# Cut locus structures on graphs

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**Abstract.** Motivated by a fundamental geometrical object, the cut locus, we introduce and study a new combinatorial structure on graphs.
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## 1 Introduction

The motivation of this work comes from a basic notion in riemannian geometry, that we shortly present in the following. In this paper, by a *surface* we always mean a complete, compact and connected 2-dimensional riemannian manifold without boundary.

The *cut locus* $C(x)$ *of* the point $x$ in the surface $S$ is the set of all extremities (different from $x$) of maximal (with respect to inclusion) shortest paths (geodesic segments) starting at $x$; for basic properties and equivalent definitions refer, for example, to [13] or [16]. The notion was introduced by H. Poincaré [15] and gain, since then, an important place in global riemannian geometry.

For surfaces $S$ is known that $C(x)$, if not a single point, is a local tree (i.e., each of its points $z$ has a neighbourhood $V$ in $S$ such that the component $K_z(V)$ of $z$ in $C(x) \cap V$ is a tree), even a tree if $S$ is homeomorphic to the sphere. A *tree* is a set $T$ any two points of which can be joined by a unique Jordan arc included in $T$.

*All our graphs are finite, connected, undirected, and may have multiple edges or loops.*

S. B. Myers [14] established that the cut locus of a real analytic surface is (homeomorphic to) a graph, and M. Buchner [3] extended the result for manifolds of arbitrary dimension. For not analytic riemannian metrics on $S$, cut loci may be quite large sets, see the work of J. Hebda [6] and of the first author [9]. Other contributions to the study of this notion were brought, among others, by M. Buchner [2], [4], H. Gluck and D. Singer [5], J. Hebda [7], J. Itoh [8], K. Shiohama and M. Tanaka [17], T. Zamfirescu [18], [19], A. D. Weinstein [20].

We show in another paper [10] that *for every graph $G$ there exists a surface $S_G$ and a point $x$ in $S$ whose cut locus $C(x)$ is isomorphic to $G$*; rephrasing, *every graph can be realized as a cut locus*.

If $G$ has an odd number $q$ of generating cycles then any surface $S_G$ realizing $G$ is non-orientable, but if $q$ is even then one cannot generally distinguish, by simply looking to the graph $G$, whether $S_G$ is orientable or not: explicit examples show that both possibilities can occur [11]. In other words, seen as a graph, *the cut locus does not encode the orientability of the ambient space*.

This is our main motivation to endow graphs with a combinatorial structure – that of cut locus structure, or shortly CL-structure.

In this paper we treat combinatorial aspects of this new notion: in Section 2 we introduce and discuss this notion, in Section 3 we give two planar representations of CL-structures, and in the last section we enumerate all such structures on "small" graphs.

In a second paper [10] we show that every CL-structure actually corresponds to a cut locus on a surface, while in a subsequent one [11] we consider the orientability of the surfaces realizing CL-structures as cut loci. In particular, any graph endowed with a CL-structure does encode the orientability of the ambient space where is lives as a cut locus. An upper bound on the number of CL-structures on a graph is given in [12].

At the end of this section we recall a few notions from graph theory, in order to fix the notation.

Let $G$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by $B$ the set of all *bridges* in the graph $G$; i.e., edges whose removal disconnects $G$. Each non-vertex component of $G \setminus B$ is called a 2-*connected component* of $G$.

A *$k$-graph* is a graph all vertices of which have degree $k$.

The power set $\mathcal{E}$ of $E$ becomes a $Z_2$-vector space over the two-element field $Z_2$ if endowed with the symmetric difference as addition. $\mathcal{E}$ can be thought of as the space of all functions $E \to Z_2$, and called the (binary) *edge space* of $G$. The (binary) *cycle space* is the subspace $\mathcal{Q}$ of $\mathcal{E}$ generated by (the edge sets of) all simple cycles of $G$. If $G$ is seen as a simplicial complex, $\mathcal{Q}$ is the space of 1-cycles of $G$ with mod 2 coefficients.

