Introduction to computed tomography

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Outline

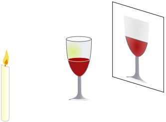
Introduction

The problem

We have a solid item to investigate. . .

- For a first look of the outside

- Next step, use a transmission image



- Cut the item in pieces ... virtually

Transmission imaging – Radiography

A ray penetrating a medium is attenuated according to Beer-Lamberts law The intensity is attenuated in the medium according to

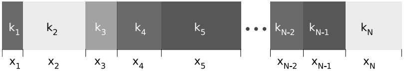
$$I = I_0 e^{\int_L k(x,y)\, dl}$$

- I - Intensity behind the sample
- I_0 - Incident intensity
- k - Attenuation coefficient,
  - μ - Linear attenuation coefficient X-rays
  - Σ - Macroscopic cross-section for neutrons
- **L** - Line through the sample.

Generalized attenuation law 1D

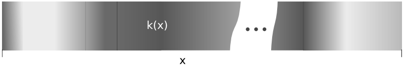
Piecewise constant sample

$$I = I_0\, e^{-\sum_i k_i\, x_i}$$



Let $x_i = \Delta x$ and $\Delta x \to 0$
We get the equation for a continuous sample

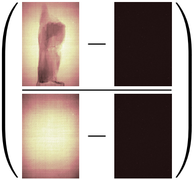
$$I = I_0\, e^{-\int_L k(x)\, dx}$$



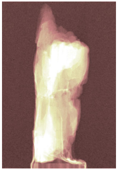
Computing an attenuation image

From Beer-Lamberts law we get

$$p = -\log\left(\frac{r - r_{DC}}{r_{OB} - r_{DC}}\right) = -\log$$



$$=$$



p Normed projection

r Measured radiogram

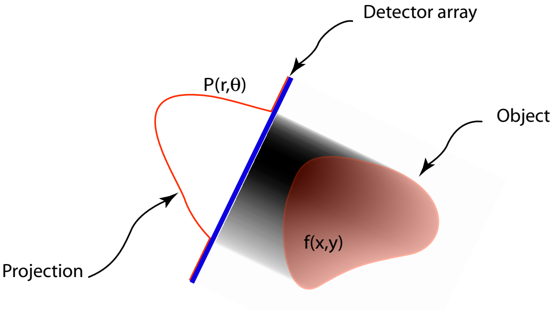
r_{DC} Dark current image (removes noise floor)

r_{OB} Open beam image, measured I_0

Each pixel represent the line integral $\int_L k(x)dx$ through the sample.

Transmission image – the projection

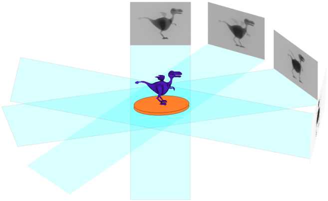
A ray illuminates a semi-transparent medium



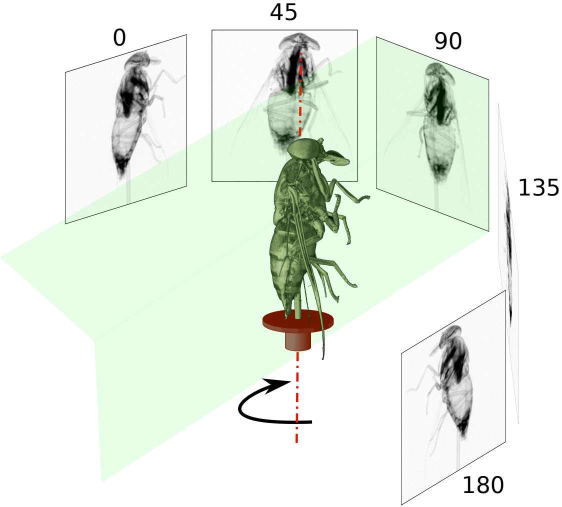
- r Position on detector
- θ Acquisition angle
- f Attenuation distribution

What is tomography?

- Free translation is slice imaging
  from Greek: *Tomos* – 'a section' or 'a cutting' and Graph – write
- A method to capture three-dimensional images.
- An indirect method using projections (radiographs) to reconstruct the inner structure of a sample.



