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Context-free pairs of groups II — Cuts, tree sets, and random walks

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Dedicated to Gert Sabidussi on the occasion of his 80th birthday

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ABSTRACT

This is a continuation of the study, begun by Ceccherini-Silberstein and Woess (2009) [5], of context-free pairs of groups and the related context-free graphs in the sense of Muller and Schupp (1985) [22]. The graphs under consideration are Schreier graphs of a subgroup of some finitely generated group, and context-freeness relates to a tree-like structure of those graphs. Instead of the cones of Muller and Schupp (1985) [22] (connected components resulting from deletion of finite balls with respect to the graph metric), a more general approach to context-free graphs is proposed via tree sets consisting of cuts of the graph, and associated structure trees. The existence of tree sets with certain "good" properties is studied. With a tree set, a natural context-free grammar is associated. These investigations of the structure of context free pairs, resp. graphs are then applied to study random walk asymptotics via complex analysis. In particular, a complete proof of the local limit theorem for return probabilities on any virtually free group is given, as well as on Schreier graphs of a finitely generated subgoup of a free group. This extends, respectively completes, the significant work of Lalley (1993, 2001) [18,20].

1. Introduction and preliminaries

This is a direct continuation of the paper of Ceccherini and Woess [5] but is to a very large extent independent of that reference.

The interplay between combinatorial group theory and formal languages is a very natural one and has implicitly or explicitly been present ever since the beginning of the study of finitely generated groups. Regarding the specific case of context-free languages, a very satisfactory theory and results were provided in the seminal work of Muller and Schupp [21,22]. For several further references, see [5], where we have undertaken a study of context-freeness in the situation of a pair (G, K), where G is a finitely generated group and K is a subgroup. The situation is best understood via Schreier graphs of such a pair, whose context-freeness is equivalent with a specific property of tree-likeness of such labelled graphs [22]. In the present paper, that notion of tree-likeness is refined and generalised. Briefly spoken, in the approach of [22] one considers *cones* in a labelled graph. These are the connected components that remain after removing any ball (with respect to the natural graph metric) around a given "root" vertex, and the graph is called context-free if there are finitely many isomorphism types of such cones as labelled graphs with finite boundaries.

In the present work, the cones are replaced by more general *cuts* (connected subgraphs with finite boundaries). We want to have a collection ("tree set") of such cuts which contains again only finitely many isomorphism classes as above, and fills up the whole graph in a uniform way. With such a tree set, we can associate a context-free grammar that generates the *word problem,* that is, the set of all words that can be read along some closed path that starts and ends at the root.

This is elaborated in detail and then applied to study the asymptotics of random walk return probabilities on such Schreier graphs in the case when the tree set has certain additional "good" properties. In this regard, one of the main aims is to derive

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such asymptotics for virtually free groups and their Schreier graphs with respect to finitely generated subgroups, thereby completing previous work.

Let us now outline the setting in more detail. We have a finitely generated group G, a subgroup K and a finite *alphabet* $\mathbf{\Sigma}$ together with a mapping $\psi : \mathbf{\Sigma} \to G$ such that $A = \psi(\mathbf{\Sigma})$ generates G as a semigroup. We call ψ a *semigroup presentation* of G. We also write ψ for the extension of this mapping as a monoid homomorphism $\psi : \mathbf{\Sigma}^* \to G$, where $\mathbf{\Sigma}^*$ consists of all words over $\mathbf{\Sigma}$, with word concatenation as the semigroup product and the empty word ϵ as the unit element. The pair (G, K) is called *context-free* if the language $L(G, K, \psi) = \{w \in \mathbf{\Sigma}^* : \psi(w) \in K\}$ is context-free. This is independent of the specific choices of $\mathbf{\Sigma}$ and ψ, see [5, Lemma 3.1].

Let us recall here that a context-free language is a subset of $\mathbf{\Sigma}^*$ that is generated by a *context-free grammar* $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$, where $\mathbf{V}$ is the (finite) set of *variables* (with $\mathbf{V} \cap \mathbf{\Sigma} = \emptyset$), the variable S is the *start symbol*, and $\mathbf{P} \subset \mathbf{V} \times (\mathbf{V} \cup \mathbf{\Sigma})^*$ is a finite set of *production rules*. We write $T \vdash u$ if $(T, u) \in \mathbf{P}$. For $v, w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$, a *rightmost derivation step* has the form $v \Rightarrow w$, where $v = v_1Tv_2$ and $w = v_1uv_2$ with $u, v_1 \in (\mathbf{V} \cup \mathbf{\Sigma})^*$, $v_2 \in \mathbf{\Sigma}^*$ and $T \vdash u$. A *rightmost derivation* is a sequence $v = w_0, w_1, \dots, w_k = w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ such that $w_{i-1} \Rightarrow w_i$; we then write $v \overset{*}{\Rightarrow} w$. Each $T \in \mathbf{V}$ generates the language $L_T = \{w \in \mathbf{\Sigma}^* : T \overset{*}{\Rightarrow} w\}$. The *language generated by* $\mathcal{C}$ is $L(\mathcal{C}) = L_S$. The grammar is called *unambiguous*, if every $w \in L(\mathcal{C})$ has a unique rightmost derivation.

Harrison [14] is an excellent source on context-free languages.

The group G itself is called context-free, if the language $L(G, \{1_G\}, \psi)$ is context-free. Muller and Schupp [21] have shown that a group is context-free if and only if it is virtually free (i.e., it has a free subgroup with finite index). Subsequently, in [22] they have introduced context-free *graphs*, which play a crucial role in the present work. For additional references, see [5]. The graphs that we are dealing with are *Schreier graphs* of (G, K), which will be explained in an instant.

In the present paper, we shall always assume that the index of K in G, as well as the graphs in consideration, are infinite.

Here are some quick reminders regarding labelled graphs.

Let $\mathbf{\Sigma}$ be a finite alphabet. A *directed graph* (X, E, ℓ) *labelled by* $\mathbf{\Sigma}$ consists of the (finite or countable) set of *vertices*, the set of *oriented, labelled edges* $E \subset X \times \mathbf{\Sigma} \times X$ and the *labelling* $\ell : E \ni (x, a, y) \mapsto a \in \mathbf{\Sigma}$. Loops are allowed. Multiple edges between the same vertices must have distinct labels. All our graphs will be *fully deterministic:* for every vertex x and label $a \in \mathbf{\Sigma}$, there is precisely one edge with label a starting at x. Our graphs will usually also be *symmetric (undirected):* there is a proper involution $a \mapsto a^{-1}$ of $\mathbf{\Sigma}$ such that for $e = (x, a, y) \in E$, also $e^{-1} = (y, a^{-1}, x)$ belongs to E.

A *path* in X is a sequence $\pi = e_1e_2 \cdots e_n$ of edges such that the terminal vertex of e_i is the initial vertex of e_{i+1} for $i = 1, \dots, n-1$. Its *label* is $\ell(\pi) = \ell(e_1)\ell(e_2) \cdots \ell(e_n) \in \mathbf{\Sigma}^*$. The *empty path* starting and ending at x has the label ϵ. We only consider graphs that are *strongly connected:* for all $x, y \in X$, there is a path from x to y. With any two vertices $x, y \in X$, we associate the language of all words that can be read along some path from x to y, that is, $L_{x,y} = \{\ell(\pi) : \pi \text{ a path from } x \text{ to } y\}$.

The *Schreier graph* $X = X(G, K, \psi)$ of (G, K) with respect to ψ has the vertex set $X = \{Kg : g \in G\}$ of all right K-cosets in G, and the set of labelled, directed edges $E = \{e = (x, a, y) : x = Kg,\ y = Kg\psi(a),\ \text{where } g \in G,\ a \in \mathbf{\Sigma}\}$. The vertex $o = K$ serves as a root of X. When ψ is *symmetric*, that is, there is a proper involution $a \mapsto a^{-1}$ of $\mathbf{\Sigma}$ such that $\psi(a^{-1}) = \psi(a)^{-1}$ then the Schreier graph is symmetric.

We clearly have that $L(G, K, \psi)$ coincides with the language $L_{o,o}$ associated with $X(G, K, \psi)$.

We now describe the concept of context-free graphs. Let X be a labelled, symmetric graph with a chosen root vertex o. By symmetry, it has its natural graph metric d. Let $B(o, n) = \{x \in X : d(x, o) \le n\}$ be the ball with radius n centred at o. Then a *cone* of X with respect to o is a connected component of $X \setminus B(o, n)$ with $n \ge 0$. Each cone is a labelled graph with its boundary consisting, by definition, of all its vertices having a neighbour that belongs to its complement in X. Then X is called a *context-free graph* in [22], if there are only finitely many isomorphism types, as labelled graphs with boundaries, of cones with respect to o.

It is shown in [5] that a fully deterministic, symmetric graph X with any chosen root vertex o is a context-free graph in the above sense if and only if $L_{o,o}$ is a context-free language. (Contrary to what one tends to believe at first glance, this is not contained in [21,22], the "if" part being the harder one.)

Let us come back to the introductory outline. In the present paper, we first introduce a more general approach to context-free graphs, via *oriented tree sets* consisting of *cuts*, and associated *structure trees* in the spirit of Dunwoody [10,11] and Dicks and Dunwoody [7]; see also Thomassen and Woess [26]. Our tree sets have *finite type* (the cuts which they contain belong to finitely many isomorphism classes) and *tesselate* the underlying graph (see below for precise explanations). We first perform a detailed study of those tree sets in our graphs, showing that they can always be modified so that they have certain connectivity and separation properties. A crucial additional property, *irreducibility*, is introduced. For Schreier graphs it is shown that irreducibility is preserved under finite index extensions of the group G. Also, the existence of a tree set with those properties is independent of the presentation map ψ. As a particular, an interesting class of examples, all those "good" properties, including irreducibility, hold when G is virtually free and K is a finitely generated free subgroup.

With a tree set with finite type that tesselates the Schreier graph of (G, K) with respect to ψ, we can associate in a natural way an unambiguous context-free grammar that generates $L(G, K, \psi)$. The above properties of the tree set translate into properties of the grammar and its *dependency digraph.*

In the final part, we change the flavour from structure-theoretic considerations to random walks and the analysis of generating functions. We start with a fully deterministic, symmetric graph and equip $\mathbf{\Sigma}$ with a probability measure μ. This induces a random walk on X, where a random step from a vertex x along any edge (x, a, y) has probability $\mu(a)$. Suppose that the graph has a tree set as above. Via the fundamental theory of Chomsky and Schutzenberger [6], the associated grammar translates into an algebraic system of equations for the generating functions associated with the transition probabilities of the random walk on our context-free graph. The "good" properties of the tree set, in particular irreducibility, guarantee that this system of equations allows us to apply the methods of complex analysis that in the monograph of Flajolet and Sedgewick [12, Section VII.6] are subsumed as the Drmota–Lalley–Woods theorem. As a matter of fact, that theorem applies directly only to the generating functions of certain restricted transitions, while the final step regarding n-step return probabilities $p^{(n)}(x,x)$ needs some additional care. As a result, we get the following alternative between three possible cases of asymptotic behaviour for all vertices x of the context-free graph.

$$p^{(n)}(x,x) \sim c_{x,x}\, R_\mu^{-n}\,, \quad \text{or} \tag{1.1}$$

$$p^{(n)}(x,x) \sim c_{x,x}\, R_\mu^{-n}\, n^{-1/2} \quad \text{or} \tag{1.2}$$

$$p^{(n)}(x,x) \sim \big(c_{x,x} + (-1)^n \bar{c}_{x,x}\big)\, R_\mu^{-n}\, n^{-3/2}, \tag{1.3}$$

as $n \to \infty$ (with n even when the graph has no odd cycles), where $0 \le |\bar{c}_{x,x}| < c_{x,x}$ and $R_\mu \ge 1$. The oscillating term in (1.3) can occur when the graph does have odd cycles, while there are none outside of some fixed finite set. The basic example exhibiting those oscillations is a reflecting random walk on the non-negative integers, as studied by Lalley [20], who also obtains the alternative between the above three types of the asymptotic behaviour for a wide class of random walks. Among further previous work, we mention Woess [27], Lalley [19] and Drmota [8].

In terms of the underlying groups G and K, we can just look at the random walk induced by μ on G itself. Then $p^{(n)}(o,o)$ is the probability that this walk, starting at 1_G, is in K at the n-th step.