# 2 Cut locus structures

**Definition 2.1** *A $G$-patch on the graph $G$ is a topological surface $P_G$ with boundary, containing (a graph isomorphic to) $G$ and contractible to it.*

**Remark 2.2** *Every boundary component of a patch is homeomorphic to a circle, as a 1-dimensional manifold without boundary.*

**Definition 2.3** *A $G$-strip (or a strip on $G$, or simply a* strip*, if the graph is clear from the context), is a $G$-patch with 1-component boundary; i.e., whose boundary is one topological circle; see Figure 1 (a).*

The next remark gives the geometrical background for the notion of cut locus structure.

**Remark 2.4** *Consider a point $x$ on a surface $S$, and a geodesic segment $\gamma : [0, l] \to S$ parameterized by arclength, with $\gamma(0) = x$ and $\gamma(l) \in C(x)$. For $\varepsilon > 0$ smaller than the injectivity radius at $x$, and hence smaller than $l$, the point $\gamma(l - \varepsilon)$ is well defined. Since $S \setminus C(x)$ is contractible to $x$ along geodesic segments, and thus homeomorphic to an open disk, the union over all $\gamma$s of those points $\gamma(l-\varepsilon)$ is homeomorphic to the unit circle, and therefore the set $\bigcup_\gamma \{\gamma(l - \mu) : 0 \leq \mu \leq \varepsilon\}$ is a $C(x)$-strip.*

**Definition 2.5** *A* cut locus structure *(shortly, a* CL-structure*) on the graph $G$ is a strip on the cyclic part $G^{cp}$ of $G$.*

**Remark 2.6** *We show in another paper* [10], *with geometrical tools, the converse to Remark 2.4: every CL-structure can be obtained (with some suitable surface and point on the surface) as described in Remark 2.4.*

**Remark 2.7** *Each $G$-strip defines a circular order around each vertex of $G$, and thus a* rotation system*. Conversely, one can alternatively define a $G$-strip as the graph associated to a rotation system, together with a 2-cell embedding having precisely one face. We choose not to follow this way, and to keep in our presentation as much as possible of the geometrical intuition.*

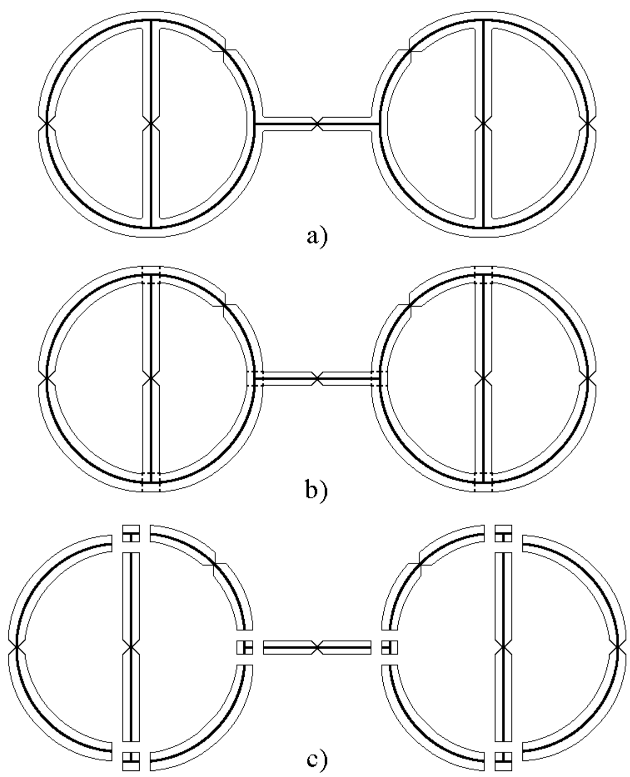


Figure 1: A strip and its elementary decomposition.

