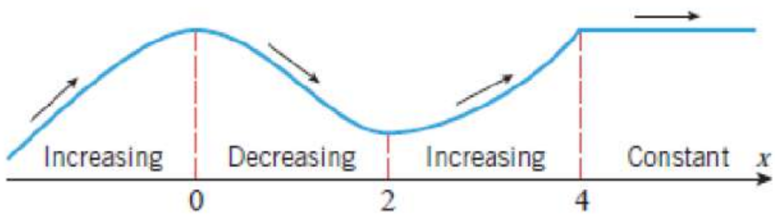
OUTLINE

Applications of differentiation

- Maximum and Minimum values
- First Derivative Test
- Applications of Derivatives

INCREASING AND DECREASING FUNCTIONS

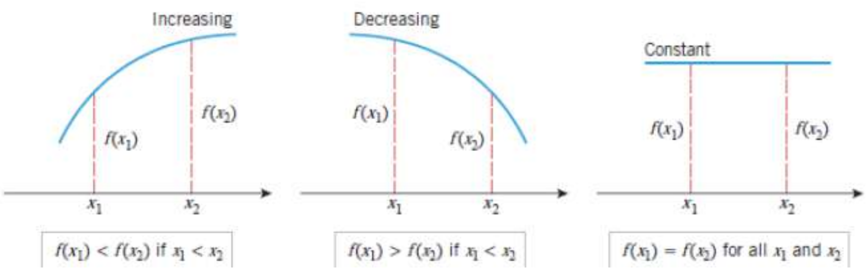
- The terms *increasing*, *decreasing*, and *constant* are used to describe the behavior of a function as we travel left to right along its graph.
- For example, the function graphed in the following figure can be described as increasing to the left of $x = 0$, decreasing from $x = 0$ to $x = 2$, increasing from $x = 2$ to $x = 4$, and constant to the right of $x = 4$.



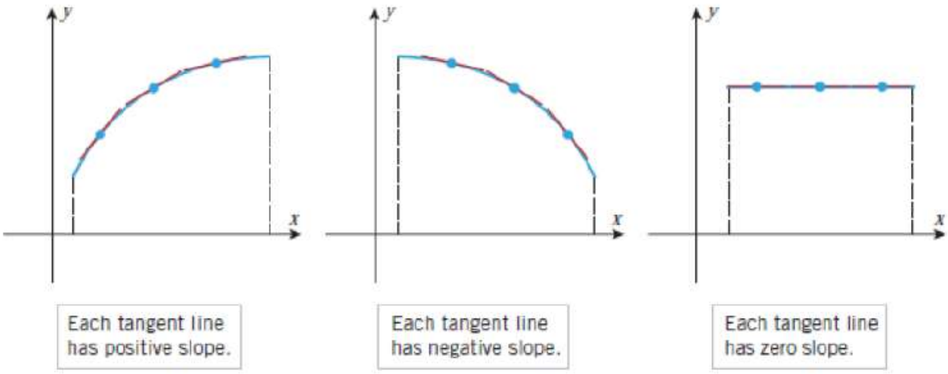
DEFINITION

Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

1. f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
2. f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
3. f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



- Following figure suggests that a differentiable function f is increasing on any interval where each tangent line to its graph has positive slope, is decreasing on any interval where each tangent line to its graph has negative slope, and is constant on any interval where each tangent line to its graph has zero slope.



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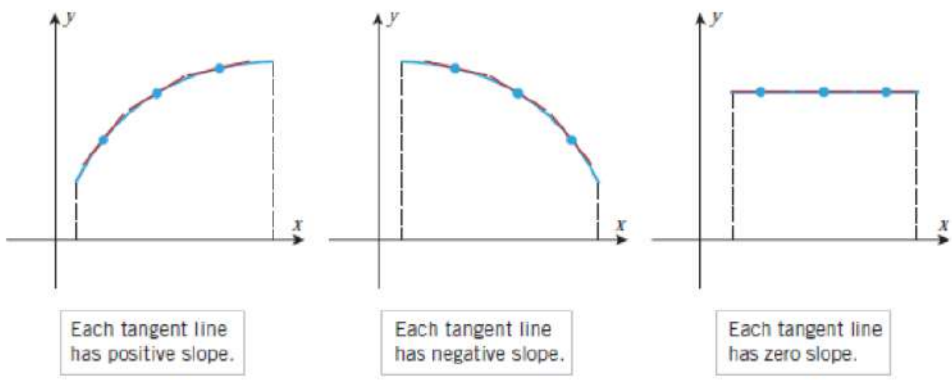


THEOREM

- Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).