In particular, when $K = \{1_G\}$ and G is virtually free, but not virtually cyclic, then we obtain the asymptotic behaviour (1.3), without oscillating term:

$$p^{(n)}(x,x) \sim c_{x,x}\, R_\mu^{-n}\, n^{-3/2}, \tag{1.4}$$

as $n \to \infty$ (and n is even when the underlying Cayley graph of G has no odd cycles) where $R_\mu > 1$ since the group is non-amenable. Note that we require that the support of the measure μ, or more precisely, the set $A = \psi(\mathbf{\Sigma})$ is a symmetric subset of G. Up to this small restriction (which can be amended by additional work), we get the general local limit theorem for finite range random walks on virtually free groups.

Behind this result, there is a long history concerning the interplay between the behaviour of random walks and the structure of the underlying groups, resp. graphs. See the survey [29] for quite complete references to work regarding asymptotics of random walks on free groups. In the present context of finite range random walks, the most significant contribution is the one of Lalley [18] regarding free groups. The use of context-free languages to derive information about random walk asymptotics on virtually free groups appears first in [28]. This is closely related to the random walks on regular languages of [20], which provide another possible approach to random walks on virtually free groups.

In addition to (1.4), we also obtain the local limit theorem for a larger class of Schreier graphs, resp. context-free graphs, and undertake a careful study of the behaviour in the periodic case, when the oscillations in (1.3) can occur. As a matter of fact, when dealing with Schreier graphs instead of just groups, such a situation is quite natural.

2. Cuts and structure trees of context-free graphs

For the study of certain aspects of context-free groups, pairs and graphs, it will be preferable to use more general types of connected subsets than the cones with respect to some root vertex. In the sequel, among other goals, we shall be interested in the property that for every pair of cones C, D, there is a cone $C' \subset C$ which is isomorphic to D. Now while it may be feasible – for example by using translation by a suitable group element – to show that C contains a subgraph isomorphic with D, it is in general not so clear how to obtain such a copy of D that is again a cone with respect to deleting a ball $B(o,n)$ for the same root vertex o as before. This is also so if one replaces – as in [5] – the root vertex with a finite reference set F for defining the cones as the components left when deleting the balls $B(F,n)$ $(n \ge 0)$.

Therefore we now introduce the setting for a more general construction of context-free graphs.

We start with an infinite symmetric, fully deterministic labelled graph (X, E, ℓ). A *cut* is an infinite, connected induced subgraph C of X such that ∂C is finite and non-empty. (The boundary ∂C is defined as the set of all vertices in C that have a neighbour outside C.) Thus, there are only finitely many edges between C and its complement C^*. With a slight deviation from the terminology of [11,7,26], two cuts C and D are said to be *non-crossing* if one of

$$C \subset D,\ D \subset C, \quad \text{or} \quad C \cap D = \emptyset \tag{2.1}$$

holds. (In the original definition, also the option $C^* \subset D$ is included.)

An *oriented tree set* in X is a non-empty family $\mathcal{E}$ of pairwise non-crossing cuts in X, which satisfies the additional requirement that

$$M = \max\{\operatorname{diam}(\partial C) : C \in \mathcal{E}\} < \infty. \tag{2.2}$$

Lemma 2.3. *There is at most one pair of cuts C, C^* such that both belong to $\mathcal{E}$, and then both C and C^* must be maximal in $\mathcal{E}$ with respect to set inclusion.*

Proof. Let C, C^* be as stated, and let $D \in \mathcal{E}$ be another cut.

If $D \subsetneq C$ then we cannot have $C \subset D^*$ (since otherwise $X = C \cup C^* \subset D^*$), nor $D^* \subset C$, nor $D^* \cap C = \emptyset$. Therefore $D^* \notin \mathcal{E}$. Similarly, one shows that if D is a cut with $C \subsetneq D$ then $D \notin \mathcal{E}$. $\square$

We mention two related properties of $\mathcal{E}$ which are not hard to prove; see e.g. [11].

$$\begin{aligned} &\text{If } C \in \mathcal{E} \text{ and } x \in C, \text{ then } \{C' \in \mathcal{E} : x \in C' \subset C\} \text{ is finite.} \\ &\text{If } C, D \in \mathcal{E} \text{ and } D \subset C, \text{ then } \{C' \in \mathcal{E} : D \subset C' \subset C\} \text{ is finite.} \end{aligned} \tag{2.4}$$

For $C, D \in \mathcal{E}$, we say that D is a *successor* of C, notation $C \to D$, if there is no $C' \in \mathcal{E}$ such that $C \supsetneq C' \supsetneq D$. In that case, C is of course a *predecessor* of D, notation $C = D^-$.

Lemma 2.5. *One of the following holds:*

(1) *Every element of $\mathcal{E}$ is contained in a cut in $\mathcal{E}$ that is maximal with respect to set inclusion, or*
(2) *There is a strictly increasing sequence (C_k) in $\mathcal{E}$ whose union is X, and for every $C \in \mathcal{E}$, there is k such $C \subset C_k$.*

Proof. Suppose that there is some element in $\mathcal{E}$ that is not contained in a maximal cut in $\mathcal{E}$. Then there must be a strictly increasing sequence $(C_k)_{k\geq 0}$ in $\mathcal{E}$. Let $x \in X$ be arbitrary. Then $d(x, \partial C_k) \to \infty$ as $k \to \infty$. Let π be a path that connects x with some element of C_0, and set $n = |\pi|$. If k is such that $d(x, \partial C_k) > n$ then π is a connected subset of X that does not intersect ∂C_k. Therefore we must have $\pi \subset \partial C_k^*$ or $\pi \subset \partial C_k$. Since π intersects $C_0 \subset C_k$, the second case must be true. In particular, $x \in C_k$.

We see that $X = \bigcup_k C_k$. Now let $C \in \mathcal{E}$ be arbitrary. Then there must be k_0 such that $C \cap C_k \neq \emptyset$ for all $k \geq k_0$. We cannot have $C_k \subset C$ for all $k \geq k_0$, since otherwise $C = X$. If $k \geq k_0$ is such that $C_k \not\subset C$, then of the three cases of (2.1), only $C \subset C_k$ remains. $\square$

Following [10,7], we construct the *(oriented) structure tree* $\mathcal{T} = \mathcal{T}_{\mathcal{E}}$. There is a slight difference, in that edges are only oriented in one way: the (oriented, non-symmetric) edge set is $\mathcal{E}$. The tree is obtained as follows: with each $C \in \mathcal{E}$, we associate its terminal vertex $\xi = \xi_C$. It coincides with the initial vertex in $\mathcal{T}$ of $D \in \mathcal{E}$ precisely when $C \to D$.

Furthermore, if $\mathcal{E}$ contains maximal elements, then we introduce one vertex ξ_0 as the initial vertex of each maximal cut in $\mathcal{E}$ (as an edge of $\mathcal{T}$).

It is clear that this defines an oriented tree. In case (1) of Lemma 2.5, we say that $\mathcal{E}$ is *rooted*. The tree has the root vertex ξ_0, and the edges are oriented away from that root. In case (2), there is no such root, but there is an end of the tree such that all edges point away from that end. Since we shall restrict our attention to case (1), we omit giving further details on the other case. We let

$$\partial_{\mathcal{E}} X = \bigcap \{C^* : C \in \mathcal{E}\}.$$

In the rooted case, this *root set* is the intersection of the complements of the maximal elements of $\mathcal{E}$. Otherwise, in case (2), $\partial_{\mathcal{E}} X$ is empty. For $C \in \mathcal{E}$, let

$$\partial_{\mathcal{E}} C = C \setminus \bigcup \{D \in \mathcal{E} : D^- = C\}.$$

Thus, X is the disjoint union of $\partial_{\mathcal{E}} X$ and all $\partial_{\mathcal{E}} C$, where $C \in \mathcal{E}$. With this partition, we can associate the *structure map* $x \mapsto \xi(x)$ from X to $\mathcal{T}_{\mathcal{E}}$. If $x \in \partial_{\mathcal{E}} X$ then $\xi(x) = \xi_0$. If $x \in \partial_{\mathcal{E}} C$ then $\xi(x) = \xi_C$, that is, C is the unique minimal cut in $\mathcal{E}$ that contains x. (If the vertices $x, y \in X$ are neighbours, then $\xi(x)$ and $\xi(y)$ either coincide or are neighbours in $\mathcal{T}_{\mathcal{E}}$.)

We say that a rooted tree set $\mathcal{E}$ *tesselates* X, if $\partial_{\mathcal{E}} X$ and $\partial_{\mathcal{E}} C$ are finite for every $C \in \mathcal{E}$. In that case, the structure tree is locally finite, that is, for every $C \in \mathcal{E}$, there are only finitely many $D \in \mathcal{E}$ with $C \to D$.

Given $\rho \in \mathbb{N}$, we say that $\mathcal{E}$ is *ρ-separated*, if $\partial_{\mathcal{E}} X \neq \emptyset$ and $d(\partial C, \partial D) \geq \rho$ whenever $C, D \in \mathcal{E}$ are distinct. When $\rho = 1$, we just say "separated". The collection of all cones with respect to a root vertex o is a rooted tree set that tesselates X and is separated.

Definition 2.6. Let X and $\mathcal{E}$ (rooted, oriented) be as above.

(a) We say that $\mathcal{E}$ has *finite type*, if it tesselates X and there are elements

$$C_1, \ldots, C_r \in \mathcal{E}$$

such that every cut $C \in \mathcal{E}$ is isomorphic with one of the C_i as a labelled graph, and that isomorphism (which is chosen and fixed for each C) maps $\partial_{\mathcal{E}} C$ to $\partial_{\mathcal{E}} C_i$. In that case, we say that C has *type i*.

(b) We say that $\mathcal{E}$ is *irreducible* if it has finite type and the cuts $C_1, \ldots, C_r$ of (a) are such that for all i, j, the cut C_i contains as a proper subset a cut in $\mathcal{E}$ that is isomorphic with C_j.

In a context-free graph X, the cones with respect to a root vertex o, or with respect to a connected, finite set F as in [5], are a special case of (a).

Note that in case of finite type as in (a), the isomorphism from C to C_i must also map ∂C to ∂C_i, since the boundaries consist precisely of those points where the respective cut is not fully deterministic. Furthermore, since $\partial_{\mathcal{E}} C$ is mapped onto $\partial_{\mathcal{E}} C_i$, the number of $D \in \mathcal{E}$ with type j for which $C \to D$, resp. $C_i \to D$ is the same.

If (a) holds, the structure tree with root ξ_0 is a tree with finitely many cone types in the sense of Nagnibeda and Woess [23]; see also [4, Section 5]. If $C \in \mathcal{E}$ is isomorphic with C_i then we say that the associated edge of $\mathcal{T}$, as well as its terminal vertex in the tree, have *type* i. The root ξ_0 has type 0.

For irreducibility it is necessary that each $C \in \mathcal{E}$ is infinite.

Similarly as in the proof of Theorem 4.2 in [5], we can encode the type structure in a finite, oriented *graph of types* Γ over the vertex set $I = \{1, \dots, r\}$, where we have $a(i,j)$ oriented edges $i \to j$ whenever there are precisely $a(i,j)$ cuts $D \in \mathcal{E}$ such that $C_i \to D$ and D has type j. This means that we enumerate the successor cuts with type j of C_i by the numbers $1, \dots, a(i,j)$, and this enumeration is carried over to any cut C with type i and its successors with type j via the isomorphism $C \to C_i$.

For our use, it will be good to have a separated tree set.

Proposition 2.7. *Let $\mathcal{E}$ be a rooted, oriented tree set consisting of cuts of the fully deterministic, symmetric labelled graph X.*

(1) *Suppose that $\mathcal{E}$ has finite type. Then for every $\rho \in \mathbb{N}$, there is a ρ-separated rooted tree set $\overline{\mathcal{E}} \subset \mathcal{E}$ that tesselates X and such that $\partial_{\overline{\mathcal{E}}} X \supset F$. (It has finite type.)*
(2) *If in addition $\mathcal{E}$ is irreducible, then $\overline{\mathcal{E}}$ can be constructed such that it is also irreducible.*
(3) *If $F \subset X$ is finite, then $\mathcal{E}$ can be modified such that $\partial_{\mathcal{E}} X \supset F$. In addition, the cuts C in $\mathcal{E}$ can be modified such that all $\partial_{\mathcal{E}} C$ as well as $\partial_{\mathcal{E}} X$ are connected. These modifications preserve ρ-separation, the property that $\partial_{\mathcal{E}} X \supset F$, and irreducibility, respectively.*

Proof. First of all, there is a finite bound on the number of elements in ∂C_i, $i = 1, \dots, r$. By (2.4), the number

$$s = \max\Big\{ \big| \{ D \in \mathcal{E} : C_i \supsetneq D \ni x \} \big| : x \in \partial C_i\,,\ i \in I \Big\}$$

is finite. When $C, D \in \mathcal{E}$ are such that as edges of $\mathcal{T}$, their initial points have distance at least s, then their boundaries do not intersect. If the initial points have distance at least ρs in $\mathcal{T}$, then their boundaries have distance at least ρ in the graph X.

