I. Coherent Sequences

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0. Introduction

A transfinite sequence $C_\xi \subseteq \xi$ ($\xi < \theta$) of sets may have a number of 'coherence properties' and the purpose of this chapter is to study some of them as well as some of their uses. 'Coherence' usually means that the C_ξ's are cho-

sen in some canonical way, beyond the natural requirement that C_ξ is closed and unbounded in ξ for all ξ. For example, choosing a canonical 'fundamental sequence' of sets $C_\xi \subseteq \xi$ for $\xi < \varepsilon_0$ relying on the specific properties of the Cantor normal form for ordinals below the first ordinal satisfying the equation $x = \omega^x$ is a basis for a number of important results in proof theory. In set theory, one is interested in longer sequences as well and usually has a different perspective in applications, so one is naturally led to use some other tools beside the Cantor normal form. It turns out that the sets C_ξ can not only be used as 'ladders' for climbing up in recursive constructions but also as tools for 'walking' from an ordinal to a smaller one. This notion of a 'walk' and the corresponding 'distance functions' constitute the main body of study in this chapter. We show that the resulting 'metric theory of ordinals' not only provides a unified approach to a number of classical problems in set theory but also has its own intrinsic interest. For example, from this theory one learns that the triangle inequality of an ultrametric

$$e(\alpha, \gamma) \leq \max\{e(\alpha, \beta), e(\beta, \gamma)\}$$

has three versions, depending on the natural ordering between the ordinals α, β and γ, that are of a quite different character and are occurring in quite different places and constructions in set theory. For example, the most frequent occurrence is the case $\alpha < \beta < \gamma$ when the triangle inequality becomes something that one can call 'transitivity' of e. Considerably more subtle is the case $\alpha < \gamma < \beta$ of this inequality. It is this case of the inequality that captures most of the coherence properties found in this article. Another thing one learns from this theory is the special role of the first uncountable ordinal in this theory. Any natural coherency requirement on the sets C_ξ ($\xi < \theta$) that one finds in this theory is satisfiable in the case $\theta = \omega_1$. The first uncountable cardinal is the only cardinal on which the theory can be carried out without relying on additional axioms of set theory. The first uncountable cardinal is the place where the theory has its deepest applications as well as its most important open problems. This special role can perhaps be explained by the fact that many set-theoretical problems, especially those coming from other fields of mathematics, are usually concerned only about the duality between the countable and the uncountable rather than some intricate relationship between two or more uncountable cardinalities. This is of course not to say that an intricate relationship between two or more uncountable cardinalities may not be a profitable detour in the course of solving such a problem. In fact, this is one of the reasons for our attempt to develop the metric theory of ordinals without restricting ourselves only to the realm of countable ordinals.

The chapter is organized as a discussion of four basic distance functions on ordinals, ρ, ρ_0, ρ_1 and ρ_2, and the reader may choose to follow the analysis of any of these functions in various contexts. The distance functions will naturally lead us to many other derived objects, most prominent of

which is the 'square-bracket operation' that gives us a way to transfer the quantifier 'for every unbounded set' to the quantifier 'for every closed and unbounded set'. This reduction of quantifiers has proven to be quite useful in constructions of various mathematical structures, some of which have been mentioned or reproduced here.

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1. Walks in the space of countable ordinals

The space ω_1 of countable ordinals is by far the most interesting space considered in this chapter. There are many mathematical problems whose combinatorial essence can be reformulated as problems about ω_1, which is in some sense the smallest uncountable structure. What we mean by 'structure' is ω_1 together with system C_α ($\alpha < \omega_1$) of fundamental sequences, i.e. a system with the following two properties:

(a) $C_{\alpha+1} = \{\alpha\}$,

(b) C_α is an unbounded subset of α of order-type ω, whenever α is a countable limit ordinal > 0.

Despite its simplicity, this structure can be used to derive virtually all known other structures that have been defined so far on ω_1. There is a natural recursive way of picking up the fundamental sequences C_α, a recursion that refers to the Cantor normal form which works well for, say, ordinals $< \varepsilon_0$[1]. For longer fundamental sequences one typically relies on some other principles of recursive definitions and one typically works with fundamental sequences with as few extra properties as possible. We shall see that the following assumption is what is frequently needed and will therefore be implicitly assumed whenever necessary:

(c) if α is a limit ordinal, then C_α does not contain limit ordinals.

1.1 Definition. A *step* from a countable ordinal β towards a smaller ordinal α is the minimal point of C_β that is $\geq \alpha$. The cardinality of the set $C_\beta \cap \alpha$, or better to say the order-type of this set, is the *weight* of the step.

1.2 Definition. A *walk* (or a *minimal walk*) from a countable ordinal β to a smaller ordinal α is the sequence $\beta = \beta_0 > \beta_1 > \ldots > \beta_n = \alpha$ such that for each $i < n$, the ordinal β_{i+1} is the step from β_i towards α.

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[1] One is tempted to believe that the recursion can be stretched all the way up to ω_1 and this is probably the way P.S. Alexandroff has found his famous Pressing Down Lemma (see [1] and [2, appendix]).

Analysis of this notion leads to several two-place functions on ω_1 that give a rich structure with many applications. So let us expose some of these functions.

1.3 Definition. The *full code* of the walk is the function $\rho_0 : [\omega_1]^2 \longrightarrow \omega^{<\omega}$ defined recursively by

$$\rho_0(\alpha, \beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

with the boundary value $\rho_0(\alpha, \alpha) = \emptyset^2$ where the symbol $^\frown$ refers to the sequence obtained by concatenating the one term sequence $\langle |C_\beta \cap \alpha| \rangle$ with the already known finite sequence $\rho_0(\alpha, \min(C_\beta \setminus \alpha))$ of integers.

Knowing $\rho_0(\alpha, \beta)$ and the ordinal β one can reconstruct two following two *traces* of the walk from β to α.

1.4 Definition. The *upper trace* of the walk from β to α is the set

$$\mathrm{Tr}(\alpha, \beta) = \{\xi : \rho_0(\xi, \beta) \sqsubseteq \rho_0(\alpha, \beta)\},$$

where $\sqsubseteq$ denotes the ordering of being an initial segment among finite sequences of ordinals. One also has the recursive definition of this trace,

$$\mathrm{Tr}(\alpha, \beta) = \{\beta\} \cup \mathrm{Tr}(\alpha, \min(C_\beta \setminus \alpha))$$

with the boundary value $\mathrm{Tr}(\alpha, \alpha) = \{\alpha\}$.

1.5 Definition. The *lower trace* of the walk from β to α is the set

$$\mathrm{L}(\alpha, \beta) = \{\lambda_\xi(\alpha, \beta) : \rho_0(\xi, \beta) \sqsubset \rho_0(\alpha, \beta)\},$$

where $\sqsubset$ is the ordering of being a strict initial segment among the finite sequences of ordinals and where for an ordinal ξ such that $\rho_0(\xi, \beta) \sqsubset \rho_0(\alpha, \beta)$,

$$\lambda_\xi(\alpha, \beta) = \max\{\max(C_\eta \cap \alpha) : \rho_0(\eta, \beta) \sqsubseteq \rho_0(\xi, \beta)\}.$$

Note the following recursive definition of this trace,

$$\mathrm{L}(\alpha, \beta) = \mathrm{L}(\alpha, \min(C_\beta \setminus \alpha)) \cup (\{\max(C_\beta \cap \alpha)\} \setminus \max(\mathrm{L}(\alpha, \min(C_\beta \setminus \alpha))))$$

with the boundary value $\mathrm{L}(\alpha, \alpha) = \emptyset$.

The following immediate facts about these notions will be frequently and very often implicitly used.

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[2]Note that while ρ_0 operates on the set $[\omega_1]^2$ of unordered pairs of countable ordinals, the formal recursive definition of $\rho_0(\alpha, \beta)$ does involves the term $\rho_0(\alpha, \alpha)$ and so we have to give it as a *boundary value* $\rho_0(\alpha, \alpha) = \emptyset$. The boundary value is always specified when such a function is recursively defined and will typically be either 0 or the empty sequence $\emptyset$. This convention will be used even when a function e with domain such as $[\omega_1]^2$ is given to us beforehand and we consider its diagonal value $e(\alpha, \alpha)$.

1.6 Lemma. *For $\alpha \le \beta \le \gamma$,*

(a) $\alpha > \mathrm{L}(\beta, \gamma)^3$ *implies that* $\rho_0(\alpha, \gamma) = \rho_0(\beta, \gamma)^\frown \rho_0(\alpha, \beta)$ *and therefore that* $\mathrm{Tr}(\alpha, \gamma) = \mathrm{Tr}(\beta, \gamma) \cup \mathrm{Tr}(\alpha, \beta)$,

(b) $\mathrm{L}(\alpha, \beta) > \mathrm{L}(\beta, \gamma)$ *implies that* $\mathrm{L}(\alpha, \gamma) = \mathrm{L}(\beta, \gamma) \cup \mathrm{L}(\alpha, \beta)$.

$\dashv$

1.7 Definition. The *full lower trace* of the minimal walk is the function $F : [\omega_1]^2 \longrightarrow [\omega_1]^{<\omega}$ defined recursively by

$$F(\alpha, \beta) = F(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup_{\xi \in C_\beta \cap \alpha} F(\xi, \alpha),$$

with the boundary value $F(\alpha, \alpha) = \{\alpha\}$ for all α.

Thus $F(\alpha, \beta)$ includes the traces of walks between any two ordinals $\le \alpha$ that have ever been referred to during the walk from β to α.

1.8 Lemma. *For all $\alpha \le \beta \le \gamma$,*

(a) $F(\alpha, \gamma) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma)$,

(b) $F(\alpha, \beta) \subseteq F(\alpha, \gamma) \cup F(\beta, \gamma)$.

Proof. The proof is by induction on γ. Let $F(\alpha, \gamma_1)$ of $F(\alpha, \gamma)$ where $\gamma_1 = \min(C_\gamma \setminus \alpha)$.

To prove (a) consider first the case $\gamma_1 < \beta$. By the inductive hypothesis,

$$F(\alpha, \gamma_1) \subseteq F(\alpha, \beta) \cup F(\gamma_1, \beta) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma),$$

since $\gamma_1 \in C_\gamma \cap \beta$, so $F(\gamma_1, \beta) \subseteq F(\beta, \gamma)$. If $\gamma_1 \ge \beta$ then $\gamma_1 = \min(C_\gamma \setminus \beta)$ and again $F(\beta, \gamma_1) \subseteq F(\beta, \gamma)$. By the inductive hypothesis

$$F(\alpha, \gamma_1) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma_1) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma).$$

Let now us consider $F(\xi, \alpha)$ for some $\xi \in C_\gamma \cap \alpha$. Using the inductive hypothesis we get

$$F(\xi, \alpha) \subseteq F(\alpha, \beta) \cup F(\xi, \beta) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma),$$

since $\xi \in C_\gamma \cap \beta$ and $F(\xi, \beta)$ is by definition included in $F(\beta, \gamma)$.

To prove (b) consider first the case $\gamma_1 < \beta$. By the inductive hypothesis:

$$F(\alpha, \beta) \subseteq F(\alpha, \gamma_1) \cup F(\gamma_1, \beta).$$

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[3] For two sets of ordinals F and G, by $F < G$ we denote the fact that every ordinal from F is smaller than every ordinal from G. When $G = \{\alpha\}$, we write $F < \alpha$ rather than $F < \{\alpha\}$.

Since $\gamma_1 \in C_\gamma \cap \beta$, the set $F(\gamma_1, \beta)$ is included in $F(\beta, \gamma)$. Similarly by the definition of $F(\alpha, \gamma)$, the set $F(\alpha, \gamma_1)$ is included in $F(\alpha, \gamma)$.

Suppose now that $\gamma_1 \geq \beta$. Note that in this case $\gamma_1 = \min(C_\gamma \cap \beta)$, so $F(\beta, \gamma_1)$ is included in $F(\beta, \gamma)$. Using the inductive hypothesis for $\alpha \leq \beta \leq \gamma_1$ we have

$$F(\alpha, \beta) \subseteq F(\alpha, \gamma_1) \cup F(\beta, \gamma_1) \subseteq F(\alpha, \gamma) \cup F(\beta, \gamma).$$

$$\dashv$$

1.9 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho_0(\alpha, \beta) = \rho_0(\min(F(\beta, \gamma) \setminus \alpha), \beta)^\frown \rho_0(\alpha, \min(F(\beta, \gamma) \setminus \alpha))$,

(b) $\rho_0(\alpha, \gamma) = \rho_0(\min(F(\beta, \gamma) \setminus \alpha), \gamma)^\frown \rho_0(\alpha, \min(F(\beta, \gamma) \setminus \alpha))$.

Proof. The proof is again by induction on γ. Let $\alpha_1 = \min(F(\beta, \gamma) \setminus \alpha)$ and $\gamma_1 = \min(C_\gamma \setminus \alpha)$. Let ξ_1 be the minimal $\xi \in (C_\gamma \cap \beta) \cup \{\min(C_\gamma \setminus \beta)\}$ such that $\alpha_1 \in F(\xi, \beta)$ (or $F(\beta, \xi)$).

Applying the inductive hypothesis to $\alpha \leq \xi_1 \leq \beta$ (or $\alpha \leq \beta \leq \xi_1$) and noting that $\alpha_1 = \min(F(\xi_1, \beta) \setminus \alpha)$ (or $\alpha_1 = \min(F(\beta, \xi_1) \setminus \alpha)$) we get

$$\rho_0(\alpha, \beta) = \rho_0(\alpha_1, \beta)^\frown \rho_0(\alpha, \alpha_1), \tag{I.1}$$

the part (a) of Lemma 1.9. Note that depending on whether $\alpha \leq \gamma_1 \leq \beta$ or $\alpha \leq \beta \leq \gamma_1$ we have that $F(\gamma_1, \beta) \subseteq F(\beta, \gamma)$ or $F(\beta, \gamma_1) \subseteq F(\beta, \gamma)$, and therefore $\alpha_2 := \min(F(\gamma_1, \beta) \setminus \alpha) \geq \alpha_1$ or $\alpha_2 := \min(F(\beta, \gamma_1) \setminus \alpha) \geq \alpha_1$. Applying the inductive hypothesis to $\alpha \leq \gamma_1 \leq \beta$ (or $\alpha \leq \beta \leq \gamma_1$) we get:

$$\rho_0(\alpha, \beta) = \rho_0(\alpha_2, \beta)^\frown \rho_0(\alpha, \alpha_2), \tag{I.2}$$

$$\rho_0(\alpha, \gamma_1) = \rho_0(\alpha_2, \gamma_1)^\frown \rho_0(\alpha, \alpha_2). \tag{I.3}$$

Comparing (I.1) and (I.2) we see that $\rho_0(\alpha, \alpha_1)$ is a tail of $\rho_0(\alpha, \alpha_2)$, so (I.3) can be rewritten as

$$\rho_0(\alpha, \gamma_1) = \rho_0(\alpha_2, \gamma_1)^\frown \rho_0(\alpha_1, \alpha_2)^\frown \rho_0(\alpha, \alpha_1) = \rho_0(\alpha_1, \gamma_1)^\frown \rho_0(\alpha, \alpha_1). \tag{I.4}$$

Finally since $\rho_0(\alpha, \gamma) = \langle|C_\gamma \cap \alpha|\rangle^\frown \rho_0(\alpha, \gamma_1) = \rho_0(\gamma_1, \gamma)^\frown \rho_0(\alpha, \gamma_1)$ holds, the equation (I.4) gives us the desired conclusion (b) of Lemma 1.9. $\dashv$

1.10 Definition. Recall that *right lexicographical ordering* on $\omega^{<\omega}$ denoted by $<_r$ refers to the total ordering defined by letting $s <_r t$ if s is an end-extension of t, or if $s(i) < t(i)$ for $i = \min\{j : s(j) \neq t(j)\}$. Using the identification between a countable ordinal α and the α-sequence $(\rho_0)_\alpha$ of elements of $\omega^{<\omega}$, the ordering $<_r$ induces the following ordering $<_c$ on ω_1:

$$\alpha <_c \beta \quad \text{iff} \quad \rho_0(\xi, \alpha) <_r \rho_0(\xi, \beta), \tag{I.5}$$

where $\xi = \Delta(\alpha, \beta) = \min\{\eta \leq \min\{\alpha, \beta\} : \rho_0(\eta, \alpha) \neq \rho_0(\eta, \beta)\}$.

1.11 Lemma. *The cartesian square of the total ordering $<_c$ of ω_1 is the union of countably many chains.*

Proof. It suffices to decompose the set of all pairs (α, β) where $\alpha < \beta$. To each such pair we associate a hereditarily finite set $p(\alpha, \beta)$ which codes the finite structure obtained from $F(\alpha, \beta) \cup \{\beta\}$ by adding relations that describe the way ρ_0 acts on it. To show that this parametrization works, suppose we are given two pairs (α, β) and (γ, δ) such that

$$p(\alpha, \beta) = p(\gamma, \delta) = p, \text{ and } \alpha <_c \gamma.$$

We must show that $\beta \leq_c \delta$. Let

$$\xi_{\alpha\beta} = \min(F(\alpha, \beta) \setminus \Delta(\alpha, \gamma)), \text{ and}$$

$$\xi_{\gamma\delta} = \min(F(\gamma, \delta) \setminus \Delta(\alpha, \gamma)).$$

Note that $F(\alpha, \beta) \cap \Delta(\alpha, \gamma) = F(\gamma, \delta) \cap \Delta(\alpha, \gamma)$ so $\xi_{\alpha\beta}$ and $\xi_{\gamma\delta}$ correspond to each other in the isomorphism of the (α, β) and (γ, δ) structures. It follows that:

$$\rho_0(\xi_{\alpha\beta}, \alpha) = \rho_0(\xi_{\gamma\delta}, \gamma)(= t_{\alpha, \gamma}), \tag{I.6}$$

$$\rho_0(\xi_{\alpha\beta}, \beta) = \rho_0(\xi_{\gamma\delta}, \delta)(= t_{\beta, \delta}). \tag{I.7}$$

Applying Lemma 1.9 we get:

$$\rho_0(\Delta(\alpha, \gamma), \alpha) = t_{\alpha\gamma}^\frown \rho_0(\Delta(\alpha, \gamma), \xi_{\alpha\beta}), \tag{I.8}$$

$$\rho_0(\Delta(\alpha, \gamma), \gamma) = t_{\alpha\gamma}^\frown \rho_0(\Delta(\alpha, \gamma), \xi_{\gamma\delta}). \tag{I.9}$$

It follows that $\rho_0(\Delta(\alpha, \gamma), \xi_{\alpha\beta}) \neq \rho_0(\Delta(\alpha, \gamma), \xi_{\gamma\delta})$. Applying Lemma 1.9 for β and δ and the ordinal $\Delta(\alpha, \gamma)$ we get:

$$\rho_0(\Delta(\alpha, \gamma), \beta) = t_{\beta\delta}^\frown \rho_0(\Delta(\alpha, \gamma), \xi_{\alpha\beta}), \tag{I.10}$$

$$\rho_0(\Delta(\alpha, \gamma), \delta) = t_{\beta\delta}^\frown \rho_0(\Delta(\alpha, \gamma), \xi_{\gamma\delta}). \tag{I.11}$$

It follows that $\rho_0(\Delta(\alpha, \gamma), \beta) \neq \rho_0(\Delta(\alpha, \gamma), \delta)$. This shows $\Delta(\alpha, \gamma) \geq \Delta(\beta, \delta)$. A symmetrical argument shows the other inequality $\Delta(\beta, \delta) \geq \Delta(\alpha, \gamma)$. It follows that

$$\Delta(\alpha, \gamma) = \Delta(\beta, \delta)(= \bar{\xi}).$$

Our assumption is that $\rho_0(\bar{\xi}, \alpha) <_r \rho_0(\bar{\xi}, \gamma)$ and since these two sequences have $t_{\alpha\gamma}$ as common initial part, this reduces to

$$\rho_0(\bar{\xi}, \xi_{\alpha\beta}) <_r \rho_0(\bar{\xi}, \xi_{\gamma\delta}). \tag{I.12}$$

On the other hand $t_{\beta\delta}$ is a common initial part of $\rho_0(\bar{\xi}, \beta)$ and $\rho_0(\bar{\xi}, \delta)$, so their lexicographical relationship depends on their tails which by (I.10) and (I.11) are equal to $\rho_0(\bar{\xi}, \xi_{\alpha\beta})$ and $\rho_0(\bar{\xi}, \xi_{\gamma\delta})$ respectively. Referring to (I.12) we conclude that indeed $\rho_0(\bar{\xi}, \beta) <_r \rho_0(\bar{\xi}, \delta)$, i.e. $\beta <_c \delta$. $\dashv$

1.12 Notation. *Well-ordered sets of rationals.* The set $\omega^{<\omega}$ ordered by the right lexicographical ordering $<_r$ is a particular copy of the rationals of the interval $(0, 1]$ which we are going to denote by $\mathbb{Q}_r$ or simply by $\mathbb{Q}$. The next lemma shows that for a fixed α, $\rho_0(\xi, \alpha)$ is a strictly increasing function from α into $\mathbb{Q}_r$. We let ρ_{0_α} or simply $\rho_{0\alpha}$ denote this function and we will identify it with its range, i.e. view it as a member of the tree $\sigma\mathbb{Q}_r$ of all well-ordered subsets of $\mathbb{Q}_r$, ordered by end-extension.

1.13 Lemma. $\rho_0(\alpha, \gamma) <_r \rho_0(\beta, \gamma)$ *whenever* $\alpha < \beta < \gamma$. $\dashv$

The sequence $\rho_{0\alpha}$ $(\alpha < \omega_1)$ of members of $\sigma\mathbb{Q}_r$ naturally determines the subtree

$$T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}.$$

Note that for a fixed α, the restriction $\rho_{0\beta} \restriction \alpha$ is determined by the way $\rho_{0\beta}$ acts on the finite set $F(\alpha, \beta)$. This is the content of Lemma 1.9. Hence all levels of $T(\rho_0)$ are countable, and therefore $T(\rho_0)$ is a particular example of an *Aronszajn tree*, a tree of height ω_1 with all levels and chains countable. We shall now see that $T(\rho_0)$ is in fact a *special Aronszajn tree*, i.e. one that admits a decomposition into countably many antichains or, equivalently, admits a strictly increasing map into the rationals.

1.14 Lemma. $\{\xi < \beta : \rho_0(\xi, \beta) = \rho_0(\xi, \gamma)\}$ *is a closed subset of β whenever* $\beta < \gamma$.

Proof. Let $\alpha < \beta$ be a given accumulation point of this set and let $\beta = \beta_0 > \ldots > \beta_n = \alpha$ and $\gamma = \gamma_0 > \ldots > \gamma_m = \alpha$ be the traces of the walks $\beta \to \alpha$ and $\gamma \to \alpha$, respectively. Using the assumption (c) of the fundamental sequences and the fact that $\alpha > 0$ is a limit ordinal we can find an ordinal $\xi < \alpha$ such that $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$ and

$$\xi > \max(C_{\beta_i} \cap \alpha) \text{ and } \xi > \max(C_{\gamma_j} \cap \alpha) \text{ for all } i < n \text{ and } j < m. \quad \text{(I.13)}$$

Note that this in particular means that the walk $\alpha \to \xi$ is a common tail of the walks $\beta \to \xi$ and $\gamma \to \xi$. Subtracting $\rho_0(\xi, \alpha)$ from $\rho_0(\xi, \beta)$ we get $\rho_0(\alpha, \beta)$ and subtracting $\rho_0(\xi, \alpha)$ from $\rho_0(\xi, \gamma)$ we get $\rho_0(\alpha, \gamma)$. It follows that $\rho_0(\alpha, \beta) = \rho_0(\alpha, \gamma)$. $\dashv$

It follows that $T(\rho_0)$ does not branch at limit levels. From this we can conclude that $T(\rho_0)$ is a special subtree of $\sigma\mathbb{Q}$ since this is easily seen to be so for any subtree of $\sigma\mathbb{Q}$ which is finitely branching at limit nodes.

1.15 Definition. Identifying the power set of $\mathbb{Q}$ with the particular copy $2^{\mathbb{Q}}$ of the Cantor set, define for every countable ordinal α,

$$G_\alpha = \{x \in 2^{\mathbb{Q}} : x \text{ end-extends no } \rho_{0\beta} \restriction \alpha \text{ for } \beta \geq \alpha\}.$$

1.16 Lemma. G_α $(\alpha < \omega_1)$ *is an increasing sequence of proper G_δ-subsets of the Cantor set whose union is equal to the Cantor set.* ⊣

1.17 Lemma. *The set $X = \{\rho_{0\beta} : \beta < \omega_1\}$ considered as a subset of the Cantor set $2^\mathbb{Q}$ has universal measure zero.*

Proof. Let μ be a given nonatomic Borel measure on $2^\mathbb{Q}$. For $t \in T(\rho_0)$, set

$$P_t = \{x \in 2^\mathbb{Q} : x \text{ end-extends } t\}.$$

Note that each P_t is a perfect subset of $2^\mathbb{Q}$ and therefore is μ-measurable. Let

$$S = \{t \in T(\rho_0) : \mu(P_t) > 0\}.$$

Then S is a downward closed subtree of $\sigma\mathbb{Q}$ with no uncountable antichains. By an old result of Kurepa (see [74]), no Souslin tree admits a strictly increasing map into the reals (as for example $\sigma\mathbb{Q}$ does). It follows that S must be countable and so we are done. ⊣

2. The coherence of maximal weights

2.1 Definition. The *maximal weight* of the walk is the two-place function $\rho_1 : [\omega_1]^2 \longrightarrow \omega$ defined recursively by

$$\rho_1(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\},$$

with the boundary value $\rho_1(\alpha, \alpha) = 0$[4]. Thus $\rho_1(\alpha, \beta)$ is simply the maximal integer appearing in the sequence $\rho_0(\alpha, \beta)$.

2.2 Lemma. *For all $\alpha < \beta < \omega_1$ and $n < \omega$,*

(a) $\{\xi \le \alpha : \rho_1(\xi, \alpha) \le n\}$ *is finite,*

(b) $\{\xi \le \alpha : \rho_1(\xi, \alpha) \ne \rho_1(\xi, \beta)\}$ *is finite.*

Proof. The proof is by induction. To prove (a) it suffices to show that for every $n < \omega$ and every $A \subseteq \alpha$ of order-type ω there is $\xi \in A$ such that $\rho_1(\xi, \alpha) > n$. Let $\eta = \sup(A)$. If $\eta = \alpha$ one chooses arbitrary $\xi \in A$ with the property that $|C_\alpha \cap \xi| > n$, so let us consider the case $\eta < \alpha$. Let $\alpha_1 = \min(C_\alpha \setminus \eta)$. By the inductive hypothesis there is $\xi \in A$ such that:

$$\xi > \max(C_\alpha \cap \eta), \tag{I.14}$$

$$\rho_1(\xi, \alpha_1) > n. \tag{I.15}$$

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[4]This is another use of the convention $\rho_1(\alpha, \alpha) = 0$ in order to facilitate the recursive definition.

Note that $\rho_0(\xi, \alpha) = \langle |C_\alpha \cap \eta| \rangle^\frown \rho_0(\xi, \alpha_1)$, and therefore

$$\rho_1(\xi, \alpha) \geq \rho_1(\xi, \alpha_1) > n.$$

To prove (b) we show by induction that for every $A \subseteq \alpha$ of order-type ω there exists $\xi \in A$ such that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$. Let $\eta = \sup(A)$ and let $\beta_1 = \min(C_\beta \setminus \eta)$. Let $n = |C_\beta \cap \eta|$ and let

$$B = \{\xi \in A : \xi > \max(C_\beta \cap \eta) \text{ and } \rho_1(\xi, \beta_1) > n\}.$$

Then B is infinite, so by the induction hypothesis we can find $\xi \in B$ such that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta_1)$. Then

$$\rho_1(\xi, \beta) = \max\{n, \rho_1(\xi, \beta_1)\} = \rho_1(\xi, \beta_1),$$

so we are done. $\dashv$

Note the following unboundedness property of ρ_1 which immediately follows from the previous lemma and which will be elaborated on in latter sections of this Chapter.

2.3 Corollary. *For every pair $\mathcal{A}$ and $\mathcal{B}$ uncountable families of pairwise disjoint finite subsets of ω_1 end every positive integer n there exist uncountable subfamilies $\mathcal{A}_0 \subseteq \mathcal{A}$ and $\mathcal{B}_0 \subseteq \mathcal{B}$ such that for every pair $a \in \mathcal{A}_0$ and $b \in \mathcal{B}_0$ such that $a < b$[5] we have $\rho_1(\alpha, \beta) > n$ for all $\alpha \in a$ and $\beta \in b$.* $\dashv$

2.4 Corollary. *The tree*

$$T(\rho_1) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1, t =^* \rho_{1\alpha}\}[6]$$

is a coherent, homogeneous, and $\mathbb{R}$-embeddable Aronszajn tree. $\dashv$

The following is an interesting property of $T(\rho_1)$ or any other coherent tree of finite-to one mappings from countable ordinals into ω.

2.5 Theorem. *The cartesian square of any lexicographically ordered coherent tree of finite-to-one mappings is the union of countably many chains.*

Proof. Let $T(a)$ be a given tree given by a coherent sequence $a_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of finite-to-one mappings. Let $<_l$ denote the lexicographical ordering of $T(a)$. We need to decompose pairs (s, t) of $T(a)$ into countably many classes that are chains in the cartesian product of $<_l$. In fact, by symmetry it suffices to decompose pairs (s, t) with $l(s) \leq l(t)$ where $l(s)$

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[5]For sets of ordinals a and b we write $a < b$ whenever $\alpha < \beta$ for all $\alpha \in a$ and $\beta \in b$.
[6]Here, $\rho_{1\alpha} : \alpha \to \omega$ are defined from ρ_1 in the usual way, $\rho_{1\alpha}(\xi) = \rho_1(\xi, \alpha)$, and $=^*$ denotes the fact that the functions agree on all but finitely many arguments.

and $l(t)$ are levels as well as domains of the mappings s and t, respectively. Given such such pair (s, t), let

$$D(s, t) = \{\xi : \xi = l(s) \text{ or } \xi < l(s) \text{ and } s(\xi) \neq t(\xi)\},$$

and let $n(s, t)$ be the maximal value s or t takes on $D(s, t)$. Let

$$F(s, t) = \{\xi : \xi = l(s) \text{ or } \xi < l(s) \text{ and } s(\xi) \leq n(s, t) \text{ or } t(\xi) \leq n(s, t)\}.$$

Note that $D(s, t) \subseteq F(s, t)$. Finally, to the pair (s, t) we associate a hereditarily finite set $p(s, t)$ which codes the behavior of the restrictions of s and t on the set $F(s, t)$.

Suppose we are given two pairs (s, t) and (u, v) such that $p(s, t) = p(u, v)$ and such that $s <_l u$. We need to show that $t \leq_l v$. By symmetry, we may assume that $l(s) \leq l(u)$ and that s and u are actually different. Let $\alpha = \Delta(s, u)$, and let us first consider the case when $\alpha < l(s)$, i.e., when s and u are incomparable. Note that by the choice of parametrization and the assumption $p(s, t) = p(u, v)$, we have

$$F(s, t) \cap \alpha = F(u, v) \cap \alpha$$

is a common initial part of $F(s, t)$ and $F(u, v)$. Since s and u agree below α, we conclude that t and v must also agree below α. If $\alpha \notin F(s, t) \cup F(u, v)$, then

$$t(\alpha) = s(\alpha) < u(\alpha) = v(\alpha),$$

so $t <_l v$, as required. If $\alpha \in F(s, t)$ but $\alpha \notin F(u, v)$ then

$$t(\alpha) \leq n(s, t) = n(u, v) < v(\alpha),$$

so $t <_l v$ also in this case. The case $\alpha \notin F(s, t)$ and $\alpha \in F(u, v)$ is impossible since

$$u(\alpha) \leq n(u, v) = n(s, t) < s(\alpha),$$

contradicts our assumption $s <_l u$. The remaining case $\alpha \in F(s, t) \cap F(u, v)$ is also impossible since the assumption $p(s, t) = p(u, v)$ would give us $s(\alpha) = u(\alpha)$ contradicting the fact that $\alpha = \Delta(s, u)$.

Suppose now that s is a strict initial segment of u. Then $F = F(s, t) \cap l(s) = F(u, v) \cap l(s)$ and

$$F(s, t) = F \cup \{l(s)\} \text{ and } F(u, v) = F \cup \{l(u)\}$$

If $l(s) < l(t)$ then $t(l(s)) \leq n(s, t) = n(u, v) < v(l(s))$. Since t and v agree bellow $l(s)$ we conclude that $t <_l v$, as required. If $l(s) = l(t)$ and since t and v agree bellow $l(s)$ we conclude that v extends t, and therefore $t <_l v$, as required. This finishes the proof. ⊣

2.6 Definition. Consider the following extension of $T(\rho_1)$:

$$\widetilde{T}(\rho_1) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1 \text{ and } t \restriction \xi \in T(\rho_1) \text{ for all } \xi < \alpha\}.$$

If we order $\widetilde{T}(\rho_1)$ by the right lexicographical ordering $<_r$ we get a complete linearly ordered set. It is not continuous as it contains jumps of the form

$$[t^\frown\langle m\rangle, t^\frown\langle m+1\rangle^\frown\vec{0}],$$

where $t \in T(\rho_1)$ and $m < \omega$. Removing the right-hand points from all the jumps we get a linearly ordered continuum which we denote by $\widetilde{A}(\rho_1)$.

2.7 Lemma. $\widetilde{A}(\rho_1)$ *is a homogeneous nonreversible ordered continuum that can be represented as the union of an increasing ω_1-sequence of Cantor sets.*

Proof. For a countable limit ordinal $\delta > 0$, set

$$\widetilde{A}_\delta(\rho_1) = \{t \in \widetilde{A}(\rho_1) : l(t) \le \delta\}.$$

Then each $\widetilde{A}_\delta(\rho_1)$ is a closed subset of $\widetilde{A}(\rho_1)$ with the level $T_\delta(\rho_1)$ being a countable order-dense subset. One easily concludes from this that $\widetilde{A}_\delta(\rho_1)$ (δ limit $< \omega_1$) is the required decomposition of $\widetilde{A}(\rho_1)$.

To show that $\widetilde{A}(\rho_1)$ is homogeneous, consider two pairs $x_0 < x_1$ and $y_0 < y_1$ of non-endpoints of $\widetilde{A}(\rho_1)$. Choose a countable limit ordinal δ strictly above the lengths of x_0, x_1, y_0 and y_1. So we have a Cantor set $\widetilde{A}_\delta(\rho_1)$, an order-dense subset $T_\delta(\rho_1)$ of it and two pairs of points $x_0 < x_1$ and $y_0 < y_1$ in

$$\widetilde{A}_\delta(\rho_1) \setminus T_\delta(\rho_1).$$

By Cantor's theorem there is an order isomorphism $\sigma : \widetilde{A}_\delta(\rho_1) \longrightarrow \widetilde{A}_\delta(\rho_1)$ such that:

$$\sigma''T_\delta(\rho_1) = T_\delta(\rho_1), \tag{I.16}$$

$$\sigma(x_i) = y_i \text{ for } i < 2. \tag{I.17}$$

Extend σ to the rest of $\widetilde{A}(\rho_1)$ by the formula

$$\sigma(t) = \sigma(t \restriction \delta)^\frown t \restriction [\delta, l(t)).$$

It is easily checked that the restriction is the required order-isomorphism.

Let us now prove that there is no order-reversing bijection

$$\pi : \widetilde{A}(\rho_1) \longrightarrow \widetilde{A}(\rho_1).$$

Since $T(\rho_1)$ with the lexicographical ordering does not have a countable order-dense subset, the image $\pi''T(\rho_1)$ must have sequences of length bigger

than any given countable ordinal. So for every limit ordinal δ we can fix t_δ in $T(\rho_1)$ of length $\geq \delta$ such that $\pi(t_\delta)$ also has length $\geq \delta$. Let

$$f(\delta) = \max\{\xi < \delta : t_\delta(\xi) \neq \pi(t_\delta)(\xi)\} + 1.$$

By the Pressing Down Lemma find an uncountable set Γ of countable limit ordinals, a countable ordinal $\bar{\xi}$ and $s, t \in T_{\bar{\xi}}(\rho_1)$ such that for all $\delta \in \Gamma$:

$$f(\delta) = \bar{\xi}, \tag{I.18}$$

$$t_\delta \restriction \bar{\xi} = s \text{ and } \pi(t_\delta) \restriction \bar{\xi} = t. \tag{I.19}$$

Moreover we may assume that

$$\{t_\delta \restriction \delta : \delta \in \Gamma\} \text{ and } \{\pi(t_\delta) \restriction \delta : \delta \in \Gamma\}$$

are antichains of the tree $T(\rho_1)$. Pick $\gamma \neq \delta$ in Γ and suppose for definiteness that $t_\gamma <_l t_\delta$. Note that the ordinal $\Delta(t_\gamma, t_\delta)$ where this relation is decided is above $\bar{\xi}$ and is smaller than both γ and δ. Similarly the ordinal $\Delta(\pi(t_\gamma), \pi(t_\delta))$ where the lexicographical ordering between $\pi(t_\gamma)$ and $\pi(t_\delta)$ is decided also belongs to the interval

$$(\bar{\xi}, \ \min\{\gamma, \delta\}).$$

It follows that $\Delta(\pi(t_\gamma), \pi(t_\delta)) = \Delta(t_\gamma, t_\delta)(= \xi)$ and that

$$\pi(t_\gamma)(\xi) = t_\gamma(\xi) < t_\delta(\xi) = \pi(t_\delta(\xi)),$$

contradicting the fact that π is order-reversing. $\dashv$

2.8 Definition.

A *homogeneous Eberlein compactum* [7]. The set $\widetilde{T}(\rho_1)$ has another natural structure, a topology generated by the family of sets of the form

$$\widetilde{V}_t = \{u \in \widetilde{T}(\rho_1) : t \subseteq u\},$$

for t a node of $T(\rho_1)$ of successor length as a clopen subbase. Let $T^0(\rho_1)$ denote the set of all nodes of $T(\rho_1)$ of successor length. Then $\widetilde{T}(\rho_1)$ can be regarded as the set of all downward closed chains of the tree $T^0(\rho_1)$ and the topology on $\widetilde{T}(\rho_1)$ is simply the topology one obtains from identifying the power set of $T^0(\rho_1)$ with the cube

$$\{0,1\}^{T^0(\rho_1)}$$

with its Tychonoff topology[8]. $\widetilde{T}(\rho_1)$ being a closed subset of the cube is compact. In fact $\widetilde{T}(\rho_1)$ has some very strong topological properties such as the property that closed subsets of $\widetilde{T}(\rho_1)$ are its retracts.

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[7]Recall that a compactum X is *Eberlein* if its function space $\mathcal{C}(X)$ can be generated by a subset which is compact in the weak topology.

[8]This is done by identifying a subset V of $T^0(\rho_1)$ with its characteristic function $\chi_V : T^0(\rho_1) \longrightarrow 2$.

2.9 Lemma. $\widetilde{T}(\rho_1)$ *is a homogeneous Eberlein compactum*

Proof. The proof that $\widetilde{T}(\rho_1)$ is homogeneous is quite similar to the corresponding part of the proof of the Lemma 2.7. To see that $\widetilde{T}(\rho_1)$ is an Eberlein compactum, i.e. that the function space $\mathcal{C}(\widetilde{T}(\rho_1))$ is weak compactly generated, let $\{X_n\}$ be a countable antichain decomposition of $T(\rho_1)$ and consider the set

$$K = \{2^{-n}\chi_{\widetilde{V}_t} : n < \omega, t \in X_n\} \cup \{\chi_{\emptyset}\}.$$

Note that K is a weakly compact subset of $\mathcal{C}(\widetilde{T}(\rho_1))$ which separates the points of $\widetilde{T}(\rho_1)$. ⊣

E. Borel's notion of *strong measure zero* is a metric notion that has several closely related topological notions due to K.Menger, F.Rothberger and others. For example *Rothberger's property C''* is a purely topological notion which in the class of metric spaces is slightly stronger than the *property C*, i.e. being of strong measure zero. It says that for every sequence $\{\mathcal{U}_n\}$ of open covers of a space X one can choose $U_n \in \mathcal{U}_n$ for each n such that

$$X = \bigcup_{n=0}^{\infty} U_n.$$

2.10 Lemma. $T(\rho_1)$ *has Rothberger's property C''.* ⊣

2.11 Remark. The subspace $T(\rho_1)$ of $\widetilde{T}(\rho_1)$ has in fact a considerably stronger covering property than C''. For example, it can be shown that its function space $\mathcal{C}(T(\rho_1))$ with the topology of pointwise convergence is a Fréchet space, or in other words, has the property that any family of continuous functions on $T(\rho_1)$ which pointwise accumulates to the constantly equal 0 function $\bar{0}$ contains a sequence $\{f_n\}$ which pointwise converges to $\bar{0}$. In fact the function space $\mathcal{C}(T(\rho_1))$ with the topology of pointwise convergence has a natural decomposition as the increasing union of an ω_1-sequence of closed separable metric vector subspaces. The separable metric vector subspaces are determined by the retraction maps of $T(\rho_1)$ onto its levels.

