Efficient Answer Enumeration in Description Logics with Functional Roles – Extended Version

Carsten Lutz, Marcin Przybyłko

Institute of Computer Science, Leipzig University, Germany {clu,przybyl}@informatik.uni-leipzig.de

Abstract

We study the enumeration of answers to ontology-mediated queries when the ontology is formulated in a description logic that supports functional roles and the query is a CQ. In particular, we show that enumeration is possible with linear preprocessing and constant delay when a certain extension of the CQ (pertaining to functional roles) is acyclic and free-connex acyclic. This holds both for complete answers and for partial answers. We provide matching lower bounds for the case where the query is self-join free.

In ontology-mediated querying, a query is combined with an ontology to enrich querying with domain knowledge and to facilitate access to incomplete and heterogeneous data (Bienvenu et al. 2014; Calvanese et al. 2009; Calì, Gottlob, and Lukasiewicz 2012). Intense research has been carried out on the complexity of ontology-mediated querying, considering in particular description logics and existential rules as the ontology languages and conjunctive queries (CQs) as the actual queries. Most of the existing studies have concentrated on the basic problem of single-testing which means to decide, given an ontology-mediated query (OMQ) Q, a database \mathcal{D} , and a candidate answer \bar{a} , whether \bar{a} is indeed an answer to Q on \mathcal{D} . From the viewpoint of many practical applications, however, the assumption that a candidate answer is provided is hardly realistic and it seems much more relevant to enumerate, given an OMQ Q and a database \mathcal{D} , all answers to Q on \mathcal{D} .

The investigation of answer enumeration for OMQs has recently been initiated in (Lutz and Przybylko 2022a) which also introduces useful new notions of minimal partial answers; such answers may contain wildcards to represent objects that are known to exist, but whose exact identity is unknown. If, for example, the ontology stipulates that

 ${\sf Researcher} \quad \sqsubseteq \quad \exists {\sf worksFor.University}$

Unversity

Academia

and the database $\mathcal D$ is {Researcher(mary)}, then there are no complete answers to the CQ

 $q(x, y) = \text{worksFor}(x, y) \land \text{Academia}(y),$

but (mary,*) is a minimal partial answer that conveys information which is otherwise lost. The ontologies in (Lutz

Copyright © 2023, Association for the Advancement of Artificial Intelligence (www.aaai.org). All rights reserved.

and Przybylko 2022a) are sets of guarded existential rules which generalize well-known description logics such as \mathcal{EL} and \mathcal{ELIH} . An important feature of description logics that is not captured by guarded rules are functionality assertions on roles, making it possible to declare that some relations must be interpreted as partial functions.

The purpose of this paper is to study answer enumeration to OMQs based on description logics with functional roles, in particular \mathcal{ELIHF} and its fragments. For the actual queries, we concentrate on CQs. We consider both the traditional complete answers and two versions of minimal partial answers that differ in which kind of wildcards are admitted. In one version, there is only a single wildcard symbol '*' while in the other version, multiple wildcards ' $*_1$ ', ' $*_2$ ', etc are admitted and multiple occurrences of the same wildcard represent the same unknown constant.

We study enumeration algorithms with a preprocessing phase that takes time linear in the size of \mathcal{D} and with constant delay, that is, in the enumeration phase the delay between two answers must be independent of \mathcal{D} . Note that we assume the OMQ Q to be fixed and of constant size, as in data complexity. If such an algorithm exists, then enumeration belongs to the complexity class $DelayC_{lin}$. If in addition the algorithm writes in the enumeration phase only a constant amount of memory, then it belons to the class CDoLin. It is not known whether $DelayC_{lin}$ and CDoLin coincide (Kazana 2013). Enumeration algorithms with these properties have been studied intensely, see e.g. (Berkholz, Gerhardt, and Schweikardt 2020; Segoufin 2015) for an overview.

It is an important result for conjunctive queries q without ontologies that enumeration is possible in CDoLin if q is acyclic and free-connex acyclic (Bagan, Durand, and Grandjean 2007), the latter meaning that q is acyclic after adding an atom that covers all answer variables. If these conditions are not met and q is self-join free (i.e., no relation symbol occurs more than once), then enumeration is not possible in DelayC_{lin} unless certain algorithmic assumptions fail that pertain to the triangle conjecture, the hyperclique conjecture, and Boolean matrix multiplication (Bagan, Durand, and Grandjean 2007; Brault-Baron 2013). In the presence of functional dependencies, which in their unary version are identical to functionality assertions in description logic, this characterization changes: enumeration in CDoLin is then possible if a certain extension q^+ of q guided by the

functional dependencies is acyclic and free-connex acyclic (Carmeli and Kröll 2020). In particular, adding functional dependencies may result in queries to become enumerable in CDoLin which were not enumerable before.

The main results presented in this paper are as follows. We consider OMQs Q where the ontology $\mathcal O$ is formulated in the description logic $\mathcal ELTHF$ and the query q is a CQ, and show that enumerating answers to Q is possible in CDoLin if q^+ is acyclic and free-connex acyclic. For complete answers, this is done by using carefully defined universal models and showing that they can be constructed in linear time via an encoding as a propositional Horn formula. For minimal partial answers (with single or multiple wildcards), we additionally make use of an enumeration algorithm that was given in (Lutz and Przybylko 2022a). Here, we only attain enumeration in DelayC_{lin}.

We also prove corresponding lower bounds for selfjoin free queries, paralleling those in (Bagan, Durand, and Grandjean 2007; Brault-Baron 2013) and (Carmeli and Kröll 2020). They concern only ontologies formulated in the fragment \mathcal{ELIF} of \mathcal{ELIHF} that disallows role inclusions. The reason is that lower bounds for OMQs with role inclusions entails a characterization of enumerability in CDoLin for CQs with self-joins, a major open problem even without ontologies. The lower bounds apply to complete and (both versions of) minimal partial answers and are conditional on the same algorithmic assumptions as in the case without ontologies. Our constructions and correctness proofs are more challenging than the existing ones in the literature since, unlike in the upper bounds, we cannot directly use the query q^+ . This is because the transition from q to q^+ changes the signature, extending the arity of relation symbols beyond two, and it is unclear how this can be reflected in the ontology.

We also study the combined complexity of single-testing for the new notions of minimal partial answers, concentrating on the important tractable description logics \mathcal{EL} and \mathcal{ELH} . It turns out that tractability is only preserved for acyclic CQs and the single wildcard version of minimal partial answers, while in all other cases the complexity increases to NP-complete or DP-complete.

1 Preliminaries

Let C, R, and K be countably infinite sets of *concept names*, role names, and *constants*. A role R is a role name $r \in R$ or an *inverse role* r^- with r a role name. If $R = r^-$, then $R^- = r$. An \mathcal{ELI} -concept is built according to the rule

$$C, D ::= A \mid C \sqcap D \mid \exists R.C$$

where A ranges over concept names and R over roles. An \mathcal{ELIHF} -ontology is a finite set of concept inclusions (CIs) $C \sqsubseteq D$ role inclusions (RIs) $R \sqsubseteq S$, and functionality assertions func(R) where (here and in what follows) C, D range over \mathcal{ELI} concepts and R, S over roles. An \mathcal{ELIF} -ontology is an \mathcal{ELIHF} -ontology that does not use RIs.

A *database* is a finite set of *facts* of the form A(c) or r(c,c') where A is a concept name or \top , r is a role name, and $c,c'\in \mathbf{K}$. We use $\mathsf{adom}(\mathcal{D})$ to denote the set of con-

stants used in database \mathcal{D} , also called its *active domain*. We may write $r^-(a,b) \in \mathcal{D}$ to mean $r(b,a) \in \mathcal{D}$.

A signature is a set of concept and role names, uniformly referred to as relation symbols. For a syntactic object O such as a concept or an ontology, we use $\operatorname{sig}(O)$ to denote the set of relation symbols used in it and ||O|| to denote its size, that is, the number of symbols needed to write it as a word using a suitable encoding.

The semantics is given in terms of interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \mathcal{I})$ where $\Delta^{\mathcal{I}}$ is a non-empty set called the domain and \mathcal{I} is the interpretation function, see (Baader et al. 2017) for details. We take the liberty to identify interpretations with non-empty and potentially infinite databases. The interpretation function \mathcal{I} is then defined as $A^{\mathcal{I}} = \{c \mid A(c) \in \mathcal{I}\}$ for concept names A and $r^{\mathcal{I}} = \{(c,c') \mid r(c,c') \in \mathcal{I}\}$ for role names r. An interpretation \mathcal{I} satisfies a CI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, a fact A(c) if $c \in A^{\mathcal{I}}$, and a fact $c \in A^{\mathcal{I}}$ if $c \in A^{\mathcal{I}}$. We thus make the standard names assumption, that is, we interpret constants as themselves. An interpretation \mathcal{I} is a model of an ontology (resp. database) if it satisfies all inclusions and assertions (resp. facts) in it.

A database \mathcal{D} is *satisfiable* w.r.t. an ontology \mathcal{O} if there is a model \mathcal{I} of \mathcal{O} and \mathcal{D} . Note that functionality assertions in an ontology \mathcal{O} can result in databases that are unsatisfiable w.r.t. \mathcal{O} . We write $\mathcal{O} \models \mathsf{func}(R)$ if every model of \mathcal{O} satisfies the functionality assertion $\mathsf{func}(R)$. In \mathcal{ELIHF} , this is decidable and ExpTIME-complete; see appendix.

Queries. A conjunctive query (CQ) is of the form $q(\bar{x}) = \exists \bar{y} \ \varphi(\bar{x}, \bar{y})$, where \bar{x} and \bar{y} are tuples of variables and $\varphi(\bar{x}, \bar{y})$ is a conjunction of concept atoms A(x) and role atoms r(x,y), with A a concept name, r a role name, and x,y variables from $\bar{x} \cup \bar{y}$. We call the variables in \bar{x} the answer variables of q, and use var(q) to denote $\bar{x} \cup \bar{y}$. We may write $\alpha \in q$ to indicate that α is an atom in q. For $V \subseteq \text{var}(q)$, we use $q|_V$ to denote the restriction of q to the atoms that use only variables in V. A homomorphism from q to an interpretation \mathcal{I} is a function $h: \text{var}(q) \to \Delta^{\mathcal{I}}$ such that $A(x) \in q$ implies $A(h(x)) \in \mathcal{I}$ and $A(x) \in q$ implies $A(h(x)) \in \mathcal{I}$ and $A(x) \in q$ implies the length of the tuple \bar{x} , is an answer to q on interpretation \mathcal{I} if there is a homomorphism h from q to \mathcal{I} with $h(\bar{x}) = \bar{d}$.

Every CQ q is associated with a canonical database \mathcal{D}_q obtained from q by viewing variables as constants and atoms as facts. We associate every database, and via \mathcal{D}_q also every CQ q, with an undirected graph $G_{\mathcal{D}} = (\mathsf{adom}(\mathcal{D}), \{\{a,b\} \mid R(a,b) \in \mathcal{D} \text{ for some role } R\}$. It is thus clear what we mean by a $path\ c_0,\ldots,c_k$ in a database and a path x_0,\ldots,x_k in a CQ. A CQ q is self-join free if every relation symbol occurs in at most one atom in q.

Ontology-Mediated Queries. An *ontology-mediated query (OMQ)* is a pair $Q = (\mathcal{O}, \Sigma, q)$ with \mathcal{O} an ontology, $\Sigma \subseteq \operatorname{sig}(\mathcal{O}) \cup \operatorname{sig}(q)$ a finite signature called the *data schema*, and q a query. We write $Q(\bar{x})$ to indicate that the answer variables of q are \bar{x} . The signature Σ expresses the promise that Q is only evaluated on Σ -databases. Let \mathcal{D} be such a database. A tuple $\bar{a} \in \operatorname{adom}(\mathcal{D})^{|\bar{x}|}$ is an *answer* to $Q(\bar{x})$ on \mathcal{D} , written $\mathcal{D} \models Q(\bar{a})$, if $\mathcal{I} \models q(\bar{a})$ for all models \mathcal{I} of \mathcal{O} and \mathcal{D} . We might alternatively write $\mathcal{D}, \mathcal{O} \models q(\bar{a})$.

With $Q(\mathcal{D})$ we denote the set of all answers to Q on \mathcal{D} . An OMQ $Q=(\mathcal{O},\Sigma,q)$ is *empty* if $Q(\mathcal{D})=\emptyset$ for every Σ -database \mathcal{D} that is satisfiable w.r.t. \mathcal{O} .

We may assume w.l.o.g. that \mathcal{ELIHF} ontologies used in OMQs are in *normal form*, that is, all CIs in it are of one of the following forms:

$$\top \sqsubseteq A$$
, $A_1 \sqcap A_2 \sqsubseteq A$, $A_1 \sqsubseteq \exists R.A_2$, $\exists R.A_1 \sqsubseteq A_2$

where A_1, A_2, A range over concept names. Every \mathcal{ELIHF} -ontology \mathcal{O} can be converted into this form in linear time without affecting the answers to OMQs (Baader et al. 2017).

With $(\mathcal{L}, \mathcal{Q})$, we denote the *OMQ language* that contains all OMQs Q in which \mathcal{O} is formulated in DL \mathcal{L} and q in query language \mathcal{Q} , such as in (\mathcal{ELIHF}, CQ) .

Partial Answers. Fix a wildcard symbol '*' that is not in **K**. A wildcard tuple for a database \mathcal{D} takes the form $(c_1,\ldots,c_n)\in(\operatorname{adom}(\mathcal{D})\cup\{*\})^n,\,n\geq 0$. For wildcard tuples $\bar{c}=(c_1,\ldots,c_n)$ and $\bar{c}'=(c'_1,\ldots,c'_n)$, we write $\bar{c}\preceq\bar{c}'$ if $c'_i\in\{c_i,*\}$ for $1\leq i\leq n$. Moreover, $\bar{c}\prec\bar{c}'$ if $\bar{c}\preceq\bar{c}'$ and $\bar{c}\neq\bar{c}'$. For example, $(a,b)\prec(a,*)$ and $(a,*)\prec(*,*)$ while (a,*) and (*,b) are incomparable w.r.t. ' \prec '. Informally, $\bar{c}\prec\bar{c}'$ expresses that tuple \bar{c} is preferred over tuple \bar{c}' as it carries more information.

A partial answer to OMQ $Q(\bar{x})=(\mathcal{O},\Sigma,q)$ on an S-database \mathcal{D} is a wildcard tuple \bar{c} for \mathcal{D} of length $|\bar{x}|$ such that for each model \mathcal{I} of \mathcal{D} and \mathcal{O} , there is a $\bar{c}'\in q(\mathcal{I})$ such that $\bar{c}'\preceq \bar{c}$. Note that some positions in \bar{c}' may contain constants from $\mathrm{adom}(\mathcal{I})\setminus \mathrm{adom}(\mathcal{D})$, and that the corresponding position in \bar{c} must then have a wildcard. A partial answer \bar{c} to Q on a Σ -database \mathcal{D} is a minimal partial answer if there is no partial answer \bar{c}' to Q on \mathcal{D} with $\bar{c}'\prec \bar{c}$. We use $Q(\mathcal{D})^*$ to denote the set of all minimal partial answers to Q on \mathcal{D} . An example is provided in the introduction. Note that $Q(\mathcal{D})\subseteq Q(\mathcal{D})^*$. To distinguish them from partial answers, we also refer to the answers in $Q(\mathcal{D})$ as complete answers.

We also define a second version of minimal partial answers where multiple wildcards are admitted, from a countably infinite set $\mathcal{W} = \{*_1, *_2, \dots\}$ disjoint from \mathbf{K} . Multiple occurrences of the same wildcard then represent the same unknown constant while different wildcards may or may not represent different constants. We use $Q(\mathcal{D})^{\mathcal{W}}$ to denote the set of minimal partial answers with multiple wildcards. A precise definition is provided in the appendix, here we only give an example.

Example 1. Let $Q(x, y, z) = (\mathcal{O}, \Sigma, q)$ where \mathcal{O} contains

Company $\sqsubseteq \exists$ hasEmployee.Person
TechCompany $\sqsubseteq Company$, CarCompany $\sqsubseteq Company$,
TechFactory $\sqsubseteq \exists$ hasOwner.TechCompany
CarFactory $\sqsubseteq \exists$ hasOwner.CarCompany

and func(hasOwner), Σ contains all symbols from \mathcal{O} , and

$$\begin{split} q(x,y,z) &= \mathsf{Person}(x) \, \wedge \\ & \mathsf{hasEmployee}(y,x) \, \wedge \, \mathsf{TechCompany}(y) \, \wedge \\ & \mathsf{hasEmployee}(z,x) \, \wedge \, \mathsf{CarCompany}(z). \end{split}$$

Further consider the database \mathcal{D} with facts

 ${\sf CarFactory}({\sf gigafactory1}), {\sf TechFactory}({\sf gigafactory1}).$

Then $Q^{\mathcal{W}}(\mathcal{D}) = \{(*_1, *_2, *_2)\}$. If we extend \mathcal{D} with hasOwner(gigafactory1, tesla), then this changes to $Q^{\mathcal{W}}(\mathcal{D}) = \{(*_1, \mathsf{tesla}, \mathsf{tesla})\}$.

Enumeration. We are interested in enumerating the complete and minimal partial answers to a given OMQ $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{L}, \mathcal{Q})$ on a given Σ -database D. An enumeration algorithm has a preprocessing phase where it may produce data structures, but no output. In the subsequent enumeration phase, it enumerates all tuples from Q(D), without repetition. Answer enumeration for an OMQ language $(\mathcal{L}, \mathcal{Q})$ is possible with linear preprocessing and constant delay, or in $\mathrm{DelayC_{lin}}$, if there is an enumeration algorithm for $(\mathcal{L}, \mathcal{Q})$ in which preprocessing takes time $f(||Q||) \cdot O(||D||)$, f a computable function, while the delay between the output of two consecutive answers depends only on ||Q||, but not on ||D||. Enumeration in $\mathrm{CD} \circ \mathrm{Lin}$ is defined likewise, except that it can use a constant total amount of extra memory in the enumeration phase.

The above definition only becomes precise when we fix a concrete machine model. We use RAMs under a uniform cost measure (Cook and Reckhow 1973), see (Grandjean 1996) for a formalization. A RAM has a one-way read-only input tape, a write-only output tape, and an unbounded number of registers that store non-negative integers of $O(\log n)$ bits, n the input size. In this model, which is standard in the $DelayC_{lin}$ context, sorting is possible in linear time and we can access in constant time lookup tables indexed by constants from adom(D) (Grandjean 1996).

We also consider *single-testing* which means to decide, given an OMQ $Q(\bar{x}) = (\mathcal{O}, \Sigma, q)$, a Σ -database D, and an answer candidate $\bar{c} \in \mathsf{adom}(D)^{|\bar{x}|}$, whether $\bar{c} \in Q(D)$.

Acyclic CQs. Let $q(\bar{x}) = \exists \bar{y} \, \varphi(\bar{x}, \bar{y})$ be a CQ. A *join tree* for $q(\bar{x})$ is an undirected tree T = (V, E) where V is the set of atoms in φ and for each $x \in \text{var}(q)$, the set $\{\alpha \in V \mid x \text{ occurs in } \alpha\}$ is a connected subtree of T. A CQ $q(\bar{x})$ is acyclic if it has a join tree. If q contains only unary and binary relations (which shall not always be the case), then q being acyclic is equivalent to $G_{\mathcal{D}_q}$ being a tree, potentially with multi-edges and self-loops. A CQ $q(\bar{x})$ is free-connex acyclic if adding a free fre

2 Upper Bounds

We identify cases that admit answer enumeration in CDoLin and DelayC_{lin}, considering both complete answers and minimal partial answers. From now on, we also use relation symbols of arity exceeding two, identifying concept names and role names with relations symbols of arity one and two.

We start with some preliminaries. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIHF}, CQ)$. Fix a linear order on the variables in q. A path y_0, \ldots, y_k in q is functional if for

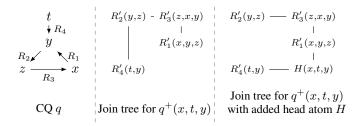


Figure 1: Illustration of Example 2

 $0 \le i < k$, there is an atom $R(y_i, y_{i+1}) \in q$ such that $\mathcal{O} \models \mathsf{func}(R)$. For a tuple \bar{x} of variables, we use \bar{x}^+ to denote the tuple $\bar{x}\bar{y}$ where \bar{y} consists of all variables y, in the fixed order, that are reachable from a variable in \bar{x} on a functional path in q and that are not part of \bar{x} .

The FA-extension of q is the CQ $q^+(\bar{x}^+)$ that contains an atom $R'(\bar{y}^+)$ for every atom $R(\bar{y})$ in q, where R' is a fresh relation symbol of arity $|\bar{y}^+|$. Note that q and q^+ are over different signatures. Moreover, q^+ may contain symbols of high arity and is self-join free. We will also consider the CQ $q^+(\bar{x})$ with a non-extended set of answer variables. Our FA-extensions are a variation of the notion of FD-extension used in (Carmeli and Kröll 2020).

Example 2. Let $\mathcal{O} = \{ \mathsf{func}(R_2^-), \mathsf{func}(R_3^-), \mathsf{func}(R_4) \}$ and $q(x,t) = R_1(x,y) \land R_2(y,z) \land R_3(z,x) \land R_4(t,y)$. Fix the variable order x < y < z < t. Then the $\mathit{CQ}\ q^+((x,t)^+)$ is

$$q^+(x,t,y) = R'_1(x,y,z), R'_2(y,z), R'_3(z,x,y), R'_4(t,y),$$

which is acyclic and free-connex acyclic while q is neither. See Figure 1 for a graphical representation.

In contrast, the version of q^+ that has a non-extended set of answer variables $\{x,t\}$ (but is otherwise identical) is acyclic, but not free-connex acyclic.

We state our upper bound for complete answers.

Theorem 1. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIHF}, CQ)$ such that $q^+(\bar{x}^+)$ is acyclic and free-connex acyclic. Then the complete answers to Q can be enumerated in CDoLin.

The proof of Theorem 1 relies on the following.

Proposition 1. Given an OMQ $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELTHF}, CQ)$ and a Σ -database \mathcal{D} that is satisfiable w.r.t. \mathcal{O} , one can compute in time $2^{\mathsf{poly}(||Q||)} \cdot O(||\mathcal{D}||)$ a database $\mathcal{U}_{\mathcal{D},Q} \supseteq \mathcal{D}$ that satisfies all functionality assertions in \mathcal{O} and such that $Q(\mathcal{D}) = q(\mathcal{U}_{\mathcal{D},Q}) \cap \mathsf{adom}(\mathcal{D})^{|\bar{x}|}$.

The database $\mathcal{U}_{\mathcal{D},Q}$ from Proposition 1 should be thought of as a univeral model for the ontology \mathcal{O} and database \mathcal{D} that is tailored specifically towards the query q. Such 'query-directed' universal models originate from (Bienvenu et al. 2013). We compute $\mathcal{U}_{\mathcal{D},Q}$ in time $2^{\text{poly}(||Q||)} \cdot O(||\mathcal{D}||)$ by constructing a suitable propositional Horn logic formula θ , computing a minimal model for θ in linear time (Dowling and Gallier 1984), and then reading off $\mathcal{U}_{\mathcal{D},Q}$ from that model. Details are provided in the appendix.

To prove Theorem 1, let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIHF}, CQ)$ with $q^+(\bar{x}^+)$ acyclic and free-connex, and

let $\mathcal D$ be a Σ -database. We first replace q by the CQ q_0 that is obtained from q by choosing a fresh concept name D and adding D(x) for every answer variable x. We also replace $\mathcal D$ by the database $\mathcal D_0=\mathcal U_{\mathcal D,Q}$ from Proposition 1 extended with D(c) for all $c\in \mathsf{adom}(\mathcal D)$. Clearly, $Q(\mathcal D)=q_0(\mathcal D_0)$ and thus it suffices to enumerate the latter.

We next replace $q_0(\bar{x})$ by $q_0^+(\bar{x}^+)$ and \mathcal{D}_0 by a database \mathcal{D}_0^+ that reflects the change in signature which comes with the transition from q_0 to q_0^+ . More precisely if atom $R(\bar{y})$ is replaced with $R'(\bar{y}^+)$ in the construction of q_0^+ and h is a homomorphism from $q_0|_{\bar{y}^+}$ to \mathcal{D}_0 , then \mathcal{D}_0^+ contains the fact $R'(h(\bar{y}^+))$. The following implies that $q_0(\mathcal{D}_0)$ is the projection of $q_0^+(\mathcal{D}_0^+)$ to the first $|\bar{x}|$ components.

Lemma 1. Every homomorphism from q_0 to \mathcal{D}_0 is also a homomorphism from q_0^+ to \mathcal{D}_0^+ and vice versa. Moreover, \mathcal{D}_0^+ can be constructed in time linear in $||\mathcal{D}||$.

To enumerate $q_0(\mathcal{D}_0)$, we may thus enumerate $q^+(\mathcal{D}_0^+)$ and project to the first $|\bar{x}|$ components. The former can be done in CDoLin: since $q^+(\bar{x}^+)$ is acyclic and free-connex, so is $q_0^+(\bar{x}^+)$, and thus we may apply the CDoLin enumeration procedure from (Bagan, Durand, and Grandjean 2007; Berkholz, Gerhardt, and Schweikardt 2020). Clearly, projection can be implemented in constant time. To argue that the resulting algorithm produces no duplicates, it remains to observe that the answers to $q_0(\mathcal{D}_0)$ and to $q^+(\mathcal{D}_0^+)$ are in a one-to-one correspondence, that is, every $\bar{c} \in q_0(\mathcal{D}_0)$ extends in a unique way to a $\bar{c}' \in q^+(\mathcal{D}_0^+)$. This, however, is an immediate consequence of Lemma 1, the definition of q_0^+ and the fact that \mathcal{D}_0 saties all functionality assertions in \mathcal{O} .

We now turn to minimal partial answers. Here, we cannot expect a result as general as Theorem 1, a counterexample is presented in Section 3. We thus resort to the stronger condition that $q^+(\bar{x})$ rather than $q^+(\bar{x}^+)$ is acyclic and freeconnex acyclic. The difference between the two conditions is related to the interplay of answer variables and functional roles. In particular, $q^+(\bar{x})$ and $q^+(\bar{x}^+)$ are identical for OMQs $Q=(\mathcal{O},\Sigma,q)$ such that answer variables have no functional edges to quantified variables, that is, for every atom R(x,y) in q, if $\mathcal{O}\models \operatorname{func}(R)$ and x is an answer variable, then y is also an answer variable.

Theorem 2. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELTHF}, CQ)$ such that $q^+(\bar{x})$ is acyclic and free-connex acyclic. Then the minimal partial answers to Q can be enumerated in $\mathsf{DelayC}_{\mathsf{lin}}$, both with multi-wildcards and with a single wildcard.