We now choose an integer $k \geq \rho s$ and modify $\mathcal{E}$ as follows.

The new tree set $\overline{\mathcal{E}}$ consists of all C in $\mathcal{E}$ whose terminal vertex (as an edge of $\mathcal{T}$) is at distance mk from ξ_0, where $m \geq 1$ (integer). This is a new tree set. The associated structure tree $\overline{\mathcal{T}}$ is obtained from $\mathcal{T}$ by keeping only the vertices of $\mathcal{T}$ at distance mk from ξ_0, where $m \geq 0$. Each of them is connected by an edge to all its descendants which are at distance $(m+1)k$ from ξ_0. Here, by a descendant of a vertex ξ of $\mathcal{T}$, we mean another vertex η with the property that ξ lies on the geodesic path from ξ_0 to η.

Since $\mathcal{E}$ tesselates X and $\mathcal{T}$ is locally finite, also $\overline{\mathcal{E}}$ tesselates X. There are only finitely many isomorphism types in $\overline{\mathcal{E}}$. Thus, $\overline{\mathcal{E}}$ is as required by statement (1).

(2) If the original $\mathcal{E}$ is irreducible, then it is not immediate that the above $\overline{\mathcal{E}}$ inherits that property, and some additional effort is needed.

Consider the adjacency matrix $A = \big(a(i,j)\big)_{i,j\in I}$ of the graph of types Γ associated with $\mathcal{E}$. Let $A^n = \big(a^{(n)}(i,j)\big)_{i,j\in I}$ be the n-th matrix power. The assumption of statement (2) says that A is an irreducible matrix: for all i,j there is $n = n_{i,j}$ such that $a^{(n)}(i,j) > 0$. By the elementary theory of irreducible non-negative matrices (see e.g. Seneta [25]), there are positive integers d (the *period* of A) and n_0 with the following properties.

- The index set has a partition $I = I_0 \cup \cdots \cup I_{\mathsf{d}-1}$ such that if $i \in I_k$ and $a(i,j) > 0$, then $j \in I_{k+1}$, where we set $I_{\mathsf{d}} = I_0$.
- $a^{(n\mathsf{d})}(i,j) > 0$ for all $n \geq n_0$ and all i,j that belong to the same block.

We choose $\bar{n} \geq n_0$ such that $\bar{\mathsf{d}} = \bar{n}\mathsf{d} > \rho s$. For each i, we can find $n_i > \rho s$ such that $a^{(n_i)}(i,j) > 0$ for some $j \in I_0$. Note that then we must have $j' \in I_0$ whenever $a^{(n_i)}(i,j') > 0$.

We now describe how to modify $\mathcal{T}$ to obtain the reduced tree $\overline{\mathcal{T}}$. Let ξ be any neighbour of ξ_0, and suppose that ξ has type i. Consider the branch of $\mathcal{T}$ that starts with ξ, that is, the subtree spanned by all descendants of ξ. Of that branch, we keep all those vertices (and their types) that are at distance $n_i + m\bar{\mathsf{d}}$ from ξ, where $m \geq 0$.

We now connect ξ_0 by a new oriented edge with those elements of the branch that are at distance n_i from ξ, and we connect each of the vertices that we have kept in our branch with each of its descendants at distance d in the original $\mathcal{T}$. We do this for every neighbour ξ of ξ_0. We obtain a new tree $\overline{\mathcal{T}}$ whose vertices as well as their types are also part of the original $\mathcal{T}$. By construction, all those vertices have their type in I_0.

The corresponding tree set $\overline{\mathcal{E}}$ consists of all $C \in \mathcal{E}$ whose terminal vertex in $\mathcal{T}$ belongs to $\overline{\mathcal{T}}$. By construction, $\overline{\mathcal{E}}$ tesselates X, is separated, and has finite type. Let $C \in \overline{\mathcal{E}}$ have type $i \in I_0$. If $j \in I_0$ then $a^{(\bar{\mathsf{d}})}(i,j) > 0$. This means that in the original $\mathcal{T}$, the cut C (as an edge) has a descendant D of type j at distance $\bar{\mathsf{d}}$ (that is, the endvertices of those edges of $\mathcal{T}$ have distance $\bar{\mathsf{d}}$). But then $D \in \overline{\mathcal{E}}$.

We now see that $\overline{\mathcal{E}}$ is irreducible, and its types are given by the set I_0.

(3) We first consider the finite set F and the structure tree $\mathcal{T} = \mathcal{T}_{\mathcal{E}}$. Recall the structure map $x \mapsto \xi(x)$ from X to $\mathcal{T}_{\mathcal{E}}$ that we have described after introducing the structure tree. We delete from $\mathcal{E}$ every cut C which, as an oriented edge of $\mathcal{T}$, lies on the geodesic path from ξ_0 to $\xi(x)$ for some $x \in F$. Only finitely many C are deleted. We obtain a new, smaller tree set $\overline{\mathcal{E}}$. The associated structure tree $\overline{\mathcal{T}}$ is obtained from $\mathcal{T}$ by contracting those geodesic paths to ξ_0, which we also consider as our root vertex of $\overline{\mathcal{T}}$. Then we get with respect to the new tree set

$$\partial_{\overline{\mathcal{E}}} C = \partial_{\mathcal{E}} C, \quad \text{if } C \in \overline{\mathcal{E}}, \quad \text{and} \quad \partial_{\overline{\mathcal{E}}} X = \partial_{\mathcal{E}} X \cup \bigcup \big\{ \partial_{\mathcal{E}} C : C \in \mathcal{E} \setminus \overline{\mathcal{E}} \big\}.$$

Thus, $\overline{\mathcal{E}}$ inherits from $\mathcal{E}$ all properties stated in the proposition, and $F \subset \partial_{\overline{\mathcal{E}}} X$.

So now let us assume that already $\mathcal{E}$ itself is such that $F \subset \partial_{\mathcal{E}} X$, and that $\mathcal{E}$ is ρ-separated, where $\rho \geq M$, with M as in (2.2).

For each $C \in \mathcal{E}$, we let $\widetilde{C} = \{x \in C : d(x, \partial C) \geq M\}$. Then $\partial \widetilde{C} = \{x \in C : d(x, \partial C) = M\}$. The subgraph $\widetilde{C}$ of X is not necessarily connected, but – having finite boundary – it will fall apart into infinite connected components $\widetilde{C}^{(1)}, \ldots, \widetilde{C}^{(k)}$, where $k = k(C)$ depends on C, plus maybe a finite number of finite components. Since ϕ is an isomorphism from C to C_i $(i \in \{1, \ldots, r\})$, it maps each $\widetilde{C}^{(l)}$ to $\widetilde{C}_i^{(l)}$ $(l = 1, \ldots, k)$ when those components are numbered accordingly.

Let $C, D \in \mathcal{E}$ with $C \to D$, and consider $\widetilde{D}^{(l')}$, where $l' \in \{1, \ldots, k(D)\}$. This is a connected subgraph of X which does not intersect $\partial \widetilde{C}$, while it does intersect $\widetilde{C}$. Therefore it must be contained in one of the components of $\widetilde{C}$, that is, in some $\widetilde{C}_i^{(l)}$ $(l \in \{1, \ldots, k(C)\})$. We infer that

$$\widetilde{\mathcal{E}} = \{\widetilde{C}^{(l)} : C \in \mathcal{E}, l = 1, \ldots, k(C)\}$$

is a ρ-separated, oriented tree set with finite type that tesselates X.

Next, suppose that $\mathcal{E}$ is irreducible. Consider $\widetilde{C}_i^{(l)}$, where $l \leq k(C_i)$, and let $j \in \{1, \ldots, r\}$. Since $\widetilde{C}_i^{(l)}$ is an infinite subgraph of X, it contains some element x that is sufficiently far from $\partial \widetilde{C}_i^{(l)}$ so that it is contained in some $D \in \mathcal{E}$ with $d(\partial D, \partial \widetilde{C}_i^{(l)}) > 0$. But then we must have $D \subset \widetilde{C}_i^{(l)}$. Now D contains some $C \in \mathcal{E}$ that is isomorphic with C_j. In turn, C contains $\widetilde{C}^{(l')}$ for all $l' \in \{1, \ldots, k(C)\}$, which are (respectively) isomorphic with $\widetilde{C}_j^{(l')}$ for $l' \in \{1, \ldots, k(C)\}$. Thus, $\widetilde{\mathcal{E}}$ is again irreducible.

Finally, we show connectedness of $\partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)} \in \widetilde{\mathcal{E}}$, where $C \in \mathcal{E}$. We fix some point x_0 in $\partial \widetilde{C}^{(l)}$ and show that every point $x \in \partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$ is connected to x_0 by a path that lies entirely within $\partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$.

Note first that – as we have seen above – when $\widetilde{C}^{(l)} \to \widetilde{D}^{(l')} \in \widetilde{\mathcal{E}}$, where $D \in \mathcal{E}$, then $\widetilde{C}^{(l)} \supset \widetilde{D}$.

(i) If $x \in D$ for such a cut $D \in \mathcal{E}$, then we have the following possibilities. Either $x \in D_M = \{y \in D : d(x, \partial D) < M\}$, in which case x is connected to some $y \in \partial D$ by a path that lies entirely within $D_M \subset \partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$. Or else x belongs to a finite connected component F of $\widetilde{D}$ (otherwise $x \notin \partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$). Then x is connected by a path within $F \subset \partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$ with some element of $\partial F \subset \partial \widetilde{D}$. The latter is connected to some element y of ∂D by a path of length M that lies within $D_M \cup \partial F$.

(ii) If $x \in \partial_{\mathcal{E}} C$ (with $d(x, \partial C) \geq M$) then x is connected to x_0 by some path that lies in $\widetilde{C}^{(l)}$. If that path does not exit from $\partial_{\mathcal{E}} C$ then it lies entirely within $\partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$, and we are done. Otherwise, it exits $\partial_{\mathcal{E}} C$ an enters some D with $C \to D \in \mathcal{E}$ and $\widetilde{C}^{(l)} \supset D$, as above. That is, x is connected to some $y \in \partial D$ by a path within $\partial_{\mathcal{E}} C$.

(iii) We are left with having to show that when $D \in \mathcal{E}$ is as above, and $y \in \partial D$, then y is connected to x_0 by a path within $\partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$. Now there is some path π from y to x_0 that lies within $\widetilde{C}^{(l)}$. This path may go back and forth between $\partial_{\mathcal{E}} C \cap \widetilde{C}^{(l)}$ and some D as above several times, and we use induction on the number of those crossings. If π goes from y directly into $\partial_{\mathcal{E}} C \cap \widetilde{C}^{(l)}$ without hitting any further point of D, then it lies within $\partial_{\mathcal{E}} \widetilde{C}^{(l)}$ as required. Otherwise, it makes one or more detours into some $D' \in \mathcal{E}$ with $C \to D'$ and $D' \subset \widetilde{C}^{(l)}$. (Initially, this may be our D for which $y \in \partial D$.) At each of those detours, π enters and exits D' at points $y', y'' \in \partial D'$, respectively. But $\mathrm{diam}(\partial D') \leq M$, so that y' and y'' are connected by a path that lies within $D'_M \subset \partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$. Thus, we may replace each of those detours with such a path within $\partial_{\widetilde{\mathcal{E}}} \widetilde{C}^{(l)}$, and in the end we get another path from y to x_0 that lies within $\partial_{\mathcal{E}} \widetilde{C}^{(l)}$, as required.

We should also show separately that with respect to $\widetilde{\mathcal{E}}$, the finite set $\partial_{\widetilde{\mathcal{E}}} X$ is connected. This follows the same reasoning as above, in particular (iii), and is omitted. □

Proposition 2.8. *Let G be finitely generated, H a subgroup with $[G : H] < \infty$, and K a subgroup of H. Furthermore, let $\psi : \mathbf{\Sigma} \to H$ and $\psi' : \mathbf{\Sigma}' \to G$ be symmetric semigroup presentations of H and G, respectively.*

Suppose that the Schreier graph $X = X(H, K, \psi)$ has a rooted, oriented tree set $\mathcal{E}$ that is irreducible. Then $X' = X(G, K, \psi')$ also has a tree set $\mathcal{E}'$ with the same properties.

Proof. Let $d(\cdot, \cdot)$ be the graph metric of $X = X(H, K, \psi)$ and $d'(\cdot, \cdot)$ the graph metric of $X' = X(G, K, \psi')$. The common "root vertex" of each of our Schreier graphs is $o = K$.