In the next application the coherent sequence $\rho_{1\alpha} : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of finite-to-one maps needs to be turned into a coherent sequence of maps that are actually one-to-one. One way to achieve this is via the following formula:

$$\bar{\rho}_1(\alpha, \beta) = 2^{\rho_1(\alpha,\beta)} \cdot (2 \cdot |\{\xi \leq \alpha : \rho_1(\xi, \beta) = \rho_1(\alpha, \beta)\}| + 1).$$

Define $\bar{\rho}_{1\alpha}$ from $\bar{\rho}_1$ just as $\rho_{1\alpha}$ was defined from ρ_1; then the $\bar{\rho}_{1\alpha}$'s are one-to-one and coherent. From ρ_1 one also has a natural sequence r_α $(\alpha < \omega_1)$ of elements of ω^ω defined as follows

$$r_\alpha(n) = |\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq n\}|.$$

Note that r_β eventually strictly dominates r_α whenever $\alpha < \beta$.

2.12 Definition. The sequences $e_\alpha = \bar{\rho}_{1\alpha}$ $(\alpha < \omega_1)$ and r_α $(\alpha < \omega_1)$ can be used in describing a functor

$$G \longmapsto G^*,$$

which to every graph G on ω_1 associates another graph G^* on ω_1 as follows:

$$\{\alpha, \beta\} \in G^* \text{ iff } \{e_\alpha^{-1}(l), e_\beta^{-1}(l)\} \in G \tag{I.20}$$

for all $l < \Delta(r_\alpha, r_\beta)$ for which these preimages are both defined and different[9].

2.13 Lemma. *Suppose that every uncountable family $\mathcal{F}$ of pairwise disjoint finite subsets of ω_1 contains two sets A and B such that $A \otimes B \subseteq G$[10]. Then the same is true about G^* provided the uncountable family $\mathcal{F}$ consists of finite cliques[11] of G^*.*

Proof. Let $\mathcal{F}$ be a given uncountable family of pairwise disjoint finite cliques of G^*. We may assume that all members of $\mathcal{F}$ are of some fixed size k.

Consider a countable limit ordinal $\delta > 0$ and an A in $\mathcal{F}$ with all its elements above δ. Let $n = n(A, \delta)$ be the minimal integer such that for all $\alpha < \beta$ in $A \cup \{\delta\}$:

$$\Delta(r_\alpha, r_\beta) < n, \tag{I.21}$$

$$e(\xi, \alpha) \neq e(\xi, \beta) \text{ for some } \xi \leq \alpha \text{ implies } e(\xi, \alpha), e(\xi, \beta) \leq n. \tag{I.22}$$

Let

$$H(\delta, A) = \{\xi \leq \omega_1 : e(\xi, \alpha) \leq n(\delta, A) \text{ for some } \alpha \geq \xi \text{ from } A \cup \{\delta\}\}.$$

Taking the transitive collapse $\bar{H}(\delta, A)$ of $H(\delta, A)$, the sequence

$$e_\alpha \upharpoonright H(\delta, A) \ (\alpha \in A)$$

collapses to a k-sequence $\bar{s}(\delta, A)$ of mappings with integer domains. Let $\bar{r}(\delta, A)$ denote the k-sequence

$$r_\alpha \upharpoonright (n(\delta, A) + 1) \ (\alpha \in A)$$

enumerated in increasing order. Hence every $A \in \mathcal{F}$ above δ generates a quadruple

$$(H(\delta, A) \cap \delta, \bar{r}(\delta, A), \bar{s}(\delta, A), n(\delta, A))$$

---

[9]As it will be clear from the proof of the following lemma the functor $G \longrightarrow G^*$ can equally be based on any other coherent sequence $e_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of one-to-one mappings and any other sequence r_α $(\alpha < \omega_1)$ of pairwise distinct reals.

[10]Here, $A \otimes B = \{\{\alpha, \beta\} : \alpha \in A, \beta \in B, \alpha \neq \beta\}$.

[11]A *clique* of G^* is a subset C of ω_1 with the property that $[C]^2 \subseteq G^*$.

[12]Recall that for $s \neq t \in \omega^\omega$, we let $\Delta(s, t) = \min\{k : s(k) \neq t(k)\}$.

of parameters. Since the set of quadruples is countable, by the assumption on the graph G we can find two distinct members A_δ and B_δ of $\mathcal{F}$ above δ that generate the same quadruple of parameters denoted by

$$(H_\delta, \bar{r}_\delta, \bar{s}_\delta, n_\delta).$$

Moreover, we choose A_δ and B_δ to satisfy the following isomorphism condition

> For every $l \le n_\delta$, $i < k$, if α is the ith member of A_δ, β the ith member of B_δ and if $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are both defined then they are G-connected to the same elements of $H_\delta \cap \delta$. $\qquad$ (I.23)

Let m_δ be the minimal integer $> n_\delta$ such that

$$\Delta(r_\alpha, r_\beta) < m_\delta \text{ for all } \alpha \in A_\delta \text{ and } \beta \in B_\delta. \qquad (\text{I.24})$$

Let

$$I_\delta = \{\xi : e(\xi, \alpha) \le m_\delta \text{ for some } \alpha \ge \xi \text{ in } A_\delta \cup B_\delta\}.$$

Let $\bar{p}_\delta$ be the k-sequence $r_\alpha \restriction m_\delta (\alpha \in A_\delta)$ enumerated in increasing order and similarly let $\bar{q}_\delta$ be the k-sequence $r_\beta \restriction m_\delta (\beta \in B_\delta)$ enumerated increasingly. Let $\bar{t}_\delta$ and $\bar{u}_\delta$ be the transitive collapse of $e_\alpha \restriction I_\delta$ ($\alpha \in A_\delta$) and $e_\beta \restriction I_\delta (\beta \in B_\delta)$, respectively. By the Pressing Down Lemma there is an unbounded $\Gamma \subseteq \omega_1$ and tuples

$$(H, \bar{r}, \bar{s}, n) \text{ and } (I, \bar{p}, \bar{q}, \bar{t}, \bar{u}, m)$$

such that for all $\delta \in \Gamma$:

$$(H_\delta, \bar{r}_\delta, \bar{s}_\delta, n_\delta) = (H, \bar{r}, \bar{s}, n), \qquad (\text{I.25})$$

$$(I_\delta \cap \delta, \bar{p}_\delta, \bar{q}_\delta, \bar{t}_\delta, \bar{u}_\delta, m_\delta) = (I, \bar{p}, \bar{q}, \bar{t}, \bar{u}, m). \qquad (\text{I.26})$$

Moreover we assume the following analogue of (I.23) where $k_1 = |I_\gamma \setminus \gamma|$ for some (equivalently all) $\gamma \in \Gamma$:

> For every $\gamma, \delta \in \Gamma$ and $i < k_1$, if η is the ith member of I_γ and if ξ is the ith member of I_δ in their increasing enumerations, then η and ξ are G-connected to the same elements of the root I, and if η happens to be the i^*th member of A_γ (or B_γ) for some $i^* < k$, then ξ must be the i^*th member of A_δ (B_δ resp.) and vice versa. $\qquad$ (I.27)

By our assumption about the graph G there exist $\gamma < \delta$ in Γ such that:

$$\max(I_\gamma) < \delta, \qquad (\text{I.28})$$

$$(I_\gamma \setminus \gamma) \otimes (I_\delta \setminus \delta) \subseteq G. \qquad (\text{I.29})$$

The proof of Lemma 2.13 is finished once we show that

$$A_\gamma \otimes B_\delta \subseteq G^*. \tag{I.30}$$

Let $i, j < k$ be given and let α be the ith member of A_γ and let β be the jth member of B_δ.

Case 1: $i \neq j$. Pick an $l < \Delta(r_\alpha, r_\beta)$. By (I.21), $l < n$. Assume that $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are both defined.

Subcase 1.1: $e_\alpha^{-1}(l) < \gamma$ and $e_\beta^{-1}(l) < \delta$. By the first choice of parameters, $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are members of the set H which is an initial part of $H(\gamma, A_\gamma)$ and $H(\delta, B_\delta)$. Therefore, we have that the behavior of e_α and e_β on H is encoded by the ith and jth term of the sequence $\bar{s}$, respectively. In particular, we have

$$e_\beta^{-1}(l) = e_{\beta''}^{-1}(l), \tag{I.31}$$

where $\beta'' =$ the jth member of A_γ. By our assumption that $[A_\gamma]^2 \subseteq G^*$ we infer that $\{\alpha, \beta''\} \in G^*$. Referring to the definition (I.20) we conclude that $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ must be G-connected if they are different.

Subcase 1.2: $e_\alpha^{-1}(l) < \gamma$ and $e_\beta^{-1}(l) \in I_\delta \setminus \delta$. Let

$$\beta' = \text{the } j\text{th member of } B_\gamma.$$

By the choice of parametrization, the position of $e_{\beta'}^{-1}(l)$ in I_γ is the same as the position of $e_\beta^{-1}(l)$ in I_δ so by (I.27) their relationship to the point $e_\alpha^{-1}(l)$ of the root I is the same. Similarly, by the choice of the first set of parameters, letting β'' be the jth member of A_γ, the position of $e_{\beta'}^{-1}(l)$ in $H(\gamma, B_\gamma)$ and $e_{\beta''}^{-1}(l)$ in $H(\gamma, A_\gamma)$ is the same, so from (I.23) we conclude that their relationship with $e_\alpha^{-1}(l)$ is the same. However, we have checked in the previous subcase that $e_\alpha^{-1}(l)$ and $e_{\beta''}^{-1}(l)$ are G-connected in case they are different.

Subcase 1.3: $e_\alpha^{-1}(l) \in I_\gamma \setminus \gamma$ and $e_\beta^{-1}(l) < \delta$. This is essentially symmetric to the previous subcase.

Subcase 1.4: $e_\alpha^{-1}(l) \in I_\gamma \setminus \gamma$ and $e_\beta^{-1}(l) \in I_\delta \setminus \delta$. The fact that $\{e_\alpha^{-1}(l), e_\beta^{-1}(l)\} \in G$ in this case follows from (I.29).

Case 2: $i = j$. Consider an $l < \Delta(r_\alpha, r_\beta)$. Note that now we have $l < m$ (see (I.24)).

Subcase 2.1: $e_\alpha^{-1}(l) < \gamma$ and $e_\beta^{-1}(l) < \delta$. Then $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are elements of I which is an initial part of both I_γ and I_δ. Therefore the jth($= i$th) term of $\bar{u}$ encodes both $e_\beta \upharpoonright I$ and $e_{\beta'} \upharpoonright I$ (see the definition of β' above). It follows that

$$e_\beta^{-1}(l) = e_{\beta'}^{-1}(l).$$

If $l \leq n$ then $e_{\beta'}^{-1}(l)$ belongs to H and since the jth($= i$th) term of $\bar{s}$ encodes both $e_\alpha \upharpoonright H$ and $e_\beta \upharpoonright H$ it follows that

$$e_\alpha^{-1}(l) = e_{\beta'}^{-1}(l).$$

If $l > n$ then $e_\alpha^{-1}(l)$ and $e_{\beta'}^{-1}(l)$ are not members of H so from (I.22) and the definition of

$$H(\gamma, A_\gamma) \cap \gamma = H = H(\gamma, B_\gamma) \cap \gamma$$

we conclude that

$$e_\gamma(e_\alpha^{-1}(l)) = e_\alpha(e_\alpha^{-1}(l)) = l = e_{\beta'}(e_{\beta'}^{-1}(l)) = e_\gamma(e_{\beta'}^{-1}(l)).$$

Since e_γ is one-to-one we conclude that

$$e_\alpha^{-1}(l) = e_{\beta'}^{-1}(l) = e_\beta^{-1}(l).$$

Subcase 2.2: $e_\alpha^{-1}(l) < \gamma$ and $e_\beta^{-1}(l) \in I_\delta \setminus \delta$. Recall that β' is the ith($= j$th) member of B_γ so by the first choice of parameters the relative position of $e_{\beta'}^{-1}(l)$ in $H(\gamma, B_\gamma)$ must be the same as the relative position of $e_\alpha^{-1}(l)$ in $H(\gamma, A_\gamma)$, i.e. it must belong to the root H. Note that if j^* is the position of β in I_δ then j^* must also be the position of β' in I_γ. It follows that the j^*th term of $\bar{u}$ encodes both

$$e_\beta \upharpoonright I_\delta \text{ and } e_{\beta'} \upharpoonright I_\gamma = e_{\beta'} \upharpoonright I.$$

So it must be that $e_\beta^{-1}(l)$ belongs to the root I, a contradiction. So this subcase never occurs.

Subcase 2.3: $e_\alpha^{-1}(l) \in I_\gamma \setminus \gamma$ and $e_\beta^{-1}(l) < \delta$. This is symmetric to the previous subcase, so it also never occurs.

Subcase 2.4: $e_\alpha^{-1}(l) \in I_\gamma \setminus \gamma$ and $e_\beta^{-1}(l) \in I_\delta \setminus \delta$. Then $\{e_\alpha^{-1}(l), e_\beta^{-1}(l)\} \in G$ follows from (I.29).

This completes the proof of Lemma 2.13. $\dashv$

2.14 Lemma. *If there is uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^2 \subseteq G^*$ then ω_1 can be decomposed into countably many sets Σ such that $[\Sigma]^2 \subseteq G$.*

Proof. Fix an uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^2 \subseteq G^*$. For a finite binary sequence s of length equal to some $l + 1$, set

$$\Gamma_s = \{\xi < \omega_1 : e(\xi, \alpha) = l \text{ for some } \alpha \text{ in } \Gamma \text{ with } s \subseteq r_\alpha\}.$$

Then the sets Γ_s cover ω_1 and $[\Gamma_s]^2 \subseteq G$ for all s. $\dashv$

2.15 Remark. Let G be the comparability graph of some Souslin tree T. Then for every uncountable family $\mathcal{F}$ of pairwise disjoint cliques of G (finite chains of T) there exist $A \neq B$ in $\mathcal{F}$ such that $A \cup B$ is a clique of G (a chain of T). However, it is not hard to see that G^* fails to have this property (i.e. the conclusion of Lemma 2.13). This shows that some assumption on the graph G in Lemma 2.13 is necessary. There are indeed many graphs that satisfy the hypothesis of Lemma 2.13. Many examples appear when one is trying to apply Martin's axiom to some Ramsey-theoretic problems. Note that the conclusion of Lemma 2.13 is simply saying that the poset of all finite cliques of G^* is ccc while its hypothesis is a bit stronger than the fact that the poset of all finite cliques of G is ccc in all of its finite powers. Applying 2.14 to the case when G is the incomparability graph of some Aronszajn tree, we see that the statement saying that all Aronszajn trees are special is a purely Ramsey-theoretic statement in the same way Souslin's hypothesis is.

We finish this section by showing that the assumption (c) on a given C-sequence C_α ($\alpha < \omega_1$) on which we base our walks and the function ρ_1 is not sufficient to give us the stronger conclusion that the tree $T(\rho_1)$ can be decomposed into countably many antichains (see Corollary 2.4). To describe an example, we need the following notation given that we have fixed one C-sequence C_α ($\alpha < \omega_1$). Let $C_\alpha(0) = 0$ and for $0 < n < \omega$, let $C_\alpha(n))$ denote the n'th element of C_α according to its increasing enumeration with the convention that $C_{\alpha+1}(n) = \alpha$ for all $n > 0$. We assume that the C-sequence is chosen so that for a limit ordinal $\alpha > 0$ and a positive integer n, the ordinal $C_{\alpha+1}(n)$ is at least $n+1$ steps away from the closest limit ordinal bellow it. For a set D of countable ordinals, let D^0 denote the set of successor ordinals in D. We also fix, for each countable ordinal α a one-to-one function e_α whose domain is the set α^0 of all successor ordinals $< \alpha$ and range included in ω which cohere in the sense that $e_\alpha(\xi) = e_\beta(\xi)$ for all but finitely many successor ordinals $\xi \in \alpha^0 \cap \beta^0$. For each $r \in ([\omega]^{<\omega})^\omega$, we associate another C-sequence C_α^r ($\alpha < \omega_1$) by letting $C_\alpha^r = C_\alpha \cap [\xi_\alpha^r, \alpha) \cup \bigcup_{n\in\omega} D_\alpha^r(n)$, where

$$D_\alpha^r(n) = \{\xi \in [C_\alpha(n), C_\alpha(n+1))^0 : e_\alpha(\xi) \in r(n)\},$$

and where $\xi_\alpha^r = \sup(\bigcup_{n\in\omega} D_\alpha^r(n))$. (This definition really takes place only when α is a limit ordinal; for successor ordinals we put $C_{\alpha+1}^r = \{\alpha\}$.) This gives us a C-sequence C_α^r ($\alpha < \omega_1$) and we can consider the corresponding $\rho_1^r : [\omega_1]^2 \longrightarrow \omega$ as above. This function will have the properties stated in Lemma 2.2, so the corresponding tree

$$T(\rho_1^r) = \{(\rho_1^r)_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}$$

is a coherent tree of finite-to-one mappings that admits a strictly increasing function into the real line. The following fact shows that $T(\rho_1^r)$ will typically not be decomposable into countably many antichains.

2.16 Lemma. *If r is a Cohen real then $T(\rho_1^r)$ has no stationary antichain.*

Proof. The family $[\omega]^{<\omega}$ of finite subsets of ω equipped with the discrete topology and its power $([\omega]^{<\omega})^\omega$ with the corresponding product topology and the given Cohen real r is to meet all dense G_δ subset of this space that one can explicitly define. Thus in particular, $\xi_\alpha^r = \alpha$, and therefore, $C_\alpha^r = \bigcup_{n\in\omega} D_\alpha^r(n)$ for every limit ordinal α. Since the tree $T(\rho_1^r)$ is coherent, to establish the conclusion of the Lemma, it suffices to show that that for every stationary $\Gamma \subseteq \omega_1$ there exists $\gamma, \delta \in \Gamma$ such that $(\rho_1^r)_\gamma \subset (\rho_1^r)_\delta$. This in turn amounts to showing that for every stationary $\Gamma \subseteq \omega_1$, the set

$$U_\Gamma = \{x \in ([\omega]^{<\omega})^\omega : (\exists \gamma, \delta \in \Gamma)\ (\rho_1^x)_\gamma \subset (\rho_1^x)_\delta\}$$

is a dense-open subset of $([\omega]^{<\omega})^\omega$. So, given a finite partial function p from ω into $[\omega]^{<\omega}$, it is sufficient to find a finite extension q of p such that the basic open subset of $([\omega]^{<\omega})^\omega$ determined by q is included in U_Γ. Let n be the minimal integer that is bigger than all integers appearing in the domain of p or any set of the form $p(j)$ for $j \in \mathrm{dom}(p)$. For $\gamma \in \Gamma$, set

$$F_n(\gamma) = \{\xi \in \gamma^0 : e_\gamma(\xi) \le n\}.$$

Applying the Pressing Down Lemma, we obtain a finite set $F \subseteq \omega_1$ and a stationary set $\Delta \subseteq \Gamma$ such that $F_n(\gamma) = F$ for all $\gamma \in \Delta$ and such that if $\alpha = \max(F) + 1$ then $e_\gamma \restriction \alpha = e_\delta \restriction \alpha$ for all $\gamma, \delta \in \Delta$. A similar application of the Pressing Down Lemma will give us an integer $m > n$ and two ordinals $\gamma < \delta$ in Δ such that $C_\gamma(j) = C_\delta(j)$ for all $j \le m$, $C_\delta(m+1) > \gamma + 1$, and

$$e_\gamma \restriction (C_\gamma(m) + 1)^0 = e_\delta \restriction (C_\gamma(m) + 1)^0.$$

Extend the partial function p to a partial function q with domain $\{0, 1, ..., m\}$ such that:

(1) $q(m) = \{e_\delta(\gamma + 1)\}$, and

(2) $q(j) = \emptyset$ for any $j < m$ not belonging to $\mathrm{dom}(p)$.

Choose any $x \in ([\omega]^{<\omega})^\omega$ extending the partial mapping q. Then from the choices of the objects n, Δ, F, m, γ, δ and q, we have that

(3) $\gamma + 1 \in C_\delta^x$, and

(4) $C_\delta^x \cap \gamma$ is an initial segment of C_γ^x.

It follows that, given a $\xi < \gamma$, the walk from δ to ξ along the C-sequence C_β^x ($\beta < \omega_1$) either leads to the same finite string of the corresponding weights as the walk from γ to ξ, or else it starts with first two steps to $\gamma + 1$ and γ, and then follows the walk from γ to ξ. Since $\rho_1^x(\xi, \delta)$ and $\rho_1^x(\xi, \gamma)$ are by definitions maximums of these two strings of weights, we conclude

that $\rho_1^x(\xi, \delta) \geq \rho_1^x(\xi, \gamma)$. On the other hand, note that by (4), in the second case, the weight $|C_\delta^x \cap \xi|$ of the first step from δ to ξ is less than or equal to the weight $|C_\gamma^x \cap \xi|$ of the first step from γ to ξ. It follows that we have also the other inequality $\rho_1^x(\xi, \delta) \leq \rho_1^x(\xi, \gamma)$. Hence we have shown that $\rho_1^x(\beta, \gamma) = \rho_1^x(\beta, \delta)$ for all $\beta < \gamma$, or in other words, that $(\rho_1^x)_\gamma \subset (\rho_1^x)_\delta$. This finishes the proof.

$\dashv$

2.17 Question. What is the condition one needs to put on a given C-sequence C_α ($\alpha < \omega_1$) that guarantees that the corresponding tree $T(\rho_1)$ can be decomposed into countably many antichains?

3. Oscillations of traces

In this section we shall consider versions of the oscillation mapping

$$\mathrm{osc} : ([\omega_1]^{<\omega})^2 \longrightarrow \mathrm{Card}$$

defined by

$$\mathrm{osc}(x, y) = |x/E(x, y)|,$$

where $E(x, y)$ is the equivalence relation on x defined by letting $\alpha E(x, y)\beta$ if and only if the closed interval determined by α and β contains no point from $y \setminus x$. So, if x and y are disjoint, $\mathrm{osc}(x, y)$ is simply the number of convex pieces the set x is split by the set y. The resulting 'oscillation theory' reproduced here in part has shown to be a quite useful and robust coding technique. We give some applications to this effect and in Section 17 below we shall extend the oscillation theory to a more general context.

3.1 Definition. For $\alpha < \beta < \omega_1$, set

$$\mathrm{osc}_0(\alpha, \beta) = \mathrm{osc}(\mathrm{Tr}(\Delta_0(\alpha, \beta), \alpha), \quad \mathrm{Tr}(\Delta_0(\alpha, \beta), \beta)),$$

where $\Delta_0(\alpha, \beta) = \min\{\xi \leq \alpha : \rho_0(\xi, \alpha) \neq \rho_0(\xi, \beta)\} - 1$[13].

The following result shows that the upper traces do realize all possible oscillations in any uncountable subset of ω_1.

3.2 Lemma. *For every uncountable subset Γ of ω_1 and every positive integer n there is an integer l such that for all $k < n$ there exist $\alpha, \beta \in \Gamma$ such that $\mathrm{osc}_0(\alpha, \beta) = l + k$.*

Proof. Fix a sequence $\mathcal{M}_i$ ($i \leq n$) of continuous $\in$-chains of length ω_1 of countable elementary submodels of $(H_{\omega_2}, \in)$ containing all the relevant objects such that $\mathcal{M}_i$ is an element of the minimal model of $\mathcal{M}_{i+1}$ for all

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[13]Note that by Lemma 1.14 this is well defined.

$i < n$. Let $C_i = \{M \cap \omega_1 : M \in \mathcal{M}_i\}$. Note that each C_i is a closed and unbounded subset of ω_1 and that for each $i < n$, C_i is an element of every model of $\mathcal{M}_{i+1}$. Let M_n be the minimal model of $\mathcal{M}_n$ and pick a $\beta \in \Gamma$ above $\delta_n = M_n \cap \omega_1$. Since $C_{n-1} \in M_n$ we can find $M_{n-1} \in \mathcal{M}_{n-1} \cap M_n$ such that $\delta_{n-1} = M_{n-1} \cap \omega_1 > \mathrm{L}(\delta_n, \beta)$. Similarly, we can find $M_{n-2} \in \mathcal{M}_{n-1} \cap M_{n-1}$ such that $\delta_{n-2} = M_{n-2} \cap \omega_1 > \mathrm{L}(\delta_{n-1}, \delta_n)$, and so on. This will give us an $\in$-chain M_i $(i \le n)$ of countable elementary submodels of $(H_{\omega_2}, \in)$ containing all relevant objects such that if we let $\delta_{n+1} = \beta$ and $\delta_i = M_i \cap \omega_1$ for $i \le n$, then $\delta_i > \mathrm{L}(\delta_{i+1}, \delta_{i+2})$ for all $i < n$. Pick a point $\xi \in C_{\delta_0}$ above $\mathrm{L}(\delta_0, \beta)$. Then (see Lemma 1.6 above), $\delta_i \in \mathrm{Tr}(\xi, \beta)$ for all $i \le n$ and in fact $\delta_0 \to \xi$ is the last step of the walk from β to ξ. Let $t = \rho_0(\xi, \beta)$, $t_i = \rho_0(\delta_i, \beta)$ for $i \le n$. The $t_n \sqsubset t_{n-1} \sqsubset ... \sqsubset t_0 \sqsubset t$. Let

$$\Gamma_0 = \{\gamma \in \Gamma : t = \rho_0(\xi, \gamma)\}.$$

Then $\Gamma_0 \in M_0$ and $\beta \in \Gamma_0$. A simple elementarity argument using models $M_0 \in M_1 \in ... \in M_n$ shows that there is an uncountable subset Σ of Γ_0 such that $\Sigma \in M_0$ and

$$\mathcal{T} = \{x \in [\omega_1]^{\le |t|} : x \sqsubseteq \mathrm{Tr}(\xi, \alpha) \text{ for some } \alpha \in \Sigma\}$$

is a tree under $\sqsubseteq$ in which a node x must either have only one immediate successor or there is $i \le n$ and uncountably many $\gamma < \omega_1$ such that $x \cup \{\gamma\} \in \mathcal{T}$ and $\mathrm{Tr}(\gamma, \alpha) = t_i$ for all $\alpha \in \Sigma$ witnessing $x \cup \{\gamma\} \in \mathcal{T}$. Note that $\{\xi\}$ is a root of $\mathcal{T}$ as well as its splitting nodes. Let Ω be the uncountable set such that $\{\xi, \gamma\} \in \mathcal{T}$ for all $\gamma \in \Omega$. Working in M_0 we find $\eta > \xi$ and uncountable $\Omega_0 \subseteq \Omega$ such that $\eta = \Delta_0(\gamma, \beta)$ for all $\gamma \in \Omega_0$ and therefore $\eta = \Delta_0(\alpha, \beta)$ for all $\alpha \in \Sigma$ witnessing $\{\xi, \gamma\} \in \mathcal{T}$. Still working in M_0 and going to an uncountable subset of Ω_0 we may assume that $\mathrm{Tr}(\eta, \gamma)$ $(\gamma \in \Omega_0)$ form a Δ-system with root x_0. Let Σ_0 be the set of all $\alpha \in \Sigma$ witnessing $\{\xi, \gamma\} \in \mathcal{T}$ for some $\gamma \in \Omega_0$. Let $s = \rho_0(\eta, \beta)$. Note that s end-extends t_0 but not necessarily t. Consider now the tree

$$\mathcal{X} = \{x \in [\omega_1]^{\le |s|} : x \sqsubseteq \mathrm{Tr}(\xi, \alpha) \text{ for some } \alpha \in \Sigma_0\}.$$

Its root is equal to x_0, the uncountably many immediate successors $x_0 \cup \{\gamma\}$ $(\gamma \in \Omega_0)$ all have the property that for some fixed initial segment s_0 of s we have that $\rho_0(\gamma, \alpha) = s_0$ for all $\alpha \in \Sigma_0$ witnessing $x_0 \cup \{\gamma\} \in \mathcal{X}$. All other splitting nodes of $\mathcal{X}$ are determined by the other initial segments $s_0 \sqsupseteq t_0 \sqsupset t_1 \sqsupset ... \sqsupset t_n$. Recall that all these objects are elements of M_0 and therefore elements of all M_i $(i \le n)$. Let $y = \mathrm{Tr}(\eta, \beta)$ and let $l = \mathrm{osc}(x_0, y)$. Fix a $k \le n$. Working in M_1 find $\gamma_0 \in \Omega_0$ such that $\mathrm{Tr}(\eta, \gamma_0) \setminus x_0 > \mathrm{Tr}(\eta, \beta) \cap \delta_1$ and find a splitting node $x_1 \sqsupseteq \mathrm{Tr}(\eta, \gamma_0)$ corresponding to t_1. Then $\Omega_1 = \{\gamma < \omega_1 : x_1 \cup \{\gamma\} \in \mathcal{X}\}$ is uncountable and $\rho_0(\gamma, \alpha) = t_1$ for all $\alpha \in \Sigma_0$ witnessing $x_1 \cup \{\gamma\} \in \mathcal{X}$. Working in M_2 find $\gamma_1 \in \Omega_1$ such

that $\gamma_1 > \mathrm{Tr}(\eta, \beta) \cap \delta_2$ and a splitting node $x_2 \sqsupseteq \mathrm{Tr}(\eta, \gamma_1)$ corresponding to t_2, and so on. Proceeding in this way we construct an increasing sequence $x_0 \sqsubset x_1 \sqsubset ... \sqsubset x_k$ of splitting nodes of $\mathcal{T}$ such that $t_i \in M_i$ and

$$x_i \setminus x_{i-1} > \mathrm{Tr}(\eta, \beta) \cap \delta_2 \text{ for all } 1 \le i \le k.$$

Find an arbitrary $\alpha \in \Sigma_0 \cap M_k$ such that $x_k \sqsubseteq \mathrm{Tr}(\eta, \alpha)$ Let $x = \mathrm{Tr}(\eta, \alpha)$. Then

$$x/E(x,y) = x_0/E(x_0,y)) \cup \{x_1 \setminus x_0, x_2 \setminus x_1, ..., x_k \setminus x_{k-1}\}.$$

This is so because the ordinal $\delta_0 \in y \setminus x$ separates the last class of $x_0/E(x_0,y)$ from the new classes $x_i \setminus x_{i-1}$ $(1 \le i \le k)$. It follows that $\mathrm{osc}_0(\alpha, \beta) = \mathrm{osc}(x,y) = l + k$, as required.

$$\dashv$$

Consider the following projection of the oscillation mapping osc_0,

$$\mathrm{osc}_0^*(\alpha, \beta) = \max\{m \in \omega : \ 2^m | \mathrm{osc}_0(\alpha, \beta)\}.$$

3.3 Corollary. *For every uncountable $\Gamma \subseteq \omega_1$ and every positive integer k there exist $\alpha, \beta \in \Gamma$ such that $\mathrm{osc}_0^*(\alpha, \beta) = k$.* $\quad \dashv$

3.4 Corollary. *Every inner model model M of set theory which correctly computes ω_1 contains a partition of ω_1 into infinitely many pairwise disjoint subsets that are stationary in the universe V of all sets.*

Proof. Our assumption about M means in particular that the C-sequence C_ξ $(\xi < \omega_1)$ can be chosen to be an element of M. It follows that the oscillation mapping osc_0^* is also an element of M. It follows that for each $\alpha < \omega_1$, the sequence of sets

$$S_{\alpha n} = \{\beta > \alpha : \mathrm{osc}_0^*(\alpha, \beta) = n\} \ (n < \omega)$$

belongs to M. So it suffices to show that there must be $\alpha < \omega_1$ such that for every $n < \omega$ the set $S_{\alpha n}$ is stationary in V. This is an immediate consequence of Corollary 3.3. $\quad \dashv$

3.5 Remark. In [46], Larson showed that Corollary 3.4 cannot be extended to partitions of ω_1 into uncountably many pairwise disjoint stationary sets. He also showed that under the Proper Forcing Axiom, for every mapping $c : [\omega_1]^2 \to \omega_1$ there is a stationary set $S \subseteq \omega_1$ such that for all $\alpha < \omega_1$,

$$\{c(\alpha, \beta) : \beta \in S \setminus (\alpha + 1))\} \ne \omega_1.$$

Note that by Corollary 3.3 this cannot be extended to mappings c with countable ranges.

One can count oscillation not only between two finite subsets of ω_1 but also between two finite partial functions from ω_1 into ω, i.e., between two finite 'weighted sets' x and y :

$$\mathrm{osc}(x,y) = |\{\xi \in \mathrm{dom}(x) : x(\xi) \leq y(\xi) \text{ but } x(\xi^+) > y(\xi^+)\}|,$$

where ξ^+ denotes the minimal ordinal in $\mathrm{dom}(x)$ above ξ if there is one.

3.6 Definition. For $\alpha < \beta < \omega_1$, set

$$\mathrm{osc}_1(\alpha,\beta) = \mathrm{osc}(\rho_{1\alpha} \restriction \mathrm{L}(\alpha,\beta), \ \ \rho_{1\beta} \restriction \mathrm{L}(\alpha,\beta)).$$

We have the following analogue of Lemma 3.2 which is now true even in its rectangular form.

3.7 Lemma. *For every pair Γ and Σ of uncountable subsets of ω_1 and every positive integer n there is an integer l such that for all $k < n$ there exist $\alpha \in \Gamma$ and $\beta \in \Sigma$ such that $\mathrm{osc}_1(\alpha,\beta) = l + k$.*

Proof. Fix a continuous $\in$-chain $\mathcal{N}$ of length ω_1 of countable elementary submodels of $(H_{\omega_2}, \in)$ containing all the relevant objects and fix also a countable elementary submodel M of $(H_{\omega_2}, \in)$ that contains $\mathcal{N}$ as an element. Let $\delta = M \cap \omega_1$. Fix $\alpha_0 \in \Gamma \setminus \delta$ and $\beta_0 \in \Sigma \setminus \delta$. Let $L_0 = \mathrm{L}(\delta, \beta_0)$. By exchanging the steps in the following construction, we may assume that $\rho_1(\xi, \alpha_0) > \rho_1(\xi, \beta_0)$ for $\xi = \max(L_0)$. Choose $\xi_0 \in C_\delta$ such that $\xi > L_0$ and such that the three mappings $\rho_{1\alpha_0}, \rho_{1\beta_0}$, and $\rho_{1\delta}$ agree on the interval $[\xi, \delta)$. By elementarity of M there will be $\delta^+ > \delta$, $\alpha_0^+ \in \Gamma \setminus \delta^+$, and $\beta_0^+ \in \Sigma \setminus \delta^+$ realizing the same type over the parameters L_0 and ξ_0. Let $L_1^+ = \mathrm{L}(\delta, \delta_0^+)$. Note that $\xi_0 \leq \min(L_1^+)$. It follows that the mappings $\rho_{1\alpha_0^+}$ and $\rho_{1\beta_0^+}$ agree on L_1^+. Choose an $N \in \mathcal{N}$ belonging to M such that $N \cap \omega_1 > L_1^+$ and choose $\xi_1 \in C_\delta \setminus N$. Then we can find $\delta_1 > \delta$ and $\alpha_1^+ \in \Gamma \setminus \delta_1$ and $\beta_1 \in \Sigma \setminus \delta_1$ realizing the same type as δ, α_0^+, and β_0^+ over the objects accumulates so far, L_0, ξ_0, L_1^+, N, and ξ_1. Let $L_1^{++} = \mathrm{L}(\delta, \delta_1)$ and let t_1^+ be the restriction of $\rho_{1\alpha_1^+}$ on $\max(L_1^+) + 1$. Then the set of $\alpha \in \Gamma$ whose $\rho_{1\alpha}$ end-extends t_1^+ belongs to the submodel N and is uncountable, since clearly it contains the ordinal α_1^+ which does not belong to N. So by the elementarity of N and Corollary 2.3 there is $\alpha_1 \in \Gamma \setminus \delta$ such that $\rho_{1\alpha_1}$ end-extends t_1^+ and

$$\rho_1(\xi, \alpha_1) > \rho_1(\xi, \beta_1) \text{ for all } \xi \in L_1^{++}. \qquad (\text{I.32})$$

Let t_1 be the restriction of $\rho_{1\alpha}$ on $\max(L_1^{++})$. Then $t_1 \in M$ and every $\alpha \in \Gamma$ whose $\rho_{1\alpha}$ end-extends t_1 satisfies I.32 and

$$\rho_1(\xi, \alpha) = \rho_1(\xi, \beta_1) \text{ for all } \xi \in L_1^+. \qquad (\text{I.33})$$

Note also that $\mathrm{L}(\delta, \beta_1) = L_0 \cup L_1^+ \cup L_1^{++}$. Using similar reasoning we can find $t_2 \in M \cap T(\rho_1)$ end-extending t_1, an $\alpha_2 \in \Gamma \setminus \delta$ whose $\rho_{1\alpha_2}$ end-extends

t_2, finite sets $\delta > \max(L_2^{++}) > \max(L_2^+) > \max(L_1^{++})$, and $\beta_2 \in \Sigma \setminus \delta$ such that $\mathrm{L}(\delta, \beta_2) = L_0 \cup L_1^+ \cup L_1^{++} \cup L_2^+ \cup L_2^{++}$ and such that the analogues of I.32 and I.33 hold for all $\alpha \in \Gamma$ whose $\rho_{1\alpha}$ end-extends t_2, and so on. This procedure gives us a sequence $L_0 < L_1^+ < L_1^{++} < ... < L_n^+ < L_n^{++}$ of finite subsets of δ, an increasing sequence $t_1 \sqsubset t_2 \sqsubset ... \sqsubset t_n$ of nodes of $T(\rho_0) \cap M$, and a sequence β_k $(1 \le k \le n)$ such that for all $1 \le k \le n$, and all $\alpha \in \Gamma$ whose $\rho_{1\alpha}$ end-extends t_k,

$$\mathrm{L}(\delta, \beta_k) = L_0 \cup L_1^+ \cup L_1^{++} \cup ... \cup L_k^+ \cup L_k^{++}, \tag{I.34}$$

$$\rho_1(\xi, \alpha) > \rho_1(\xi, \beta_k) \text{ for all } \xi \in L_k^{++}, \tag{I.35}$$

$$\rho_1(\xi, \alpha) = \rho_1(\xi, \beta_k) \text{ for all } \xi \in L_k^+. \tag{I.36}$$

Choose $\alpha \in \Gamma \cap M$ whose $\rho_{1\alpha}$ end-extends t_n such that the set $L_n = \mathrm{L}(\alpha, \delta)$ lies above L_n^{++} and such that all the $\rho_{1\beta_k}$'s agree on L_n. Note that by Lemma 1.6 for each $1 \le k \le n$,

$$\mathrm{L}(\alpha, \beta_k) = L_0 \cup L_1^+ \cup L_1^{++} \cup ... \cup L_k^+ \cup L_k^{++} \cup L_n. \tag{I.37}$$

Let $l = \mathrm{osc}(\rho_{1\alpha} \restriction (L_0 \cup L_n), \ \rho_{1\beta} \restriction (L_0 \cup L_n))$ where β is equal to one of the β_k's. Note that l does not depend on which β_k we choose and that according to I.35, I.36 and I.37, for each $1 \le k \le n$, we have that $\mathrm{osc}_1(\alpha, \beta_k) = l + k$, as required. $\dashv$

Note that in the above proof the set Γ can be replaced by any uncountable family of pairwise disjoint sets giving us the following slightly more general conclusion.

3.8 Lemma. *For every uncountable family $\mathcal{G}$ of pairwise disjoint finite subsets of ω_1 all of some fixed size m, every uncountable subset Σ of ω_1, and every positive integer n there exist integers l_i $(i < m)$ and $a \in \mathcal{G}$ such that for all $k < n$ there exist $\beta \in \Sigma$ such that $\mathrm{osc}_1(a(i), \beta) = l_i + k$ for all $i < m$.*[14] $\dashv$

Having in mind an application we define a projection of osc_1 as follows. For a real x, let $[x]$ denote the greatest integer not bigger than x. Choose a mapping $* : \mathbb{R} \to \omega$ such that $*(x) = n$ if $x - [x] \in \mathrm{I}_n$, where $\mathrm{I}_n = (\frac{1}{n+3}, \frac{1}{n+2}]$. For x in $\mathbb{R}$ we use the notation x^* for $*(x)$. Choose a one-to-one sequence x_ξ $(\xi < \omega_1)$ of elements of the first half of the unit interval $\mathrm{I} = [0, 1]$ which together with the point 1 form a set that is linearly independent over the field of rational numbers. Finally, set

$$\mathrm{osc}_1^*(\alpha, \beta) = (\mathrm{osc}_1(\alpha, \beta) \cdot x_\alpha)^*.$$

Then Lemma 3.8 turns into the following statement about osc_1^*.

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[14] Here $a(i)$ denotes the ith member of a according to the increasing enumeration of a.