**Definition 2.8** *An elementary strip is an edge-strip (arc-strip) or a point-strip; i.e., a strip defined by the graph with precisely one edge (arc) of different extremities, respectively by the graph consisting of one single vertex.*

**Definition 2.9** *An elementary decomposition of a $G$-patch $P_G$ is a decomposition of $P_G$ into elementary strips such that:*
*- each edge-strip corresponds to precisely one edge of $G$;*
*- each point-strip corresponds to precisely one vertex of $G$; see Figure 1 (b) and (c).*

**Remark 2.10** *Our notion of "$G$-patch" is equivalent to that of "fibered surface" introduced by M. Bestvina and M. Handel:* "a fibered surface is a compact surface $F$ with boundary which is decomposed into arcs and into polygons that are modeled on $k$-junctions, $k = 1, 2, 3, ....$ The components of the subsurface fibered by arcs are strips. Shrinking the decomposition elements to points produces a graph $G$, where vertices (of valence $k$) correspond to ($k$-) junctions and strips to edges. We can think of $G$ as being embedded in $F$, representing the spine of $F$" [1]. *We choose the most (in our opinion) appropriate name for our purpose, and thus different from theirs.*

In order to easier handle a CL-structure, we associate to it an object of combinatorial nature. To this goal, denote by $\mathcal{P}$ and $\mathcal{A}$ the set of all point-strips, respectively edge-strips, of a CL-structure $\mathcal{C}$ on the graph $G$.

**Definition 2.11** *Consider an elementary decomposition of the $G$-strip $P_G$ such that each elementary strip has a distinguished face, labeled $\bar{0}$. The face opposite to the distinguished face will be labeled $\bar{1}$. Here, $\bar{0}$ and $\bar{1}$ are the elements of the 2-element group $(Z_2, \oplus)$.*

*To each pair $(v, e) \in V \times E$ consisting of a vertex $v$ and an edge $e$ incident to $v$, we associate the $Z_2$-sum $\bar{s}(v, e)$ of the labels of the elementary strips $\nu \in \mathcal{P}$, $\varepsilon \in \mathcal{A}$ associated to $v$ and $e$; i.e., $\bar{s}(v, e) = \bar{0}$ if the distinguished faces of $\nu$ and $\varepsilon$ agree to each other, and $\bar{1}$ otherwise. Therefore, to any cut locus structure $\mathcal{C}$ we can associate a function $s_{\mathcal{C}} : E \to \{\bar{0}, \bar{1}\}$,*

$$s_{\mathcal{C}}(e) = \bar{s}(v, e) \oplus \bar{s}(v', e), \tag{1}$$

*where $v$ and $v'$ are the vertices of the edge $e \in E$.*

*We call the function $s_{\mathcal{C}}$ defined by (1) the* companion function *of $\mathcal{C}$.*

The value $s_{\mathcal{C}}(e)$ above can be thought of as the *switch* of the edge $e$.

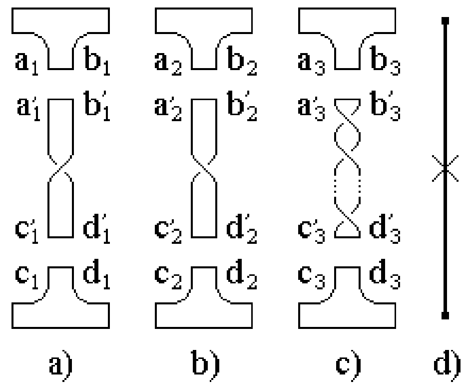


Figure 2: Equivalent CL-structures (a), (b) and (c), and schematic representation (d). The edge-strip at (a) corresponds to a rectangular band whose base is $\pi$-rotated "to the left" with respect to the top; the edge-strip at (b) corresponds to a rectangular band whose base is $\pi$-rotated "to the right" with respect to the top; the edge-strip at (c) corresponds to a rectangular band whose base is $(2k+1)\pi$-rotated "to the left" with respect to the top.