Inspecting the sample from different views



A first attempt to reconstruction: Algebraic solution

Observations

$$\begin{array}{ccl} 2 & 3 & \rightarrow 5 \\ 1 & 4 & \rightarrow 5 \\ \downarrow & \downarrow & \\ 3 & 7 & \end{array}$$

Equation system

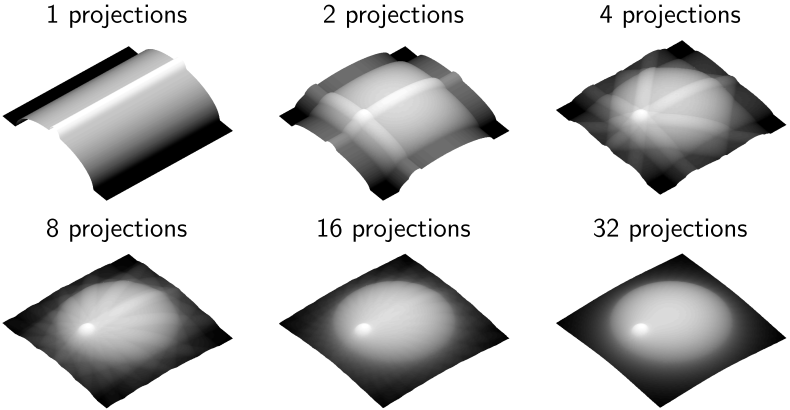
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= y_1 \\ a_{21}x_3 + a_{22}x_4 &= y_2 \\ a_{11}x_1 + a_{21}x_3 &= y_3 \\ a_{12}x_2 + a_{22}x_4 &= y_4 \\ &\vdots \end{aligned}$$

$$\Rightarrow Ax = y$$

solve the equation system for x

Many equations, sparse matrix A, no unique solution...

A first attempt: Basic back-projection



The solution is too smooth. . . something is missing!!!

History

Johann Radon (1887–1956)

Developed the foundation for the inversion required by tomography in 1917. He found the analytical solution to the inverse of the projection of a sample.

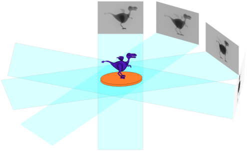
Sir Godfrey N. Hounsfield (1919–2004)

Constructed the first tomograph. He was awarded with the Nobel Prize in 1979

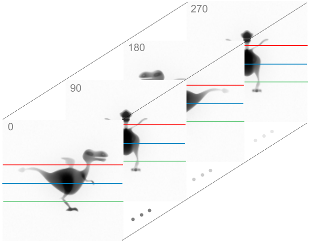
Reconstruction

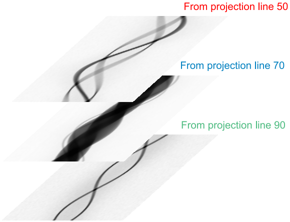
Acquistion and rearranging the projection data

To reconstruct a slice you need information from all projections



- This is called a sinogram
- Each line in the sinogram represent an acquisition angle



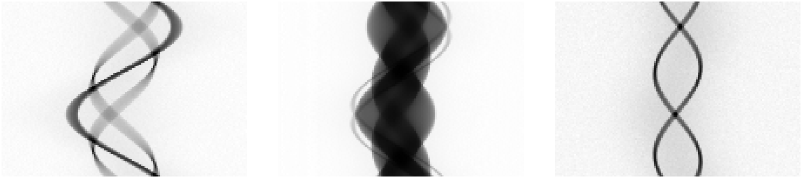


The Radon Transform and the sinogram

A projection I acquired at angle θ

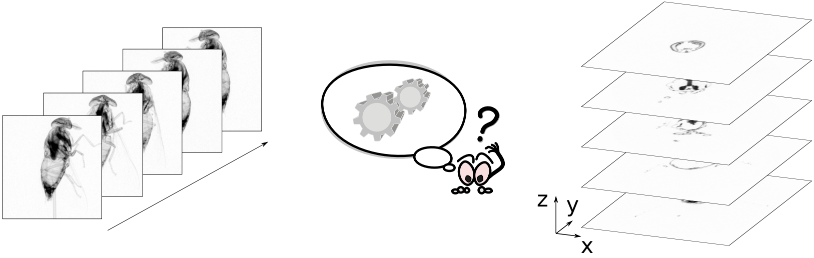
$$p = -\ln\left(\frac{I(u,\theta)}{I_0(u)}\right) = \int_{-\infty}^{\infty} \underbrace{k(x,y)}_{\text{Wanted}} \; \underbrace{\delta(x\,\cos\,\theta + y\,\sin\,\theta - u)}_{\text{Observation ray}}\, dx\, dy$$

This is the acquisition space.



The reverse process – reconstruction

The scanning provides projection data. . .