To prove Theorem 2, we make use of a recent result regarding OMQs in which the ontologies are sets of guarded existential rules. We refer to the class of such ontologies as \mathbb{G} . It was shown in (Lutz and Przybylko 2022a) that minimal partial answers to OMQs $Q=(\mathcal{O},\Sigma,q)\in(\mathbb{G},\mathrm{CQ})$ can be enumerated in DelayClin if q is acyclic and free-connex acyclic, both with multi-wildcards and with a single wildcard. The enumeration algorithms presented in (Lutz and Przybylko 2022a) are non-trivial and we use them as a blackbox. To achieve this, we need a slightly more 'low-level' formulation of the results from (Lutz and Przybylko 2022a). In what follows, we restrict our attention to minimal partial

answers with a single wildcard. The multi-wildcard case is analogous, details are in the appendix.

Fix a countably infinite set ${\bf N}$ of *nulls* that is disjoint from ${\bf K}$ and does not contain the wildcard symbol '*'. In what follows, we assume that databases may use nulls in place of constants. Let ${\mathcal D}$ be a database and $q(\bar x)$ a CQ. For an answer $\bar a \in q({\mathcal D})$, we use $\bar a_{\bf N}^*$ to denote the unique wildcard tuple for ${\mathcal D}$ obtained from $\bar a$ by replacing all nulls with '*'. We call $\bar a_{\bf N}^*$ a *partial answer* to q on ${\mathcal D}$ and say that it is *minimal* if there is no $\bar b \in q({\mathcal D})$ with $\bar b_{\bf N}^* \prec \bar a_{\bf N}^*$. With $q({\mathcal D})_{\bf N}^*$, we denote the set of minimal partial answers to q on ${\mathcal D}$.

A database \mathcal{E} is *chase-like* if there are databases $\mathcal{D}_1, \dots, \mathcal{D}_n$ such that

- 1. $\mathcal{E} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n$,
- 2. \mathcal{D}_i contains exactly one fact that uses no nulls, and that fact contains all constants in $\mathsf{adom}(\mathcal{D}_i) \setminus \mathbf{N}$,
- 3. $\operatorname{adom}(\mathcal{D}_i) \cap \operatorname{adom}(\mathcal{D}_j) \cap \mathbf{N} = \emptyset$ for $1 \leq i < j \leq n$.

We call $\mathcal{D}_1,\ldots,\mathcal{D}_n$ a witness for \mathcal{E} being chase-like. The term 'chase-like' refers to the chase, a well-known procedure for constructing universal models (Johnson and Klug 1982). The query-directed universal models $\mathcal{U}_{\mathcal{D},Q}$ from Proposition 1 are chase-like when the elements of $N=\operatorname{adom}(\mathcal{U}_{\mathcal{D},Q})\setminus\operatorname{adom}(\mathcal{D})$ are viewed as nulls. A witness $\mathcal{D}_1,\ldots,\mathcal{D}_n$ is obtained by removing from $\mathcal{U}_{\mathcal{D},Q}$ all atoms r(a,b) with $a,b\in\operatorname{adom}(\mathcal{D})$ and taking the resulting maximally connected components. The domain sizes $|\operatorname{adom}(\mathcal{D}_i')|$ then only depend on Q, but not on \mathcal{D} . The following is Proposition E.1 in (Lutz and Przybylko 2022b).

Theorem 3. For every $CQ(q(\bar{x}))$ that is acyclic and free-connex ayclic, enumerating the answers $q(\mathcal{D})_{\mathbf{N}}^*$ is in $\mathsf{DelayC}_{\mathsf{lin}}$ for databases \mathcal{D} and sets of nulls $N \subseteq \mathsf{adom}(\mathcal{D})$ such that \mathcal{D} is chase-like with witness $\mathcal{D}_1, \ldots, \mathcal{D}_n$ where $|\mathsf{adom}(\mathcal{D}_i)|$ does not depend on \mathcal{D} for $1 \le i \le n$.

The strategy for proving Theorem 2 is now similar to the case of complete answers. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELTHF}, CQ)$ with $q^+(\bar{x})$ acyclic and free-connex acyclic, and let \mathcal{D} be a Σ -database. It is shown in the appendix that the query-directed universal model $\mathcal{U}_{\mathcal{D},Q}$ is also universal for partial answers with a single wildcard in the sense that $Q(\mathcal{D})^* = q(\mathcal{U}_{\mathcal{D},Q})^*_{\mathbf{N}}$. We thus first replace \mathcal{D} with $\mathcal{U}_{\mathcal{D},Q}$, aiming to enumerate $Q(\mathcal{D})^* = q(\mathcal{U}_{\mathcal{D},Q})^*_{\mathbf{N}}$. We next replace $q(\bar{x})$ with $q^+(\bar{x})$ and $\mathcal{D}_0 = \mathcal{U}_{\mathcal{D},Q}$ with \mathcal{D}_0^+ . It follows from Lemma 1 that $q(\mathcal{D}_0)^*_{\mathbf{N}} = q^+(\mathcal{D}_0^+)^*_{\mathbf{N}}$. Note that no projection is needed since q^+ has answer variables \bar{x} here, in contrast to answer variables \bar{x}^+ in the case of complete answers. It remains to invoke Theorem 3.

3 Lower Bounds

The main aim of this section is to establish lower bounds that (partially) match the upper bounds stated in Theorems 1 and 2. First, however, we show that Theorem 2 cannot be strengthened by using $q^+(\bar{x}^+)$ in place of $q^+(\bar{x})$. All results presented in this section are conditional on algorithmic conjectures and assumptions. One of the conditions concerns Boolean matrix multiplication.

A Boolean $n \times n$ matrix is a function $M : [n]^2 \to \{0, 1\}$ where [n] denotes the set $\{1, \dots, n\}$. The *product* of two

Boolean $n \times n$ matrices M_1, M_2 is the Boolean $n \times n$ matrix $M_1M_2 := \sum_{c=1}^n M_1(a,c) \cdot M_2(c,b)$ where sum and product are interpreted over the Boolean semiring. In (nonsparse) Boolean matrix multiplication (BMM), one wants to compute M_1M_2 given M_1 and M_2 as $n \times n$ arrays. In sparse Boolean matrix multiplication (spBMM), input and output matrices M are represented as lists of pairs (a,b) with M(a,b) = 1. The currently best known algorithm for BMM achieves running time $n^{2.37}$ (Alman and Williams 2021) and it is open whether running time n^2 can be achieved; this would require dramatic advances in algorithm theory. Regarding spBMM, it is open whether running time $O(|M_1| + |M_2| + |M_1M_2|)$ can be attained, that is, time linear in the size of the input and the output (represented as lists). This clearly implies BMM in time n^2 , but the converse is not known.

The following implies that Theorem 2 cannot be strengthened by using $q^+(\bar{x}^+)$ in place of $q^+(\bar{x})$.

Theorem 4. There is an OMQ $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIF}, CQ)$ such that $q^+(\bar{x}^+)$ is acyclic and free-connex acyclic, but the minimal partial answers to Q cannot be enumerated in $DelayC_{lin}$ unless spBMM is possible in time $O(|M_1| + |M_2| + |M_1M_2|)$. This holds both for single wildcards and multi-wildcards.

Proof. Let
$$Q(\bar{x}) = (\mathcal{O}, \Sigma, q)$$
 where
$$\mathcal{O} = \{A \sqsubseteq \exists f^-. \top, \mathsf{func}(f)\}$$

$$\Sigma = \{A, r_1, r_2, f\}$$

$$q(x, z, y) = r_1(x, u_1) \land f(z, u_1) \land f(z, u_2) \land r_2(u_2, y).$$

It is easy to see that $q^+(\bar{x}^+)$ is just q, except that now all variables are answer variables. Thus, $q^+(\bar{x}^+)$ is acyclic and free-connex acyclic, as required.

Assume to the contrary of what is to be shown that there is an algorithm that given a database \mathcal{D} , enumerates $Q(\mathcal{D})^*$ in DelayC_{lin} (the case of multi-wildcards is identical). Then this algorithm can be used, given two Boolean matrices M_1M_2 in list representation, to compute M_1M_2 in time $O(|M_1| + |M_2| + |M_1M_2|)$. This is done as follows.

Given M_1 and M_2 , we construct a database \mathcal{D} by adding facts $r_1(a,c)$ and A(c) for every (a,c) with $M_1(a,c)=1$ and facts $r_2(c,b)$ and A(c) for every (c,b) with $M_2(c,b)=1$. It is easy to verify that $Q(\mathcal{D})^*=\{(a,*,b)\mid (a,b)\in M_1M_2\}$. We may thus construct a list representation of M_1M_2 by enumerating $Q(\mathcal{D})^*$. Since $|\mathcal{D}|=|M_1|+|M_2|$ and we can enumerate in DelayC_{lin}, the overall time spent is $O(|M_1|+|M_2|+|M_1M_2|)$.

We now consider lower bounds for Theorems 1 and 2. Here, we need two additional algorithmic conjectures that are closely related, both from fine-grained complexity theory. Recall that a k-regular hypergraph is a pair H=(V,E) where V is a finite set of vertices and $E\subseteq 2^V$ contains only sets of cardinality k. Consider the following problems:

• The triangle detection problem is to decide, given an undirected graph G = (V, E) as a list of edges, whether G contains a 3-clique (a "triangle").

The (k + 1, k)-hyperclique problem, for k ≥ 3, is to decide whether a given k-uniform hypergraph H contains a hyperclique of size k + 1, that is, a set of k + 1 vertices such that each subset of size k forms a hyperedge in H.

The *triangle conjecture* states that there is no algorithm for triangle detection that runs in linear time (Abboud and Williams 2014) and the *hyperclique conjecture* states that every algorithm that solves the (k+1,k)-hyperclique problem, for some $k \geq 3$, requires running time at least $n^{k+1-o(1)}$ with n the number of vertices (Lincoln, Williams, and Williams 2018). Note that triangle detection is the same as (k+1,k)-hyperclique for k=2, but the formulation of the two conjectures differs in that the former refers to the number of edges and the latter to the number of nodes. The following theorem summarizes our lower bounds.

Theorem 5. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIF}, CQ)$ be non-empty with q self-join free and connected.

- 1. If q^+ is not acyclic, then enumerating complete answers to Q is not in DelayC_{lin} unless the triangle conjecture fails or the hyperclique conjecture fails.
- 2. If $q^+(\bar{x}^+)$ is acyclic, but not free-connex acyclic, then enumerating complete answers to Q is not in DelayC_{lin} unless spBMM is possible in time $O(|M_1| + |M_2| + |M_1M_2|)$.

The same is true for least partial answers, both with a single wildcard and with multi-wildcards.

Recall that we use different versions of q^+ , namely $q^+(\bar x^+)$ and $q^+(\bar x)$ in Theorems 1 and 2. The difference is moot for Point 1 of Theorem 2 as $q^+(\bar x^+)$ is acyclic if and only if $q^+(\bar x)$ is.

The proof of Theorem 5 is inspired by proofs from (Bagan, Durand, and Grandjean 2007; Brault-Baron 2013; Carmeli and Kröll 2020) and uses similar ideas. However, the presence of ontologies and the fact that we want to capture minimal partial answers makes our proofs much more subtle. In particular, the constructions in (Carmeli and Kröll 2020) first transition from q to q^+ and then work purely on q^+ , but we cannot do this due to the presence of the ontology, which is formulated in the signature of q, not of q^+ . We (partially) present the proof of Point 1 and refer to the appendix for full detail.

The proof of Point 1 of Theorem 5 splits into two cases. Recall that the Gaifman graph of a CQ q is the undirected graph that has the atoms of q as its nodes and an edge between any two nodes/atoms that share a variable. It is known that if q is not acyclic, then its Gaifman graph is not chordal or not conformal (Beeri et al. 1983). Here, chordal means that every cycle of length at least 4 has a chord and conformal means that for every clique C in the Gaifman graph, there is an atom in q that contains all variables in C. The first case of the proof of Point 1 of Theorem 5 is as follows.

Lemma 2. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIF}, CQ)$ be nonempty such that q is self-join free and connected and the hypergraph of q^+ is not chordal. Then enumerating complete answers to Q is not in $\mathsf{DelayC}_{\mathsf{lin}}$ unless the triangle conjecture fails. The same is true for least partial answers, both with a single wildcard and with multi-wildcards. The second case is formulated similarly, but refers to nonconformality and the hyperclique conjecture. We give the proof of Lemma 2.

Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIF}, \mathbf{CQ})$ be as in Lemma 2, and let y_0, \ldots, y_k be a chordless cycle in the Gaifman graph of q^+ that has length at least 4. Let $Y = \{y_0, \ldots, y_k\}$ and for easier reference let $y_{k+1} = y_0$. For every variable x in q, we use Y_x to denote the set of variables $y \in Y$ such that q contains a functional (possibly empty) path from x to y.

Let G=(V,E) be an undirected graph. We may assume w.l.o.g. that G does not contain isolated vertices. Our aim is to construct a database \mathcal{D} , in time linear in |E|, such that G contains a triangle if and only if $Q(\mathcal{D}) \neq \emptyset$. Clearly, a DelayC_{lin} enumeration algorithm for Q lets us decide the latter in linear time and thus we have found an algorithm for triangle detection that runs in time linear in |E|, refuting the triangle conjecture.

The construction proceeds in two steps. In the first step, we define a database \mathcal{D}_0 that encodes the graph G. The constants in \mathcal{D}_0 are pairs $\langle x,f\rangle$ with $x\in \text{var}(q)$ and f a partial function from Y to V. For every variable x in q and word $w=a_0\ldots a_k\in V^*$ we use f_x^w denote the function that maps each variable $y_i\in Y_x$ to a_i and is undefined on all other variables. We may treat E as a symmetric (directed) relation, writing e.g. $(a,b)\in E$ and $(b,a)\in E$ if $\{a,b\}\in E$. For every atom r(x,y) in q with $r\in \Sigma$, add the following facts to \mathcal{D}_0 :

- 1. if $y_0 \in Y_x \cup Y_y$: $r(\langle x, f_x^{ab^k} \rangle, \langle y, f_y^{ab^k} \rangle)$ for all $(a, b) \in E$,
- $2. \ \ \text{if} \ y_k \in Y_x \cup Y_y : r(\langle x, f_x^{a^kb} \rangle, \langle y, f_y^{a^kb} \rangle) \ \text{for all} \ (a,b) \in E,$
- 3. if neither is true: $r(\langle x, f_x^{b^{k+1}} \rangle, \langle y, f_y^{b^{k+1}} \rangle)$ for all $a \in V$.

In addition, we add the fact A(c) for every concept name $A \in \Sigma$ and every constant c introduced above.

To provide an intuition for the reduction, let us start with a description that is relatively simple, but inaccurate. Consider a homomorphism h from q to \mathcal{D}_0 . It can be shown that h must map every variable y_i to a constant of the form $\langle y_i, f_{y_i} \rangle$ and that the domain of the function f_{y_i} is $\{y_i\}$. Since $f_{y_i}(y_i)$ is a node from G, the homomorphism h thus identifies a sequence of nodes a_0,\ldots,a_k from G, with $a_i=f_{y_i}(y_i)$. The construction of \mathcal{D}_0 ensures that $a_1=\cdots=a_{k-1}$ and a_0,a_1,a_k forms a triangle in G. Conversely, every triangle in G gives rise to a homomorphism from q to \mathcal{D}_0 of the described form. For other variables x from y, the use of the function y in constants y serves the purpose of ensuring that all functionality assertions in y are satisfied in y. A concrete example for the construction of y0 is provided in the appendix.

The above description is inaccurate for several reasons. First, instead of homomorphisms into \mathcal{D}_0 , we need to consider homomorphisms into the universal model $\mathcal{U}_{\mathcal{D}_0,\mathcal{O}}$ (defined in the appendix). Then variables y_i need not be mapped to a constant $\langle y_i, f_{y_i} \rangle$, but can also be mapped to elements outside of $\operatorname{adom}(\mathcal{D}_0)$. This does not break the reduction but complicates the correctness proof. Another difficulty arises from the fact that \mathcal{O} and q may use symbols that do not occur in Σ and and we need these to be derived by \mathcal{O} at the

relevant points in \mathcal{D}_0 . This is achieved in the second step of the construction of \mathcal{D}_0 , described next.

Informally, we want $\mathcal O$ to derive, at every constant $c\in \mathsf{adom}(\mathcal D_0)$, anything that it could possibly derive at any constant in any database. This is achieved by attaching certain tree-shaped databases to every constant in $\mathcal D_0$. We next make this precise. Let $\mathcal R_\Sigma$ be the set of all role names from Σ and their inverses. The infinite tree-shaped Σ -database D_ω has as its active domain $\mathsf{adom}(\mathcal D_\omega)$ the set of all (finite) words over alphabet $\mathcal R_\Sigma$ and contains the following facts:

- A(w) for all $w \in \mathsf{adom}(\mathcal{D}_\omega)$ and concept names $A \in \Sigma$;
- r(w, w') for all $w, w' \in \mathsf{adom}(\mathcal{D}_\omega)$ with w' = wr;
- r(w', w) for all $w, w' \in \mathsf{adom}(\mathcal{D}_{\omega})$ with $w' = wr^-$.

We cannot directly use \mathcal{D}_{ω} in the construction of \mathcal{D} since it is infinite. Consider all concept names A such that $\mathcal{D}_{\omega}, \mathcal{O} \models A(\varepsilon)$. We prove in the appendix that these are precisely the concept names A that are non-empty, that is, $\mathcal{D}, \mathcal{O} \models A(c)$ for some database \mathcal{D} and some $c \in \text{adom}(\mathcal{D})$. Clearly the number of such concept names A is finite. By compactness, there is thus a finite database $\mathcal{D}_{\text{tree}} \subseteq \mathcal{D}_{\omega}$ such that $\mathcal{D}_{\text{tree}}, \mathcal{O} \models A(\varepsilon)$ for all non-empty concept names A. We may w.l.o.g. assume that $\mathcal{D}_{\text{tree}}$ is the initial piece of \mathcal{D}_{ω} of some finite depth $k \geq 1$.

In principle, we would like to attach a copy of $\mathcal{D}_{\text{tree}}$ at every constant in \mathcal{D}_0 . This, however, might violate functionality assertions in \mathcal{O} and thus we have to be a bit more careful. For a role $R \in \{r, r^-\}$ with $r \in \Sigma$, let $\mathcal{D}_R \subseteq \mathcal{D}_{\text{tree}}$ be the database that consists of the fact $R(\varepsilon, R)$ and the subtree in $\mathcal{D}_{\text{tree}}$ rooted at R. Now, the final database \mathcal{D} used in the reduction is obtained from \mathcal{D}_0 as follows: for every $c \in \text{adom}(\mathcal{D}_0)$ and every role $R \in \{r, r^-\}$ with $r \in \Sigma$ such that there is no fact $R(c, c') \in \mathcal{D}_0$, add a disjoint copy of \mathcal{D}_R , glueing the copy of ε to c.

It is easy to see that $\mathcal D$ can be computed in time O(||E||). In particular, the database $\mathcal D_{\mathsf{tree}}$ can be constructed (in time independent of $\mathcal D$) by generating initial pieces of $\mathcal D_\omega$ of increasing depth and checking whether all non-empty concept names are implied at ε . In the appendix, we show that $\mathcal D$ satisfies all functionality assertions in $\mathcal O$ and is derivation complete at $\mathsf{adom}(\mathcal D_0)$. We then use a rather subtle analysis to prove the following.

Lemma 3.

TD1 If there is a minimal partial answer to Q on \mathcal{D} (with a single wildcard or with multiple wildcards), then there is a triangle in G.

TD2 If there is a triangle in G then there is a complete answer to Q on \mathcal{D} .

4 Combined Complexity of Single-Testing

The results on enumeration provide a (mild) indication that partial answers can be computationally more challenging than complete ones: the condition used in Theorem 1 is weaker than that in Theorem 2, and Theorem 1 achieves CDoLin while Theorem 2 achieves only DelayC_{lin}. Other cases in point may be found in (Lutz and Przybylko 2022a). This situation prompts us to study the effect of answer partiality on the combined complexity of single-testing.

We concentrate on the fragments \mathcal{EL} and \mathcal{ELH} of \mathcal{ELTHF} that do not admit inverse roles and functionality assertions and, in the case of \mathcal{EL} , also no role inclusions. These DLs bear special importance as single-testing complete answers to OMQs $Q=(\mathcal{O},\Sigma,q)\in(\mathcal{ELH},\mathrm{CQ})$ is in PTIME if q acyclic and NP-complete otherwise, both in combined complexity, and thus no harder than without ontologies (Krötzsch, Rudolph, and Hitzler 2007; Bienvenu et al. 2013). We will show that making answers partial may have an adverse effect on these complexities, starting, however, with a positive result. It is proved by a Turing-reduction to single-testing complete answers.

Theorem 6. For OMQs $Q = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELH}, CQ)$ with q acyclic, single-testing minimal partial answers with a single-wildcard is in PTIME in combined complexity.

Partial answers with multi-wildcards are less well-behaved. The lower bound in the next result is proved by a reduction from 1-in-3-SAT and only needs a very simple ontology that consists of a single CI of the form $A \sqsubseteq \exists r. \top$.

Theorem 7. For OMQs $Q = (\mathcal{O}, \Sigma, q) \in (\mathcal{EL}, CQ)$ with q acyclic, single-testing minimal partial answers with multi-wildcards is NP-complete in combined complexity. The same is true in (\mathcal{ELH}, CQ)

We now move from acyclic to unrestricted CQs. This makes the complexity increase further, and the difference between single and multi-wildcards vanishes.

Theorem 8. For OMQs $Q = (\mathcal{O}, \Sigma, q) \in (\mathcal{EL}, CQ)$ single-testing minimal partial answers is DP-complete in combined complexity. This is true both for single wildcards and multiwildcards, and the same holds also in (\mathcal{ELH}, CQ) .

For most other OMQ languages, we do not expect a difference in complexity between single-testing complete answers and single-testing partial answers. As an example, we consider \mathcal{ELIHF} where single-testing complete answers is ExpTime-complete (Eiter et al. 2008).

Theorem 9. In (\mathcal{ELIHF}, CQ) , single-testing minimal partial answers is EXPTIME-complete in combined complexity, both with single wildcards and multi-wildcards.

We remark that the *data* complexity of single-testing minimal partial answers in (\mathcal{ELIHF}, CQ) is in PTIME, both with a single wildcard and with multi-wildcards. This can be shown by using essentially the same arguments as in the proof of Theorem 6.

5 Conclusion

It would be interesting to extend our results to \mathcal{ELTHF} with local functionality assertions, that is, with concepts of the form $(\leqslant 1\ R)$ or even $(\leqslant 1\ R\ C)$. This is non-trivial as it is unclear how to define the CQ extension q^+ . It would also be interesting and non-trivial to get rid of self-join freeness in the lower bounds, see (Berkholz, Gerhardt, and Schweikardt 2020; Carmeli and Segoufin 2022). Another natural question is whether answers can be enumerated in some given order, see e.g. (Carmeli et al. 2021). Note that it was observed in (Lutz and Przybylko 2022a) that when enumerating $Q(\mathcal{D})^*$ or $Q(\mathcal{D})^{\mathcal{W}}$, it is possible to enumerate the complete answers before the truely partial ones.

Acknowledgements

The authors were supported by the DFG project LU 1417/3-1 QTEC.

References

- Abboud, A.; and Williams, V. V. 2014. Popular Conjectures Imply Strong Lower Bounds for Dynamic Problems. In *Proceedings of FOCS 2014*, 434–443. IEEE Computer Society.
- Alman, J.; and Williams, V. V. 2021. A Refined Laser Method and Faster Matrix Multiplication. In *SODA 2021*, 522–539. SIAM.
- Baader, F.; Horrocks, I.; Lutz, C.; and Sattler, U. 2017. *An Introduction to Description Logic*. Cambridge University Press.
- Bagan, G.; Durand, A.; and Grandjean, E. 2007. On Acyclic Conjunctive Queries and Constant Delay Enumeration. In *Proceedings of CSL 2007*, volume 4646 of *LNCS*, 208–222. Springer.
- Beeri, C.; Fagin, R.; Maier, D.; and Yannakakis, M. 1983. On the Desirability of Acyclic Database Schemes. *J. ACM*, 30: 479–513.
- Berkholz, C.; Gerhardt, F.; and Schweikardt, N. 2020. Constant delay enumeration for conjunctive queries: a tutorial. *ACM SIGLOG News*, 7(1): 4–33.
- Bienvenu, M.; Ortiz, M.; Simkus, M.; and Xiao, G. 2013. Tractable Queries for Lightweight Description Logics. In *Proc. of IJCAI13*, 768–774. IJCAI/AAAI.
- Bienvenu, M.; ten Cate, B.; Lutz, C.; and Wolter, F. 2014. Ontology-Based Data Access: A Study through Disjunctive Datalog, CSP, and MMSNP. *ACM Trans. Database Syst.*, 39(4): 33:1–33:44.
- Brault-Baron, J. 2013. De la pertinence de l'énumération: complexité en logiques propositionnelle et du premier ordre. (The relevance of the list: propositional logic and complexity of the first order). Ph.D. thesis, University of Caen Normandy, France.
- Calì, A.; Gottlob, G.; and Lukasiewicz, T. 2012. A general Datalog-based framework for tractable query answering over ontologies. *J. Web Semant.*, 14: 57–83.
- Calvanese, D.; Giacomo, G. D.; Lembo, D.; Lenzerini, M.; Poggi, A.; Rodriguez-Muro, M.; and Rosati, R. 2009. Ontologies and Databases: The DL-Lite Approach. In *Proc. of Reasoning Web*, volume 5689 of *LNCS*, 255–356. Springer.
- Carmeli, N.; and Kröll, M. 2020. Enumeration Complexity of Conjunctive Queries with Functional Dependencies. *Theory Comput. Syst.*, 64(5): 828–860.
- Carmeli, N.; and Segoufin, L. 2022. Conjunctive queries with self-joins, towards a fine-grained complexity analysis. *CoRR*, abs/2206.04988.
- Carmeli, N.; Tziavelis, N.; Gatterbauer, W.; Kimelfeld, B.; and Riedewald, M. 2021. Tractable Orders for Direct Access to Ranked Answers of Conjunctive Queries. In *Proc. of PODS*, 325–341. ACM.
- Cook, S. A.; and Reckhow, R. A. 1973. Time Bounded Random Access Machines. *J. Comput. Syst. Sci.*, 7(4): 354–375.

- Dowling, W. F.; and Gallier, J. H. 1984. Linear-time algorithms for testing the satisfiability of propositional horn formulae. *The Journal of Logic Programming*, 1(3): 267–284.
- Eiter, T.; Gottlob, G.; Ortiz, M.; and Simkus, M. 2008. Query Answering in the Description Logic Horn- \mathcal{SHIQ} . In *Proc. of JELIA*, volume 5293 of *LNCS*, 166–179. Springer.
- Grandjean, E. 1996. Sorting, linear time and the satisfiability problem. *Annals of Mathematics and Artificial Intelligence*, 16: 183–236.
- Johnson, D. S.; and Klug, A. C. 1982. Testing Containment of Conjunctive Queries Under Functional and Inclusion Dependencies. In *Proc. of PODS*, 164–169. ACM.
- Kazana, W. 2013. *Query evaluation with constant delay.* (*L'évaluation de requêtes avec un délai constant*). Ph.D. thesis, École normale supérieure de Cachan, Paris, France.
- Krötzsch, M.; Rudolph, S.; and Hitzler, P. 2007. Conjunctive Queries for a Tractable Fragment of OWL 1.1. In *Proc. of ISWC*, volume 4825 of *LNCS*, 310–323. Springer.
- Lincoln, A.; Williams, V. V.; and Williams, R. R. 2018. Tight Hardness for Shortest Cycles and Paths in Sparse Graphs. In *SODA 2018*, 1236–1252. SIAM.
- Lutz, C.; and Przybylko, M. 2022a. Efficiently Enumerating Answers to Ontology-Mediated Queries. In *Proc. of PODS*, 277–289. ACM.
- Lutz, C.; and Przybylko, M. 2022b. Efficiently Enumerating Answers to Ontology-Mediated Queries. *CoRR*, abs/2203.09288.
- Segoufin, L. 2015. Constant Delay Enumeration for Conjunctive Queries. *SIGMOD Rec.*, 44(1): 10–17.