3.9 Lemma. *For every uncountable family $\mathcal{G}$ of pairwise disjoint finite subsets of ω_1 all of some fixed size m, every uncountable subset Σ of ω_1, and every mapping $h : m \to \omega$ there exist $a \in \mathcal{G}$ and $\beta \in \Sigma$ such that $\mathrm{osc}_1^*(a(i), \beta) = h(i)$ for all $i < m$.*

Proof. Let ε be half of the length of the shortest interval of the form $\mathrm{I}_{h(i)}$ $(i < m)$. Note that for $a \in \mathcal{G}$, the points $1, x_{a(0)}, ..., x_{a(m-1)}$ are rationally independent, so a standard fact from number theory (see for example [32]; XXIII) gives us that there is an integer n_a such that for every $y \in \mathrm{I}^m$ there exist $k < n_a$ such that $|k \cdot x_{a(i)} - y_i| < \varepsilon$ (mod 1) for all $i < m$. Going to an uncountable subfamily of $\mathcal{G}$, we may assume that there is n such that $n_a = n$ for all $a \in \mathcal{G}$. By Lemma 3.8 there exits $a \in \mathcal{G}$ and l_i $(i < m)$ such that for every $k < n$ there is $\beta_k > a$ such that $\mathrm{osc}_1(a(i), \beta_k) = l_i + k$ for all $i < m$. For $i < m$, let $y_i = l_i \cdot x_{a(i)}$ and let z_i be the middle-point of the interval $\mathrm{I}_{h(i)}$. Find $k < n$ such that $|k \cdot x_{a(i)} - (z_i - y_i)| < \varepsilon$ (mod 1) for all $i < m$. Then $\mathrm{osc}_1^*(a(i), \beta_k) = h(i)$ for all $i < m$, as required.   ⊣

3.10 Corollary. *There is a regular hereditarily Lindelöf space which is not separable.*

Proof. For $\beta < \omega_1$, let $f_\beta \in \{0,1\}^{\omega_1}$ be defined by letting $f_\beta(\beta) = 1$, $f_\beta(\gamma) = 0$ for $\gamma > \beta$, and $f_\beta(\alpha) = \min\{1, \mathrm{osc}_1^*(\alpha, \beta)\}$ for $\alpha < \beta$. Consider $F = \{f_\beta : \beta < \omega_1\}$ as a subspace of $\{0,1\}^{\omega_1}$. Clearly F is not separable. That F is hereditarily Lindelöf follows easily from Lemma 3.9.   ⊣

3.11 Remark. The projection osc_0^* of the oscillation mapping osc_0 appears in [78] as historically first such a map with more than four colors that takes all of its values on every symmetric square of an uncountable subset of ω_1. The variations osc_1 and osc_1^* are on the other hand very recent and are due to J.T. Moore [54] who made them in order to obtain the conclusion of Corollary 3.10. Concerning Corollary 3.10 we note that the dual implication behaves quite differently since assuming the Proper Forcing Axiom all hereditarily separable regular spaces are hereditarily Lindelöf (see [80]).

4. Triangle inequalities

In this section we study a characteristic of the minimal walk that satisfies certain triangle inequalities reminiscent of those found in an ultrametric space. We also present several applications of the corresponding metric-like theory of ω_1. This leads us to the following variation of the oscillation mapping.

4.1 Definition. Define $\rho : [\omega_1]^2 \longrightarrow \omega$ by recursion as follows

$$\rho(\alpha, \beta) \quad = \quad \max\{|C_\beta \cap \alpha|, \rho(\alpha, \min(C_\beta \setminus \alpha)),$$
$$\rho(\xi, \alpha) : \xi \in C_\beta \cap \alpha\},$$

with the boundary condition $\rho(\alpha, \alpha) = 0$ for all α.

4.2 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leq \alpha : \rho(\xi, \alpha) \leq n\}$ *is finite,*

(b) $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$,

(c) $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$.

Proof. Note that $\rho(\alpha, \beta) \geq \rho_1(\alpha, \beta)$, so (a) follows from the corresponding property of ρ_1. The proof of (b) and (c) is simultaneous by induction on α, β and γ:

To prove (b), consider $n = \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$, and let

$$\xi_\alpha = \min(C_\gamma \setminus \alpha) \text{ and } \xi_\beta = \min(C_\gamma \setminus \beta).$$

We have to show that $\rho(\alpha, \gamma) \leq n$.

Case 1^b: $\xi_\alpha = \xi_\beta$. Then by the inductive hypothesis,

$$\rho(\alpha, \xi_\alpha) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \xi_\beta)\}.$$

From the definition of $\rho(\beta, \gamma)$ we get that $\rho(\beta, \xi_\beta) \leq \rho(\beta, \gamma) \leq n$, so replacing $\rho(\beta, \xi_\beta)$ by $\rho(\beta, \gamma)$ in the above inequality we get $\rho(\alpha, \xi_\alpha) \leq n$. Consider a $\xi \in C_\gamma \cap \alpha = C_\gamma \cap \beta$. By the inductive hypothesis

$$\rho(\xi, \alpha) \leq \max\{\rho(\xi, \beta), \rho(\alpha, \beta)\}.$$

From the definition of $\rho(\beta, \gamma)$ we see that $\rho(\xi, \beta) \leq \rho(\beta, \gamma)$, so replacing $\rho(\xi, \beta)$ with $\rho(\beta, \gamma)$ in the last inequality we get that $\rho(\xi, \alpha) \leq n$. Since $|C_\gamma \cap \alpha| = |C_\gamma \cap \beta| \leq \rho(\beta, \gamma) \leq n$, referring to the definition of $\rho(\alpha, \gamma)$ we conclude that $\rho(\alpha, \gamma) \leq n$.

Case 2^b: $\xi_\alpha < \xi_\beta$. Then $\xi_\alpha \in C_\gamma \cap \beta$, so

$$\rho(\xi_\alpha, \beta) \leq \rho(\beta, \gamma) \leq n.$$

By the inductive hypothesis

$$\rho(\alpha, \xi_\alpha) \leq \max\{\rho(\alpha, \beta), \rho(\xi_\alpha, \beta)\} \leq n.$$

Similarly, for every $\xi \in C_\gamma \cap \alpha \subseteq C_\gamma \cap \beta$,

$$\rho(\xi, \alpha) \leq \max\{\rho(\xi, \beta), \rho(\alpha, \beta)\} \leq n.$$

Finally $|C_\gamma \cap \alpha| \leq |C_\gamma \cap \beta| \leq \rho(\beta, \gamma) \leq n$. Combining these inequalities we get the desired conclusion $\rho(\alpha, \gamma) \leq n$.

To prove (c), consider now $n = \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$. We have to show that $\rho(\alpha, \beta) \leq n$. Let ξ_α and ξ_β be as above and let us consider the same two cases as above.

Case 1^c: $\xi_\alpha = \xi_\beta = \bar{\xi}$. Then by the inductive hypothesis

$$\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \bar{\xi}), \rho(\beta, \bar{\xi})\}.$$

This gives the desired bound $\rho(\alpha, \beta) \leq n$, since $\rho(\alpha, \xi_\alpha) \leq \rho(\alpha, \gamma) \leq n$ and $\rho(\beta, \xi_\beta) \leq \rho(\beta, \gamma) \leq n$.

Case 2^c: $\xi_\alpha < \xi_\beta$. Applying the inductive hypothesis again we get

$$\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \xi_\alpha), \rho(\xi_\alpha, \beta)\} \leq n.$$

This completes the proof.   ⊣

4.3 Remark. Note the following two alternate definitions of ρ:

$$\rho(\alpha, \beta) = \max\{|C_\eta \cap \xi| : \xi, \eta \in F(\alpha, \beta) \cup \{\beta\} \text{ and } \xi < \eta\}.$$

$$\rho(\alpha, \beta) = \max\{\rho_1(\xi, \eta) : \xi, \eta \in F(\alpha, \beta) \cup \{\beta\} \text{ and } \xi < \eta\}.$$

These two definitions make it more transparent (in light of Lemma 1.9) that ρ is a subadditive function. The proof of Lemma 4.2 is given, however, because of some later generalizations of this function to higher ordinals.

The following fact shows that the function ρ has a considerably finer coherence property than ρ_1.

4.4 Lemma. *If $\alpha < \beta < \gamma$ and if $\rho(\alpha, \beta) > \rho(\beta, \gamma)$, then $\rho(\alpha, \gamma) = \rho(\alpha, \beta)$.*

Proof. Applying Lemma 4.2,

$$\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\} = \rho(\alpha, \beta),$$

$$\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\} = \rho(\alpha, \gamma).$$

Note that $\max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$ must be equal to $\rho(\alpha, \gamma)$ rather than $\rho(\beta, \gamma)$ since by the assumption $\rho(\alpha, \beta)$, which is bounded by the maximum, is bigger than $\rho(\beta, \gamma)$.   ⊣

4.5 Remark. Referring to the standard definitions of *metrics* and *ultrametrics* the properties 4.2(b) and (c) could more properly be called *ultrasubadditivities* though we shall keep calling them *subadditivity* properties of the function ρ. Recall the notion of a full lower trace $F(\alpha, \beta)$ given in Definition 1.7 above. The function $d : [\omega_1]^2 \longrightarrow \omega$ defined by

$$d(\alpha, \beta) = |F(\alpha, \beta)|$$

is an example of a truly subadditive function, since by Lemma 1.8 we have the following inequalities whenever $\alpha < \beta < \gamma$:

(a) $d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma)$,

(b) $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\beta, \gamma)$.

4.6 Definition. Define $\bar{\rho} : [\omega_1]^2 \longrightarrow \omega$ as follows

$$\bar{\rho}(\alpha, \beta) = 2^{\rho(\alpha, \beta)} \cdot (2 \cdot |\{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}| + 1).$$

4.7 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$,*

(a) $\bar{\rho}(\alpha, \gamma) \neq \bar{\rho}(\beta, \gamma)$,

(b) $\bar{\rho}(\alpha, \gamma) \leq \max\{\bar{\rho}(\alpha, \beta), \bar{\rho}(\beta, \gamma)\}$,

(c) $\bar{\rho}(\alpha, \beta) \leq \max\{\bar{\rho}(\alpha, \gamma), \bar{\rho}(\beta, \gamma)\}$.

Proof. Only (b) and (c) require some argument. Consider first (b). If $\rho(\alpha, \beta) \leq \rho(\beta, \gamma)$ then $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\} = \rho(\beta, \gamma)$, so

$$\{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \gamma)\} \subseteq \{\xi \leq \beta : \rho(\xi, \beta) \leq \rho(\beta, \gamma)\}.$$

Therefore in this case $\bar{\rho}(\alpha, \gamma) \leq \bar{\rho}(\beta, \gamma)$. On the other hand, if $\rho(\beta, \gamma) \leq \rho(\alpha, \beta)$, then $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\} = \rho(\alpha, \beta)$, so

$$\{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \gamma)\} \subseteq \{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}.$$

So in this case $\bar{\rho}(\alpha, \gamma) \leq \bar{\rho}(\alpha, \beta)$. This checks (b). Checking (c) is similar.
⊣

4.8 Definition. For $p \in \omega^{<\omega}$ define a binary relation $<_p$ on ω_1 by letting $\alpha <_p \beta$ iff $\alpha < \beta$, $\bar{\rho}(\alpha, \beta) \in |p|$ and

$$p(\bar{\rho}(\xi, \alpha)) = p(\bar{\rho}(\xi, \beta)) \text{ for any } \xi < \alpha \text{ such that } \bar{\rho}(\xi, \alpha) < |p|. \qquad (\text{I.38})$$

4.9 Lemma.

(a) $<_p$ *is a tree ordering on ω_1 of height $\leq |p| + 1$,*

(b) $p \subseteq q$ *implies* $<_p \subseteq <_q$.

Proof. This follows immediately from Lemma 4.7. ⊣

4.10 Definition. For $x \in \omega^\omega$, set

$$<_x = \bigcup\{<_{x \restriction n} : n < \omega\}.$$

4.11 Lemma. *For every $p \in \omega^{<\omega}$ there is a partition of ω_1 into finitely many pieces such that if $\alpha < \beta$ belong to the same piece then there is $q \supseteq p$ in $\omega^{<\omega}$ such that $\alpha <_q \beta$.*

Proof. For $\alpha < \omega_1$, the set

$$F_{|p|}(\alpha) = \{\xi \le \alpha : \bar{\rho}(\xi, \alpha) \le |p|\}$$

has size $\le |p| + 1$ so if we enrich $F_{|p|}(\alpha)$ to a structure that would code the behavior of $\bar{\rho}$ on $F_{|p|}(\alpha)$ one can realize only finitely many different isomorphism types. Thus it suffices to show that if $\alpha < \beta < \omega_1$ with $F_{|p|}(\alpha)$, $F_{|p|}(\beta)$ isomorphic, then there is an extension q of p such that $\alpha <_q \beta$.

Consider the graph G of all unordered pairs of integers$< \bar{\rho}(\alpha, \beta)$ of the form

$$\{\bar{\rho}(\xi, \alpha), \bar{\rho}(\xi, \beta)\}$$

for some $\xi < \alpha$. It suffices to show that every connected component of G has at most one point $< |p|$. For if this is true, extend p to a mapping $q : \bar{\rho}(\alpha, \beta) \longrightarrow \omega$ which is constant on each component of G. Clearly, $\alpha <_q \beta$ for any such q.

Consider an integer $n < |p|$ and a G-path $n = n_0, n_1, \ldots, n_k$ which starts with n. Pick ordinals $\xi_0, \xi_1, \ldots, \xi_{k-1} < \alpha$ and $\gamma_i \in \{\alpha, \beta\}$ for $i \le 2k - 1$ which witness the connections, i.e.

$$\rho(\xi_0, \gamma_0) = n_0 \text{ and } \rho(\xi_{k-1}, \gamma_{2k-1}) = n_k, \tag{I.39}$$

$$\rho(\xi_{i-1}, \gamma_{2i-1}) = \rho(\xi_i, \gamma_{2i}) = n_i \text{ for } 0 < i < k, \tag{I.40}$$

$$\gamma_i \ne \gamma_{i+1} \text{ for all } i \le 2k - 2. \tag{I.41}$$

By our assumption about the isomorphism between the structures of $F_{|p|}(\gamma_0)$ and $F_{|p|}(\gamma_1)$ the ordinals $\xi_0 \in F_{|p|}(\gamma_0)$ does not belong to the root

$$F_{|p|}(\gamma_0) \cap F_{|p|}(\gamma_1).$$

It follows that $n_1 > |p|$. Now the following general fact about $\bar{\rho}$ (together with the properties (I.40)-(I.41)) shows that the sequence $n_0, n_1, \ldots, n_k$ is strictly increasing, finishing thus the proof that a given $n < |p|$ is not G-connected to any other integer $< |p|$.

4.12 Lemma. *Suppose $\eta_\alpha \ne \eta_\beta < \min\{\alpha, \beta\}$ and $\bar{\rho}(\eta_\alpha, \alpha) = \bar{\rho}(\eta_\beta, \beta) = n$. Then $\bar{\rho}(\eta_\alpha, \beta), \bar{\rho}(\eta_\beta, \alpha) > n$.*

Proof. Suppose for example that $\eta_\alpha < \eta_\beta$. First note that $\rho(\eta_\alpha, \eta_\beta) \not\le \rho(\eta_\beta, \beta)$ or else $\{\xi \le \eta_\alpha : \rho(\xi, \eta_\alpha) \le \rho(\eta_\alpha, \alpha)\}$ would be a proper initial part of $\{\xi \le \eta_\beta : \rho(\xi, \eta_\beta) \le \rho(\eta_\beta, \beta)\}$ contradicting the fact that $\bar{\rho}(\eta_\alpha, \alpha) = \bar{\rho}(\eta_\beta, \beta)$ yields that the two sets are of the same cardinality. Thus $\rho(\eta_\alpha, \eta_\beta) > \rho(\eta_\beta, \beta) = \rho(\eta_\alpha, \alpha)$ which together with the subadditivity properties of ρ gives us that

$$\{\xi \le \eta_\alpha : \rho(\xi, \eta_\alpha) \le \rho(\eta_\alpha, \alpha)\} \subseteq \{\xi \le \eta_\alpha : \rho(\xi, \eta_\alpha) \le \rho(\eta_\alpha, \eta_\beta)\}.$$

This gives the inequality $\bar{\rho}(\eta_\alpha, \eta_\beta) > \bar{\rho}(\eta_\alpha, \alpha) = n$. Combining this inequality with the subadditivities of $\bar{\rho}$ we get:

$$n < \bar{\rho}(\eta_\alpha, \eta_\beta) \leq \max\{\bar{\rho}(\eta_\alpha, \alpha), \rho(\eta_\beta, \alpha)\} = \bar{\rho}(\eta_\beta, \alpha),$$

$$n < \bar{\rho}(\eta_\alpha, \eta_\beta) \leq \max\{\bar{\rho}(\eta_\alpha, \beta), \rho(\eta_\beta, \beta)\} = \bar{\rho}(\eta_\alpha, \beta).$$

The case $\eta_\beta \leq \eta_\alpha$ is considered similarly. This completes the proof. $\dashv$

$\dashv$

4.13 Theorem. *For every infinite subset* $\Gamma \subseteq \omega_1$, *the set*

$$G_\Gamma = \{x \in \omega^\omega : \alpha <_x \beta \text{ for some } \alpha, \beta \in \Gamma\}$$

is a dense open subset of the Baire space.

Proof. This is an immediate consequence of Lemma 4.11. $\dashv$

4.14 Definition. For $\alpha < \beta < \omega_1$, let $\alpha <_{\bar{\rho}} \beta$ denote the fact that $\bar{\rho}(\xi, \alpha) = \bar{\rho}(\xi, \beta)$ for all $\xi < \alpha$. Then $<_{\bar{\rho}}$ is a tree ordering on ω_1 obtained by identifying α with the member $\bar{\rho}(\cdot, \alpha)$ of the tree $T(\bar{\rho})$. Note that $<_{\bar{\rho}} \subseteq <_x$ for all $x \in \omega^\omega$ and that there exists $x \in \omega^\omega$ such that $<_x = <_{\bar{\rho}}$ (e.g., one such x is the identity map $\text{id} : \omega \longrightarrow \omega$).

4.15 Theorem. *If* Γ *is an infinite* $<_{\bar{\rho}}$*-antichain, the set*

$$H_\Gamma = \{x \in \omega^\omega : \alpha \not<_x \beta \text{ for some } \alpha < \beta \text{ in } \Gamma\}$$

is a dense open subset of the Baire space.

Proof. Let $p \in \omega^{<\omega}$ be given. A simple application of Ramsey's theorem gives us a strictly increasing sequence γ_i $(i < \omega)$ of members of Γ such that $\Delta(\gamma_i, \gamma_j) \leq \Delta(\gamma_j, \gamma_k)$[15] for all $i < j < k$. Going to a subsequence, we may assume that either $\Delta(\gamma_i, \gamma_{i+1}) < \Delta(\gamma_{i+1}, \gamma_{i+2})$ for all i or else for some ordinal ξ and all $i < j$, $\Delta(\gamma_i, \gamma_j) = \xi$. In the first case, by Lemma 4.7(a), we can find i such that one of the integers $\bar{\rho}(\Delta(\gamma_i, \gamma_{i+1}), \gamma_i)$ or $\bar{\rho}(\Delta(\gamma_i, \gamma_{i+1}), \gamma_{i+1})$ does not belong to the domain of p, so there will be $q \supseteq p$ such that $\bar{\rho}(\gamma_i, \gamma_{i+1}) < |q|$ and $\gamma_i \not<_q \gamma_{i+1}$. This guarantees that $\gamma_i \not<_x \gamma_{i+1}$ for every $x \supseteq q$. In the second case, $\bar{\rho}(\xi, \gamma_i)$ $(i < \omega)$ is a one-to-one sequence of integers so there will be $i < j$ such that neither of the integers $\bar{\rho}(\xi, \gamma_i)$ or $\bar{\rho}(\xi, \gamma_j)$ belongs to the domain of p, so there is $q \supseteq p$ such that $\bar{\rho}(\gamma_i, \gamma_j) < |q|$ and $\gamma_i \not<_q \gamma_j$. Hence, $\gamma_i \not<_x \gamma_j$ for every $x \supseteq q$. $\dashv$

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[15] For $\alpha < \beta < \omega_1$, $\Delta(\alpha, \beta) = \min\{\xi \leq \alpha : \bar{\rho}(\xi, \alpha) \neq \bar{\rho}(\xi, \beta)\}$.

4.16 Definition. For a family $\mathcal{F}$ of infinite $<_{\bar{\rho}}$-antichains, we say that a real $x \in \omega^\omega$ is *$\mathcal{F}$-Cohen* if $x \in G_\Gamma \cap H_\Gamma$ for all $\Gamma \in \mathcal{F}$. We say that x is *$\mathcal{F}$-Souslin* if no member of $\mathcal{F}$ is a $<_x$-chain or a $<_x$-antichain. We say that a real $x \in \omega^\omega$ is *Souslin* if the tree ordering $<_x$ on ω_1 has no uncountable chains nor antichains, i.e. when x is $\mathcal{F}$-Souslin for $\mathcal{F}$ equal to the family of all uncountable subsets of ω_1.

Note that since every uncountable subset of ω_1 contains an uncountable $<_{\bar{\rho}}$-antichain, if a family $\mathcal{F}$ refines the family of all uncountable $<_{\bar{\rho}}$-antichains, then every $\mathcal{F}$-Souslin real is Souslin. The following fact summarizes Theorems 4.13 and 4.15 and connects the two kinds of reals.

4.17 Theorem. *If $\mathcal{F}$ is a family of infinite $<_{\bar{\rho}}$-antichains, then every $\mathcal{F}$-Cohen real is $\mathcal{F}$-Souslin.* $\dashv$

4.18 Corollary. *If the density of the family of all uncountable subsets of ω_1 is smaller than the number of nowhere dense sets needed to cover the real line, then there is a Souslin tree.* $\dashv$

4.19 Remark. Recall that the *density* of a family $\mathcal{F}$ of infinite subsets of some set S is the minimal size of a family $\mathcal{F}_0$ of infinite subsets of S with the property that every member of $\mathcal{F}$ is refined by a member of $\mathcal{F}_0$. A special case of Corollary 4.18, when the density of the family of all uncountable subsets of ω_1 is equal to $\aleph_1$, was first observed by T. Miyamoto (unpublished).

4.20 Corollary. *Every Cohen real is Souslin.*

Proof. Every uncountable subset of ω_1 in the Cohen extension contains an uncountable subset from the ground model. So it suffices to consider the family $\mathcal{F}$ of all infinite $<_{\bar{\rho}}$-antichains from the ground model. $\dashv$

If ω_1 is a successor cardinal in the constructible subuniverse, then $\bar{\rho}$ can be chosen to be coanalytic and so the transformation $x \longmapsto <_x$ will transfer combinatorial notions of Souslin, Aronszajn or special Aronszajn trees into the corresponding classes of reals that lie in the third level of the projective hierarchy. This transformation has been explored in several places in the literature (see, e.g. [4],[33]).

4.21 Remark. We have just seen how the combination of the subadditivity properties (4.7(b),(c)) of the coherent sequence $\bar{\rho}_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of one-to-one mappings can be used in controlling the finite disagreement between them. It turns out that in many contexts the coherence and the subadditivities are essentially equivalent restrictions on a given sequence $e_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$. For example, the following construction shows that this is so for any sequence of finite-to-one mappings $e_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$.

4.22 Definition. Given a coherent sequence $e_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of finite-to-one mappings, define $\tau_e : [\omega_1]^2 \longrightarrow \omega$ as follows

$$\tau_e(\alpha, \beta) = \max\{\max\{e(\xi, \alpha), e(\xi, \beta)\} : \xi \leq \alpha, e(\xi, \alpha) \neq e(\xi, \beta)\}^{16}.$$

4.23 Lemma. *For every* $\alpha < \beta < \gamma < \omega_1$,

(a) $\tau_e(\alpha, \beta) \geq e(\alpha, \beta)$,

(b) $\tau_e(\alpha, \gamma) \leq \max\{\tau_e(\alpha, \beta), \tau_e(\beta, \gamma)\}$,

(c) $\tau_e(\alpha, \beta) \leq \max\{\tau_e(\alpha, \gamma), \tau_e(\beta, \gamma)\}$.

Proof. To see (a) note that it trivially holds if $e(\alpha, \beta) = 0$ so we can concentrate on the case $e(\alpha, \beta) > 0$. Applying the convention $e(\alpha, \alpha) = 0$ we see that $e(\xi, \alpha) \neq e(\xi, \beta)$ for $\xi = \alpha$. It follows that $\tau_e(\alpha, \beta) \geq \max\{\max\{e(\alpha, \alpha), e(\alpha, \beta)\} = e(\alpha, \beta)$.

To see (b) let $n = \max\{\tau_e(\alpha, \beta), \tau_e(\beta, \gamma)\}$ and let $\xi \leq \alpha$ be such that $e(\xi, \alpha) \neq e(\xi, \gamma)$. We need to show that $e(\xi, \alpha) \leq n$ and $e(\xi, \gamma) \leq n$. If $e(\xi, \alpha) > n$ then since $\tau_e(\alpha, \beta) \leq n$ we must have $e(\xi, \alpha) = e(\xi, \beta))$ and so $e(\xi, \beta) \neq e(\xi, \gamma)$. It follows that $\tau_e(\beta, \gamma) \geq e(\xi, \beta)) > n$, a contradiction. Similarly, $e(\xi, \gamma) > n$ yields $e(\xi, \beta) = e(\xi, \gamma) \neq e(\xi, \alpha)$ and therefore $\tau_e(\alpha, \beta) \geq e(\xi, \beta)) > n$, a contradiction.

The proof of (c) is similar. ⊣

4.24 Definition. We say that $a : [\omega_1]^2 \longrightarrow \omega$ is *transitive* if for $\alpha < \beta < \gamma < \omega_1$

$$a(\alpha, \gamma) \leq \max\{a(\alpha, \beta), a(\beta, \gamma)\}.$$

Transitive maps occur quite frequently in set-theoretic constructions. For example, given a sequence A_α $(\alpha < \omega_1)$ of subsets of ω that increases relative to the ordering $\subseteq^*$ of inclusion modulo a finite set, the mapping $a : [\omega_1]^2 \longrightarrow \omega$ defined by

$$a(\alpha, \beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}$$

is a transitive map. The transitivity condition by itself is not nearly as useful as its combination with the other subadditivity property (4.7(c)). Fortunately, there is a general procedure that produces a subadditive dominant to every transitive map.

4.25 Definition. For a transitive $a : [\omega_1]^2 \longrightarrow \omega$ define $\rho_a : [\omega_1]^2 \longrightarrow \omega$ recursively as follows:

$$\rho_a(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, a(\min(C_\beta \setminus \alpha), \beta), \rho_a(\alpha, \min(C_\beta \setminus \alpha)),$$
$$\rho_a(\xi, \alpha) : \xi \in C_\beta \cap \alpha\}.$$

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[16] Note the occurence of the boundary value $e(\alpha, \alpha) = 0$ in this formula as well.

4.26 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leq \alpha : \rho_a(\xi, \alpha) \leq n\}$ *is finite,*

(b) $\rho_a(\alpha, \gamma) \leq \max\{\rho_a(\alpha, \beta), \rho_a(\beta, \gamma)\}$,

(c) $\rho_a(\alpha, \beta) \leq \max\{\rho_a(\alpha, \gamma), \rho_a(\beta, \gamma)\}$,

(d) $\rho_a(\alpha, \beta) \geq a(\alpha, \beta)$.

Proof. The proof of (a),(b),(c) is quite similar to the corresponding part of the proof of Lemma 4.2. This comes of course from the fact that the definition of ρ and ρ_a are closely related. The occurrence of the factor $a(\min(C_\beta \setminus \alpha), \beta)$ complicates a bit the proof that ρ_a is subadditive, and the fact that a is transitive is quite helpful in getting rid of the additional difficulty. The details are left to the interested reader. Given $\alpha < \beta$, for every step $\beta_n \to \beta_{n+1}$ of the minimal walk $\beta = \beta_0 > \beta_1 > \ldots > \beta_k = \alpha$, we have $\rho_a(\alpha, \beta) \geq \rho_a(\beta_n, \beta_{n+1}) \geq a(\beta_n, \beta_{n+1})$ by the very definition of ρ_a. Applying the transitivity of a to this path of inequalities we get the conclusion (d). ⊣

4.27 Lemma. $\rho_a(\alpha, \beta) \geq \rho_a(\alpha+1, \beta)$ *whenever $0 < \alpha < \beta$ and α is a limit ordinal.*

Proof. Recall the assumption (c) about the fixed C-sequence C_ξ ($\xi < \omega_1$) on which all our definitions are based: if ξ is a limit ordinal > 0, then no point of C_ξ is a limit ordinal. It follows that if $0 < \alpha < \beta$ and α is a limit ordinal, then the minimal walk $\beta \to \alpha$ must pass through $\alpha + 1$ and therefore $\rho_a(\alpha, \beta) \geq \rho_a(\alpha + 1, \beta)$. ⊣

Let us now give an application of ρ_a to a classical phenomenon of occurrence of gaps in the quotient algebra $\mathcal{P}(\omega)/\text{fin}$.

4.28 Definition. A *Hausdorff gap* in $\mathcal{P}(\omega)/\text{fin}$ is a pair of sequences A_α ($\alpha < \omega_1$) and B_α ($\alpha < \omega_1$) such that

(a) $A_\alpha \subseteq^* A_\beta \subseteq^* B_\beta \subseteq^* B_\alpha$ whenever $\alpha < \beta$, but

(b) there is no C such that $A_\alpha \subseteq^* C \subseteq^* B_\alpha$ for all α.

The following straightforward reformulation shows that a Hausdorff gap is just another instance of a nontrivial coherent sequence

$$f_\alpha : A_\alpha \longrightarrow 2 \ (\alpha < \omega_1)$$

where the domain A_α of f_α is not the ordinal α itself but a subset of ω and that the corresponding sequence of domains A_α ($\alpha < \omega_1$) is a realization of ω_1 inside the quotient $\mathcal{P}(\omega)/\text{fin}$ in the sense that $A_\alpha \subseteq^* A_\beta$ whenever $\alpha < \beta$.

4.29 Lemma. *A pair of ω_1-sequences A_α $(\alpha < \omega_1)$ and B_α $(\alpha < \omega_1)$ form a Hausdorff gap iff the pair*

$$\bar{A}_\alpha = A_\alpha \cup (\omega \setminus B_\alpha) \ (\alpha < \omega_1) \ \text{and} \ \bar{B}_\alpha = \omega \setminus B_\alpha \ (\alpha < \omega_1)$$

has the following properties:

(a) *$\bar{A}_\alpha \subseteq^* \bar{A}_\beta$ whenever $\alpha < \beta$,*

(b) *$\bar{B}_\alpha =^* \bar{B}_\beta \cap \bar{A}_\alpha$ whenever $\alpha < \beta$,*

(c) *there is no B such that $\bar{B}_\alpha =^* B \cap \bar{A}_\alpha$ for all α.*

<div align="right">⊣</div>

From now on we fix a strictly $\subseteq^*$-increasing chain A_α $(\alpha < \omega_1)$ of infinite subsets of ω and let $a : [\omega_1]^2 \longrightarrow \omega$ be defined by

$$a(\alpha, \beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}.$$

Let $\rho_a : [\omega_1]^2 \longrightarrow \omega$ be the corresponding subadditive dominant of a defined above. For $\alpha < \omega_1$, set

$$D_\alpha = A_{\alpha+1} \setminus A_\alpha.$$

4.30 Lemma. *The sets $D_\alpha \setminus \rho_a(\alpha, \gamma)$ and $D_\beta \setminus \rho_a(\beta, \gamma)$ are disjoint whenever $0 < \alpha < \beta < \gamma$ and α and β are limit ordinals.*

Proof. This follows immediately from Lemmas 4.26 and 4.27. <div align="right">⊣</div>

We are in a position to define a partial mapping $m : [\omega_1]^2 \longrightarrow \omega$ by

$$m(\alpha, \beta) = \min(D_\alpha \setminus \rho_a(\alpha, \beta)),$$

whenever $\alpha < \beta$ and α is a limit ordinal.

4.31 Lemma. *The mapping m is coherent, i.e., $m(\alpha, \beta) = m(\alpha, \gamma)$ for all but finitely many limit ordinals $\alpha < \min\{\beta, \gamma\}$.*

Proof. This is by the coherence of ρ_a and the fact that $\rho_a(\alpha, \beta) = \rho_a(\alpha, \gamma)$ already implies $m(\alpha, \beta) = m(\alpha, \gamma)$. <div align="right">⊣</div>

4.32 Lemma. *$m(\alpha, \gamma) \neq m(\beta, \gamma)$ whenever $\alpha \neq \beta < \gamma$ and α, β limit.*

Proof. This follows from Lemma 4.30. <div align="right">⊣</div>

For $\beta < \omega_1$, set

$$B_\beta = \{m(\alpha, \beta) : \alpha < \beta \ \text{and} \ \alpha \ \text{limit}\}.$$

4.33 Lemma. *$B_\beta =^* B_\gamma \cap A_\beta$ whenever $\beta < \gamma$.*

Proof. By the coherence of m. $\dashv$

Note the following immediate consequence of Lemma 4.30 and the definition of m.

4.34 Lemma. $m(\alpha, \beta) = \max(B_\beta \cap D_\alpha)$ *whenever* $\alpha < \beta$ *and* α *limit.* $\dashv$

4.35 Lemma. *There is no* $B \subseteq \omega$ *such that* $B \cap A_\beta =^* B_\beta$ *for all* β.

Proof. Suppose that such a B exists and for a limit ordinal α let us define $g(\alpha) = \max(B \cap D_\alpha)$. Then by Lemma 4.34, $g(\alpha) = m(\alpha, \beta)$ for all $\beta < \omega_1$ and all but finitely many limit ordinals $\alpha < \beta$. By Lemma 4.32, it follows that g is a finite-to-one map, a contradiction. $\dashv$

4.36 Theorem. *For every strictly* $\subset^*$*-increasing chain* A_α ($\alpha < \omega_1$) *of subsets of* ω, *there is a sequence* B_α ($\alpha < \omega_1$) *of subsets of* ω *such that:*

(a) $B_\alpha =^* B_\beta \cap A_\alpha$ *whenever* $\alpha < \beta$,

(b) *there is no* B *such that* $B_\alpha = B \cap A_\alpha$ *for all* α.

$\dashv$

4.37 Remark. For a given countable ordinal α let $h_\alpha : A_\alpha \longrightarrow 2$ be such that $h_\alpha^{-1}(1) = B_\alpha$. Then by Lemma 4.33, the corresponding sequence h_α ($\alpha < \omega_1$) of partial functions is *coherent* in the sense that for every pair $\alpha < \beta < \omega_1$, the set $D_h(\alpha, \beta) = \{n \in A_\alpha \cap A_\beta : h_\alpha(n) \neq h_\beta(n)\}$ is finite. By Lemma 4.35, the sequence is *nontrivial* in the sense that there is no $g : \omega \longrightarrow 2$ such that $h_\alpha =^* g \upharpoonright A_\alpha$ for all $\alpha < \omega_1$. Hausdorff's original sequence, when reformulated in this way, has the property that the maps $d_h(\cdot, \beta) : \beta \longrightarrow \omega$ defined by

$$d_h(\alpha, \beta) = |D_h(\alpha, \beta)|$$

are finite-to-one mappings. A general principle, the P-ideal dichotomy, (see 4.45 below) asserts that every coherent and nontrivial sequence must contain a subsequence with Hausdorff's property. We shall now embark on a general construction of a very similar sequence of coherent partial functions domains included in ω_1 rather than in ω.

4.38 Definition. The *number of steps* of the minimal walk is the two-place function $\rho_2 : [\omega_1]^2 \longrightarrow \omega$ defined recursively by

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1,$$

with the convention that $\rho_2(\gamma, \gamma) = 0$ for all γ.

This is an interesting mapping which is particularly useful on higher cardinalities and especially in situations where the more informative mappings ρ_0, ρ_1 and ρ lack their usual coherence properties. Later on we shall devote a whole section to ρ_2 but here we need it only to succinctly express the following mapping.

4.39 Definition. The *last step function* of the minimal walk is the two-place map $\rho_3 : [\omega_1]^2 \longrightarrow 2$ defined by letting

$$\rho_3(\alpha, \beta) = 1 \text{ iff } \rho_0(\alpha, \beta)(\rho_2(\alpha, \beta) - 1) = \rho_1(\alpha, \beta).$$

In other words, we let $\rho_3(\alpha, \beta) = 1$ just in case the last step of the walk $\beta \to \alpha$ comes with the maximal weight.

4.40 Lemma. $\{\xi < \alpha : \rho_3(\xi, \alpha) \neq \rho_3(\xi, \beta)\}$ *is finite for all* $\alpha < \beta < \omega_1$.

Proof. It suffices to show that for every infinite $\Gamma \subseteq \alpha$ there exists $\xi \in \Gamma$ such that $\rho_3(\xi, \alpha) = \rho_3(\xi, \beta)$. Shrinking Γ we may assume that for some fixed $\bar{\alpha} \in F(\alpha, \beta)$ and all $\xi \in \Gamma$:

$$\bar{\alpha} = \min(F(\alpha, \beta) \setminus \xi), \tag{I.42}$$

$$\rho_1(\xi, \alpha) = \rho_1(\xi, \beta), \tag{I.43}$$

$$\rho_1(\xi, \alpha) > \rho_1(\bar{\alpha}, \alpha), \tag{I.44}$$

$$\rho_1(\xi, \beta) > \rho_1(\bar{\alpha}, \beta). \tag{I.45}$$

It follows (see 1.9) that for every $\xi \in \Gamma$:

$$\rho_0(\xi, \alpha) = \rho_0(\bar{\alpha}, \alpha)^\frown \rho_0(\xi, \bar{\alpha}), \tag{I.46}$$

$$\rho_0(\xi, \beta) = \rho_0(\bar{\alpha}, \beta)^\frown \rho_0(\xi, \bar{\alpha}). \tag{I.47}$$

So for any $\xi \in \Gamma$, $\rho_3(\xi, \alpha) = 1$ iff the last term of $\rho_0(\xi, \bar{\alpha})$ is its maximal term iff $\rho_3(\xi, \beta) = 1$. $\dashv$

The sequence $(\rho_3)_\alpha : \alpha \longrightarrow 2$ $(\alpha < \omega_1)$[17] is therefore coherent in the sense that $(\rho_3)_\alpha =^* (\rho_3)_\beta \upharpoonright \alpha$ whenever $\alpha < \beta$. We need to show that the sequence is not trivial, i.e. that it cannot be uniformized by a single total map from ω_1 into 2. In other words, we need to show that ρ_3 still contains enough information about the C-sequence C_α $(\alpha < \omega_1)$ from which it is defined. For this it will be convenient to assume that C_α $(\alpha < \omega_1)$ satisfies the following natural condition:

(d) If α is a limit ordinal > 0 and if ξ occupies the nth place in the increasing enumeration of C_α (that starts with $\min(C_\alpha)$ on its 0th place), then $\xi = \lambda + n + 1$ for some limit ordinal λ (possibly 0).

---

[17]Recall the way one always defines the fiber-functions from a two-variable function applied to the context of ρ_3: $(\rho_3)_\alpha(\xi) = \rho_3(\xi, \alpha)$.

4.41 Definition. Let Λ denote the set of all countable limit ordinals and for an integer $n \in \omega$, let $\Lambda + n = \{\lambda + n : \lambda \in \Lambda\}$.

4.42 Lemma. $\rho_3(\lambda + n, \beta) = 1$ *for all but finitely many n with $\lambda + n < \beta$.*

Proof. Clearly we may assume that $\alpha = \lambda + \omega \leq \beta$. Then there is $n_0 < \omega$ such that for every $n \geq n_0$ the walk $\beta \to \lambda + n$ passes through α. By (d) we know that $C_\alpha = \{\lambda + n : 0 < n < \omega\}$ so in any such walk $\beta \to \lambda + n$ there is only one step from α to $\lambda + n$. So choosing $n \geq n_0, \rho_1(\alpha, \beta)$ we will ensure that the last step of $\beta \to \lambda + n$ comes with the maximal weight, i.e. $\rho_3(\lambda + n, \beta) = 1$. $\dashv$

4.43 Lemma. *For all $\beta < \omega_1$, $n < \omega$, the set $\{\lambda \in \Lambda : \lambda + n < \beta$ and $\rho_3(\lambda + n, \beta) = 1\}$ is finite.*

Proof. Given an infinite subset Γ of $(\Lambda + n) \cap \beta$ we need to find a $\lambda + n \in \Gamma$ such that $\rho_3(\lambda + n, \beta) = 0$. Shrinking Γ if necessary assume that $\rho_1(\lambda + n, \beta) > n + 2$ for all $\lambda + n \in \Gamma$. So if $\rho_3(\lambda + n, \beta) = 1$ for some $\lambda + n \in \Gamma$ then the last step of $\beta \to \lambda + n$ would have to be of weight$> n + 2$ which is impossible by our assumption (d) about C_α ($\alpha < \omega_1$). $\dashv$

The meaning of these properties of ρ_3 perhaps is easier to comprehend if we reformulate them in a way that resembles the original formulation of the existence of Hausdorff gaps.

4.44 Lemma. *Let $B_\alpha = \{\xi < \alpha : \rho_3(\xi, \alpha) = 1\}$ for $\alpha < \omega_1$. Then:*

1. $B_\alpha =^* B_\beta \cap \alpha$ *for $\alpha < \beta$,*

2. $(\Lambda + n) \cap B_\beta$ *is finite for all $n < \omega$ and $\beta < \omega_1$,*

3. $\{\lambda + n : n < \omega\} \subseteq^* B_\beta$ *whenever $\lambda + \omega \leq \beta$.*

$\dashv$

In particular, there is is no uncountable $\Gamma \subseteq \omega_1$ such that $\Gamma \cap \beta \subseteq^* B_\beta$ for all $\beta < \omega_1$. On the other hand, the P-ideal[18] $\mathfrak{I}$ generated by B_β ($\beta < \omega_1$) is large as it contains all intervals of the form $[\lambda, \lambda + \omega)$. The following general dichotomy about P-ideals shows that here indeed we have quite a canonical example of a P-ideal on ω_1.