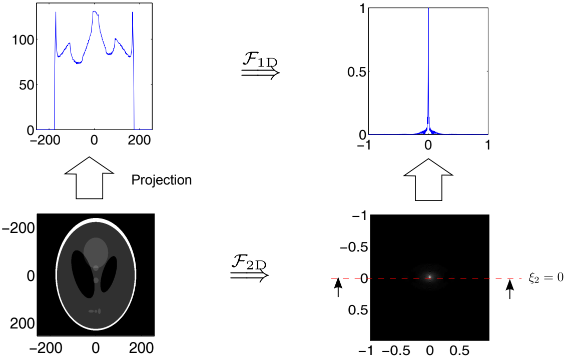
. . . but we want to find the cross section which caused the projection.

Inversion – Fourier slice theorem

Theorem

The Fourier transform of a parallel projection $p(x)$ of an object $f(x,y)$ obtained at an angle θ equals a line through origin in the 2D Fourier transform of $f(x,y)$ at the same angle.

$$f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy \qquad f(\xi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-j2\pi\xi x}\,dx\,dy$$



$$f(x,y) \qquad F(\xi_1,\xi_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-j2\pi\,(x\xi_1+y\xi_2)}\,dxdy$$

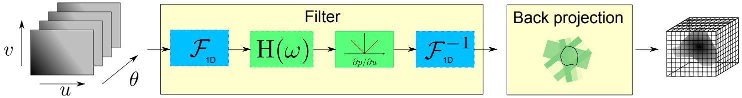
Analytical solution

Reconstruction in the frequency domain:

$$k(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} |\omega| P(\omega,\theta)\, e^{j2\pi\omega(x\cos\theta + y\sin\theta)}\, d\omega d\theta$$

Reconstruction in the spatial domain:

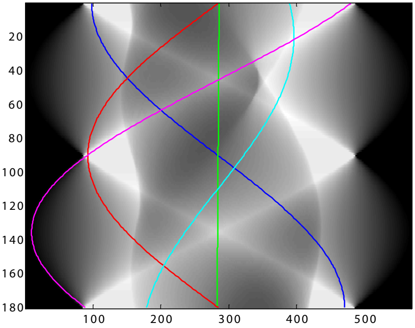
$$k(x,y) = \frac{1}{2\pi^2} \int_0^{\pi} \int_{-\infty}^{\infty} \underbrace{\partial p/\partial u(u,\theta)}_{Convolution} \underbrace{[x\cos\theta + y\sin\theta - u]^{-1}}_{Rotation}\, du d\theta$$



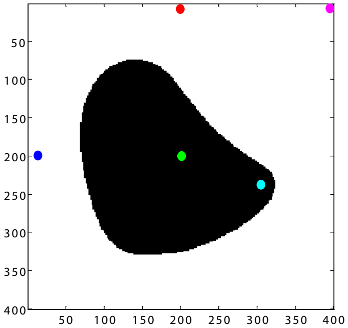
Some line integrals in the sinogram

Each spot is reconstructed by summing up along sine curves

Sinogram



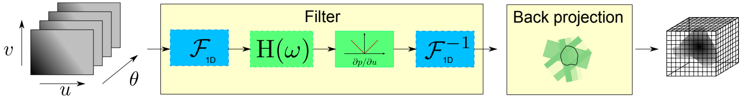
Cross section



The reconstruction filter

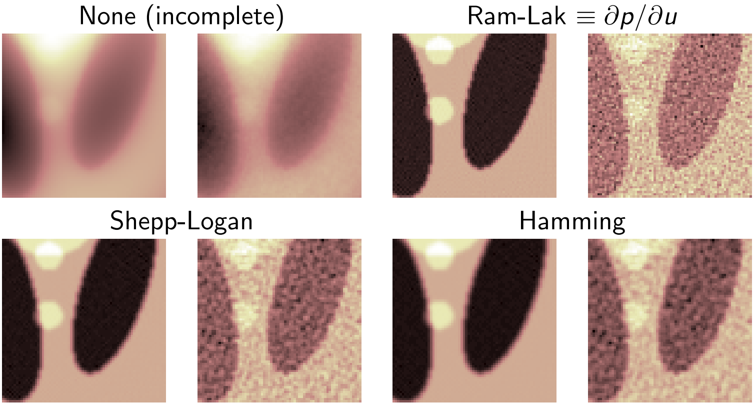
The filter has two parts:

- A derivative: $\partial p/\partial u(u,\theta) \equiv \mathcal{F}^{-1}(|\omega| \cdot \mathcal{F}(p))$
- Apodization: Shepp-Logan, Hamming, *etc*



Reconstruction filter

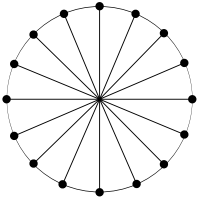
The high level of discretization makes the inversion noise sensitive.