A Additional Details for Section 1

We formally define the notion of partial answers with multiwildcards. Fix a countably infinite set of *multi-wildcards* $\mathcal{W} = \{*_1, *_2, \dots\}$, disjoint from \mathbf{K} . A *multi-wildcard tuple* for a database \mathcal{D} is a tuple $(c_1, \dots, c_n) \in (\operatorname{adom}(\mathcal{D}) \cup \mathcal{W})^n$, $n \geq 0$, such that if $c_i = *_j$ with j > 1, then there is an i' < i with $c_{i'} = *_{j-1}$. For example, $(a, *_1, b, *_2, *_1, a)$ is a multi-wildcard tuple. Occurrences of the same wildcard represent occurrences of the same unknown constant while different wildcards represent constants that may or may not be different. For multi-wildcard tuples $\bar{c} = (c_1, \dots, c_n)$ and $\bar{c}' = (c'_1, \dots, c'_n)$, we write $\bar{c} \preceq \bar{c}'$ if

1. $c_i = c_i'$ or $c_i' \in \mathcal{W}$ for $1 \le i \le n$ and

2. $c'_i = c'_j$ implies $c_i = c_j$ for $1 \le i, j \le n$.

Then, $\bar{c} \prec \bar{c}'$ if $\bar{c} \preceq \bar{c}'$ and $\bar{c} \neq \bar{c}'$. For example, $(*_1, a) \prec (*_1, *_2)$ and $(a, *_1, *_2, *_1) \prec (a, *_1, *_2, *_3)$. Informally, $\bar{c} \prec \bar{c}'$ means that \bar{c} is strictly more informative than \bar{c}' .

A partial answer with multi-wildcards to OMQ $Q(\bar{x})=(\mathcal{O},\Sigma,q)$ on a Σ -database \mathcal{D} is a multi-wildcard tuple \bar{c} for \mathcal{D} of length $|\bar{x}|$ such that for each model \mathcal{I} of \mathcal{O} and \mathcal{D} , there is a $\bar{c}'\in q(\mathcal{I})$ such that $\bar{c}'\preceq\bar{c}$. A partial answer with multi-wildcards \bar{c} to Q on a Σ -database \mathcal{D} is a minimal partial answer if there is no partial answer with multi-wildcards \bar{c}' to Q on \mathcal{D} with $\bar{c}'\prec\bar{c}$. The minimal partial answer evaluation of $Q(\bar{x})$ on \mathcal{D} with multi-wildcards, denoted $Q(\mathcal{D})^*$, is the set of all minimal partial answers to Q on \mathcal{D} with multi-wildcards. Note that $Q(\mathcal{D})\subseteq Q(\mathcal{D})^{\mathcal{W}}$.

Throughout the paper, we assume that given an \mathcal{ELTHF} ontology $\mathcal O$ and roles R,S, it is decidable whether $\mathcal O \models R \sqsubseteq S$ and whether $\mathcal O \models \operatorname{func}(R)$. It is in fact not hard to see that these problems are EXPTIME-complete. We describe the upper bounds.

To decide whether $\mathcal{O} \models R \sqsubseteq S$, one constructs the database $\mathcal{D} = \{R(a,b)\}$ and then checks whether $\mathcal{D}, \mathcal{O} \models S(x,y)$, which is possible in EXPTIME (Eiter et al. 2008).

To decide whether $\mathcal{O} \models \mathsf{func}(R)$, one constructs the database $\mathcal{D} = \{R(a,b_1), R(a,b_2)\}$ and then checks whether \mathcal{D} is satisfiable w.r.t. \mathcal{O} which is the case if and only if $\mathcal{D}, \mathcal{O} \not\models \exists x \, A(x)$ with A a fresh concept name.

Throughout the appendix, we say that a CQ q is a *tree* if the undirected graph $G_{\mathcal{D}_q}$ is a tree and q contains no reflexive atoms r(x,y). Note that multi-edges $r(x,y), s(x,y) \in q$, with $r \neq s$, are admitted.

B Simulations

We introduce the notion of a simulation, which is closely linked to the expressive power of \mathcal{ELI} and will be used throughout the appendix. Let $\mathcal I$ and $\mathcal J$ be interpretations. A *simulation* from $\mathcal I$ to $\mathcal J$ is a relation $S\subseteq\mathsf{adom}(\mathcal I)\times\mathsf{adom}(\mathcal J)$ such that

Sim1 $A(c) \in \mathcal{I}$ and $(c, c') \in S$ implies $A(c') \in \mathcal{J}$ and **Sim2** $R(c_1, c_2) \in \mathcal{I}$ and $(c_1, c'_1) \in S$ implies that there is a $c' \in \operatorname{adom}(\mathcal{I})$ such that $R(c', c') \in \mathcal{I}$ and

is a $c_2' \in \mathsf{adom}(\mathcal{J})$ such that $R(c_1', c_2') \in \mathcal{J}$ and $(c_2, c_2') \in S$.

If there is a simulation S from \mathcal{I} to \mathcal{J} such that $(c, c') \in S$, then we write $(\mathcal{I}, c) \preceq (\mathcal{J}, c')$.

C Universal Models

We show how to construct a universal model $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ of a database \mathcal{D} and \mathcal{ELTHF} -ontology \mathcal{O} , using a somewhat unusual combination of chase and 'direct definition' that is tailored towards the needs of our subsequent proofs and constructions.

In a nutshell, we use chase-like rule applications for the database part of the universal model and a 'direct definition' for the parts of the universal model generated by existential quantifiers in \mathcal{O} . One advantage over a pure chase approach is that this avoids ever having to identify generated objects. To see why such identifications might be necessary, consider an ontology that includes

$$\mathcal{O} = \{A_1 \sqsubseteq \exists r_1.B_1, A_2 \sqsubseteq \exists r_2.B_2, A \sqsubseteq \exists s.B \\ r_1 \sqsubseteq s, r_2 \sqsubseteq s, \mathsf{func}(s)\}$$

and a database that includes $A_2(c)$. Chasing would generate $r_1(c,c_1), B_1(c_1)$ and $r_2(c,c_2), B_2(c_2)$ with with c_1,c_2 fresh constants. If now other parts of the chase generate A(c), then c_2 has to be merged into c_1 and $s(c,c_1)$ has to be added.

Let \mathcal{O} be an \mathcal{ELIHF} -ontology in normal form and \mathcal{D} a database that is satisfiable w.r.t. \mathcal{O} . For each non-empty set ρ of roles and set M of concept names, define a CQ

$$q^1_{\rho,M}(x) = \exists y \bigwedge \{R(x,y) \mid R \in \rho\} \land \bigwedge \{A(y) \mid A \in M\}$$

and let $q_{\rho,M}^2(x,y)$ denote the same CQ, but with y an additional answer variable rather than a quantified variable. For the chase part of our construction, we use the following chase rules:

- **R1** If $A_1(c), \ldots, A_n(c) \in \mathcal{D}$, $\mathcal{O} \models A_1 \sqcap \cdots \sqcap A_n \sqsubseteq A$, and $A(c) \notin \mathcal{D}$, then add A(c) to \mathcal{D} ;
- **R2** If $A(c_1), R(c_2, c_1) \in \mathcal{D}$, $\exists R.A \sqsubseteq B \in \mathcal{O}$, and $B(c_2) \notin \mathcal{D}$, then add $B(c_2)$ to \mathcal{D} ;
- **R3** If $R(c_1, c_2) \in \mathcal{D}$, $R \subseteq S \in \mathcal{O}$, and $S(c_1, c_2) \notin \mathcal{D}$, then add $S(c_1, c_2)$ to \mathcal{D} ;
- **R4** If $A_1(c_1),\ldots,A_n(c_1),R(c_1,c_2)\in\mathcal{D},\ \mathrm{func}(R)\in\mathcal{O},\ \{A_1(c_1),\ldots,A_n(c_n)\},\mathcal{O}\models q_{\rho,M}^1(c_1),\ S,R\in\rho,\ A\in M,\ \mathrm{and}\ S(c_1,c_2)\notin\mathcal{D}\ \mathrm{or}\ A(c_2)\notin\mathcal{D},\ \mathrm{then}\ \mathrm{add}\ S(c_1,c_2)\ \mathrm{and}\ A(c_2)\ \mathrm{to}\ \mathcal{D}.$

In rule **R1**, we may have n=0 to capture CIs of the form $T \subseteq A$. Rule **R4** is best understood in the light of the example above. For a database $\mathcal D$ and ontology $\mathcal O$, we use $\mathrm{ch}_{\mathcal O}(\mathcal D)$ to denote the result of exhaustively applying the above rules to $\mathcal D$. Note that no rule introduces fresh constants, and thus rule application terminates. It is also easy to see that the final result does not depend on the order in which the rules are applied.

To construct the universal model $\mathcal{U}_{\mathcal{D},\mathcal{O}}$, we first build the chase $\mathrm{ch}_{\mathcal{O}}(\mathcal{D})$ and then proceed with the 'semantic' part of the construction as follows. Let $c \in \mathrm{adom}(\mathcal{D})$. We use M_c to denote the set of concept names A with $A(c) \in \mathrm{ch}_{\mathcal{O}}(\mathcal{D})$. Moreover, we use \mathcal{D}_M to denote the database $\{A(\widehat{c}) \mid A \in M\}$ where \widehat{c} is an arbitrary, but fixed constant. If $M = M_c$, then we take $\widehat{c} = c$. For a non-empty set ρ of roles and set M of concept names, we write $c \leadsto_{\mathcal{D},\mathcal{O}}^{\rho} M$ if

1.
$$\mathcal{D}_{M_c}, \mathcal{O} \models q_{\rho,M}^1(c),$$

- 2. ρ and M are maximal with this property, that is, there are no $\rho' \supseteq \rho$ and $M' \supseteq M$ with $(\rho', M') \neq (\rho, M)$ such that $\mathcal{D}_{M_c}, \mathcal{O} \models q^1_{\rho', M'}(c)$, and
- 3. there is no $R(c,c')\in {\rm ch}_{\mathcal O}(\mathcal D)$ such that $R\in \rho$ and ${\rm func}(R)\in \mathcal O.$

Similarly, for a non-empty set ρ of roles and sets M_1, M_2 of concept names, we write $M_1 \leadsto_{\mathcal{D},\mathcal{O}}^{\rho} M_2$ if for $1 \leq i < n$:

- 4. $\mathcal{D}_{M_i}, \mathcal{O} \models q^1_{\rho_{i+1}, M_{i+1}}(\widehat{c})$ and
- 5. ρ_{i+1} and M_{i+1} are maximal with this property.

A trace for \mathcal{D} and \mathcal{O} is a sequence

$$t = c\rho_1 M_1 \rho_2 M_2 \dots \rho_n M_n, \ n \ge 0,$$

where $c \in \mathsf{adom}(\mathcal{D})$, ρ_1, \ldots, ρ_n are sets of roles that occur in \mathcal{O} , and M_1, \ldots, M_n are sets of concept names that occur in \mathcal{O} such that for $1 \leq i < n$:

- 6. $c \leadsto_{\mathcal{D} \mathcal{O}}^{\rho_1} M_1$,
- 7. $M_i \leadsto_{\mathcal{O}}^{\rho_{i+1}} M_{i+1}$,
- 8. there is no role name R with $R^- \in \rho_i$, $R \in \rho_{i+1}$, and func $(R) \in \mathcal{O}$, for $1 \le i < n$.

Let T denote the set of all traces for \mathcal{D} and \mathcal{O} . Then, the *universal model* of \mathcal{D} and \mathcal{O} is the following database with active domain T, viewed as an interpretation:

$$\mathcal{U}_{\mathcal{D},\mathcal{O}} = \mathsf{ch}_{\mathcal{O}}(\mathcal{D}) \cup \{ A(t\rho M) \mid t\rho M \in \mathbf{T} \text{ and } A \in M \} \cup \{ R(t,t\rho M) \mid t\rho M \in \mathbf{T} \text{ and } R \in \rho \}.$$

The following lemma summarizes the most important properties of universal models.

Lemma 4.

- 1. $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ is a model of \mathcal{D} and \mathcal{O} .
- 2. If \mathcal{I} is a model of \mathcal{D} and \mathcal{O} , then there is a homomorphism from $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ to \mathcal{I} that is the identity on $\mathsf{adom}(\mathcal{D})$.
- 3. $Q(\mathcal{D}) = q(\mathcal{U}_{\mathcal{D},\mathcal{O}}) \cap \mathsf{adom}(\mathcal{D})^{|x|}$.

To prove Lemma 4, let us first give two technical lemmas that will also be useful later on.

Lemma 5. Let \mathcal{O} be an \mathcal{ELIF} ontology in normal form, $\mathcal{D}_1, \mathcal{D}_2$ be databases that are satisfiable w.r.t. \mathcal{O} , and $c_i \in \mathsf{adom}(\mathcal{D}_i)$ for $i \in \{1,2\}$. Then $(\mathcal{D}_1, c_1) \preceq (\mathcal{D}_2, c_2)$ and $\mathcal{D}_1, \mathcal{O} \models A(c_1)$ implies $\mathcal{D}_2, \mathcal{O} \models A(c_2)$ for every concept name A.

Proof. Consider the chase procedure used as the first part of the construction of universal models. Started on \mathcal{D}_1 , it constructs a finite sequence

$$\mathcal{D}_1 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k = \mathsf{ch}_{\mathcal{O}}(\mathcal{D}_1).$$

Since $(\mathcal{D}_1, c_1) \preceq (\mathcal{D}_2, c_2)$, there is a simulation S from \mathcal{D}_1 to \mathcal{D}_2 with $(c_1, c_2) \in S$. We prove by induction on i that

(†) S is a simulation from I_i to $ch_{\mathcal{O}}(\mathcal{D}_2)$.

The induction start is trivial. For the induction step assume that we have already shown that S is a simulation from I_i to $\mathrm{ch}_{\mathcal{O}}(\mathcal{D}_2)$. We need to verify that Conditions **Sim1** and **Sim2** are also satisfied for S as a simulation from I_{i+1} to $\mathrm{ch}_{\mathcal{O}}(\mathcal{D}_2)$.

Recall that I_{i+1} was obtained from I_i by application of one of the rules $\mathbf{R1}$ to $\mathbf{R4}$ from Section C. Since $\mathcal O$ is formulated in $\mathcal {ELIF}$ (and thus contains no role inclusions) rule $\mathbf{R3}$ cannot be used. Similarly, no new binary facts R(c,c') are added by the rules and, thus, $\mathbf{Sim2}$ holds by inductive assumption.

For **Sim1**, let $(c,c') \in S$ and $A(c) \in I_{i+1}$. If $A(c) \in I_i$ then $A(c') \in \mathcal{U}_{\mathcal{D}_2,\mathcal{O}}$ by the inductive assumption. Otherwise, A(c) was added by one of the above rules. Thus, A(c) is a consequence of some facts $S = \{A_1(c), \ldots, A_\ell(c), R_1(c, c_1), \ldots, R_k(c, c_k)\} \subseteq I_i$ and the ontology, i.e. $S, \mathcal{O} \models A(c)$. By inductive assumption, there are constants c'_i , for $1 \leq j \leq k$ such that $S' = \{A_1(c'), \ldots, A_\ell(c), R_1(c, c'_1), \ldots, R_k(c, c'_k)\} \subseteq \mathcal{U}_{\mathcal{D}_2,\mathcal{O}}$. Clearly, we have that $S', \mathcal{O} \models A(c')$. Since $\mathcal{U}_{\mathcal{D}_2,\mathcal{O}}$ is a model of \mathcal{O} , we infer that $A(c') \in \mathcal{U}_{\mathcal{D}_2,\mathcal{O}}$. This finishes the proof of (\dagger) .

Now assume that $\mathcal{D}_1, \mathcal{O} \models A(c_1)$. Then Point 3 of Lemma 4 yields $A(c_1) \in \mathcal{U}_{\mathcal{D}_1,\mathcal{O}}$. Since the construction of $\mathcal{U}_{\mathcal{D}_1,\mathcal{O}}$ from $\mathsf{ch}_{\mathcal{O}}(\mathcal{D}_1)$ adds no facts A(c) with $c \in \mathsf{adom}(\mathcal{D}_1)$, this yields $A(c_1) \in \mathsf{ch}_{\mathcal{O}}(\mathcal{D}_1)$. Now (\dagger) yields $A(c_2) \in \mathsf{ch}_{\mathcal{O}}(\mathcal{D}_2)$, thus $A(c_2) \in \mathcal{U}_{\mathcal{D}_2,\mathcal{O}}$, implying $\mathcal{D}_2, \mathcal{O} \models A(c_2)$.

For $c \in \text{dom}(\mathcal{D})$, we use $\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}$ to denote the restriction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ to the domain that consists of all traces starting with c. Note that, by construction of $\mathcal{U}_{\mathcal{D},\mathcal{O}},\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}$ takes the shape of a tree without multi-edges and reflexive loops. We also refer to $\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}$ as the tree in $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ rooted in c.

Lemma 6. Let \mathcal{O} be an \mathcal{ELIHF} -ontology and let $\mathcal{D}_1, \mathcal{D}_2$ be databases that are satisfiable w.r.t. \mathcal{O} and c_1, c_2 be constants such that $\mathcal{D}_1, \mathcal{O} \models A(c_1)$ implies $\mathcal{D}_2, \mathcal{O} \models A(c_2)$ for every concept name A. Further, let \mathcal{J} be a model of \mathcal{D}_2 and \mathcal{O} . Then there is a homomorphism h from $\mathcal{U}_{\mathcal{D}_1,\mathcal{O}}^{\downarrow c_1}$ to \mathcal{J} with $h(c_1) = c_2$.

Proof. For brevity, let $\mathcal{I} = \mathcal{U}_{\mathcal{D}_1,\mathcal{O}}^{\downarrow c_1}$. Let \mathbf{T}_n , $n \geq 0$, be the set of traces in \mathcal{I} of length n. In particular $\mathbf{T}_0 = \{c_1\}$.

For all $n \geq 0$ we construct a homomorphism h_n from $\mathcal{I}|_{\mathbf{T}_n}$ to \mathcal{J} such that $h_n(c_1) = c_2$. The desired homomorphism h is then obtained in the limit, that is, $h = \bigcup_{n \geq 0} h_n$.

The homomorphism h_0 is simply the identity on c_1 . Since $\mathcal{D}_1, \mathcal{O} \models A(c_1)$ implies $\mathcal{D}_2, \mathcal{O} \models A(c_2)$ for every concept name A, h_0 is a homomorphism.

For the inductive step, let us assume that we have already defined h_n . We show how to use it to define h_{n+1} .

We first take $h_{n+1} = h_n$. Now, let $w \rho M \in \mathbf{T}_{n+1} \setminus \mathbf{T}_n$ be a trace in the next layer of the tree. Since $w \in \mathbf{T}_n$, $h_{n+1}(w)$ is already defined. Let $N = \{A \mid A(w) \in \mathcal{I}\}$ and $N' = \{A \mid A(w) \in \mathcal{J}\}$. Since h_n is a homomorphism, $N \subseteq N'$. By construction of the universal model $\mathcal{U}_{\mathcal{D}_1,\mathcal{O}}$ we have $M = \{A \mid A(w \rho M) \in \mathcal{I}\}$. Moreover, $D_N, \mathcal{O} \models q_{0,M}^1(w)$.

Since \mathcal{J} is a model of \mathcal{D}_2 and \mathcal{O} , and $N\subseteq N'$, we infer that $D_{N'}, \mathcal{O}\models q^1_{\rho,M}(h(w))$. Therefore, there is a constant $d\in \mathsf{dom}(\mathcal{J})$ such that $\mathcal{J}, \mathcal{O}\models q^2_{\rho,M}(h(w),d)$. We set

 $(\dagger) \ h_{n+1}(w\rho M) = d.$

We now show that h_{n+1} is a homomorphism from $\mathcal{I}|_{\mathbf{T}_{n+1}}$ to \mathcal{J} . For $w,w'\in\mathbf{T}_n$, we have $h_{n+1}(w)=h_n(w)$ and $h_{n+1}(w')=h_n(w')$. Thus, by inductive assumption, $A(w)\in\mathcal{I}$ implies $A(h_{n+1}(w))\in\mathcal{J}$ and $R(w,w')\in\mathcal{I}$ implies $R(h_{n+1}(w),h_{n+1}(w'))\in\mathcal{J}$.

Thus, let $w \in \mathbf{T}_{n+1} \setminus \mathbf{T}_n$. By definition of $\mathcal{U}_{\mathcal{D}_1,\mathcal{O}}$ there is a unique $w' \in \mathbf{T}_n$ such that $w = w' \rho M$ where $M = \{A \mid A(w) \in \mathcal{U}_{\mathcal{D}_1,\mathcal{O}}\}$, and $\rho = \{R \mid R(w',w) \in \mathcal{U}_{\mathcal{D}_1,\mathcal{O}}\}$. Moreover, since \mathcal{I} is a tree, we have that for all $w'' \in \mathbf{T}_{n+1}$ if R(w'',w) for some role name R then w'' = w'.

Now, if $A(w) \in \mathcal{I}$ then $A \in M$ and $A(h_{n+1}(w)) \in \mathcal{I}$ by the choice (\dagger) of $h_{n+1}(w)$. Similarly, if $R(w,w'') \in \mathcal{I}$ for some $w'' \in \mathbf{T}_{n+1}$ then w'' = w' and $R(h_{n+1}(w),h_{n+1}(w'')) \in \mathcal{I}$ by, again, the choice (\dagger) of $h_{n+1}(w)$.

We now give the proof of Lemma 4.

Proof of Lemma 4. We start with Point 1. It is clear by construction that $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ is a model of \mathcal{D} .

We next show that $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ also satisfies all CIs in \mathcal{O} , making a case distinction according to the form of the CI:

- $\top \sqsubseteq A \in \mathcal{O}$.
- For all $c \in \operatorname{adom}(\mathcal{D})$, Rule **R1** ensures that $A(c) \in \operatorname{ch}_{\mathcal{O}}(\mathcal{D}) \subseteq \mathcal{U}_{\mathcal{D},\mathcal{O}}$. For traces $t = t'\rho M$, by construction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ it suffices to show that $A \in M$. But this is a consequence of the definition of traces, in particular of Conditions 2 and 5 above which from now on we refer to as the 'maximality conditions'.
- $A_1 \sqcap A_2 \sqsubseteq A \in \mathcal{O}$. Assume that $t \in A_1^{\mathcal{U}_{\mathcal{D},\mathcal{O}}} \cap A_2^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$. If $t = c \in \operatorname{adom}(\mathcal{D})$, then $t \in A_1^{\operatorname{ch}_{\mathcal{O}}(\mathcal{D})} \cap A_2^{\operatorname{ch}_{\mathcal{O}}(\mathcal{D})}$ and rule $\mathbf{R1}$ ensures that $A(c) \in \mathcal{U}_{\mathcal{D},\mathcal{O}}$. If $t = c\rho M$ or $t = t'\rho M$, then $t \in A_1^{\mathcal{U}_{\mathcal{D},\mathcal{O}}} \cap A_2^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ implies $A_1, A_2 \in M$, thus also $A \in M$ by the maximality conditions, which yields $t \in A^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ by construction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$.
- $A_1 \sqsubseteq \exists R. A_2 \in \mathcal{O}$. Let $t \in A_1^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$. First assume that $t = c \in \mathsf{adom}(\mathcal{D})$. Further assume that
- $\begin{array}{lll} \text{(*) there are } S(c,c_2) \in \mathcal{D} \text{ and } \rho,M \text{ such that } \\ & \text{func}(S) \in \mathcal{O}, \ \mathcal{D}_{M_c},\mathcal{O} \models q^1_{\rho,M}(c), \ S,R \in \rho, \text{ and } \\ & A_2 \in M. \end{array}$

Then Rule **R4** makes sure that $R(c,c_2)$ and $A_2(c_2)$ are in $\mathcal{U}_{\mathcal{D},\mathcal{O}}$, and thus $c\in(\exists R.A_2)^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ as required. Now assume that (*) does not hold. Clearly $t\in A_1^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ implies $A_1(t)\in\mathsf{ch}_{\mathcal{O}}(\mathcal{D})$. Thus $A_1\in M_c$ and consequently there must be some set of roles ρ and of concept names M such that $R\in\rho$, $A_2\in M$, and \mathcal{D}_{M_c} , $\mathcal{O}\models q_{\rho,M}^1(c)$. We may assume w.l.o.g. that ρ and M are maximal with this property. Thus Conditions 1 and 2 of $c\leadsto_{\mathcal{D},\mathcal{O}}^{\rho}M$ are satisfied, and it is not hard to show that Condition 3 must also be satisfied. If, in fact, Condition 3 is violated then there is

an $R'(c,c') \in \mathsf{ch}_{\mathcal{O}}(\mathcal{D})$ with $R \in \rho$ and $\mathsf{func}(R) \in \mathcal{O}$. But then (*) is satisfied for ρ, M and S = R.

Now assume that $t = c\rho M$. Then $A_1 \in M$ and $\mathcal{D}_{M_c}, \mathcal{O} \models q_{\rho,M}^1(c)$. If $\mathcal{D}_{M_c} \models A_2(c)$ and $R^- \in \rho$, then an easy semantic argument shows \mathcal{D}_{M_c} , $\mathcal{O} \models A_2(c)$ and thus rule **R1** ensures that $A_2(c) \in \mathcal{U}_{\mathcal{D},\mathcal{O}}$ and the construction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ yields $(t,c) \in R^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$, thus we are done. Now assume that $\mathcal{D}_{M_c} \not\models A_2(c)$ or $R^- \notin \rho$. Since $A_1 \in M$, there must be some set of roles ρ' and of concept names M' such that $R \in \rho$, $A_2 \in M$, and $\mathcal{D}_M, \mathcal{O} \models q^1_{\rho',M'}(\widehat{c})$. We may assume w.l.o.g. that ρ' and M' are maximal with this property. Thus $M \leadsto_{\mathcal{D},\mathcal{O}}^{\rho'} M'$. Using the fact that $\mathcal{D}_{M_c}, \mathcal{O} \not\models A_2(c)$ or $R^- \not\in \rho$ one can prove that Condition 8 of traces is satisfied for $t' = t\rho' M'$, based on a straightforward model-theoretic argument. Since all other conditions of traces are trivially satisfied for t', we obtain $t' \in \mathbf{T}$ and the construction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ yields $(t,t') \in R^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ and $t' \in A_2^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$.