We start by observing that there are integers $M_1, M_2 \geq 1$, $N \geq 0$ that depend only on (H, ψ) and (G, ψ'), such that the following properties hold.

$$\begin{aligned} &\text{For every } x \in X' \text{ there is } y \in X \text{ with } d'(x, y) \leq N, \quad \text{and} \\ &\text{for all } y_1, y_2 \in X, \text{ one has } d'(y_1, y_2)/M_1 \leq d(y_1, y_2) \leq M_2\, d'(y_1, y_2). \end{aligned} \tag{2.9}$$

For the sake of clarity, we subdivide the proof in two steps.

Step 1. $H = G$. Then X and X' have the same vertex set, but different labelled edges which come from the two different symmetric semigroup presentations ψ, ψ' with associated alphabets $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$, respectively. In this case, $N = 0$.

We can assume that $\mathcal{E}$ is R-separated, where $R = 2M_2$.

Let C_i, $i \in I$, be the finitely many representatives of the isomorphism types of $\mathcal{E}$.

For any $C \in \mathcal{E}$, we now let $\widetilde{C} = \{x \in C : d(x, \partial C) \geq M_2/2\}$, where ∂C is the boundary with respect to the graph structure of X. We now consider $\widetilde{C}$ as an induced subgraph of X', that is, with those labelled edges that it inherits from X'.

Assume that C has type i, and let $\phi : C \to C_i$ be the corresponding isomorphism between the two as labelled subgraphs of X. We claim that the restriction $\widetilde{\phi}$ of ϕ to $\widetilde{C}$ is an isomorphism between $\widetilde{C}$ and $\widetilde{C}_i$ as labelled subgraphs of X'. It is clear that it is a bijection. Let $x, x' \in \widetilde{C}$ be connected by the edge (x, b, x') of X', where $b \in \mathbf{\Sigma}'$. Write $\bar{x} = \phi(x)$ and $\bar{x}' = \phi(x')$. There is a word $v_b \in \mathbf{\Sigma}^+$ with $|v_b| \leq M_2$ such that $\psi(v_b) = \psi'(b)$ in G. The unique path[1] $\pi_x(v_b)$ in X that starts at x and has label v_b has length at most M_2, so that it lies entirely within C. Therefore ϕ maps this path to the path $\pi_{\bar{x}}(v_b)$ which ends at $\bar{x}'$. But then we have the edge $(\bar{x}, b, \bar{x}')$ in the edge set of X'. This may become clearer in terms of cosets. We can write $\bar{x} = Kg$, $\bar{x}' = Kg'$ with $g, g' \in G$. If $\pi_{\bar{x}}(v_b)$ ends in $\bar{x}'$ then $Kg\psi'(b) = Kg\psi(v_b) = Kg'$.

Thus, $\widetilde{\phi} : \widetilde{C} \to \widetilde{C}_i$ is indeed an isomorphism.

Let (x, b, x') again be an edge of X'. Suppose that $x \in \widetilde{C}$ and $d(x, \partial C) \geq 3M_2/2$. Then $d(x', x) \leq M_2$, whence $d(x', \partial C) \geq M_2/2$. Therefore $x' \in C$. We conclude that the boundary $\partial'\widetilde{C}$ of $\widetilde{C}$ in X' is contained in the finite set $\{x \in C : M_2/2 \leq d(x, \partial C) < 3M_2/2\}$.

We continue in a way similar to the proof of Proposition 2.7(3). The subgraph $\widetilde{C}$ of X' is not necessarily connected, but – having a finite boundary – it will fall apart into infinite connected components $\widetilde{C}^{(1)}, \ldots, \widetilde{C}^{(k)}$, where $k = k(C)$ depends on C, plus maybe a finite number of finite components. Since $\widetilde{\phi}$ is an isomorphism from $\widetilde{C}$ to $\widetilde{C}_i$, it maps each $\widetilde{C}^{(l)}$ to $\widetilde{C}_i^{(l)}$ $(l = 1, \ldots, k)$ when those components are numbered accordingly.

Suppose that $C, D \in \mathcal{E}$ with $C \to D$. Let $\widetilde{D}^{(l')}$ be one of the components of $\widetilde{D}$ according to the above construction. This is a connected subgraph of $\widetilde{C}$. Therefore it is contained in one of the components of $\widetilde{C}$.

At this point, we see that

$$\mathcal{E}' = \{\widetilde{C}^{(l)} : C \in \mathcal{E},\ 1 \leq l \leq k(C)\}$$

is a rooted, oriented tree set of X' that has finite type.

Now suppose that $\mathcal{E}$ is irreducible. We show that also $\mathcal{E}'$ is irreducible. Consider $\widetilde{C}_i^{(l)}$. Since it is infinite, we can choose $x \in \widetilde{C}_i^{(l)}$ sufficiently far from $\partial'\widetilde{C}_i^{(l)}$ such that it is contained in some $C \in \mathcal{E}$ with $d(C, \partial'\widetilde{C}_i^{(l)}) \geq d(\partial C, \partial'\widetilde{C}_i^{(l)}) > M_1M_2$. Then, by (2.9), $d'(C, \partial'\widetilde{C}_i^{(l)}) > M_1$. Thus, the set $C^{M_1} = \{x \in X' : d'(x, C) \leq M_1\}$ induces a connected subgraph of X' that does not intersect $\partial'\widetilde{C}_i^{(l)}$, while it does intersect $\widetilde{C}_i^{(l)}$. Therefore it must be contained in D. In particular, $\widetilde{C}_i^{(l)} \supset C$ as sets. Given C_j with $j \in I$, by assumption there is $D \in \mathcal{E}$ such that $C \supsetneq D$, and D is isomorphic with C_j as a labelled subgraph of X. By construction, D contains each set $\widetilde{D}_j^{l'}$ $(l' = 1, \ldots, k(D))$. Above, we have shown that via the isomorphism $D \to C_j$ in X, the new cut $\widetilde{D}_j^{(l')}$ is isomorphic with $\widetilde{C}_j^{(l')}$ as a labelled subgraph of X'. Therefore $\mathcal{E}'$ is irreducible.

Step 2. We now assume that $X = X(H, K, \psi)$ is as stated in the proposition. By Step 1, we only need to show that $X' = X(G, K, \psi')$ has a tree set with the required properties for *some* symmetric semigroup presentation ψ' of G. Our choice of ψ' is as follows. We let $g_0 = 1_G, g_1, \ldots, g_m$ be representatives of the right cosets Hg $(g \in G)$ of H in G.

Then we define $\mathbf{\Sigma}' = \mathbf{\Sigma} \uplus \{b_1, b_1^{-1}, \ldots, b_m, b_m^{-1}\}$ and set $\psi'(a) = \psi(a)$, if $a \in \mathbf{\Sigma}$ and $\psi'(b_i^{\pm 1}) = g_i^{\pm 1}$ $(i = 1, \ldots, m)$. This is a symmetric semigroup presentation of G. For X and X', we have $N = 0$ and $M_1 = 1$ in (2.9). We write E and E' for the edge sets of X and X', respectively.

We work with a tree set $\mathcal{E}$ that satisfies the same assumptions as in Step 1, and is ρ-separated with $\rho = 6M_2$.

For $C \in \mathcal{E}$, we first define $\check{C} = \{x \in C : d(x, \partial C) \geq 3M_2/2\}$, and let

$$\widetilde{C} = \biguplus_{k=0}^{m} \{Khg_k : Kh \in \check{C}\} = \check{C} \cup \biguplus_{k=1}^{m} \{x \in X' : (y, b_k, x) \in E' \text{ for some } y \in \check{C}\}.$$

Now suppose again that $\phi : C \to C_i$ is an isomorphism of those cuts as labelled subgraphs of X. We construct $\widetilde{\phi} : \widetilde{C} \to \widetilde{C}_i$ in the only possible way. Namely, $\widetilde{\phi}(x) = \phi(x)$, if $x \in \check{C}$, while if x is such that $(y, b_k, x) \in E'$, where $y \in \check{C}$, then $\widetilde{\phi}(x) = \bar{x}$ is the unique element of X' such that $\big(\phi(y), b_k, \bar{x}\big) \in E'$. In terms of K-cosets, this means

$$\widetilde{\phi}(Khg_k) = K\bar{h}g_k, \quad \text{if } y = Kh \in \check{C} \text{ and } \phi(y) = K\bar{h}.$$

It is a straightforward exercise that $\widetilde{\phi}$ is well-defined and bijective. Now let $x, x' \in \widetilde{C}$ with $(x, a', x') \in E'$, where $a' \in \mathbf{\Sigma}'$. We need to show that also $(\bar{x}, a', \bar{x}') \in E'$, where $\bar{x} = \widetilde{\phi}(x)$ and $\bar{x}' = \widetilde{\phi}(x')$.

[1] It is here that we use crucially the assumption that X is fully deterministic: in this case, for every vertex x and every word $v \in \mathbf{\Sigma}^*$, there is a unique path $\pi_x(v)$ that starts at x and has label v.

There are $y, y' \in \check{C}$ and $k, l \in \{0, \dots, m\}$ such that $(y, b_k, x), (y', b_l, x') \in E'$ when $k \geq 1$, resp. $l \geq 1$, and $y = x$ when $k = 0$, resp. $y' = x'$ when $l = 0$. In particular, $g_k\, \psi'(\bar{a})\, g_l^{-1} \in H$, and there is a word $v \in \mathbf{\Sigma}^*$ such that $|v| \leq 3M_2$ and $\psi(v) = g_k\, \psi'(\bar{a})\, g_l^{-1}$. Let $\bar{y} = \phi(y) = \widetilde{\phi}(y)$ and $\bar{y}' = \phi(y') = \widetilde{\phi}(y')$. The path $\pi_y(v)$ from y to y' lies entirely within C. Therefore it is mapped by ϕ to the path $\pi_{\bar{y}}(v)$ from $\bar{y}$ to $\bar{y}'$, which lies entirely within C_i. By the construction of $\widetilde{\phi}$, we now get that in $\widetilde{C}_i$ we have the edge $(x, \bar{a}, x')$. Again, this can also be seen in terms of cosets: let $\bar{y} = K\bar{h}$. Then we have shown that $\bar{y}' = K\bar{h}'$, where $\bar{h}' = \bar{h}\, \psi(v)$, so that

$$\bar{x} = K\bar{h}\, g_k \quad \text{and} \quad \bar{x}' = K\bar{h}'\, g_l = K\bar{h}\, \psi(v)\, g_l = K\bar{h}\, g_k\, \psi'(\bar{a}).$$

The rest of the proof evolves almost precisely as in Step 1 and is omitted. □

We obtain the following important class of graphs that have "good" tree sets.

Theorem 2.10. *Suppose that $X = X(G, K, \psi)$, where G is virtually free, $\mathbb{K}$ is a finitely generated free subgroup of G, and ψ is is any symmetric semigroup presentation of G. Then X has a rooted, oriented tree set that is irreducible.*

Proof. The group G has a free subgroup $\mathbb{F}$ with $[G : \mathbb{F}] < \infty$ that contains $\mathbb{K}$. This follows from the discussion by Scott [24], or also from Bogopol'ski [1, Theorem 9.1 and proof].

Now, we have observed in [5, Corollary 5.6] that with respect to ψ given by the standard free generators of the free group $\mathbb{F}$ and their inverses, the Schreier graph $X(\mathbb{F}, \mathbb{K})$ is such that outside a finite set F, its cones are isomorphic with the cones of the standard Cayley graph of the free group. In this way, we obtain a tree set which consists just of those cones and inherits irreducibility from the tree associated with the free group.

We can now apply Proposition 2.8 to conclude the proof. □

3. Grammars associated with tree sets, and their dependency di-graphs

The *dependency di-graph* $\mathcal{D} = \mathcal{D}(\mathcal{C})$ of a context-free grammar $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ is an oriented graph with vertex set $\mathbf{V}$, with an edge from T to U (notation $T \to U$) if in $\mathbf{P}$ there is a production $T \vdash u$ with u containing U. (Compare e.g. with Kuich [16].) We write $T \xrightarrow{*} U$ if in $\mathcal{D}$ there is an oriented path of length ≥ 0 from T to U.

Consider the equivalence relation on $\mathbf{V}$ where $T \sim U$ if $T \xrightarrow{*} U$ and $U \xrightarrow{*} T$. The equivalence classes, denoted $\mathbf{V}(T)$ $(T \in \mathbf{V})$ are called the *strong components* of $\mathcal{D}(\mathcal{C})$. The strong components are partially ordered: $\mathbf{V}(T) \preccurlyeq \mathbf{V}(U)$ if there is an oriented path in $\mathcal{D}$ from T to U. A strong component is called *essential*, if it is maximal in the above order. The graph is called *strongly connected*, if the whole of $\mathbf{V}$ is a single strong component. (These notions make sense for any oriented graph.)