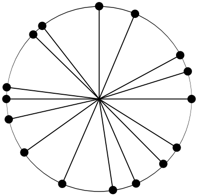
Low-pass filters are used to smooth the solution

When the analytical solution has problems

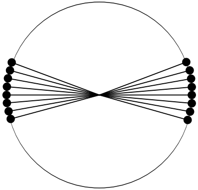
Few projections



Irregularly distributed



Limited view



Low SNR/Low contrast

Iterative methods overview

Algebraic methods

- ART
- SIRT
- TV
- etc...

Statistical methods

- Maximum likelihood
- Penalized ML
- etc..

Pros & cons

+ Sparse, irregularly sampled projection data
  - Limited angle
  - Few views
+ Physical model can be included

- Requires prior information for best performance.
- Time consuming

Algebraic Reconstruction Method (ART)

Problem to solve

We want to solve the equation $A\,x = y$,
where A is the forward projection operator, a large sparse matrix

Kaczmarz method (ART)

$$x^{k+1} = x^k + \lambda_k \frac{y_i - \langle a_i, x^k \rangle}{\|a_i\|^2}\, a_i$$

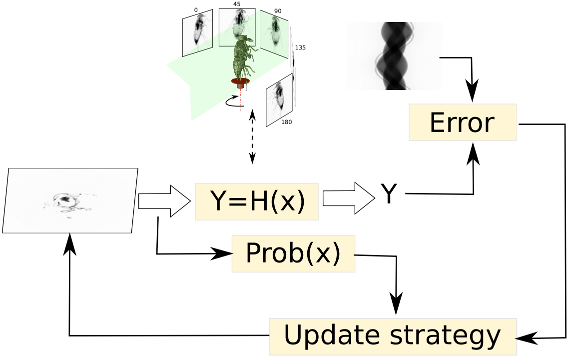
- a_i the i^{th} row of the system matrix A.
- x^k the reconstructed image at the k^{th} iteration.
- y_i the i^{th} element of the sinogram
- λ_k relaxation parameter

Statistic reconstruction methods

Problem to solve

We want to solve the equation $A\,x = y + N$,

Iteration scheme

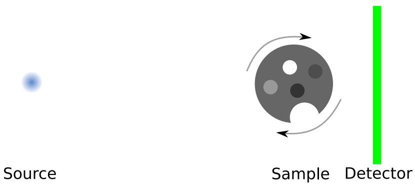


Beam geometry

Different beamline configurations

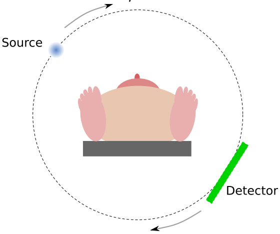
Static beamline

Mainly scientific/engineering

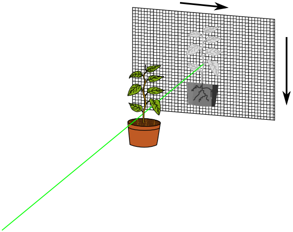


Rotating beamline

Medical/scientific

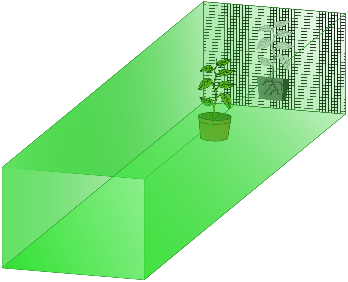


Pencil-beam



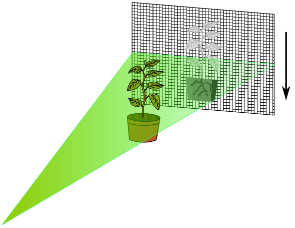
- Simples beam geometry
- Single pixels are scanned
- The 'Hounsfield-approach'

Parallel beam



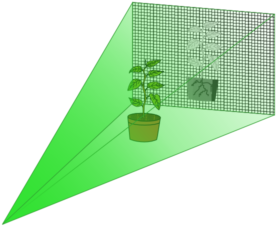
- Produces 2D projections
- No geometric unsharpness
- Simple reconstruction, filtered back projection [?]