Now assume that $t=t'\rho M$ with $t'=t''\rho' M'$. Then $A_1\in M$. If $A_2\in M'$ and $R^-\in \rho$, then $(t,t')\in R^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ and $t'\in A_2^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ and we are done. Now assume that $A_2\notin M'$ or $R^-\notin \rho$. Since $A_1\in M$, there must be some set of roles $\widehat{\rho}$ and of concept names \widehat{M} such that $R\in \widehat{\rho},\,A_2\in \widehat{M},\,$ and $\mathcal{D}_M,\mathcal{O}\models q_{\widehat{\rho},\widehat{M}}^1(\widehat{c}).$ We may assume w.l.o.g. that $\widehat{\rho}$ and \widehat{M} are maximal with this property. Thus $M\leadsto_{\mathcal{D},\mathcal{O}}^{\widehat{\rho}}\widehat{M}.$ The fact that $A_2\notin M'$ or $R^-\notin \rho$ can be used to prove model-theoretically that Condition 8 of traces is satisfied for $\widehat{t}=t\widehat{\rho}\widehat{M}.$ Since all other conditions of traces are trivially satisfied for $\widehat{t},$ we obtain $\widehat{t}\in \mathbf{T}$ and the construction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ yields $(t,\widehat{t})\in R^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ and $\widehat{t}\in A_2^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}.$

• $\exists R.A_1 \sqsubseteq A_2 \in \mathcal{O}$. Let $(t,t') \in R^{\mathcal{U}_{\mathcal{D}},\mathcal{O}}$ and $t' \in A_1^{\mathcal{U}_{\mathcal{D}},\mathcal{O}}$. We distinguish several cases.

t,t' are both from $adom(\mathcal{D})$, then Rule **R2** yields $t \in A_2^{\mathcal{U}_{\mathcal{D}},\mathcal{O}}$, as required.

t is from $\operatorname{adom}(\mathcal{D})$ and $t'=t\rho M$. Then $A_1\in M, R\in \rho$, and $\mathcal{D}_{M_c},\mathcal{O}\models q^1_{\rho,M}(t')$. Consequently, $\mathcal{D}_{M_c},\mathcal{O}\models A_2(t)$, implying $t\in A_2^{\mathcal{U}_{\mathcal{D}},\mathcal{O}}$.

 $\begin{array}{ll} t' \ \textit{is from} \ \mathsf{adom}(\mathcal{D}) \ \textit{and} \ t = t' \rho M. \ \mathsf{Then} \ \mathcal{D}, \mathcal{O} \models A_1(t'), \\ R^- \in \ \rho, \ \mathsf{and} \ \rho \ \mathsf{and} \ M \ \mathsf{are \ maximal \ with} \ \mathcal{D}, \mathcal{O} \models \\ q^1_{\rho,M}(t). \ \mathsf{Thus \ clearly} \ A_2 \in M, \ \mathsf{implying} \ t \in A_2^{\mathcal{U}_{\mathcal{D},\mathcal{O}}} \\ \mathsf{due} \ \mathsf{to} \ \mathsf{Rule} \ \mathbf{R1}. \end{array}$

t,t' are both not from $\operatorname{adom}(\mathcal{D})$ and $t'=t\rho'M'$. Let $t=c\rho M$ or $t=M_0\rho M$. Then $A_1\in M',\,R\in\rho'$, and $\mathcal{D}_M,\mathcal{O}\models q^1_{\rho',M'}(\widehat{c})$. It is easy to see that this implies $\mathcal{D}_M,\mathcal{O}\models A_2(\widehat{c})$. Since M is maximal with $\mathcal{D}_{M_c},\mathcal{O}\models q^1_{\rho,M}(c)$ or $\mathcal{D}_{M_0}\models q^1_{\rho,M}(\widehat{c})$, it follows that $A_2\in M$. This implies that $t\in A_2^{U_{\mathcal{D},\mathcal{O}}}$.

t,t' are both not from $\operatorname{adom}(\mathcal{D})$ and $t=t'\rho M$. Let $t'=c\rho'M'$ or $t'=M_0\rho'M'$. Then $A_1\in M'$, $R^-\in \rho$, and M is maximal with $\mathcal{D}_{M'},\mathcal{O}\models q^1_{\rho,M}(\widehat{c})$. Consequently, $A_2\in M$, implying $t\in A_2^{\mathcal{U}_{\mathcal{D}},\mathcal{O}}$.

We next show that the RIs in $\mathcal O$ are satisfied. Thus let $R \sqsubseteq S \in \mathcal O$. If $R(c,c') \in \mathcal U_{\mathcal D,\mathcal O}$ with $c,c' \in \operatorname{adom}(\mathcal D)$, then $R(c,c') \in \operatorname{ch}_{\mathcal O}(\mathcal D)$ and Rule $\operatorname{\bf R3}$ yields $(c,c') \in S^{\mathcal U_{\mathcal D,\mathcal O}}$. It remains to deal with the cases $(c,c\rho M) \in R^{\mathcal U_{\mathcal D,\mathcal O}}$, $(t,t\rho M) \in R^{\mathcal U_{\mathcal D,\mathcal O}}$, and $(t\rho M,t) \in R^{\mathcal U_{\mathcal D,\mathcal O}}$. In the first two cases, we must have $R \in \rho$, implying $S \in \rho$ by the maximality conditions, and thus also $(c,c\rho M) \in S^{\mathcal U_{\mathcal D,\mathcal O}}$ and $(t,t\rho M) \in S^{\mathcal U_{\mathcal D,\mathcal O}}$, respectively. In the last case, we must have $R^- \in \rho$. Clearly $R \sqsubseteq S \in \mathcal O$ implies $\mathcal O \models R^- \sqsubseteq S^-$. We can thus argue similarly that $(t\rho M,t) \in S^{\mathcal U_{\mathcal D,\mathcal O}}$.

It remains to show that all functionality assertions are satisfied. We first note that it is straightforward to show the following, essentially by induction on the number of rule applications used to construct $ch_{\mathcal{O}}(\mathcal{D})$:

- (i) $c \in A^{\mathsf{ch}_{\mathcal{O}}(\mathcal{D})}$ implies $\mathcal{D}, \mathcal{O} \models A(c)$;
- (ii) $(c, c') \in R^{\mathsf{ch}_{\mathcal{O}}(\mathcal{D})}$ implies $\mathcal{D}, \mathcal{O} \models R(c, c')$.

We may replace the precondition in (ii) with the equivalent condition $(c,c') \in R^{\mathcal{U}_{\mathcal{D},\mathcal{O}}}$ (and likewise for (i), but this is not going to be important).

Assume to the contrary of what we want to show that there are $R(t,t_1), R(t,t_2) \in \mathcal{U}_{\mathcal{D},\mathcal{O}}$ with func $(R) \in \mathcal{O}$ and $t_1 \neq t_2$. We distinguish several cases:

 t, t_1, t_2 are all from $\mathsf{adom}(\mathcal{D})$. It then follows from (ii) (with precondition $(c, c') \in R^{\mathcal{U}_{\mathcal{D}, \mathcal{O}}}$) that \mathcal{D} is not satisfiable w.r.t. \mathcal{O} , a contradiction.

 t,t_1 are from $\mathrm{adom}(\mathcal{D})$ and $t_2=t\rho_2M_2$. Then $t\leadsto^{\rho_2}_{\mathcal{D},\mathcal{O}}M_2$ and $R\in\rho_2$. Since $R(t,t_1)\in\mathcal{U}_{\mathcal{D},\mathcal{O}}$, also $R(t,t_1)\in\mathrm{ch}_{\mathcal{O}}(\mathcal{D})$. Together with $\mathrm{func}(R)\in\mathcal{O}$, we thus obtain a contradiction against Condition 3 of $t\leadsto^{\rho_2}_{\mathcal{D},\mathcal{O}}M_2$.

 $t_1=t\rho_1M_1$, and $t_2=t\rho_2M_2$. Then $t\leadsto_{\mathcal{D},\mathcal{O}}^{\rho_1}M_1$ and $t\leadsto_{\mathcal{D},\mathcal{O}}^{\rho_2}M_2$. Moreover, $R\in\rho_1\cap\rho_2$. The maximality conditions thus yield $\rho_1=\rho_2$ and $M_1=M_2$, in contradiction to $t_1\neq t_2$.

 $t=t_1\rho_1M_1$, and $t_2=t\rho_2M_2$. Then $R^-\in\rho_1$ and $R\in\rho_2$, contradicting Condition 8 of paths for t_2 .

For Point 2, let \mathcal{I} be a model of \mathcal{D} and \mathcal{O} . We construct a homomorphism from $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ to \mathcal{I} that is the identity on $\mathsf{adom}(\mathcal{D})$ in two steps.

We first show that $\operatorname{ch}_{\mathcal{O}}(\mathcal{D}) \subseteq \mathcal{I}$. This gives us a homomorphism h_0 from $\mathcal{U}_{\mathcal{D},\mathcal{O}}|_{\operatorname{adom}(\mathcal{D})} = \operatorname{ch}_{\mathcal{O}}(\mathcal{D})$ to \mathcal{I} that is the identity on $\operatorname{adom}(\mathcal{D})$. Then, we extend h_0 to a homomorphism from $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ to \mathcal{I} using Lemma 6.

Let $\mathcal{D} = \mathcal{N}_0, \mathcal{N}_1, \dots, \mathcal{N}_\ell = \operatorname{ch}_\mathcal{O}(\mathcal{D})$ be the databases obtained by applying rules **R1-R4** in the construction of the chase. To show that $\operatorname{ch}_\mathcal{O}(\mathcal{D}) \subseteq \mathcal{I}$, it suffices to show that $\mathcal{N}_i \subseteq \mathcal{I}$ for all $0 \le i \le \ell$. The base case, i = 0, is trivial as \mathcal{I} is a model of $\mathcal{D} = \mathcal{N}_0$.

For the inductive step, let us assume that $\mathcal{N}_i \subseteq \mathcal{I}$, $i < \ell$. We show that $\mathcal{N}_{i+1} \subseteq \mathcal{I}$. By definition, \mathcal{N}_{i+1} was constructed from \mathcal{N}_i by applying one of the rules **R1-R4**.

Let **R1** be the applied rule. It has added a single new fact A(c). Since the rule has been used, there are facts $A_1(c),\ldots,A_n(c)\in\mathcal{N}_i$. By the inductive assumption we have that $A_1(c),\ldots,A_n(c)\in\mathcal{I}$. Since \mathcal{I} is a model of \mathcal{O} and $\mathcal{O}\models A_1\sqcap\cdots\sqcap A_n\sqsubseteq A$ we infer that $A(c)\in\mathcal{I}$. This

implies that $\mathcal{N}_{i+1} = \mathcal{N}_i \cup \{A(c)\} \subseteq \mathcal{I}$ and ends the inductive step for rule **R1**. This ends the first step of the proof. Let h_0 be the identity homomorphism from $\mathrm{ch}_{\mathcal{O}}(\mathcal{D})$ to \mathcal{I} .

For the second step note that $\mathrm{ch}_{\mathcal{O}}(\mathcal{D}) \subseteq \mathcal{I}$ implies that $(\mathrm{ch}_{\mathcal{O}}(\mathcal{D}),c) \preceq (\mathcal{I},c)$ for every $c \in \mathrm{dom}(\mathcal{D})$. It thus follows from Lemmas 5 and 6 that there is a homomorphism h_c from $\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}$ to \mathcal{I} . We show that $h=h_0 \cup \bigcup_{c \in \mathrm{dom}(\mathcal{D})} h_c$ is the desired homomorphism from $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ to \mathcal{I} .

The function h is well defined as h_0 is identity on $adom(\mathcal{D})$ and for all $c \in adom(\mathcal{D})$ we have that $h_c(c) = c$. This also implies that h is identity on $adom(\mathcal{D})$.

To show that h is a homomorphism observe the following. For concept names, if $A(d) \in \mathcal{U}_{\mathcal{D},\mathcal{O}}$ then $h(d) = h_0(d)$ or $h(d) = h_c(d)$ for some $c \in \text{dom}(\mathcal{D})$. Thus, $A(h(c)) \in \mathcal{I}$ as h_0 and h_c are homomorphisms.

Similarly, for role names, if R(c,d) is a fact in $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ then either $c,d\in \mathsf{dom}(\mathcal{D})$ and $h(c)=h_0(c),h(d)=h_0(d)$ or there is $c'\in \mathsf{dom}(\mathcal{D})$ such that c and d are traces in $\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}$. In the former case we have that $R(h(c),h(d))\in\mathcal{I}$ as h_0 is a homomorphism from $\mathsf{ch}_{\mathcal{O}}(\mathcal{D})$ to \mathcal{I} , and in the latter we have that $R(h(c),h(d))\in\mathcal{I}$ as h_c is a homomorphism from $\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}$ to \mathcal{I} .

Finally, Point 3 is a consequence of Point 2 and the definitions of $Q(\mathcal{D})$ and $q(\mathcal{U}_{\mathcal{D},\mathcal{O}})$.

C.1 Query-Directed Universal Models

Let $Q=(\mathcal{O},\Sigma,q)\in(\mathcal{ELTHF},\operatorname{CQ})$ with \mathcal{O} in normal form, and let $n=|\operatorname{var}(q)|.$ We use $\operatorname{cl}(Q)$ to denote the set of all Boolean tree CQs that use at most n variables, only symbols from $\operatorname{sig}(\mathcal{O})\cup\operatorname{sig}(q),$ and satisfy all functionality assertions in $\mathcal{O}.$ We may assume that the CQs in $\operatorname{cl}(Q)$ use only variables from a fixed set of size n, and thus $\operatorname{cl}(Q)$ is finite.

Let \mathcal{D} be a Σ -database that is satisfiable w.r.t. \mathcal{O} . The *Quiversal extension of* \mathcal{D} , denoted $\mathcal{U}_{\mathcal{D},Q}$ is constructed in three steps as follows.

Step 1. Start with $ch_{\mathcal{O}}(\mathcal{D})$.

Step 2. Extend $\operatorname{ch}_{\mathcal{O}}(\mathcal{D})$ to the restriction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ to traces of length n. Informally, this attaches trees of depth n to constants $c \in \operatorname{adom}(\mathcal{D})$. This can be done following exactly the definition of traces, and it involves deciding questions of the form \mathcal{D}_M , $\mathcal{O} \models q_{\rho,M'}^1(c)$.

Step 3. For every $c \in \mathsf{adom}(\mathcal{D})$, let again M_c denote the set of concept names A with $A(c) \in \mathsf{ch}_{\mathcal{O}}(\mathcal{D})$. Then include, for every $p \in \mathsf{cl}(Q)$ such that $\mathcal{D}_{M_c}, \mathcal{O} \models p$ for some $c \in \mathsf{adom}(\mathcal{D})$, a copy of \mathcal{D}_p that uses only fresh constants.

It should be clear that $\mathcal{U}_{\mathcal{D},Q}$ satisfies all functionality assertions in \mathcal{O} .

Lemma 7. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIHF}, CQ)$ and let \mathcal{D} be a Σ -database. Then $Q(\mathcal{D}) = q(\mathcal{U}_{\mathcal{D},Q}) \cap \operatorname{adom}(\mathcal{D})^{|\bar{x}|}$. What is more, for every $Q'(\bar{x}') = (\mathcal{O}, \Sigma, q')$ where $|\operatorname{var}(q')| \leq |\operatorname{var}(q)|$, we have $Q'(\mathcal{D}) = q'(\mathcal{U}_{\mathcal{D},Q}) \cap \operatorname{adom}(\mathcal{D})^{|\bar{x}|}$.

Proof. It clearly suffices to prove the 'what is more' part of the lemma.

The ' \supseteq ' direction is simple. Assume that $\bar{c} \in q'(\mathcal{U}_{\mathcal{D},Q}) \cap$ adom $(\mathcal{D})^{|\bar{x}'|}$. Using the construction of $\mathcal{U}_{\mathcal{D},Q}$, it is not hard to see that there is a homomorphism from $\mathcal{U}_{\mathcal{D},Q}$ to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ that is the identity on $\mathrm{adom}(\mathcal{D})$. Thus $\bar{c} \in q'(\mathcal{U}_{\mathcal{D},\mathcal{O}})$ and it follows from Point 3 of Lemma 4 that $\bar{c} \in Q'(\mathcal{D})$.

For the ' \subseteq ' direction, assume that $\bar{c} \in Q'(\mathcal{D})$. Then there is a homomorphism h from q' to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ such that $h(\bar{x}') = \bar{c}$. Let q_1,\ldots,q_k be the maximal connected components of q' such that $h(y) \notin \text{dom}(\mathcal{D})$ for all variables y that occur in them. Note that by definition, q_1,\ldots,q_k are Boolean. Further let q_0 be the disjoint union of all remaining maximal connected components of q'.

Let $n=|\mathrm{var}(q)|$ and let $\mathcal{U}_{\mathcal{D},\mathcal{O},n}$ denote the restriction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ to traces of length at most n. It is not hard to verify that $h(y)\in \mathrm{adom}(\mathcal{U}_{\mathcal{D},\mathcal{O},n})$ for all $y\in \mathrm{var}(q_0)$, and thus the restriction h' of h to the variables in q_0 is a homomorphism from q_0 to $\mathcal{U}_{\mathcal{D},\mathcal{O},n}\subseteq\mathcal{U}_{\mathcal{D},Q}$. To show that $\bar{c}\in q'(\mathcal{U}_{\mathcal{D},Q})\cap \mathrm{adom}(\mathcal{D})^{|\bar{x}|}$, it remains to extend h' to q_1,\ldots,q_k .

Let $1 \leq i \leq k$ and let q_i' be the CQ obtained from q_i by identifying variables $x,y \in \text{var}(q)$ if h(x) = h(y). Clearly, it suffices to show that we can extend h' to q_i' in place of q_i . Since $h(y) \notin \text{adom}(\mathcal{D})$ for all $y \in \text{var}(q_i)$ and due to the shape of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$, the CQ q_i' must be a tree CQ. Consequently, it is in cl(Q).

Since the restriction h_i of h to the variables in q_i' is a homomorphism from q_i' to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ that does not have any constants from $\mathsf{adom}(\mathcal{D})$ in its range, there is an $e_i \in \mathsf{adom}(\mathcal{D})$ such that the range of h_i consists only of traces that start with e_i . The construction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ then yields $\mathcal{D}_{M_{e_i}},\mathcal{O} \models q_i()$. Consequently, a copy \mathcal{D}'_{q_i} of \mathcal{D}_{q_i} was added during the construction of $\mathcal{U}_{\mathcal{D},\mathcal{Q}}$. It is thus easy to extend h' to q_i' , as desired.

We now show that the Q-universal database extension $\mathcal{U}_{\mathcal{D},Q}$ is also universal for answers with wildcards when we view all elements of $\mathsf{adom}(\mathcal{U}_{\mathcal{D},Q}) \setminus \mathsf{adom}(\mathcal{D})$ as nulls.

Lemma 8.

1.
$$Q(\mathcal{D})^* = q(\mathcal{U}_{\mathcal{D},Q})_{\mathbf{N}}^*$$
 and
2. $Q(\mathcal{D})^{\mathcal{W}} = q(\mathcal{U}_{\mathcal{D},Q})_{\mathbf{N}}^{\mathcal{W}}$.

Proof. We only consider Point 2. The proof of Point 1 is similar. When we speak of partial answers, we thus generally mean partial answers with multi-wildcards. We first observe that it suffices to prove the following:

- (a) every minimal partial answer to Q on \mathcal{D} is a partial answer to q on $\mathcal{U}_{\mathcal{D},\mathcal{O}}$;
- (b) every partial answer to q on $\mathcal{U}_{\mathcal{D},Q}$ if a partial answer to Q on \mathcal{D} .

For assume that (a) and (b) have been shown. First let $\bar{a}^{\mathcal{W}} \in Q(\mathcal{D})^{\mathcal{W}}$. By (a), $\bar{a}^{\mathcal{W}}$ is a partial answer to q on $\mathcal{U}_{\mathcal{D},Q}$. Assume to the contrary of what we have to show that $\bar{a}^{\mathcal{W}}$ is not minimal, that is, there is a partial answer $\bar{b}^{\mathcal{W}}$ to q on $\mathcal{U}_{\mathcal{D},Q}$ with $\bar{b}^{\mathcal{W}} \prec \bar{a}^{\mathcal{W}}$. By (b), $\bar{b}^{\mathcal{W}}$ is a partial answer to Q on \mathcal{D} , contradicting minimality of $\bar{a}^{\mathcal{W}}$.

Conversely, let $\bar{a}^{\mathcal{W}} \in q(\mathcal{U}_{\mathcal{D},Q})^{\mathcal{W}}_{\mathbf{N}}$. By (b), $\bar{a}^{\mathcal{W}}$ is a partial answer to Q on \mathcal{D} . Assume to the contrary of what we have

to show that $\bar{a}^{\mathcal{W}}$ is not minimal, that is, there is a partial answer $\bar{b}^{\mathcal{W}}$ to Q on \mathcal{D} with $\bar{b}^{\mathcal{W}} \prec \bar{a}^{\mathcal{W}}$. We may assume w.l.o.g. that $\bar{b}^{\mathcal{W}}$ is minimal. Thus (a) implies that $\bar{b}^{\mathcal{W}}$ is a partial answer to q on $\mathcal{U}_{\mathcal{D},Q}$, contradicting minimality of $\bar{a}^{\mathcal{W}}$.

It thus remains to prove (a) and (b). We start with the former. Assume that $\bar{a}^{\mathcal{W}} \in Q(\mathcal{D})^{\mathcal{W}}$. Let $\bar{x} = x_1, \ldots, x_n$, let $\bar{a}^{\mathcal{W}} = a_1, \ldots, a_n$, and let the wildcards from \mathcal{W} that occur in $\bar{a}^{\mathcal{W}}$ be $*_1, \ldots, *_\ell$. Consider the CQ $q'(\bar{x}')$ obtained from q in the following way:

- introduce fresh quantified variables z_1, \ldots, z_ℓ ;
- if a_i = *_j, then replace in q' the answer variable x_i with quantified variable z_j.

Further let \bar{a} be obtained from $\bar{a}^{\mathcal{W}}$ by removing all wild-cards

By a simple semantic argument, it follows from the fact that $\bar{a}^{\mathcal{W}}$ is a partial answer to Q on \mathcal{D} that $\bar{a} \in Q'(\mathcal{D})$ where $Q'(\bar{x}') = (\mathcal{O}, \Sigma, q')$. Note that $|\mathsf{var}(q')| \leq |\mathsf{var}(q)|$. Applying Lemma 7 thus yields $\bar{a} \in q'(\mathcal{U}_{\mathcal{D},Q}) \cap \mathsf{adom}(\mathcal{D})|^{\bar{x}'}|$. Consequently, there is a homomorphism h' from q' to $\mathcal{U}_{\mathcal{D},Q}$ such that $h'(\bar{x}') = \bar{a}$. We observe the following:

(*)
$$h'(z_i) \notin \mathsf{adom}(\mathcal{D}) \text{ for } 1 \leq i \leq \ell.$$

For assume to the contrary that there is a z_p with $h'(z_p) \in \mathsf{adom}(\mathcal{D})$. Consider

- the CQ $q''(\bar{x}'')$ obtained from q in exactly the same way as q' except that if $a_i = *_p$, then x_i is *not* replaced with z_p ,
- the tuple $\bar{b}^{\mathcal{W}}$ obtained from $\bar{a}^{\mathcal{W}}$ in the following way: if $a_i = *_p$, then replace a_i with $h'(z_p)$, and
- the tuple \bar{b} obtained from $\bar{b}^{\mathcal{W}}$ by removing all wildcards.

The homomorphism h' witnesses that $\bar{b} \in q''(\mathcal{U}_{\mathcal{D},Q}) \cap \operatorname{adom}(\mathcal{D})^{|\bar{x}''|}$. From Lemma 7 we thus obtain $\bar{b} \in Q''(\mathcal{D})$ where $Q''(\bar{x}'') = (\mathcal{O}, \Sigma, \underline{q}'')$. An easy semantic argument shows that, consequently, $\bar{b}^{\mathcal{W}}$ is a partial answer to Q on \mathcal{D} , contradicting the minimality of $\bar{a}^{\mathcal{W}}$. Thus (*) is shown.

We may obtain from h' a homomorphism h from q to $\mathcal{U}_{\mathcal{D},Q}$ by setting $h(x_i) = z_j$ whenever x_i was replaced by z_j in the construction of q'. Using h and (*), it is now easy to show that $\bar{a}^{\mathcal{W}}$ is a partial answer to q on $\mathcal{U}_{\mathcal{D},Q}$, as required.

For (b), let $\bar{a}^{\mathcal{W}}$ be a partial answer to q on $\mathcal{U}_{\mathcal{D},\mathcal{O}}$. Let q' and \bar{a} be constructed exactly as above. Then $\bar{a} \in q'(\mathcal{U}_{\mathcal{D},Q}) \cap$ adom $(\mathcal{D})^{|\bar{x}'|}$. Lemma 7 yields $\bar{a} \in Q'(\mathcal{D})$ where again $Q'(\bar{x}') = (\mathcal{O}, \Sigma, q')$. An easy semantic argument shows that, consequently, $\bar{a}^{\mathcal{W}}$ is a partial answer to Q on \mathcal{D} , as required.

C.2 Query-Directed Universal Models in Linear Time

Proposition 2. Let $Q = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIHF}, CQ)$ and \mathcal{D} a Σ -database that is satisfiable w.r.t. \mathcal{O} . Then the Q-universal extension $\mathcal{U}_{\mathcal{D},Q}$ can be computed in time linear in $||\mathcal{D}||$, more precisely in time $2^{\mathsf{poly}(||Q||)} \cdot ||\mathcal{D}||$.

To prove Proposition 2, we make use of the fact that minimal models for propositional Horn formulas can be computed in linear time (Dowling and Gallier 1984). More precisely, we derive a satisfiable propositional Horn formula θ from $\mathcal O$ and $\mathcal D$, compute a minimal model of θ in linear time, and read off $\mathrm{ch}_{\mathcal O}(\mathcal D)$ from that model. It then remains to construct $\mathcal U_{\mathcal D,\mathcal Q}$ from $\mathrm{ch}_{\mathcal O}(\mathcal D)$, exactly as described in the previous section.

An edge in \mathcal{D} is a pair (c_1, c_2) such that $R(c_1, c_2) \in \mathcal{D}$ for some role R. Note that if (c_1, c_2) is an edge in \mathcal{D} , them so is (c_2, c_1) . We introduce the following propositional variables:

- x_{A(c)} for every concept name A ∈ sig(O) and every c ∈ adom(D);
- $x_{r(c,c')}$ for every edge (c,c') in \mathcal{D} and every role name $r \in \operatorname{sig}(\mathcal{O})$.

We may write $x_{r-(c',c)}$ on place of $x_{r(c,c')}$. Clearly, the number of introduced variables is bounded by $||\mathcal{D}|| \cdot ||\mathcal{O}||$.