The grammar $\mathcal{C}$ (and the language it generates) is called *ergodic*, if $\mathcal{D}(\mathcal{C})$ is strongly connected. This notion was introduced by Ceccherini and Woess [3].

Assumptions 3.1. We consider an infinite symmetric, fully deterministic graph X that admits a rooted tree set $\mathcal{E}$ with finite type. By Proposition 2.7 we may assume without loss of generality that $\mathcal{E}$ is separated and that $\partial_{\mathcal{E}} X$ and all $\partial_{\mathcal{E}} C$ $(C \in \mathcal{E})$ are connected. We let $I = \{1, \dots, r\}$ be the set of types of cuts in $\mathcal{E}$. In addition, we set $C_0 = X$ and $\partial C_0 = \partial_{\mathcal{E}} C_0 = \partial_{\mathcal{E}} X$ and consider all maximal $C \in \mathcal{E}$ as successors of C_0 (that is, $C_0 \to C$).

In the proof of [5, Theorem 4.2], we have constructed a deterministic pushdown automaton that accepts L_{x_0,y_0}, where $x_0, y_0 \in F$, and F is the finite reference set with respect to which the cones are defined (not necessarily a single root vertex). Replacing F by $\partial_{\mathcal{E}} X$ and the cones with respect to F by the cuts in $\mathcal{E}$, we now construct an unambiguous grammar for the same language. We introduce the sets of variables

$$\mathbf{V}_0 = \{T_{x,y_0} : x \in \partial_{\mathcal{E}} C_0\}, \qquad \mathbf{V}_i = \{T_{x,y} : x \in \partial_{\mathcal{E}} C_i,\, y \in \partial C_i\} \quad (i \geq 1), \quad \text{and} \qquad \mathbf{V} = \bigcup_{i=0}^{r} \mathbf{V}_i\,. \tag{3.2}$$

For each $C \in \mathcal{E}$, there are $i \in \{1, \dots, r\}$ and an isomorphism $\phi_C : C \to C_i$. For every $x \in X \setminus \partial_{\mathcal{E}} X$, there is a unique $C \in \mathcal{E}$ such that $x \in \partial_{\mathcal{E}} C$. We set $\phi(x) = \phi_C(x) \in \bigcup_{i=1}^{r} \partial_{\mathcal{E}} C_i$.

The set of production rules $\mathbf{P}$ of our grammar is as follows. For all $a, b \in \mathbf{\Sigma}$, all $i, j \in \{0, \dots, r\}$ with $j \geq 1$, $x, x', x'' \in \partial_{\mathcal{E}} C_i$, $y \in \partial C_i$ (with $y = y_0$ when $i = 0$) and $\bar{x}, \bar{y} \in \partial C$, where $C_i \to C$ in $\mathcal{E}$ and C is isomorphic with C_j,

$$\begin{aligned} &T_{y,y} \vdash \epsilon \\ &T_{x,y} \vdash aT_{x',y} && \text{whenever } (x, a, x') \in E, \\ &T_{x,y} \vdash aT_{\phi(\bar{x}),\phi(\bar{y})} bT_{x'',y} && \text{whenever } (x, a, \bar{x}),\ (\bar{y}, b, x'') \in E. \end{aligned} \tag{3.3}$$

In passing, we observe that this grammar has the so-called operator normal form (the right hand sides of the productions contain no subword in $\mathbf{V}^2$).

Lemma 3.4. (a) *Let the start symbol be $T_{x,y} \in \mathbf{V}_i$, where $i \in \{0, \dots, r\}$. Then the grammar $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, T_{x,y})$, with $\mathbf{V}$ as in (3.2) and $\mathbf{P}$ as in (3.3), is unambiguous and generates the language*

$$L_{x,y}(C_i) = \{\ell(\pi) : \pi \in \Pi_{x,y} \text{ and } \pi \subset C_i\}.$$

(b) *The dependency digraph $\mathcal{D}(\mathcal{C})$ has the following properties.*

(1) *If $x, x' \in \partial_{\mathcal{E}} C_i$, $y \in \partial C_i$ (with $y = y_0$ when $i = 0$) then there is an oriented path in $\mathcal{D}(\mathcal{C})$ from $T_{x,y}$ to $T_{x',y}$.*
(2) *If $C_i \to C \in \mathcal{E}$, where C has type j, and if $x \in \partial_{\mathcal{E}} C_i$, $y \in \partial C_i$, $x' \in \partial_{\mathcal{E}} C_j$ and $y' \in \partial C_j$, then there isan oriented path in $\mathcal{D}(\mathcal{C})$ from $T_{x,y}$ to $T_{x',y'}$.*

(c) *In particular, if $\mathcal{E}$ is also irreducible, then the strong components of $\mathcal{D}(\mathcal{C})$ are $\mathbf{V}_0$ and $\mathbf{V}_{ess} = \mathbf{V}_1 \cup \cdots \cup \mathbf{V}_s$, and $\mathbf{V}_0 \preccurlyeq \mathbf{V}_{ess}$.*

Proof. (a) The language generated by $T_{x,y}$ is $L_{x,y}(C_i)$, because each non-trivial path from x to y within C_i (where $x \in \partial_{\mathcal{E}} C_i$ and $y \in \partial C_i$) decomposes uniquely in the following way.

- Either its first edge goes from $x \in \partial_{\mathcal{E}} C_i$ to some $x' \in \partial_{\mathcal{E}} C_i$, and it is followed by a path from x' to y within C_i,
- or else the first edge goes to a vertex $\bar{x} \in \partial C$, where $C_i \to C$ (and C is isomorphic with some C_j), this is followed by a path within C from $\bar{x}$ to some $\bar{y} \in \partial C$ and an edge from $\bar{x}$ to some $x'' \in \partial_{\mathcal{E}} C_i$, and then it terminates with a path from x'' to y within C_i. The reader is invited to draw a figure.

The grammar is unambiguous by its construction.

(b) Regarding (1), since $\partial_{\mathcal{E}} C_i$ is connected, there is a path in the graph X from x to x' that lies within $\partial_{\mathcal{E}} C_i$. Using the second type of the production rules in (3.3), we see that this translates into a path in the graph $\mathcal{D}(\mathcal{E})$ from $T_{x,y}$ to $T_{x',y}$. It lies within $\mathbf{V}_i$.

(2) Let C be a successor cone of C_i that has type j, and let $x' \in \partial_{\mathcal{E}} C_j$ and $y' \in \partial C_j$ be as stated. By (1), it is sufficient to consider only the case when $x' \in \partial C_j$. Let $\bar{x}, \bar{y} \in \partial C$ be the (unique) elements for which $\phi(\bar{x}) = x'$ and $\phi(\bar{y}) = y'$. There must be $x'', x''' \in \partial_{\mathcal{E}} C_i$ and $a, b \in \mathbf{\Sigma}$ such that $(x'', a, \bar{x})$, $(\bar{y}, b, x''') \in E$.

By (1), in $\mathcal{D}(\mathcal{C})$ there is a path from $T_{x,y}$ to $T_{x'',y}$. In our grammar, there is the production $T_{x'',y} \vdash aT_{x',y'}bT_{x''',y}$. Thus, in $\mathcal{D}(\mathcal{C})$ there is an edge from $T_{x'',y}$ to $T_{x',y'}$, and we also get a path from $T_{x,y}$ to $T_{x',y'}$. This proves (2) and completes (b).

(c) is an immediate consequence of (b). □

Corollary 3.5. *Let $\mathcal{E}$ be a rooted, oriented tree set consisting of cuts of the fully deterministic, symmetric labelled graph X. Suppose that $\mathcal{E}$ tesselates X and has finite type.*

Then $L_{x,y}$ is context-free for every $x, y \in X$.

In particular, X is a context-free graph in the sense of [22].

Remark 3.6. When we apply the above to a Schreier graph $X(G, K, \psi)$, the assumption of symmetry of ψ is not crucial. Suppose the graph has a tree set with the above properties for some (and hence every) symmetric semigroup presentation. If ψ is not symmetric then we can symmetrise it by adding further elements to $\mathbf{\Sigma}$ and defining an appropriate symmetric extension $\overline{\psi}$ of ψ such that each oriented edge of $X(G, K, \psi)$ is also an edge of $X(G, K, \overline{\psi})$, and for every pair of inverse edges of $X(G, K, \overline{\psi})$, at least one is an edge of $X(G, K, \psi)$. If we have a tree set for the symmetrised graph, then we can also use it for the original graph. That is, while our cuts come from $X(G, K, \overline{\psi})$, in the resulting grammar (3.3), we refer to the edge set E of $X(G, K, \psi)$. This grammar will generate the required languages $L_{x,y}$ for the original non-symmetric graph.

One problem, however, is that when the tree set of $X(G, K, \overline{\psi})$ is irreducible, it is not necessarily true that the grammar associated with $X(G, K, \psi)$ has the properties derived in Lemma 3.4(c). This is the primary obstacle when one wants to extend the results of the next sections directly to the non-symmetric case.

We can adopt the more general definition of a fully deterministic, symmetric graph X to be context-free when it has a tree set of finite type that tesselates X. We see that this is in reality equivalent with the definition in terms of cones with respect to a finite set F, or more specifically, with respect to a single point o.

The reason why we have embarked on the fatigue regarding cuts and tree sets is that we want to have a "good" grammar, that is, one as in Lemma 3.4(c). This will have an interesting application, as we shall see in the next section. The collection of all cones with respect to a given set F is of course a special case of a tree set. Our main problem was that when working just with cones, it is by no means clear how to show that irreducibility can in some suitable way be transferred to the cones of a symmetric Schreier graph $X(G, K, \psi')$ from those of $X(H, K, \psi)$, where $[G : H] < \infty$. For tree sets, we can do this, so that we get a "good" grammar, for example, for all symmetric Schreier graphs $X(G, \mathbb{K}, \psi)$, where G is a virtually free group and $\mathbb{K}$ is a finitely generated free subgroup of G. This applies, in particular, to Cayley graphs of virtually free groups.

4. Generating functions, tree sets, and random walks

Let $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ be a *non-ambiguous* context-free grammar. With each element $a \in \mathbf{\Sigma}$, we associate a positive real number $\mu(a)$. Furthermore, let z be a complex variable. We also associate a complex variable y_T with every $T \in \mathbf{V}$ and let $\mathbf{y} = (y_T)_{T \in \mathbf{V}}$, a row vector. Then we define

$$\begin{aligned} &\zeta(a) = \mu(a)z \quad \text{for } a \in \mathbf{\Sigma}, \qquad \zeta(T) = y_T \quad \text{for } T \in \mathbf{V}, \quad \text{and} \\ &\zeta(u) = \zeta(u_1)\cdots\zeta(u_n), \quad \text{if } u = u_1\cdots u_n \in (\mathbf{\Sigma} \cup \mathbf{V})^*. \end{aligned} \tag{4.1}$$

With the language L_T generated by $T \in \mathbf{V}$, we associate the generating function

$$f_T(z) = \sum_{w \in L_T} \zeta(w). \tag{4.2}$$

This defines an analytic function in a neighbourhood of the origin in $\mathbb{C}^{|\mathbf{\Sigma}|}$. Indeed, if $\mu(a)|z| < 1/|\mathbf{\Sigma}|$ for every $a \in \mathbf{\Sigma}$ then the series converges absolutely. With T, we also associate the polynomial

$$\mathfrak{P}_T(z; \mathbf{y}) = \sum_{u \,:\, T \vdash u} \zeta(u) \tag{4.3}$$

in $\mathbf{y}$ and z.

We can assume without loss of generality (up to a simple modification of the production rules that preserves unambiguity) that our grammar has no chain rules (productions of the form $T \vdash U$, where $T, U \in \mathbf{V}$).

Then the fundamental theory of Chomsky and Schutzenberger [6] implies that the functions $f_T(z)$ satisfy the system of equations

$$f_T(z) = \mathfrak{P}_T\big(z; f_U(z), U \in \mathbf{V}\big), \quad T \in \mathbf{V}, \tag{4.4}$$

where each variable y_U has been replaced by the function $f_U(z)$. Compare also with Kuich and Salomaa [17, Section 14].