Fan beam



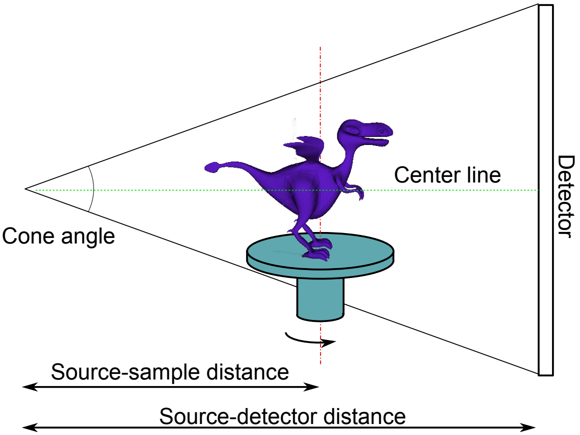
- Line-wise scan
- Beam incidence must be perpendicular to detector plane
- Magnifying in one direction
- Sample or detector must be vertically translated

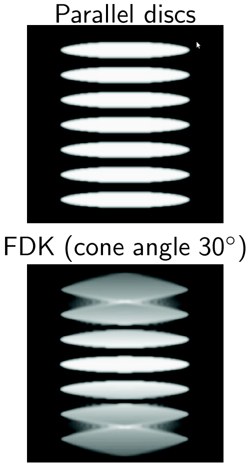
Cone beam



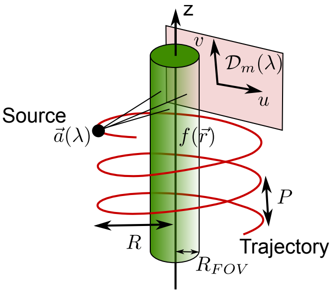
+ Uses 2D-projections.
+ Magnifying due to beam divergence.
- Non-trivial reconstruction using [?].
- Only in the central slice is exact.

Problems with cone-beam



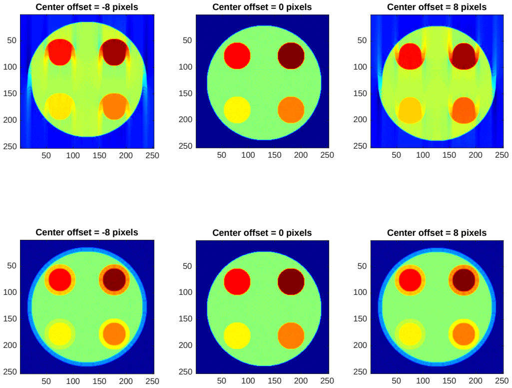


Helical scans



- Exact 3D solution
- Long objects
- Reconstruction using Katsevich[**?**]

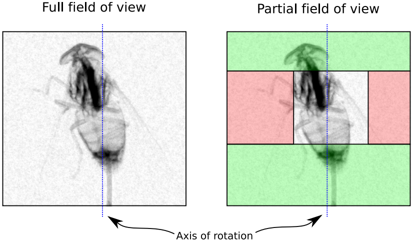
The importance of the center of rotation



Large samples – The problem

Requirement

Projections from at least 180° + sample must always be visible.



Two options to handle samples larger than the field of view

- Translate the COR and use a 360° orbit.
- Truncated reconstruction

Truncated or Local tomography

$\boxed{\text{A truncated tomography has incomplete data support.}}$

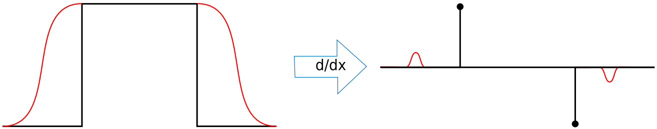
Effects of truncation

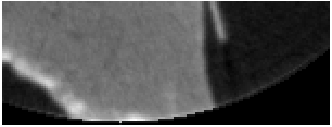
1. Some attenuation information is missing $\rightarrow$ bias
   The shadow contains more attenuation than the projection data shows.
2. Truncation gives spikes on the edges.
   The derivative in the reconstruction formula produce edge artifacts.

