$\begin{aligned} \mathrm{Prob}(x=0) &\leqslant \mathrm{Prob}(|x - E(x)| \geqslant E(x)) \\ &\leqslant \frac{\sigma^2(x)}{E^2(x)} \\ &= \frac{1}{n} \end{aligned} \tag{2}$$

thus as n goes to infinity, $\mathrm{Prob}(x=0)$ goes to zero. □

Exercise 2. Consider a model of random subset $N(n,\, p)$ of integers $\{1,\, 2,\, ...,\, n\}$ where, $N(n,\, p)$ is the set obtained by independently at random including each of $\{1, 2, ..., n\}$ into the set with probability p. Define what an "increasing property" of $N(n, p)$ means. Prove that every increasing property of $N(n, p)$ has a threshold.

Answer.

Definition.

Q is an increasing property of N if when a subset N has the property any subset obtained by adding element to N must also have the property.

Lemma. *Every increasing porperty of $N(n, p)$ has a threshold.*

Proof.

***Lemma* 1.** *If Q is an increasing property and $0 \leqslant p \leqslant q \leqslant 1$, then the probability that $N(n,p)$ has property Q is less than or equal to the probability that $N(n,p)$ has property Q.*

Proof.

generate $N(n,q)$ in this way: first generate $N(n,p)$. Then generate independently another subset $N\left(n, \frac{q-p}{1-p}\right)$ and take the union by putting on an element iff at least one of the two subsets has that element. We call this union operation replication, and call this subset obtained H.

We claim that H has the same distrubition as $N(n, q)$. This follows since the probability that an element is in H is $p + (1-p)\frac{q-p}{1-p} = q$ and clearly the elements of H are in/out independently. Now the lemma follows since whenever $N(n,p)$ has the property Q, H also has the property Q. $\square$

***Lemma* 2.** *Any monotone property Q of $N(n,p)$ has a phase transition at $p(n)$, where for each n, $p(n)$ is the minimum real number a for which the probability that $N(n,p)$ has property Q is $\frac{1}{2}$.*

Proof.

Suppose $p_0(n)$ is any function such that

$$\lim_{n \to \infty} \frac{p_0(n)}{p(n)} = 0$$

We will show that almost surely $N(n, p_0)$ does not have property Q. Suppose this is false. Then the probability that $N(n, p_0)$ has the property Q does not converge to zero, so there must be a positive real number ε such that the probability that $N(n,p_0)$ has property Q is at least ε on an infinite subsequence I of n.

Let $m = \left\lceil \left(\frac{1}{\varepsilon}\right) \right\rceil$. Let H be the m-fold replication of $N(n,p_0)$. For all $n \in I$, we have

$$\begin{aligned} \mathrm{Prob}(N(n, m\,p_0) \text{ does not have } Q) &\leqslant \mathrm{Prob}(H \text{ does not have } Q) \\ &\leqslant (\mathrm{Prob}(N(n,p_0) \text{ does not have } Q))^m \\ &\leqslant (1-\varepsilon)^m \\ &\leqslant e^{-1} \\ &\leqslant \frac{1}{2} \end{aligned} \tag{3}$$

So for these n, since $p(n)$ is the minimum real number a for which the probability that $N(n,p)$ has property Q is $\frac{1}{2}$, $m\,p_0 \geqslant p(n)$. This implies that $\frac{p_0(n)}{p(n)}$ is at least $\frac{1}{m}$ infinitely contradicting the hypothesis that $\lim\limits_{n \to \infty} \frac{p_0(n)}{p(n)} = 0$.

$\square$

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