The propositional Horn formula θ consists of the following conjuncts:

- 1. $x_{A(c)}$ for every $A(c) \in \mathcal{D}$ and $x_{r(c,c')}$ for every $r(c,c') \in \mathcal{D}$;
- 2. $x_{A_1(c)} \wedge \cdots \wedge x_{A_n(c)} \to x_{A(c)}$ for all $c \in \mathsf{adom}(\mathcal{D})$ and all concept names A_1, \ldots, A_n, A such that $\mathcal{O} \models A_1 \sqcap \cdots \sqcap A_n \sqsubseteq A$;
- 3. $x_{A(c_1)} \wedge x_{R(c_2,c_1)} \rightarrow x_{B(c_2)}$ for all edges (c_1,c_2) of $\mathcal D$ and all $\exists R.A \sqsubseteq B \in \mathcal O$;
- 4. $x_{R(c_1,c_2)} \to x_{S(c_1,c_2)}$ for all edges (c_1,c_2) of $\mathcal D$ and all $R \sqsubseteq S \in \mathcal O$;
- 5. $x_{A(c_1)} \wedge x_{R(c_1,c_2)} \rightarrow x_{S(c_1,c_2)}$ and $x_{A(c_1)} \wedge x_{R(c_1,c_2)} \rightarrow x_{B(c_2)}$ for all $A \sqsubseteq \exists S.B \in \mathcal{O}$ such that $S \sqsubseteq_{\mathcal{O}}^* R$ and func $(R) \in \mathcal{O}$.

The size of θ is bounded by $2^{O(||\mathcal{O}||)} \cdot ||\mathcal{D}||$ and θ can be constructed in time $2^{O(||\mathcal{O}||)} \cdot ||\mathcal{D}||$.

Since θ contains no negative literals, it is clearly satisfiable and thus has a unique minimal model. Let V be the truth assignment that represents this minimal model. We construct the database \mathcal{D}_{θ} by including all A(c) such that $V(x_{A(c)}) = 1$ and all $r(c_1, c_2)$ such that $V(x_{r(c_1, c_2)}) = 1$. It is clear that the construction of \mathcal{D}_{θ} is possible in time $2^{O(||\mathcal{O}||)} \cdot ||\mathcal{D}||$. Moreover, it is easy to verify that $\mathcal{D}_{\theta} = \operatorname{ch}_{\mathcal{O}}(\mathcal{D})$ given that the implications in θ exactly parallel the rules of the chase given in Section C.

We may now construct $\mathcal{U}_{\mathcal{D},\mathcal{Q}}$ from $\mathrm{ch}_{\mathcal{O}}(\mathcal{D})$ exactly as in the previous section, using Steps 2 and 3 described there. In Step 2, we may iterate over all $c \in \mathrm{adom}(\mathcal{D})$ and then add the fragment of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ that consists of traces of length at most n starting with c, following exactly the definition of such traces, that is, proceeding in the order of increasing trace length. It is easy to see that the number of traces that start with c is bounded by $2^{\mathrm{poly}(||\mathcal{Q}||)}$. This involves deciding questions of the form \mathcal{D}_M , $\mathcal{O} \models q^1_{\rho,M'}(c)$ which is possible in time $2^{\mathrm{poly}(||\mathcal{Q}||)}$ (Eiter et al. 2008). Thus, Step 2 can be implemented in time $2^{\mathrm{poly}(||\mathcal{Q}||)} \cdot ||\mathcal{D}||$.

For Step 3, first note that the number of CQs in \mathcal{Q} is bounded by $2^{\text{poly}(||\mathcal{Q}||)}$. We may iterated over all $c \in$

 $\operatorname{adom}(D)$ and all $p \in \mathcal{Q}$ and check whether $\mathcal{D}_{M_c}, \mathcal{O} \models p$, and if so, add a disjoint copy of p. Again the described check is possible in time $2^{\operatorname{poly}(||\mathcal{Q}||)}$ (Eiter et al. 2008) and the overall time needed is $2^{\operatorname{poly}(||\mathcal{Q}||)} \cdot ||\mathcal{D}||$.

D Proofs for Section 2

Lemma 1. Every homomorphism from q_0 to \mathcal{D}_0 is also a homomorphism from q_0^+ to \mathcal{D}_0^+ and vice versa. Moreover, \mathcal{D}_0^+ can be constructed in time linear in $||\mathcal{D}||$.

Proof. First assume that h is a homomorphism from q_0 to \mathcal{D}_0 . Let $R'(\bar{y}^+)$ be an atom in q_0^+ , and let $R(\bar{y})$ be the atom in q_0 that gave rise to it during the construction of q_0^+ . The restriction $h|_{\bar{y}^+}$ of h to the variables in \bar{y}^+ is a homomorphism from $q_0|_{\bar{y}^+}$ to \mathcal{D}_0 . By definition, \mathcal{D}_0^+ thus contains the atom $R'(h(\bar{y}^+))$, as required.

Conversely, assume that h is a homomorphism from q_0^+ to \mathcal{D}_0^+ and let $R(\bar{y})$ be an atom in q_0 . Then q_0^+ contains a corresponding atom $R'(\bar{y}^+)$ and $R'(h(\bar{y}^+)) \in \mathcal{D}_0^+$. By construction of \mathcal{D}_0^+ , there is a homomorphism g from $q_0|_{\bar{y}^+}$ to \mathcal{D}_0 that is identical to the restriction of h to the variables in \bar{y}^+ . It follows that $R(h(\bar{y})) \in \mathcal{D}_0$, as required.

We sketch the computation of $||\mathcal{D}_0^+||$ in linear time. We first construct a lookup table that allows us to find in constant time, given a $c \in \mathsf{adom}(\mathcal{D}_0)$ and a role R with $\mathsf{func}(R) \in \mathcal{O}$, the unique $c' \in \mathsf{adom}(\mathcal{D}_0)$ with $R(c,c') \in \mathcal{D}_0$ (if existant). This relies on the use of the RAM model and can be achieved by a single scan of \mathcal{D}_0 . We then iterate over every atom $R(\bar{y})$ of q_0 . If $R(\bar{y})$ is replaced with $R'(\bar{y}^+)$ in q_0^+ , then we need to add to \mathcal{D}_0^+ a fact $R'(h(\bar{y}^+))$ for every homomorphism h from $q_0|_{\bar{y}^+}$ to \mathcal{D}_0 . To find these homomorphisms h, we consider all facts $R(\bar{c}) \in \mathcal{D}_0$ as candidates for $R(h(\bar{y}))$. We can find them by a single scan of \mathcal{D}_0 . Every variable z in \bar{y}^+ is reachable from a variable y in \bar{y} by a functional path in q_0 . Following this functional paths in \mathcal{D}_0 , starting at h(y)and using the lookup table, we can find the unique possible target h(z) (if existant). It then remains to check whether the constructed h is really a homomorphism. This is possible in constant time if we have previously created lookup tables that tell us for every k-ary $R \in \Sigma$ and $\bar{c} \in \mathsf{adom}(\mathcal{D}_0)^k$ whether $R(\bar{c}) \in \mathcal{D}_0$.

We next establish Theorem 2 for the multi-wildcard case. Let \mathcal{D} be a database that may use nulls in place of constants, and let $q(\bar{x})$ be a CQ. For an answer $\bar{a} \in q(\mathcal{D})$, we use $\bar{a}_{\mathbf{N}}^{\mathcal{W}}$ to denote the (unique) multi-wildcard tuple for \mathcal{D} obtained from \bar{a} by consistently replacing nulls from \mathbf{N} with wildcards using exactly a prefix of the ordered set $\mathcal{W} = \{*_1, *_2, \ldots\}$ and respecting the order, that is, if the first occurrence of a null c_1 in \bar{a} is before the first occurrence of a null c_2 , c_1 is replaced with $*_i$, and c_2 with $*_j$, then i < j. With $q(\mathcal{D})_{N}^{\mathcal{W}}$. We call such an $\bar{a}_{N}^{\mathcal{W}}$ a partial answer to q on \mathcal{D} (with multi-wildcards) and say that it is minimal if there is no $\bar{a} \in q(\mathcal{D})$ with $\bar{b}_{N}^{\mathcal{W}} \prec \bar{a}_{N}^{\mathcal{W}}$. We use $q(\mathcal{D})_{N}^{\mathcal{W}}$ to denote the set of minimal partial answers with multi-wildcards to q on \mathcal{D} .

As already mentioned, In the query-directed universal models $\mathcal{U}_{\mathcal{D},Q}$ from Proposition 1, we may view all elements of $N = \mathsf{adom}(\mathcal{U}_{\mathcal{D},Q}) \setminus \mathsf{adom}(\mathcal{D})$ as nulls. We then have $Q(\mathcal{D})^{\mathcal{W}} = q(\mathcal{U}_{\mathcal{D},Q})^{\mathcal{W}}_{\mathbf{N}}$.

To prove Theorem 2 in the multi-wildcard case, we replace Theorem 3 with the following result, also from (Lutz and Przybylko 2022b) (Proposition F.1).

Theorem 10. For every CQ $q(\bar{x})$ that is acyclic and free-connex ayclic, enumerating the answers $q(\mathcal{D})_{\mathbf{N}}^{\mathcal{W}}$ is in $\mathsf{DelayC}_{\mathsf{lin}}$ for databases \mathcal{D} and sets of nulls $N \subseteq \mathsf{adom}(\mathcal{D})$ such that \mathcal{D} is chase-like with witness $\mathcal{D}_1, \ldots, \mathcal{D}_n$ where $|\mathsf{adom}(\mathcal{D}_i)|$ does not depend on \mathcal{D} for $1 \le i \le n$.

We now proceed as in the single-wildcard case. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELTHF}, \operatorname{CQ})$ with $q^+(\bar{x})$ acyclic and free-connex acyclic, and let \mathcal{D} be a Σ -database. By Lemma 8, the query-directed universal model $\mathcal{U}_{\mathcal{D},Q}$ is also universal for partial answers with multi-wildcards in the sense that $Q(\mathcal{D})^{\mathcal{W}} = q(\mathcal{U}_{\mathcal{D},Q})^{\mathcal{W}}_{\mathbf{N}}$. We thus first replace \mathcal{D} with $\mathcal{U}_{\mathcal{D},Q}$, aiming to enumerate $q(\mathcal{U}_{\mathcal{D},Q})^{\mathcal{W}}_{\mathbf{N}}$. We next replace $q(\bar{x})$ with $q^+(\bar{x})$ and $\mathcal{D}_0 = \mathcal{U}_{\mathcal{D},Q}$ with \mathcal{D}_0^+ . It follows from Lemma 1 that $q(\mathcal{D}_0)^{\mathcal{W}}_{\mathbf{N}} = q^+(\mathcal{D}_0^+)^{\mathcal{W}}_{\mathbf{N}}$ and thus we can apply the algorithm promised by Theorem 10 to enumerate $q^+(\mathcal{D}_0^+)^{\mathcal{W}}_{\mathbf{N}}$.

E Proofs for Section 3

We provide proofs for Section 3, starting with several preliminaries.

E.1 Derivation completeness

Recall that we have defined in the main part of the paper a database \mathcal{D}_{ω} and a finite subset $\mathcal{D}_{\text{tree}}$. We will use the latter in all of our lower bound proofs. In this section, we estbalish some fundamental properties related to it.

We say that a Σ -database $\mathcal D$ is derivation complete at $c \in \mathsf{adom}(\mathcal D)$ if for every Σ -database $\mathcal D'$ that is satisfiable w.r.t. $\mathcal O$ and every $c' \in \mathsf{adom}(\mathcal D')$, there is a homomorphism h from $\mathcal U_{\mathcal D',\mathcal O}^{\downarrow c'}$ to $\mathcal U_{\mathcal D,\mathcal O}$ with h(c') = c. Informally, derivation completeness at c means that $\mathcal O$ derives at c anything that it could possibly derive at any constant in any database. For a set $\Gamma \subseteq \mathsf{adom}(\mathcal D)$, we say that $\mathcal D$ is derivation complete at Γ if it is derivation complete at every $c \in \Gamma$.

The databases \mathcal{D}_{ω} and $\mathcal{D}_{\mathsf{tree}}$ are both derivation complete. Moreover, in all three reductions the construction of the final databases \mathcal{D} makes sure that for every $c \in \mathsf{adom}(\mathcal{D}_0)$ there is a simulation S from $\mathcal{D}_{\mathsf{tree}}$ to \mathcal{D} such that $(\varepsilon, c) \in S$. This guarantees that \mathcal{D} is derivation complete at c.

To show all of this formally, let us define a weaker version of derivation completeness that only pertains to concept names, c.f. the construction of $\mathcal{D}_{\text{tree}}$. A concept name A is non-empty if there is a Σ -database \mathcal{D} and a $c \in \text{adom}(\mathcal{D})$ such that $\mathcal{D}, \mathcal{O} \models A(c)$. Now, a Σ -database \mathcal{D} is derivation complete for concept names at $c \in \text{adom}(\mathcal{D})$ if $\mathcal{D}, \mathcal{O} \models A(c)$ for every non-empty concept name A.

Lemma 9. \mathcal{D}_{ω} is derivation complete for concept names at ε .

Proof. Let A be a non-empty concept name. Then there is a Σ -database \mathcal{D} and a $c \in \mathsf{dom}(\mathcal{D})$ with $\mathcal{D}, \mathcal{O} \models A(c)$. Clearly, we have $(\mathcal{D}, c) \preceq (\mathcal{D}_{\omega}, \varepsilon)$. Hence, Lemma 5 implies $\mathcal{D}_{\omega}, \mathcal{O} \models A(\varepsilon)$, as required.

It follows that, by construction, $\mathcal{D}_{\text{tree}}$ is also derivation complete for concept names. By the following lemma, $\mathcal{D}_{\text{tree}}$ is derivation complete in general and what is more, simulation from $(\mathcal{D}_{\text{tree}}, \varepsilon)$ guarantees derivation completeness.

Lemma 10. Let \mathcal{D} be a Σ -database that is satisfiable w.r.t. \mathcal{O} and $c \in \mathsf{adom}(\mathcal{D})$ such that $(\mathcal{D}_{\mathsf{tree}}, \varepsilon) \preceq (\mathcal{D}, c)$. Then \mathcal{D} is derivation complete at c.

Proof. Let \mathcal{D} and c be as in the lemma. Further let \mathcal{D}' be any Σ -database and $c' \in \mathsf{adom}(\mathcal{D}')$. We have to show that there is a homomorphism h from $\mathcal{U}^{\downarrow c'}_{\mathcal{D}',\mathcal{O}}$ to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ with h(c') = c. From the fact that $\mathcal{D}_{\mathsf{tree}}$ is derivation complete for concept names at ε , it follows that $\mathcal{D}',\mathcal{O} \models A(c')$ implies $\mathcal{D}_{\mathsf{tree}},\mathcal{O} \models A(\varepsilon)$. From $(\mathcal{D}_{\mathsf{tree}},\varepsilon) \preceq (\mathcal{D},c)$ and Lemma 5, we obtain $\mathcal{D},\mathcal{O} \models A(c)$. Lemma 6 thus yields the desired homomorphism h.

E.2 Additional Preliminaries

A path π in a database \mathcal{D} is a sequence of database constants a_0, \ldots, a_k such that for $0 \leq i < k$ there is an edge $\{a_i, a_{i+1}\}$ in the Gaifman graph $G_{\mathcal{D}}$.

We say that a path is simple if no constant repeats, that it is chordless if there is no edge $\{a_i,a_j\}$ such that |i-j|>1 except for $\{i,j\}=\{1,k\}$, and that it is a cycle if $k\geq 1$ and $\{a_0,a_k\}$ is an edge or if k=1 and $\mathcal D$ contains two distinct facts that both mention a_0 and a_1 . We say that constants $c,c'\in \mathsf{adom}(\mathcal D)$ are connected in $\mathcal D$ if $\mathcal D$ contains a path that starts at c and ends at c'.

A path in a CQ q is a sequence y_0, \ldots, y_k of variables from q that forms a path in \mathcal{D}_q . The path is functional if for $0 \le i < k$ there is an atom $r(y_i, y_{i+1}) \in q$ such that func $(r) \in \mathcal{O}$.

E.3 Cyclic Queries - The Case of Non-Chordality

We finish the proof of Lemma 2 that we have started in the main body of the paper. First, however, we give an example of the construction of \mathcal{D}_0 in Figure 2. We use the query q and graph G shown in the top part of the figure and assume that the role names f_1 and f_2 are declared functional by \mathcal{O} .

Now back to the proof of Lemma 2. We first observe some basic properties of \mathcal{D}_0 .

Claim 1.

- 1. for every variable x in q, we have $Y_x \subseteq \{y_i, y_{i+1}\}$ for some $0 \le i \le k$.
- 2. $Y_{y_i} = \{y_i\} \text{ for } 0 \le i \le k$.

Proof. Both points follow from the chordlessness of the cycle y_0, \ldots, y_k .

For Point 1, let x be a variable in q and assume that $Y_x\supseteq\{y_j,y_i\}$ and $y_j\notin\{y_{i-1},y_{i+1}\}$ with $y_{-1}=y_k$. By definition of Y_x , there is then a functional path in q from x

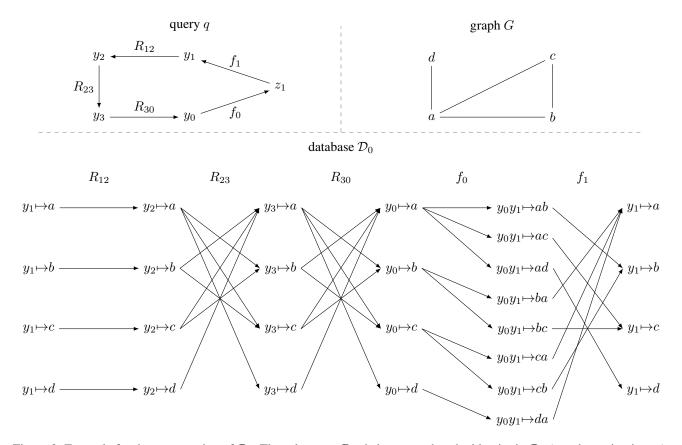


Figure 2: Example for the construction of \mathcal{D}_0 . The role name R_{01} is interpreted as the identity in \mathcal{D}_0 (not shown in picture).

to y_j and from x to y_i . Thus, by construction of q^+ the Gaifman graph of q^+ contains the edge $\{y_j,y_i\}$, in contradiction to y_0,\ldots,y_k being chordless.

For Point 2, first note that $y_i \in Y_{y_i}$ by definition of Y_{y_i} . By Point 1, this leaves as the only possible options $Y_{y_i} = \{y_i\}$, $Y_{y_i} = \{y_i, y_{i-1}\}$ (with $y_{-1} = y_k$), and $Y_{y_i} = \{y_i, y_{i+1}\}$. The latter two options, however, are not possible. We show this exemplarily for the last option. By construction of q^+ , the fact that $\{y_i, y_{i-1}\}$ is an edge in the Gaifman graph of q^+ implies that q contains an atom α that contains variables u, v such that y_{i-1} is reachable on a functional path from y_i Since $y_{i+1} \in Y_{y_i}$, there is also a functional path from y_i to y_{i+1} and thus we find a functional path from v to v to v to v to v to v the Gaifman graph of v contains the edge v construction of v the Gaifman graph of v contains the edge v construction to the chordfreeness of v contains the edge v contains the edge v contains the characteristic points.

Lemma 11. The database \mathcal{D}

- 1. can be computed in time O(||E||).
- 2. satisfies all functionality assertions in \mathcal{O} .
- 3. is derivation complete at $adom(\mathcal{D}_0)$.

Proof. Point 1 is clear from the construction.

For Point 2, take a role R such that $\operatorname{func}(R) \in \mathcal{O}$ and facts $R(c,d), R(c,d') \in \mathcal{D}$. We show that d=d'. If one of c,d or d' belongs to a copy of $\mathcal{D}_{\mathsf{tree}}$ then clearly d=d'. Thus, let us assume that $c,d,d' \in \mathsf{adom}(\mathcal{D}_0)$. Then the facts R(c,d) and

R(c,d') were added to \mathcal{D}_0 because of an atom R(x,y) in q and, due to the self-join freeness of q we have $c=\langle x,f_c\rangle$, $d=\langle y,f_d\rangle$, and $d'=\langle y,f_{d'}\rangle$ for functions $f_c,f_d,f_{d'}$. From func $(R)\in\mathcal{O}$ we obtain $Y_y\subseteq Y_x$. No matter whether fact R(c,d) was added in Step 1, 2, or 3 of the construction of \mathcal{D}_0 , there is a word w such that $f_c=f_x^w$ and $f_d=f_y^w$. It thus follows from $Y_y\subseteq Y_x$ that for all $y_i\in Y_y$, we have $f_d(y_i)=f_c(y_i)$. The same is true for the fact R(c,d'), and thus we obtain $f_d=f_{d'}$, which in turn implies d=d'.

By construction of \mathcal{D} , it is easy to see that $(\mathcal{D}_{\mathsf{tree}}, \varepsilon) \leq (\mathcal{D}, c)$ for all $c \in \mathsf{adom}(\mathcal{D}_0)$. By Lemma 10, it follows that \mathcal{D} is derivation complete at $\mathsf{adom}(\mathcal{D}_0)$.

Claim 2. Let h be a homomorphism from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$. Then

- 1. for every simple path u_0, \ldots, u_m in q, m > 0, $h(u_0) \in \mathsf{adom}(\mathcal{D}_0)$ and $h(u_m) \in \mathsf{adom}(\mathcal{D}_0)$ implies $h(u_i) \in \mathsf{adom}(\mathcal{D}_0)$ for $0 \le i \le m$;
- 2. for every cycle u_0, \ldots, u_m in q, we have $h(u_i) \in \operatorname{adom}(\mathcal{D}_0)$ for $0 \le i \le m$.

Proof. We only prove Point 1, the proof of Point 2 is similar. Assume towards a contradiction that $h(u_j) \notin \operatorname{adom}(\mathcal{D}_0)$ for some j with $0 \le j \le m$ and additionally let j be smallest with this property. Moreover, let $\ell > j$ be smallest such that $h(u_\ell) \in \operatorname{adom}(\mathcal{D}_0)$. Note that $0 < j < \ell < m$. By definition of a path, q must contain atoms

 $R_j(u_{j-1}, u_j), \dots, R_\ell(u_{\ell-1}, u_\ell)$. Since q is self-join free, the relation symbols R_j, \dots, R_ℓ are all distinct.

Next observe that by construction of \mathcal{D} and by definition of universal models, the interpretation \mathcal{I} obtained from $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ by removing all edges r(a,b) with $a,b\in \mathsf{adom}(\mathcal{D}_0)$ is a collection of trees without multi-edges. More precisely, these are the trees added in the second step in the construction of \mathcal{D} and the trees $\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}$ of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$. We view as the roots of these trees the elements of $\mathsf{adom}(\mathcal{D}_0)$.

Then h maps $R_j(u_{j-1},u_j),\ldots,R_\ell(u_{\ell-1},u_\ell)$ to some tree of this collection, and both u_{j-1} and u_ℓ are mapped to the root of the tree. This, however, clearly contradicts the fact that the relation symbols R_j,\ldots,R_ℓ are all distinct and the tree has no multi-edges. \square

The next lemma states soundness and completeness of the reduction.

Lemma 3.

TD1 If there is a minimal partial answer to Q on \mathcal{D} (with a single wildcard or with multiple wildcards), then there is a triangle in G.

TD2 If there is a triangle in G then there is a complete answer to Q on \mathcal{D} .

We now prove Claim 3, which finishes the lower bound proof. To show **TD2**, let a,b,c be a triangle in G. To show that there is a complete answer to Q on \mathcal{D} , it suffices to exhibit a homomorphism h from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ such that $h(x) \in \mathsf{adom}(\mathcal{D}_0)$ for every answer variable x.

Since Q is non-empty, there is a database \mathcal{D}' that is satisfiable w.r.t. \mathcal{O} and a homomorphism h' from q to $\mathcal{U}_{\mathcal{D}',\mathcal{O}}$ such that $h'(\bar{x})$ contains only elements of $\operatorname{adom}(\mathcal{D}')$. We use h' to guide the construction of h.

Let W_0 be the set of variables $x \in \text{var}(q)$ with $h'(x) \in \text{adom}(\mathcal{D}')$ and for each $c \in \text{adom}(\mathcal{D}')$, let W_c be the set of variables $x \in \text{var}(q)$ such that h'(x) is a trace that starts with c, that is, h'(x) is c or located in the tree $\mathcal{U}^{\downarrow c}_{\mathcal{D}',\mathcal{O}}$ in $\mathcal{U}_{\mathcal{D}',\mathcal{O}}$ that is rooted in c.

We first note that for all $c \in \operatorname{adom}(\mathcal{D}')$, the restriction q_c of q to the variables in W_c is a disjoint union of trees (without multi-edges and reflexive loops) in the sense that the Gaifman graph of q_c is a tree. Let p be a connected component of q_c . Clearly, p has no multi-edges and reflexive loops since the same is true for $\mathcal{U}^{\downarrow c}_{\mathcal{D}',\mathcal{O}}$. It remains to show that p is a tree. Assume to the contrary that p contains atoms $R_1(x_0,x_1),\ldots,R_m(x_{m-1},x_m)$ with $x_0=x_m$. Since $\mathcal{U}^{\downarrow c}_{\mathcal{D}',\mathcal{O}}$ is a tree, h' must map two distinct atoms from this list to the same fact in $\mathcal{U}^{\downarrow c}_{\mathcal{D}',\mathcal{O}}$. This contradicts the selfjoin freeness of q, which implies that R_1,\ldots,R_m are pairwise distinct.

We start the construction of h by setting $h(x) = \langle x, f_x^{ab^{k-1}c} \rangle$ for all $x \in W_0$. Let q_0 be the restriction of q to the variables in W_0 . It can be verified that h is indeed a homomorphism from q_0 to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$. In fact, binary atoms R(x,x') in q_0 are preserved: since $h'(x),h'(x') \in \mathsf{adom}(\mathcal{D}')$ and the construction of $\mathcal{U}_{\mathcal{D}',\mathcal{O}}$ from \mathcal{D}' does not add any binary facts R(a,a') with $a,a' \in \mathsf{adom}(\mathcal{D}')$, we

must have $R \in \Sigma$; we thus obtain $R(h(x), h(x')) \in \mathcal{D}_0$ from the construction of \mathcal{D}_0 and the fact that a,b,c is a triangle in G. Unary atoms A(x) with $A \in \Sigma$ are trivially preserved by construction of \mathcal{D} . And unary atoms A(x) with $A \notin \Sigma$ are preserved since \mathcal{D} is derivation complete at $\operatorname{adom}(\mathcal{D}_0)$. More precisely, if A(x) is an atom in q_0 , then non-emptiness of q implies that A is non-empty, and since derivation completeness (trivially) implies derivation completeness for concept names it follows that $\mathcal{O}, \mathcal{D} \models A(c)$ for all $c \in \operatorname{adom}(\mathcal{D}_0)$.

