Assignment for

Computer Science Theory for the Information Age

Day 1

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Exercise 1. Project the surface area of a sphere of radius $\sqrt{d}$ in d-dimensions onto a line through the center. For d equal 2 and 3, derive an explicit formula for how the projected surface area changes as we move along the line. For large d, argue (intuitively) that the projected surface area should behave like a Gaussian.

Answer.

Suppose we pick the line with $(x_2 = x_3 = \ldots = x_d = 0)$, calculate the derivate surface area parallel to this line, so that we have

$$d S_p(x_1) = A_{d-1}\left(\sqrt{d - x_1^2}\right) d x_1 \tag{1}$$

where $A_d(r)$ is the surface area of a d-dimensional sphere.

Notice that the integral of this notion is not equal to the surface area of the sphere.

In particular, the case of 2-dimensional sphere

$$d S_p(x_1) = 2\, d x_1 \tag{2}$$

the case of 3-dimensional sphere

$$d S_p(x_1) = 2\,\pi \sqrt{d - x^2}\, d x_1 \tag{3}$$

And for the cases of high-dimensional situations, take[1]

$$A_d(r) = \frac{2\,\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} r^{d-1} \tag{4}$$

we have

$$\frac{d S_p(x_1)}{d x_1} = \frac{2\,\pi^{\frac{d}{2}-1}}{\Gamma\left(\frac{d}{2}-1\right)} (d - x_1^2)^{\frac{d}{2}-1} \tag{5}$$

consider the Stirling's formula for the gamma function[2]

$$\Gamma(x) = \sqrt{\frac{2\,\pi}{x}} \left(\frac{x}{e}\right)^x \tag{6}$$

1. *Computer Science Theory for the Information Age*, p10, Lemma 2.1

we have

$$\frac{d S_p(x_1)}{d x_1} = \frac{2\,\pi^{\frac{d}{2}-1}}{\sqrt{\frac{4\,\pi}{d-2}}\left(\frac{d-2}{2e}\right)^{\frac{d}{2}-1}}\,(d-x_1^2)^{\frac{d}{2}-1} = \frac{(2\,\pi\,e)^{\frac{d}{2}-1}}{\sqrt{\pi}}\left(\frac{d-x_1^2}{d-2}\right)^{\frac{d}{2}-1} \tag{7}$$

take $d \rightarrow +\infty$, we have

$$\frac{d S_p(x_1)}{d x_1} = \frac{(2\,\pi\,e)^{\frac{d}{2}-1}}{\sqrt{\pi}}\,e^{1-\frac{x_1^2}{2}} \tag{8}$$

Note that this notion contains factor $e^{-\frac{x^2}{2}}$, which is liner related with the Guassian.

Exercise 2. For what value of d is the volume, $V(d)$, of a d-dimonsional unit sphere maximum?

Answer.

Take[3]

$$V(d) = \frac{2}{d}\,\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{9}$$

where gamma function

$$\Gamma(x) = (x-1)\,\Gamma(x-1) \tag{10}$$

Consider the ratio

$$r(d) = \frac{V(d)}{V(d-2)} = \frac{\frac{2}{d}\,\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}}{\frac{2}{d-2}\,\frac{\pi^{\frac{d}{2}-1}}{\Gamma\left(\frac{d}{2}-1\right)}} = \frac{2\,\pi}{d} \tag{11}$$

for $d \geqslant 7$, $r(d) < 1$, so the maximum of volume must be with $d=6$ or $d=5$.

$$V(6) = \frac{2\,\pi}{6}\,V(4) = \frac{\pi^2}{6}\,V(2) = \frac{\pi^3}{6} \tag{12}$$

$$V(5) = \frac{2\,\pi}{5}\,V(3) = \frac{8\,\pi^2}{15} \tag{13}$$

Notice that

$$V(6) < V(5) \tag{14}$$

So the maximum volume of all d-dimensional unit sphere is

$$V_m = V(5) = \frac{8\,\pi^2}{15} \approx 5.26379 \tag{15}$$

2. *Wikipedia*, *Stirling's approximation*, en.wikipedia.org/wiki/Stirling%27s_approximation
3. *Computer Science Theory for the Information Age*, p10, Lemme 2.1