**Definition 2.12** *Assume first that the graph $G$ is 2-connected. Two CL-structures $\mathcal{C}$, $\mathcal{C}'$ on $G$ are called equivalent if their companion functions are equivalent: i.e., $s_{\mathcal{C}}$ and $s_{\mathcal{C}'}$ are equal, up to a simultaneous change of the distinguished face for all elementary strips in $G$ (i.e., either $s_{\mathcal{C}} = s_{\mathcal{C}'}$, or $s_{\mathcal{C}} = \bar{1} \oplus s_{\mathcal{C}'}$).*

*If $G$ is not 2-connected, the CL-structures $\mathcal{C}$, $\mathcal{C}'$ on $G$ are called* equivalent *if their companion functions are equivalent on every 2-connected component of $G$. See Figure 2.*

**Definition 2.13** *An edge-strip $P_e$ (or simply an edge $e$) in a CL-structure $\mathcal{C}$ is called* switched *if $s_{\mathcal{C}}(e) = \bar{1}$.*

**Proposition 2.14** *If two CL-structures on the same graph $G$ are equivalent then the corresponding $G$-strips are homeomorphic surfaces.*

*Proof*: We may assume that $G$ is cyclic.

Assume, moreover, that we have two CL-structures on $G$, whose companion functions are equivalent on every 2-connected component of $G$. The desired homeomorphism can be constructed inductively, extending it with each new "gluing" of an elementary strip, see Figure 2. $\square$

## 3 Representations of CL-structures

We propose two ways to planary represent a CL-structure $\mathcal{C}$ on the graph $G$.

**Definition 3.1** *The* graph representation *of* $\mathcal{C}$ *starts with some planar representation of* $G$*, and afterward points out the CL-structure, see Figure 3 (a).*

**Definition 3.2** *The* natural representation *of* $\mathcal{C}$ *starts by representing in the plane each vertex-strip such that its distinguished face is "up", and afterward connects the vertex-strips by edge-strips. The idea is illustrated by Figure 3 (b) and (c).*

**Remark 3.3** *Consider the natural representation of a CL-structure on a cubic graph. We shall overwrite an "*x*" to the drawn image of an edge if its strip is switched, and an "*=*" to the drawn image of an edge if its strip is not-switched. See Figures 4 and 3.*

**Remark 3.4** *Neither the natural representation, nor the graph representation, of a CL-structure on a graph is unique.*

**Proposition 3.5** *For any planar cubic graph* $G$ *and any CL-structure on* $G$*, the natural representation and the graph representation coincide, up to planar homeomorphisms.*

*Proof:* This follows from the definitions above. □

**Example 3.6** *If the* 3*-graph* $G$ *is not planar, Proposition 3.5 is not true. An easy example, obtained from a flat torus of rectangular fundamental domain (see the procedure described in Remark 2.4), is illustrated by Figure 5.*

Directly from the definitions we have the following.

**Lemma 3.7** *In any natural representation of a strip, each cycle-patch contains at least one switched edge-strip.*

We can give four open questions.

