THE DERIVATIVE FUNCTION

- The function f' defined by the formula

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit exists.

- The term "derivative" is used because the function f' is *derived* from the function f by a limiting process.

- https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-1-new/ab-2-2/e/the-formal-and-alternate-form-of-the-derivative

- https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-1-new/ab-2-2/e/the-formal-and-alternate-form-of-the-derivative

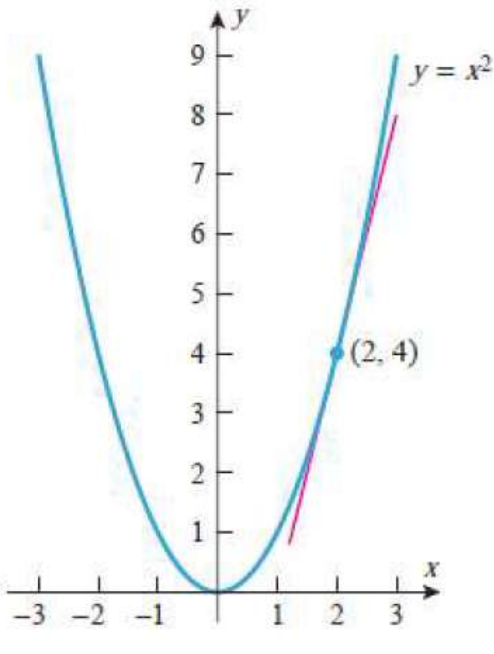
EXAMPLE

- Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{2xh}{h} + \frac{h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

- Therefore, the slope of the tangent line to $y = x^2$ at $x = 2$ is $f'(2) = 4$.
- Since $y = 4$ if $x = 2$, the point slope form of the tangent line is $y - 4 = 4(x - 2)$.
- Similarly $y = 4x - 4$.



FINDING THE EQUATION OF TANGENT LINE OF A FUNCTION AT A GIVEN POINT

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$.

1. Evaluate $f(x_0)$; the given point of tangency is $(x_0, f(x_0))$.
2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.
3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or equivalently

$$y = f(x_0) + f'(x_0)(x - x_0)$$

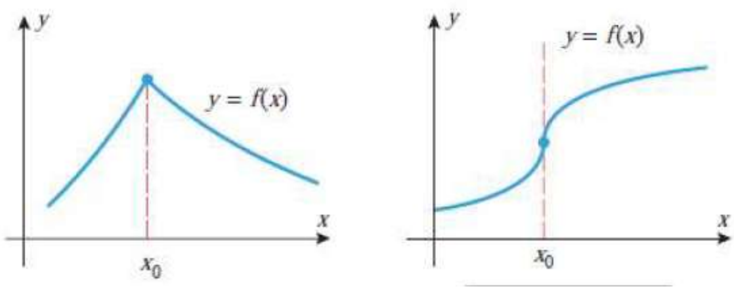
DIFFERENTIABILITY

- It is possible that the limit that defines the derivative of a function f may not exist at certain points in the domain of f. At such points the derivative is undefined.
- A function f is said to be **differentiable at $\boldsymbol{x_0}$** if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \text{ exists.}$$

- If f is differentiable at each point of the open interval (a, b), then we say that it is **differentiable on $(\boldsymbol{a}, \boldsymbol{b})$**, and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$ and $(-\infty, +\infty)$.

- Geometrically, a function f is differentiable at x_0 if the graph of f has a tangent line at x_0.
- At a corner point, the slopes of the secant lines have different limits from the left and from the right, and hence the **two-sided** limit that defines the derivative does not exist.
- At a point of vertical tangency the slopes of the secant lines approach $+\infty$ or $-\infty$ from the left and from the right so again the limit that defines the derivative does not exist.



THE RELATIONSHIP BETWEEN DIFFERENTIABILITY AND CONTINUITY

Theorem

- If a function f is differentiable at x_0, then f is continuous at x_0.