We return to the Assumptions 3.1 regarding the graph X and the tree set $\mathcal{E}$, which is also assumed to be irreducible. From now on, the grammar $\mathcal{C}$ will always be the one of (3.3). We let $\mathbf{\Sigma}$ be the (symmetric) label alphabet of X. In (4.1), we assume that $\mu(a) > 0$ for all $a \in \mathbf{\Sigma}$. It is no loss of generality to require that $\mu(\mathbf{\Sigma}) = \sum_{a \in \mathbf{\Sigma}} \mu(a) = 1$; otherwise, we can just replace z with $z/\mu(\mathbf{\Sigma})$. Then μ gives rise to a *random walk* on X. This is the time-homogeneous Markov chain $(Z_n)_{n \geq 0}$ whose state space is X and whose one-step transition probabilities are

$$p(x, y) = \sum_{a \in \mathbf{\Sigma} \,:\, (x,a,y) \in E} \mu(a). \tag{4.5}$$

See the monograph by Woess [30] for an outline of the theory of random walks on infinite graphs and groups. We use the terminology of that reference.

We write $p^{(n)}(x, y) = \Pr[Z_n = y \mid Z_0 = x]$ for the probability that the random walk starting at x is in y after n steps. This is just the (x, y)-entry of the matrix power P^n, where $P = \big(p(x, y)\big)_{x,y \in X}$ is the transition matrix. The *Green function* of the random walk is

$$G(x, y \mid z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)\, z^n = G(x, y \mid z), \quad z \in \mathbb{C}. \tag{4.6}$$

It is a well known consequence of connectedness of X (i.e., irreducibility of the transition matrix) that its radius of convergence

$$R_\mu = 1/\limsup_n p^{(n)}(x, y)^{1/n} \tag{4.7}$$

is the same for all $x, y \in X$ and that either $G(x, y \mid R_\mu) < \infty$ for all x, y or $G(x, y \mid R_\mu) = \infty$ for all x, y.

For the variables of the grammar $\mathcal{C}$, the associated generating functions now have the following interpretations:

(1) If $T = T_{x,y_0} \in \mathbf{V}_0$, where $x, y_0 \in \partial_{\mathcal{E}} X$ then

$$f_T(z) = G(x, y_0 \mid z).$$

(2) If $T = T_{x,y} \in \mathbf{V}_{\text{ess}}$, where $x \in \partial_{\mathcal{E}} C_i$, $y \in \partial C_i$ with $i \geq 1$ then

$$f_T(z) = \sum_{n=0}^{\infty} p_{C_i}^{(n)}(x, y)\, z^n,$$

where $p_{C_i}^{(n)}(x, y)$ is the probability that the random walk starting at x is in y at the n-th step, without having left C_i .

For $T \in \mathbf{V}$, we write $\delta_T = 1$ if there is a production $T \vdash \epsilon$, and $\delta_T = 0$, otherwise. Then we have for our grammar

$$\mathfrak{P}_T(z; \mathbf{y}) = \delta_T + \sum_{T \vdash aU} \mu(a)\, z\, y_U + \sum_{T \vdash aVbU} \mu(a)\, \mu(b)\, z^2\, y_V\, y_U\,, \tag{4.8}$$

where the sums range over all productions of (3.3) whose left hand side is $T \in \mathbf{V}$.

We study analytic properties of the associated generating functions according to (4.1) and (4.2).

We first restrict the grammar to $\mathbf{V}_{\text{ess}}$, as defined in Lemma 3.4. That is, we keep only those productions whose left hand sides belong to $\mathbf{V}_{\text{ess}}$. Note that all variables occurring in its right hand sides also belong to $\mathbf{V}_{\text{ess}}$. We get a grammar $\mathcal{C}_{\text{ess}}$ that is unambiguous and ergodic, and there are productions whose right hand sides contain two variables (the grammar is *non-linear*).

Consider the associated system of Eqs. (4.4). Equations of this type occur quite frequently. We appeal to methods that have been developed in the context of random walks on free groups and trees by Lalley [18,20]; compare also with the exposition in Woess [30, Section 19.B] and with Nagnibeda and Woess [23]. In the more general context of analytic combinatorics, see the monographs of Flajolet and Sedgewick [12, Section VII.6] and Drmota [9, Section 2.2.5]. For the setting of generating functions associated with grammars, see also [3]. Appealing to [12, Theorem VII.6], we get the following.

Proposition 4.9. (a) *All the power series $f_T(z)$, $T = T_{x,y} \in \mathbf{V}_{ess}$, have the same radius of convergence R.*

(b) *One has $f_T(R) < \infty$, and $z = R$ is a simple (i.e., quadratic) branching point of each $f_T(z)$. There are functions $g_T(z)$ and $h_T(z)$, which near $z = R$ are analytic and real-valued for real z, such that $h_T(R) > 0$ and the identity*

$$f_T(z) = g_T(z) - h_T(z)\sqrt{R-z}$$

is valid in a neighbourhood of the point R in $\mathbb{C}$, except for real $z > R$.

Now that we have good information about the generating functions associated with the variables in $\mathbf{V}_{\mathrm{ess}}$, let us investigate how they determine the "remaining" functions $f_T(z)$, $T \in \mathbf{V}_0$. For such T, the polynomial $\mathfrak{P}_T(z; \mathbf{y})$ of (4.8) is such that of the variables U, V appearing there, one has $U \in \mathbf{V}_0$ and $V \in \mathbf{V}_{\mathrm{ess}}$. That is, the column vector

$$\mathbf{f}(z) = \big(f_T(z)\big)_{T\in\mathbf{V}_0}$$

is determined by the linear system

$$\mathbf{f}(z) = \mathbf{e} + Q(z)\mathbf{f}(z), \tag{4.10}$$

where $\mathbf{e} = \big(\delta_T\big)_{T\in\mathbf{V}_0}$ (another column vector), and $Q(z) = \big(q_{T,U}(z)\big)_{T,U\in\mathbf{V}_0}$ is the matrix over $\mathbf{V}_0$ with entries

$$q_{T,U}(z) = \sum_{T\vdash aU} \mu(a)\, z + \sum_{T\vdash aVbU} \mu(a)\,\mu(b)\, z^2 f_V(z), \tag{4.11}$$

where the first sum ranges over all $a \in \mathbf{\Sigma}$ such that we have the production $T \vdash aU$, and the second sum ranges over all $a, b \in \mathbf{\Sigma}$ and $V \in \mathbf{V}_{\mathrm{ess}}$ such that we have the production $T \vdash aUbV$. Note that there is at least one term of this last type, that is, $Q(z)$ contains at least one of the functions $f_V(z)$, $V \in \mathbf{V}_{\mathrm{ess}}$, in one of its terms.

Lemma 4.12. *For $0 < z \le R$, the non-negative matrix $Q(z)$ is irreducible and depends continuously on z (analytically for $0 < z < R$). Furthermore, in a complex neighbourhood of the point R, except for real $z > R$, we can decompose*

$$Q(z) = A(z) - \sqrt{R-z}\,B(z)$$

where the matrices $A(z)$ and $B(z)$ are analytic functions of z near R, and $B(R)$ is a non-negative matrix that does not vanish.

Proof. It is clear that our matrix is continuous for $0 \le z \le R$, resp. analytic for $0 \le z < R$. Now recall that each variable in $\mathbf{V}_0$ has the form $T = T_{x,y_0}$, where $x \in \partial_{\mathcal{E}} X$. We may as well replace each T with the corresponding x and index $\mathbf{f}(z)$, $\mathbf{e}(z)$ and $Q(z)$ accordingly. In particular, when $T = T_{x,y_0}$ and $U = T_{x',y_0}$ then $q_{T,U}(z) = q_{x,x'}(z)$, and we see that in this notation,

$$q_{x,x'}(z) \ge \bar{p}_{x,x'} z, \quad \text{where } \bar{p}_{x,x'} = \sum_{a\,:\,(x,a,x')\in E} \mu(a), \quad \text{if } x, x' \in \partial X. \tag{4.13}$$

But already the matrix $\overline{P} = \big(\bar{p}_{x,x'}\big)_{x,x'\in\partial_{\mathcal{E}} X}$ is irreducible, since by our construction, $\partial_{\mathcal{E}} X$ is a connected subgraph of X.

The decomposition of $Q(z)$ is an immediate consequence of (4.11) and Proposition 4.9. □

We shall need the following "side-product" of the Perron–Frobenius theory of non-negative matrices.

Lemma 4.14. *Let $M(z)$, $z \in (\alpha, \beta) \subset \mathbb{R}$, be irreducible, non-negative square matrices of the same dimension whose entries are real-analytic functions of z. Let $\lambda(z)$ be the largest positive eigenvalue of $M(z)$, and let $\mathbf{v}(z)^t$ and $\mathbf{w}(z)$ be the (strictly positive) left and right Perron–Frobenius eigenvectors, that is*

$$\mathbf{v}(z)^t M(z) = \lambda(z)\cdot\mathbf{v}(z)^t \quad \textit{and} \quad M(z)\mathbf{w}(z) = \lambda(z)\cdot\mathbf{w}(z),$$

normalised such that $\langle \mathbf{v}(z), \mathbf{1}\rangle = \langle \mathbf{v}(z), \mathbf{w}(z)\rangle = 1$. (Here, $\langle\cdot,\cdot\rangle$ is the inner product, and $\mathbf{1}$ is the vector with all entries $= 1$.)

Then $\lambda(z)$, $\mathbf{v}(z)$ and $\mathbf{w}(z)$ are analytic functions of $z \in (\alpha, \beta)$, and

$$\lambda'(z) = \mathbf{v}(z)^t M'(z)\mathbf{w}(z).$$

(Derivatives with respect to z.)

Proof. The fact that $\lambda(z)$, $\mathbf{v}(z)$ and $\mathbf{w}(z)$ are analytic functions is straightforward; compare e.g. with [20, Lemma 7.6]. Let $z, z_0 \in (\alpha, \beta)$. Then

$$\mathbf{v}(z_0)^t\big(M(z) - M(z_0)\big)\mathbf{w}(z) = \lambda(z) - \lambda(z_0).$$

Dividing by $z - z_0$ and letting $z \to z_0$, we get the formula for $\lambda'(z)$. □

Proposition 4.15. *For the Green function associated with the random walk, there is the following alternative between cases* (a), (b) *or* (c), *where R is as in* Proposition 4.9(a).

(a) *$R_\mu < R$, and for every pair $x, y \in X$, the singularity $z = R_\mu$ is a simple pole of $G(x, y \mid z)$.*

(b) *$R_\mu = R$, and for every pair $x, y \in X$, there are functions $g_{x,y}(z)$ and $h_{x,y}(z)$, which near $z = R$ are analytic and real-valued for real z, such that $h_{x,y}(R) > 0$ and the identity*

$$G(x, y \mid z) = g_{x,y}(z) + h_{x,y}(z)/\sqrt{R_\mu - z}$$

is valid in a complex neighbourhood of R_μ in $\mathbb{C}$, except for real $z > R_\mu$.

(c) *$R_\mu = R$, and for every pair $x, y \in X$, there are functions $g_{x,y}(z)$ and $h_{x,y}(z)$, which near $z = R$ are analytic and real-valued for real z, such that $h_{x,y}(R) > 0$ and the identity*

$$G(x, y \mid z) = g_{x,y}(z) - h_{x,y}(z)\,\sqrt{R_\mu - z}$$

is valid in a complex neighbourhood of R_μ in $\mathbb{C}$, except for real $z > R_\mu$.

Proof. The elements of the vector $\mathbf{f}(z)$ in Eq. (4.10) are the functions $G(x, y_0 \mid z)$, where $x \in \partial_{\mathcal{E}} X$. We shall apply the last lemma to $Q(z)$. For $0 \le z \le R$, let $\lambda(z)$ be its largest real eigenvalue. If $z = 0$ then $Q(z)$ is the zero matrix, so that $\lambda(0) = 0$. Otherwise, by Lemma 4.12, the Perron–Frobenius theorem applies, and $\lambda(z)$ is a simple zero of the characteristic polynomial $\chi(\lambda, z) = \det\big(\lambda \cdot I - Q(z)\big)$, where I is the identity matrix over $\mathbf{V}_0$.

For $z \in (0,\ R)$, we can write

$$\chi(1, z) = \chi(1, z) - \chi\big(\lambda(z), z\big) = \big(1 - \lambda(z)\big)\,\eta(z),$$

where $\eta(z)$ is analytic and non-zero.

Now let $A(z)$ be the "adjunct" matrix of $I - Q(z)$, whose (T, U)-entry is

$$(-1)^{\pm 1} \det\big(I - Q(z) \mid U, T\big),$$

with the row of U and column of T deleted from the matrix $I - Q(z)$, and the sign according to the parity of the position of (U, T). Then $A(z)$ is real-analytic for $z \in (0,\ R)$, and $\big(I - Q(z)\big)A(z) = \chi(1, z) \cdot I$.