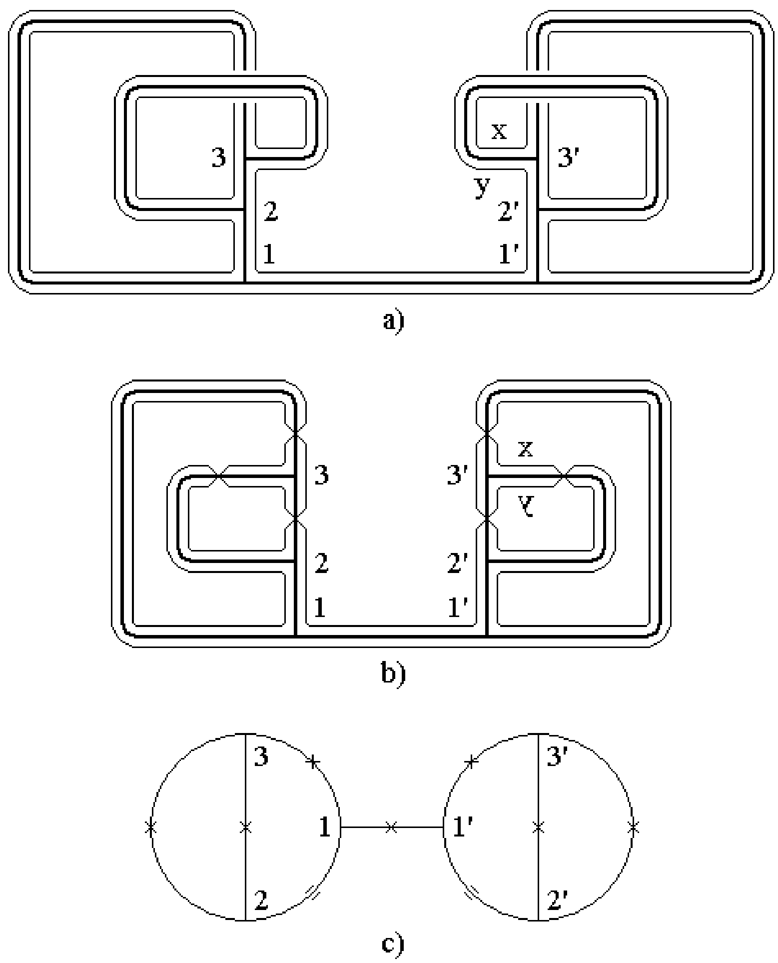


Figure 3: Representations of CL-structures. a) *Graph representation* of a strip. b) Intermediate step to obtain (c). c) *Natural representation* for the strip at (a). Additional points $x, y$ are indicated to make clear the transformation.

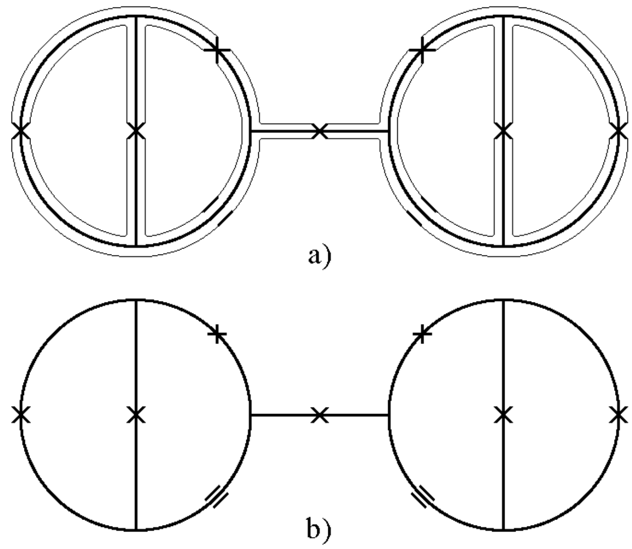


Figure 4: Schematic representation of the strip in Figure 1 (a).

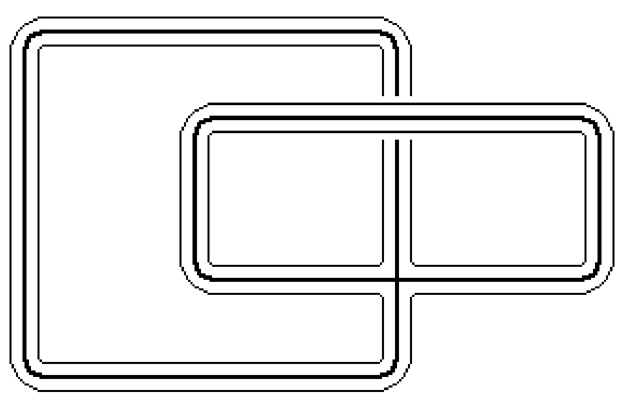

Figure 5: CL-structure obtained from a flat torus of rectangular fundamental domain.