Removing truncation artifacts

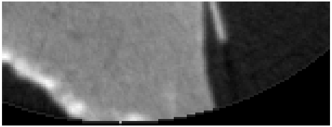
Origin The derivative of the truncated edge is steep

Solution Add a smooth transition from edge to zero





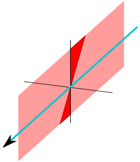
Original



Padded

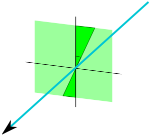
Tilted acquistion axis

Along the beam



- Hard to correct
- Requires vector based reconstructor

Across the beam



Small angles corrected with COR shifts

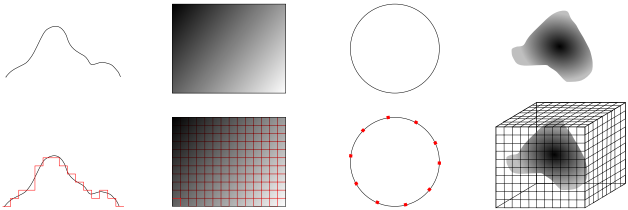
Large angles corrected with rotation

Sampling

Discretizing the reconstruction formula

The inversion formula is impractical since it would require infinite amount of equations to solve.

- The projections are digital images
    - Intensity sampling [bits/pixel]
    - Spatial sampling [pixels/mm]
- The rotation is done in steps
- The reconstruction is done on a finite matrix

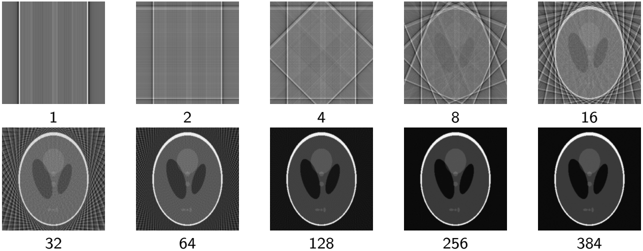


How many projections are needed?

The number of projections is determined by the sampling theorem [?].

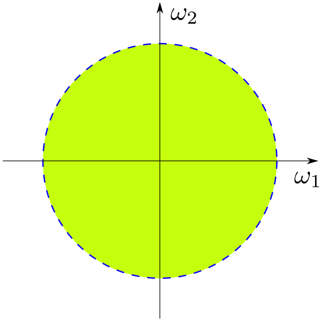
$$N_{projections} = \frac{\pi}{2} N_u$$

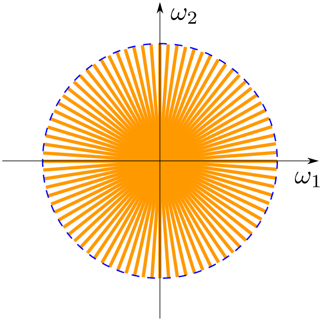
N_u Number of pixels in the direction perpendicular to the axis of rotation.



Intuitive proof of the sampling theorem

Basic idea The unit circle in the Fourier domain must be filled.

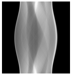
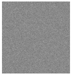
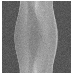
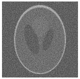




Noise and Dose

Noise

Noise is an additive statistical phenomenon.

$\mathcal{R}^{-1}\Big\{$  $\Big\} + \mathcal{R}^{-1}\Big\{$  $\Big\} = \mathcal{R}^{-1}\Big\{$  $\Big\} \rightarrow$ 

Noise sources:

- Noise induced by the radiation source.
- Thermal noise from the electronics.
- Algorithmic, rounding errors, interpolation model etc.

Dose

The dose is the amount of radiation events hitting the detector. More events improve the SNR (the law of great numbers).

Noise, exposure time, and number of projections

The noise level of a slice is directly connected to the dose used. The dose is defined as

$$Dose = Flux \times Time$$

The signal to noise ratio can be improved by increasing

- the beam intensity,
- the exposure time,
- the number of projections,
- detector exchange.

Contrast

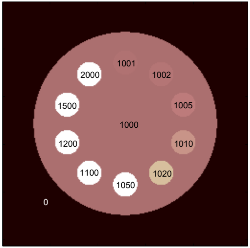
What influences the contrast?

$$C_{slice}\, W_{sample} = C_{projection}\, N_{projections}$$

C_{slice}	Slice contrast
$C_{projection}$	Projection contrast (Open beam - darkest region)
$N_{projections}$	Number of projections
W_{sample}	Largest width of the sample in pixels

Contrast experiment

The phantom

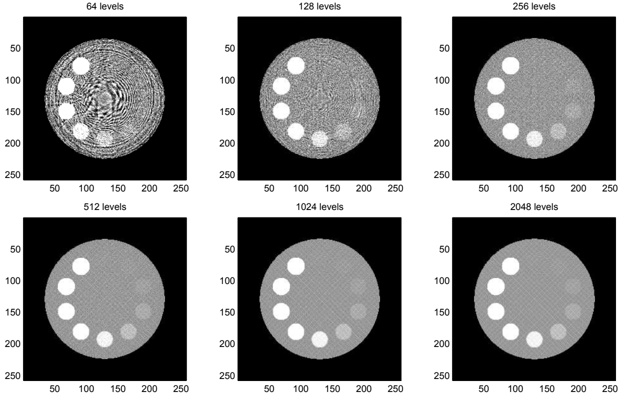


Parameters

- w=192
- $N_{projections}$=288
- $C_{projection}$=6, 7, 8, 9, 10,11, 12, 13 bits
- Contrast ratio: 1000:1, . . . , 1:2
- Noise free

What can be seen?