To extend h to the remaining variables in q, consider each set W_c separately (clearly, those sets are disjoint and contain all remaining variables). As established above, the restriction q_c of q to the variables in W_c is a disjoint union of trees. Let p be any such tree. Since q is connected, p contains a (unique) variable x with h'(x) = c. Since \mathcal{D} is derivation complete at c, there is a homomorphism h_c from $\mathcal{U}_{\mathcal{D}',\mathcal{O}}^{\downarrow c}$ to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ with $h_c(c) = h(x)$. Then set $h(y) = h_c(h'(y))$ for every $y \in \text{var}(p)$. This finishes the construction of h.

To show **TD1**, assume that there is a minimap partial answer to Q on \mathcal{D} . Then there is a homomorphism h from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$. Ideally, we would like to show that for $i \in \{0,1,k\}$, $h(y_i)$ is of the form $\langle y_i,f_i\rangle$ and $f_0(y_0),f_1(y_1),f_k(y_k)$ forms a triangle in G. Unfortunately, it may be the case that $h(y_i) \notin \mathsf{adom}(\mathcal{D}_0)$ and thus we need to argue in a slightly more complex way. We first show the following.

Claim 3. There is a variable u_i such that $h(u_i) \in \text{adom}(\mathcal{D}_0)$ and $y_i \in Y_{u_i}$, for $0 \le i \le k$.

Proof. Let $0 \le i \le k$. Recall that y_0, \dots, y_k is a cycle in q^+ and let $y_{k+1} = y_0$. By construction of q^+ , this means that for $0 \le j \le k$, q contains an atom α_j that contains (possibly identical) variables u_j, v_j such that q contains a functional path π_j from u_j to y_j and a functional path π'_j from v_j to y_{j+1} . The concatenation $\pi_0^-\pi_0\cdots\pi_k^-\pi_k$ is a cycle $C=z_0,\dots,z_\ell$ in q where - denotes path reversal. It may be that C is not a simple cycle, but we can use it to find a simple cycle in q that contains a variable u_i with $y_i \in Y_{u_i}$.

Let u_i be the last variable on the longest common prefix of the paths ${\pi'}_{i-1}^-\pi_{i-1}$ and $\pi_i^-\pi_i'$. This u_i must be in the $({\pi'}_{i-1})^-$ prefix of the first path and in the π_i^- prefix of the second path because all variables z in the π_{i-1} suffix of the first path satisfy $y_{i-1} \in Y_z$ and all variables z in the π_i' suffix of the second path satisfy $y_{i+1} \in Y_z$, c.f. Claim 1. This clearly implies that $y_i \in Y_{u_i}$. We can now obtain a cycle C' by starting with C and

- 1. cutting out the prefixes of ${\pi'}_{i-1}^-$ and π_i^- up to (excluding) u_i and
- 2. removing subpaths that end in the same variable (and do not contain u_i) from the remaining cycle to make it simple.

By Point 2 of Claim 2,
$$h(u_i) \in \mathsf{adom}(\mathcal{D}_0)$$
.

¹One possible and particularly simple case is that α_j takes the form $R(y_i, y_{j+1})$, $\pi_i = y_i$, and $\pi'_i = y_{j+1}$.

Claim 4. Let y, y' be variables in q such that $h(y), h(y') \in \mathsf{adom}(\mathcal{D}_0)$. Then, $h(y) = \langle y, f_y \rangle$ and $h(y) = \langle y', f_{y'} \rangle$ for some functions f_y and $f_{y'}$. Moreover, if $y_i \in Y_y \cap Y_{y'}$ then $f_y(y_i) = f_{y'}(y_i)$.

Proof. Since q is connected, there is a simple path z_0,\dots,z_ℓ in q from $z_0=y$ to $z_\ell=y'$. Since $h(y),h(y')\in \operatorname{adom}(\mathcal{D}_0)$ and by Point 1 of Claim 2, all variables on this path are mapped to $\operatorname{adom}(\mathcal{D}_0)$. Thus, there are atoms $R(y,z_1)$ and $S(z_{\ell-1},y')$ such that $R(h(y),h(z_1))\in\mathcal{D}_0$ and $S(h(z_{\ell-1}),h(y'))\in\mathcal{D}_0$. It now follows from the construction of \mathcal{D}_0 and the self-join freeness of q that $h(y)=\langle y,f_y\rangle$ and $h(y)=\langle y',f_{y'}\rangle$.

For the second part of the claim observe that if $y_i \in Y_y \cap Y_{y'}$ then there are a simple functional path from y to y_i and a simple functional path from y' to y_i . Hence, there is a simple path from y to y' such that for every variable u on this path we have $y_i \in Y_u$. By Claim 10, this implies $f_y(y_i) = f_{y'}(y_i)$.

Let $f = \bigcup_{0 \le i \le k} h(u_i)$. By Claim 4, f is a function. We argue that $a = f(y_0)$, $b = f(y_1)$, and $c = f(y_k)$ form a triangle in G.

First for the edge $\{a,b\}$. We start with observing that qcontains a path π from u_0 to u_1 such that $Y_z \cap \{y_0, y_1\} \neq \emptyset$ for all variables z on π . We can find π by concatenating the functional path from u_0 to y_0 with a path from y_0 to y_1 that contains only variables t with $Y_z \cap \{y_0, y_1\} \neq \emptyset$ (which exists since $\{y_0, y_1\}$ is an edge in G_{q^+} and then again with the reverse of the functional path from u_1 to y_1 . We can convert π into a simple path by cutting out subpaths. Since $h(u_0), h(u_1) \in \mathsf{adom}(\mathcal{D}_0)$ and by Point 1 of Claim 2, all variables on this path are mapped to $adom(\mathcal{D}_0)$. It now follows from $y_0 \in Y_{u_0}$, $y_1 \in Y_{u_1}$, and Claim 1 that there are two consecutive variables z_0, z_1 on π such that $y_0 \in Y_{z_0}$ and $y_1 \in Y_{z_1}$. In summary, q contains an atom $R(z_0, z_1)$ with $y_0 \in Y_{z_0}, y_1 \in Y_{z_1}, \text{ and } h(z_0), h(z_1) \in \mathsf{adom}(\mathcal{D}_0).$ Then $R(h(z_0), h(z_1)) \in \mathcal{D}_0$ and Claim 4 yields $h(z_0) = \langle z_0, f_{z_0} \rangle$ with $f_{z_0}(y_0) = f(y_0)$ and $h(z_1) = \langle z_1, f_{z_1} \rangle$ with $f_{z_1}(y_1) = \langle z_1, f_{z_2} \rangle$ $f(y_1)$. Since q is self-join free, the fact $R(h(z_0), h(z_1))$ must have been added to \mathcal{D}_0 due to the atom $R(z_0, z_1) \in q$. Since $Y_{z_1}\cup Y_{z_2}=\{y_0,y_1\}$, the fact must have been added in Case 1. Thus $\{a,b\}\in E$, as required.

The case of the edge $\{b,c\}$ is analogous, with u_0,y_0 replaced by u_{k-1},y_{k-1} and u_1,y_1 by u_k,y_k . Moreover, it uses Case 2 of the construction of \mathcal{D}_0 in place of Case 1.

The case of the edge $\{c,b\}$ is also analogous, with u_0,y_0 replaced by u_k,y_k and u_1,y_1 by u_0,y_0 . The fact $R(h(z_0),h(z_1))$ was then added by Case 1 and Case 2 simultaneously (they simply produce the same fact), which yields $\{c,b\} \in E$.

E.4 Cyclic Queries - The Case of Non-Conformity

We now consider the second case in the proof of Point 1 of Theorem 5. The basic idea of our reduction goes back to (Brault-Baron 2013), but the details differ. Recall that a $CQ \ q$ is conformal if for every clique C in the Gaifman graph of q, there is an atom in q that contains all variables in C.

Lemma 12. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELI}, CQ)$ be non-empty such that q is self-join free and connected and the hypergraph of q^+ is non-conformal. Then enumerating complete answers to Q is not in $DelayC_{lin}$ unless the hyperclique conjecture fails. The same is true for least partial answers, both with a single wildcard and with multi-wildcards.

Let $Q(\bar{x})=(\mathcal{O},\Sigma,q)\in(\mathcal{ELI},\mathrm{CQ})$ be as in Lemma 12, and let $Y=\{y_0,\ldots,y_k\}, k\geq 2$, be a clique in the Gaifman graph of q^+ . Further assume that every proper subset of Y is contained in an edge in q^+ . MP: what if k=2?

Let G=(V,E) be a k-regular hypergraph, that is, $V=\{v_1,\ldots,v_n\}$ is a set of vertices and E is a set of subsets of V of size k called the hyperedges. Our aim is to construct a database \mathcal{D} , in time n^k , such that G contains a hyperclique of size k+1 if and only if $Q(\mathcal{D})\neq\emptyset$. Clearly, a DelayC_{lin} enumeration algorithm for Q lets us decide the latter in linear time and thus we have found an algorithm for the (k+1,k)-hyperclique problem that runs in time $O(n^k)$, refuting the hyperclique conjecture.

The reduction shares a lot of ideas, intuition, and notation with the previous one. In particular, for every variable x in q the set Y_x is defined as in the previous reduction and we again define the desired database \mathcal{D} by starting with a 'core part' \mathcal{D}_0 that encodes the hyperclique problem and then adding trees in a second step.

We start with the construction of \mathcal{D}_0 , where we use the same functions f_x^w as in the previous reduction. For every atom r(x, y) in q with $r \in \Sigma$, add the following facts to \mathcal{D}_0 :

• $r(\langle x, f_x^w \rangle, \langle y, f_y^w \rangle)$ for every hyperedge $e = \{a_1, \dots, a_k\} \in E$ and every serialization $w = a_1 \cdots a_k$ of e.

The construction of \mathcal{D} from \mathcal{D}_0 is exactly as in the previous reduction, based on the database $\mathcal{D}_{\text{tree}}$ and its fragments \mathcal{D}_R . More precisely, we extend the database \mathcal{D}_0 to the desired database \mathcal{D} as follows: for every $c \in \text{adom}(\mathcal{D}_0)$ and every role $R \in \{r, r^-\}$ with $r \in \Sigma$ such that there is no fact $R(c, c') \in \mathcal{D}_0$, add a disjoint copy of \mathcal{D}_R , glueing the copy of ε to c.

Finally, for every unary relation A symbol in Σ and every constant c in the so-far constructed database we add fact A(c).

Lemma 13. The database \mathcal{D}

- 1. can be computed in time $O(n^k)$,
- 2. satisfies all functional assertions in \mathcal{O} ,
- *3.* is derivation complete at $adom(\mathcal{D}_0)$.

Point 1 is clear from the construction and Points 2 and 3 can be proved in the same way as the corresponding points in Claim 16.

We observe the following counterpart of Claim 2. The proof is omitted.

Claim 5. Let h be a homomorphism from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$. Then

- 1. for every simple path u_0, \ldots, u_m in q, m > 0, $h(u_0) \in \mathsf{adom}(\mathcal{D}_0)$ and $h(u_m) \in \mathsf{adom}(\mathcal{D}_0)$ implies $h(u_i) \in \mathsf{adom}(\mathcal{D}_0)$ for $0 \le i \le m$;
- 2. for every cycle u_0, \ldots, u_m in q, we have $h(u_i) \in \mathsf{adom}(\mathcal{D}_0)$ for $0 \le i \le m$.

To finish the proof of Lemma 12, it remains to show the following.

Lemma 14.

HD1 if there is a minimal partial answer to Q on \mathcal{D} then there is a (k+1)-hyperclique in the Gaifman graph of G, **HD2** if there is a (k+1)-hyperclique in the Gaifman graph of G then there is a complete answer to Q on \mathcal{D} .

To show **HD2**, let v_0, \ldots, v_k be a hyperclique in G. To show that there is a complete answer to Q on \mathcal{D} , it suffices to exhibit a homomorphism h from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ such that $h(x) \in$ $adom(\mathcal{D}_0)$ for every answer variable x. The construction of h is very similar to the proof of the **TD2** part of Claim 3 and we provide only an outline.

Since Q is non-empty, there is a database \mathcal{D}' that is satisfiable w.r.t. \mathcal{O} and a homomorphism h' from q to $\mathcal{U}_{\mathcal{D}'}$ \mathcal{O} such that $h'(\bar{x})$ contains only elements of $\mathsf{adom}(\mathcal{D}')$. Let W_0 be the set of variables $x \in \text{var}(q)$ with $h'(x) \in \text{adom}(\mathcal{D}')$.

We may construct the desired homomorphism h by setting $h(x) = \langle x, f_x^{v_0 \cdots v_k} \rangle$ for all $x \in W_0$, and then extending to the remaining variables in q (which are part of tree-shaped subqueries of q) exactly as in the proof of the **TD2** part of

To prove **HD1**, we start with observing an analogue of Claim 3.

Claim 6. There is a variable u_i such that $h(u_i) \in$ $adom(\mathcal{D}_0)$ and $y_i \in Y_{u_i}$, for $0 \le i \le k$.

Proof. We prove the claim for the case $y_i = y_1$, the other cases are symmetric. By Point 2 of Claim 2, it suffices to show that q contains a simple cycle and a variable u_i on the cycle with $y_i \in Y_{u_i}$.

Consider the variables y_1, y_2, y_3 . We first show that qcontains atoms $R_1(v_1^2, v_1^3)$, $R_2(v_2^1, v_2^3)$, and $R_3(v_3^1, v_3^2)$ such that for every variable v_i^j :

- 1. $y_j \in Y_{v_i^j}$ via a functional path π_i^j in q from v_i^j to y_j ,
- 2. $y_i \notin Y_{v_i^j}$ (and thus $y_i \notin Y_x$ for any variable x on path π_i^j),
- 3. the concatenated path $(\pi_i^{j_1})^- \pi_i^{j_2}$ is simple where $\{i, j_1, j_2\} = \{1, 2, 3\}$ and \cdot denotes path reversal.

In fact, by choice of Y, q^+ contains atoms $\alpha_1, \alpha_2, \alpha_3$ such that for $i, j \in \{1, 2, 3\}$ α_i contains y_i iff $i \neq j$. This implies in particular that $y_i \notin Y_z$ for any variable z in α_i . By construction of q^+ , we find in q (unary or binary atoms) β_1,β_2,β_3 such that β_i contains only variables from α_i and every variable in α_i is reachable in a functional path in qfrom a variable in β_i . Thus $y_i \notin Y_z$ for any variable z in β_i . Moreover, for $\{i, j_1, j_2\} = \{1, 2, 3\}$ the atom β_i contains (possibly identical) variables w_i, w'_i such that q contains a functional path π from w_i to y_{j_1} and π' from w_i' to y_{j_2} . If these two paths share no variables, then we can choose $R_i(v_i^{j_1}, v_i^{j_2})$ to be β_i , $\pi_i^{j_1}$ to be π , and $\pi_i^{j_2}$ to be π' . Otherwise, we choose $v_i^{j_1}$ to be the last variable shared by the two paths, $R_i(v_i^{j_1}, v_i^{j_2})$ to be some atom that connects $v_i^{j_1}$ with

its successor node on π' , $\pi_i^{j_1}$ to be the suffix of π starting at $v_i^{j_1}$, and $\pi_i^{j_2}$ to be the suffix of π' starting at $v_i^{j_2}$.

Let z_1 be the first variable shared by paths π_3^1 and π_2^1 , z_2 the first variable on π_1^2 and π_3^2 , and z_3 the first variable on π_2^3 and π_1^3 . This is illustrated in Figure 3. Now consider the following cycle C:

- path π^1_3 from v^1_3 to z_1 , reverse of path π^1_2 from z_1 to v^1_2 , path π^2_3 from v^3_2 to z_3 , reverse of path π^3_1 from z_3 to v^3_1 , path π^1_1 from v^1_1 to z_2 , reverse of path π^2_3 from z_2 to v^2_3 .

If C is simple, then we can choose $u_i = v_3^1$ and are done. Otherwise, we can extract from C a simple cycle that still contains $u_i = v_3^1$, as follows.

Let v be the first variable on C (regarding the order in which C is presented above) that occurs more than once on C, let its first appearance be on (possibly reversed) π_i^j and the last appearance on (possibly reversed) π_{ℓ}^{m} . Note that

- (a) $\ell \neq j$ because v being on π_i^j implies that $y_j \in Y_v$ while v being on π_{ℓ}^m implies $y_{\ell} \notin Y_v$ by Point 2 above;
- (b) $\ell \neq i$ due to Point 3 above.

Points (a) and (b) imply that the first appearance of v on C is before z_3 . It also implies that if the first appearance is before z_3 , then any other appearance (including the last) is after z_3 . We can thus remove the subcycle between the first and last occurrence without removing $u_i = v_3^1$.

The resulting cycle is simple. To see this, assume to the contrary that there is a variable w with repeated appearance. Then $w \neq v$ and the first appearance of w cannot be before the (only) appearance of v by choice of v. It can also not be after the appearance of v since then all appearances of w would be after z_3 in the original cycle C and Points (a) and (b) above imply that this is impossible.

Now we can prove **HD1**. Let h be a homomorphism from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$. By Claim 6 for $0 \leq i \leq k$ there is a variable u_i in q such that $y_i \in Y_{u_i}$ and $h(u_i) \in \mathsf{adom}(\mathcal{D}_0)$. The following claim is proved in exactly the same way as Claim 4, details are omitted. The proof is identical and omitted.

Claim 7. Let y, y' be variables in q such that $h(y), h(y') \in$ $adom(\mathcal{D}_0)$. Then, $h(y) = \langle y, f_y \rangle$ and $h(y) = \langle y', f_{y'} \rangle$ for some functions f_y and $f_{y'}$. Moreover, if $y_i \in Y_y \cap Y_{y'}$ then $f_y(y_i) = f_{y'}(y_i).$

Let $f = \bigcup_{0 \le i \le k} f_{u_i}$. It follows from Claim 7 that f is a function from \overline{Y} to V. For brevity, let $v_i = f(y_i)$ for $0 \le 1$ $i \leq k$. To end the proof it suffices to show that for every set $I \subseteq \{0, \dots, k\}$ of size k, the set $V_I = \{v_i \in V \mid i \in I\}$ is a hyperedge in G, i.e. $V_I \in E$.

Since $Y_I = \{y_i \in Y \mid i \in I\} \subseteq Y$, there is an atom $R_I^+(\bar{z})$ in q^+ that contains all variables in Y_I . We show that q contains an atom R(z,z') such that $Y_I \subseteq Y_z \cup Y_{z'}$ and $h(z), h(z') \in \mathsf{adom}(\mathcal{D}_0)$. Before we prove the existence of this atom, we show how this implies that $V_I \in E$. It follows from $h(z), h(z') \in \mathsf{adom}(\mathcal{D}_0)$ that $R(h(z), h(z')) \in \mathcal{D}_0$. By Claim 7 and the definition of \mathcal{D}_0 , there is thus a word $w = b_1 \cdots b_k$ such that $h(z) = \langle z, f_z^w \rangle, h(z') = \langle z', f_{z'}^w \rangle,$

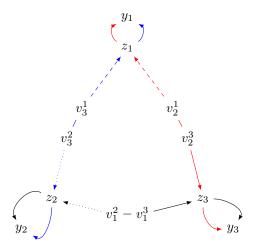


Figure 3: Illustration of the proof of existence of a simple cycle containing u_i . The directed edges symbolise functional paths between variables. Paths composed of the same colour sections (red, blue, black) or of the same structure sections (solid, dashed, dotted) are simple.

and w is a permutation of vertices in an edge in G. The definition of f_z^w and $f_{z'}^w$ together with the fact that $Y_I \subseteq Y_z \cup Y_{z'}$ yields $v_i = f(y_i) = b_i$ for all $i \in I$. Thus, $V_I = \{v_i\}_{i \in I} = \{b_1, \ldots, b_k\}$. This proves that V_I is an edge in G, as desired.

Now we prove the existence of the atom R(z,z'). By the construction of q^+ , q contains a (unary or binary) atom α and two (not necessarily distinct) variables w,w' in α such that for every $y_i \in Y_I$, there is a functional path in q from w to y_i . Since q is connected, every variable occurs in a binary atom, and thus we may actually assume that α is of the form $R'(z_0,z'_0)$. Then $Y_I \subseteq Y_{z_0} \cup Y_{z'_0}$, that is, for every variable $y_i \in Y_I$ there is a simple functional path π_i in q from z_0 to y_i or there is a simple functional path π'_i in q from z'_0 to y_i .

We next show that there is a variable v such that $h(v) \in \mathsf{adom}(\mathcal{D}_0)$ and q contains a path from z_0 to v such that for all variables x on the path, $Y_{z_0} \subseteq Y_x$. Likewise, there is a variable v' such that $h(v') \in \mathsf{adom}(\mathcal{D}_0)$ and q contains a path from z'_0 to v' such that for all variables x on the path, $Y_{z'_0} \subseteq Y_x$. We only do the case of v explicitly, the case of v' is symmetric.

If $Y_{z_0} = \{y_i\}$ is a singleton, we choose $v = u_i$ and are done. Note that there is a functional path π from z_0 to y_i and a functional path π' from u_i to y_i , and the required path from z_0 to v is $\pi\pi'$.

Otherwise, let $Y_{z_0} = \{y_{i_1}, \dots, y_{i_m}\}, j \geq 2$. We prove the following by induction on i.

Claim 8. For $2 \le i \le m$, there is variable w_i in q such that $\{y_{i_1}, \ldots, y_{i_i}\} \subseteq Y_{w_i}$, $h(w_i) \in \mathsf{adom}(\mathcal{D}_0)$, and there is a simple functional path π_i from z_0 to w_i .

We may then choose $v = w_m$.

Proof. The induction start is i=2. Since $y_{i_1},y_{i_2}\in Y_{z_0}$, we find a simple functional path τ_1 from z_0 to y_{i_1} and a simple functional path τ_2 from z_0 to y_{i_2} . Clearly, all variables x on τ_1 satisfy $y_{i_1}\in Y_x$ and all variables x on τ_2

satisfy $y_{i_2} \in Y_x$. We choose w_2 to be the last variable on the longest common prefix of τ_1 and τ_2 (note that these paths end in y_{i_1} and y_{i_2} , which are distinct), and π_2 to be this prefix. Clearly, $\{y_{i_1}, y_{i_2}\} \subseteq Y_{w_2}$ and thus it remains to show that $h(w_2) \in \operatorname{adom}(\mathcal{D}_0)$.

Let ω_1, ω_2 be the parts of τ_1 and τ_2 that remain after the common prefix π_2 . Further recall that there is a simple functional path ρ_1 from u_{i_1} to y_{i_1} and a simple functional path ρ_2 from u_{i_2} to y_{i_2} . Now consider the path $\mu = \rho_1^- \omega_1^- \omega_2 \rho_2$ from u_{i_1} to u_{i_2} . We can make the $\rho_1^- \omega_1^-$ prefix simple by removing subpaths and we can make the $\tau_2 \omega_2$ suffix simple in the same way.

First assume that the resulting path μ' , which must still contain w_2 , is simple. Since $h(u_{i_1}), h(u_{i_2}) \in \mathsf{adom}(\mathcal{D}_0)$, Point 1 of Claim 5 yields $w_2 \in \mathsf{adom}(\mathcal{D}_0)$ and we are done.

Now assume that μ' is not simple. Then some variable $x \neq w_2$ must occur on the path $\omega_1 \rho_1$ and on the path $\omega_2 \rho_2$. This gives rise to a cycle C in q, starting from w_2 on $\omega_1 \rho_1$ to x and back on $\rho_2 \omega_2$ to w_2 . Note that this must indeed be a cycle since it contains w_2 as well as the successors of w_2 on the paths τ_1 and τ_2 , and these three variables must be pairwise distinct by choice of w_2 and since τ_1, τ_2 are simple paths. Point 2 of Claim 5 yields $w_2 \in \text{adom}(\mathcal{D}_0)$ and we are done.

The induction step, where i>2, is essentially identical to the induction start except that we now start from the simple functional path $\tau_1=\pi_{i-1}$ from z_0 to w_{i-1} in place of the path τ_1 from z_0 to y_{i_1} . All the remaining arguments are identical.

Hence, q contains a simple path π from z_0 to v such that $h(v) \in \mathsf{adom}(\mathcal{D}_0)$ and all variables x on the path satisfy $Y_{z_0} \subseteq Y_x$, and a simple path π' from z_0' to v' such that $h(v') \in \mathsf{adom}(\mathcal{D}_0)$ and all variables x on the path satisfy

²If the alledged cycle had only length two, then we would need more than one atom between the two involved variables, c.f. the definition of cycles.

 $Y_{z_0'}\subseteq Y_x$. First assume that π and π' share a variable and let z be the first variable on π^- shared with π' . Consider the path τ obtained by traveling on π^- from v to z and then on π from z to v'. Clearly, τ is simple and thus Point 1 of Claim 5 together with the fact that $h(v), h(v') \in \operatorname{adom}(\mathcal{D}_0)$ implies that $h(z) \in \operatorname{adom}(\mathcal{D}_0)$. Moreover we have $Y_I \subseteq h(z)$. Let z' be the successor of z on τ . Then q must contain an atom R(z,z') which is the atom that we have been looking for. Now assume that π and π' are disjoint. Then $\pi^-\pi'$ is a simple path from v to v' that contains both z_0 and z_0' . Point 1 of Claim 5 yields $R(h(z_0),h(z_0')) \in \mathcal{D}_0$ and $R(z_0,z_0')$ is the promised atom.

E.5 Acyclic Queries that are not Free-Connex Acyclic

We now prove Point 2 of Theorem 5.

Lemma 15. Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELIF}, CQ)$ be a self-join free, non-empty, connected OMQ such that q^+ is acyclic but not free-connex acyclic. Then enumerating the complete answers to Q is not in DelayC_{lin} unless one can multiply Boolean matrices M_1, M_2 in time $O(|M_1| + |M_2| + |M_1M_2|)$. The same is true for least partial answers, both with a single wildcard and with multi-wildcards.

Let $Q(\bar{x}) = (\mathcal{O}, \Sigma, q)$ be as in Lemma 15. It is well known that if a CQ is acyclic but not free-connex acyclic, then it contains a bad path, that is, a path y_0, \ldots, y_k that is simple, chordless, and not a cycle, and such that y_0, y_k are answer variables while y_1, \ldots, y_{k-1} are quantified variables. See e.g. (Berkholz, Gerhardt, and Schweikardt 2020) for a proof. Thus q^+ contains a bad path y_0, \ldots, y_k . Note that y_0 and y_k are answer variables in q^+ , but not necessarily in q. By construction of q^+ , however, there is an answer variable in q that has a functional path to y_0 and a different one that has a functional path to y_k . We can assume w.l.o.g. that these answer variables in q are the first two answer variables x_1 and x_2 in the tuple \bar{x} .

Let M_1, M_2 be $n \times n$ -matrices. Our aim is to construct a database \mathcal{D} , in time $O(|M_1|+|M_2|)$, such that a DelayC_{lin} algorithm for enumerating the (complete or partial) answers to Q on \mathcal{D} allows us to compute M_1M_2 in time $O(|M_1|+|M_2|+|M_1M_2|)$. The reduction shares ideas, intuition, and notation with the previous two reductions. In particular, for every variable x in q the set Y_x is defined as in the previous reduction and we again define the desired database \mathcal{D} by starting with a 'core part' \mathcal{D}_0 that encodes matrix multiplication and then add trees in a second step. Before we give details, we observe the following properties of the sets Y_x .