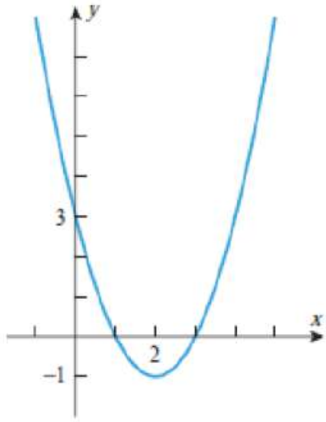
I. If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

II. If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.

III. If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

Example:

Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.



$f(x) = x^2 - 4x + 3$

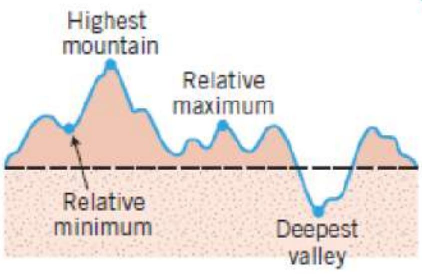
- The graph of f in the following figure suggests that f is decreasing for $x \leq 2$ and increasing for $x \geq 2$.
- To confirm this, we analyze the sign of f'.
- The derivative of f is $f'(x) = 2x - 4 = 2(x - 2)$.
- It follows that $f'(x) < 0$ if $x < 2$

  $f'(x) > 0$ if $2 < x$
- Since f is continuous everywhere, f is decreasing on $(-\infty, 2]$

  f is increasing on $[2, +\infty)$

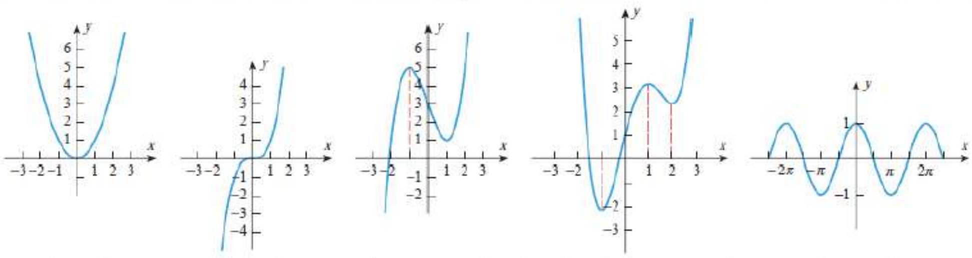
RELATIVE MAXIMA AND MINIMA (LOCAL MAXIMA OR MINIMA)

- A function f is said to have a **relative maximum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval.
- A function f is said to have a **relative minimum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval.
- If f has either a relative maximum or a relative minimum at x_0, then f is said to have a **relative extremum** at x_0.



Example

- $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima.
- $f(x) = x^3$ has no relative extrema.
- $f(x) = x^3 - 3x + 3$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.
- $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ has relative minima at $x = -1$ and $x = 2$ and a relative maximum at $x = 1$.
- $f(x) = \cos x$ has relative maxima at all even multiples of π and relative minima at all odd multiples of π.



$y = x^2$ $\quad$ $y = x^3$ $\quad$ $y = x^3 - 3x + 3$ $\quad$ $y = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ $\quad$ $y = \cos x$

GLOBAL EXTREMA

Global Maximum

A function has an **absolute maximum** (or **global maximum**) at c if $f\ (x)\ \leq\ f\ (c)$ for all x in the domain D of f.

The value $f(c)$ is called the **maximum value** of f on D .

Global Minimum

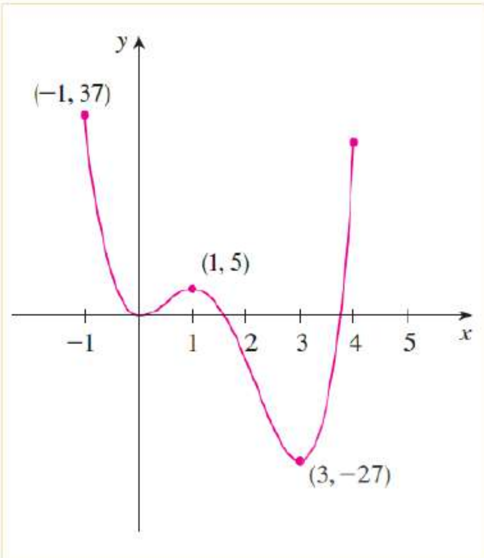
A function has an **absolute minimum** (or **global minimum**) at c if $f\ (x)\ \geq\ f\ (c)$ for all x in the domain D of f.

