Assignment for

Computer Science Theory for the Information Age

Day 7

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Exercise 1. Suppose $\boldsymbol{A}$ is an $n \times n$ matrix with block diagonal structure with k equal size blocks where all entries of the i^{th} block are a_i with $a_1 > a_2 > \ldots > a_k > 0$. Show that $\boldsymbol{A}$ has exactly k nonzero sigular vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ where $\boldsymbol{v}_i$ has the value $\left(\frac{k}{n}\right)^{1/2}$ in the coordinates corresponding to the i^{th} block and 0 elsewhere. In other words, the singualr vectors exactly identify the blocks of the diagonal. What happens if $a_1 = a_2 = \ldots = a_k$? In the case where the a_i are equal, what is the structure of the set of all possible singular vectors?

Answer.

Let $\boldsymbol{v}_1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, then

$$\begin{aligned}
|\boldsymbol{A}\,\boldsymbol{v}_1| &= \sqrt{\frac{n}{k}\left(\sum_{i=1}^{\frac{n}{k}} a_1 x_i\right)^2 + \frac{n}{k}\left(\sum_{i=\frac{n}{k}+1}^{2\frac{n}{k}} a_2 x_i\right)^2 + \cdots + \frac{n}{k}\left(\sum_{i=(k-1)\frac{n}{k}+1}^{k\frac{n}{k}} a_k x_i\right)^2} \\
&\leqslant \frac{n}{k}\sqrt{a_1^2\left(\sum_{i=1}^{\frac{n}{k}} x_i^2\right) + a_2^2\left(\sum_{i=\frac{n}{k}+1}^{2\frac{n}{k}} x_i^2\right) + \cdots + a_k^2\left(\sum_{i=(k-1)\frac{n}{k}+1}^{k\frac{n}{k}} x_i^2\right)} \\
&\leqslant \frac{n}{k}\sqrt{a_1^2} \\
&= \frac{n\,a_1}{k}
\end{aligned}$$

Then when $x_1 = x_2 = \cdots = x_{\frac{n}{k}} = \sqrt{\frac{n}{k}}$ and for the else $x_i = 0$ the $|\boldsymbol{A}\,\boldsymbol{v}_1|$ reaches its maximum. So we have $\boldsymbol{v}_1 = \begin{pmatrix} \sqrt{\frac{n}{k}} \\ \sqrt{\frac{n}{k}} \\ \vdots \\ \sqrt{\frac{n}{k}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with $\sigma_1 = \frac{n\,a_1}{k}$.

Then let $\boldsymbol{v}_2 = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ be another unit vector perpendicular to $\boldsymbol{v}_1$ so with

$$\begin{aligned}
y_1^2 + y_2^2 + \cdots + y_n^2 &= 1 \\
y_1 + y_2 + \cdots + y_{\frac{n}{k}} &= 0
\end{aligned}$$

Then

$$
\begin{aligned}
|\boldsymbol{A}\,\boldsymbol{v}_2| &= \sqrt{\frac{n}{k}\left(\sum_{i=1}^{\frac{n}{k}} a_1\,y_i\right)^2 + \frac{n}{k}\left(\sum_{i=\frac{n}{k}+1}^{2\frac{n}{k}} a_2\,y_i\right)^2 + \cdots + \frac{n}{k}\left(\sum_{i=(k-1)\frac{n}{k}+1}^{k\frac{n}{k}} a_k\,y_i\right)^2} \\
&= \sqrt{0 + \frac{n}{k}\left(\sum_{i=\frac{n}{k}+1}^{2\frac{n}{k}} a_2\,y_i\right)^2 + \cdots + \frac{n}{k}\left(\sum_{i=(k-1)\frac{n}{k}+1}^{k\frac{n}{k}} a_k\,y_i\right)^2} \\
&\leqslant \frac{n}{k}\sqrt{a_2^2\left(\sum_{i=\frac{n}{k}+1}^{2\frac{n}{k}} y_i^2\right) + a_3^2\left(\sum_{i=2\frac{n}{k}+1}^{3\frac{n}{k}} y_i^2\right) + \cdots + a_k^2\left(\sum_{i=(k-1)\frac{n}{k}+1}^{k\frac{n}{k}} y_i^2\right)} \\
&\leqslant \frac{n}{k}\sqrt{a_2^2} \\
&= \frac{n\,a_2}{k}
\end{aligned}
$$

Then when $y_{\frac{n}{k}+1} = y_{\frac{n}{k}+2} = \cdots = y_{2\frac{n}{k}} = \sqrt{\frac{n}{k}}$ and for the else $y_i = 0$ the $|\boldsymbol{A}\,\boldsymbol{v}_2|$ reaches its maximum.

So we have $\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{\frac{n}{k}} \\ \sqrt{\frac{n}{k}} \\ \vdots \\ \sqrt{\frac{n}{k}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with $\sigma_2 = \frac{n\,a_2}{k}$.

