# Assignment for

## Computer Science Theory for the Information Age

## Day 6

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**Exercise 1.** Modify the proof that every monotone property has a threshold for $G(n,p)$ to apply to the 3-CNF satisfiability problem.

**Answer.**

We'll use $F(n,m)$ to represent a 3-CNF formula with $n$ variables, $m$ clauses.

**Definition 1.** *The increasing property of unsatisfiability.*

*$Q$ is the increasing property of $F(n,m)$ if when a formula $F$ has the property any formula obtained by adding clauses to $F$ must also have the property.*

**Definition 2.** *D-fold replication of $F(n,m)$*

*To make an M-fold repilcation $F(n,m')$ of $F(n,m)$, first make $d$ copies of $F(n,m)$ with clauses generated independently, then include every clause present in any of the $d$ copies into $F(n,m')$.*

**Lemma 3.** *Let $Q$ be an increasing property of $F(n,m)$. for $m_1 < m_2$*

$$\mathrm{Prob}(F(n,m_1)\,\mathrm{has}\,Q) \leqslant \mathrm{Prob}(F(n,m_2)\,\mathrm{has}\,Q)$$

***Proof.*** *First we generated $F(n,m_1)$, then we generate $F(n,m_2-m_1)$, finally we include every clause present in either $F(n,m_!)$ or $F(n,m_2-m_1)$ to get the $F(n,m_2)$. By this mean we know that if $F(n,m_1)$ has property $Q$ $F(n,m_2)$ must has property $Q$ as well.* □

**Theorem.** *Every increasing property $Q$ of $F(n,m(n))$ has a threshold at $m(n)=a$, where for each $n$, $a$ is the minimum real number such that*

$$\mathrm{Prob}(F(n,a)\,\mathrm{has}\,Q) = \frac{1}{2}$$

***Proof.***

*Let $m_0(n)$ be any function such that $\lim_{n\to\infty}\frac{m_0(n)}{m(n)}=0$. We need to prove that*

$$\lim_{n\to\infty}\mathrm{Prob}(F(n,m_0(n))\,\mathrm{has}\,Q)=0 \tag{1}$$

*Assume that* $\mathrm{Prob}(F(n,m_0(n))\,\mathrm{has}\,Q)$ *does not converge to zero as n goes to infinity. Then there must be a infinity subsequence $I$ of $n$ such that*

$$\exists\varepsilon>0\forall n\in I\,\mathrm{Prob}(F(n,m_0(n))\,\mathrm{has}\,Q)>\varepsilon \tag{2}$$

*Let* $d = \lceil \frac{1}{\varepsilon} \rceil$*, and let* $F(n, m'(n))$ *be the d-fold replication of* $F(n, m_0(n))$*. Similar proof to Lemma 3 we have*

$$\text{Prob}(F(n, m'(n)) \,\text{dose not have}\, Q) \leqslant (\text{Prob}(F(n, m_0(n)) \,\text{does not have}\, Q))^d \tag{3}$$

*while*

$$m'(n) = 1 - (1 - m_0(n))^d \leqslant d\, m_0(n) \tag{4}$$

*so that*

$$\begin{aligned} \text{Prob}(F(n, d\, m_0(n)) \,\text{does not have}\, Q) &\leqslant (\text{Prob}(F(n, m_0(n)) \,\text{does not have}\, Q))^d \\ &\leqslant (1 - \varepsilon)^d \\ &\leqslant \frac{1}{e} \\ &\leqslant \frac{1}{2} \end{aligned} \tag{5}$$

*For infinity many n, since* $m(n)$ *is the minimum that* $\text{Prob}(F(n,\, m(n))$ has $Q) = \frac{1}{2}$*, by Lemma 3,* $d\, m_0(n) \geqslant m(n)$*, which contradicts the hypothesis that* $\lim_{n \to \infty} \frac{m_0(n)}{m(n)} = 0$*.*

□

**Exercise 2.** Verify that the sum of r rank-one matrices $\sum_{i=1}^{r} \sigma_i\, \boldsymbol{u}_i\, \boldsymbol{v}_i^T$ can be written as $\boldsymbol{U} \boldsymbol{D} \boldsymbol{V}^T$, where the $\boldsymbol{u}_i$ are the columns of $\boldsymbol{U}$ and $\boldsymbol{v}_i$ are the columns of $\boldsymbol{V}$. To do this, first verify that for any two matrices $\boldsymbol{P}$ and $\mathbf{Q}$, we have

$$\boldsymbol{P} \boldsymbol{Q}^T = \sum_i \boldsymbol{p}_i\, \boldsymbol{q}_i^T$$

where $\boldsymbol{p}_i$ is the $i^{\text{th}}$ column of $\boldsymbol{P}$ and $\boldsymbol{q}_i$ is the $i^{\text{th}}$ column of $\boldsymbol{Q}$.

**Answer.**

First we prove $\boldsymbol{P} \boldsymbol{Q}^T = \sum_i \boldsymbol{p}_i\, \boldsymbol{q}_i^T$.

**Proof.**

Let $\boldsymbol{A} = \boldsymbol{P} \boldsymbol{Q}^T$ and $\boldsymbol{B} = \sum_i \boldsymbol{p}_i\, \boldsymbol{q}_i^T$. Then

$$a_{ij} = \sum_{k=1}^{r} p_{ik}\, q_{jk} \tag{6}$$

and

$$\begin{aligned} b_{ij} &= \sum_{i=1}^{r} (\boldsymbol{p}_i\, \boldsymbol{q}_i^T)_{ij} \\ &= \sum_{k=1}^{r} p_{ik}\, q_{jk} \end{aligned} \tag{7}$$

so we have

$$a_{ij} = b_{ij} \tag{8}$$

which is

$$\boldsymbol{P} \boldsymbol{Q}^T = \sum_i \boldsymbol{p}_i\, \boldsymbol{q}_i^T \tag{9}$$

$\square$

Then we prove $\sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T$.

**Proof.**

Let $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T$ and $\boldsymbol{B} = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$. Then

$$\begin{aligned} b_{ij} &= \sum_{i=1}^{r} (\sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T)_{ij} \\ &= \sigma_i \sum_{k=1}^{r} u_{ik} v_{jk} \end{aligned}$$

so if we let $\boldsymbol{D}$ be a diagonal matrix with

$$d_{ii} = \sigma_i \tag{10}$$

we can find that

$$a_{ij} = \sum_{k=1}^{r} u_{ik} \sigma_i v_{jk} \tag{11}$$

which is

$$\boldsymbol{A} = \boldsymbol{B} \tag{12}$$

$\square$