The value $f(c)$ is called the **minimum value** of f on D.

EXAMPLE

$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \le x \le 4$

- $f(1) = 5$ is a local maximum.
- $f(-1) = 37$ is the global maximum.
- $f(0) = 0$ is a local minimum.
- $f(3) = -27$ is both a local and global minimum.
- No local or global maximum at $x = 4$.



CRITICAL POINTS

Definition

A **critical point** of a function is a point in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

- To distinguish between the two types of critical points we call x a **stationary point** of f if $f'(x) = 0$.

Example:

1. Let $f(x) = 2x^2 - 3x + 1$, then $f'(x) = 4x - 3$. Hence $x_0 = 3/4$ is a critical point.

2. Let $g(x) = |x|$, Here $x_0 = 0$ is a critical point as $g(x)$ is not differentiable at $x_0 = 0$.

EXERCISE

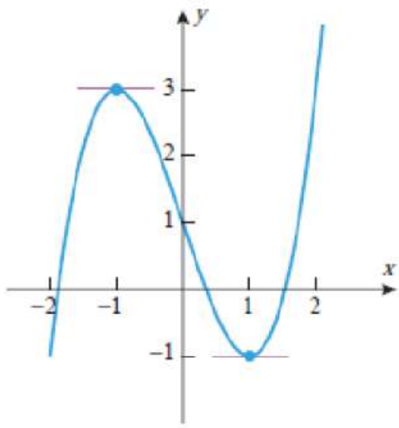
- Find all critical points of $f(x) = x^3 - 3x + 1$.

First derivative of $f(x)$is $f'(x) = 3x^2 - 3$

Take $f'(x) = 3x^2 - 3 = 0$

By solving above equation we get $x = 1$ and $x = -1$.

Therefore the critical points of $f(x)$ are $x = 1$ and x= -1.



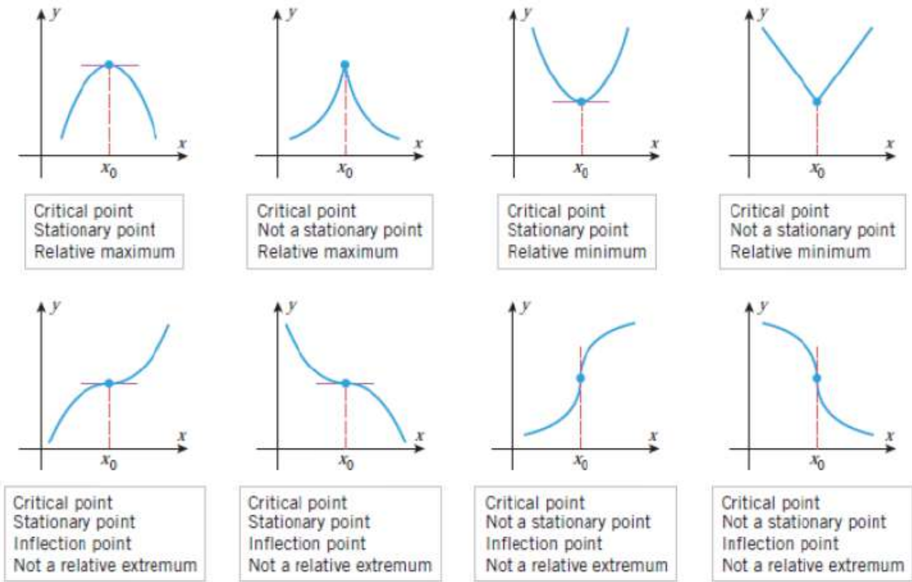
$y = x^3 - 3x + 1$

FIRST DERIVATIVE TEST

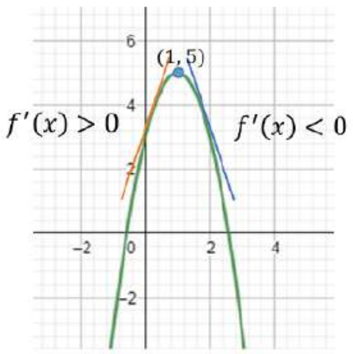
Suppose that c is a critical point of a continuous function $f(x)$.