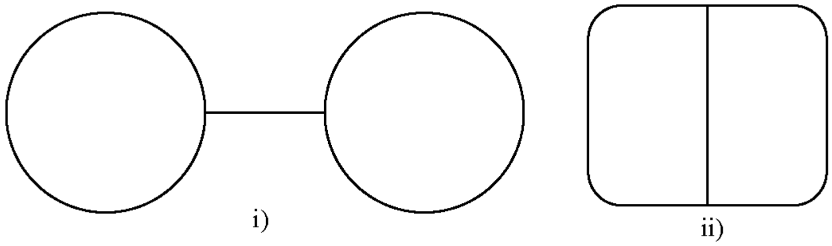


Figure 6: All 3-graphs with 2 generating cycles.

**Question 3.8** *Characterize the companion functions of CL-structures in the set* $\mathcal{S} = \{s : E \to \{\bar{0}, \bar{1}\}\}$.

**Question 3.9** *A planar graph is, by definition, a graph which can be represented in the plane without crossings (self-intersections). As we have seen in Example 3.6, there are CL-structures on (not cubic) planar graphs whose natural representasions in the plane necessarily produce crossings. What is the minimal number of such crossings which guarantees a planar natural representation?*

*The same question can be asked for non planar graphs too, where the (minimal number of necessary) crossings of the graphs is a new parameter.*

**Question 3.10** *How many CL-structures can coexist on the same graph?*

Some (not sharp) upper bound will be given in [11].

**Question 3.11** *Which of the graphs with* $q$ *generating cycles has the largest number of different CL-structures?*

We shall address in the following section the last two questions above, for graphs with two and three generating cycles.

# 4 CL-structures on small graphs

We present in this section all distinct cut locus structures on 3-graphs with $q = 2, 3$ generating cycles.

The following statement can be obtained by straightforward inductive constructions.

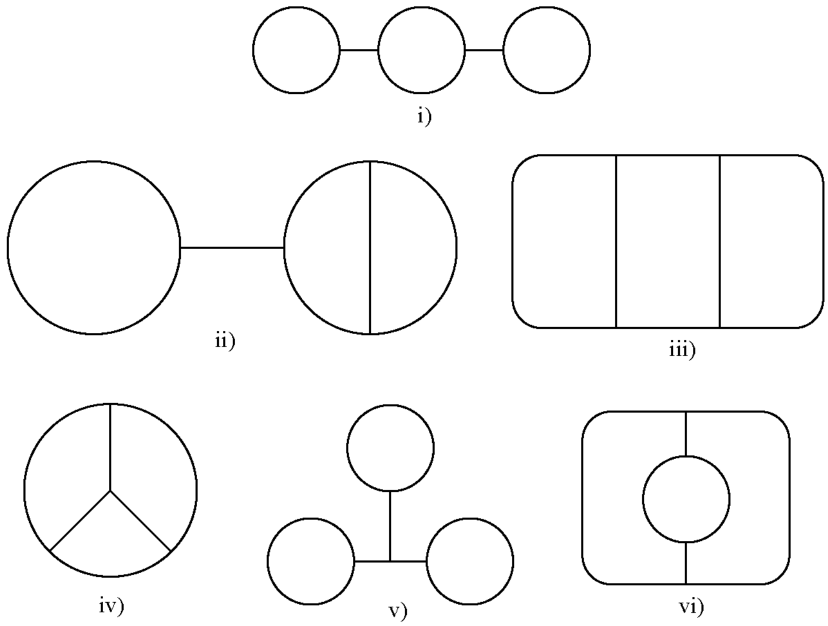


Figure 7: All 3-graphs with 3 generating cycles.

**Lemma 4.1** *There are precisely* 2*, respectively* 6*, distinct* 3*-graphs with* 2*, respectively* 3*, generating cycles, see Figures 6 and 7.*

**Theorem 4.2** *a) There are precisely* 3 *non-equivalent CL-structures on the* 3*-graphs with* 2 *generating cycles, see Figures 8 and 9.*

*b) There are precisely* 17 *non-equivalent CL-structures on the* 3*-graphs with* 3 *generating cycles, see Figures 10 – 15.*

*Proof*: We employ the natural representation of CL-structures. It is straightforward to generate all patches on the graphs in Figures 6 – 7, to keep only

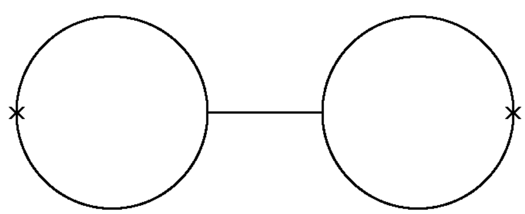

Figure 8: Unique CL-structure on the graph in Figure 6 (i).