4.45 Definition.
  The P-ideal dichotomy. For every P-ideal $\mathfrak{I}$ of countable subsets of some set S either:

1. there is uncountable $X \subseteq S$ such that $[X]^\omega \subseteq \mathfrak{I}$, or

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[18]Recall that an ideal $\mathfrak{I}$ of subsets of some set S is a *P-ideal* if for every sequence A_n ($n < \omega$) of elements of $\mathfrak{I}$ there is B in $\mathfrak{I}$ such that $A_n \setminus B$ is finite for all $n < \omega$. A set X is *orthogonal* to $\mathfrak{I}$ if $X \cap A$ is finite for all A in $\mathfrak{I}$.

2. S can be decomposed into countably many sets orthogonal to $\mathfrak{I}$.

4.46 Remark. It is known that the P-ideal dichotomy is a consequence of the Proper Forcing Axiom and moreover that it does not contradict the Continuum Hypothesis (see [89]). This is an interesting dichotomy which will be used in this article for testing various notions of coherence as we encounter them. For example, let us consider the following notion of coherence, already encountered above at several places, and see how it is influenced by the P-ideal dichotomy.

5. Conditional weakly null sequences

In this section we mention some applications of the concept of ρ-function on ω_1 satisfying the inequalities of Lemma 4.2. As we shall see any such a semi-distance function ρ can be used as a coding procedure in constructions of long weakly null sequences in Banach spaces with no unconditional basic subsequences.

5.1 Example. *A weakly null sequence with no unconditional basic subsequence.* We describe a weakly null normalized sequence $(x_\xi)_{\xi<\omega_1}$ of members of some Banach space X which contains no infinite unconditional basic subsequence[19]. Fix an $0 < \varepsilon < 1$ and an infinite set M of positive integers such that $1 \in M$ and

$$\sum \{(\frac{m}{n})^{1/2} : m, n \in M, m < n\} \le \varepsilon. \qquad (\text{I.48})$$

Fix also a natural bijection $\ulcorner . \urcorner : \mathrm{HF} \to M$, where HF denotes the family of all hereditarily finite sets, so that in particular $\ulcorner \emptyset \urcorner = 1$. The *$\rho$-number* od a finite set $F \subseteq \omega_1$ is the integer

$$\rho(F) = \max\{\rho(\alpha, \beta) : \alpha, \beta \in F, \alpha \le \beta\}.$$

For $F \subseteq \omega_1$ and $k \in \omega$ let

$$(F)_k = \{\alpha \le \max(F) : \rho(\alpha, \beta) \le k \text{ for some } \beta \in F \text{ with } \beta \ge \alpha\}.$$

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[19]Recall that a *basic sequence* in a Banach space X is a seminormalized sequence $(x_n)_{n\in\omega}$ with the property that every x in its closed linear span $\overline{\mathrm{span}}\{x_n : n \in \omega\}$ can be uniquely represented as a sum $\sum_{n\in\omega} a_n x_n$. Note that this is equivalent to saying that the norms of projection operators $P_k : \overline{\mathrm{span}}\{x_n : n \in \omega\} \to \overline{\mathrm{span}}\{x_n : n < k\}$ are uniformly bounded. When the norms of projections $P_M : \overline{\mathrm{span}}\{x_n : n \in \omega\} \to \overline{\mathrm{span}}\{x_n : n \in M\}$ over all subsets M of ω are uniformly bounded, the sequence (x_n) is said to be *unconditional*. The natural basis (e_n) of any of the sequence-spaces ℓ_p for $p \in [1, \infty]$ is unconditional. A typical example of a basic sequence that is not unconditional is the *summing basis* $e_0, e_0+e_1, e_0+e_1+e_2, ...$ of c_0 or the summing basis $e_0+e_1+e_2+...+e_n+...,$ $e_1 + e_2 + ... + e_n + ..., e_2 + ... + e_n + ..., ...$ in c.

We say that F is *k-closed* if $(F)_k = F$. The *ρ-closure* of F is the set $(F)_\rho = (F)_k$, where $k = \rho(F)$. This leads us to the function

$$\sigma : [[\omega_1]^{<\omega}]^{<\omega} \longrightarrow M$$

defined by letting $\sigma(\emptyset) = \ulcorner\emptyset\urcorner$ and

$$\sigma(F_0, F_1, ..., F_n) = \ulcorner((\pi_E"F_0, \pi_E"F_1, ..., \pi_E"F_n), \rho(\bigcup_{i=0}^{n} F_i))\urcorner,$$

where $E = (\bigcup_{i=0}^{n} F_i)_\rho$ and where $\pi_E : E \to |E|$ is the unique order-preserving map between the sets of ordinals E and $|E| = \{0, 1, ..., |E| - 1\}$. We say that a sequence $(E_i)_{i<n}$ of finite subsets of ω_1 is *special* if

1. $E_i < E_j$[20] for $i < j < n$,

2. $|E_0| = 1$ and $|E_j| = \sigma(E_0, E_1, ..., E_{j-1})$ for $0 < j < n$.

Let $c_{00}(\omega_1)$ be the normed space of all finitely supported maps from ω_1 into the reals. Let $(e_\xi)_{\xi<\omega_1}$ be the basis of $c_{00}(\omega_1)$ and let $(e_\xi)^*_{\xi<\omega_1}$ be the corresponding sequence of biorthogonal functionals. To a finite set $E \subseteq \omega_1$, we associate the following vector and functional on $c_{00}(\omega_1)$,

$$x_E = \frac{1}{|E|^{1/2}} \sum_{\alpha \in E} e_\alpha \text{ and } \phi_E = \frac{1}{|E|^{1/2}} \sum_{\alpha \in E} e_\alpha^*. \tag{I.49}$$

Given a special sequence $(E_i)_{i<n}$ of finite subsets of ω_1, the corresponding *special functional* is defined by $\sum_{i<n} \phi_{E_i}$. Let $\mathcal{F}$ be the family of all special functionals of $c_{00}(\omega_1)$. This family induces the following norm on $c_{00}(\omega_1)$,

$$\| x \| = \sup\{\langle f, x \rangle : f \in \mathcal{F}\}. \tag{I.50}$$

The Banach space X is the completion of the normed space $(c_{00}(\omega_1), \| . \|)$. Note that $\| e_\alpha \| = 1$ and that $(e_\alpha)_{\alpha<\omega_1}$ is a weakly null sequence in X. To see this note that given a finite set G with $|G| \in M$ and a special sequence $(E_i)_{i<n}$, if $\phi = \sum_{i<n} \phi_{E_i}$ is the corresponding special functional, then the inner product between ϕ and the average $|G|^{-1} \sum_{\alpha \in G} e_\alpha = |G|^{-1/2} x_G$ is not bigger than $|G|^{-1/2}(1 + \sum_{|G| \neq m \in M} \min((m/|G|)^{1/2}, (|G|/m)^{1/2}))$ and therefore not bigger than $|G|^{-1/2}(1 + \varepsilon)$ by our assumption I.48. It follows that the averages $\| |G|^{-1} \sum_{\alpha \in G} e_\alpha \|$ tend to 0 as $|G| \in M$ goes to infinity, so $(e_\alpha)_{\alpha<\omega_1}$ must be weakly null in X. To show that no infinite subsequence of $(e_\alpha)_{\alpha<\omega_1}$ is unconditional it suffices to show that for every set $G \subseteq \omega_1$ of

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[20]Recall, that for two sets of ordinals E and F, by $E < F$ we denote the fact that every ordinal from E is smaller from every ordinal from F. Sequences of sets of ordinals of this sort are called *block sequences*.

order type ω there is seminormalized *block-subsequence*[21] of $(e_\alpha)_{\alpha \in G}$ which represents the summing basis in c. More precisely, we shall show that if $(E_i)_{i < \omega}$ is any infinite special sequence of finite subsets of G and if for $i < \omega$ we let $v_i = x_{E_i}$ as defined in I.49, then for every $n < \omega$ and every sequence $(a_i)_{i \leq n} \subseteq [-1, +1]$ of scalars.

$$\max_{0 \leq k \leq n} \mid \sum_{i=0}^{k} a_i \mid \leq \| \sum_{i=0}^{n} a_i v_i \| \leq (3 + \varepsilon) \max_{0 \leq k \leq n} \mid \sum_{i=0}^{k} a_i \mid . \qquad (\text{I.51})$$

To see the first inequality, note that according to I.49 the absolute value of the inner product between the spacial functional $\phi = \sum_{i=0}^{n} \phi_{E_i}$ and the vector $x = \sum_{i=0}^{n} a_i v_i$ is equal to $\mid \sum_{i=0}^{n} a_i \mid$ giving us the first inequality. To see the second inequality, consider a special functional $\psi = \sum_{i=0}^{m} \phi_{F_i}$ where $(F_i)_{i \leq m}$ is some other special sequence of finite sets. In order to estimate the inner product $\langle \psi, x \rangle$ we need to know how the two special sequences interact and this requires properties of the function ρ that controls them. To see this, let $l \leq \min(m, n)$ be maximal with the property that $|E_l| = |F_l|$, and let $E = (\bigcup_{i < l} E_i)_\rho$ and $F = (\bigcup_{i < l} F_i)_\rho$. By the definition of a special sequence and $|E_l| = |F_l|$, we know that:

1. $\rho(\bigcup_{i<l} E_i) = \rho(\bigcup_{i<l} F_i)$,

2. $\pi_E"E_i = \pi_E"F_i$ for $i < l$.

It follows that E and F have the same cardinality and the same ρ-numbers and therefore by the properties 4.1 of ρ, their intersection $H = E \cap F$ is their initial segment. Hence, π_E and π_E agree on H. Let k be the maximal integer such that $E_k \subseteq H$. Then $E_i = F_i$ for all $i \leq k$, and using the properties 4.1 of ρ again, $E_i \cap F_j = \emptyset$ for $k + 1 < i, j < l$. Combining all this we see that $|\langle \psi, x \rangle|$ is bounded by

$$\mid \sum_{i=0}^{k} a_i + a_{k+1} \langle \phi_{F_{k+1}}, x_{E_{k+1}} \rangle + a_l \langle \phi_{F_l}, x_{E_l} \rangle \mid + \sum \{ (\frac{m}{n})^{1/2} : m, n \in M, m < n \}.$$

Referring to I.48 we get that $|\langle \psi, x \rangle|$ is bounded by the right hand side of the second inequality of I.51.

5.2 Remark. The first example of a weakly null sequence (of length ω) with no unconditional basic subsequence was constructed by Maurey and Rosenthal [51]. Sixteen years latter Gowers and Maurey [27] have solved the basic unconditional sequence problem by constructing an infinite-dimensional separable reflexive Banach space with no infinite unconditional basic sequence. The following result of [3] relates to the Example 5.1 the same way the result of [27] relates to the example of [51].

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[21]i.e., a seminormalized sequence $(x_i)_{i < \omega}$ of finitely supported vectors of the linear span of $(e_\alpha)_{\alpha \in G}$ such that $\text{supp}(x_i) < \text{supp}(x_j)$ for $i < j$.

5.3 Example. *A reflexive Banach space X_{ω_1} with a Schauder basis of length ω_1 containing no unconditional basic sequence.* The space X_{ω_1} is the completion of $c_{00}(\omega_1)$ under norm given by the formula I.50, where now $\mathcal{F} = \mathcal{F}_{\omega_1}$ is chosen as the minimal set of finitely supported functionals on $c_{00}(\omega_1)$ such that:

1. $\mathcal{F} \supseteq \{e_\alpha^* : \alpha < \omega_1\}$ and $\mathcal{F}$ is symmetric and closed under restriction on intervals of ω_1.

2. For every $(\phi_i)_{i<n_{2k}} \subseteq \mathcal{F}$ such that $\text{supp}(\phi_i) < \text{supp}(\phi_j)$ for $i < j < n_{2k}$, the functional $m_{2k}^{-1} \sum_{i<n_{2k}} \phi_i$ belongs to $\mathcal{F}$.

3. For every *special sequence* $(\phi_i)_{i<n_{2k+1}} \subseteq \mathcal{F}$ the corresponding *special functional* $m_{2k+1}^{-1} \sum_{i<n_{2k+1}} \phi_i$ belongs to $\mathcal{F}$.

4. $\mathcal{F}$ is closed under rational convex combinations.

The increasing sequences (m_k) and (n_k) of positive integers are defined by: $m_k = 2^{4^k}$ and $n_0 = 4$ and $n_{k+1} = (4n_k)^{\log_2 m_{k+1}^3}$. The notion of a *special sequence of functionals* is defined analogously to that in the Example 5.1 as follows. First of all we say that $\phi \in \mathcal{F}$ is of type 0 if $\phi = \pm e_\alpha^*$ for some α, of type *I* if it obtained from a simpler ones as the sum $m_k^{-1} \sum_{i<n_k} \phi_i$ restricted to some interval and we call the integer m_k, the *weight* of ϕ, and denote it $w(\phi)$. Note that a given functional of type *I* can have more than one representation and therefore more than one weight. We say that ϕ is of type *II* if it is a rational convex combination of functionals of type 0 and type *I*. The definition of a special sequence of functionals will be again based on ρ and an injection $\ulcorner . \urcorner : \text{HF} \to \{2k : k \text{ odd }\}$, with the property that for every finite sequence $(\phi_i, w_i, p_i)_{i<j}$ where $w_i, p_i \in \omega$ and where ϕ_i's are finite partial maps from ω into $\mathbb{Q}$ such that $\ulcorner (\phi_i, w_i, p_i)_{i<j} \urcorner$ is bigger than the maximum of support of any of the ϕ_i's, the square of any integer of the form w_i or p_i, and the ratio of the square of any nonzero value of a ϕ_i. A sequence $(\phi_i)_{i<n_{2k+1}}$ of members of $\mathcal{F}$ is a *special sequence of functionals*, if

1. $\text{supp}(\phi_i) < \text{supp}(\phi_j)$ for $i < j < n_{2k+1}$,

2. each ϕ_i is of type *I* and of some fixed weight $w(\phi_i) = m_{2k_i}$ with k_0 even and satisfying $m_{2k_0} > n_{2k+1}^2$,

3. for $0 < j < n_{2k+1}$ there is an increasing sequence $(p_i)_{i<j}$ of integers such that $p_i \geq \rho(\bigcup_{i'<i} \text{supp}(\phi_{i'}))$ and $w(\phi_j) = m_{\ulcorner (\pi_E \text{"} \phi_i, w(\phi_i), p_i)_{i<j} \urcorner}$, where $E = (\bigcup_{i<j} \text{supp}(\phi_i))_\rho$.

(The collapse $\psi_i = \pi_E \text{"} \phi_i$ is defined by the formula $\psi(\pi_E(\alpha)) = \phi(\alpha)$.) The condition 1. from the definition of $\mathcal{F}$ provides that $(e_\alpha)_{\alpha<\omega_1}$ is a bimonotone Schauder basis of X_{ω_1}, the condition 3. is responsible for the conditional

structure of the norm of X_{ω_1} in a similar manner as in the Example 5.1, but in order to come to the point the idea can be applied, one needs the unconditional structure given but the closure property 2. of $\mathcal{F}$. This requires a rather deep tree-analysis of the functionals of $\mathcal{F}$ that gives the required norm estimates, an analysis that is beyond the scope of this presentation. We refer the reader to [3] for the details. Here we just list some of the most interesting properties of X_{ω_1}. For example, the space X_{ω_1} is reflexive, has no infinite unconditional basic sequence, and is not isomorphic to any proper subspace or a nontrivial quotient. The basis $(e_\alpha)_{\alpha<\omega_1}$ allows us to talk about diagonal operators of X_{ω_1}. It turns out that all bounded diagonal operators D of X_{ω_1} are given by a sequence $(\lambda_\alpha)_{\alpha<\omega_1}$ of eigenvalues that have the property that $\lambda_\alpha = \lambda_\beta$ whenever $\alpha \leq \beta < \alpha + \omega$. Moreover there is only countably many different eigenvalues among $(\lambda_\alpha)_{\alpha<\omega_1}$. Recall, than an operator between two Banach spaces is *strictly singular* if it is not an isomorphism when restricted to any infinite-dimensional subspace of its domain. It turns out that every bounded operator T on X_{ω_1} has the form $D+S$, where D is a diagonal-step operator with countably many eigenvalues and where X is a strictly singular operator. It follows that in particular that every bounded linear operator on X_{ω_1} is a multiple of the identity plus the operator with separable range[22]. It turns out that we can actually identify the space $\mathcal{D}(X_{\omega_1})$ of bounded diagonal operators as the dual of the James space over a particular reflexive Banach space T_0 and ω_1 as the corresponding ordered set (see [3] for the definition of this functor). One can similarly characterize the operator space of arbitrary closed subspace of X_{ω_1} as well obtaining thus some interesting new phenomena about operator spaces. For example one can find a closed subspace Y of X_{ω_1} such that every bounded operator $T : Y \to Y$ is a strictly singular perturbation of a multiple of identity while operator space $\mathcal{L}(Y, X_{\omega_1})$ is quite rich in the sense that

$$\mathcal{L}(Y, X_{\omega_1}) = J_{T_0}^* \oplus \mathcal{S}(Y, X_{\omega_1}).$$

Taking the restrictions $X_I = \overline{\mathrm{span}}(\{e_\alpha : \alpha \in I\})$ to infinite intervals $I \subseteq \omega_1$ one obtains an uncountable family of separable reflexive space which, with an appropriate adjustment on ρ, are 'asymptotic versions' of each other. This is the new phenomenon in the separable theory not revealed by the previous methods. It essential says that any finite-dimensional subspace of one X_I can be moved into any other X_J by an operator T such that $\| T \| \| T^{-1} \| \leq 4 + \varepsilon$.

5.4 Question. What is the minimal cardinal θ_u with property that every weakly null sequence of length θ_u contains an infinite unconditional subsequence?

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[22] A considerably easier example of a space with this property will be given in some detail in 7.16 below.

We have seen above that $\theta_u \geq \omega_2$. From a result of Ketonen [39] we infer that θ_u is not bigger than the first Ramsey cardinal. Not much more about θ_u seems to be known.

6. An ultrafilter from a coherent sequence

6.1 Definition. A mapping $a : [\omega_1]^2 \longrightarrow \omega$ is *coherent* if for every $\alpha < \beta < \omega_1$ there exist only finitely many $\xi < \alpha$ such that $a(\xi, \alpha) \neq a(\xi, \beta)$, or in other words, $a_\alpha =^* a_\beta \restriction \alpha^{23}$. We say that a is *nontrivial* if there is no $h : \omega_1 \longrightarrow \omega$ such that $h \restriction \alpha =^* a_\alpha$ for all $\alpha < \omega_1$.

Note that the existence of a coherent and nontrivial $a : [\omega_1]^2 \longrightarrow 2$ (such as, for example, the function ρ_3 defined above) is something that corresponds to the notion of a *Hausdorff gap* in this context. Notice moreover, that this notion is also closely related to the notion of an *Aronszajn tree* since

$$T(a) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1 \text{ and } t =^* a_\alpha\}$$

is such an Aronszajn tree whenever $a : [\omega_1]^2 \longrightarrow \omega$ is coherent and non-trivial[24]. In fact, we shall call an arbitrary Aronszajn tree T *coherent* if T is isomorphic to $T(a)$ for some coherent and nontrivial $a : [\omega_1]^2 \longrightarrow \omega$. In case the range of the map a is actually smaller than ω, e.g. equal to some integer k, then it is natural to let $T(a)$ be the collection of all $t : \alpha \longrightarrow k$ such that $\alpha < \omega_1$ and $t =^* a_\alpha$. This way, we have coherent binary, ternary, etc. Aronszajn trees rather than only ω-ary coherent Aronszajn trees.

6.2 Definition. The *support* of a map $a : [\omega_1]^2 \longrightarrow \omega$ is the sequence $\mathrm{supp}(a_\alpha) = \{\xi < \alpha : a(\xi, \alpha) \neq 0\}$ $(\alpha < \omega_1)$ of subsets of ω_1. A set Γ is *orthogonal* to a if $\mathrm{supp}(a_\alpha) \cap \Gamma$ is finite for all $\alpha < \omega_1$. We say that $a : [\omega_1]^2 \longrightarrow \omega$ is *nowhere dense* if there is no uncountable $\Gamma \subseteq \omega_1$ such that $\Gamma \cap \alpha \subseteq^* \mathrm{supp}(a_\alpha)$ for all $\alpha < \omega_1$.

Note that ρ_3 is an example of a nowhere dense coherent map for the simple reason that ω_1 can be covered by countably many sets $\Lambda + n$ $(n < \omega)$ that are orthogonal to ρ_3. The following immediate fact shows that ρ_3 is indeed a prototype of a nowhere dense and coherent map $a : [\omega_1]^2 \longrightarrow \omega$.

6.3 Proposition. *Under the P-ideal dichotomy, for every nowhere dense and coherent map* $a : [\omega_1]^2 \longrightarrow \omega$ *the domain* ω_1 *can be decomposed into countably many sets orthogonal to* a. $\dashv$

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[23]A mapping $a : [\omega_1]^2 \longrightarrow \omega$ is naturally identified with a sequence a_α $(\alpha < \omega_1)$, where $a_\alpha : \alpha \longrightarrow \omega$ is defined by $a_\alpha(\xi) = a(\xi, \alpha)$.

[24]Similarities between the notion of a Hausdorff gap and the notion of an Aronszajn tree have been further explained recently in the two papers of Talayco ([72], [73]), where it is shown that they naturally correspond to first cohomology groups over a pair of very similar spaces.

6.4 Notation. To every $a : [\omega_1]^2 \longrightarrow \omega$ associate the corresponding Δ-function $\Delta_a : [\omega_1]^2 \longrightarrow \omega$ as follows:

$$\Delta_a(\alpha, \beta) = \min\{\xi < \alpha : a(\xi, \alpha) \neq a(\xi, \beta)\}$$

with the convention that $\Delta_a(\alpha, \beta) = \alpha$ whenever $a(\xi, \alpha) = a(\xi, \beta)$ for all $\xi < \alpha$. Given this notation, it is natural to let

$$\Delta_a(\Gamma) = \{\Delta_a(\alpha, \beta) : \alpha, \beta \in \Gamma, \alpha < \beta\}$$

for an arbitrary set $\Gamma \subseteq \omega_1$.

6.5 Lemma. *Suppose that a is nontrivial and coherent and that every uncountable subset of $T(a)$ contains an uncountable antichain. Then for every pair Σ, Ω of uncountable subsets of ω_1 there exists an uncountable subset Γ of ω_1 such that $\Delta_a(\Gamma) \subseteq \Delta_a(\Sigma) \cap \Delta_a(\Omega)$.*

Proof. For each limit ordinal $\xi < \omega_1$ we fix $\alpha(\xi) \in \Sigma$ and $\beta(\xi) \in \Omega$ above ξ and let

$$F_\xi = \{\eta < \xi : a(\eta, \alpha(\xi)) \neq a(\eta, \xi) \text{ or } a(\eta, \beta(\xi)) \neq a(\eta, \xi)\}.$$

By the Pressing Down Lemma there is a stationary set $\Gamma \subseteq \omega_1$ and a finite set F such that $F_\xi = F$ for all $\xi \in \Gamma$. Let $\eta_0 = \max(F) + 1$. By the assumption about $T(a)$, find an uncountable $\Gamma_0 \subseteq \Gamma$ such that a_ξ ($\xi \in \Gamma_0$) is an antichain of $T(a)$. Moreover, we may assume that for some s, t and u in $T(a)$:

$$a_{\alpha(\xi)} \upharpoonright \eta_0 = s, a_{\beta(\xi)} \upharpoonright \eta_0 = t \text{ and } a_\xi \upharpoonright \eta_0 = u \text{ for all } \xi \in \Gamma_0. \qquad (\text{I}.52)$$

It follows that for all $\xi < \eta$ in Γ_0:

$$\Delta_a(\alpha(\xi), \alpha(\eta)) = \Delta_a(\xi, \eta) = \Delta_a(\beta(\xi), \beta(\eta)). \qquad (\text{I}.53)$$

So in particular, $\Delta_a(\Gamma_0) \subseteq \Delta_a(\Sigma) \cap \Delta_a(\Omega)$. $\qquad \dashv$

6.6 Notation. For $a : [\omega_1]^2 \longrightarrow \omega$, set

$$\mathcal{U}(a) = \{A \subseteq \omega_1 : A \supseteq \Delta_a(\Gamma) \text{ for some uncountable } \Gamma \subseteq \omega_1\}.$$

By Lemma 6.5, $\mathcal{U}(a)$ is a uniform filter on ω_1 for every nontrivial coherent $a : [\omega_1]^2 \longrightarrow \omega$ for which $T(a)$ contains no Souslin subtrees. It turns out that under some very mild assumption, $\mathcal{U}(a)$ is in fact a uniform ultrafilter on ω_1.

6.7 Theorem. *Under MA_{ω_1}, the filter $\mathcal{U}(a)$ is an ultrafilter for every nontrivial and coherent $a : [\omega_1]^2 \longrightarrow \omega$.*

Proof. Let A be a given subset of ω_1 and let $\mathcal{P}$ be the collection of all finite subsets p of ω_1 such that $\Delta_a(p) \subseteq A$. If $\mathcal{P}$ is a ccc poset then an application of MA_{ω_1} would give us an uncountable $\Gamma \subseteq \omega_1$ such that $\Delta_a(\Gamma) \subseteq A$, showing thus that A belongs to $\mathcal{U}(a)$. So let us consider the alternative when $\mathcal{P}$ is not a ccc poset. Let $\mathcal{X}$ be an uncountable subset of $\mathcal{P}$ consisting of pairwise incompatible conditions of $\mathcal{P}$. Going to an uncountable Δ-subsystem of $\mathcal{X}$ and noticing that the root does not contribute to the incompatibility, one sees that without loss of generality, we may assume that the members of $\mathcal{X}$ are in fact pairwise disjoint. Thus for each limit ordinal $\xi < \omega_1$ we can fix p_ξ in $\mathcal{X}$ lying entirely above ξ. For $\xi < \omega_1$, let

$$F_\xi = \{\eta < \xi : a(\eta, \xi) \neq a(\eta, \alpha) \text{ for some } \alpha \in p_\xi\}.$$

By the Pressing Down Lemma find a stationary set $\Gamma \subseteq \omega_1$, a finite set $F \subseteq \omega_1$, an integer n and $t_0, \ldots, t_{n-1}$ in $T(a)$ of height $\eta_0 = \max(F) + 1$ such that for all $\xi \in \Gamma$:

$$F_\xi = F, \tag{I.54}$$

$$a_\alpha \restriction \eta_0 = t_i \text{ with } \alpha \text{ the } i\text{th member of } p_\xi \text{ for some } i < n. \tag{I.55}$$

By MA_{ω_1} the tree $T(a)$ is special, so going to an uncountable subset of Γ we may assume that a_ξ ($\xi \in \Gamma$) is an antichain of $T(a)$, and moreover that for every $\xi < \eta$ in Γ:

$$a_\alpha \restriction \xi \nsubseteq a_\beta \restriction \eta \text{ for all } \alpha \in p_\xi \text{ and } \beta \in p_\eta. \tag{I.56}$$

It follows that for every $\xi < \eta$ in Γ:

$$\Delta_a(\alpha, \beta) = \Delta_a(\xi, \eta) \text{ for all } \alpha \in p_\xi \text{ and } \beta \in p_\eta. \tag{I.57}$$

By our assumption that conditions from $\mathcal{X}$ are incompatible in $\mathcal{P}$, we conclude that for every $\xi < \eta$ in Γ, there exist $\alpha \in p_\xi$ and $\beta \in p_\eta$ such that $\Delta_a(\alpha, \beta) \notin A$. So from (I.57) we conclude that $\Delta_a(\Gamma) \cap A = \emptyset$, showing thus that the complement of A belongs to $\mathcal{U}(a)$. $\dashv$

6.8 Remark. One may find Theorem 6.7 a bit surprising in view of the fact that it gives us an ultrafilter $\mathcal{U}(a)$ on ω_1 that is Σ_1-definable over the structure $(H_{\omega_2}, \in)$. It is well-known that there is no ultrafilter on ω that is Σ_1-definable over the structure $(H_{\omega_1}, \in)$.

It turns out that the transformation $a \longmapsto \mathcal{U}(a)$ captures some of the essential properties of the corresponding and more obvious transformation $a \longmapsto T(a)$. To state this we need some standard definitions.

6.9 Definition. For two trees S and T, by $S \leq T$ we denote the fact that there is a strictly increasing map $f : S \longrightarrow T$. Let $S < T$ whenever $S \leq T$ and $T \nleq S$ and let $S \equiv T$ whenever $S \leq T$ and $T \leq S$. In general,

the equivalence relation $\equiv$ on trees is very far from the finer relation $\cong$, the isomorphism relation. However, the following fact shows that in the realm of trees $T(a)$, these two relations may coincide and moreover, that the mapping $T(a) \longmapsto \mathcal{U}(a)$ reduces $\equiv$ and $\cong$ to the equality relation among ultrafilters on ω_1 (see [90]).

6.10 Theorem. *Assuming* MA_{ω_1}, *for every pair of coherent and nontrivial mappings* $a : [\omega_1]^2 \longrightarrow \omega$ *and* $b : [\omega_1]^2 \longrightarrow \omega$, *the trees* $T(a)$ *and* $T(b)$ *are isomorphic iff* $T(a) \equiv T(b)$ *iff* $\mathcal{U}(a) = \mathcal{U}(b)$.

Proof. We only prove that $T(a) \equiv T(b)$ is equivalent to $\mathcal{U}(a) = \mathcal{U}(b)$, referring the reader to [90] for the remaining part of the argument.

Choose a pair of strictly increasing mappings $f : T(a) \longrightarrow T(b)$ and $g : T(b) \longrightarrow T(a)$. Replacing f by the mapping $t \longmapsto f(t) \upharpoonright \mathrm{dom}(t)$ and g by the mapping $t \longmapsto g(t) \upharpoonright \mathrm{dom}(t)$, we may assume that f and g are also level-preserving. Recall our notation $a_\delta(\xi) = a(\xi, \delta)$ and $b_\delta(\xi) = b(\xi, \delta)$ that gives us particular representatives a_δ and b_δ of the δth level of $T(a)$ and $T(b)$ respectively. For $\delta \in \Lambda$, set

$$F_\delta = \{\xi < \delta : a_\delta(\xi) \neq g(f(a_\delta))(\xi) \text{ or } b_\delta(\xi) \neq f(a_\delta)(\xi)\}.$$

Then F_δ is a finite subset of δ. Find a stationary subset Σ of Λ and $F \subseteq \omega_1$ such that $F_\delta = F$ for all $\delta \in \Sigma$. Find now an uncountable $\Gamma \subseteq \Sigma$ such that a_δ $(\delta \in \Gamma)$ is an antichain of $T(a)$, b_δ $(\delta \in \Gamma)$ is an antichain of $T(b)$, and the mapping

$$\delta \longmapsto (a_\delta \upharpoonright \bar{\xi}, f(a_\delta) \upharpoonright \bar{\xi}, g(f(a_\delta)) \upharpoonright \bar{\xi}, b_\delta \upharpoonright \bar{\xi})$$

is constant on Γ, where $\bar{\xi} = \max(F) + 1$. It follows that for $\gamma \neq \delta$ in Γ,

$$\Delta(a_\gamma, a_\delta) = \Delta(g(f(a_\gamma)), g(f(a_\delta))), \qquad (\mathrm{I}.58)$$

$$\Delta(b_\gamma, b_\delta) = \Delta(f(a_\gamma), f(a_\delta)). \qquad (\mathrm{I}.59)$$

Since f is strictly increasing, we must have $\Delta(a_\gamma, a_\delta) \leq \Delta(f(a_\gamma), f(a_\delta)) = \Delta(b_\gamma, b_\delta)$. Since g is strictly increasing, we have that

$$\Delta(b_\gamma, b_\delta) = \Delta(f(a_\gamma), f(a_\delta)) \leq \Delta(g(f(a_\gamma)), g(f(a_\delta))) = \Delta(a_\gamma, a_\delta).$$

It follows that $\Delta(a_\gamma, a_\delta) = \Delta(b_\gamma, b_\delta)$ for all $\gamma \neq \delta$ in Γ. From the proof of Lemma 6.5 we conclude that $\Delta_a(\Omega)$ $(\Omega \subseteq \Gamma, \Omega$ uncountable$)$ generates the filter $\mathcal{U}(a)$ and similarly that $\Delta_b(\Omega)$ $(\Omega \subseteq \Gamma, \Omega$ uncountable$)$ generates the ultrafilter $\mathcal{U}(b)$. It follows that $\mathcal{U}(a) = \mathcal{U}(b)$.

Suppose now that $\mathcal{U}(a) = \mathcal{U}(b)$. By symmetry, it suffices to show that $T(a) \leq T(b)$. Note again that the argument in the proof of Lemma 6.5 shows that there is a stationary set $\Gamma \subseteq \Lambda$ such that Δ_a and Δ_b coincide on $[\Gamma]^2$. Let $\mathcal{P}$ be the poset of all finite partial strictly increasing level-preserving

mappings p from $T(a)$ into $T(b)$ which (uniquely) extend to strictly increasing level-preserving mappings on the downward-closures of their domains. By MA_{ω_1}, it suffices to show that $\mathcal{P}$ satisfies the countable chain condition. So let p_δ ($\delta \in \Gamma$) be a given sequence of conditions of $\mathcal{P}$. For $\delta \in \Gamma$, set

$$E_\delta = \{\xi < \delta : a_\delta(\xi) \neq t(\xi) \text{ or } b_\delta(\xi) \neq p(t)(\xi)$$

$$\text{for some } t \in \mathrm{dom}(p_\delta) \text{ of height} \geq \delta\}.$$

Then E_δ is a finite subset of δ for each $\delta \in \Gamma$. Find a stationary set $\Sigma \subseteq \Gamma$ and $E \subseteq \omega_1$ such that $E_\delta = E$ for all $\delta \in \Sigma$. Shrinking Σ, we may assume that the mapping $\delta \longmapsto p_\delta \restriction \delta$ is constant on Σ. Let $\bar\eta$ be the minimal ordinal strictly above $\max(E)$ and the height of every node of $\mathrm{dom}(p_\delta) \restriction \delta$ for some (all) $\delta \in \Sigma$. For $\delta \in \Sigma$, let $\hat{p}_\delta$ be the extension of p_δ on the downward-closure of its domain in $T(a)$. Find a stationary $\Omega \subseteq \Sigma$ such that each of the projections $\delta \longmapsto \hat{p}_\delta \restriction \bar\eta$, $\delta \longmapsto a_\delta \restriction \bar\eta$ and $\delta \longmapsto b_\delta \restriction \bar\eta$ is constant on Ω. Find $\gamma < \delta$ in Ω such that $a_\gamma \neq a_\delta \restriction \gamma$, $b_\gamma \neq b_\delta \restriction \gamma$ and $\hat{p}_\gamma$ and $\hat{p}_\delta$ are isomorphic via the isomorphism that fixes their common restriction to the $\bar\eta$th levels of $T(a)$ and $T(b)$. It suffices to check that $q = p_\gamma \cup p_\delta$ satisfies

$$\Delta(q(x), q(y)) \geq \Delta(x, y) \tag{I.60}$$

for every $x \in \mathrm{dom}(p_\gamma)$ and $y \in \mathrm{dom}(p_\delta)$. If x and y correspond to each other in the isomorphism between $\hat{p}_\gamma$ and $\hat{p}_\delta$, then either they belong to the common part of p_γ and p_δ in which case (I.60) follows by the assumption that p_γ and p_δ are strictly increasing level-preserving, or we have

$$\Delta(x, y) = \Delta(a_\gamma, a_\delta) = \Delta(b_\gamma, b_\delta) = \Delta(p_\gamma(x), p_\delta(y)). \tag{I.61}$$

If x is different from the copy $\bar{y} \in \mathrm{dom}(p_\gamma)$ of $y \in \mathrm{dom}(p_\delta)$, then $\Delta(x, y) = \Delta(x, \bar{y})$ and $\Delta(p_\gamma(x), p_\delta(y)) = \Delta(p_\gamma(x), p_\gamma(y))$, so we conclude that (I.60) follows from the fact that p_γ is strictly increasing level-preserving. This finishes the proof. $\dashv$

6.11 Remark. It turns out that the ultrafilters of the form $\mathcal{U}(a)$ for $a : [\omega_1]^2 \longrightarrow \omega$ nontrivial and coherent are all Rudin-Keisler equivalent. The proof of this fact can be found in [90].

6.12 Definition. The *shift* of $a : [\omega_1]^2 \longrightarrow \omega$ is defined to be the mapping $a^{(1)} : [\omega_1]^2 \longrightarrow \omega$ determined by

$$a^{(1)}(\alpha, \beta) = a(\alpha + 1, \hat{\beta}),$$

where $\hat{\beta} = \min\{\lambda \in \Lambda : \lambda \geq \beta\}$. The n-fold iteration of the shift operation is defined recursively by the formula $a^{(n+1)} = (a^{(n)})^{(1)}$.

The following fact about this shift operation, together with Lemma 4.44, can be used to show that Aronszajn trees are not well-quasi-ordered under the quasi-ordering $\leq$ (see [90]).

6.13 Theorem. *If a is nontrivial, coherent and orthogonal to Λ, then it holds that $T(a) > T(a^{(1)})$.*

Proof. By the assumption $a \perp \Lambda$, the subtree S of $T(a)$ consisting of all t's that take the value 0 at every limit ordinal in their domains is an uncountable initial part of $T(a)$. Note that every node of $T(a)$ is equal to the restriction of some finite change of a node from S. For a node t at some limit level λ of S, let $t^{(1)} : \lambda \longrightarrow \omega$ be determined by the formula $t^{(1)}(\alpha) = t(\alpha + 1)$. Let $S^{(1)} = \{t^{(1)} : t \in S \restriction \lambda\}$. Every node of $T(a^{(1)})$ is equal to the restriction of some finite change of a node of $S^{(1)}$. Note also that $t \longmapsto t^{(1)}$ is a one-to-one map on $S \restriction \Lambda$ and that its inverse naturally extends to a strictly increasing map from $T(a^{(1)})$ into T. This shows that $T(a^{(1)}) \leq T(a)$.

It remains to establish $T(a) \not\leq T(a^{(1)})$. Note that if there is a strictly increasing $g : T(a) \longrightarrow T(a^{(1)})$, then there is one that is moreover level-preserving. For example $f(t) = g(t) \restriction |t|$ is one such map. So let $f : T(a) \longrightarrow T(a^{(1)})$ be a given level-preserving map. For a countable limit ordinal ξ, set

$$F_\xi = (\mathrm{supp}(a_\xi) \cap \Lambda) \cup \{\alpha < \xi : a_\xi^{(1)}(\alpha) \neq f(a_\xi)(\alpha)\}. \qquad \text{(I.62)}$$

Then F_ξ is a finite set of ordinals $< \xi$, so by the Pressing Down Lemma there is a stationary set $\Gamma \subseteq \Lambda$ and $F \subseteq \omega_1$ such that $F_\xi = F$ for all $\xi \in \Gamma$. Let $\alpha_0 = \max(F) + 1$. Shrinking Γ if necessary, assume that for some $s, t : \alpha_0 \longrightarrow 2$,

$$a_\xi \restriction \alpha_0 = s \text{ and } f(a_\xi) \restriction \alpha_0 = t \text{ for all } \xi \in \Gamma. \qquad \text{(I.63)}$$

Choose $\xi < \eta$ in Γ such that $a_\xi \neq a_\eta \restriction \xi$ and let $\delta = \Delta(a_\xi, a_\eta)(= \min\{\alpha < \xi : a_\xi(\alpha) \neq a_\eta(\alpha)\})$. Then by (I.62) and (I.63), δ is not a limit ordinal and

$$\delta - 1 = \Delta(a_\xi^{(1)}, a_\eta^{(1)}) = \Delta(f(a_\xi), f(a_\eta)). \qquad \text{(I.64)}$$

It follows that a_ξ and a_η split in $T(a)$ after their images $f(a_\xi)$ and $f(a_\eta)$ split in $T(a^{(1)})$, which cannot happen if f is strictly increasing. This finishes the proof. $\dashv$

6.14 Corollary. *If a is nontrivial, coherent and orthogonal to $\Lambda + n$ for all $n < \omega$, then $T(a^{(n)}) > T(a^{(m)})$ whenever $n < m < \omega$.*

Proof. Note that if a is orthogonal to $\Lambda + n$ for all $n < \omega$, then so is every of its finite shifts $a^{(m)}$. $\dashv$

6.15 Corollary. $T(\rho_3^{(n)}) > T(\rho_3^{(m)})$ *whenever* $n < m < \omega$. $\dashv$

Somewhat unexpectedly, with very little extra assumptions we can say much more about $\leq$ in the domain of coherent Aronszajn trees (see [90]).

6.16 Theorem. *Under* MA_{ω_1}, *the family of coherent Aronszajn trees is totally ordered under* $\leq$.

Proof. Given two nontrivial coherent sequences a and b, we need to show that either $T(a) \leq T(b)$ or $T(b) \leq T(a)$. It turns out that these two alternatives correspond to the following two cases.