1. If $f'(x)$ changes from **positive to negative at c**, then $f(x)$ has a **local maximum at c.**
2. If $f'(x)$ changes from **negative to positive at c**, then $f(x)$ has a **local minimum at c .**
3. If $f'(x)$ **does not change sign at c** (for example, if $f'(x)$ is positive on both sides of c or negative on both sides), then $f(x)$ has **no local maximum or minimum at c.**

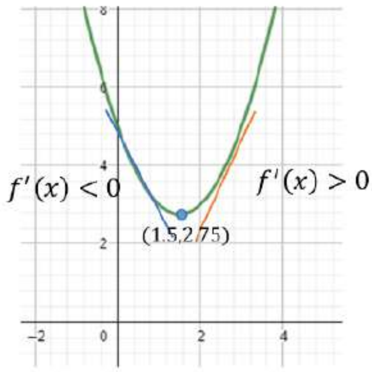
EXPLANATION



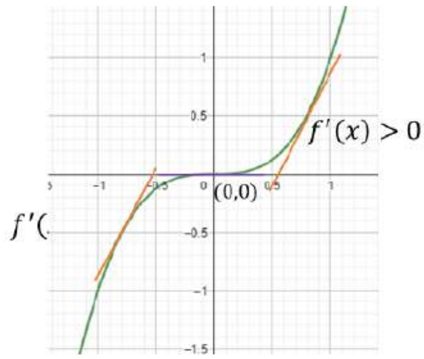
Examples:



Graph of y $= -2x^2 + 4x + 3$



Graph of y $= x^2 - 3x + 5$



Graph of y $= x^3$

Local Maximum

Local Minimum

Neither Local Maximum nor Local Minimum

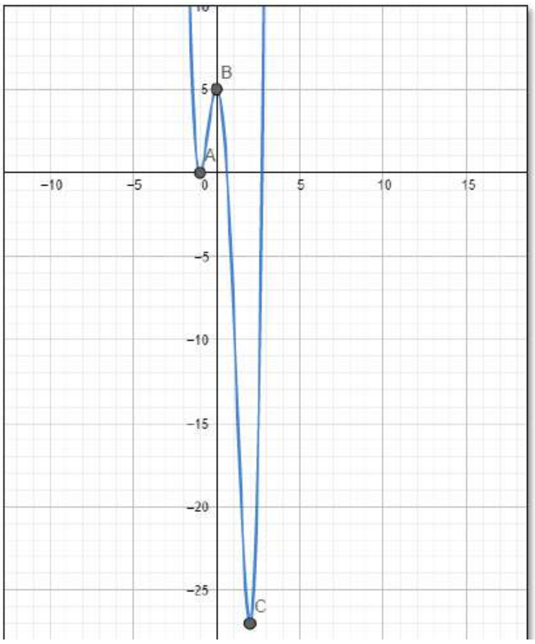
- Example: $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
- $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$ Therefore critical points: $x_0 = -1, 0, 2$

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1,0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0,2)$
$2 < x$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

From the First Derivative Test,

- $f'(x)$ changes from negative to positive **at -1**, so $f(-1)$ is a **local minimum** value.
- $f'(x)$ changes from positive to negative **at 0**, so $f(0)$ is a **local maximum** value.
- $f'(x)$ changes from negative to positive **at 2**, so $f(2)$ is a **local minimum** value.

- Graph of $y = 3x^4 - 4x^3 - 12x^2 + 5$



APPLICATIONS OF DERIVATIVES

- Assume that the utility (y) that a consumer obtains from the consumption of different units of a good (x) is given by the function $y = f(x) = \frac{1}{x}$. Find the marginal utility function using the definition of the derivative. Also find the marginal utility when $x = x_0 = 5$.

How to get the answer:

Step 1: Apply the definition of definition of derivative. (It will give us the marginal utility function as mentioned in the question.)

Step 2: Evaluate the value of the marginal utility function at $x = 5$.

APPLICATIONS OF DERIVATIVES

- Assume that the total utility that an individual obtains from the consumption of different units of a good is given by the function $U = f(x) = \ln x$, where U denotes the total utility obtained and x denotes the units of the good consumed. Find the individual's marginal utility, and the rate of change of marginal utility and determine how marginal utility behaves.

- How to get the answer:

Step 1: Marginal utility is obtained by differentiating the total utility function with respect to x.

Step 2: The rate of change of marginal utility is given by the second derivative of the total utility function.

- Interpretation: State how the marginal utility of the individual varies as x increases.

- State how total utility that the individual obtains varies as x increases (whether the rate is diminishing).