Changing projection contrast with constant number of projections



The reconstruction noise decrease with increasing dynamics

Artifacts

Common artifacts

Rings are caused by stuck or dead pixels. They have the same value for all projections

Lines are caused by single pixels or groups pixels in a single projection

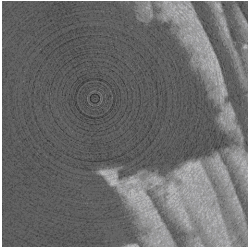
High contrast these artifacts appear as starlike streaks originating from the high contrast object.

Motion when the sample changes during acquisition.

Beam hardening Polychromatic beam

Scattering The beam is scattered

Ring artifacts



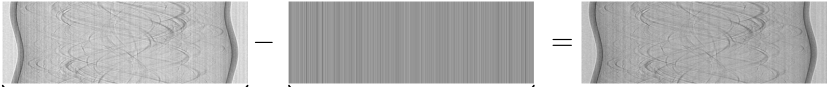
- Ring artifacts are very common artifacts in tomography.
- They are caused by a stuck, dead, or hot pixels.
- They appear as:
    - Lines in the sinogram
    - Concentric rings in the CT slices

Correction in the Radon space

Projections Identify and remove spots that persists through projections.

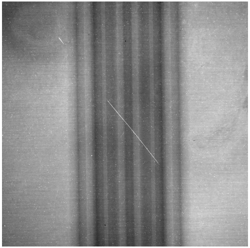
Sinograms Identify lines parallel to the θ-axis

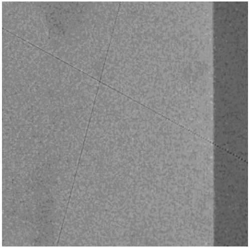
- Subtract first derivative of average projection form sinogram.
- Filter sinogram in Fourier domain (notch filter or wavelet filter).



$$\underbrace{\quad}_{s(u,\theta)} - \underbrace{\quad}_{1^T \cdot (E_u[s] - median_N(E_u[s]))} =$$

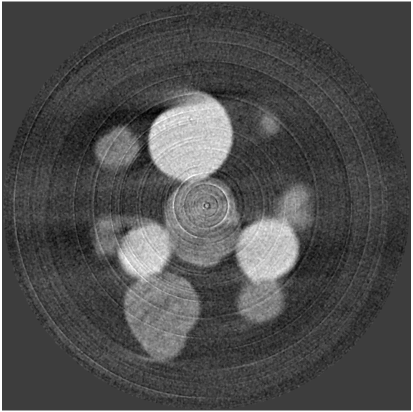
Line artifacts


Projection

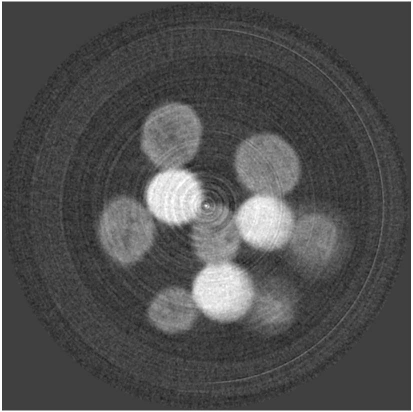

Reconstructed slice

- Line artifacts are more common with neutrons
- The origin of a line is a local spot in the sinogram.
- The orientation and position depends on when the spot was registered.
- Correction method provided in image proc lecture.

Motion artifacts



Sequential acquisition



Golden ratio acquisition

Suppressing the effect of motion

Dynamic processes are hard to observe with CT

- CT needs long scan times.
- If the interfaces move more than 1 pixel during the scan motion artifacts will appear.