   Case 1: There is uncountable $\Omega \subseteq \omega_1$ such that Δ_a is dominated by Δ_b on $[\Omega]^2$. Let $\mathcal{P}$ be the poset of finite partial level-preserving functions p from $T(a)$ into $T(b)$ that can be extended to strictly increasing level-preserving maps from $\hat{D}_p$ into $T(b)$, where $\hat{D}_p$ is the downward closure of the domain D_p of p. Let R_p denote the range of p and $\hat{R}_p$ its downward closure in $T(b)$. If $\mathcal{P}$ is a ccc poset, then applying MA_{ω_1} to $\mathcal{P}$ and the natural sequence of dense open subsets of $\mathcal{P}$ would give us the desired strictly increasing level-preserving map from $T(a)$ into $T(b)$. So let p_ξ $(\xi < \omega_1)$ be a given one-to-one sequence of members of $\mathcal{P}$. We may assume that the p_ξ's form a Δ-system with root r. Let $D_\xi = D_{p_\xi} \setminus D_r$ and $R_\xi = R_{p_\xi} \setminus R_r$ for $\xi < \omega_1$. For limit ξ fix $g(\xi) < \xi$ and $h(\xi) \in \Omega \setminus \xi$ such that for every $t \in D_\xi$ with height $\geq \xi$:

$$t \equiv a_\xi \equiv a_{h(\xi)} \quad \text{on } [g(\xi), \xi), \tag{I.65}$$

$$p_\xi(t) \equiv b_\xi \equiv b_{h(\xi)} \quad \text{on } [g(\xi), \xi). \tag{I.66}$$

By the Pressing Down Lemma, there is a stationary $\Sigma \subseteq \omega_1$ such that g is constantly equal to η_0 on Σ and moreover that the projection maps of $D_\xi \cup \{a_\xi, a_{h(\xi)}\}$ onto the η_0th level of $T(a)$ and $R_\xi \cup \{b_\xi, b_{h(\xi)}\}$ onto the η_0th level of $T(b)$ are constant as long as ξ belongs to Σ. Moreover we may assume for $\xi \neq \eta$, every node of $D_\xi \cup \{a_\xi, a_{h(\xi)}\}$ is incomparable to every node of $D_\eta \cup \{a_\eta, a_{h(\eta)}\}$, and similarly, every node of $R_\xi \cup \{b_\xi, b_{h(\xi)}\}$ is incomparable to every node of $R_\eta \cup \{b_\eta, b_{h(\eta)}\}$. Note that we may also assume that every node of D_ξ, for $\xi \in \Sigma$, has height $\geq \xi$. So pick $\xi \neq \eta$ in Σ. Then for all $s \in D_\xi$ and $t \in D_\eta$:

$$\Delta(s, t) = \Delta(a_{h(\xi)}, a_{h(\eta)}) = \Delta_a(h(\xi), h(\eta)), \tag{I.67}$$

$$\Delta(p_\xi(s), p_\eta(t)) = \Delta(b_{h(\xi)}, b_{h(\eta)}) = \Delta_b(h(\xi), h(\eta)). \tag{I.68}$$

Since $\{h(\xi), h(\eta)\}$ belongs to $[\Omega]^2$, we conclude that for all $s \in D_\xi$ and $t \in D_\eta$, s and t split before $p_\xi(s)$ and $p_\eta(t)$. So $p_\xi \cup p_\eta$ extends to a strictly increasing level-preserving function from the downward closure of $D_{p_\xi} \cup D_{p_\eta}$ into the downward closure of $R_{p_\xi} \cup R_{p_\eta}$. This finishes the proof that $\mathcal{P}$ is a ccc forcing relation.

Case 2: For every uncountable $\Omega \subseteq \omega_1$ there exists $\{\alpha, \beta\}$ in $[\Omega]^2$ such that $\Delta_a(\alpha, \beta) > \Delta_b(\alpha, \beta)$. Let $\mathcal{Q}$ be the poset of all finite subsets q of ω_1 such that Δ_a dominates Δ_b on $[q]^2$. If $\mathcal{Q}$ is a ccc poset, then an application of MA_{ω_1} would give us an uncountable subset Ω of ω_1 such that Δ_a dominates Δ_b on $[\Omega]^2$, so by Case 1 we would conclude that $T(b) \leq T(a)$. To show that $\mathcal{Q}$ is a ccc poset, let q_ξ ($\xi < \omega_1$) be a given one-to-one sequence of members of $\mathcal{Q}$. We may assume the q_ξ's form a Δ-system. Note that the root does not contribute to the incompatibility of two conditions of the Δ-system and so removing it, we assume that all q_ξ's are in fact pairwise disjoint. For a limit $\xi < \omega_1$, fix $g(\xi) < \xi$ such that for all $\alpha \in q_\xi \setminus \xi$:

$$a_\alpha \equiv a_\xi \text{ and } b_\alpha \equiv b_\xi \text{ on } [g(\xi), \xi). \tag{I.69}$$

Applying the Pressing Down Lemma and working as above, we find a stationary set $\Omega \subseteq \omega_1$ such that for all $\xi \neq \eta$ in Ω, and $\alpha \in q_\xi$, $\beta \in q_\eta$:

$$\Delta_a(\alpha, \beta) = \Delta_a(\xi, \eta) \text{ and } \Delta_b(\alpha, \beta) = \Delta_b(\xi, \eta). \tag{I.70}$$

By the assumption of Case 2, we can find $\xi \neq \eta$ in Ω such that $\Delta_a(\xi, \eta) > \Delta_b(\xi, \eta)$. It follows that Δ_a dominates Δ_b on $p_\xi \cup p_\eta$. This completes the proof that $\mathcal{Q}$ is a ccc poset and therefore the proof of Theorem 6.16. $\dashv$

6.17 Remark. While under MA_{ω_1}, the class of coherent Aronszajn trees is totally ordered by $\leq$, Corollary 6.15 gives us that this chain of trees is not well-ordered. This should be compared with an old result of Ohkuma [56] that the class of all scattered trees is well-ordered by $\leq$ (see also [47]). It turns out that the class of all Aronszajn trees is not totally ordered under $\leq$, i.e. there exist Aronszajn trees S and T such that $S \not\leq T$ and $T \not\leq S$. The reader is referred to [90] for more information on this and other related results that we chose not to reproduce here.

7. The trace and the square-bracket operation

Recall the notion of a *minimal walk* from a countable ordinal β to a smaller ordinal α along the fixed C-sequence C_ξ ($\xi < \omega_1$): $\quad \beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha$ where $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$. Recall also the notion of a *trace*

$$\mathrm{Tr}(\alpha, \beta) = \{\beta_0, \beta_1, \cdots, \beta_n\},$$

the finite set of places visited in the minimal walk from β to α. The following simple fact about the trace lies at the heart of all known definitions of square-bracket operations not only on ω_1 but also at higher cardinalities.

7.1 Lemma. *For every uncountable subset Γ of ω_1 the union of $\mathrm{Tr}(\alpha, \beta)$ for $\alpha < \beta$ in Γ contains a closed and unbounded subset of ω_1.*

Proof. It suffices to show that the union of traces contains every countable limit ordinal δ such that $\sup(\Gamma \cap \delta) = \delta$. Pick an arbitrary $\beta \in \Gamma \setminus \delta$ and let

$$\beta = \beta_0 > \beta_1 > \cdots > \beta_k = \delta$$

be the minimal walk from β to δ. Let $\gamma < \delta$ be an upper bound of all sets of the form $C_{\beta_i} \cap \delta$ for $i < k$. By the choice of δ there is $\alpha \in \Gamma \cap \delta$ above γ. Then the minimal walk from β to α starts as $\beta_0 > \beta_1 > \cdots > \beta_k$, so in particular δ belongs to $\mathrm{Tr}(\alpha, \beta)$. $\dashv$

We shall now see that it is possible to pick a single place $[\alpha\beta]$ in $\mathrm{Tr}(\alpha, \beta)$ so that Lemma 7.1 remains valid with $[\alpha\beta]$ in place of $\mathrm{Tr}(\alpha, \beta)$. Recall that by Lemma 1.14,

$$\Delta(\alpha, \beta) = \min\{\xi \leq \alpha : \rho_0(\xi, \alpha) \neq \rho_0(\xi, \beta)\}$$

is a successor ordinal. We shall be interested in its predecessor,

7.2 Definition. $\Delta_0(\alpha, \beta) = \Delta(\alpha, \beta) - 1$.

Thus, if $\xi = \Delta_0(\alpha, \beta)$, then $\rho_0(\xi, \alpha) = \rho_0(\xi, \beta)$ and so there is a natural isomorphism between $\mathrm{Tr}(\xi, \alpha)$ and $\mathrm{Tr}(\xi, \beta)$. We shall define $[\alpha\beta]$ by comparing the three sets $\mathrm{Tr}(\alpha, \beta), \mathrm{Tr}(\xi, \alpha)$ and $\mathrm{Tr}(\xi, \beta)$.

7.3 Definition. The *square-bracket operation* on ω_1 is defined as follows:

$$
\begin{aligned}
[\alpha\beta] &= \min(\mathrm{Tr}(\alpha, \beta) \cap \mathrm{Tr}(\Delta_0(\alpha, \beta), \beta)) \\
&= \min(\mathrm{Tr}(\Delta_0(\alpha, \beta), \beta) \setminus \alpha).
\end{aligned}
$$

Next, recall the function $\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$ defined from the C-sequence C_ξ $(\xi < \omega_1)$ and the corresponding tree $T(\rho_0)$. For $\gamma < \omega_1$ let $(\rho_0)_\gamma$ be the fiber-mapping $: \gamma \to \omega^{<\omega}$ defined by $(\rho_0)_\gamma(\alpha) = \rho_0(\alpha, \gamma)$.

7.4 Lemma. *For every uncountable subset Γ of ω_1 the set of all ordinals of the form $[\alpha\beta]$ for some $\alpha < \beta$ in Γ contains a closed and unbounded subset of ω_1.*

Proof. For $t \in T(\rho_0)$ let

$$\Gamma_t = \{\gamma \in \Gamma : (\rho_0)_\gamma \text{ end-extends } t\}.$$

Let S be the collection of all $t \in T(\rho_0)$ for which Γ_t is uncountable. Clearly, S is a downward closed uncountable subtree of T. The Lemma is proved once we prove that every countable limit ordinal $\delta > 0$ with the following two properties can be represented as $[\alpha\beta]$ for some $\alpha < \beta$ in Γ:

$$\sup(\Gamma_t \cap \delta) = \delta \text{ for every } t \in S \text{ of length } < \delta, \qquad (\mathrm{I.71})$$

every $t \in S$ of length $< \delta$ has two incomparable successors in S both of length $< \delta$. $\qquad$ (I.72)

Fix such a δ and choose $\beta \in \Gamma \setminus \delta$ such that $(\rho_0)_\beta \restriction \delta \in S$ and consider the minimal walk from β to δ:

$$\beta = \beta_0 > \beta_1 > \cdots > \beta_k = \delta.$$

Let $\gamma < \delta$ be an upper bound of all sets of the form $C_{\beta_i} \cap \delta$ for $i < k$. Since the restriction $t = (\rho_0)_\beta \restriction \gamma$ belongs to S, by (I.72) we can find one of its end-extensions s in S which is incomparable with $(\rho_0)_\beta$. It follows that for $\alpha \in \Gamma_s$, the ordinal $\Delta_0(\alpha, \beta)$ has the fixed value

$$\xi = \min\{\xi < |s| : s(\xi) \neq \rho_0(\xi, \beta)\} - 1.$$

Note that $\xi \geq \gamma$, so the walk $\beta \to \delta$ is a common initial part of walks $\beta \to \xi$ and $\beta \to \alpha$ for every $\alpha \in \Gamma_s \cap \delta$. Hence if we choose $\alpha \in \Gamma_s \cap \delta$ above $\min(C_\delta \setminus \xi)$ (which we can by (I.71)), we get that the walks $\beta \to \xi$ and $\beta \to \alpha$ never meet after δ. In other words for any such α, the ordinal δ is the minimum of $\mathrm{Tr}(\alpha, \beta) \cap \mathrm{Tr}(\xi, \beta)$, or equivalently δ is the minimum of $\mathrm{Tr}(\xi, \beta) \setminus \alpha$. $\qquad \dashv$

It should be clear that the above argument can easily be adjusted to give us the following slightly more general fact about the square-bracket operation.

7.5 Lemma. *For every uncountable family A of pairwise disjoint finite subsets of ω_1, all of the same size n, the set of all ordinals of the form $[a(1)b(1)] = [a(2)b(2)] = \cdots = [a(n)b(n)]$ [25] for some $a \neq b$ in A contains a closed and unbounded subset of ω_1.* $\qquad \dashv$

It turns out that the square-bracket operation can be used in constructions of various mathematical objects of complex behaviour where all known previous constructions needed the Continuum Hypothesis or stronger enumeration principles. The usefulness of $[\cdot\cdot]$ in these constructions is based on the fact that $[\cdot\cdot]$ reduces the quantification over uncountable subsets of ω_1 to the quantification over closed unbounded subsets of ω_1. For example composing $[\cdot\cdot]$ with a unary operation $* : \omega_1 \longrightarrow \omega_1$ which takes each of the values stationary many times one gets the following fact about the mapping $c(\alpha, \beta) = [\alpha\beta]^*$.

7.6 Theorem. *There is a mapping $c : [\omega_1]^2 \longrightarrow \omega_1$ which takes all the values from ω_1 on any square $[\Gamma]^2$ of some uncountable subset Γ of ω_1.* $\qquad \dashv$

---

[25] For a finite set x of ordinals of size n we use the notation $x(1), x(2), \ldots, x(n)$ or $x(0), x(1), \ldots, x(n-1)$, depending on the context, for the enumeration of x according to the natural ordering on the ordinals.

This result gives a definitive limitation on any form of a Ramsey Theorem for the uncountable which tries to say something about the behaviors of restrictions of colorings on an uncountable square. Recall that in Section 3.1 we have seen that there are mappings of this sort with complex behavior on sets of the form

$$\Gamma_0 \otimes \Gamma_1 = \{\{\alpha_0, \alpha_1\} : \alpha_0 \in \Gamma_0, \alpha_1 \in \Gamma_1, \alpha_0 < \alpha_1\},$$

where Γ_0 and Γ_1 are two uncountable subsets of ω_1. It turns out that the square bracket operation has a simple behavior on $\Gamma_0 \times \Gamma_1$ for every pair Γ_0 and Γ_1 of sufficiently thin uncountable subsets of ω_1. This can be made precise as follows.

7.7 Lemma. *For every uncountable family A of pairwise disjoint subsets of ω_1, all of the same size n, there exist a partition $I_1, \ldots, I_k$ of $\{1, \ldots, n\}$, an uncountable $B \subseteq A$, and an $h_b : n \times n \longrightarrow \omega_1$ for each $b \in B$ such that for all $a < b$ [26] in B, $[a(p)b(q)] = h_b(p, q)$ whenever $p \in I_i$, $q \in I_j$ and $(i, j) \in k \times k \setminus \Delta$ [27]. Moreover the set of ordinals δ for which there exist $a < b$ in B such that $[a(p)b(q)] = \delta$ for all $p, q \in I_i$ and all $i = 1, \ldots, k$ contains a closed and unbounded subset of ω_1.*

Proof. Identifying an ordinal α with $(\rho_0)_\alpha$, we can think of A as a family of n-tuples of nodes of the tree $T(\rho_0)$. For a limit ordinal $\delta < \omega_1$, choose a_δ in A such that every node from a_δ has height $\geq \delta$. Let c_δ be the projection of a_δ on the δth level of $T(\rho_0)$. Let $h(\delta) < \delta$ be such that the projection of c_δ on the $h(\delta)$th level has size equal to the size of c_δ, i.e.

$$\Delta(s, t) < h(\delta) \text{ for all } s \neq t \text{ in } c_\delta. \tag{I.73}$$

Find a stationary set $\Gamma \subseteq \omega_1$ such that h takes a constant value γ_0 on Γ and that the γ_0th projection takes the constant value on c_δ ($\delta \in \Gamma$). Fix $\delta < \varepsilon$ in Γ and consider $\alpha \in a_\delta$ and $\beta \in a_\varepsilon$ that project to different elements of the γ_0th level of $T(\rho_0)$. Then $\Delta(\alpha, \beta)$ is independent of α and β or δ and ε as it is equal to $\Delta((\rho_0)_\alpha \upharpoonright \gamma_0, (\rho_0)_\beta \upharpoonright \gamma_0)$. So the trace $\mathrm{Tr}(\Delta_0(\alpha, \beta), \beta)$ depends only on β and the projection $(\rho_0)_\alpha \upharpoonright \gamma_0$. Thus, going to a subset of Γ, we may assume that its size depends only on the position of β in some a_ε and that the minimal point of $\mathrm{Tr}(\Delta_0(\alpha, \beta), \beta) \setminus \alpha$ is the same for all α in some a_δ, for $\delta < \varepsilon$, having a fixed projection on the γ_0th level of $T(\rho_0)$. This is the content of the first part of the conclusion of Lemma 7.7. To see the second part of the conclusion, we shrink Γ even further to assure that the union of c_δ ($\delta \in \Gamma$) is an antichain of $T(\rho_0)$. Now note that for two typical $\delta < \varepsilon$ in Γ, $[\alpha\beta] = [\alpha'\beta']$ for all $\alpha, \alpha' \in a_\delta$ and $\beta, \beta' \in a_\varepsilon$ with the property that $(\rho_0)_\alpha \upharpoonright \gamma_0 = (\rho_0)_{\alpha'} \upharpoonright \gamma_0 = (\rho_0)_\beta \upharpoonright \gamma_0 = (\rho_0)_{\beta'} \upharpoonright \gamma_0$. So the rest of the proof reduces to the proof of Lemma 7.5 above. ⊣

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[26] For a pair a and b of sets of ordinals, $a < b$ denotes the fact that $\alpha < \beta$ for all $\alpha \in a$ and $\beta \in b$.

[27] By Δ, we denote the diagonal of $k \times k$.

7.8 Remark. To see the point of Lemma 7.7, consider the projection $[\alpha\beta]^*$ of the square-bracket operation generated by a mapping $*$ from ω_1 onto ω rather than ω_1. Then the sequence $h_b^* : n \times n \longrightarrow \omega$ ($b \in B$) of corresponding compositions will have the same constant value $h : n \times n \longrightarrow \omega$ modulo, of course, shrinking B to some uncountable subset. Noting that the proof of Lemma 7.7 applied to a typical A will actually result in the partition $I_i = \{i\}$ ($i = 1, \ldots, n$), we conclude that if α and β are coming from two different positions of some a_δ and a_ε in B, then $[\alpha\beta]^*$ will depend on the positions themselves rather than α and β.

7.9 Remark. It turns out that there is a natural variation of the square-bracket operation (see the mapping b of [78, p.287]), where, in Lemma 7.7, we always have the finest possible partition $I_i = \{i\}$ ($i = 1, \ldots, n$) for *every* uncountable family A of pairwise disjoint n-tuples of countable ordinals. This property of the projection of the square-bracket operation has an interesting application in the Quadratic Form Theory, given by Baumgartner-Spinas [6] and Shelah-Spinas [67].

Note that the basic C-sequence C_α ($\alpha < \omega_1$) which we have fixed at the beginning of this chapter can be used to actually define a unary operation $* : \omega_1 \longrightarrow \omega_1$ which takes each of the ordinals from ω_1 stationarily many times. So the projection $[\alpha\beta]^*$ can actually be defined in our basic structure $(\omega_1, \omega, \vec{C})$. We are now at the point to see that our basic structure is actually rigid.

7.10 Lemma. *The algebraic structure $(\omega_1, [\cdot\cdot], *)$ has no nontrivial automorphisms.*

Proof. Let h be a given automorphism of $(\omega_1, [\cdot\cdot], *)$. If the set Γ of fixed points of h is uncountable, h must be the identity map. To see this, consider a $\xi < \omega_1$. By the property of the map $c(\alpha, \beta) = [\alpha\beta]^*$ stated in Theorem 7.6 there exist $\gamma < \delta$ in Γ such that $[\gamma\delta]^* = \xi$. Applying h to this equation we get

$$h(\xi) = h([\gamma\delta]^*) = (h([\gamma\delta]))^* = [h(\gamma)h(\delta)]^* = [\gamma\delta]^* = \xi.$$

It follows that $\Delta = \{\delta < \omega_1 : h(\delta) \neq \delta\}$ is in particular uncountable. Shrinking Δ and replacing h by h^{-1}, if necessary, we may safely assume that $h(\delta) > \delta$ for all $\delta \in \Delta$. Consider a $\xi < \omega_1$ and let S_ξ be the set of all $\alpha < \omega_1$ such that $\alpha^* = \xi$. By our choice of $*$ the set S_ξ is stationary. By Lemma 7.5 applied to the family $A = \{\{\delta, h(\delta)\} : \delta \in \Delta\}$ we can find $\gamma < \delta$ in Δ such that $[\gamma\delta] = [h(\gamma)h(\delta)]$ belongs to S_ξ, or in other words,

$$[\gamma\delta]^* = [h(\gamma)h(\delta)]^* = \xi.$$

Since $[h(\gamma)h(\delta)]^* = h([\gamma\delta]^*)$ we conclude that $h(\xi) = \xi$. Since ξ was an arbitrary countable ordinal, this shows that h is the identity map. $\dashv$

We give now an application of this rigidity result to a problem in Model Theory about the quantifier $Qx =$'there exist uncountably many x' and its higher dimensional analogues $Q^n x_1 \cdots x_n =$'there exist an uncountable n-cube many $x_1, \cdots, x_n$'. By a result of Ebbinghaus and Flum [19] (see also [57]) every model of every sentence of $L(Q)$ has nontrivial automorphisms. However we shall now see that this is no longer true about the quantifier Q^2.

7.11 Example. The sentence ϕ will talk about one unary relation N, one binary relation $<$ and two binary functional symbols C and E. It is the conjunction of the following seven sentences

(ϕ_1) $Qx \quad x = x$,

(ϕ_2) $\neg Qx \quad N(x)$,

(ϕ_3) $<$ is a total ordering,

(ϕ_4) E is a symmetric binary operation,

(ϕ_5) $\forall x < y \quad N(E(x, y))$,

(ϕ_6) $\forall x < y < z \quad E(x, z) \neq E(y, z)$,

(ϕ_7) $\forall x \forall n \{N(n) \rightarrow \neg Q^2 uv [\exists u' < u \exists v' < v(u' \neq v' \wedge E(u', u) = E(v', v) = n) \wedge \forall u' < u \; \forall v' < v(E(u', u) = E(v', v) = n \rightarrow (C(u', v') \neq x \vee C(u, v) \neq x))]\}$.

The model of ϕ that we have in mind is the model $(\omega_1, \omega, <, c, e)$ where $c(\alpha, \beta) = [\alpha\beta]^*$ and $e : [\omega_1]^2 \longrightarrow \omega$ is any mapping such that $e(\alpha, \gamma) \neq e(\beta, \gamma)$ whenever $\alpha < \beta < \gamma$ (e.g. we can take $e = \bar{\rho}_1$ or $e = \bar{\rho}$). The sentence ϕ_7 is simply saying that for every $\xi < \omega_1$ and every uncountable family A of pairwise disjoint unordered pairs of countable ordinals there exist $a \neq b$ in A such that

$$c(\min a, \min b) = c(\max a, \max b) = \xi.$$

This is a consequence of Lemma 7.5 and the fact that $S_\xi = \{\alpha : \alpha^* = \xi\}$ is a stationary subset of ω_1. These are the properties of $[\cdot\cdot]$ and $*$ which we have used in the proof of Lemma 7.10 in order to prove that $(\omega_1, [\cdot\cdot], *)$ is a rigid structure. So a quite analogous proof will show that any model $(M, N, <, C, E)$ of ϕ must be rigid. $\dashv$

The crucial property of $[\cdot\cdot]$ stated in Lemma 7.5 can also be used to provide a negative answer to the basis problem for uncountable graphs by constructing a large family of pairwise orthogonal uncountable graphs.

7.12 Definition. For a subset Γ of ω_1, let $\mathcal{G}_\Gamma$ be the graph whose vertex-set is ω_1 and whose edge-set is equal to

$$\{\{\alpha, \beta\} : [\alpha\beta] \in \Gamma\}.$$

7.13 Lemma. *If the symmetric difference between Γ and Δ is a stationary subset of ω_1, then the corresponding graphs $\mathcal{G}_\Gamma$ and $\mathcal{G}_\Delta$ are orthogonal to each other, i.e. they do not contain uncountable isomorphic subgraphs.* $\dashv$

We have seen above that comparing $[\cdot\cdot]$ with a map $\pi : \omega_1 \longrightarrow I$ where I is some set of mathematical objects/requirements in such a way that each object/requirement is given a stationary preimage, gives us a way to meet each of these objects/requirements in the square of any uncountable subset of ω_1. This observation is the basis of all known applications of the square-bracket operation. A careful choice of I and $\pi : \omega_1 \longrightarrow I$ gives us a projection of the square-bracket operation that can be quite useful. So let us illustrate this on yet another example.

7.14 Definition. Let $\mathcal{H}$ be the collection of all maps $h : 2^n \longrightarrow \omega_1$ where n is a positive integer denoted by $n(h)$. Choose a mapping $\pi : \omega_1 \longrightarrow \mathcal{H}$ which takes each value from $\mathcal{H}$ stationarily many times. Choose also a one-to-one sequence r_α $(\alpha < \omega_1)$ of elements of the Cantor set 2^ω. Note that both these objects can actually be defined in our basic structure $(\omega_1, \omega, \vec{C})$. Consider the following projection of the square-bracket operation:

$$[\![\alpha\beta]\!] = \pi([\alpha\beta])(r_\alpha \restriction n(\pi([\alpha\beta]))).$$

It is easily checked that the property of $[\cdot\cdot]$ stated in Lemma 7.5 corresponds to the following property of the projection $[\![\alpha\beta]\!]$:

7.15 Lemma. *For every uncountable family A of pairwise disjoint finite subsets of ω_1, all of the same size n, and for every n-sequence $\xi_1, \ldots, \xi_n$ of countable ordinals there exist a and b in A such that $[\![a(i)b(i)]\!] = \xi_i$ for $i = 1, \ldots, n$.* $\dashv$

This projection of $[\cdot\cdot]$ leads to an interesting example of a Banach space with 'few' operators, which we will now describe.

7.16 Example. A nonseparable reflexive Banach space E with the property that every bounded linear operator $T : E \longrightarrow E$ can be expressed as $T = \lambda I + S$ where λ is a scalar, I the identity operator of E, and S an operator with separable range.

Let $I = 3 \times [\omega_1]^{<\omega}$ and let us identify the index-set I with ω_1, i.e. pretend that $[\![\cdot\cdot]\!]$ takes its values in I rather than ω_1. Let $[\![\cdot\cdot]\!]_0$ and $[\![\cdot\cdot]\!]_1$ be the two projections of $[\![\cdot\cdot]\!]$.

$$\mathcal{G} = \{G \in [\omega_1]^{<\omega} : [\![\alpha\beta]\!]_0 = 0 \text{ for all } \{\alpha, \beta\} \in [G]^2\},$$

$$\mathcal{H} = \{H \in [\omega_1]^{<\omega} : [\![\alpha\beta]\!]_0 = 1 \text{ for all } \{\alpha, \beta\} \in [H]^2\}.$$

Let $\mathcal{K}$ be the collection of all finite sets $\{\{\alpha_i, \beta_i\} : i < k\}$ of pairs of countable ordinals such that for all $i < j < k$:

$$\max\{\alpha_i, \beta_i\} < \min\{\alpha_j, \beta_j\}, \tag{I.74}$$

$$[\![\alpha_i\alpha_j]\!]_0 = [\![\beta_i\beta_j]\!]_0 = 2, \tag{I.75}$$

$$[\![\alpha_i\alpha_j]\!]_1 = [\![\beta_i\beta_j]\!]_1 = \{\alpha_l : l < i\} \cup \{\beta_l : l < i\}. \tag{I.76}$$

The following properties of $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ should be clear:

$\mathcal{G}$ and $\mathcal{H}$ contain all the singletons, are closed under subsets and they are 1-orthogonal to each other in the sense that $\mathcal{G} \cap \mathcal{H}$ contains no doubleton. (I.77)

$\mathcal{G}$ and $\mathcal{H}$ are both 2-orthogonal to the family of the unions of members of $\mathcal{K}$. (I.78)

If K and L are two distinct members of $\mathcal{K}$, then there are no more than 5 ordinals α such that $\{\alpha, \beta\} \in K$ and $\{\alpha, \gamma\} \in L$ for some $\beta \neq \gamma$. (I.79)

For every sequence $\{\alpha_\xi, \beta_\xi\}$ ($\xi < \omega_1$) of pairwise disjoint pairs of countable ordinals there exist arbitrarily large finite sets $\Gamma, \Delta \subseteq \omega_1$ such that $\{\alpha_\xi : \xi \in \Gamma\} \in \mathcal{G}$, $\{\beta_\xi : \xi \in \Gamma\} \in \mathcal{H}$ and $\{\{\alpha_\xi, \beta_\xi\} : \xi \in \Delta\} \in \mathcal{K}$. (I.80)

For a function x from ω_1 into $\mathbb{R}$, set

$$\|x\|_{\mathcal{H},2} = \sup\{(\sum_{\alpha \in H} x(\alpha)^2)^{\frac{1}{2}} : H \in \mathcal{H}\},$$

$$\|x\|_{\mathcal{K},2} = \sup\{(\sum_{\{\alpha,\beta\} \in K} (x(\alpha) - x(\beta))^2)^{\frac{1}{2}} : K \in \mathcal{K}\}.$$

Let $\|\cdot\| = \max\{\|\cdot\|_\infty, \|\cdot\|_{\mathcal{H},2}, \|\cdot\|_{\mathcal{K},2}\}$ and define $\bar{E}_2 = \{x : \|x\| < \infty\}$. Let 1_α be the characteristic function of $\{\alpha\}$. Finally, let E_2 be the closure of the linear span of $\{1_\alpha : \alpha \in \omega_1\}$ inside $(\bar{E}_2, \|\cdot\|)$. The following facts about the norm $\|\cdot\|$ are easy to establish using the properties of the families $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ listed above

If x is supported by some $G \in \mathcal{G}$, then $\|x\| \leq 2 \cdot \|x\|_\infty$. (I.81)

If x is supported by $\bigcup K$ for some K in $\mathcal{K}$, then $\|x\| \leq 10 \cdot \|x\|_\infty$. (I.82)

The role of the seminorm $\|\cdot\|_{\mathcal{H},2}$ is to ensure that every bounded operator $T : E_2 \longrightarrow E_2$ can be expressed as $D+S$, where D is a diagonal operator relative to the basis[28] 1_α $(\alpha < \omega_1)$ and where S has separable range. Namely, if this were not so, we could find a sequence $\{\alpha_\xi, \beta_\xi\}$ $(\xi < \omega_1)$ of pairwise disjoint pairs from ω_1 such that $T(1_{\alpha_\xi})(\beta_\xi) \neq 0$ for all $\xi < \omega_1$. Thinning the sequence, we may assume that for some fixed $r > 0$ and all $\xi < \omega_1$, $|T(1_{\alpha_\xi})(\beta_\xi)| \geq r$. By (I.80) for every $n < \omega$ we can find a subset Γ of ω_1 of size n such that $G = \{\alpha_\xi : \xi \in \Gamma\} \in \mathcal{G}$ and $H = \{\beta_\xi : \xi \in \Gamma\} \in \mathcal{H}$. Then, by (I.81), $2 \cdot \|T\| \geq \|T(\chi_G)\| \geq (\sum_{\beta \in H} T(\chi_G)(\beta)^2)^{\frac{1}{2}} \geq r \cdot \sqrt{n}$. Since n is arbitrary, this contradicts the fact that T is bounded. The role of $\|\cdot\|_{\mathcal{K},2}$ is to represent every bounded diagonal operator $D(x)(\xi) = \lambda_\xi x(\xi)$ as $\lambda I + S$ for some scalar λ and operator S with separable range. This reduces to the fact that the sequence λ_ξ $(\xi < \omega_1)$ is eventually constant. If this were false, we would be able to extract a sequence $\{\alpha_\xi, \beta_\xi\}$ $(\xi < \omega_1)$ of pairwise disjoint pairs of countable ordinals and an $r > 0$ such that $|\lambda_{\alpha_\xi} - \lambda_{\beta_\xi}| \geq r$ for all $\xi < \omega_1$. By (I.80), for every $n < \omega$, we can find a subset Δ of ω_1 of size n such that $K = \{\{\alpha_\xi, \beta_\xi\} : \xi \in \Delta\}$ belongs to $\mathcal{K}$. Then, by (I.82),

$$
\begin{aligned}
10 \cdot \|D\| &\geq \|D(\chi_{\bigcup K})\| \\
&\geq \Big(\sum_{\xi \in \Delta} (D(\chi_{\bigcup K})(\alpha_\xi) - D(\chi_{\bigcup K})(\beta_\xi))^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{\xi \in \Delta} (\lambda_{\alpha_\xi} - \lambda_{\beta_\xi})^2\Big)^{\frac{1}{2}} \\
&\geq r \cdot \sqrt{n}.
\end{aligned}
$$

Since n was arbitrary this contradicts the fact that D is a bounded operator.

Note that $\|x\| \leq 2\|x\|_2$ for all $x \in \ell_2(\omega_1)$. It follows that $\ell_2(\omega_1) \subseteq E_2$ and the inclusion is a bounded linear operator. Note also that $\ell_2(\omega_1)$ is a dense subset of E_2. Therefore E_2 is a weak compactly generated space. For example, $W = \{x \in \ell_2(\omega_1) : \|x\|_2 \leq 1\}$ is a weakly compact subset of E_2 and its linear span is dense in E_2. To get a reflexive example out of E_2 one uses an interpolation method of Davis, Figiel, Johnson and Pelczynski [11] as follows. Let p_n be the Minkowski functional of the set $2^n W + 2^{-n} \mathrm{Ball}(E_2)$.[29] Let

$$
E = \{x \in E_2 : \|x\|_E = (\sum_{n=0}^{\infty} p_n(x)^2)^{\frac{1}{2}} < \infty\}.
$$

---

[28] Indeed it can be shown that 1_α $(\alpha < \omega_1)$ is a 'transfinite basis' of E_2 in the sense of [70]. So every vector x of E_2 has a unique representation as $\Sigma_{\alpha < \omega_1} x(\alpha) 1_\alpha$ and the projection operators $P_\beta : E_2 \to E_2 \upharpoonright \beta$ $(\beta < \omega_1)$ are uniformly bounded.

[29] I.e. $p_n(x) = \inf\{\lambda > 0 : x \in \lambda B\}$, where B denotes this set.

By [11, Lemma 1], E is a reflexive Banach space and $\ell_2(\omega_1) \subseteq E \subseteq E_2$ are continuous inclusions. Note that $p_n(x) < r$ iff $x = y + z$ for some $y \in E_2$ and $z \in \ell_2(\omega_1)$ such that $\|y\| < 2^{-n}r$ and $\|z\|_2 < 2^n r$. The following two facts about E correspond to the facts (I.81) and (I.82) and they are used in the proof that every bounded operator $T : E \longrightarrow E$ has the form $\lambda I + S$ in a very similar way.

> If x is the vector supported by some $G \in \mathcal{G}$ which can be split into n blocks such that the k-th one has size 2^{4k} and x takes the value 2^{-2k} on each term of the k-th block, then $\|x\|_E \le 2$.   (I.83)

> If $\{\{\alpha_\xi, \beta_\xi\} : \xi \in \Delta\}$ is a member of K so that Δ can be split into n blocks such that the k-th one Δ_k has size 2^{4k} and if x is the vector supported by $\bigcup K$ for $\xi \in \Delta_k, x(\alpha_\xi) = 2^{-2k}$ and $x(\beta_\xi) = 2^{-2k}$, then $\|x\|_E \le 12$.   (I.84)

We leave the checking of (I.83) and (I.84) as well as the corresponding proof that E has 'few' operators to the interested reader.   ⊣

7.17 Remark. The above example is reproduced from Wark [94] who based his example on a previous construction due to Shelah and Steprans [68]. The reader is referred to these sources for more information.

We only mention yet another interesting application of the square-bracket operation, given recently by Erdős, Jackson and Mauldin [21]:

7.18 Example. For every positive integer n there exist collections $\mathcal{H}$ and X of hyperplanes and points of $\mathbb{R}^n$, respectively, and a coloring $P : \mathcal{H} \longrightarrow \omega$ such that:

> any n hyperplanes of distinct colors meet in at most one point, and   (I.85)

> there is no coloring $Q : X \longrightarrow \omega$ such that for every $H \in \mathcal{H}$ there exist at most $n - 1$ points x in $X \cap H$ such that $Q(x) = P(H)$.   (I.86)

Finally we mention yet another projection of the square-bracket operation.

7.19 Definition. Let $\mathcal{H}$ now be the collection of all maps $h : 2^n \times 2^n \longrightarrow \omega_1$ where $n = n(h) < \omega$ and let π be a mapping from ω_1 onto $\mathcal{H}$ that takes each of the values stationarily many times. Define a new operation on ω_1 by

$$|\alpha\beta| = \pi([\alpha\beta])(r_\alpha \restriction n(\pi([\alpha\beta])), r_\beta \restriction n(\pi([\alpha\beta]))).   (I.87)$$

7.20 Lemma. *For every positive integer n, every uncountable subset Γ of ω_1 and every symmetric $n \times n$-matrix M of countable ordinals there is a one-to-one $\phi : n \longrightarrow \Gamma$ such that $|\phi(i)\phi(j)| = M(i,j)$ for $i,j < n$.*

Proof. Let Γ be an uncountable subset of ω_1 and let M be a given $n \times n$ matrix on countable ordinals. Then for each α we can find t_α in the αth level of the tree $T(\rho_0)$ and a finite set $F_\alpha \subseteq \Gamma \setminus \alpha$ of size n such that $t_\alpha \subseteq (\rho_0)_\gamma$ for all $\gamma \in F_\alpha$. Let n_α be the minimal integer such that $r_\gamma \upharpoonright n_\alpha \neq r_\delta \upharpoonright n_\alpha$ whenever $\gamma \neq \delta$ are members of F_α. For a limit ordinal α let $\xi(\alpha) < \alpha$ be an upper bound of all sets of the form $C_{\gamma_i} \cap \alpha$ where $\gamma \in F_\alpha$, $i < k < \omega$ and $\gamma = \gamma_0 > \ldots > \gamma_k = \alpha$ is the minimal walk from γ to α along the C-sequence fixed at the beginning of this chapter. Then there is a stationary set $\Delta \subseteq \omega_1$, $m < \omega$, $s_0, \ldots, s_{n-1} \in 2^m$ and $\xi < \omega_1$ such that:

$$n_\alpha = m \text{ and } \xi_\alpha = \xi \text{ for all } \alpha \in \Delta, \tag{I.88}$$

$$t_\alpha \ (\alpha \in \Delta) \text{ form an antichain of } T(\rho_0), \tag{I.89}$$

$$\begin{aligned}&r_\gamma \upharpoonright m = s_i \text{ whenever } i < n \text{ and } \gamma \text{ occupies the } i\text{th place in} \\ &\text{some } F_\alpha \text{ for } \alpha \in \Delta.\end{aligned} \tag{I.90}$$

Let $h : 2^m \times 2^m \longrightarrow \omega_1$ be an arbitrary map subject to the requirement that $h : (s_i, s_j) = M(i,j)$ for all $i,j < n$. Let $\Sigma_h = \{\delta < \omega_1 : \pi(\delta) = h\}$. By the choice of π the set Σ_h is stationary in ω_1. By the proof of the basic property 7.4 of the square bracket operation we can find $\nu \in \Sigma_h$, $\alpha \in \Delta \cap \nu$ and $\beta \in \Delta \setminus \nu$ such that $t_\alpha \upharpoonright \xi = t_\beta \upharpoonright \xi$, $F_\alpha \subseteq \nu$ and $[\alpha\beta] = \nu$. We claim that

$$[\gamma\delta] = \nu \text{ for all } \gamma \in F_\alpha \text{ and } \delta \in F_\beta. \tag{I.91}$$

To see this, first note that $\Delta_0(\gamma, \delta) = \Delta_0(\alpha, \beta) = \sigma$ for all $\gamma \in F_\alpha$ and $\delta \in F_\beta$. By the choice of ξ and the fact that $\sigma \geq \xi$ the walk from some $\delta \in F_\beta$ to σ goes through β, so for any $\gamma \in F_\alpha$ and $\delta \in F_\beta$ we have

$$[\gamma\delta] = \min(\mathrm{Tr}(\sigma, \delta) \setminus \gamma) = \min(\mathrm{Tr}(\sigma, \beta) \setminus \alpha) = [\alpha\beta].$$

In fact the proof of Lemma 7.4 shows that we can find $\nu_0 \in \Sigma_h$, $\alpha_0 \in \Delta \cap \nu$ and an uncountable set $\Delta_1 \subseteq \Delta \setminus \nu_0$ such that $F_{\alpha_0} \subseteq \nu_0$, $t_{\alpha_0} \upharpoonright \xi = t_\beta \upharpoonright \xi$ and $[\alpha_0\beta] = \gamma_0$ for all $\beta \in \Delta_1$. It follows that

$$[\gamma\delta] = \nu_0 \text{ for all } \gamma \in F_{\alpha_0}, \delta \in F_\beta \text{ and } \beta \in \Delta_1. \tag{I.92}$$

So we can repeat the procedure for Δ_1 in place of Δ and get $\nu_1 \in \Sigma_h$, $\alpha_1 \in \Delta_1 \cap \nu_0$ and uncountable $\Delta_2 \subseteq \Delta_1 \setminus \gamma_1$ such that $[\alpha_1\beta] = \nu_1$ for all $\beta \in \Delta_2$, and so on. Repeating this n times we get a sequence $\alpha_0, \ldots, \alpha_{n-1}$ such that for all $i < j < n$,

$$\pi([\gamma\delta]) = h \text{ for all } \gamma \in F_{\alpha_i} \text{ and } \delta \in F_\beta \text{ with } \beta \in \Delta_j. \tag{I.93}$$

Define $\phi : n \longrightarrow \omega_1$ by letting $\phi(i)$ be the ith member of F_{α_i}. Going back to the definition of the projection $|\cdot\cdot|$ of $[\cdot\cdot]$ one sees that indeed $|\phi(i)\phi(j)| = M(i,j)$ holds for all $i < j < n$. $\dashv$

8. A square-bracket operation on a tree

In this section we try to show that the basic idea of the square-bracket operation on ω_1 can perhaps be more easily grasped by working on an arbitrary special Aronszajn tree rather than $T(\rho_0)$. So let $T = \langle T, <_T \rangle$ be a fixed special Aronszajn tree and let $a : T \longrightarrow \omega$ be a fixed map witnessing this, i.e. a mapping with the property that $a^{-1}(n)$ is an antichain of T for all $n < \omega$. We shall assume that for every $s, t \in T$ the greatest lower bound $s \wedge t$ exists in T. For $t \in T$ and $n < \omega$, set

$$F_n(t) = \{s \leq_T t : s = t \text{ or } a(s) \leq n\}.$$

Finally, for $s, t \in T$ with $ht(s) \leq ht(t)$, let

$$[st]_T = \min\{v \in F_{a(s \wedge t)}(t) : ht(v) \geq ht(s)\}.$$

(If $ht(s) \geq ht(t)$ we let $[st]_T = [ts]_T$.)