PRACTICE PROBLEMS

- https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-1-new/ab-2-4/e/differentiability-at-a-point-graphical
- https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-1-new/ab-2-4/e/differentiability-at-a-point-algebraic

DERIVATIVE NOTATIONS

- The process of finding a derivative is called **differentiation**.
- When the independent variable is x, the differentiation operation is commonly denoted by

$$f'(x) = \frac{d}{dx}[f(x)] \qquad \text{or} \qquad f'(x) = D_x[f(x)]$$

- In the case where there is a dependent variable $y = f(x)$, the derivative is commonly denoted by

$$f'(x) = y'(x) \qquad \text{or} \qquad f'(x) = \frac{dy}{dx}$$

DERIVATIVE AT A POINT

- The value of the derivative at a point x_0 can be expressed as

$$f'(x_0) = \frac{d}{dx}[f(x)]\,|_{x=x_0} \qquad f'(x_0) = D_x[f(x)]\,|_{x=x_0}$$

$$f'(x_0) = y'(x_0) \qquad f'(x_0) = \frac{dy}{dx}|_{x=x_0}$$

DERIVATIVE AT A POINT

- The value of the derivative at a point x_0 can be expressed as

$$f'(x_0) = \frac{d}{dx}[f(x)]\,\Big|_{x=x_0} \qquad f'(x_0) = D_x[f(x)]\,\Big|_{x=x_0}$$

$$f'(x_0) = y'(x_0) \qquad f'(x_0) = \frac{dy}{dx}\Big|_{x=x_0}$$

DERIVATIVE OF A CONSTANT

Theorem

- The derivative of a constant function is 0; that is, if c is any real number, then $\frac{d}{dx}[c] = 0$.

Exercise: Prove the above theorem using the definition of derivatives.

Examples:

- $\frac{d}{dx}[5] = 0$
- $\frac{d}{dx}[\pi] = 0$
- $\frac{d}{dx}\left[-\sqrt{2}\right] = 0$

DERIVATIVES OF POWER FUNCTIONS

Theorem

- If n is a positive integer, then $\frac{d}{dx}[x^n] = nx^{n-1}$.

Examples

1. $\frac{d}{dx}[x] = 1$
2. $\frac{d}{dx}[x^2] = 2x$
3. $\frac{d}{dx}[x^{20}] = 20x^{19}$

EXTENDED POWER RULE

Theorem

- If r is any real number, then $\frac{d}{dx}[x^r] = rx^{r-1}$.

Examples

1. $\frac{d}{dx}[x^{\pi}] = \pi x^{\pi-1}$
2. $\frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$
3. $\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}\left[x^{\frac{1}{3}}\right] = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}}$

DERIVATIVE OF A CONSTANT TIMES A FUNCTION

Theorem (Constant Multiple Rule)

- If f is differentiable at x and c is any real number, then cf is also differentiable at x and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Exercise

Prove the above theorem using the definition of derivatives.

Examples:

1. $\frac{d}{dx}[5x^4] = 5\frac{d}{dx}[x^4] = 5 \times 4x^3 = 20x^3$
2. $\frac{d}{dx}[\pi x^2] = \pi\frac{d}{dx}[x^2] = \pi \times 2x = 2\pi x$

SUM AND DIFFERENCE RULES

Theorem

- If f and g are differentiable at x, then so are $f + g$ and $f - g$ and

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$
$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Examples

1) $\frac{d}{dx}[3x + 4] = \frac{d}{dx}[3x] + \frac{d}{dx}[4] = 3\frac{d}{dx}[x] + 0 = 3 \cdot 1 = 3$

2) $\frac{d}{dx}[x^2 - 6x^7] = \frac{d}{dx}[x^2] - \frac{d}{dx}[6x^7] = 2x - 7x^6$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$

<u>Exercise</u>

- Using the definition of the derivative obtain $\frac{d}{dx}(\sin x) = \cos x$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

<u>Exercise</u>

Use quotient rule and prove that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

We will establish that $f(x) = \ln x$ is differentiable for $x > 0$ by applying the derivative definition to $f(x)$.

- $\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad x > 0$
- $\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \qquad x > 0$

DERIVATIVE OF EXPONENTIAL FUNCTIONS

- To obtain a derivative formula for b^x we rewrite $y = b^x$ as $x = \log_b y$ and differentiate and obtain

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ and replacing y by b^x we have $\frac{dy}{dx} = y \ln b = b^x \ln b$.

- Thus we have $\frac{d}{dx}[b^x] = b^x \ln b$
- When $b = e$ we have $\ln e = 1$.
- $\frac{d}{dx}[e^x] = e^x$