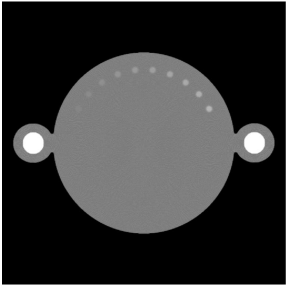
The solution

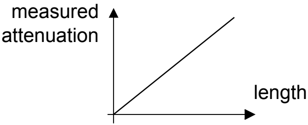
- Increment the acquisition angle by the Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$
- The sample will always be observed at 'orthogonal' angles.

[?][?]

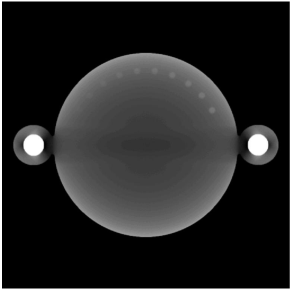
Beam hardening

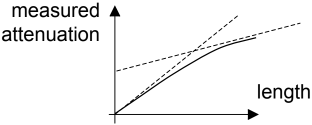
Monochromatic





Polychromatic

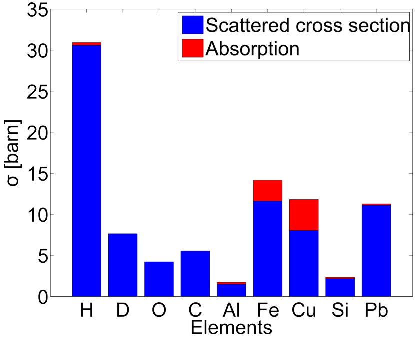




Attenuation for neutrons

The attenuation law assumes the intensity to be absorbed. . .

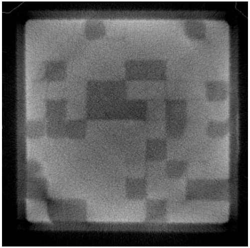
This is not true for neutrons!!!



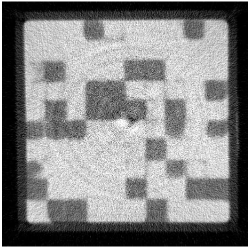
Scatter

Scatter is bad for

- Quantitative imaging
- Segmentation algorithms



Uncorrected



Corrected by QNI [?]

Summary

Summary

- Tomography is an indirect acquisition method
- Different sources can be used
- The perfect tomography needs
  - many projections
  - well illuminated projections
- Artifacts may and will appear but can mostly be corrected.

References I

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Filtered back-projection (Proof)

Image function:

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi_1,\xi_2)\, e^{j2\pi(x\xi_1+y\xi_2)}\, d\xi_1 d\xi_2$$

Coordinate transform $\{\xi_1,\xi_2\}=\{\omega\cos\theta,\omega\sin\theta\}$

$$f(x,y)=\int_{0}^{2\pi}\int_{-\infty}^{\infty} F(\omega\cos\theta,\omega\sin\theta)\, e^{j2\pi\omega(x\cos\theta+y\sin\theta)}\, d\omega d\theta$$

Fourier slice theorem:

$$f(x,y)=\int_{0}^{2\pi}\int_{-\infty}^{\infty} P(\omega,\theta)\, e^{j2\pi\omega(x\cos\theta+y\sin\theta)}\omega\, d\omega d\theta$$

Symmetry properties:

$$P(\omega,\theta+\pi)=P(-\omega,\theta)$$

Rotated coordinates:

$$f(x,y)=\int_{0}^{\pi}\int_{-\infty}^{\infty} P(\omega,\theta)\, e^{j2\pi\omega(x\cos\theta+y\sin\theta)}|\omega|\, d\omega d\theta$$

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A basic back-projection algorithm

```
pProj  : pointer to line in sinogram
pSlice : pointer to slice matrix

for (float line=0; line<nProjections; line++) {                // Loop over projections i
    for (size_t y=0; y<SizeY; y++) {                          // Loop over matrix in y
        const size_t cfStartX = mask[y].first;                // Get x-coordinates
        const size_t cfStopX  = mask[y].second;
        fStartU += cos(theta[line]);                          // Compute first proj. pos.
        float fPosU=fStartU-sin(theta[line])*cfStartX;

        for (size_t x=cfStartX; x<cfStopX; x++) {             // Loop over matrix in x
            int nPosU=static_cast<int>(fPosU-=sin(theta[line]));  // Compute position

            const float interpB=fPosU-nPosU;                  // Interpolation weight right
            const float interpA=1.0f-interpB;                 // Interpolation weight left

            pSlice[x+y*sizeX]+=interpA*pProj[nPosU]+interpB*pProj[nPosU+1];
        }
    }
}
```