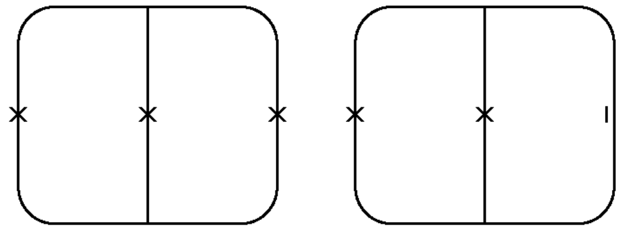

Figure 9: 2 CL-structures on the graph in Figure 6 (ii).

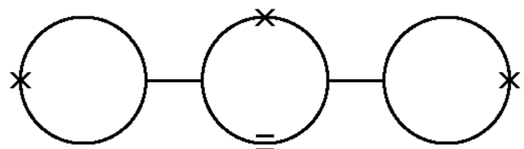

Figure 10: Unique CL-structure on the graph in Figure 7 (i).

the strips (by the use of Lemma 3.7), and to use Definition 2.12 and the symmetries of the graphs to identify equivalent CL-structures. $\square$

Our last result shows that the case of CL-structures on 3-graphs is, in some sense, sufficient. For, define the *degree of a graph* as the maximal degree of its vertices.

**Theorem 4.3** *Any CL-structures on a graph with $q$ generating cycles and degree larger than 3 can be obtained from CL-structures on 3-graphs with $q$ generating cycles, by contracting non-switched edge-strips.*

*Proof*: Fix $q$; we consider only graphs with $q$ generating cycles, and proceed by induction over the number of vertices of degree larger that 3. Denote by $D(G)$ this number for the graph $G$.

Assume the cyclic graph $G$ has $D(G) \geq 1$, and choose a vertex $v$ in $G$ with $\deg(v) = d > 3$.

Let $\mathcal{C}$ be a CL-structure on $G$, and denote by $v_1, ..., v_d$ the neighbours of $v$ in $G$, and by $T$ the subtree of $G$ rooted at $v$, with leaves $v_1, ..., v_d$. Let $G^-$ be the complement of $T$ in $G$, and $\mathcal{C}^-$ be the union of patches naturally induced by $\mathcal{C}$ on $G^-$. Let $s_{\mathcal{C}}^-$ be the restriction of the companion function $s_{\mathcal{C}}$ of $\mathcal{C}$ to $G^-$.

Replace $T$ in $G$ by a tree $T_3$ of leaves $v_1, ..., v_d$, all of whose internal vertices have degree 3 ($T_3$ is generally not unique), and denote by $G^v$ the

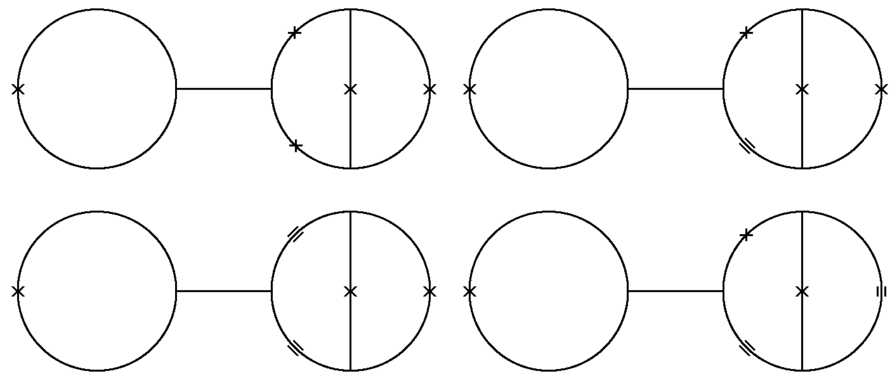

Figure 11: 4 CL-structures on the graph in Figure 7 (ii).

