CSE 221: Algorithms

Order statistics

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References

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Contents

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- Using $O(n \lg n)$ algorithm (such as heapsort or mergesort, but *not* quicksort), it runs in $O(n \lg n) + \Theta(1)$ or $\boxed{\Theta(n \lg n)}$ time in the worst case.

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- Can we do better?

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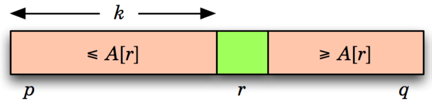
Idea behind the selection algorithm

Basic idea

1. Partition the sequence around a pivot, which returns the location of the pivot k after partitioning.



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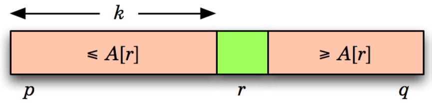

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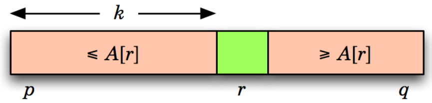

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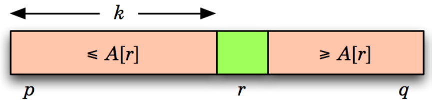

2. If $k = i$, then return the value at that index.
3. Otherwise, if $k > i$, then the i^{th} value must be *left* of k, so recursively partition the subarray left of k until $k = i$.
4. Otherwise, if $k > i$, then the i^{th} value must be *right* of k, so recursively partition the subarray left of k until $k = i - k$.

Partition based selection example

Given the following sequence (and using the first element as the *pivot*):

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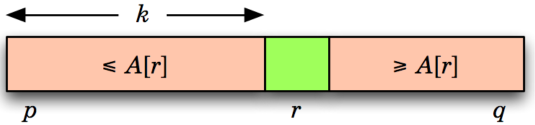
The selection algorithm

RANDOMIZED-SELECT(A, p, q, i) $\triangleright$ i^{th} smallest of $A[p\,..\,q]$

if $p = q$
  then return $A[q]$
$r \leftarrow$ RANDOMIZED-PARTITION(A, p, q)
$k \leftarrow r - p + 1$ $\triangleright$ $k = rank(A[r])$
if $i = k$
  then return $A[r]$
if $i < k$
  then return RANDOMIZED-SELECT$(A, p, r - 1, i)$
  else return RANDOMIZED-SELECT$(A, r + 1, q, i - k)$



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- There is however a $\Theta(n)$ algorithm in the worst case, but it has a large hidden constant (*by Blum, Floyd, Pratt, Rivest and Tarjan in 1973*).

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- There is however a $\Theta(n)$ algorithm in the worst case, but it has a large hidden constant (*by Blum, Floyd, Pratt, Rivest and Tarjan in 1973*).
- Given this large constant before the n in the linear-time algorithm, the randomized algorithm works better in practice.