(a) If $\lambda(R) > 1$ then there is a unique $z_0 \in (0,\ R)$ such that $\lambda(z_0) = 1$. For $z < z_0$, the matrix $I - Q(z)$ is invertible, whence $\mathbf{f}(z)$ is analytic.

Now note that $Q'(z) \ge \overline{P}$, where $\overline{P}$ is the irreducible matrix defined in the proof of Lemma 4.12. It is the restriction of the transition matrix of our random walk to $\partial_{\mathcal{E}} X$.

In particular, $\lambda'(z_0) > 0$ by Lemma 4.14. Therefore $1 - \lambda(z) = (z_0 - z)\gamma(z)$, where $\gamma(z)$ is analytic near $z = z_0$ and $\gamma(z_0) > 0$. We can write for z near z_0

$$\begin{aligned}
\big(I - Q(z)\big)^{-1} &= \frac{1}{\chi(1, z)} A(z) = \frac{1}{1 - \lambda(z)} \frac{1}{\eta(z)} A(z) \\
&= \frac{1}{z_0 - z} B(z), \quad \text{where} \\
B(z) = \frac{1}{\gamma(z)\eta(z)} A(z).
\end{aligned}$$

If z is close to z_0 then $A(z)$ is close to $A(z_0)$. This last matrix is strictly positive in each entry, since $A(z_0) = \frac{\partial}{\partial\lambda}\chi(1, z_0) \cdot \mathbf{w}(z_0)\mathbf{v}(z_0)^t$, where $\mathbf{v}(z_0)_t$ and $\mathbf{w}(z_0)$ are the left and right Perron–Frobenius eigenvectors of $Q(z_0)$, see [25]. Therefore, for real z close to z_0, the matrix $B(z)$ has all entries strictly positive. We obtain near z_0

$$\mathbf{f}(z) = \frac{1}{z_0 - z} B(z)\mathbf{e}.$$

Since R_μ is the smallest positive singularity of each of the functions $G(x, y \mid z)$, we see that $R_\mu = z_0$, which is a simple pole of $G(x, y_0 \mid z)$, where $x \in \partial_{\mathcal{E}} X$.

We shall argue at the end of this proof that the same must be true for all $x, y \in X$; compare once more with [20].

(b) Suppose next that $\lambda(R) = 1$. Then $I - Q(z)$ is invertible for all $z \in [0,\ R)$, and $\mathbf{f}(z)$ is analytic for all those z. Since $R_\mu \le R$ always, we have $R_\mu = R$.

Now we substitute $\mathfrak{z} = \sqrt{R - z}$ (for complex z, unless $z > R$ is real). Then the expansion of Proposition 4.9(b), for $V \in \mathbf{V}_{\mathrm{ess}}$ instead of T, can be written as

$$f_V(z) = \widetilde{f}_V\left(\sqrt{R - z}\right), \quad \text{where } \widetilde{f}_V(\mathfrak{z}) = \widetilde{g}_V(\mathfrak{z}) - \mathfrak{z}\,\widetilde{h}_V(R - \mathfrak{z}^2) \text{ with}$$

$$\widetilde{g}_V(\mathfrak{z}) = g_V(R - \mathfrak{z}^2) \quad \text{and} \quad \widetilde{h}_V(\mathfrak{z}) = h_V(R - \mathfrak{z}^2).$$

$\widetilde{f}_V(\mathfrak{z})$ is an analytic function near the origin, and $\widetilde{f}_V'(0) = -h_T(R) < 0$. When we carry over this substitution to the matrix $Q(z)$, then we find near $z = R$ that $Q(z) = \widetilde{Q}\left(\sqrt{R-z}\right)$, where the matrix $\widetilde{Q}(\mathfrak{z})$ is analytic near $\mathfrak{z} = 0$, non-negative irreducible for $\mathfrak{z} > 0$, and

$$\widetilde{Q}'(0) = \left(\widetilde{q}'_{T,U}(0)\right)_{T,U\in\mathbf{V}_0} \quad \text{with } \widetilde{q}'_{T,U}(0) = -\sum_{T\vdash aVbU} \mu(a)\mu(b)\, R^2\, h_V(R).$$

As above, the last sum is over all productions $T \vdash aVbU$ with $a, b \in \mathbf{\Sigma}$, $V \in \mathbf{V}_{\text{ess}}$. Since there is at least one such production for some $T, U \in \mathbf{V}_0$, we see that $\widetilde{Q}'(0)$ is non-positive and strictly negative in some entry. Lemma 4.14 implies that $\widetilde{\lambda}'(0) < 0$, where $\widetilde{\lambda}(\mathfrak{z})$ is the largest positive eigenvalue of the non-negative matrix $\widetilde{Q}(\mathfrak{z})$ for $\mathfrak{z} > 0$.

Arguing precisely as in case (a), we now deduce that for $\mathfrak{z}$ near 0,

$$\left(I - \widetilde{Q}(\mathfrak{z})\right)^{-1} = \frac{1}{\mathfrak{z}}\widetilde{B}(\mathfrak{z}),$$

where $\widetilde{B}(\mathfrak{z})$ is a matrix with analytic entries that are strictly positive when $\mathfrak{z} \geq 0$. Re-substituting $\mathfrak{z} = \sqrt{R-z}$, we obtain the desired expansion for $G(x, y_0 \mid z)$ for $x \in \partial_{\mathcal{E}} X$.

(c) If $\lambda(R) < 1$ then $R_\mu = R$ by the same reason as in case (b). We make the same substitution as in case (b) but observe that $I - Q(z)$ is invertible for all $z \in [0, R]$, so that $I - \widetilde{Q}(\mathfrak{z})$ is invertible for all $\mathfrak{z}$ close to 0, and

$$\left(I - \widetilde{Q}(\mathfrak{z})\right)^{-1} = \widetilde{B}(\mathfrak{z}) = \sum_{n=0}^{\infty} \widetilde{Q}(\mathfrak{z})^n.$$

This is a strictly positive matrix with analytic entries, and once more $\mathbf{f}(z) = \widetilde{B}\left(\sqrt{R-z}\right)\mathbf{e}$. In order to make sure that in the proposed singular expansion, we really have $h_{x,y_0}(R) > 0$, we must check that the matrix $\widetilde{B}'(0)$ is strictly negative in each entry:

$$\widetilde{B}'(0) = \left(\sum_{n=1}^{\infty} n\widetilde{Q}(0)^{n-1}\right)\widetilde{Q}'(0). \tag{4.16}$$

The matrix sum in the parentheses is absolutely convergent and, by irreducibility of $\widetilde{Q}(0)$, strictly positive in each entry. We have seen above that $\widetilde{Q}'(0)$ is non-positive and strictly negative in some entry.

We conclude that the alternative between cases (a)–(c) holds for all Green functions $G(x, y_0 \mid z)$, where $x \in \partial_{\mathcal{E}} X$. Given arbitrary $x, y \in X$, we know that we can modify $\mathcal{E}$ so that $x, y \in \partial_{\mathcal{E}} X$, and we can set $y_0 = x$.

Thus, for every individual choice of $x, y \in X$, one of the three singular expansions of (a), (b) or (c) is valid. For the following argument, compare once more with [20].

Let $x, y, x', y' \in X$. Then, by connectedness of X, there are $k, l \in \mathbb{N}$ such that $p^{(k)}(x, x') > 0$ and $p^{(l)}(y', y) > 0$, and $p^{(k+n+l)}(x, y) \geq p^{(k)}(x, x')p^{(n)}(x', y')p^{(l)}(y', y)$ for all n. Therefore, setting $C = C(x, y, x', y') = p^{(k)}(x, x')p^{(l)}(y', y)/2^{k+l}$, we have $C > 0$ and

$$G(x, y \mid z) \geq C\, G(x', y' \mid z) \quad \text{for } 1/2 \leq z < R.$$

An analogous inequality holds when x, y is exchanged with x', y'. That is,

$$0 < \lim_{z\to R} G(x, y \mid z)/G(x', y' \mid z) < \infty.$$

Thus, $G(x, y \mid z)$ and $G(x', y' \mid z)$ cannot have different expansions among the three possible types. □

Set $\mathsf{d} = 2$ if X is a bipartite graph (i.e., it has no odd cycles ≡ closed paths with odd length), and $\mathsf{d} = 1$, otherwise. This is the *period* of the random walk. Then for every $x, y \in X$, we have that $p^{(n)}(x, y) > 0$ for all but finitely many n for which d divides $n - d(x, y)$. The *strong period* is

$$\mathsf{d}_s = \gcd\{n \in \mathbb{N} : \inf_{x\in X} p^{(n)}(x, x) > 0\},$$

if the latter set is non-empty. In our case, we always have $p^{(2)}(x, x) \geq \sum_{a\in\mathbf{\Sigma}} \mu(a)\mu(a^{-1})$, so that $\mathsf{d}_s \in \{1, 2\}$.

Lemma 4.17. *We have $\mathsf{d}_s = 1$ if and only if there is a cut $C_i \in \mathcal{E}$ $(i \geq 1)$ that contains an odd cycle.*

Proof. If $\mathsf{d}_s = 1$ then there is an odd m such that every vertex lies on a cycle with length m. If x lies in C_i and $d(x, \partial C_i) \geq m/2$ then that cycle must lie within C_i.

Conversely, assume that some $C_i \in \mathcal{E}$ $(i \geq 1)$ contains an odd cycle. Let r be its length. There is m_1 such that every $y \in \partial C_i$ is at distance at most m_1 from that cycle.

Since $\mathcal{E}$ is irreducible and tesselates X, there is m_2 such that for every $x \in X$ there is a cone C of type i such that $d(x, \partial C) \leq m_2$. We can construct a path from x to a copy of our cycle within C which has length $s \leq m_1 + m_2$. Thus,

x lies on a cycle (possibly with repeated edges) with length $r + 2s$. By going back and forth along some edge, this can be extended to a cycle with length $m = r + 2(m_1 + m_2)$, which is odd. But then $p^{(m)}(x, x) \geq \big(\min_{a\in\Sigma} \mu(a)\big)^m$ for every x. Therefore $\mathsf{d}_s = 1$. □

Thus, we have three possible cases: $\mathsf{d} = \mathsf{d}_s = 1$ (in the situation of Lemma 4.17), $\mathsf{d} = \mathsf{d}_s = 2$ (when X is a bipartite graph), or $\mathsf{d} = 1$ and $\mathsf{d}_s = 2$.

A result of Cartwright [2] applies, see [30, Theorem 9.4 and page 96]. It yields the following.

Lemma 4.18. *The only singularities of each of the functions $G(x, y \mid z)$ on the circle of convergence $\{z : |z| = R_\mu\}$ of these power series are R_μ and possibly $-R_\mu$. In addition,*

(a) *if $\mathsf{d} = \mathsf{d}_s = 1$ then R_μ is the only singularity, and*
(b) *If $\mathsf{d} = \mathsf{d}_s = 2$ then both R_μ and $-R_\mu$ are singularities.*

In the second of those cases, $G(x, y \mid -z) = (-1)^{d(x,y)}\, G(x, y \mid\ z)$, so that the singular expansion of $G(x, y \mid z)$ at $z = -R_\mu$ is immediate from the one at $z = R_\mu$, as described in Proposition 4.15. In the remaining case, this is more complicated.

Proposition 4.19. *Suppose that $\mathsf{d} = 1$ and $\mathsf{d}_s = 2$ for our random walk. Then for all $x, y \in X$, the Green function has the following expansion at $z = -R_\mu$.*

(1) *If $R_\mu < R$, then $G(x, y \mid z)$ is analytic at $z = -R_\mu$.*
(2) *If $R_\mu = R$, then there are functions $\bar g_{x,y}(z)$ and $\bar h_{x,y}(z)$, which near $z = -R_\mu$ are analytic and real-valued for real z, such that the identity*
$$G(x, y \mid z) = \bar g_{x,y}(z) - \bar h_{x,y}(z)\sqrt{R_\mu + z}$$
is valid in a complex neighbourhood of $-R_\mu$ in $\mathbb{C}$, except for real $z < -R_\mu$.

Furthermore, in case (c) *of Proposition 4.15, one has $|\bar h_{x,y}(-R_\mu)| < h_{x,y}(R_\mu)$.*

Proof. We continue to use the notation of the proof of Proposition 4.15.