The following fact corresponds to Lemma 7.4 when $T = T(\rho_0)$.

8.1 Lemma. *If X is an uncountable subset of T, the set of nodes of T of the form $[st]_T$ for some $s, t \in X$ intersects a closed and unbounded set of levels of T.*   ⊣

We do not give a proof of this fact as it is almost identical to the proof of Lemma 7.4 which deals with the special case $T = T(\rho_0)$. But one can go further and show that $[\cdot\cdot]_T$ shares all the other properties of the square-bracket operation $[\cdot\cdot]$ described in the previous section. Some of these properties, however, are easier to visualize and prove in the general context. For example, consider the following fact which in the case $T = T(\rho_0)$ is the essence of Lemma 7.20.

8.2 Lemma. *Suppose $A \subseteq T$ is an uncountable antichain and that for each $t \in A$ be given a finite set F_t of its successors. Then for every stationary set $\Gamma \subseteq \omega_1$ there exists an arbitrarily large finite set $B \subseteq A$ such that the height of $[xy]_T$ belongs to Γ whenever $x \in F_s$ and $y \in F_t$ for some $s \neq t$ in B.*   ⊣

Let us now examine in more detail the collection of graphs $\mathcal{G}_\Gamma(\Gamma \subseteq \omega_1)$ of 7.12 but in the present more general context.

8.3 Definition. For $\Gamma \subseteq \omega_1$, let

$$K_\Gamma = \{\{s, t\} \in [T]^2 : ht([st]_T) \in \Gamma\}.$$

Working as in 7.13 one shows that (T, K_Γ) and (T, K_Δ) have no isomorphic uncountable subgraph whenever the symmetric difference between Γ and Δ is a stationary subset of ω_1, i.e. whenever they represent different members of the quotient algebra $\mathcal{P}(\omega_1)/NS$. In particular, K_Γ contains no square $[X]^2$ of an uncountable set $X \subseteq T$ whenever Γ contains no closed and unbounded subset of ω_1. The following fact is a sort of converse to this.

8.4 Lemma. *If Γ contains a closed and unbounded subset of ω_1 then there is a proper forcing notion introducing an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$.*

Proof. The forcing notion $\mathbb{P}$ is defined to be the set of all pairs $p = \langle X_p, \mathcal{N}_p \rangle$ where

$\mathcal{N}_p$ is a finite $\in$-chain of countable elementary submodels of H_{ω_2} containing all the relevant objects. (I.94)

X_p is a finite subset of T such that $ht([st]_T) \in \Gamma$ for every pair $\{s, t\}$ of elements of X_p. (I.95)

for all $s \neq t$ in X_p there exist $N \in \mathcal{N}_p$ such that $s \in N$ iff $t \notin N$. (I.96)

for every $s \neq t$ in X_p, $N \in \mathcal{N}_p$, if $t \notin N$ then the height of $\min(F_{a(s \wedge t)}(t) \setminus N)$ belongs to Γ. (I.97)

The ordering of $\mathbb{P}$ is the coordinatewise inclusion. To show that $\mathbb{P}$ is a proper forcing notion, let M be a given countable elementary submodel of some large enough H_θ containing all the relevant objects and let $p \in \mathbb{P} \cap M$. We shall show that

$$q = \langle X_p, \mathcal{N}_p \cup \{M \cap H_{\omega_2}\} \rangle$$

is an M-generic condition extending p. So let $\mathcal{D} \in M$ be a dense-open subset of $\mathbb{P}$ and let r be a given extension of q. Extending r we may assume that $r \in \mathcal{D}$. Let $p_0 = r \cap M$. Note that $p_0 \in \mathbb{P} \cap M$. Let m_0 be the maximum of the union of the following two finite sets of integers

$$\{a(s \wedge t) : s, t \in X_r\},$$

$$\{a(t \restriction \delta) : t \in X_r, \delta = N \cap \omega_1 \text{ for some } N \in \mathcal{N}_p \text{ such that } t \notin N\}.$$

Let k be the size of the projection of the set $X_r \setminus M$ on the level $T_{M \cap \omega_1}$ of the tree T. Let $\mathcal{F}$ be the family of all k-element subsets F of T for which one can find $\bar{r} \in \mathcal{D}$ realizing the same type as r over p_0 such that, if N is the minimal model of $\mathcal{N}_{\bar{r}} \setminus \mathcal{N}_{p_0}$ then

$$F = \{x \restriction (N \cap \omega_1) : x \in X_{\bar{r}} \setminus X_{p_0}\}.$$

Since the projection of $X_{\bar{r}} \setminus M$ onto $T_{M \cap \omega_1}$ belongs to $\mathcal{F}$, the family is a rather large subset of $[T]^k$. It follows that the family $\mathcal{E}$ of all k-element subsets E of T for which we can find $\delta < \omega_1$ such that

$$\mathcal{F}_E = \{F \in \mathcal{F} : \{x \restriction \delta : x \in F\} = E\}$$

is uncountable, is also quite large. In particular, we can find an ordinal $\alpha_0 \in M \cap \omega_1$ above the heights of all nodes from the set

$$\bigcup \{M \cap F_{m_0}(t) : t \in X_r\}$$

and $E_1 \in \mathcal{E} \cap M$ such that

$$\{x \restriction \alpha_0 : x \in E_1\} = \{x \restriction \alpha_0 : x \in X_r \setminus M\}$$

and such that every node of $X_r \setminus M$ is incomparable with every node of E_1. Let

$$m^+ = \max\{a(x \wedge y) : x \in X_r \setminus M, y \in E_1\},$$

$$m^- = \min\{a(x \wedge y) : x \in X_r \setminus M, y \in E_1 \text{ and } a(x \wedge y) > m_0\}.$$

Let β_0 be an ordinal of $M \cap \omega_1$ above α_0 which bounds the heights of all nodes from the set

$$\bigcup \{M \cap F_{m^+}(t) : t \in X_r \setminus M\}.$$

By the definitions of $\mathcal{E}$ and $\mathcal{F}$ there is $\bar{r} \in \mathcal{D} \cap M$ realizing the same type over p_0 as r such that if N is the minimal model of $\mathcal{N}_{\bar{r}} \setminus \mathcal{N}_{p_0}$ then $N \cap \omega_1 \geq \beta_0$ and the projection

$$F = \{x \restriction (N \cap \omega_1) : x \in X_{\bar{r}} \setminus X_{p_0}\}$$

dominates the set E_1. We claim that

$$\bar{q} = \langle X_{\bar{r}} \cup X_r, \mathcal{N}_{\bar{r}} \cup \mathcal{N}_r \rangle$$

is a condition of $\mathbb{P}$. This will finish the proof of properness of $\mathbb{P}$ since clearly $\bar{q}$ extends both $\bar{r}$ and r. Clearly, only (I.95) and (I.97) for $\bar{q}$ require some argumentation. We check first (I.97), so let s and t be a pair of distinct members of $X_{\bar{r}} \cup X_r$ and let $N \in \mathcal{N}_{\bar{r}} \cup \mathcal{N}_r$ be a model such that $t \notin N$. Let $m = a(s \wedge t)$. If $t \in X_{\bar{r}}$ and if there is $s' \in X_{\bar{r}}$ such that $s' \neq t$ and $s' \wedge t = s \wedge t$ the conclusion that the height of $\min(F_m(t) \setminus N)$ belongs to Γ follows from the fact that $\bar{r}$ satisfies (I.97). Similarly one considers the case when $t \in X_r \setminus M$ and when $s' \wedge t = s \wedge t$ for some $s' \in X_r$, $s' \neq t$. For if N belongs to $\mathcal{N}_r$ the conclusion follows from the fact that r satisfies (I.97) and if $N \in \mathcal{N}_{\bar{r}} \setminus \mathcal{N}_r$ then by the choice of $\bar{r}$,

$$F_m(t) \setminus N = F_m(t) \setminus M,$$

and so the conclusion that the height of $\min(F_m(t) \setminus N)$ belongs to Γ follows from the fact that r satisfies (I.97) for s' and t from X_r and $M \cap H_{\omega_2}$ from $\mathcal{N}_r$. Let us now consider the case $t \in X_r \setminus M$, $s \in X_{\bar{r}} \setminus X_r$ and $s' \wedge t \neq s \wedge t$ for all $s' \in X_r \setminus \{t\}$. Note that in this case we have the following inequalities:

$$m_0 < m^- \leq m = a(s \wedge t) \leq m^+.$$

So by the choice of m_0 the set $F_m(t)$ contains $t \restriction (N' \cap \omega_1)$ for every $N' \in \mathcal{N}_r$. Since every ordinal of the form $N' \cap \omega_1$ ($N' \in \mathcal{N}_r$) belongs to Γ we are done in case N belongs to $\mathcal{N}_r$. So suppose $N \in \mathcal{N}_{\bar{r}} \setminus \mathcal{N}_r$. Then by the choice of the bound β_0 and the choice of $\bar{r}$,

$$F_m(t) \setminus N = F_m(t) \setminus M$$

so the conclusion of (I.97) follows again from the fact that $M \cap H_{\omega_2}$ belongs to $\mathcal{N}_r$. Consider now the remaining case $t \in X_{\bar{r}} \setminus X_r$ and $s \in X_r \setminus X_{\bar{r}}$. Note that in this case the model N must belong to $\mathcal{N}_r$ so if $s' \wedge t = s \wedge t$ for some $s' \in X_{\bar{r}}$, $s \neq t$ the conclusion follows from the fact that $\bar{r}$ satisfies the condition (I.97). So let us consider the subcase when $s' \wedge t \neq s \wedge t$ for all $s' \in X_{\bar{r}}$, $s' \neq t$. Thus, $s \wedge t$ is the new splitting so by the choice of α_0 and E_0 we have the inequalities:

$$m_0 < m^- \leq m = a(s \wedge t).$$

Recall that $\bar{r}$ was chosen to realize the same type over p_0 as r so we in particular have that $F_{m_0}(t)$ and therefore its superset $F_m(t)$ contains the projection $t \restriction (N \cap \omega_1)$. It follows that $\min(F_m(t) \setminus N) = t \restriction (N \cap \omega_1)$, so its height is indeed a member of Γ (as $\Gamma \in N$ and Γ contains a closed and unbounded subset of ω_1).

It remains to check (I.95) for $\bar{q}$. So let s and t be a given pair of distinct members of $X_{\bar{r}} \cup X_r$. We need to show that the height of $[st]_T$ belongs to Γ. Clearly the only nontrivial case is when, say, $s \in X_{\bar{r}} \setminus X_r$ and $t \in X_r \setminus X_{\bar{r}}$. Suppose first that $s \wedge t = s' \wedge t$ for some $s' \in X_r$, $s' \neq t$. Since $M \cap H_{\omega_2}$ belongs to $\mathcal{N}_r$ and since r satisfies (I.97), the height of $\min(F_{a(s' \wedge t)}(t) \setminus M)$ belongs to Γ. Since the height of s is above the bound β_0 for $F_{m^+}(t) \cap M$,

$$[st]_T = \min(F_{a(s \wedge t)}(t) \setminus M) = \min(F_{a(s' \wedge t)}(t) \setminus M),$$

so we are done in this subcase. Suppose now that $s \wedge t$ is a new splitting, i.e. that

$$m_0 < m = a(s \wedge t) \leq m^+.$$

So again we know that in this subcase $F_m(t)$ contains the projection $t \restriction (M \cap \omega_1)$. By our choice of the bound β_0 and the condition $\bar{r}$ we know that this projection is the minimal point of $F_m(t)$ whose height is bigger or

equal to the height of s. By the definition of the square-bracket operation, we have

$$[st]_T = t \restriction (M \cap \omega_1).$$

Since $M \cap \omega_1$ belongs to Γ this finishes our checking that $\bar{q}$ satisfies (I.95). Moreover, this also finishes the proof that the forcing notion $\mathbb{P}$ is proper.

It is clear that $\mathbb{P}$ forces that the square of the union $\dot{X}$ of the X_p's for p belonging to the generic filter is included in K_Γ. What is not so clear is that $\mathbb{P}$ forces that the set $\dot{X}$ is uncountable as promised. To ensure this we replace $\mathbb{P}$ with its restriction $\mathbb{P}(\leq r)$ to the condition

$$r = \langle \{t\}, \{M \cap H_{\omega_2}\} \rangle,$$

where M is an arbitrary countable elementary submodel of some large enough H_θ containing all the relevant objects and where t is any node of T of height at least $M \cap \omega_1$. Note that the above proof of properness of $\mathbb{P}$ starting from $p = \langle \emptyset, \emptyset \rangle$ shows that

$$q = \langle \emptyset, \{M \cap H_{\omega_2}\} \rangle$$

is an M-generic condition of $\mathbb{P}$ so in particular its extension r is also M-generic. It follows that if we let $M[\dot{G}]$ be the name for the set formed by interpreting all $\mathbb{P}$-names belonging to M itself. Then r forces that $\dot{X}$ is an element of the countable elementary submodel $M[\dot{G}]$ of $H_\theta[\dot{G}]$ which contains a point t of height above $M[\dot{G}] \cap \omega_1$. It follows that r forces $\dot{X}$ to be uncountable. This finishes the proof of Lemma 8.4. $\qquad \dashv$

8.5 Corollary. *The graph K_Γ contains the square of some uncountable subset of T in some ω_1-preserving forcing extension if and only if Γ is a stationary subset of ω_1.*

Proof. If Γ is disjoint from a closed and unbounded subset then in any ω_1-preserving forcing extension its complement $\Delta = \omega_1 \setminus \Gamma$ will be a stationary subset of ω_1. So by the basic property 8.1 of the square-bracket operation no such a forcing extension will contain an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. On the other hand, if Γ is a stationary subset of ω_1, going first to some standard ω_1-preserving forcing extension in which Γ contains a closed and unbounded subset of ω_1 and then applying Lemma 8.4, we get an ω_1-preserving forcing extension having an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. $\qquad \dashv$

8.6 Remark. Corollary 8.5 gives us a further indication of the extreme complexity of the class of graphs on the vertex-set ω_1. It also bears some relevance to the recent work of Woodin [98] who, working in his $\mathbb{P}_{\max}$-forcing extension, was able to associate a stationary subset of ω_1 to any partition of $[\omega_1]^2$ into two pieces. So one may view Corollary 8.5 as some sort of

converse to this since in the $\mathbb{P}_{\max}$-extension one is able to get a sufficiently generic filter to the forcing notion $\mathbb{P} = \mathbb{P}_\Gamma$ of Lemma 8.4 that would give us an uncountable $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. In other words, under a bit of PFA or Woodin's axiom (*), a set $\Gamma \subseteq \omega_1$ contains a closed and unbounded subset of ω_1 if and only if K_Γ contains $[X]^2$ for some uncountable $X \subseteq T$.

9. Special trees and Mahlo cardinals

One of the most basic questions frequently asked about set-theoretical trees is the question whether they contain any *cofinal branch*, a branch that intersects each level of the tree. The fundamental importance of this question has already been realized in the work of Kurepa [45] and then later in the works of Erdős and Tarski in their respective attempts to develop the theory of partition calculus and large cardinals (see [23]). A tree T of height equal to some regular cardinal θ may not have a cofinal branch for a very special reason as the following definition indicates.

9.1 Definition. For a tree $T = \langle T, <_T \rangle$, a function $f : T \to T$ is *regressive* if $f(t) <_T t$ for every $t \in T$ that is not a minimal node of T. A tree T of height θ is *special* if there is a regressive map $f : T \longrightarrow T$ with the property that the f-preimage of every point of T can be written as the union of $< \theta$ antichains of T.

This definition in case $\theta = \omega_1$ reduces indeed to the old definition of special tree, a tree that can be decomposed into countably many antichains. More generally we have the following:

9.2 Lemma. *If θ is a successor cardinal then a tree T of height θ is special if and only if T is the union of $< \theta$ antichains.* ⊣

The new definition, however, seems to be the right notion of speciality as it makes sense even if θ is a limit cardinal.

9.3 Definition. A tree T of height θ is *Aronszajn* if T has no cofinal branches and if every level of T has size $< \theta$.

Recall the well-known characterization of weakly compact cardinals due to Tarski and his collaborators: a strongly inaccessible cardinal θ is weakly compact if and only if there are no Aronszajn trees of height θ. We supplement this with the following:

9.4 Theorem. *The following are equivalent for a strongly inaccessible cardinal θ:*

(1) θ is Mahlo.

(2) there are no special Aronszajn trees of height θ.

Proof. Suppose θ is a Mahlo cardinal and let T be a given tree of height θ all of whose levels have size $< \theta$. To show that T is not special let $f : T \longrightarrow T$ be a given regressive mapping. By our assumption of θ there is an elementary submodel M of some large enough structure H_κ such that $T, f \in M$ and $\lambda = M \cap \theta$ is a regular cardinal $< \theta$. Note that $T \upharpoonright \lambda$ is a subset of M and since this tree of height λ is clearly not special, there is $t \in T \upharpoonright \lambda$ such that the preimage $f^{-1}(t)$ is not the union of $< \lambda$ antichains. Using the elementarity of M we conclude that $f^{-1}(t)$ is actually not the union of $< \theta$ antichains.

The proof that (2) implies (1) uses the method of minimal walks in a rather crucial way. So suppose to the contrary that our cardinal contains a closed and unbounded subset C consisting of singular strong limit cardinals. Using C, we choose a C-sequence C_α ($\alpha < \theta$) such that: $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ for α limit such that $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$ but if $\alpha = \sup(C \cap \alpha)$ then take C_α such that:

$$\mathrm{tp}(C_\alpha) = \mathrm{cf}(\alpha) < \min(C_\alpha), \tag{I.98}$$

$$\xi = \sup(C_\alpha \cap \xi) \text{ implies } \xi \in C, \tag{I.99}$$

$$\xi \in C_\alpha \text{ and } \xi > \sup(C_\alpha \cap \xi) \text{ imply } \xi = \eta + 1 \text{ for some } \eta \in C. \tag{I.100}$$

Given the C-sequence C_α ($\alpha < \theta$) we have the notion of minimal walk along the sequence and various distance functions defined above. In this proof we are particularly interested in the function ρ_0 from $[\theta]^2$ into the set $\mathbb{Q}_\theta$ of all finite sequences of ordinals from θ:

$$\rho_0(\alpha, \beta) = \langle \mathrm{tp}(C_\beta \cap \alpha) \rangle ^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha))$$

where we stipulate that $\rho_0(\gamma, \gamma) = 0$ for all γ. We would like to show that the tree

$$T(\rho_0) = \{(\rho_0)_\beta \upharpoonright \alpha : \alpha \leq \beta < \theta\}$$

is a special Aronszajn tree of height θ. Note that the size of the αth level $(T(\rho_0))_\alpha$ of $T(\rho_0)$ is controlled in the following way:

$$|(T(\rho_0))_\alpha| \leq |\{C_\beta \cap \alpha : \alpha \leq \beta < \theta\}| + |\alpha| + \aleph_0. \tag{I.101}$$

So under the present assumption that θ is a strongly inaccessible cardinal, all levels of $T(\rho_0)$ do indeed have size $< \theta$. It remains to define the regressive map

$$f : T(\rho_0) \longrightarrow T(\rho_0)$$

that will witness speciality of $T(\rho_0)$. Note that it really suffices defining f on all levels whose index belong to our club C of singular cardinals. So let $t = (\rho_0)_\beta \upharpoonright \alpha$ be a given node of T such that $\alpha \in C$ and $\alpha \leq \beta < \theta$. Note that by our choice of the C-sequence every term of the finite sequence of

ordinals $\rho_0(\alpha, \beta)$ is strictly smaller than α. So, if we let $f(t) = t \upharpoonright \ulcorner \rho_0(\alpha, \beta) \urcorner$, where $\ulcorner \cdot \urcorner$ is a standard coding of finite sequences of ordinals by ordinals, we get a regressive map. To show that f is one-to-one on chains of $T(\rho_0)$, which would be more than sufficient, suppose $t_i = (\rho_0)_{\beta_i} \upharpoonright \alpha_i$ $(i < 2)$ are two nodes such that $t_0 \subsetneq t_1$. Our choice of the C-sequence allows us to deduce the following general fact about the corresponding ρ_0-function as in the case $\theta = \omega_1$ dealt with above in Lemma 1.14.

> If $\alpha \leq \beta \leq \gamma$, α is a limit ordinal, and if $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$ for all $\xi < \alpha$, then $\rho_0(\alpha, \beta) = \rho_0(\alpha, \gamma)$. $\hfill$ (I.102)

Applying this to the triple of ordinals α_0, β_0 and β_1 we conclude that $\rho_0(\alpha_0, \beta_0) = \rho_0(\alpha_0, \beta_1)$. Now observe another fact about the ρ_0-function whose proof is identical to that of $\theta = \omega_1$ dealt with above in Lemma 1.13.

$$\text{If } \alpha \leq \beta \leq \gamma \text{ then } \rho_0(\alpha, \gamma) <_r \rho_0(\beta, \gamma). \qquad (\text{I.103})$$

Applying this to the triple $\alpha_0 < \alpha_1 \leq \beta_1$ we in particular have that $\rho_0(\alpha_0, \beta_1) \neq \rho_0(\alpha_1, \beta_1)$. Combining this with the above equality gives us that $\rho_0(\alpha_0, \beta_0) \neq \rho_0(\alpha_1, \beta_1)$ and therefore that $f(t_0) \neq f(t_1)$. $\hfill \dashv$

This leads us to the following Ramsey-theoretic characterization of Mahlo cardinals.

9.5 Theorem. *A cardinal θ is a Mahlo cardinal if and only if every regressive map f defined on a cube $[C]^3$ of a closed and unbounded subset of θ has an infinite min-homogeneous set $X \subseteq C$.*[30]

Proof. Again, the direct implication is simple so we leave it to the interested reader. For the converse, note that the statement about regressive maps easily implies that θ must be a strongly inaccessible cardinal. So assume θ is a strongly inaccessible non-Mahlo cardinal. Let C be a closed and unbounded subset of θ consisting of strong limit singular cardinals. We shall find a regressive map on $[C]^3$ without an infinite min-homogeneous set. This will give us an opportunity to further expose the tree $T(\rho_0)$ and the regressive map $f : T(\rho_0) \upharpoonright C \longrightarrow T(\rho_0)$ defined in the proof of Theorem 9.4. We have already observed that the length of every branching node t must be of the form $\alpha + 1$ for some α and in fact the following more precise description is true:

> If t is a branching node of $T(\rho_0)$ then its length is equal to $\alpha + 1$ for some $\alpha \in C$. $\hfill$ (I.104)

Let

$$\alpha^- = \max(C \cap (\alpha + 1)) \text{ and } \alpha^+ = \min(C \setminus (\alpha + 1)).$$

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[30]Recall, that X is min-homogeneous for f if $f(\alpha, \beta, \gamma) = f(\alpha', \beta', \gamma')$ for every pair $\alpha < \beta < \gamma$ and $< \alpha' < \beta' < \gamma'$ of triples of elements of X such that $\alpha = \alpha'$.

Let $\beta, \gamma > \alpha + 1$ be two ordinals such that $(\rho_0)_\beta$ and $(\rho_0)_\gamma$ split at t. Let $\beta = \beta_0 > \ldots > \beta_k = \alpha$ and $\gamma = \gamma_0 > \ldots > \gamma_l = \alpha$ be the walks from β to α and γ to α respectively. From the fact that $(\rho_0)_\beta$ and $(\rho_0)_\gamma$ agree below α we conclude that $k = l$ and $C_{\beta_i} \cap \alpha = C_{\gamma_i} \cap \alpha$ for all $i \leq k$. By the choice of the C-sequence and the inequality $\alpha > \alpha^-$ we can deduce a sequence of stronger agreements $C_{\beta_i} \cap \alpha^+ = C_{\gamma_i} \cap \alpha^+$ for all $i \leq k$. However, that would mean that $\rho_0(\alpha + 1, \beta) = \rho_0(\alpha + 1, \gamma)$, contrary to our assumption that $(\rho_0)_\beta$ and $(\rho_0)_\gamma$ split at t. It follows that $\alpha = \alpha^-$ which was to be shown.

Going back to the proof of Theorem 9.5, recall the regressive map $f : T(\rho_0) \upharpoonright C \longrightarrow T(\rho_0)$ defined in 9.4:

$$f((\rho_0)_\beta \upharpoonright \alpha) = (\rho_0)_\beta \upharpoonright \ulcorner \rho_0(\alpha, \beta) \urcorner.$$

We extend f to the whole tree by letting $f(t) = t \upharpoonright \alpha$ where α is the maximal element of C that is less than or equal to the length of t. Then we know that f is one-to-one on any chain whose nodes are separated by C. From f we shall derive three other regressive maps $p : [C]^3 \longrightarrow \theta$, $q : [C]^3 \longrightarrow \omega$ and $r : [C]^2 \longrightarrow \theta$. First note that for $\alpha \neq \beta$ in C the functions $(\rho_0)_\alpha$ and $(\rho_0)_\beta$ are incomparable, so it is natural to denote by $\alpha \wedge \beta$ the node of $T(\rho_0)$ where they split. For $\alpha < \beta < \gamma < \theta$, let

$$p(\alpha, \beta, \gamma) = \text{length}(f^n(\beta \wedge \gamma)), \text{ where } n \text{ is the minimal integer such that } f^n(\beta \wedge \gamma) \subseteq (\rho_0)_\alpha. \tag{I.105}$$

$$q(\alpha, \beta, \gamma) = \min\{n : f^n(\beta \wedge \gamma) = \emptyset\}. \tag{I.106}$$

$$r(\alpha, \beta) = \ulcorner \langle \rho_0(\xi, \alpha), \rho_0(\xi, \beta) \rangle \urcorner, \text{ where } \xi = \text{length}(\alpha \wedge \beta). \tag{I.107}$$

Now we claim that no infinite subset of C can be simultaneously min-homogeneous with respect to all three regressive maps. For, given an infinite increasing sequence $\{\alpha_i\}$ of elements of C using Ramsey's theorem we may assume that either

$$\alpha_i \wedge \alpha_j = \alpha_k \wedge \alpha_l \text{ for all } i \neq j \text{ and } k \neq l \text{ in } \omega, \text{ or} \tag{I.108}$$

$$\alpha_i \wedge \alpha_{i+1} \subsetneqq \alpha_{i+1} \wedge \alpha_{i+2} \text{ for all } i < \omega. \tag{I.109}$$

Note that (I.108) would violate min-homogeneity with respect to the mapping r. If (I.109) holds and if the sequence is min-homogeneous with respect to p and q another application of Ramsey's theorem would give us an integer m and an ordinal ξ such that for all $i < j < k < \omega$:

$$m = \min\{n : f^n(\alpha_j \wedge \alpha_k) \subseteq (\rho_0)_{\alpha_i}\}, \tag{I.110}$$

$$f^m(\alpha_j \wedge \alpha_k) = (\rho_0)\alpha_0 \upharpoonright \xi (= t). \tag{I.111}$$

By (I.104) the chain $\{\alpha_i \wedge \alpha_{i+1} : i \geq 1\}$ is separated by C, so using the basic property of f we conclude that $m > 1$. Then $f^{-1}(t)$ includes the set $\{f^{m-1}(\alpha_i \wedge \alpha_{i+1}) : i \geq 1\}$. By (I.110) we have,

$$\alpha_i \wedge \alpha_{i+1} \subsetneqq f^{m-1}(\alpha_{i+1} \wedge \alpha_{i+2}) \subseteq \alpha_{i+1} \wedge \alpha_{i+2}, \qquad \text{(I.112)}$$

so $\{f_{m-1}(\alpha_i \wedge \alpha_{i+1}) : i \geq 1\}$ is an infinite chain of $T(\rho_0)$ separated by C, a contradiction. This finishes the proof of Theorem 9.5. $\dashv$

Starting from the case $n = 1$ one can now easily deduce the following characterization of n-Mahlo cardinals.

9.6 Theorem. *The following are equivalent for an uncountable cardinal θ and a positive integer n:*

(1) *θ is n-Mahlo.*

(2) *Every regressive map defined on $[C]^{n+2}$ for some closed and unbounded subset C of θ has an infinite min-homogeneous subset.*

Proof. For a set X of ordinals and $n \in \omega$, let $W(X,n)$ denote the statement that there exist $f : [X]^{n+2} \longrightarrow \omega$ and regressive $g[X]^{n+3} \longrightarrow \sup(X)$ such that if $H \subseteq X$ is min-homogeneous for g and $f"[H]^{n+2} = \{k\}$ for some $k \in \omega$, then $|H| \leq n + k + 5$. Note that in the previous proof we have established $W(C,0)$ for every closed unbounded subset C of some limit ordinal θ such that C contains no inaccessible cardinals (the mapping q is really a map from $[C]^2$ into ω). The proof of the Theorem is finished once we show that in general $W(C,n)$ holds for every set of ordinals C of limit order type that is closed in its supremum and which contains no n-Mahlo cardinals. So let C be a given closed and unbounded subset of some limit ordinal θ and suppose that C contains no $n + 1$-Mahlo cardinals. We need to show that $W(C, n + 1)$ holds. Note that for this it suffices to show by induction that $W(C \cap \delta, n)$ holds for all $\delta \in C$ since there is a natural way to combine pairs of maps $f_\delta : [C]^{n+2} \longrightarrow \omega$ and $g_\delta : [C]^{n+3} \longrightarrow \theta$ for $\delta \in C$ into maps $f : [C]^{n+3} \longrightarrow \omega$ and $g : [C]^{n+4} \longrightarrow \theta$ witnessing $W(C, n + 1)$. The successor steps of the induction follow from the easily checked fact that $W(X,n)$ implies $W(Y,n)$ for every end-extension Y of X of cardinality at most $2^{|\sup(X)|}$. Assume now that δ is a limit point of C and that $W(C \cap \gamma, n)$ holds for all $\delta \in C \cap \delta$. Since δ is not an $n+1$-Mahlo cardinal there is a closed and unbounded set $D \subseteq C \cap \delta$ containing no n-Mahlo cardinals. By the induction hypothesis, $W(D, n)$ holds. Now, there is a very natural way to combine maps $f_\gamma : [C]^{n+2} \longrightarrow \omega$ and $g_\gamma : [C]^{n+3} \longrightarrow \theta$ for $\gamma \in C \cap \delta$ witnessing $W(C \cap \gamma, n)$ and maps $f_\delta : [D]^{n+2} \longrightarrow \omega$ and $g_\delta : [D]^{n+3} \longrightarrow \theta$ witnessing $W(D, n)$ into maps $F_\delta : [C \cap \delta]^{n+2} \longrightarrow \omega$ and $G_\delta : [C \cap \delta]^{n+3} \longrightarrow \theta$ witnessing $W(C \cap \delta)$. $\dashv$

9.7 Remark. Theorems 9.5 and 9.6 are due to Hajnal, Kanamori and Shelah [30] who were improving an earlier characterization of this sort due to Schmerl [64]. Using Theorem 9.6 one can also prove the following characterization theorem essentially given in [64](see also [30]) which has some interesting metamathematical applications (see [26] and [38]).

9.8 Theorem. *The following are equivalent for an uncountable cardinal θ and a positive integer n:*

(1) *θ is n-Mahlo.*

(2) *Every regressive map defined on $[C]^{n+3}$ for some closed and unbounded subset C of θ has a min-homogeneous set of size $n + 5$.*

$\dashv$

We finish this section by showing how the idea of proof of Theorem 9.4 leads us naturally to the following well-known fact, first established by Silver (see [52]) when θ is a successor of a regular cardinal.

9.9 Theorem. *If θ is a regular uncountable cardinal which is not Mahlo in the constructible universe, then there is a constructible special Aronszajn tree of height θ.*

Proof. Working in L we choose a closed and unbounded subset C of θ consisting of singular ordinals and a C-sequence C_α $(\alpha < \theta)$ such that $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ when $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$, while if α is a limit point of C we take C_α to have the following properties:

$$\xi = \sup(C_\alpha \cap \xi) \text{ implies } \xi \in C \qquad\qquad (\text{I.113})$$

$$\xi > \sup(C_\alpha \cap \xi) \text{ implies } \xi = \eta + 1 \text{ for some } \eta \in C. \qquad (\text{I.114})$$

We choose the C-sequence to also have the following crucial property:

$$|\{C_\alpha \cap \xi : \xi \leq \alpha < \theta\}| \leq |\xi| + \aleph_0 \text{ for all } \xi < \theta. \qquad (\text{I.115})$$

It is clear then that the tree $T(\rho_0)$, where ρ_0 is the ρ_0-function of C_α $(\alpha < \theta)$, is a constructible special Aronszajn tree of height θ. $\dashv$

We are also in a position to deduce the following well-known fact.

9.10 Theorem. *The following are equivalent for a successor cardinal θ:*

(a) *there is a special Aronszajn tree of height θ.*

(b) *there is a C-sequence C_α $(\alpha < \theta)$ such that $\mathrm{tp}(C_\alpha) \leq \theta^-$ for all α and such that $\{C_\alpha \cap \xi : \alpha < \theta\}$ has size $\leq \theta^-$ for all $\xi < \theta$.*

Proof. If C_α ($\alpha < \theta$) is a C-sequence satisfying (b) and if ρ_0 is the associated ρ_0-function then $T(\rho_0)$ is a special Aronszajn tree of height θ. Suppose $<_T$ is a special Aronszajn tree ordering on θ such that $[\theta^- \cdot \alpha, \theta^- \cdot (\alpha+1))$ is its αth level. Let C be the club of ordinals $< \theta$ divisible by θ^-. Let $f : \theta \longrightarrow \theta^-$ be such that the f-preimage of every ordinal $< \theta^-$ is an antichain of the tree $(\theta, <_T)$. We choose a C-sequence C_α ($\alpha < \theta$) such that $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ for α limit with the property that $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$, but if α is a limit point of C we take C_α more carefully as follows: $C_\alpha = \{\alpha_\xi : \xi < \eta\}$ where

$$\alpha_\lambda = \sup\{\alpha_\xi : \xi < \lambda\} \text{ for } \lambda \text{ limit } < \eta, \tag{I.116}$$

$$\alpha_0 = \text{the } <_T\text{-predecessor of } \alpha \text{ with minimal } f\text{-image}, \tag{I.117}$$

$$\alpha_{\xi+1} = \text{the } <_T\text{-predecessor of } \alpha \text{ with minimal } f\text{-image subject to the requirement that } f(\alpha_{\xi+1}) > f(\alpha_{\zeta+1}) \text{ for all } \zeta < \xi, \tag{I.118}$$

$$\eta \text{ is the limit ordinal } \leq \theta^- \text{ where the process stops, i.e.} \\ \sup\{f(\alpha_{\xi+1}) : \xi < \eta\} = \theta^-. \tag{I.119}$$

Note that if α and β are two limit points of C and if $\gamma <_T \alpha, \beta$ then $C_\alpha \cap \gamma = C_\beta \cap \gamma$. From this one concludes that the C-sequence is locally small, i.e. that $\{C_\alpha \cap \gamma : \gamma \leq \alpha < \theta\}$ has size $\leq \theta^-$ for all $\gamma < \theta$. $\dashv$

9.11 Corollary. *If $\theta^{<\theta} = \theta$ then there exists a special Aronszajn tree of height θ^+.* $\dashv$

9.12 Corollary. *In the constructible universe, special Aronszajn trees of any regular uncountable non-Mahlo height exist.* $\dashv$

9.13 Remark. In a large portion of the literature on this subject the notion of a special Aronszajn tree of height equal to some successor cardinal θ^+ is somewhat weaker, equivalent to the fact that the tree can be embedded inside the tree $\{f : \alpha \longrightarrow \theta : \alpha < \theta \,\&\, f \text{ is } 1-1\}$. One would get our notion of speciality by restricting the tree on successor ordinals losing thus the frequently useful property of a tree that different nodes of the same limit height have different sets of predecessors. The result 9.11 in this weaker form is due to Specker [71], while the result 9.12 is essentially due to Jensen [35].

10. The weight function on successor cardinals

In this section we assume that $\theta = \kappa^+$ and we fix a C-sequence C_α ($\alpha < \kappa^+$) such that

$$\text{tp}(C_\alpha) \leq \kappa \text{ for all } \alpha < \kappa^+. \tag{I.120}$$

Let $\rho_1 : [\kappa^+]^2 \longrightarrow \kappa$ be defined recursively by

$$\rho_1(\alpha, \beta) = \max\{\mathrm{tp}(C_\beta \cap \alpha), \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}$$

where we stipulate that $\rho_1(\gamma, \gamma) = 0$ for all γ.

10.1 Lemma. $|\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq \nu\}| \leq |\nu| + \aleph_0$ *for all* $\alpha < \kappa^+$ *and* $\nu < \kappa$.

Proof. Let ν^+ be the first infinite cardinal above the ordinal ν. The proof of the conclusion is by induction on α. So let $\Gamma \subseteq \alpha$ be a given set of order-type ν^+. We need to find $\xi \in \Gamma$ such that $\rho_1(\xi, \alpha) > \nu$. This will clearly be true if there is $\xi \in \Gamma$ such that $\mathrm{tp}(C_\alpha \cap \xi) > \nu$. So, we may assume that

$$\mathrm{tp}(C_\alpha \cap \xi) \leq \nu \text{ for all } \xi \in \Gamma. \tag{I.121}$$

Then there must be an ordinal $\alpha_1 \in C_\alpha$ such that

$$\Gamma_1 = \{\xi \in \Gamma : \alpha_1 = \min(C_\alpha \setminus \xi)\}$$

has size ν^+. By the inductive hypothesis there is $\xi \in \Gamma_1$ such that (see (I.121))

$$\rho_1(\xi, \alpha_1) > \nu \geq \mathrm{tp}(C_\alpha \cap \xi). \tag{I.122}$$

It follows that

$$\rho_1(\xi, \alpha) = \max\{\mathrm{tp}(C_\alpha \cap \xi), \rho_1(\xi, \alpha_1)\} = \rho_1(\xi, \alpha_1) > \nu.$$

This finishes the proof. $\dashv$

10.2 Lemma. *If* κ *is regular, then* $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}$ *has size* $< \kappa$ *for all* $\alpha < \beta < \kappa^+$.

Proof. The proof is by induction on α and β. Let $\Gamma \subseteq \alpha$ be a given set of order-type κ. We need to find $\xi \in \Gamma$ such that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$. Let $\gamma = \sup(\Gamma)$ and

$$\gamma_0 = \max(C_\beta \cap \gamma), \beta_0 = \min(C_\beta \setminus \gamma). \tag{I.123}$$

Note that by our assumption on κ and the C-sequence, these two ordinals are well-defined and

$$\gamma_0 < \gamma \leq \beta_0 < \beta. \tag{I.124}$$

By Lemma 10.1 and the inductive hypothesis there is ξ in $\Gamma \cap (\gamma_0, \gamma)$ such that

$$\rho_1(\xi, \alpha) = \rho_1(\xi, \beta_0) > \mathrm{tp}(C_\beta \cap \gamma). \tag{I.125}$$

It follows that $C_\beta \cap \gamma = C_\beta \cap \xi$ and $\beta_0 = \min(C_\beta \setminus \xi)$, and so

$$\rho_1(\xi, \beta) = \max\{\mathrm{tp}(C_\beta \cap \xi), \rho_1(\xi, \beta_0)\} = \rho_1(\xi, \beta_0) = \rho_1(\xi, \alpha).$$

This completes the proof. $\dashv$

10.3 Remark. The assumption about the regularity of κ in Lemma 10.2 is essential. For example, it can be seen (see [7, p.72]) that the conclusion of this lemma fails if κ is a singular limit of supercompact cardinals.