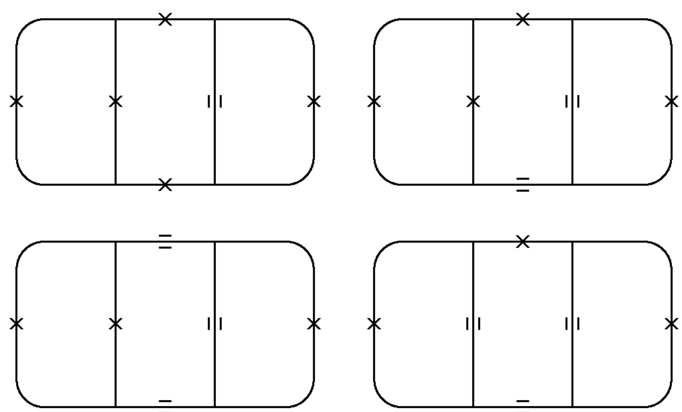

Figure 12: 4 CL-structures on the graph in Figure 7 (iii).

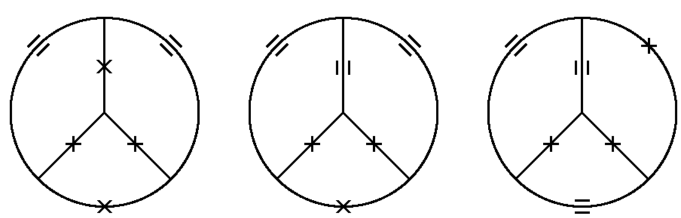

Figure 13: 3 CL-structures on the graph in Figure 7 (iv).

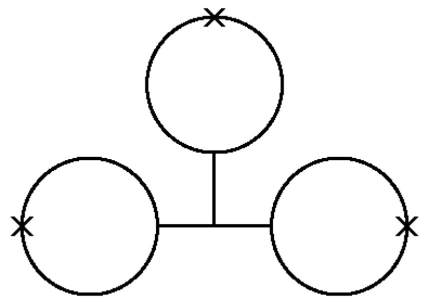

Figure 14: Unique CL-structure on the graph in Figure 7 (v).

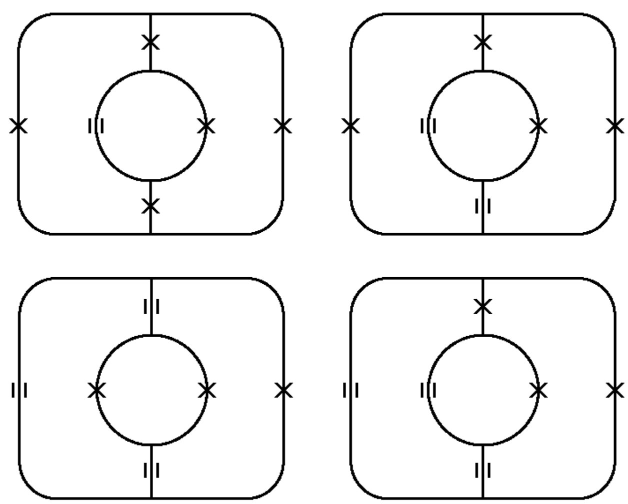

Figure 15: 4 CL-structures on the graph in Figure 7 (vi).

new graph. Now complete $\mathcal{C}^-$ to a CL-structure $\mathcal{C}^v$ on $G^v$, by extending $s_{\mathcal{C}}^-$ on the internal edges of $T_3$ with $\bar{0}$, and on the external edges of $T_3$ with the original values of $s_{\mathcal{C}}$. Observe that $\mathcal{C}^v$ is indeed a CL-structure on $G^v$, and $D(G^v) = D(G) - 1$, so the proof is complete. $\square$

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