Let $x, y \in X$. We know from Lemma 4.17 that each cut C_i , $i \geq 1$, is bipartite, while there is some odd cycle in X. By Proposition 2.7(3), we can assume without loss of generality that $\mathcal{E}$ is such that this odd cycle as well as x, y are contained in $\partial_{\mathcal{E}}X$. We choose $y_0 = y$.

Claim. *For every $z \in \mathbb{C}$ with $|z| \leq R$ and $z \neq |z|$, each eigenvalue λ of $Q(z)$ satisfies $|\lambda| < \lambda(|z|)$.*

Proof. Let z be as stated. From (4.11), we see that $|q_{T,U}(z)| \leq q_{T,U}(|z|)$ for all $T, U \in \mathbf{V}_0$. It follows that $|\lambda| \leq \lambda(|z|)$ for every eigenvalue λ of $Q(z)$.

So let $\lambda \in \mathbb{C}$ with $|\lambda| = \lambda(|z|)$. The claim will be proved when we show that $\lambda I - Q(z)$ is invertible.

Consider the matrix $\overline{P}$ of (4.13). Since $\partial_{\mathcal{E}}X$ is connected and contains an odd cycle, this is a *primitive* matrix, that is, there is n_0 such the $\overline{P}^n$ is strictly positive in each entry for all $n \geq 0$. Let $\bar\lambda$ be the Perron–Frobenius eigenvalue of $\overline{P}$. Once more by the Perron–Frobenius theorem, all its other eigenvalues have absolute value $< \bar\lambda$.

For all z, we can decompose $Q(z) = z\,\overline{P} + \overline{Q}(z)$, where the matrix $\overline{Q}(z)$ is non-zero and non-negative for $z > 0$. We have $|z|\,\overline{P} \leq Q(|z|)$ elementwise, and the inequality is strict in some entries. Therefore $|z|\bar\lambda < \lambda(|z|)$, and $\lambda I - z\,\overline{P}$ is invertible, with
$$\big(\lambda I - z\,\overline{P}\big)^{-1} = \sum_{n=0}^{\infty} \frac{z^n}{\lambda^{n+1}}\overline{P}^n.$$
Since all entries of $\overline{P}^n$ are >0 for $n \geq n_0$, we have the strict triangle inequality
$$\Big|\big(\lambda I - z\,\overline{P}\big)^{-1}\Big| < \sum_{n=0}^{\infty} \frac{|z|^n}{|\lambda|^{n+1}}\overline{P}^n = \big(\lambda(|z|)\, I - |z|\,\overline{P}\big)^{-1}.$$
Here and below, the absolute value is meant to be taken matrix-element-wise. Now we write
$$\lambda I - Q(z) = \big(\lambda I - z\,\overline{P}\big)\Big(I - \big(\lambda I - z\,\overline{P}\big)^{-1}\overline{Q}(z)\Big).$$
By the above strict inequality,
$$\Big|\big(\lambda I - z\,\overline{P}\big)^{-1}\overline{Q}(z)\Big| < \big(\lambda(|z|)\, I - |z|\,\overline{P}\big)^{-1}\overline{Q}(|z|),$$
a positive matrix whose Perron–Frobenius eigenvalue is 1 (and the associated eigenvector is $\mathbf{w}(|z|)$, the Perron–Frobenius eigenvector of $Q(|z|)$). We see that every eigenvalue of $\big(\lambda I - z\,\overline{P}\big)^{-1}\overline{Q}(z)$ has absolute value <1. Thus, $\lambda I - Q(z)$ as a product of two invertible matrices is invertible, proving the claim.

We conclude that for $z \in [-R_\mu, 0]$, every eigenvalue λ of $Q(z)$ satisfies $|\lambda| < 1$. Therefore $I - Q(z)$ is invertible, and the inverse matrix is $\sum_n Q(z)^n$.

In case (1), it follows that $\big(I - Q(z)\big)^{-1}$ is analytic at $z = -R_\mu$, whence (4.10) yields that each function $G(x, y \mid z)$, where $x \in \partial_{\mathcal{E}} X$ and $y = y_0$, is analytic at $-R_\mu$. Arguing as at the end of the proof of Proposition 4.15, it follows that the same is true for arbitrary $x, y \in X$.

In case (2), we substitute $\mathfrak{z} = \sqrt{R+z}$ (for complex z, unless $z < -R$ is real). Then, again as in the the proof of Proposition 4.15, and using the same notation, we have for these z that $Q(z) = \overset{\approx}{Q}\big(\sqrt{R+z}\big)$, where the matrix $\overset{\approx}{Q}(\mathfrak{z})$ is analytic and $I - \overset{\approx}{Q}(\mathfrak{z})$ is invertible near $\mathfrak{z} = 0$. We get

$$\Big(I - \overset{\approx}{Q}(\mathfrak{z})\Big)^{-1} = \overset{\approx}{B}(\mathfrak{z}) = \sum_{n=0}^{\infty} \overset{\approx}{Q}(\mathfrak{z})^n,$$

a matrix with analytic entries, and once more $\mathbf{f}(z) = \overset{\approx}{B}\big(\sqrt{R+z}\big)\,\mathbf{e}$. This yields the proposed singular expansion at $-R$.

In order to show that $|\bar{h}_{x,y}(-R_\mu)| < h_{x,y}(R_\mu)$ in case (c) of Proposition 4.15, it is sufficient to show that one has the element-wise strict inequality $\Big|\overset{\approx}{B}{}'(0)\Big| < -\widetilde{B}'(0)$, where $\widetilde{B}'(0)$ is as in (4.16). Observe that the functions $f_V(z)$, $V \in \mathbf{V}_{\mathrm{ess}}$ are either even or odd, because each $C \in \mathcal{E}$ is bipartite, that is, $f_V(-z) = \epsilon_V f_V(z)$ with $\epsilon_V \in \{\pm 1\}$. It is then a straightforward task to compute

$$\overset{\approx}{q}{}'_{T,U}(0) = -\sum_{T \vdash aVbU} \mu(a)\mu(b)\, R^2\, \epsilon_V\, h_V(R).$$

By primitivity of $\widetilde{Q}(0) = Q(R)$, and since $\overset{\approx}{Q}(0) = Q(-R)$, we have once more the strict triangle inequality

$$\left|\sum_{n=1}^{\infty} n\, \overset{\approx}{Q}(0)^{n-1}\right| < \sum_{n=1}^{\infty} n\widetilde{Q}(0)^{n-1}.$$

The required inequality follows.

So far, this proof applies to $G(x, y \mid z)$ for all $x \in \partial_{\mathcal{E}} X$ and $y = y_0$. Since the alternative between the three cases of Proposition 4.15 has been shown to hold for *all* $x, y \in X$, we get that also the alternative for the expansions at $-R_\mu$ is the same for all x, y. $\square$

Now that we know the behaviour of $G(x, y \mid z)$ at all its singularities (i.e., at R_μ and possibly $-R_\mu$) on the circle of convergence, the classical method of Darboux implies the following, compare with [18] (and various other references).

Theorem 4.20. *Suppose that the fully deterministic, symmetric labelled graph X admits a rooted, oriented tree set $\mathcal{E}$ which tesselates X and is irreducible. Let μ be a probability measure supported by Σ. Then one of the following alternatives holds for the associated random walk.*

(1) $p^{(n)}(x, y) \sim c_{x,y}\, R_\mu^{-n}$ *for all* $x, y \in X$, *or*

(2) $p^{(n)}(x, y) \sim c_{x,y}\, R_\mu^{-n}\, n^{-1/2}$ *for all* $x, y \in X$, *or*

(3) $p^{(n)}(x, y) \sim \big(c_{x,y} + (-1)^n \bar{c}_{x,y}\big)\, R_\mu^{-n}\, n^{-3/2}$ *for all* $x, y \in X$,

as $n \to \infty$ *and* d *divides* $n - d(x, y)$.

Here, $\sim$ is asymptotic equivalence of sequences, that is, quotients tend to 1. We have $c_{x,y} > 0$ and $|\bar{c}_{x,y}| \le c_{x,y}$. Furthermore, $\bar{c}_{x,y} = 0$ when $\mathsf{d}_s = \mathsf{d}$.

We remark that one will be able to obtain analogous random walk asymptotics on pairs of groups whose symmetric Schreier graphs have irreducible tree sets, also when the semigroup presentation ψ and the set $S = \psi(\mathbf{\Sigma})$ are *not symmetric*. There are some technical points that appear to make this quite laborious and space-consuming when one follows the tree set approach, compare with Remark 3.6. On the other hand, here we have made an additional effort in order to clarify what happens when the random walk is not necessarily strongly aperiodic (the latter means that $\mathsf{d}_s = 1$). In fact, those situations can arise quite naturally when one considers general Schreier graphs.

Theorem 2.10 yields the following.

Corollary 4.21. *If ψ is a symmetric semigroup presentation of the virtually free group G, and K is a finitely generated subgroup of G, then for every random walk* (4.5) *on the Schreier graph $X(G, K, \psi)$, the transition probabilities satisfy* (1), (2) *or* (3) *of Theorem* 4.20.

Proof. Let $\mathbb{F}$ be a free subgroup of G with $[G : \mathbb{F}] < \infty$, and let $\mathbb{K} = \mathbb{F} \cap K$. Then $k = [K : \mathbb{K}] < \infty$, and $\mathbb{K}$ is free. By Theorem 2.10, the Schreier graph $X = X(G, \mathbb{K}, \psi)$ satisfies all hypotheses of Theorem 4.20. Let $\overline{X} = X(G, K, \psi)$. Write $o = \mathbb{K}$ and $\bar{o} = K$ for the root vertices of X and $\overline{X}$, respectively. For $x = \mathbb{K}g \in X$, we have the natural projection $\sigma : X \to \overline{X}$, $x \mapsto \bar{x} = Kg \in \overline{X}$, which is k-to-one. Let $g_1, \ldots, g_k$ be representatives of the right $\mathbb{K}$-cosets in K.

Then we have

$$L_{\bar{o},\bar{x}} = \{w \in \mathbf{\Sigma}^* : \psi(w) \in K\} = \{w \in \mathbf{\Sigma}^* : \psi(w) \in Kg_i \text{ for some } i \le k\} = \biguplus_{x \in X\,:\,\sigma(x)=\bar{x}} L_{o,x}.$$

Therefore the transition probabilities $\bar{p}^{(n)}(\bar{o}, \bar{x})$ of the random walk on $\bar{X}$ and $p^{(n)}(o, x)$ of the random walk on X are related by

$$\bar{p}^{(n)}(\bar{o}, \bar{x}) = \sum_{x \in X\,:\,\sigma(x)=\bar{x}} p^{(n)}(o, x),$$

a sum over k elements. Thus, $p^{(n)}(\bar{o}, \bar{x})$ inherits the asymptotic behaviour of the $p^{(n)}(o, x)$, which has one of the three above types. Since we can use any vertex as the root (which just means passing to a conjugate of K), the result follows. □

It is an easy exercise to construct examples (G, K) where $X(G, K)$ satisfies the assumptions of Theorem 4.20, but K is not finitely generated.

By abuse of notation, we also consider μ as a probability measure on G. In reality, in the setting that we have chosen, this is the image under the mapping ψ of the measure μ on $\mathbf{\Sigma}$. On G, the support is the symmetric set $A = \psi(\mathbf{\Sigma})$, but μ itself is not necessarily symmetric.

Corollary 4.22. *Let G be a finitely generated, virtually free group that is not virtually cyclic, and μ a probability measure on G whose support is finite, symmetric, and generates G. Then $R_\mu > 1$, and the transition probabilities of the associated random walk satisfy*

$$p^{(n)}(x, y) \sim c_{x,y}\, R_\mu^{-n}\, n^{-3/2} \quad \textit{for all } x, y \in G,$$

as $n \to \infty$ (taking into account the parity of n, when the period is $\mathsf{d} = 2$).

Proof. The fact that $R_\mu > 1$ follows from the non-amenability of G via a famous theorem of Kesten [15]. See e.g. [30, Corollary 12.5].

It is known from a theorem of Guivarc'h [13] that the asymptotics of (1) and (2) in Theorem 4.20 cannot occur on a virtually free group that is not virtually cyclic. See e.g. [30, Theorem 7.8]. □

As mentioned in the Introduction, this is closely related to the random walks on regular languages of [20]. Indeed, it is quite clear that a random walk on a virtually free group can be interpreted in these terms, since those groups have a normal form that is regular. (Underneath, there is a relation between context-free graphs and finite state automata, compare also with [4, Section 5].) In order to apply the result of [20] to prove Theorem 4.20, one needs to provide a suitable regular normal form and then to work out that the specific irreducibility hypotheses of [20] are satisfied, a task that appears to have the same quintessence as proving the existence of a tree set which is irreducible, as we have done.

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