Claim 9.

- 1. For every variable x in q, $Y_x \subseteq \{y_i, y_{i+1}\}$ for some i with $0 \le i < k$.
- 2. For every atom R(x,y) in q, $Y_x \cup Y_y \subseteq \{y_i,y_{i+1}\}$ for some i with $0 \le i < k$.
- 3. $Y_{y_i} = \{y_i\} \text{ for } 0 \le i \le k$.
- 4. For every answer variable x' in $q, Y_{x'} \in \{\emptyset, \{y_0\}, \{y_k\}\}.$

Proof. All points follow from the fact that the bad path y_0, \ldots, y_k is chordless and not a cycle.

For Point 1, let x be a variable in q and assume that $Y_x \supseteq \{y_j, y_i\}$ and $y_j \notin \{y_{i-1}, y_{i+1}\}$ with $y_{-1} = y_k$. By definition of Y_x , there is then a functional path in q from x to y_j and from x to y_i . Thus, by construction of q^+ the Gaifman graph of q^+ contains the edge $\{y_j, y_i\}$, in contradiction to y_0, \ldots, y_k being chordless.

The proof of Point 2 is similar.

For Point 3, first note that $y_i \in Y_{y_i}$ by definition of Y_{y_i} . By Point 1, this leaves as the only possible options $Y_{y_i} = \{y_i\}$, $Y_{y_i} = \{y_i, y_{i-1}\}$ (with $y_{-1} = y_k$), and $Y_{y_i} = \{y_i, y_{i+1}\}$. The latter two options, however, are not possible. We show this exemplarily for the last option. By construction of q^+ , the fact that $\{y_i, y_{i-1}\}$ is an edge in the Gaifman graph of q^+ implies that q contains an atom α that contains variables u, v such that y_{i-1} is reachable on a functional path from y_i . Since $y_{i+1} \in Y_{y_i}$, there is also a functional path from y_i to y_{i+1} and thus we find a functional path from v to v construction of v the Gaifman graph of v contains the edge v construction of v to the Caifman graph of v contains the edge v construction to the chordfreeness of v contains the edge v contains the edge v contains the contains the edge v contain

For Point 4 observe that the variables y_1,\ldots,y_{k-1} are quantified. Thus, by construction of q^+ (which is $q^+(\bar{x}^+)$), q contains no functional path from an answer variable in q to any of these variables. Consequently, if x' is an answer variable in q, then $Y_{x'}\subseteq\{y_0,y_k\}$. But by construction of $q^+,Y_{x'}=\{y_0,y_k\}$ would imply that the Gaifman graph of q^+ contains the edge $\{y_0,y_k\}$, in contradiction to y_1,\ldots,y_k not being a cycle in q^+ . Thus $Y_{x'}\subseteq\{y_0,y_k\}$ and we are

To give an intuition of the reduction, consider the CQ $q_0(y_0,y_2)=\exists y_1\,R_1(y_0,y_1)\wedge R_2(y_1,y_2)$, which represents a 'paradigmatic' bad path, and assume that the ontology is empty. To compute the product M_1M_2 , we can use the database

$$\mathcal{D} = \{R_1(\langle y_0, a \rangle, \langle y_1, b \rangle) \mid (a, b) \in M_1\} \cup \{R_2(\langle y_1, b \rangle, \langle y_2, c \rangle) \mid (b, c) \in M_2\}.$$

Clearly, the answers to q_0 on \mathcal{D} , projected to the second component, are exactly the 1-entries in M_1M_2 . The above database \mathcal{D} may be viewed as the direct product of q_0 with the database $\{R_i(a,b) \mid (a,b) \in M_i, i \in \{1,2\}\}$.

Also in general (and as in the previous two reductions), the constructed database \mathcal{D} may be viewed as a variation of the direct product. A homomorphism h from q to \mathcal{D} that maps every $\hat{y_i} \in Y$ to a constant of the form $\langle y_i, f_{y_i} \rangle$ gives rise to a pair $(a, b) \in M_i$ as follows. By Point 3 of Claim 9, the domain of the function f_{y_i} is $\{y_i\}$, and by construction of \mathcal{D} $f_{y_i}(y_i)$ will be an element of [n]. The homomorphism h thus identifies a sequence of elements $a_0, \ldots, a_k \in [n]$, with $a_i = f_{y_i}(y_i)$. The construction of \mathcal{D} ensures that then $(a_0, a_1) \in M_1, a_1 = \cdots = a_{k-1}, \text{ and } (a_{k-1}, a_k) \in M_2,$ and thus $(a_0, a_k) \in M_1 M_2$. As in the previous two reductions, however, we cannot ensure that h maps every y_i to a constant of the form $\langle y_i, f_{y_i} \rangle$. This leads to similar technical complications as in the previous reductions. Conversely, all pairs $(a,c) \in M_1$ and $(c,b) \in M_2$ give rise to a homomorphism from q to \mathcal{D} .

We now construct the database \mathcal{D} .

Step 1: Encoding the product of M_1 and M_2 . We use the same functions f_x^w also employed in the two previous reductions. The database \mathcal{D}_0 contains the following facts, for every atom r(x,y) in q with $r \in \Sigma$:

MR1 if $y_0 \in Y_x \cup Y_y$, the fact $r(\langle x, f_x^{ab^k} \rangle, \langle y, f_y^{ab^k} \rangle)$ for all $(a, b) \in M_1$,

MR2 if $y_k \in Y_x \cup Y_y$, the fact $r(\langle x, f_x^{b^k c} \rangle, \langle y, f_y^{b^k c} \rangle)$ for all $(b, c) \in M_2$,

MR3 otherwise, the fact $r(\langle x, f_x^{b^{k+1}} \rangle, \langle y, f_y^{b^{k+1}} \rangle)$ for all b such that $(a,b) \in M_1$ or $(b,c) \in M_2$.

It also contains A(c) for every concept name $A \in \Sigma$ and every constant $c \in \mathsf{adom}(\mathcal{D}_0)$.

Step 2: Adding trees. This step is the same as in the previous two reductions. We extend the database \mathcal{D}_0 to the desired database \mathcal{D} as follows: for every $c \in \mathsf{adom}(\mathcal{D}_0)$ and every role $R \in \{r, r^-\}$ with $r \in \Sigma$ such that there is no fact $R(c, c') \in \mathcal{D}_0$, add a disjoint copy of \mathcal{D}_R , glueing the copy of ε to c.

The following is immediate from the construction of \mathcal{D}_0 , by a straightforward analysis of the Cases 1-3 in the construction of \mathcal{D}_0 that may introduce an atom R(a,b).

Claim 10. Let $R(a,b) \in \mathcal{D}_0$ with $a = \langle x, f_x \rangle$ and $b = \langle y, f_y \rangle$. Then $f_x(y_i) = f_y(y_i)$ for all $y_i \in Y_x \cap Y_y$.

We next summarize some important properties of the database \mathcal{D} .

Lemma 16. The database \mathcal{D}

- 1. can be computed in time $O(|M_1| + |M_2|)$.
- 2. satisfies the functional assertions in O.
- 3. is derivation complete at $adom(\mathcal{D}_0)$.

Point 1 should be clear by construction of \mathcal{D} and Points 2 and 3 can be proved in the same way as the corresponding points in Claim 16.

The following lemma makes precise the connection between answers to Q and the matrix product M_1M_2 . Recall that x_1 and x_2 are the first two answer variables of q. For any complete or partial answer \bar{a} , we use a_1 and a_2 to denote the first two constants in \bar{a} .

Lemma 17.

MM1 For every minimal partial answer \bar{a} to Q on \mathcal{D} , if $a_1 = \langle x_1, f_{x_1} \rangle \in \mathsf{adom}(\mathcal{D}_0)$ and $a_2 = \langle x_2, f_{x_2} \rangle \in \mathsf{adom}(\mathcal{D}_0)$ then $(f_{x_1}(y_0), f_{x_2}(y_k)) \in M_1M_2$.

MM2 For all $(a,b) \in M_1M_2$, there is a complete answer \bar{a} to Q on \mathcal{D} such that $a_1 = \langle x_1, f_{x_1} \rangle$, $a_2 = \langle y_x, f_{x_2} \rangle$ with $f_{x_1}(y_0) = a$ and $f_{x_2}(y_k) = b$.

MM3 There are no more than $O(|M_1| + |M_2| + |M_1M_2|)$ minimal partial answers to Q on \mathcal{D} .

This holds both for minimal partial answers with a single wildcard and with multi-wildcards.

Before we prove the lemma, we let us explain how it is used to prove Lemma 15.

Assume that the complete or minimal partial answers to Q (with a single or multiple wildcards) answers can be enumerated in DelayC_{lin}. We can then multiply matrices M_1 and

 M_2 in time $O(|M_1|+|M_2|+|M_1M_2|)$ as follows. We build the database $\mathcal D$ as described above. Then, we enumerate the answers to Q on $\mathcal D$. For every answer $\bar a$ with $a_1=\langle x_1,f_{x_1}\rangle$ and $a_2=\langle x_2,f_{x_2}\rangle$ we extract the pair $(f_{x_1}(y_0),f_{x_2}(y_k))\in [n]\times [n]$. By **MM1** and **MM2**, the result of this extraction is exactly the set of pairs in M_1M_2 . Since $\mathcal D$ can be constructed in time $O(|M_1|+|M_2|)$ and by **MM3** there are no more than $O(|M_1|+|M_2|+|M_1M_2|)$ answers, this algorithm produces M_1M_2 in time $O(|M_1|+|M_2|+|M_1M_2|)$.

To prepare for proof of Lemma 17, we make some technical observations.

Claim 11. Let h be a homomorphism from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ and u_0,\ldots,u_m a simple path in q,m>0. If $h(u_0)\in \mathsf{adom}(\mathcal{D}_0)$ and $h(u_m)\in \mathsf{adom}(\mathcal{D}_0)$, then $h(u_i)\in \mathsf{adom}(\mathcal{D}_0)$ for $0\leq i\leq m$.

Proof. Assume towards a contradiction that $h(u_j) \notin \operatorname{adom}(\mathcal{D}_0)$ for some j with $0 \leq j \leq m$ and additionally let j be smallest with this property. Moreover, let $\ell > j$ be smallest such that $h(u_\ell) \in \operatorname{adom}(\mathcal{D}_0)$. Note that $0 < j < \ell < m$. By definition of a path, q must contain atoms $R_j(u_{j-1}, u_j), \ldots, R_\ell(u_{\ell-1}, u_\ell)$. Since q is self-join free, the relation symbols R_j, \ldots, R_ℓ are all distinct.

Next observe that by construction of \mathcal{D} and by definition of universal models, the interpretation \mathcal{I} obtained from $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ by removing all edges r(a,b) with $a,b\in \mathsf{adom}(\mathcal{D}_0)$ is a collection of trees without multi-edges. More precisely, these are the trees added in the second step in the construction of \mathcal{D} and the trees $\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}$ of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$. We view as the roots of these trees the elements of $\mathsf{adom}(\mathcal{D}_0)$.

Then h maps $R_j(u_{j-1},u_j),\ldots,R_\ell(u_{\ell-1},u_\ell)$ to some tree of this collection, and both u_{j-1} and u_ℓ are mapped to the root of the tree. This, however, clearly contradicts the fact that the relation symbols R_j,\ldots,R_ℓ are all distinct and the tree has no multi-edges.

Claim 12. Let y, y' be variables in q such that $y \neq y'$ and $h(y), h(y') \in \mathsf{adom}(\mathcal{D}_0)$. Then, $h(y) = \langle y, f_y \rangle$ and $h(y) = \langle y', f_{y'} \rangle$ for some functions $f_y \colon Y_y \to [n]$ and $f_{y'} \colon Y_{y'} \to [n]$. Moreover, if $y_i \in Y_y \cap Y_{y'}$ then $f_y(y_i) = f_{y'}(y_i)$.

Proof. Since q is connected, there is a simple path z_0,\ldots,z_ℓ in q from $z_0=y$ to $z_\ell=y'$. Since $h(y),h(y')\in {\sf adom}(\mathcal{D}_0)$, by the previous claim, all variables on this path are mapped to ${\sf adom}(\mathcal{D}_0)$. Thus, there are atoms $R_0(y,z_1)$ and $R_2(z_{\ell-1},y')$ such that $R_0(h(y),h(z_1))\in\mathcal{D}_0$ and $R_2(h(z_{\ell-1}),h(y'))\in\mathcal{D}_0$. Those facts have been added by one of the rules **MR1**, **MR2**, or **MR3** in the construction of \mathcal{D}_0 and, thus, $h(y)=\langle y,f_y\rangle$ and $h(y)=\langle y',f_{y'}\rangle$.

For the second part of the claim observe that if $y_i \in Y_y \cap Y_{y'}$ then there are a simple functional path from y to y_i and a simple functional path from y' to y_i . Hence, there is a simple path from y to y' such that for every variable u on this path we have that $y_i \in Y_u$. By Claim 10, this implies that $f_u(y_i) = f_{u'}(y_i)$.

We now prove Lemma 17, starting with **MM1**. In fact, we show the following slightly more general statement that will be useful also in the proof of **MM3**.

Claim 13. Let h be a homomorphism from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ and $z,z' \in \text{var}(q)$ such that $y_0 \in Y_z$, $y_k \in Y_{z'}$, $h(z) = \langle z, f_z \rangle$, and $h(z') = \langle z', f_{z'} \rangle$. Then $(f_z(y_0), f_{z'}(y_k)) \in M_1M_2$.

We may infer MM1 by applying the claim for $z=x_1$ and $z^\prime=x_2$.

To prove Claim 13, let h be a homomorphism from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ and $z,z' \in \text{var}(q)$ such that $y_0 \in Y_z, y_k \in Y_{z'}, h(z) = \langle z, f_z \rangle$, and $h(z') = \langle z', f_{z'} \rangle$.

Recall that y_0,\ldots,y_k is a bad path in q^+ . Thus, for every pair $y_i,y_{i+1},0\leq i< k$, there is an atom $R^+(\bar{z})$ in q^+ that uses both y_i and y_{i+1} . Hence, there is a simple path π_i from y_i to y_{i+1} in q such that for every variable x on the path we have that $Y_x\neq\emptyset$. Indeed, since there is an atom in q^+ that uses both y_i and y_{i+1} , by definition of q^+ there is an atom R(v,v') in q such that there are a simple functional path ν from v to y_i in q and a simple functional path ν' from v' to y_{i+1} in q. Let v'' be the last variable on the path ν that also belongs to ν' . The desired simple path is the concatenation of the reverse of the suffix from v'' to y_i of the path ν and the suffix from v'' to y_{i+1} of the path ν' .

Moreover, since $y_0 \in Y_z$ and $y_k \in Y_{z'}$, there are a simple functional path π_{-1} from z to y_0 in q and a simple functional path π_{k+1}^- from z' to y_k in q. Concatenating paths $\pi_i, -1 \le i \le k+1$, we obtain a path from z to z' such that $Y_u \ne \emptyset$ for every variable u on this path. Since $z \ne z'$, we can refine it to a simple path z_0, \ldots, z_ℓ from $z_0 = z$ to $z_\ell = z'$ in q, by removing internal cycles. We call this path π . Notice that $y_0 \in Y_{z_0}, y_k \in Y_{z_\ell}$, and for every variable u on this path the set Y_u is not empty.

Let $R_j(z_j, z_{j+1})$, $0 \le j < \ell$, be a sequence of atoms on this path. Then, for every $0 \le i < k$ there is an atom $R_i(v_i, v_i')$ such that $\{y_i, y_{i+1}\} \subseteq Y_{v_i} \cup Y_{v_i'}$.

To show this let $S_j = Y_{z_j} \cup Y_{z_{j+1}}$ for $0 \le j < \ell$. Now, recall that by Claim 9 for every $0 \le j < \ell$ there is $0 \le o < k$ such that $S_j \subseteq \{y_o, y_{o+1}\}$ and, by the choice of π , we have that $S_j \cap S_{j+1} \ne \emptyset$. Moreover $y_0 \in S_0$ and $y_k \in S_{\ell-1}$. Now assume that there is $0 \le i < k$ such that $\{y_i, y_{i+1}\} \not\subseteq S_j$ for all $0 \le j < \ell$. Then, by a simple inductive argument on the length of the sequence of sets S_j , we can show that $y_o \notin S_j$ for all o > i. Thus, since $y_k \in S_\ell$, if $\{y_i, y_{i+1}\} \not\subseteq S_j$ for all $0 \le j < \ell$ then i > k. This implies that for every $0 \le i < k$ there is an atom $R_i(v_i, v_i')$ such that $\{y_i, y_{i+1}\} \subseteq Y_{z_j} \cup Y_{z_{j+1}}$.

 $Y_{z_{j+1}}$. Since the path π is simple and its ends are mapped to $\mathsf{adom}(\mathcal{D}_0)$, i.e. $h(z_0), h(z_\ell) \in \mathsf{adom}(\mathcal{D}_0)$, by Claim 11, we have that $h(u) \in \mathsf{adom}(\mathcal{D}_0)$ for every variable u in π . Thus, by Claim 12, we know that $h(v_i) = \langle v_i, f_{v_i} \rangle$ and $h(v_i') = \langle v_i', f_{v_i'} \rangle$ for some partial functions $f_{v_i'}, f_{v_i}$ from Y to [n].

Let $f = \bigcup_{0 \le i < k} f_{v_i} \cup f_{v'_i}$. We claim that f is a function from Y to [n]. The existence of atoms $R_i(v_i, v'_i)$ shows that for every $y_i \in Y$ there is $d \in [n]$ such that $(y_i, d) \in f$. On the other hand, Claim 12 shows that the value d is uniquely defined. Thus, f is a function from Y to [n].

Let $f(y_0) = a$, $f(y_1) = c$, and $f(y_k) = b$. Then

ME1 $f(y_0) = f_z(y_0) = a$,

ME2 for 0 < i < k we have that $f(y_i) = c$,

ME3 and $f(y_k) = f_{z'}(y_k) = b$.

ME4 Finally, $(a, c) \in M_1$, $(c, b) \in M_2$.

For **ME1** recall that $y_0 \in Y_z$. Hence, there is a simple functional path from z to y_0 . For all vertices u on this path we have that $y_0 \in Y_u$ and, moreover, $h(u) \in \mathsf{adom}(\mathcal{D}_0)$. Thus $f(y_0) = f_z(y_0)$.

ME3 is shown in the same way. For **ME2** observe that for 0 < i < k-1 we have that $Y_{v_i} \cup Y_{v_i'} = \{y_i, y_{i+1}\}$ thus fact $R_i(\langle v_i, f_{v_i} \rangle, \langle v_i', f_{v_i'} \rangle)$ was added by the third rule **MR3** in construction of \mathcal{D}_0 . Thus, there is a word $w = c_1 e^{k-1} c_2$, for some $c, c_1, c_2 \in [n]$, such that $f_{v_i}^w = f_{v_i}$ and $f_{v_i'}^w = f_{v_i'}$. By the definition of f_x^w we obtain $f(y_i) = f(y_{i+1}) = c$.

To show $(a,c) \in M_1$ observe that $Y_{v_0} \cup Y_{v_0'} = \{y_0,y_1\}$. Thus, the fact $R_0(\langle v_0,f_{v_0}\rangle,\langle v_0',f_{v_0'}\rangle)$ was added by the first rule (**MR1**) in construction of \mathcal{D}_0 . Thus, $f_{v_0}^w = f_{v_0}$ and $f_{v_0'}^w = f_{v_0'}$ for $w = ac^k$. By definition of **MR1**, this implies that $(a,c) \in M_1$. A similar argument shows that $(c,b) \in M_2$.

From **ME4** we infer immediately that $(a,b) \in M_1M_2$, which shows that $(f_z(y_0), f_{z'}(y_k)) \in M_1M_2$ and ends the proof of **MM1**.

MM2 Let $(a,b) \in M_1M_2$ and choose a d such that $(a,d) \in M_1$ and $(d,b) \in M_2$. To prove MM2, it suffices to identify a homomorphism h from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ such that $h(\bar{x}) = \bar{a}$.

Since Q is non-empty, there is a database \mathcal{D}' that is satisfiable w.r.t. \mathcal{O} and a homomorphism h' from q to $\mathcal{U}_{\mathcal{D}',\mathcal{O}}$ such that $h'(\bar{x})$ contains only elements of $\mathsf{adom}(\mathcal{D}')$. Let W_0 be the set of variables $x \in \mathsf{var}(q)$ with $h'(x) \in \mathsf{adom}(\mathcal{D}')$.

We may construct the desired homomorphism h by setting $h(x) = \langle x, f_x^{ad^{k-1}b} \rangle$ for all $x \in W_0$, and then extending to the remaining variables in q (which are part of tree-shaped subqueries of q) exactly as in the proof of the **TD2** part of Claim 3.

MM3 Let $\bar{a} \in Q^*(\mathcal{D})$ be a minimal partial answer and $x \in \text{var}(q)$. We say that \bar{a} is *misguided* and that x is *misguided* to $c \in \text{adom}(\mathcal{D}_0)$ for \bar{a} if there is a homomorphism h from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ such that $h(\bar{x}) = \bar{a}$ and h(x) = c has the form $\langle x', f_x \rangle$ with $x' \neq x$. We first show the following.

Claim 14. There are no more than

$$|\bar{x}| \cdot |\mathsf{adom}(\mathcal{D}_0)| \cdot (|\mathsf{adom}(\mathcal{D}_{\mathsf{tree}})| + 1)^{|\bar{x}|}$$

misguided minimal partial answers.

Note that the above count is $O(|M_1| + |M_2|)$ since $|\mathsf{adom}(\mathcal{D}_0)|$ is $O(|M_1| + |M_2|)$ and $|\bar{x}|$ and $|\mathsf{adom}(\mathcal{D}_{\mathsf{tree}})|$ are constants.

Proof. It clearly suffices to show that for every $x \in \text{var}(q)$ and $c \in \text{adom}(\mathcal{D}_0)$, there are at most $(|\text{adom}(\mathcal{D}_{\text{tree}})|+1)^{|\bar{x}|}$ minimal partial answers $\bar{a} \in Q^*(\mathcal{D})$ such that x is misguided to c for \bar{a} . Thus fix x and $c = \langle x', f_x \rangle$ with $x' \neq x$. It suffices to show that if h is a homomorphism from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ such that $h(\bar{x}) = \bar{a}$ and h(x) = c, then for every answer variable x_i , the corresponding constant a_i in \bar{a} is either a constant from a tree attached in Step 2 of the construction

of \mathcal{D} to c (which includes the possibility that $h(x_i) = c$) or it is '*'.

Let x_i be an answer variable. Since q is connected, q contains a simple path z_1,\ldots,z_ℓ from $x=z_1$ to $x_i=z_\ell$. There are thus atoms $R_1(z_1,z_2),\ldots,R_{\ell-1}(z_{\ell-1},z_\ell)\in q$. Consider the fact $R_1(h(x),h(z_2))\in \mathcal{D}$. We argue that $R_1(h(x),h(z_2))\notin \mathcal{D}_0$. Assume for a contradiction that the opposite is true. From $h(x)=\langle x',f_x\rangle$ and the construction of \mathcal{D}_0 , it follows that q contains a fact of the form $R_1(x',y)$. This contradicts q being self-join free and containing the atom $R_1(x,z_2)$ with $x\neq x'$. From $R_1(h(x),h(z_2))\notin \mathcal{D}_0$, it follows that $h(z_2)\notin \mathrm{adom}(\mathcal{D}_0)$, that is, $h(z_2)$ is a constant from a tree attached in Step 2 of the construction of \mathcal{D} to c or an element of $\mathrm{adom}(\mathcal{U}_{\mathcal{D},\mathcal{O}})\setminus \mathrm{adom}(\mathcal{D})$.

To show that a_i is a constant from a tree attached in Step 2 of the construction of \mathcal{D} to c or '*', it suffices to show that $h(z_j) \notin \mathsf{adom}(\mathcal{D}_0)$ for $2 \leq j \leq \ell$, or in other words: after leaving $\mathsf{adom}(\mathcal{D}_0)$ in the first step, entering a tree-shaped part of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$, the path $R_1(z_1,z_2),\ldots,R_{\ell-1}(z_{\ell-1},z_{\ell})$ never leaves that tree-shaped part again. Assume to the contrary that $h(z_j) \in \mathsf{adom}(\mathcal{D}_0)$ with $2 \leq j \leq \ell$. Then Claim 11 applied to the simple path z_1,\ldots,z_j yields $h(z_1) \in \mathsf{adom}(\mathcal{D}_0)$, a contradiction.

It remains to analyse the number of minimal partial answers \bar{a} that are not misguided.

We first establish a technical claim that analyses the facts $R(a,b) \in \mathcal{D}$, which 'cross' from the \mathcal{D}_0 -part of \mathcal{D} into the tree part added in the second step of the construction.

Claim 15. Let $R(x,y) \in q$, $R(a,b) \in \mathcal{D}$, $a = \langle x, f \rangle \in \mathsf{adom}(\mathcal{D}_0)$, and $b \notin \mathsf{adom}(\mathcal{D}_0)$. Then

1.
$$\{y_0, y_k\} \cap Y_x = \emptyset$$
 and
2. $\{y_0, y_k\} \cap Y_y \neq \emptyset$.

Proof. Let R(x,y) and R(a,b) be as in the claim. Since $b \notin \mathsf{adom}(\mathcal{D}_0)$, the fact R(a,b) was added to the database \mathcal{D} in Step 2 of its construction. Since $a \in \mathsf{adom}(\mathcal{D}_0)$, it follows that there is no $d \in \mathsf{adom}(\mathcal{D}_0)$ such that $R(a,d) \in \mathcal{D}_0$.

Since $a \in \operatorname{adom}(\mathcal{D}_0)$, there is a fact $R'(a,c) \in \mathcal{D}_0$. Thus, by construction of \mathcal{D}_0 there is a a word $w = w_0 \cdots w_k \in [n]^*$ such that $f = f_x^w$ and $(w_0, w_1) \in M_1$ or $(w_{k-1}, w_k) \in M_2$.

For Point 1, assume to the contrary of what is to be shown that $y_0 \in Y_x$. From $R(x,y) \in q$ and the construction of \mathcal{D}_0 it then follows that $R(\langle x, f_x^w \rangle, \langle y, f_y^w \rangle) \in \mathcal{D}_0$. This is a contradiction to that fact that there is no $d \in \mathsf{adom}(\mathcal{D}_0)$ such that $R(a,d) \in \mathcal{D}_0$. We can show similarly that $y_k \notin Y_x$.

For Point 2, assume to the contrary that $\{y_0, y_k\} \cap (Y_x \cup Y_y) = \emptyset$. Then, by Point 3 of the construction of \mathcal{D}_0 we have $R(\langle x, f_x^w \rangle, \langle y, f_y^w \rangle) \in \mathcal{D}_0$ which again yields a contradiction.