10.4 Definition. Define $\bar{\rho}_1 : [\kappa^+]^2 \longrightarrow \kappa$ by

$$\bar{\rho}_1(\alpha, \beta) = 2^{\rho_1(\alpha,\beta)} \cdot (2 \cdot \mathrm{tp}\{\xi \leq \alpha : \rho_1(\xi, \beta) = \rho_1(\alpha, \beta)\} + 1).$$

The following fact shows that this stretching of ρ_1 keeps the basic coherence property stated in Lemma 10.2 but it gives us also the injectivity property that could be quite usefull.

10.5 Lemma. *If κ is a regular cardinal then*

  (a) $\bar{\rho}_1(\alpha, \gamma) \neq \bar{\rho}_1(\beta, \gamma)$ *whenever* $\alpha < \beta < \gamma < \kappa^+$,

  (b) $|\{\xi \leq \alpha : \bar{\rho}_1(\xi, \alpha) \neq \bar{\rho}_1(\xi, \beta)\}| < \kappa$ *whenever* $\alpha < \beta < \kappa^+$.

$\dashv$

10.6 Remark. Note that Lemma 10.5 gives an alternative proof of Corollary 9.11 since under the assumption $\kappa^{<\kappa} = \kappa$ the tree $T(\bar{\rho}_1)$ will have levels of size at most κ. It should be noted that the coherent sequence $(\bar{\rho})_\alpha$ $(\alpha < \kappa^+)$ of one-to-one mappings is an object of independent interest which can be particularly useful in stepping-up combinatorial properties of κ to κ^+. It is also an object that has interpretations in such areas as the theory of Čech-Stone compactifications of discrete spaces (see, e.g. [95], [10], [60], [44], [16]). We have already noted that if κ is singular then we may no longer have the coherence property of Lemma 10.2. To get this property, one needs to make some additional assumption on the C-sequence C_α $(\alpha < \kappa^+)$, an assumption about the coherence of the C-sequence. This will be subject of some of the following chapters where we will concentrate on the finer function ρ rather than ρ_1.

11. The number of steps

The purpose of this section is to isolate a condition on C-sequences C_α $(\alpha < \theta)$ on regular uncountable cardinals θ as weak as possible subject to a requirement that the corresponding function

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1$$

is in some sense nontrivial, i.e. far from being constant. Without doubt the C-sequence $C_\alpha = \alpha$ $(\alpha < \theta)$ is the most trivial choice and the corresponding ρ_0-function gives no information whatsoever about the cardinal θ. The following notion of the triviality of a C-sequence on θ seems to be only marginally different.

11.1 Definition. A C-sequence C_α $(\alpha < \theta)$ on a regular uncountable cardinal θ is *trivial* if there is a closed and unbounded set $C \subseteq \theta$ such that for every $\alpha < \theta$ there is $\beta \geq \alpha$ with $C \cap \alpha \subseteq C_\beta$.

11.2 Theorem. *The following are equivalent for any C-sequence C_α $(\alpha < \theta)$ on a regular uncountable cardinal θ and the corresponding function ρ_2:*

(i) *C_α $(\alpha < \theta)$ is nontrivial.*

(ii) *For every family A of θ pairwise disjoint finite subsets of θ and every integer n there is a subfamily B of A of size θ such that $\rho_2(\alpha, \beta) > n$ for all $\alpha \in a$, $\beta \in b$ and $a \neq b$ in B.*

Proof. Note that if a closed and unbounded set $C \subseteq \theta$ witnesses the triviality of the C-sequence and if for $\alpha < \theta$ we let $\beta(\alpha) \geq \alpha$ be the minimal β such that $C \cap \alpha \subseteq C_{\beta(\alpha)}$, then any disjoint subfamily of $A = \{\{\alpha, \beta(\alpha)\} : \alpha \in C\}$ of size θ violates (ii) for $n = 1$.

Conversely, assume we are given a nontrivial C-sequence C_α $(\alpha < \theta)$. Consider the following statement:

(iii) *For every pair Γ and Δ of unbounded subsets of θ and every integer n there exist a tail subset Γ_0 of Γ, an unbounded subset Δ_0 of Δ and a regressive strictly increasing map $f : \Delta_0 \longrightarrow \theta$ such that $\rho_2(\alpha, \beta) > n$ for all $\alpha \in \Gamma_0$ and $\beta \in \Delta_0$ with $\alpha < f(\beta)$.*

To get (ii) from (iii) one assumes that the family A consists of finite sets of some fixed size m and performs m^2 successive applications of (iii): this will give us an unbounded $\bar{A} \subseteq A$ and regressive strictly increasing maps $f_{ij} : \pi_j[\bar{A}] \longrightarrow \theta$ $(i, j < m)$, where π_j is the projection mapping that takes b to the jth member of b. We have then that for all $a, b \in \bar{A}$ and $i, j < m$:

$$\text{if } \max(a) < f_{ij}(\pi_j(b)), \text{ then } \rho_2(\pi_i(a), \pi_j(b)) > n.$$

Going from there, we get a B as in (ii) by thinning out $\bar{A}$ once again.

Let us prove (iii) from (i) by induction on n. So suppose Γ and Δ are given two unbounded subsets of θ and that (iii) be true for $n - 1$. Let M_ξ $(\xi < \theta)$ be a continuous $\in$-chain of submodels of H_{θ^+} containing all relevant objects such that $\delta_\xi = M_\xi \cap \theta \in \theta$. Let

$$C = \{\xi < \theta : \delta_\xi = \xi\}.$$

Then C is a closed and unbounded subset of θ, so by our assumption (i) on C_α $(\alpha < \theta)$ there is $\delta \in C$ such that $C \cap \delta \nsubseteq C_\beta$ for all $\beta \geq \delta$. Pick an arbitrary $\beta \in \Delta$ above δ. Then there is $\xi \in C \cap \delta$ such that $\xi \notin C_\beta$. Then $\bar{\alpha} = \sup(C_\beta \cap \xi) < \xi$. Using the elementary of the submodel M_ξ we conclude that for every $\alpha \in \Gamma \setminus \bar{\alpha}$ there is $\beta(\alpha) \in \Delta \setminus (\alpha + 1)$ such that $\sup(C_{\beta(\alpha)} \cap \alpha) = \bar{\alpha}$. Applying the inductive hypothesis to the two sets

$$\Gamma \setminus \bar{\alpha} \quad \text{and} \quad \{\min(C_{\beta(\alpha)} \setminus \alpha) : \alpha \in \Gamma \setminus \bar{\alpha}\}$$

we get a tail subset Γ_0 of $\Gamma \setminus \bar{\alpha}$, an unbounded subset Γ_1 of $\Gamma \setminus \bar{\alpha}$ and a strictly increasing map $g : \Gamma_1 \longrightarrow \theta$ such that $g(\gamma) < \min(C_{\beta(\gamma)} \setminus \gamma)$ and

$$\rho_2(\alpha, \min(C_{\beta(\gamma)} \setminus \gamma)) > n - 1 \text{ for all } \alpha \in \Gamma_0, \gamma \in \Gamma_1 \text{ with } \alpha < g(\gamma). \quad \text{(I.126)}$$

Let $\Delta_0 = \{\beta(\gamma) : \gamma \in \Gamma_1\}$ and $f : \Delta_0 \longrightarrow \theta$ be defined by

$$f(\beta(\gamma)) = \min\{\gamma, g(\gamma)\}.$$

Then for $\alpha \in \Gamma_0$ and $\gamma \in \Gamma_1$ with $\alpha < f(\beta(\gamma)) \le \gamma < \beta(\gamma)$ we have

$$
\begin{aligned}
\rho_2(\alpha, \beta(\gamma)) &= \rho_2(\alpha, \min(C_{\beta(\gamma)} \setminus \alpha)) + 1 \\
&= \rho_2(\alpha, \min(C_{\beta(\gamma)} \setminus \gamma)) + 1 \\
&> (n-1) + 1 = n.
\end{aligned}
$$

This finishes the proof. $\dashv$

11.3 Corollary. *Suppose that C_α $(\alpha < \theta)$ is a nontrivial C-sequence and let $T(\rho_0)$ be the corresponding tree (see Section 9 above). Then every subset of $T(\rho_0)$ of size θ contains an antichain of size θ.*

Proof. Consider a subset K of $[\theta]^2$ of size θ which gives us a subset of $T(\rho_0)$ of size θ as follows: $\rho_0(\cdot, \beta) \upharpoonright (\alpha + 1)$ $(\{\alpha, \beta\} \in K)$. Here, we are assuming without loss of generality that the set consists of successor nodes of $T(\rho_0)$. Clearly, we may also assume that the set takes at most one point from a given level of $T(\rho_0)$. Shrinking K further, we obtain that ρ_2 is constant on K. Let n be the constant value of $\rho_2 \upharpoonright K$. Applying 11.2(ii) to K and n, we get $K_0 \subseteq K$ of size θ such that $\rho_2(\alpha, \delta) > n$ for all $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ from K_0 with properties $\alpha < \beta$, $\gamma < \delta$ and $\alpha < \gamma$. Then $\rho_0(\cdot, \beta) \upharpoonright (\alpha + 1)$ $(\{\alpha, \beta\} \in K_0)$ is an antichain in $T(\rho_0)$. $\dashv$

11.4 Remark. It should be clear that nontrivial C-sequences exist on any successor cardinal. Indeed, with very little extra work one can show that nontrivial C-sequences exist for some inaccessible cardinals quite high in the Mahlo-hierarchy. To show how close this is to the notion of weak compactness, we will give the following characterization of it which is of independent interest. [31]

11.5 Theorem. *The following are equivalent for an inaccessible cardinal θ:*

(i) *θ is weakly compact.*

(ii) *For every C-sequence C_α $(\alpha < \theta)$ there is a closed and unbounded set $C \subseteq \theta$ such that for all $\alpha < \theta$ there is $\beta \ge \alpha$ such that $C_\beta \cap \alpha = C \cap \alpha$.*

---

[31]It turns out that every C-sequence on θ being trivial is not quite as strong as the weak compactness of θ. As pointed out to us by Donder and König, one can show this using a model of Kunen [43, §3].

Proof. To see the implication from (i) to (ii), let us assume that C_α ($\alpha < \theta$) is a C-sequence and let $\rho_0 : [\theta]^2 \longrightarrow \mathbb{Q}_\theta$ be the corresponding ρ_0-function. Set

$$T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \le \beta < \theta\}.$$

Since θ is inaccessible, the levels of $T(\rho_0)$ have size $< \theta$, so by the tree-property of θ (see [37]) the tree $T(\rho_0)$ must contain a cofinal branch b. For each $\alpha < \theta$ fix $\beta(\alpha) \ge \alpha$ such that the restriction of $(\rho_0)_{\beta(\alpha)}$ to α belongs to b. For limit $\alpha < \theta$ let $\gamma(\alpha)$ be the largest member γ appearing in the walk

$$\beta(\alpha) = \beta_0(\alpha) > \beta_1(\alpha) > \ldots > \beta_{k(\alpha)}(\alpha) = \alpha$$

along the C-sequence with the property that $C_\gamma \cap \alpha$ is unbounded in α. (Thus $\gamma = \alpha$ or $\gamma = \beta_{k(\alpha)-1}(\alpha)$.) By the Pressing Down Lemma there is a stationary set $\Gamma \subseteq \theta$ and an integer k, a sequence t in $\mathbb{Q}_\theta$ and an ordinal $\bar\alpha < \theta$ such that for all $\alpha \in \Gamma$:

$k(\alpha) = k$ and $\sup(C_{\beta_i(\alpha)} \cap \alpha) \le \bar\alpha$ for all $i < k$ for which this set is bounded in α.   (I.127)

$\mathrm{tp}(C_{\beta_i(\alpha)} \cap \alpha) = t(i)$ for all $i < k$ for which this set is bounded in α.   (I.128)

Shrinking Γ we may assume that either for all $\alpha \in \Gamma$, $\gamma(\alpha) = \alpha$ in which case t is chosen to be a k-sequence, or for all $\alpha \in \Gamma$, $\gamma(\alpha) = \beta_{k-1}(\alpha)$ in which case t is chosen to be a $(k-1)$-sequence. It follows that

$$\rho_0(\xi, \beta(\alpha)) = t^\frown \rho_0(\xi, \gamma(\alpha)) \text{ for all } \alpha \in \Gamma \text{ and } \xi \in [\bar\alpha, \alpha). \qquad (\text{I.129})$$

It follows that for $\alpha < \alpha'$ in Γ the mappings $(\rho_0)_{\gamma(\alpha)}$ and $(\rho_0)_{\gamma(\alpha')}$ have the same restrictions on the interval $[\bar\alpha, \alpha)$. So in particular we have that

$$C_{\gamma(\alpha)} \cap [\bar\alpha, \alpha) = C_{\gamma(\alpha')} \cap [\bar\alpha, \alpha) \text{ whenever } \alpha < \alpha' \text{ in } \Gamma. \qquad (\text{I.130})$$

Going to a stationary subset of Γ we may assume that all $C_{\gamma(\alpha)}$ ($\alpha \in \Gamma$) have the same intersection with $\bar\alpha$. It is now clear that the union of $C_{\gamma(\alpha)} \cap \alpha$ ($\alpha \in \Gamma$) is a closed and unbounded subset of θ satisfying the conclusion of (ii).

To prove the implication from (ii) to (i), let $T = (\theta, <_T)$ be a tree on θ whose levels are intervals of θ or more precisely, there is a closed and unbounded set Δ of limit ordinals in θ such that if δ_ξ ($\xi < \theta$) is its increasing enumeration then $[\delta_\xi, \delta_{\xi+1})$ is equal to the ξth level of T. Choose a C-sequence C_α ($\alpha < \theta$) as follows. First of all let $C_{\alpha+1} = \{\alpha\}$ and $C_\alpha = (\bar\alpha, \alpha)$ if α is limit and $\bar\alpha = \sup(\Delta \cap \alpha) < \alpha$. If α is a limit point of Δ, let

$$C_\alpha = \overline{\{\xi < \alpha : \xi <_T \alpha\}}.$$

By (ii) there is a closed and unbounded set $C \subseteq \theta$ such that for all $\alpha < \theta$ there is $\beta(\alpha) \ge \alpha$ with $C_{\beta(\alpha)} \cap \alpha = C \cap \alpha$. Let Γ be the set of all successor ordinals $\xi < \theta$ such that $C \cap [\delta_\xi, \delta_{\xi+1}) \ne \emptyset$. For $\xi \in \Gamma$ let $t_\xi = \min(C \cap [\delta_\xi, \delta_{\xi+1}))$. Then it is seen that $\{t_\xi : \xi \in \Gamma\}$ is a chain of T of size θ. This finishes the proof. $\dashv$

We have already remarked that every successor cardinal $\theta = \kappa^+$ admits a nontrivial C-sequence C_α $(\alpha < \theta)$. It suffices to take the C_α's to be all of order-type $\leq \kappa$. It turns out that for such a C-sequence the corresponding ρ_2-function has a property that is considerably stronger than 11.2(ii).

11.6 Theorem. *For every infinite cardinal κ there is a C-sequence on κ^+ such that the corresponding ρ_2-function has the following unboundedness property: for every family A of κ^+ pairwise disjoint subsets of κ^+, all of size $< \kappa$, and for every $n < \omega$ there exists $B \subseteq A$ of size κ^+ such that $\rho_2(\alpha, \beta) > n$ whenever $\alpha \in a$ and $\beta \in b$ for some $a \neq b$ in B.*

Proof. We shall show that this is true for every C-sequence C_α $(\alpha < \kappa^+)$ with the property that $\text{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$. Let us first consider the case when κ is a regular cardinal. The proof is by induction on n. Suppose the conclusion is true for some integer n and let A be a given family of κ^+ pairwise disjoint subsets of κ^+, all of size $< \kappa^+$. Let M_ξ $(\xi < \kappa^+)$ be a continuous $\in$-chain of elementary submodels of $H_{\kappa^{++}}$ such that for every ξ the M_ξ contains all the relevant objects and has the property that $\delta_\xi = M_\xi \cap \kappa^+ \in \kappa^+$. Let $C = \{\delta_\xi : \xi < \kappa^+\}$. Choose $\xi = \delta_\xi \in C$ of cofinality κ and $b \in A$ such that $\beta > \xi$ for all $\beta \in b$. Then $\gamma = \sup\{\max(C_\beta \cap \xi) : \beta \in b\} < \xi$. By the elementarity of M_ξ we conclude that for every $\eta \in (\gamma, \kappa^+)$ there is b_η in A above η with $\gamma = \sup\{\max(C_\beta \cap \eta) : \beta \in b_\eta\}$. Consider the following family of subsets of κ^+: $\hat{a}_\eta = b_\eta \cup \{\min(C_\beta \setminus \eta) : \beta \in b_\eta\}$ $(\eta \in (\gamma, \kappa^+))$. By the inductive hypothesis there is an unbounded $\Gamma \subseteq \kappa^+$ such that for all $\eta < \zeta$ in Γ, $b_\eta \subseteq \zeta$ and $\rho_2(\alpha, \beta) > n$ for all $\alpha \in \hat{a}_\eta$ and $\beta \in \hat{a}_\zeta$. Let $B = \{b_\eta : \eta \in \Gamma\}$ and let b_η and b_ζ for $\eta < \zeta$ be two given members of B. Then $\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1 \geq n + 1$ for all $\alpha \in b_\eta$ and $\beta \in b_\zeta$. This completes the inductive step and therefore the proof of the Theorem when κ is a regular cardinal.

If κ is a singular cardinal, going to a subfamily of A, we may assume that there is a regular cardinal $\lambda < \kappa$ and an unbounded set $\Gamma \subseteq \kappa^+$ with the property that A can be enumerated as a_η $(\eta \in \Gamma)$ such that for all $\eta \in \Gamma$:

$$|a_\eta| < \lambda, \tag{I.131}$$

$$\beta > \eta \text{ for all } \beta \in a_\eta, \tag{I.132}$$

$$|C_\beta \cap \eta| < \lambda \text{ for all } \beta \in a_\eta. \tag{I.133}$$

Choose $\xi = \delta_\xi \in C$ of cofinality λ and $\hat{\eta} \in \Gamma$ above ξ and proceed as in the previous case. This completes the inductive step and the proof of the theorem 11.6. $\dashv$

Note that the combination of the ideas from the proofs of Theorem 11.2 and 11.6 gives us the following variation.

11.7 Theorem. *Suppose that a regular uncountable cardinal θ supports a nontrivial C-sequence and let ρ_2 be the associated function. Then for every integer n and every pair of θ-sized families A_0 and A_1, where the members of A_0 are pairwise disjoint bounded subsets of θ and the members of A_1 are pairwise disjoint finite subsets of θ, there exist $B_0 \subseteq A_0$ and $B_1 \subseteq A_1$ of size θ such that $\rho_2(\alpha, \beta) > n$ whenever $\alpha \in a$ and $\beta \in b$ for some $a \in B_0$ and $b \in B_1$ such that $\sup(a) < \min(b)$.* ⊣

An interesting application of this result has recently been given by Gruenhage [28]. To state his result, given a map from a space X onto a space Y, let $\sharp(f)$ be the order-type of the shortest chain $\mathcal{C}$ of closed subsets of X such that $f''H = Y$ for all $H \in \mathcal{C}$, while $f''(\bigcap \mathcal{C}) \neq Y$. If such a chain does not exist, set $\sharp(f) = \infty$.

11.8 Theorem. *Let $f : X \longrightarrow Y$ be a given closed surjection. Assume that the space X has the property that every open cover of X has a point countable open refinement and that the space Y has the property that every open subspace of Y either contains an isolated point or a sequence of open sets whose intersection is not open. Then $\sharp(f)$ does not support a nontrivial C-sequence.* ⊣

This shows that under some very mild assumption on a pair of spaces X and Y, if their sizes are smaller than the first regular uncountable cardinal θ not supporting a nontrivial C-sequence, then for every closed map f from X onto Y, there is a closed subspace H of X on which f is *irreducible*, i.e. $f''K \neq Y$ for every proper closed subset K of H.

12. Square sequences

12.1 Definition. A C-sequence C_α $(\alpha < \theta)$ is a *square-sequence* if and only if it is *coherent*, i.e. it has the property that $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit point of C_β.

Note that the nontriviality conditions appearing in Definition 11.1 and Theorem 11.5 coincide in the realm of square-sequences:

12.2 Lemma. *A square-sequence C_α $(\alpha < \theta)$ is trivial if and only if there is a closed and unbounded subset C of θ such that $C_\alpha = C \cap \alpha$ whenever α is a limit point of C.* ⊣

To a given square-sequence C_α $(\alpha < \theta)$ one naturally associates a tree ordering $<^2$ on θ as follows:

$$\alpha <^2 \beta \text{ if and only if } \alpha \text{ is a limit point of } C_\beta. \tag{I.134}$$

The triviality of C_α ($\alpha < \theta$) is then equivalent to the statement that the tree $(\theta, <^2)$ has a chain of size θ. In fact, one can characterize the tree orderings $<_T$ on θ for which there exists a square sequence C_α ($\alpha < \theta$) such that for all $\alpha < \beta < \theta$:

$$\alpha <_T \beta \text{ if and only if } \alpha \text{ is a limit point of } C_\beta. \tag{I.135}$$

12.3 Lemma. *A tree ordering $<_T$ on θ admits a square sequence C_α ($\alpha < \theta$) satisfying (I.135) if and only if*

(i) *$\alpha <_T \beta$ can hold only for limit ordinals α and β such that $\alpha < \beta$,*

(ii) *$P_\beta = \{\alpha : \alpha <_T \beta\}$ is a closed subset of β, which is unbounded in β whenever $\mathrm{cf}(\beta) > \omega$ and*

(iii) *minimal as well as successor nodes of the tree $<_T$ on θ are ordinals of cofinality ω.*

Proof. For each ordinal $\alpha < \theta$ of countable cofinality we fix a subset $S_\alpha \subseteq \alpha$ of order-type ω cofinal with α. Given a tree ordering $<_T$ on θ with properties (i)-(iii), for a limit ordinal $\beta < \theta$ let P_β^+ be the set of all successor nodes from $P_\beta \cup \{\beta\}$ including the minimal one. For $\alpha \in P_\beta^+$ let α^- be its immediate predecessor in P_β. Finally, set

$$C_\beta = P_\beta \cup \bigcup \{S_\alpha \cap [\alpha^-, \alpha) : \alpha \in P_\beta^+\}.$$

It is easily checked that this defines a square-sequence C_β ($\beta < \theta$) with the property that $\alpha <_T \beta$ holds if and only if α is a limit point of C_β. $\dashv$

12.4 Remark. It should be clear that the proof of Lemma 12.3 shows that the exact analogue of this result is true for any cofinality $\kappa < \theta$ rather than ω.

An important result about square sequences is the following result that can be deduced on the basis of the well-known construction of square sequences in the constructible universe due to Jensen [35] (see 1.10 of [78]).

12.5 Theorem. *If a regular uncountable cardinal θ is not weakly compact in the constructible subuniverse then there is a nontrivial square sequence on θ which is moreover constructible.* $\dashv$

12.6 Corollary. *If a regular uncountable cardinal θ is not weakly compact in the constructible subuniverse then there is a constructible Aronszajn tree on θ.*

Proof. Let C_α $(\alpha < \theta)$ be a fixed nontrivial square sequence which is constructible. Changing the C_α's a bit, we may assume that if β is a limit ordinal with $\alpha = \min C_\beta$ or if $\alpha \in C_\beta$ but $\sup(C_\beta \cap \alpha) < \alpha$ then α must be a successor ordinal in θ. Consider the corresponding function $\rho_0 : [\theta]^2 \longrightarrow \mathbb{Q}_\theta$

$$\rho_0(\alpha, \beta) = \operatorname{tp}(C_\beta \cap \alpha)^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

where $\rho_0(\gamma, \gamma) = \emptyset$ for all $\gamma < \theta$. Consider the tree

$$T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \le \beta < \theta\}.$$

Clearly $T(\rho_0)$ is constructible. By (I.101) the αth level of $T(\rho_0)$ is bounded by the size of the set $\{C_\beta \cap \alpha : \beta \ge \alpha\}$. Since the intersection of the form $C_\beta \cap \alpha$ is determined by its maximal limit point modulo a finite subset of α, we conclude that the αth level of $T(\rho_0)$ has size $\le |\alpha| + \aleph_0$. Since the sequence C_α $(\alpha < \theta)$ is nontrivial, the proof of Theorem 11.5 shows that $T(\rho_0)$ has no cofinal branches. $\dashv$

12.7 Lemma. *Suppose C_α $(\alpha < \theta)$ is a square sequence on θ, $<^2$ the associated tree ordering on θ and $T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \le \beta < \theta\}$ where $\rho_0 : [\theta]^2 \longrightarrow \mathbb{Q}_\theta$ is the associated ρ_0-function. Then $\alpha \longmapsto (\rho_0)_\alpha$ is a strictly increasing map from the tree $(\theta, <^2)$ into the tree $T(\rho_0)$.*

Proof. If α is a limit point of C_β then $C_\alpha = C_\beta \cap \alpha$ so the walks $\alpha \to \xi$ and $\beta \to \xi$ for $\xi < \alpha$ get the same code $\rho_0(\xi, \alpha) = \rho_0(\xi, \beta)$. $\dashv$

The purpose of this section, however, is to analyze a family of ρ-functions associated with a square sequence C_α $(\alpha < \theta)$ on some regular uncountable cardinal θ, both fixed from now on. Recall that an ordinal α *divides* an ordinal γ if there is β such that $\gamma = \alpha \cdot \beta$, i.e. γ can be written as the union of an increasing β-sequence of intervals of type α. Let $\kappa \le \theta$ be a fixed infinite regular cardinal. Let $\Lambda_\kappa : [\theta]^2 \longrightarrow \theta$ be defined by

$$\Lambda_\kappa(\alpha, \beta) = \max\{\xi \in C_\beta \cap (\alpha + 1) : \kappa \text{ divides } \operatorname{tp}(C_\beta \cap \xi)\}. \qquad (\text{I.136})$$

Finally we are ready to define the main object of study in this section:

$$\rho_\kappa : [\theta]^2 \longrightarrow \kappa \qquad (\text{I.137})$$

defined recursively by

$$
\begin{aligned}
\rho_\kappa(\alpha, \beta) \;=\; &\sup\{\operatorname{tp}(C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)), \rho_\kappa(\alpha, \min(C_\beta \setminus \alpha)), \\
&\quad \rho_\kappa(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)\},
\end{aligned}
$$

where we stipulate that $\rho_\kappa(\gamma, \gamma) = 0$ for all γ.

The following consequence of the coherence property of C_α $(\alpha < \theta)$ will be quite useful.

12.8 Lemma. *If α is a limit point of C_β then $\rho_\kappa(\xi, \alpha) = \rho_\kappa(\xi, \beta)$ for all $\xi < \alpha$.* ⊣

Note that ρ_κ is something that corresponds to the function $\rho : [\omega_1]^2 \longrightarrow \omega$ considered in Definition 4.1 (see also Section 14) and that the ρ_κ's are simply various *local versions* of the key definition. We shall show that they all have the crucial subadditive properties.

12.9 Lemma. *If $\alpha < \beta < \gamma < \theta$ then*

(a) $\rho_\kappa(\alpha, \gamma) \leq \max\{\rho_\kappa(\alpha, \beta), \rho_\kappa(\beta, \gamma)\}$,

(b) $\rho_\kappa(\alpha, \beta) \leq \max\{\rho_\kappa(\alpha, \gamma), \rho_\kappa(\beta, \gamma)\}$.

Proof. The proof is by induction an α, β and γ. We first prove the inductive step for (a). Let $\nu = \max\{\rho_\kappa(\alpha, \beta), \rho_\kappa(\beta, \gamma)\}$. We need to show that $\rho_\kappa(\alpha, \gamma) \leq \nu$. To this end, let

$$\gamma_\alpha = \min(C_\gamma \setminus \alpha) \text{ and } \gamma_\beta = \min(C_\gamma \setminus \beta). \tag{I.138}$$

Case 1^a: $\alpha < \Lambda_\kappa(\beta, \gamma)$. Then by Lemma 12.8, $\rho_\kappa(\alpha, \gamma) = \rho_\kappa(\alpha, \Lambda_\kappa(\beta, \gamma))$. By the definition of $\rho_\kappa(\beta, \gamma)$ we have

$$\rho_\kappa(\Lambda_\kappa(\beta, \gamma), \beta) \leq \rho_\kappa(\beta, \gamma) \leq \nu. \tag{I.139}$$

Applying the inductive hypothesis (b) for the triple $\alpha < \Lambda_\kappa(\beta, \gamma) \leq \beta$ we conclude that $\rho_\kappa(\alpha, \Lambda_\kappa(\beta, \gamma)) \leq \nu$.

Case 2^a: $\alpha \geq \Lambda_\kappa(\beta, \gamma)$. Then $\Lambda_\kappa(\beta, \gamma) = \Lambda_\kappa(\alpha, \gamma) = \lambda$.

Subcase 2^a.1: $\gamma_\alpha = \gamma_\beta = \bar\gamma$. By the inductive hypothesis (a) for the triple $\alpha < \beta \leq \bar\gamma$ we get

$$\rho_\kappa(\alpha, \bar\gamma) \leq \max\{\rho_\kappa(\alpha, \beta), \rho_\kappa(\beta, \bar\gamma)\} \leq \nu, \tag{I.140}$$

since $\rho_\kappa(\beta, \bar\gamma) \leq \rho_\kappa(\beta, \gamma)$. Note that $[\lambda, \alpha) \cap C_\gamma$ is an initial part of $[\lambda, \beta) \cap C_\gamma$ so its order-type is bounded by $\rho_\kappa(\beta, \gamma)$ and therefore by ν. Consider a $\xi \in [\lambda, \alpha) \cap C_\gamma$. Applying the inductive hypothesis (b) for the triple $\xi < \alpha < \beta$ we conclude that

$$\rho_\kappa(\xi, \alpha) \leq \max\{\rho_\kappa(\xi, \beta), \rho_\kappa(\alpha, \beta)\} \leq \nu, \tag{I.141}$$

since $\rho_\kappa(\xi, \beta) \leq \rho_\kappa(\beta, \gamma)$ by the definition of $\rho_\kappa(\beta, \gamma)$. This shows that all ordinals appearing in the formula for $\rho_\kappa(\alpha, \gamma)$ are $\leq \nu$ and so $\rho_\kappa(\alpha, \gamma) \leq \nu$.

Subcase 2^a.2: $\gamma_\alpha < \gamma_\beta$. Then γ_α belongs to $C_\gamma \cap [\lambda, \beta)$ and therefore $\rho_\kappa(\gamma_\alpha, \beta) \leq \rho_\kappa(\beta, \gamma) \leq \nu$. Applying the inductive hypothesis (b) for $\alpha \leq \gamma_\alpha < \beta$ gives us

$$\rho_\kappa(\alpha, \gamma_\alpha) \leq \max\{\rho_\kappa(\alpha, \beta), \rho_\kappa(\gamma_\alpha, \beta)\} \leq \nu. \tag{I.142}$$

Given $\xi \in C_\gamma \cap [\lambda, \alpha)$ using the inductive hypothesis (b) for $\xi < \alpha < \beta$ we get that

$$\rho_\kappa(\xi, \alpha) \leq \max\{\rho_\kappa(\xi, \beta), \rho_\kappa(\alpha, \beta)\} \leq \nu, \qquad (I.143)$$

since $\rho_\kappa(\xi, \beta) \leq \rho_\kappa(\beta, \gamma)$. Since $\mathrm{tp}(C_\gamma \cap [\lambda, \alpha)) \leq \mathrm{tp}(C_\gamma \cap [\lambda, \beta)) \leq \rho_\kappa(\beta, \gamma) \leq \nu$ we see again that all the ordinals appearing in the formula for $\rho_\kappa(\alpha, \gamma)$ are $\leq \nu$.

Let us now concentrate on the inductive step for (b). This time let $\nu = \max\{\rho_\kappa(\alpha, \gamma), \rho_\kappa(\beta, \gamma)\}$ and let γ_α and γ_β be as in (I.138). We need to show that $\rho_\kappa(\alpha, \beta) \leq \nu$.

Case 1^b: $\alpha < \Lambda_\kappa(\beta, \gamma)$. By Lemma 12.8,

$$\rho_\kappa(\alpha, \Lambda_\kappa(\beta, \gamma)) = \rho_\kappa(\alpha, \gamma) \leq \nu. \qquad (I.144)$$

From the formula for $\rho_\kappa(\beta, \gamma)$ we see that $\rho_\kappa(\Lambda_\kappa(\beta, \gamma), \beta) \leq \rho_\kappa(\beta, \gamma) \leq \nu$. Applying the inductive hypothesis (a) for the triple $\alpha < \Lambda_\kappa(\beta, \gamma) \leq \beta$ we get

$$\rho_\kappa(\alpha, \beta) \leq \max\{\rho_\kappa(\alpha, \Lambda_\kappa(\beta, \gamma)), \rho_\kappa(\Lambda_\kappa(\beta, \gamma), \beta)\} \leq \nu. \qquad (I.145)$$

Case 2^b: $\alpha \geq \Lambda_\kappa(\beta, \gamma)$.

Subcase 2^b.1: $\gamma_\alpha = \gamma_\beta = \bar{\gamma}$. Then $\rho_\kappa(\alpha, \bar{\gamma}) \leq \rho_\kappa(\alpha, \gamma) \leq \nu$ and $\rho_\kappa(\beta, \bar{\gamma}) \leq \rho_\kappa(\beta, \gamma) \leq \nu$. Applying the inductive hypothesis (b) for $\alpha < \beta < \bar{\gamma}$ we get

$$\rho_\kappa(\alpha, \beta) \leq \max\{\rho_\kappa(\alpha, \bar{\gamma}), \rho_\kappa(\beta, \bar{\gamma})\} \leq \nu. \qquad (I.146)$$

Subcase 2^b.2: $\gamma_\alpha < \gamma_\beta$. Then $\gamma_\alpha \in C_\gamma \cap [\lambda, \beta)$ and so $\rho_\kappa(\gamma_\alpha, \beta)$ appears in the formula for $\rho_\kappa(\beta, \gamma)$ and so we conclude that $\rho_\kappa(\gamma_\alpha, \beta) \leq \rho_\kappa(\beta, \gamma) \leq \nu$. Similarly, $\rho_\kappa(\alpha, \gamma_\alpha) \leq \rho_\kappa(\alpha, \gamma) \leq \nu$. Applying the inductive hypothesis (a) for the triple $\alpha \leq \gamma_\alpha < \beta$ we get

$$\rho_\kappa(\alpha, \beta) \leq \max\{\rho_\kappa(\alpha, \gamma_\alpha), \rho_\kappa(\gamma_\alpha, \beta)\} \leq \nu. \qquad (I.147)$$

This finishes the proof. $\dashv$

The following is an immediate consequence of the fact that the definition of ρ_κ is closely tied to the notion of a walk along the fixed square sequence.

12.10 Lemma. $\rho_\kappa(\alpha, \gamma) \geq \rho_\kappa(\alpha, \beta)$ *whenever* $\alpha \leq \beta \leq \gamma$ *and* β *belongs to the trace of the walk from* γ *to* α. $\qquad \dashv$

12.11 Lemma. *Suppose* $\beta \leq \gamma < \theta$ *and that* β *is a limit ordinal* > 0. *Then* $\rho_\kappa(\alpha, \gamma) \geq \rho_\kappa(\alpha, \beta)$ *for coboundedly many* $\alpha < \beta$.

Proof. Let $\gamma = \gamma_0 > \gamma_1 > \ldots > \gamma_{n-1} > \gamma_n = \beta$ be the trace of the walk from γ to β. Let $\bar{\gamma} = \gamma_{n-1}$ if β is a limit point of $C_{\gamma_{n-1}}$, otherwise let $\bar{\gamma} = \beta$. Note that by Lemma 12.8, in any case we have that

$$\rho_\kappa(\alpha, \beta) = \rho_\kappa(\alpha, \bar{\gamma}) \text{ for all } \alpha < \beta. \qquad (I.148)$$

Let $\bar{\beta} < \beta$ be an upper bound of all $C_{\gamma_i} \cap \beta$ $(i < n)$ which are bounded in β. Then $\bar{\gamma}$ is a member of the trace of any walk from γ to some ordinal α in the interval $[\bar{\beta}, \beta)$. Applying Lemma 12.10 to this fact gives us

$$\rho_\kappa(\alpha, \gamma) \geq \rho_\kappa(\alpha, \bar{\gamma}) \text{ for all } \alpha \in [\bar{\beta}, \beta). \tag{I.149}$$

Since $\rho_\kappa(\alpha, \bar{\gamma}) = \rho_\kappa(\alpha, \beta)$ for all $\alpha < \beta$ (see (I.148)), this gives us the conclusion of the Lemma. $\dashv$

12.12 Lemma. *The set $P_\nu^\kappa(\beta) = \{\xi < \beta : \rho_\kappa(\xi, \beta) \leq \nu\}$ is a closed subset of β for every $\beta < \theta$ and $\nu < \kappa$.*

Proof. The proof is by induction on β. So let $\alpha < \beta$ be a limit point of $P_\nu^\kappa(\beta)$. Let $\beta_1 = \min(C_\beta \setminus \alpha)$ and let $\lambda = \Lambda_\kappa(\alpha, \beta)$. If $\lambda = \alpha$ then $\rho_\kappa(\alpha, \beta) = 0$ and therefore $\alpha \in P_\nu^\kappa(\beta)$. So we may assume that $\lambda < \alpha$. Then

$$\Lambda_\kappa(\xi, \beta) = \lambda \text{ for every } \xi \in [\lambda, \alpha). \tag{I.150}$$

Case 1: $\bar{\alpha} = \sup(C_\beta \cap \alpha) < \alpha$. Then $\beta_1 = \min(C_\beta \setminus \xi)$ for every $\xi \in P_\nu^\kappa(\beta) \cap (\bar{\alpha}, \alpha)$. It follows that

$$P_\nu^\kappa(\beta) \cap (\bar{\alpha}, \alpha) \subseteq P_\nu^\kappa(\beta_1), \tag{I.151}$$

so α is also a limit point of $P_\nu^\kappa(\beta_1)$. By the inductive hypothesis $\alpha \in P_\nu^\kappa(\beta_1)$, i.e. $\rho_\kappa(\alpha, \beta_1) \leq \nu$. Since $\Lambda_\kappa(\xi, \beta) = \lambda$ for all $\xi \in [\lambda, \alpha)$, we get

$$\operatorname{tp}(C_\beta \cap [\lambda, \alpha)) = \sup\{\operatorname{tp}(C_\beta \cap [\lambda, \xi)) : \xi \in P_\nu^\kappa(\beta) \cap [\lambda, \alpha)\} \leq \nu. \tag{I.152}$$

Consider $\eta \in C_\beta \cap [\lambda, \alpha)$. Then $\eta \in C_\beta \cap [\lambda, \xi)$ for some $\xi \in P_\nu^\kappa(\beta) \cap [\bar{\alpha}, \alpha)$ so by the definition of $\rho_\kappa(\xi, \beta)$ we conclude that $\rho_\kappa(\eta, \xi) \leq \rho_\kappa(\xi, \beta) \leq \nu$. By (I.151) we also have $\rho_\kappa(\xi, \beta_1) \leq \nu$ so by 12.9(a), $\rho_\kappa(\eta, \beta_1) \leq \nu$. Applying 12.9(b) to $\eta < \alpha < \beta_1$, we conclude that

$$\rho_\kappa(\eta, \alpha) \leq \max\{\rho_\kappa(\eta, \beta_1), \rho_\kappa(\alpha, \beta_1)\} \leq \nu. \tag{I.153}$$

This shows that all the ordinals appearing in the formula for $\rho_\kappa(\alpha, \beta)$ are $\leq \nu$ and therefore that $\rho_\kappa(\alpha, \beta) \leq \nu$.

Case 2: α is a limit point of C_β. As in the previous case, to show that $\rho_\kappa(\alpha, \beta) \leq \nu$ we need to have $\operatorname{tp}(C_\beta \cap [\lambda, \alpha))$, $\rho_\kappa(\alpha, \beta_1)$ and $\rho_\kappa(\eta, \alpha)$ with $\eta \in C_\beta \cap [\lambda, \alpha)$ all $\leq \nu$. Note that $\beta_1 = \alpha$ so $\rho_\kappa(\alpha, \beta_1) = 0$. Similarly as in the first case we conclude $\operatorname{tp}(C_\beta \cap [\lambda, \alpha)) \leq \nu$. By Lemma 12.8,

$$\rho_\kappa(\xi, \alpha) = \rho_\kappa(\xi, \beta) \leq \nu \text{ for all } \xi \in P_\nu^\kappa(\beta) \cap \alpha. \tag{I.154}$$

Consider an $\eta \in C_\beta \cap [\lambda, \alpha)$. Since α is a limit point of $P_\nu^\kappa(\beta)$ there is ξ from this set above η. By (I.150) we have that $\eta \in C_\beta \cap [\Lambda_\kappa(\xi, \beta), \xi)$ so $\rho_\kappa(\eta, \xi) \leq \rho_\kappa(\xi, \beta) \leq \nu$. Applying 12.9(a) to $\eta < \xi < \alpha$ we get

$$\rho_\kappa(\eta, \alpha) \leq \max\{\rho_\kappa(\eta, \xi), \rho_\kappa(\xi, \alpha)\} \leq \nu. \tag{I.155}$$

This finishes the proof. $\dashv$

For $\alpha < \beta < \theta$ and $\nu < \kappa$ set

$$\alpha <_\nu^\kappa \beta \text{ if and only if } \rho_\kappa(\alpha, \beta) \leq \nu. \qquad (\text{I.156})$$

12.13 Lemma.

(1) $<_\nu^\kappa$ is a tree ordering on θ,

(2) $<_\nu^\kappa \subseteq <_\mu^\kappa$ whenever $\nu < \mu < \kappa$,

(3) $\in{\restriction}\,\theta = \bigcup_{\nu<\kappa} <_\nu^\kappa$.