Similarly for all the else $\boldsymbol{v}_i$, when the $\left((i-1)\,\frac{n}{k}+1\right)^{\text{th}} \sim \left(i\,\frac{n}{k}\right)^{\text{th}}$ component if $\boldsymbol{v}_i$ is $\sqrt{\frac{n}{k}}$ and all the other compoents are zero, the $|\boldsymbol{A}\,\boldsymbol{v}_i|$ reaches its maximum with $\sigma_i = \frac{n\,a_i}{k}$.

When under the situation that $a_1 = a_2 = \ldots = a_k = a_0$, let $\boldsymbol{v}_i = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, then

$$
\begin{aligned}
|\boldsymbol{A}\,\boldsymbol{v}_i| &= \sqrt{\frac{n}{k}\left(\sum_{i=1}^{\frac{n}{k}} a_1\,x_i\right)^2 + \frac{n}{k}\left(\sum_{i=\frac{n}{k}+1}^{2\frac{n}{k}} a_2\,x_i\right)^2 + \cdots + \frac{n}{k}\left(\sum_{i=(k-1)\frac{n}{k}+1}^{k\frac{n}{k}} a_k\,x_i\right)^2} \\
&= a_0\sqrt{\frac{n}{k}}\sqrt{\left(\sum_{i=1}^{\frac{n}{k}} x_i\right)^2 + \left(\sum_{i=\frac{n}{k}+1}^{2\frac{n}{k}} x_i\right)^2 + \cdots + \left(\sum_{i=(k-1)\frac{n}{k}+1}^{k\frac{n}{k}} x_i\right)^2} \\
&\leqslant \frac{n\,a_0}{k}\sqrt{\sum_{i=1}^{n} x_i^2} \\
&= \frac{n\,a_0}{k}
\end{aligned}
$$

When $\boldsymbol{v}_i$ has some constant $\left((i-1)\,\frac{n}{k}+1\right)^{\text{th}} \sim \left(i\,\frac{n}{k}\right)^{\text{th}}$ components with equal vaule and all the nonzero components' square value add up to 1, $|\boldsymbol{A}\,\boldsymbol{v}_i|$ reaches its maxmimum. So all possible singular vectors are a set of orthonormal basis on the space formed by $\boldsymbol{v}_1 \sim \boldsymbol{v}_k$ given below.

Exercise 2. Computer the singlar valued decomposition of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Answer.

We caculate the engenvectors and engenvalues of $\boldsymbol{A}^T\boldsymbol{A}$.

$$\begin{aligned}\boldsymbol{A}^T\boldsymbol{A} &= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}\end{aligned}$$

By

$$|\lambda I - \boldsymbol{A}^T\boldsymbol{A}| = \left|\begin{pmatrix} \lambda - 10 & -14 \\ -14 & \lambda - 20 \end{pmatrix}\right| = 0$$

We have $\sigma_1 = \lambda_1 = \sqrt{15 + \sqrt{221}}$ and $\sigma_2 = \lambda_2 = \sqrt{15 - \sqrt{221}}$.

By caculating the engenvector

$$(\lambda I - \boldsymbol{A}^T\boldsymbol{A})\,\boldsymbol{v} = \boldsymbol{0}$$

We have $\boldsymbol{v}_1 = \begin{pmatrix} \sqrt{\frac{\sqrt{221} - 5}{2\sqrt{221}}} \\ \sqrt{\frac{\sqrt{221} + 5}{2\sqrt{221}}} \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} -\sqrt{\frac{\sqrt{221} + 5}{2\sqrt{221}}} \\ \sqrt{\frac{\sqrt{221} - 5}{2\sqrt{221}}} \end{pmatrix}$.

And so $\boldsymbol{u}_1 = \frac{\boldsymbol{A}\,\boldsymbol{v}_1}{\sigma_1} = \begin{pmatrix} \sqrt{\frac{\sqrt{221} - 10}{2\sqrt{221}}} \\ \sqrt{\frac{\sqrt{221} + 10}{2\sqrt{221}}} \end{pmatrix}$ and $\boldsymbol{u}_2 = \frac{\boldsymbol{A}\,\boldsymbol{v}_2}{\sigma_2} = \begin{pmatrix} \sqrt{\frac{\sqrt{221} + 10}{2\sqrt{221}}} \\ -\sqrt{\frac{\sqrt{221} - 10}{2\sqrt{221}}} \end{pmatrix}$.

So

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T \\ &= \begin{pmatrix} \sqrt{\frac{\sqrt{221} - 10}{2\sqrt{221}}} & \sqrt{\frac{\sqrt{221} + 10}{2\sqrt{221}}} \\ \sqrt{\frac{\sqrt{221} + 10}{2\sqrt{221}}} & -\sqrt{\frac{\sqrt{221} - 10}{2\sqrt{221}}} \end{pmatrix}\begin{pmatrix} \sqrt{15 + \sqrt{221}} & 0 \\ 0 & \sqrt{15 - \sqrt{221}} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{\sqrt{221} - 5}{2\sqrt{221}}} & \sqrt{\frac{\sqrt{221} + 5}{2\sqrt{221}}} \\ -\sqrt{\frac{\sqrt{221} + 5}{2\sqrt{221}}} & \sqrt{\frac{\sqrt{221} - 5}{2\sqrt{221}}} \end{pmatrix}\end{aligned}$$