Recall that the restriction of $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ to domain $\mathsf{dom}(\mathcal{U}_{\mathcal{D},\mathcal{O}})\setminus \mathsf{adom}(\mathcal{D}_0)$ is a disjoint union of trees. These trees consists of the constants that have been added in Step 2 of the construction of \mathcal{D} and of the nulls that have been added in the construction of the universal model. Let \mathfrak{T}' be the set of all those trees and let \mathfrak{T} be the subset

of trees in \mathfrak{T}' that contain at least one constant in \mathcal{D} , i.e. $\mathfrak{T} = \{ \mathcal{T} \in \mathfrak{T}' \mid \mathsf{dom}(\mathcal{T}) \cap \mathsf{dom}(\mathcal{D}) \neq \emptyset \}.$

For a constant $c \in \mathsf{dom}(\bigcup \mathfrak{T})$, let $\mathsf{root}(c)$ be the root of the tree $\mathcal{T} \in \mathfrak{T}$ that contains c, i.e. $\mathsf{root}(c)$ is the unique constant in \mathcal{T} such that there is a fact $R(\mathsf{root}(c),d) \in \mathcal{D}$ with $d \in \mathsf{adom}(\mathcal{D}_0)$. Moreover, let $T_c^{\mathcal{U}}$ be the domain of the tree that contains c. Clearly, for all $c \in \mathcal{U}_{\mathcal{D},\mathcal{O}}$ we have $|T_c^{\mathcal{U}} \cap \mathsf{dom}(\mathcal{D})| \leq |\mathsf{adom}(\mathcal{D}_{\mathsf{tree}})|$.

Further let $\operatorname{adom}(\mathcal{D}_0)^{\mathcal{U}} = \bigcup_{c \in \operatorname{adom}(\mathcal{D}_0)} \operatorname{dom}(\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow c}).$

It is not hard to verify that the set $\operatorname{adom}(\mathcal{D}_0)^{\mathcal{U}}$ and the sets $T_c^{\mathcal{U}}$, for $c \in \operatorname{dom}(\mathcal{D}) \setminus \operatorname{adom}(\mathcal{D}_0)$, form a partition of $\operatorname{adom}(\mathcal{U}_{\mathcal{D},\mathcal{O}})$.

The next claim analyses the relationship between the homomorphism targets of answer variables in q. Recall that by Claim 9, Y_{x_i} is $\{y_0\}$, $\{y_k\}$, or \emptyset for each answer variable x_i . Also recall that $Y_{x_1} = \{y_0\}$ and $Y_{x_2} = \{y_k\}$. Intuitively, part **MH1** implies that for homomorphisms from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ that are not misguided either (**MH1a**) every variable x such that $y_0 \in Y_x$ is mapped to a trace below a constant in $\operatorname{adom}(\mathcal{D}_0)$ or (**MH1b**) there is a single constant $d \in \operatorname{dom}(\mathcal{D}) \setminus \operatorname{adom}(\mathcal{D}_0)$ such that every every variable x such that $y_0 \in Y_x$ is mapped to the tree $T_d^{\mathcal{U}}$. A similar observation can be made for variables x such that $y_k \in Y_x$. Finally, for variables x such that $Y_x = \emptyset$, the claim implies, part **MH2**, that either x is mapped to a trace below a constant from $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ or is mapped to one of the trees above.

Claim 16. Let h be a homomorphism from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ that is not misguided. Then the following hold:

MH1 Let x be an answer variable such that $Y_x = \{y_0\}$ or $Y_x = \{y_k\}$. For every variable x' such that $Y_x = Y_{x'}$: **MH1a** if $h(x) \in \operatorname{adom}(\mathcal{D}_0)^{\mathcal{U}}$ then $h(x') \in \operatorname{adom}(\mathcal{D}_0)^{\mathcal{U}}$,

MH1b if $h(x) \notin \operatorname{adom}(\mathcal{D}_0)^{\mathcal{U}}$ then $h(x') \in T^{\mathcal{U}}_{root(h(x))}$.

MH2 Let x be an answer variable such that $Y_x = \emptyset$. If $h(x) \notin \mathsf{adom}(\mathcal{D}_0)^{\mathcal{U}}$ then there is an answer variable x' with $Y_{x'} \neq \emptyset$ such that $h(x) \in T^{\mathcal{U}}_{root(h(x'))}$.

Proof. For **MH1**, let x, x' be as in the claim. Since $Y_x = Y'_x = \{y\}$ for some $y \in \{y_0, y_k\}$, there are a simple functional path from x to y and a simple functional path from x' to y. Moreover, since the paths are functional and $y \in Y_y$, for any variable u on either of those paths we have that $y \in Y_u$. Thus, there is a simple path z_1, \ldots, z_ℓ from $z_1 = x$ to $z_\ell = x'$ such that for every variable z_i we have that $y \in Y_{z_i}$.

For **MH1a**, let us assume, by contradiction, that $h(x) \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$ and $h(x') \notin \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$. Thus, $\mathsf{root}(h(x'))$ is well defined. Let $d = \mathsf{root}(h(x'))$. By construction of \mathcal{D} , there is z_j such that $h(z_j) = d$. Thus, there are z_{j-1} and an atom $R(z_{j-1}, z_j)$ such that $h(z_{j-1}) = c \in \mathsf{adom}(\mathcal{D}_0)$, $h(z_j) = d \notin \mathsf{adom}(\mathcal{D}_0)$, and $R(c, d) \in \mathcal{D}$. By Claim 15, we have that $y \notin Y_{z_{j-1}}$, but this is impossible as we know that for every z_i on the path we have that $y \in Y_{z_i}$. This proves **MH1a**.

For **MH1b**, let $h(x) \notin \operatorname{adom}(\mathcal{D}_0)^{\mathcal{U}}$. Thus, $\operatorname{root}(h(x))$ is well defined. Let $d = \operatorname{root}(h(x))$. By contradiction, let us assume that $h(x') \notin T_d^{\mathcal{U}}$. Then, $h(z_1), \dots, h(z_\ell)$ is a path in \mathcal{D} . Moreover, since $h(z_\ell) \notin T_d^{\mathcal{U}}$, by construction of \mathcal{D}

there is a constant $h(z_{j+1}) = c \in \mathsf{adom}(\mathcal{D}_0)$ and an atom $R(z_j, z_{j+1}) \in q$ such that $h(z_j) = d$. Since $c, d \in \mathsf{dom}\mathcal{D}$, we have $R(c, d) \in \mathcal{D}$ and, by Claim 15, we infer that $y \notin Y_{z_{j+1}}$. Again, this is impossible as we know that for every z_i on the path we have that $y \in Y_{z_i}$. This proves **MH1b**.

For MH2, let x be an answer variable with $Y_x = \emptyset$ and $h(x) \notin \operatorname{adom}(\mathcal{D}_0)$. Then, $\operatorname{root}(h(x))$ is well defined. Let $d = \operatorname{root}(h(x))$. If $h(x_1) \in T_d^{\mathcal{U}}$, then we are done, so let us assume that $h(x_1) \notin T_d^{\mathcal{U}}$. Let z_1, \ldots, z_ℓ be a simple path from $z_1 = x$ to $z_\ell = x_1$. Since $h(x_1) \notin T_d^{\mathcal{U}}$ and $h(x) \in T_d^{\mathcal{U}}$, by construction of \mathcal{D} there is a variable z_i on this path such that $h(z_i) = d$ and $h(z_{i+1}) \in \operatorname{adom}(\mathcal{D}_0)$. Thus, there is an atom $R(z_i, z_{i+1}) \in q$ and a fact $R(h(z_i), h(z_{i+1}))$. Thus, by Claim 15, we have that $\{y_0, y_k\} \cap Y_{h(z_i)} \neq \emptyset$. By MH1a $y_0 \notin Y_{h(z_i)}$. Thus $y_k \in Y_{h(z_i)}$ and, by MH1b, $h(x_2) \in T_d^{\mathcal{U}}$. This ends the proof of MH2.

For every answer $\bar{a} \in Q(\mathcal{D})^{\mathcal{W}}$, choose a homomorphism h from q to $\mathcal{U}_{\mathcal{D},\mathcal{O}}$ with $h(\bar{x}) = \bar{a}$ that is not misguided. We analyse four cases according to where exactly h maps the answer variables in q, and determine the maximum number of minimal partial answers in each case. Note that by Claim 9, the cases are exhaustive.

Case 1: There is no variable x such that $Y_x = \{y_0\}$ or $Y_x = \{y_k\}$ and $h(x) \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$. We will show that there are $O(|M_1| + |M_2|)$ minimal partial answers consistent with this condition.

By Claim 16 part **MH1b**, there is a constant d_1 such that all answer variables x with $Y_x = \{y_0\}$ are mapped to the tree $T_{d_1}^{\mathcal{U}}$, and likewise for variables with $Y_x = \{y_k\}$ and some constant d_2 . To be more precise, let $d_1 = \operatorname{root}(h(x_1))$ and $d_2 = \operatorname{root}(h(x_2))$. Since $Y_{x_1} = \{y_0\}$ and $Y_{x_2} = \{y_k\}$, Point **MH1b** of Claim 16 implies that $h(x) \in T^{\mathcal{U}}_{d_1}$ for every answer variable x with $Y_x=\{y_0\}$ and $h(x)\in T_{d_2}^\mathcal{U}$ for every answer variable x with $Y_x=\{y_k\}$. Moreover, by MH2, by Claim 9, and by fact that h is not misguided, for every answer variable x with $Y_x = \emptyset$ we have $h(x) \in$ $T_{d_1}^{\mathcal{U}} \cup T_{d_2}^{\mathcal{U}} \cup \{\langle x, \emptyset \rangle\}$. This implies that, for every pair of constants $d_1, d_2 \in dom(\mathcal{D})$, for any answer variable x with $h(x) \in \text{dom}(\mathcal{D})$ we have $h(x) \in T_{d_1}^{\mathcal{U}} \cup T_{d_2}^{\mathcal{U}} \cup \{\langle x, \emptyset \rangle\}$. This gives us $2|\mathsf{adom}(\mathcal{D}_\mathsf{tree})| + 1$ possible values of h(x). Since for every answer variable x_i such that $h(x_i) \notin dom(\mathcal{D})$ the value a_i is one of $|\bar{x}|$ wildcards, for a fixed pair (d_1, d_2) we have no more than $(2|\mathsf{adom}(\mathcal{D}_{\mathsf{tree}})| + 1 + |\bar{x}|)^{|\bar{x}|}$ minimal partial answers, which is independent of $||\mathcal{D}||$. Thus, to show that the number of possible minimal partial answers is $O(|M_1| + |M_2|)$, it is enough to show that the number of possible pairs (d_1, d_2) is $O(|M_1| + |M_2|)$.

 atoms in q. By Claim 15 we have that $Y_{v_1} \cap \{y_0, y_k\} \neq \emptyset$. Since all variables x with $y_k \in Y_x$ are mapped to the tree $T_{d_2}^{\mathcal{U}}$, which by assumption is different from $T_{d_1}^{\mathcal{U}}$, we infer that $y_0 \in Y_{v_1}$. Thus, by Claim 15 and the fact that there is an i with $Y_x \cup Y_{x'} \subseteq \{y_i, y_{i+1}\}$ for every atom $R(x, x') \in q$, cf. Point 1 of Claim 9, we infer that $Y_{z_1} \subseteq \{y_1\}$. Indeed, since $y_0 \in Y_{v_1}$ and $R_1(z_1, v_1) \in q$, $Y_{v_1} \cup Y_{z_1} \subseteq \{y_0, y_1\}$. Thus, Point 1 of Claim 15 implies that $Y_{z_1} \subseteq \{y_1\}$. By a similar argument, $Y_{z_2} \subseteq \{y_{k-1}\}$.

Now, following a similar argument as in the **ME2** statement of **MM1**, we can show that $f_{z_1}(y_1) = f_{z_2}(y_{k-1})$. Thus, if $c_1 \neq c_2$ then $f_{z_1} \subseteq \{(y_0, a)\}$ and $f_{z_2} \subseteq \{(y_k, a)\}$ for some a used in M_1 or in M_2 , i.e. there is a pair $(a, b) \in M_1 \cup M_2$ or a pair $(b, a) \in M_1 \cup M_2$.

Therefore, there are no more than $8|\text{var}(q)|^2(|M_1|+|M_2|)$ possible pairs $c_1 \neq c_2$. Indeed, there is no more than $2(|M_1|+|M_2|)$ possible values of a and, thus, no more than $4 \cdot 2(|M_1|+|M_2|)$ possible pairs (f_{z_1},f_{z_2}) . Since there is no more than $|\text{var}(q)|^2$ possible pairs (z_1,z_2) we can bound the number of pairs (c_1,c_2) as above. Now, observe that for i=1,2 and every possible constant c_i , there is no more than $|\Sigma|$ different possible candidates for d_i . Thus, if $c_1 \neq c_2$ then there is no more than

$$8|\text{var}(q)|^2(|M_1|+|M_2|)|\Sigma| = O(|M_1|+|M_2|)$$

possible pairs (d_1, d_2) . Combining the $c_1 \neq c_2$ and the $c_1 = c_2$ case, in total there are no more than

$$|\Sigma| \cdot |\mathsf{adom}(\mathcal{D}_0)| + 8|\Sigma| \cdot |\mathsf{var}(q)|^2 (|M_1| + |M_2|)$$

possible pairs (d_1,d_2) . Since, by Claim 16 Point 1, $|\mathsf{adom}(\mathcal{D}_0)| = O(|M_1| + |M_2|)$, we infer that there are no more than $O(|M_1| + |M_2|)$ possible minimal partial answers in $Case\ 1$.

Case 2: There is a variable x such that $Y_x = \{y_0\}$ and $h(x) \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$, but no variable y such that $Y_x = \{y_k\}$ and $h(y) \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$.

Let $d = \operatorname{root}(h(x_2))$. We show that there is a pair $(a, b) \in M_1$ such that the following holds:

- 1. for all answer variables x' such that $Y_{x'} = \{y_k\}$ we have $h(x') \in T_d^{\mathcal{U}};$
- 2. for all answer variables x' such that $Y_{x'} = \{y_0\}$ we have either $h(x') \notin \text{adom}(\mathcal{D})$ or $h(x') = \langle x', f_{x'} \rangle$ with $f_{x'}(y_0) = a$; note that in the latter case, $f_{x'}$ is only defined on the variable y_0 and thus h(x') is uniquely determined:
- 3. for all answer variables x' such that $Y_{x'} = \emptyset$ we have either $h(x') \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$ or $h(x') \in T_d^\mathcal{U}$.

It follows that in this case there are no more than

$$|M_2| \cdot (|\mathsf{adom}(\mathcal{D}_{\mathsf{tree}})| + 1)^{|\bar{x}|}$$

possible partial answers.

Point 1 follows from **MH1b** in Claim 16, which implies that for all answer variables x' with $y_k \in Y_{x'}$ we have $h(x') \in T_d^{\mathcal{U}}$.

Now for Point 2. By **MH1a** in Claim 16, we have $h(x') \in \text{adom}(\mathcal{D}_0)^{\mathcal{U}}$ for all answer variables x' with $y_0 \in Y_{x'}$. If

 $h(x') \notin \mathsf{adom}(\mathcal{D})$ for all such x', then we are done. Thus assume otherwise, that is, for some answer variable x' with $y_0 \in Y_{x'}$ we have $h(x') \in \mathsf{adom}(\mathcal{D}_0)$.

Since $y_0 \in Y_{x'}$, there is a simple functional path z_1,\ldots,z_ℓ from $x'=z_1$ to $y_0=z_\ell$. Thus, there are atoms $R_1(z_1,z_2),\ldots,R_{\ell-1}(z_{\ell-1},z_\ell)\in q$. Moreover, there is a variable $z=z_j$ such that $h(y_0)\in \text{dom}(\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow h(z)})$. Thus, $h(z)=\langle z,f_z\rangle\in \text{adom}(\mathcal{D}_0)$ and, by construction of \mathcal{D}_0 , $f_z(y_0)=a$ for some $(a,b)\in M_1$.

Since $h(y_0) \in \text{dom}(\mathcal{U}_{\mathcal{D},\mathcal{O}}^{\downarrow h(z)})$, for every answer variable x' with $y_0 \in Y_{x'}$ and for every simple functional path v_1, \ldots, v_m from $x' = z_1$ to $y_0 = z_m$, which has to exists as $y_0 \in Y_{x'}$, there is a variable $v = v_j$ on this path such that $h(v) = \langle v, f_v \rangle = \langle z, f_z \rangle = h(z)$. Since $y_0 \in Y_{v_i}$ or all variables v_i on this path, by Claim 10 we infer that $f_{x'}(y_0) = f_z(y_0) = a$. This proves Point 2.

Point 3 follows immediately from **MH2** in Claim 16.

Case 3: There is a variable x such that $Y_x = \{y_k\}$ and $h(x) \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$, but no variable y such that $Y_x = \{y_0\}$ and $h(y) \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$.

By an argument smmetric to the one for Case 2, we may show that there are no more than $|M_2| \cdot (|\mathsf{adom}(\mathcal{D}_{\mathsf{tree}})| + 1)^{|\bar{x}|}$ possible partial answers in this case.

Case 4; There is a variable x such that $Y_x = \{y_0\}$ and $h(x) \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$ and a variable y such that $Y_y = \{y_k\}$ and $h(y) \in \mathsf{adom}(\mathcal{D}_0)^\mathcal{U}$.

Then, by **MH1a** of Claim 16, for every answer variable z we have $h(z) \in \mathsf{adom}(\mathcal{D}_0)^{\mathcal{U}}$.

If $h(x) \in \mathsf{adom}(\mathcal{D}_0)$ for some answer variable x with $Y_x = \{y_0\}$ then we can argue as in Case 2 that for every answer variable x' with $Y_{x'} = \{y_0\}$ and $h(x') \in \mathsf{adom}(\mathcal{D}_0)$ we have $h(x') = \langle x', f_x \rangle$.

Similarly, if $h(y) \in \mathsf{adom}(\mathcal{D}_0)$ for some answer variable y with $Y_y = \{y_k\}$, then we can argue as in Case 3 that for every answer variable y' with $Y_{y'} = \{y_k\}$ and $h(y') \in \mathsf{adom}(\mathcal{D}_0)$ we have $h(y') = \langle y', f_x \rangle$.

Moreover, if both such x and y exist then Claim 13 gives $(f_x(y_0), f_y(y_k)) \in M_1M_2$. It follows that we have no more than

$$O(|M_1| + |M_2| + |M_1M_2|) \cdot 2^{|\bar{x}|}$$

possible partial answers of this kind.

MP: the below should be here, true? To end the proof of MM3 we recall that every minimal partial answer is either misguided or falls under one of the four cases Case 1-4. above. We have shown that there is no more than $O(|M_1| + |M_2|)$ misguided minimal partial answers. Furthermore, every of the four cases admits $O(|M_1| + |M_2| + |M_1M_2|)$ minimal partial answers. Thus, MM3 holds.

F Proofs for Section 4

Theorem 6. For OMQs $Q = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELH}, CQ)$ with q acyclic, single-testing minimal partial answers with a single-wildcard is in PTIME in combined complexity.

Proof. We first establish Theorem 6 in the version where 'minimal partial answers' are replaced with 'partial answers'. Let an OMQ $Q(\bar{x}) = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELH}, CQ)$ be

given with q acyclic, as well as a Σ -database $\mathcal D$ and a tuple $\bar a^* \in (\mathsf{adom}(\mathcal D) \cup \{*\})^{|\bar x|}$. We simultaneously modify q and $\bar a^*$ as follows:

- let $q'(\bar{x})$ be obtained from $q(\bar{x})$ by quantifying all variables that have a '*' in the corresponding position of \bar{a}^* ;
- let \bar{a} be obtained from \bar{a}^* by removing all wildcards.

We then check whether $\bar{a} \in Q'(\mathcal{D})$ where $Q' = (\mathcal{O}, \Sigma, q')$ and return the answer. This can be done in PTIME (Bienvenu et al. 2013).

For the case of minimal partial answers, we first check using the above procedure whether \bar{a}^* is a partial answer and return 'no' if this is not the case. We then consider all tuples $\bar{b}^* \in (\mathsf{adom}(\mathcal{D}) \cup \{*\})^{|\bar{x}|}$ that may be obtained from \bar{a}^* by replacing a single occurrence of '*' with a constant from $\mathsf{adom}(\mathcal{D})$. For each such tuple, we check whether it is a partial answer using the above procedure. If any of the checks succeeds, return 'no'. Otherwise, return 'yes'.

Theorem 7. For OMQs $Q = (\mathcal{O}, \Sigma, q) \in (\mathcal{EL}, CQ)$ with q acyclic, single-testing minimal partial answers with multi-wildcards is NP-complete in combined complexity. The same is true in (\mathcal{ELH}, CQ)

Proof. The upper bound is established essentially as in the proof of Theorem 6, with two differences. First, when constructing q' we now introduce one quantified variable per wildcard $*_i$ in \bar{a}^* , and thus the resulting CQ is not necessarily acyclic. Checking whether \bar{a}^* is a partial answer is thus only possible in NP now, rather than in PTIME. And second, we now consider all tuples tuples $\bar{b}^* \in (\mathsf{adom}(\mathcal{D}) \cup \{*\})^{|\bar{x}|}$ such that one of the following holds:

- 1. \bar{b}^* is obtained from \bar{a}^* by choosing a wildcard $*_i$ and replacing all occurrences of $*_i$ with a constant from $adom(\mathcal{D})$;
- 2. \bar{b}^* is obtained from \bar{a}^* by choosing two distinct wildcard $*_i$ and $*_j$, replacing all occurrences of $*_j$ with $*_i$, and renaming the remaining wildcard to ensure that they are numbered consecutively.

In summary, we then obtain an NP upper bound.

For the lower bound, we use a reduction from positive 1-in-3-SAT. Let φ be a 3-formula with clauses c_1,\ldots,c_n and variables z_1,\ldots,z_m . All literals are positive and we are interested in whether φ has a 1-in-3 assignment, that is, an assignment that satisfies exactly one variable per clause.

Let \mathcal{O} be the \mathcal{EL} -ontology $\{V \sqsubseteq \exists r. \top\}$ and q the acyclic CQ that contains one connected component for each clause c_i of φ , as follows:

$$v_1(y_i, y_{i,1}) \wedge v_2(y_i, y_{i,2}) \wedge v_3(y_i, y_{i,3}) \wedge r(y_{i,1}, x_{i,1}) \wedge r(y_{i,2}, x_{i,2}) \wedge r(y_{i,3}, x_{i,3}).$$

The answer variables are the variables $x_{i,j}$ with $1 \le i \le n$ and $1 \le j \le 3$. The OMQ Q is now (\mathcal{O}, Σ, q) where Σ is the set of relation symbols in \mathcal{O} and q.

The tuple $\bar{a}^{\mathcal{W}}$ to be tested uses wildcards $*_1, \ldots, *_m$. In the position for each answer variable $x_{i,j}$, it has wildcard $*_{\ell}$ if the j-th variable in clause i is z_{ℓ} . Thus every wildcard

is used multiple times, which ensures that occurrences of the same variable in different clauses receive the same truth value.

The database \mathcal{D} contains the following facts:

$$\begin{array}{lll} V(a_T) & V(a_F) \\ v_1(a_1,a_T) & v_2(a_1,a_F) & v_3(a_1,a_F) \\ v_1(a_2,a_F) & v_2(a_2,a_T) & v_3(a_2,a_F) \\ v_1(a_3,a_F) & v_2(a_3,a_F) & v_3(a_3,a_T) \end{array}$$

Informally, each variable y_i from each clause may be mapped to each of a_1, a_2, a_3 , representing the options of making the first, second, or third variable in c_i true while making all other variables false.

It can be verified that φ has a 1-in-3 assignment if and only if $\bar{a}^{\mathcal{W}} \in Q(\mathcal{D})^{\mathcal{W}}$.

Theorem 8. For OMQs $Q = (\mathcal{O}, \Sigma, q) \in (\mathcal{EL}, CQ)$ single-testing minimal partial answers is DP-complete in combined complexity. This is true both for single wildcards and multi-wildcards, and the same holds also in (\mathcal{ELH}, CQ) .

Proof. For the upper bound, we use essentially the same strategy as in the proofs of Theorems 6 and 7. We first observe that in (\mathcal{ELH}, CQ) , single-testing (not necessarily minimal) partial answers with multi-wildcards is in NP. Let us call this problem P_1 . To implement the strategy followed in the proofs of Theorems 6 and 7 in DP, we consider the intersection of P_1 with the following CoNP-problem P_2 : given an OMQ $Q = (\mathcal{O}, \Sigma, q) \in (\mathcal{ELH}, CQ)$, a database \mathcal{D} , and a list of tuples $\bar{a}_1^*, \ldots, \bar{a}_n^* \in (\mathsf{adom}(\mathcal{D}) \cup \{*\})^{|\bar{x}|}$, whether $(Q, \mathcal{D}, \bar{a}_i^*)$ is a no-instance of P_1 , for some i with $1 \leq i \leq n$.

To prove hardness, we provide a polynomial time reduction of the DP-complete problem 3COL-no3COL, defined as follows: given a pair (G_1,G_2) of undirected graphs, decide whether G_1 is 3-colorable and G_2 is not.

Let (G_1,G_2) be an instance of 3COL-no3COL. We construct in polynomial time an OMQ $Q \in (\mathcal{EL}, CQ)$, a database \mathcal{D} , and a wildcard tuple \bar{a}^* such that $\bar{a}^* \in Q(\mathcal{D})^*$ if and only if G_1 is 3-colorable and G_2 is not. Assume that $G_1 = ([n], E_1)$ and $G_2 = ([m], E_2)$. Define $Q(x) = (\mathcal{O}, \Sigma, q)$ where

$$\mathcal{O} = \{B \sqsubseteq \exists R.A\}$$

$$\Sigma = \{A, B, r, s, t\}$$

$$q(x) = \bigwedge_{(i,j) \in E_1} r(x_i, x_j) \wedge \bigwedge_{(i,j) \in E_2} s(y_i, y_j) \wedge t(y_1, x) \wedge A(x).$$

Further, define a Σ -database $\mathcal D$ with domain $\{a,b,c,d,e\}$ and facts

- r(a,b), r(b,c), r(c,a), r(b,a), r(c,b), r(a,c),
- s(a,b), s(b,c), s(c,a), s(b,a), s(c,b), s(a,c),
- t(a,d), A(d),
- s(e, e), B(e).

It should be clear that the first conjunction in q checks for 3-colorability of G_1 as the r-part of \mathcal{D} is simply the CSP-template for 3-colorability. The remaining conjunction of q

check for non-3-colorability of G_2 in the following way: the answer variable x can be mapped to the constant d if and only if G_2 is 3-colorable and it can always be mapped to an anonymous element generated by \mathcal{O} . It is this easy to prove the following (for the multi-wildcard version, '*' is simply replaced with '*1').

Claim 17. $Q(\mathcal{D})^* = \{*\}$ if and only if G_1 is 3-colorable and G_2 is not.

Theorem 9. In (\mathcal{ELIHF}, CQ) , single-testing minimal partial answers is EXPTIME-complete in combined complexity, both with single wildcards and multi-wildcards.

Proof. We may once more use the strategy from the proofs of Theorems 6 and 7. It amounts to polynomially many calls to an EXPTIME problem, thus resulting in EXPTIME overall complexity.