Proof. This follows immediately from Lemma 12.9. $\dashv$

Recall the notion of a special tree of height θ from Section 9, a tree T for which one can find a T-regressive map $f : T \longrightarrow T$ with the property that the preimage of any point is the union of $< \theta$ antichains. By a *tree on θ* we mean a tree of the form $(\theta, <_T)$ with the property that $\alpha <_T \beta$ implies $\alpha < \beta$.

12.14 Lemma. *If a tree T on θ is special, then there is an ordinal-regressive map $f : \theta \longrightarrow \theta$ and a closed and unbounded set $C \subseteq \theta$ such that f is one-to-one on all chains separated by C.*

Proof. Let $g : \theta \longrightarrow \theta$ be a T-regressive map such that for each $\xi < \theta$ the preimage $g^{-1}(\xi)$ can be written as a union of a sequence $A_\delta(\xi)$ $(\delta < \lambda_\xi)$ of antichains, where $\lambda_\xi < \theta$. Let C be the collection of all limit $\alpha < \theta$ with the property that $\lambda_\xi < \alpha$ for all $\xi < \alpha$. Let $\ulcorner .,. \urcorner$ denote the Gödel pairing function which in particular has the property that $\ulcorner \alpha, \beta \urcorner < \delta$ for $\delta \in C$ and α, β, δ. Define ordinal-regressive map $f : \theta \longrightarrow \theta$ by letting $f(\alpha) = \max(C \cup \alpha)$ if $\alpha \notin C$ and for $\alpha \in C$, let $f(\alpha) = \ulcorner g(\alpha), \gamma \urcorner$ where $\gamma < \lambda_{g(\alpha)} (< \alpha)$ is minimal ordinal with property that $\alpha \in A_\gamma(g(\alpha))$. Then f is as required. $\dashv$

By Lemma 12.13 we have a sequence $<_\nu^\kappa$ $(\nu < \kappa)$ of tree orderings on θ. The following Lemma tells us that they are frequently quite large orderings.

12.15 Lemma. *If $\theta > \kappa$ is not a successor of a cardinal of cofinality κ then there must be $\nu < \kappa$ such that $(\theta, <_\nu^\kappa)$ is a nonspecial tree on θ.*

Proof. Suppose to the contrary that all trees are special. By Lemma 12.14 we may choose ordinal-regressive maps $f_\nu : \theta \longrightarrow \theta$ for all $\nu < \kappa$ and a single closed and unbounded set $C \subseteq \theta$ such that each of the maps f_ν is one-to-one on $<_\nu^\kappa$-chains separated by C. Using the Pressing Down Lemma we find a stationary set Γ of cofinality κ^+ ordinals $< \theta$ and $\lambda < \theta$ such that $f_\nu(\gamma) < \lambda$ for all $\gamma \in \Gamma$ and $\nu < \kappa$. If $|\lambda|^+ < \theta$, let $\Delta = \lambda$, $\Gamma = \Gamma_0$ and if $|\lambda|^+ = \theta$, represent λ as the increasing union of a sequence Δ_ξ $(\xi < \mathrm{cf}(|\lambda|))$

of sets of size $< |\lambda|$. Since $\kappa \neq \mathrm{cf}(|\lambda|)$ there is a $\bar{\xi} < \mathrm{cf}(|\lambda|)$ and a stationary $\Gamma_0 \subseteq \Gamma$ such that for all $\gamma \in \Gamma_0$, $f_\nu(\gamma) \in \Delta_{\bar{\xi}}$ for κ many $\nu < \kappa$. Let $\Delta = \Delta_{\bar{\xi}}$. This gives us subsets Δ and Γ_0 of θ such that

$$|\Delta|^+ < \theta \text{ and } \Gamma_0 \text{ is stationary in } \theta, \tag{I.157}$$

$$\Sigma_\gamma = \{\nu < \kappa : f_\nu(\gamma) \in \Delta\} \text{ is unbounded in } \kappa \text{ for all } \gamma \in \Gamma_0. \tag{I.158}$$

Let $\bar{\theta} = \kappa^+ \cdot |\Delta|^+$. Then $\bar{\theta} < \theta$ and so we can find $\beta \in \Gamma_0$ such that $\Gamma_0 \cap C \cap \beta$ has size $\bar{\theta}$. Then there will be $\nu_0 < \kappa$ and $\Gamma_1 \subseteq \Gamma_0 \cap C \cap \beta$ of size $\bar{\theta}$ such that $\rho_\kappa(\alpha, \beta) \leq \nu_0$ for all $\alpha \in \Gamma_1$. By (I.158) we can find $\Gamma_2 \subseteq \Gamma_1$ of size $\bar{\theta}$ and $\nu_1 \geq \nu_0$ such that $f_{\nu_1}(\alpha) \in \Delta$ for all $\alpha \in \Gamma_2$. Note that Γ_2 is a $<_{\nu_1}^\kappa$-chain separated by C, so f_{ν_1} is one-to-one on Γ_2. However, this gives us the desired contradiction since the set Δ, in which f_{ν_1} embeds Γ_2 has size smaller than the size of Γ_2. This finishes the proof. $\dashv$

It is now natural to ask the following question: under which assumption on the square sequence C_α $(\alpha < \theta)$ can we conclude that neither of the trees $(\theta, <_\nu^\kappa)$ will have a branch of size θ?

12.16 Lemma. *If the set* $\Gamma_\kappa = \{\alpha < \theta : \mathrm{tp}\, C_\alpha = \kappa\}$ *is stationary in* θ, *then none of the trees* $(\theta, <_\nu^\kappa)$ *has a branch of size* θ.

Proof. Assume that B is a $<_\nu^\kappa$-branch of size θ. By Lemma 12.12, B is a closed and unbounded subset of θ. Pick a limit point β of B which belongs to Γ_κ. Pick $\alpha \in B \cap \beta$ such that $\mathrm{tp}(C_\beta \cap \alpha) > \nu$. By definition of $\rho_\kappa(\alpha, \beta)$ we have that $\rho_\kappa(\alpha, \beta) \geq \mathrm{tp}(C_\beta \cap \alpha) > \nu$ since clearly $\Lambda_\kappa(\alpha, \beta) = 0$. This contradicts the fact that $\alpha <_\nu^\kappa \beta$ and finishes the proof. $\dashv$

12.17 Definition. A square sequence on θ is *special* if the corresponding tree $(\theta, <^2)$ is special, i.e. there is a $<^2$-regressive map $f : \theta \longrightarrow \theta$ with the property that the f-preimage of every $\xi < \theta$ is the union of $< \theta$ antichains of $(\theta, <^2)$.

12.18 Theorem. *Suppose* $\kappa < \theta$ *are regular cardinals such that* θ *is not a successor of a cardinal of cofinality* κ. *Then to every square sequence* C_α $(\alpha < \theta)$ *for which there exist stationarily many* α *such that* $\mathrm{tp}\, C_\alpha = \kappa$, *one can associate a sequence* $C_{\alpha\nu}$ $(\alpha < \theta, \nu < \kappa)$ *such that:*

(i) $C_{\alpha\nu} \subseteq C_{\alpha\mu}$ *for all* α *and* $\nu \leq \mu$,

(ii) $\alpha = \bigcup_{\nu < \kappa} C_{\alpha\nu}$ *for all limit* α,

(iii) $C_{\alpha\nu}$ $(\alpha < \theta)$ *is a nonspecial (and nontrivial) square sequence on* θ *for all* $\nu < \kappa$.

Proof. Fix $\nu < \kappa$ and define $C_{\alpha\nu}$ by induction on $\alpha < \theta$. So suppose β is a limit ordinal $< \theta$ and that $C_{\alpha\nu}$ is defined for all $\alpha < \beta$. If $P_{\nu}^{\kappa}(\beta)$ is bounded in β, let $\bar{\beta}$ be the maximal limit point of $P_{\nu}^{\kappa}(\beta)$ ($\bar{\beta} = 0$ if the set has no limit points) and let

$$C_{\beta\nu} = C_{\bar{\beta}\nu} \cup P_{\nu}^{\kappa}(\beta) \cup (C_{\beta} \cap [\max(P_{\nu}^{\kappa}(\beta)), \beta)). \qquad (\text{I}.159)$$

If $P_{\nu}^{\kappa}(\beta)$ is unbounded in β, let

$$C_{\beta\nu} = P_{\nu}^{\kappa}(\beta) \cup \bigcup \{C_{\alpha\nu} : \alpha \in P_{\nu}^{\kappa}(\beta) \ \& \ \alpha = \sup(P_{\nu}^{\kappa}(\beta) \cap \alpha)\}. \qquad (\text{I}.160)$$

By Lemmas 12.9 and 12.12, $C_{\beta\nu}$ $(\beta < \theta)$ is well defined and it forms a square sequence on θ. The properties (i) and (ii) are also immediate. To see that for each $\nu < \kappa$ the sequence $C_{\beta\nu}$ $(\beta < \theta)$ is nontrivial, one uses Lemma 12.16 and the fact that if α is a limit point of $C_{\beta\nu}$ occupying a place in $C_{\beta\nu}$ that is divisible by κ, then $\alpha <_{\nu}^{\kappa} \beta$. By Lemma 12.15, or rather its proof, we conclude that there is $\bar{\nu} < \kappa$ such that $C_{\beta\nu}$ $(\beta < \theta)$ is nonspecial for all $\nu \geq \bar{\nu}$. This finishes the proof. $\qquad \dashv$

12.19 Lemma. *For every pair of regular cardinals $\kappa < \theta$, every special square sequence C_{α} $(\alpha < \theta)$ can be refined to a special square sequence $\bar{C}_{\alpha}$ $(\alpha < \theta)$ with the property that* tp $\bar{C}_{\alpha} = \kappa$ *for stationarily many $\alpha < \theta$.*

Proof. Let $<^2$ be the tree ordering associated with C_{α} $(\alpha < \theta)$ and let $f : \theta \longrightarrow \theta$ be a $<^2$-regressive map such that $f^{-1}(\xi)$ is the union of $< \theta$ antichains of $(\theta, <^2)$ for all $\xi < \theta$.

Case 1: $\kappa = \omega$. By the Pressing Down Lemma there is a stationary set Γ of cofinality ω ordinals $< \theta$ on which f takes some constant value. Thus Γ can be decomposed into $< \theta$ antichains of $(\theta, <^2)$ and so Γ contains a stationary subset Γ_0 of pairwise $<^2$-incomparable ordinals. For $\alpha \in \Gamma_0$ let $\bar{C}_{\alpha}$ be any ω-sequence of successor ordinals converging to α. If α is a limit ordinal not in Γ_0 but there is (a unique!) $\bar{\alpha} <^2 \alpha$ in Γ_0, let $\bar{C}_{\alpha} = C_{\alpha} \setminus \bar{\alpha}$. In all other cases, let $\bar{C}_{\alpha} = C_{\alpha}$. It is easily seen that $\bar{C}_{\alpha}$ $(\alpha < \theta)$ is as required.

Case 2: $\kappa > \omega$. By Lemma 12.14 we may find a closed and unbounded subset C of θ and an ordinal-regressive map $g : C \longrightarrow \theta$ which is one-to-one on $<^2$-chains included in C (and which has the property that for $\alpha \in C$ $g(\alpha)$ is the Gödel pairing of $f(\alpha)$ and the index of the antichain which contains α in the fixed antichain-decomposition of the preimage $f^{-1}(f(\alpha))$. By the argument from Case 1 there is a stationary set Γ of limit points of C consisting of cofinality κ ordinals $< \theta$ such that Γ is an antichain of $(\theta, <^2)$. Given $\alpha \in \Gamma$, let $h(\alpha)$ be the minimal ordinal $\lambda < \theta$ such that $g(\xi) < \lambda$ for unboundedly many limit points ξ of C_{α} that also belong to C. Since g is regressive and since κ, the cofinality of α, is uncountable, the Pressing Down Lemma gives us that $h(\alpha) < \alpha$. Choose a stationary set $\Gamma_0 \subseteq \Gamma$ such that h takes some constant value λ on Γ_0. Note that since g is one-to one

on $<^2$-chains included in C the constant value λ cannot be a successor ordinal. It follows that λ must be a limit ordinal of cofinality equal to κ. Let λ_ν $(\nu < \kappa)$ be a strictly increasing sequence which converges to λ.

We first define $\bar{C}_\alpha$ for $\alpha \in \Gamma_0$ as a strictly increasing continuous sequence $c_\nu(\alpha)$ $(\nu < \kappa)$ of limit points of C_α that also belong to C and satisfy the following requirements at each $\nu < \kappa$:

$$c_\mu(\alpha) = \sup_{\nu < \mu} c_\nu(\alpha) \text{ for } \mu \text{ limit} \tag{I.161}$$

$$\begin{aligned} &c_{\nu+1}(\alpha) > c_\nu(\alpha) \text{ is minimal subject to being a limit point} \\ &\text{of } C_\alpha, \text{ belonging to } C, \text{ and being } > \gamma \text{ for every limit point} \\ &\gamma \text{ of } C_\alpha \text{ such that } \gamma \in C \text{ and } g(\gamma) < \lambda_{\nu+1}. \end{aligned} \tag{I.162}$$

If α is a limit point of $\bar{C}_\beta$ for some $\beta \in \Gamma_0$, set $\bar{C}_\alpha = \bar{C}_\beta \cap \alpha$. Note that by the canonicity of the definition of $c_\nu(\beta)$ $(\nu < \kappa)$ for $\beta \in \Gamma_0$, we would get the same result, as $\bar{C}_\beta \cap \alpha = \bar{C}_{\bar\beta} \cap \alpha$ for any other $\bar\beta \in \Gamma_0$ for which α is a limit point of $\bar{C}_{\bar\beta}$.

If α is a limit ordinal which is not a limit point of any $\bar{C}_\beta$ for $\beta \in \Gamma_0$ and there is (a unique!) $\bar\alpha <^2 \alpha$ in Γ_0, let $\bar{C}_\alpha = C_\alpha \setminus \bar\alpha$.

If α is a limit ordinal such that $\alpha <^2 \bar\alpha$ for some $\bar\alpha \in \Gamma_0$ but α is not a limit point on any $\bar{C}_\beta$ for $\beta \in \Gamma_0$, let $\bar{C}_\alpha = C_\alpha \setminus \max(\bar{C}_{\bar\alpha} \cap \alpha)$. Note again that there is no ambiguity in this definition, as

$$\bar{C}_\beta \cap \alpha = \bar{C}_\gamma \cap \alpha$$

holds for every β and γ in Γ_0 such that $\alpha <^2 \beta$ and $\alpha <^2 \gamma$. In all other cases set $\bar{C}_\alpha = C_\alpha$. It should be clear that $\bar{C}_\alpha$ $(\alpha < \theta)$ is a nontrivial square sequence satisfying the conclusion of Lemma 12.19. $\dashv$

Finally we can state the main result of this Section which follows from Theorem 12.18 and Lemma 12.19.

12.20 Theorem. *If regular uncountable cardinal $\theta \neq \omega_1$ carries a nontrivial square sequence then it also carries a nontrivial square-sequence which is moreover nonspecial.* $\dashv$

12.21 Corollary. *If a regular uncountable cardinal $\theta \neq \omega_1$ is not weakly compact in the constructible subuniverse then there is a nonspecial Aronszajn tree of height θ.*

Proof. By Theorem 12.5, θ carries a nontrivial square sequence C_α $(\alpha < \theta)$. By Theorem 12.20 we may assume that the sequence is moreover nonspecial. Let ρ_0 be the associated ρ_0-function and consider the tree $T(\rho_0)$. As in Corollary 12.6 we conclude that $T(\rho_0)$ is an Aronszajn tree of height θ. By Lemma 12.7 there is a strictly increasing map from $(\theta, <^2)$ into $T(\rho_0)$, so $T(\rho_0)$ must be nonspecial. $\dashv$

12.22 Remark. The assumption $\theta \neq \omega_1$ in Theorem 12.20 is essential as there is always a nontrivial square sequence on ω_1 but it is possible to have a situation where all Aronszajn trees on ω_1 are special. For example MA_{ω_1} implies this. In [48], Laver and Shelah have shown that any model with a weakly compact cardinal admits a forcing extension satisfying CH and the statement that all Aronszajn trees on ω_2 are special. A well-known open problem in this area asks whether one can have GCH rather than CH in a model where all Aronszajn trees on ω_2 are special.

13. The full lower trace of a square sequence

In this section θ is a regular uncountable cardinal and C_α $(\alpha < \theta)$ is a nontrivial square sequence on θ. Recall the function $\Lambda = \Lambda_\omega : [\theta]^2 \longrightarrow \theta$:

$$\Lambda(\alpha, \beta) = \text{maximal limit point of } C_\beta \cap (\alpha + 1).$$

($\Lambda(\alpha, \beta) = 0$ if $C_\beta \cap (\alpha + 1)$ has no limit points.)

The purpose of this section is to study the following recursive trace formula, describing a mapping $F : [\theta]^2 \longrightarrow [\theta]^{<\omega}$:

$$F(\alpha, \beta) = F(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup \{F(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\},$$

where $F(\gamma, \gamma) = \{\gamma\}$ for all γ.

13.1 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $F(\alpha, \gamma) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma)$,

(b) $F(\alpha, \beta) \subseteq F(\alpha, \gamma) \cup F(\beta, \gamma)$.

Proof. The proof of the Lemma is by simultaneous induction. We first check (a). Let $\gamma_\alpha = \min(C_\gamma \setminus \alpha)$, $\gamma_\beta = \min(C_\gamma \setminus \beta)$ and $\lambda = \Lambda(\beta, \gamma)$. If $\alpha < \lambda$ then $C_\lambda = C_\gamma \cap \lambda$ and therefore $F(\alpha, \lambda) = F(\alpha, \gamma)$. Applying the inductive hypothesis (b) for $\alpha < \lambda \leq \beta$ we get the required conclusion

$$F(\alpha, \gamma) = F(\alpha, \lambda) \subseteq F(\alpha, \beta) \cup F(\lambda, \beta) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma). \qquad (\text{I}.163)$$

So we may assume that $\lambda \leq \alpha$. Then $\Lambda(\alpha, \gamma) = \lambda$. If $\gamma_\alpha < \beta$, applying the inductive hypothesis (b) for $\alpha \leq \gamma_\alpha < \beta$, we have

$$F(\alpha, \gamma_\alpha) \subseteq F(\alpha, \beta) \cup F(\gamma_\alpha, \beta) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma), \qquad (\text{I}.164)$$

since $\gamma_\alpha \in C_\gamma \cap [\Lambda(\beta, \gamma), \beta)$. If $\gamma_\alpha \geq \beta$ then $\gamma_\alpha = \gamma_\beta$, and applying the inductive hypothesis (a) for $\alpha < \beta \leq \gamma_\alpha = \gamma_\beta$, we have

$$F(\alpha, \gamma_\alpha) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma_\beta) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma). \qquad (\text{I}.165)$$

Consider now $\xi \in C_\gamma \cap [\lambda, \alpha)$. By the inductive hypothesis (b) for $\xi < \alpha \leq \beta$ we have

$$F(\xi, \alpha) \subseteq F(\alpha, \beta) \cup F(\xi, \beta) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma), \qquad (\text{I}.166)$$

since $\xi \in C_\gamma \cap [\Lambda(\beta, \gamma), \beta)$. It follows that all the factors of $F(\alpha, \gamma)$ are included in the union $F(\alpha, \beta) \cup F(\beta, \gamma)$, so this completes the checking of (a).

To prove (b) define γ_α, γ_β and λ as above and consider first the case $\lambda \geq \alpha$. Then $C_\lambda = C_\gamma \cap \lambda$ and therefore $F(\alpha, \lambda) = F(\alpha, \gamma)$. Applying the inductive hypothesis (a) for $\alpha < \lambda \leq \beta$ we get

$$F(\alpha, \beta) \subseteq F(\alpha, \lambda) \cup F(\lambda, \beta) \subseteq F(\alpha, \gamma) \cup F(\beta, \gamma). \qquad (\text{I}.167)$$

So we may assume that $\lambda < \alpha$ in which case we also know that $\Lambda(\alpha, \gamma) = \lambda$. Consider first the case $\gamma_\alpha < \beta$. Applying the inductive hypothesis (a) for $\alpha \leq \gamma_\alpha < \beta$ we have

$$F(\alpha, \beta) \subseteq F(\alpha, \gamma_\alpha) \cup F(\gamma_\alpha, \beta) \subseteq F(\alpha, \gamma) \cup F(\beta, \gamma), \qquad (\text{I}.168)$$

since $\gamma_\alpha \in C_\gamma \cap [\Lambda(\beta, \gamma), \beta)$. Suppose now that $\gamma_\alpha \geq \beta$. Then $\gamma_\alpha = \gamma_\beta$ and applying the inductive hypothesis (b) for the triple $\alpha \leq \beta \leq \gamma_\alpha = \gamma_\beta$ we have

$$F(\alpha, \beta) \subseteq F(\alpha, \gamma_\alpha) \cup F(\beta, \gamma_\beta) \subseteq F(\alpha, \gamma) \cup F(\beta, \gamma). \qquad (\text{I}.169)$$

This completes the proof. $\dashv$

13.2 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho_0(\alpha, \beta) = \rho_0(\min(F(\beta, \gamma) \setminus \alpha), \beta)^\frown \rho_0(\alpha, \min(F(\beta, \gamma) \setminus \alpha))$,

(b) $\rho_0(\alpha, \gamma) = \rho_0(\min(F(\beta, \gamma) \setminus \alpha), \gamma)^\frown \rho_0(\alpha, \min(F(\beta, \gamma) \setminus \alpha))$.

Proof. The proof is by induction on α, β and γ. Let

$$\lambda = \Lambda(\beta, \gamma), \gamma_1 = \min(C_\gamma \setminus \alpha) \text{ and } \alpha_1 = \min(F(\beta, \gamma) \setminus \alpha).$$

Pick $\xi \in \{\min(C_\gamma \setminus \beta)\} \cup (C_\gamma \cap [\lambda, \beta))$ such that $\alpha_1 \in F(\xi, \beta)$. Then $\alpha_1 = \min(F(\xi, \beta) \setminus \alpha)$, so applying the inductive hypothesis (b) for $\alpha \leq \xi < \beta$ we get

$$\rho_0(\alpha, \beta) = \rho_0(\alpha_1, \beta)^\frown \rho_0(\alpha, \alpha_1) \qquad (\text{I}.170)$$

which checks (a) for $\alpha \leq \beta \leq \gamma$. Suppose first that $\lambda > \alpha$. Note that $F(\lambda, \beta)$ is a subset of $F(\beta, \gamma)$ so $\alpha_2 = \min(F(\lambda, \beta) \setminus \alpha) \geq \alpha_1$. Applying the inductive hypothesis for $\alpha \leq \lambda \leq \beta$ we get

$$\rho_0(\alpha, \beta) = \rho_0(\alpha_2, \beta)^\frown \rho_0(\alpha, \alpha_2), \qquad (\text{I}.171)$$

$$\rho_0(\alpha, \lambda) = \rho_0(\alpha_2, \lambda)^\frown \rho_0(\alpha, \alpha_2). \qquad (I.172)$$

Combining (I.170) and (I.171) we infer that $\rho_0(\alpha, \alpha_1)$ is a tail of $\rho_0(\alpha, \alpha_2)$, so (I.172) can be written as

$$\rho_0(\alpha, \lambda) = \rho_0(\alpha_2, \lambda)^\frown \rho_0(\alpha_1, \alpha_2)^\frown \rho_0(\alpha, \alpha_1) = \rho_0(\alpha_1, \lambda)^\frown \rho_0(\alpha, \alpha_1). \quad (I.173)$$

Since λ is a limit point of C_γ, we have $\rho_0(\alpha, \gamma) = \rho_0(\alpha, \lambda)$, so (I.173) gives us the conclusion (b) of Lemma 13.2.

Consider now the case $\lambda \leq \alpha$. Then γ_1 is either equal to $\min(C_\gamma \setminus \beta)$ or it belongs to $[\lambda, \beta) \cap C_\gamma$. In any case we have that $F\{\gamma_1, \beta\} \subseteq F(\beta, \gamma)$, so

$$\gamma_2 = \min(F\{\gamma_1, \beta\} \setminus \alpha) \geq \alpha_1. \qquad (I.174)$$

Applying the inductive hypothesis for α, γ_1 and β we get

$$\rho_0(\alpha, \beta) = \rho_0(\gamma_2, \beta)^\frown \rho_0(\alpha, \gamma_2), \qquad (I.175)$$

$$\rho_0(\alpha, \gamma_1) = \rho_0(\gamma_2, \gamma_1)^\frown \rho_0(\alpha, \gamma_2). \qquad (I.176)$$

Combining (I.170) and (I.175) we see that $\rho_0(\alpha, \alpha_1)$ is a tail of $\rho_0(\alpha, \gamma_2)$, so (I.176) can be rewritten as

$$\rho_0(\alpha, \gamma_1) = \rho_0(\gamma_2, \gamma_1)^\frown \rho_0(\alpha_1, \gamma_2)^\frown \rho_0(\alpha, \alpha_1) = \rho_0(\alpha_1, \gamma_1)^\frown \rho_0(\alpha, \alpha_1).$$
$$(I.177)$$

Since $\rho_0(\alpha, \gamma) = \operatorname{tp}(C_\gamma \cap \alpha)^\frown \rho_0(\alpha, \gamma_1)$, (I.177) gives us the conclusion (b) of Lemma 13.2. This finishes the proof. $\dashv$

Recall the function $\rho_2 : [\theta]^2 \longrightarrow \omega$ which counts the number of steps in the walk along the fixed C-sequence C_α $(\alpha < \theta)$ which in this Section is assumed to be moreover a square sequence:

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1,$$

where we let $\rho_2(\gamma, \gamma) = 0$ for all γ. Thus $\rho_2(\alpha, \beta) + 1$ is simply equal to the cardinality of the trace $\operatorname{Tr}(\alpha, \beta)$ of the minimal walk from β to α.

13.3 Lemma. $\sup_{\xi < \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \infty$ *for all* $\alpha < \beta < \theta$.

Proof. By Lemma 13.2,

$$\sup_{\xi < \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| \leq \sup_{\xi \in F(\alpha, \beta)} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|.$$

$\dashv$

13.4 Definition. Set $\mathcal{I}$ to be the set of all countable $\Gamma \subseteq \theta$ such that $\sup_{\xi \in \Delta} \rho_2(\xi, \alpha) = \infty$ for all $\alpha < \theta$ and infinite $\Delta \subseteq \Gamma \cap \alpha$.

13.5 Lemma. $\mathcal{I}$ *is a P-ideal of countable subsets of* θ.

Proof. Let Γ_n $(n < \omega)$ be a given sequence of members of $\mathcal{I}$ and fix $\beta < \theta$ such that $\Gamma_n \subseteq \beta$ for all n. For $n < \omega$ set $\Gamma_n^* = \{\xi \in \Gamma_n : \rho_2(\xi, \beta) \geq n\}$. Since Γ_n belongs to $\mathcal{I}$, Γ_n^* is a cofinite subset of Γ_n. Let $\Gamma_\infty = \bigcup_{n < \omega} \Gamma_n^*$. Then Γ_∞ is a member of $\mathcal{I}$ such that $\Gamma_n \setminus \Gamma_\infty$ is finite for all n. $\dashv$

13.6 Theorem. *The P-ideal dichotomy implies that a nontrivial square sequence can exist only on $\theta = \omega_1$.*

Proof. Applying the P-ideal dichotomy on $\mathcal{I}$ from 13.4 we get the two alternatives (see 4.45):

there is an uncountable $\Delta \subseteq \theta$ such that $[\Delta]^\omega \subseteq \mathcal{I}$, or $\qquad$ (I.178)

there is a decomposition $\theta = \bigcup_{n < \omega} \Sigma_n$ such that $\Sigma_n \perp \mathcal{I}$ for all n. $\qquad$ (I.179)

By Lemma 13.3, if (I.178) holds, then $\Delta \cap \alpha$ must be countable for all $\alpha < \theta$ and so the cofinality of θ must be equal to ω_1. Since we are working only with regular uncountable cardinals, we see that (I.178) gives us that $\theta = \omega_1$ must hold. Suppose now (I.179) holds and pick $k < \omega$ such that Σ_k is unbounded in θ. Since $\Sigma_k \perp \mathcal{I}$ we have that $(\rho_2)_\alpha$ is bounded on $\Sigma_k \cap \alpha$ for all $\alpha < \theta$. So there is an unbounded set $\Gamma \subseteq \theta$ and an integer n such that for each $\alpha \in \Gamma$ the restriction of $(\rho_2)_\alpha$ on $\Sigma_k \cap \alpha$ is bounded by n. By Theorem 11.2 we conclude that the square sequence C_α $(\alpha < \theta)$ we started with must be trivial. $\dashv$

13.7 Definition. By S_θ we denote the *sequential fan* with θ edges, i.e. the space on $(\theta \times \omega) \cup \{*\}$ with $*$ as the only nonisolated point, while a typical neighborhood of $*$ has the form $\mathcal{U}_f = \{(\alpha, n) : n \geq f(\alpha)\} \cup \{*\}$ where $f : \theta \longrightarrow \omega$.

13.8 Theorem. *If there is a nontrivial square sequence on θ then the square of the sequential fan S_θ has tightness [32] equal to θ.*

The proof will be given after a sequence of definitions and lemmas.

13.9 Definition. Given a square sequence C_α $(\alpha < \theta)$ and its number of steps function $\rho_2 : [\theta]^2 \longrightarrow \omega$ we define $d : [\theta]^2 \longrightarrow \omega$ by letting

$$d(\alpha, \beta) = \sup_{\xi \leq \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|.$$

13.10 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $d(\alpha, \beta) \geq \rho_2(\alpha, \beta)$,

---

[32] The *tightness* of a point x in a space X is equal to θ if θ is the minimal cardinal such that, if a set $W \subseteq X \setminus \{x\}$ accumulates to x, then there is a subset of W of size $\leq \theta$ that accumulates to x.

(b) $d(\alpha, \gamma) \le d(\alpha, \beta) + d(\beta, \gamma)$,

(c) $d(\alpha, \beta) \le d(\alpha, \gamma) + d(\beta, \gamma)$.

Proof. The conclusion (a) follows from the fact that we allow $\xi = \alpha$ in the definition of $d(\alpha, \beta)$. The conclusions (b) and (c) are consequences of the triangle inequalities of the ℓ_∞-norm and the fact that in both inequalities we have that the domain of functions on the left-hand side is included in the domain of functions on the right-hand side.   ⊣

13.11 Definition. For $\gamma \le \theta$, let

$$W_\gamma = \{((\alpha, d(\alpha, \beta)), (\beta, d(\alpha, \beta))) : \alpha < \beta < \gamma\}.$$

The following Lemma establishes that the tightness of the point $(*, *)$ of S_θ^2 is equal to θ, giving us the proof of Theorem 13.8.

13.12 Lemma. $(*, *) \in \bar{W}_\theta$ *but* $(*, *) \notin \bar{W}_\gamma$ *for all* $\gamma < \theta$.

Proof. To see that W_θ accumulates to $(*, *)$, let $\mathcal{U}_f^2$ be a given neighborhood of $(*, *)$. Fix an unbounded set $\Gamma \subseteq \theta$ on which f is constant. By Theorem 11.2 and Lemma 13.10(a) there exist $\alpha < \beta$ in Γ such that $d(\alpha, \beta) \ge f(\alpha) = f(\beta)$. Then $((\alpha, d(\alpha, \beta)), (\beta, d(\alpha, \beta)))$ belongs to the intersection $W_\theta \cap \mathcal{U}_f^2$. To see that for a given $\gamma < \theta$ the set W_γ does not accumulate to $(*, *)$, choose $g : \theta \longrightarrow \omega$ such that

$$g(\alpha) = 2d(\alpha, \gamma) + 1 \text{ for } \alpha < \gamma.$$

Suppose $W_\gamma \cap \mathcal{U}_g^2$ is nonempty and choose $((\alpha, d(\alpha, \beta)), (\beta, d(\alpha, \beta)))$ from this set. Then

$$d(\alpha, \beta) \ge 2d(\alpha, \gamma) + 1 \text{ and } d(\alpha, \beta) \ge 2d(\beta, \gamma) + 1. \qquad (\text{I}.180)$$

It follows that $d(\alpha, \beta) \ge d(\alpha, \gamma) + d(\beta, \gamma) + 1$ contradicting Lemma 13.10(c).
⊣

Since $\theta = \omega_1$ admits a nontrivial square sequence, Theorem 13.8 leads to the following result of Gruenhage and Tanaka [29].

13.13 Corollary. *The square of the sequential fan with ω_1 edges is not countably tight.*   ⊣

13.14 Question. What is the tightness of the square of the sequential fan with ω_2 edges?

13.15 Corollary. *If a regular uncountable cardinal θ is not weakly compact in the constructible subuniverse then the square of the sequential fan with θ edges has tightness equal to θ.*   ⊣

The way we have proved that S_θ^2 has tightness equal to θ is of independent interest. We have found a point $(*, *)$ of S_θ^2 and a set W_θ of isolated points which accumulates to $(*, *)$ but has no other accumulation points and moreover, no subset of W_θ of size $< \theta$ has accumulation points at all. This is interesting in view of the following result of Dow and Watson [17], where $\mathfrak{C}$ denotes the category of spaces that contain the converging sequence $\omega + 1$ and is closed under finite products, quotients and arbitrary topological sums.

13.16 Theorem. *If for every regular cardinal θ there is a space X in $\mathfrak{C}$ which has a set of isolated points of size θ witnessing tightness θ of its unique accumulation point in X, then every space belongs to $\mathfrak{C}$.* $\dashv$

13.17 Corollary. *If every regular uncountable cardinal supports a nontrivial square sequence then every topological space X can be obtained from the converging sequence after finitely many steps of taking (finite) products, quotients, and arbitrary topological sums.*

Proof. This follows from combining (the proof of) Theorem 13.8 and Theorem 13.16. $\dashv$

If κ is a strongly compact cardinal then the tightness of the product of every pair of spaces of tightness $< \kappa$ is $< \kappa$ and so $\mathfrak{C}$ contains no space of tightness κ or larger. It has been shown in [17] that if the Lebesque measure extends to a countably additive measure on all sets of reals then there even is a countable space which does not belong to $\mathfrak{C}$. On the other hand, by Theorem 12.5 and Corollary 13.17, if no regular cardinal above ω_1 is weakly compact in the constructible subuniverse, then every space can be generated by the converging sequence by taking finite products, quotients and arbitrary topological sums.

13.18 Remark. We have seen that the case $\theta = \omega_1$ is quite special when one considers the problem of existence of various nontrivial square sequences on θ. It should be noted that a similar result about the problem of the tightness of S_θ^2 is not available. In particular, it is not known whether the P-ideal dichotomy or a similar consistent hypothesis of set theory implies that the tightness of the square of, say, S_{ω_2} is smaller than ω_2. It is interesting that considerably more is known about the dual question, the question of initial compactness of the Tychonoff cube $\mathbb{N}^\theta$. For example, if one defines $B_{\alpha\beta} = \{f \in \mathbb{N}^\theta : f(\alpha), f(\beta) \leq d(\alpha, \beta)\}$ $(\alpha < \beta < \theta)$ one gets an open cover of $\mathbb{N}^\theta$ without a subcover of size $< \theta$. However, for small θ such as $\theta = \omega_2$ one is able to find such a cover of $\mathbb{N}^\theta$ without any additional set-theoretic assumption and in particular without the assumption that θ carries a nontrivial square sequence.

14. Special square sequences

The following well-known result of Jensen [35] supplements the corresponding result for weakly compact cardinals listed above as Theorem 12.5.

14.1 Theorem. *If a regular uncountable cardinal θ is not Mahlo in the constructible subuniverse then there is a special square sequence on θ which is moreover constructible.* $\dashv$

Today we know many more inner models with sufficient amount of fine structure necessary for building special square sequences. So the existence of special square sequences, especially at successors of strong-limit singular cardinals, is tied to the existence of some other large cardinal axioms. The reader is referred to the relevant chapters of this Handbook for the specific information. In this section we give the combinatorial analysis of walks along special square sequences and the corresponding distance functions. Let us start by restating some results of Section 12.

14.2 Theorem. *Suppose $\kappa < \theta$ are regular cardinals and that θ carries a special square sequence. Then there exist $C_{\alpha\nu}$ ($\alpha < \theta, \nu < \kappa$) such that:*

(1) $C_{\alpha\nu} \subseteq C_{\alpha\mu}$ for all α and $\nu < \mu$,

(2) $\alpha = \bigcup_{\nu<\kappa} C_\alpha$ for all limit α,

(3) $C_{\alpha\nu}$ ($\alpha < \theta$) is a nontrivial square sequence on θ for all $\nu < \kappa$.

Moreover, if θ is not a successor of a cardinal of cofinality κ then each of the square sequences can be chosen to be nonspecial. $\dashv$

14.3 Theorem. *Suppose $\kappa < \theta$ are regular cardinals and that θ carries a special square sequence. Then there exist $<_\nu$ ($\nu < \kappa$) such that:*

(i) $<_\nu$ is a closed tree ordering of θ for each $\nu < \kappa$,

(ii) $\in \restriction \theta = \bigcup_{\nu<\kappa} <_\nu$,

(iii) no tree $(\theta, <_\nu)$ has a chain of size θ.

$\dashv$

14.4 Lemma. *The following are equivalent when θ is a successor of some cardinal κ:*

(1) there is a special square sequence on θ,

(2) there is a square sequence C_α ($\alpha < \theta$) such that $\mathrm{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \theta$.

Proof. Let D_α $(\alpha < \kappa^+)$ be a given special square sequence. By Lemma 9.2 the corresponding tree $(\kappa^+, <^2)$ can be decomposed into κ antichains so let $f : \kappa^+ \longrightarrow \kappa$ be a fixed map such that $f^{-1}(\xi)$ is a $<^2$-antichain for all $\xi < \kappa$. Let $\alpha < \kappa^+$ be a given limit ordinal. If D_α has a maximal limit point $\bar{\alpha} < \alpha$, let $C_\alpha = D_\alpha \setminus \bar{\alpha}$. Suppose now that $\{\xi : \xi <^2 \alpha\}$ is unbounded in α and define a strictly increasing continuous sequence $c_\alpha(\xi)$ $(\xi < \nu(\alpha))$ of its elements as follows. Let $c_\alpha(0) = \min\{\xi : \xi <^2 \alpha\}$, $c_\alpha(\eta) = \sup_{\xi < \eta} c_\alpha(\xi)$ for η limit, and $c_\alpha(\xi + 1)$ is the minimal $<^2$-predecessor γ of α such that $\gamma > c_\alpha(\xi)$ and has the minimal f-image among all $<^2$-predecessors that are $> c_\alpha(\xi)$. The ordinal $\nu(\alpha)$ is defined as the place where the process stops, i.e. when $\alpha = \sup_{\xi < \nu(\alpha)} c_\alpha(\xi)$. Let $C_\alpha = \{c_\alpha(\xi) : \xi < \nu(\alpha)\}$. It is easily checked that this gives a square sequence C_α $(\alpha < \kappa^+)$ with the property that $\mathrm{tp}(C_\alpha) \le \kappa$ for all $\alpha < \kappa^+$. $\dashv$

Square sequences C_α $(\alpha < \kappa^+)$ that have the property $\mathrm{tp}(C_\alpha) \le \kappa$ for all $\alpha < \kappa^+$ are usually called $\square_\kappa$-sequences. So let C_α $(\alpha < \kappa^+)$ be a $\square_\kappa$-sequence fixed from now on. Let

$$\Lambda(\alpha, \beta) = \text{maximal limit point of } C_\beta \cap (\alpha + 1) \qquad (\mathrm{I}.181)$$

when such a limit point exists; otherwise $\Lambda(\alpha, \beta) = 0$. The purpose of this section is to analyze the following distance function:

$$\rho : [\kappa^+]^2 \longrightarrow \kappa \text{ defined recursively by} \qquad (\mathrm{I}.182)$$

$$
\begin{aligned}
\rho(\alpha, \beta) \;=\; & \max\{\mathrm{tp}(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)), \\
& \quad \rho(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\},
\end{aligned}
$$

where we stipulate that $\rho(\gamma, \gamma) = 0$ for all γ. Clearly $\rho(\alpha, \beta) \ge \rho_1(\alpha, \beta)$, so by Lemma 10.1 we have

14.5 Lemma. $|\{\xi \le \alpha : \rho(\xi, \alpha) \le \nu\}| \le |\nu| + \aleph_0$ *for all $\alpha < \kappa^+$ and $\nu < \kappa$.*
$\dashv$

The following two crucial subadditive properties of ρ have proofs that are almost identical to the proof of the corresponding properties of, say, the function ρ_ω discussed above in Section 12.

14.6 Lemma. *For all $\alpha \le \beta \le \gamma$,*

(a) $\rho(\alpha, \gamma) \le \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$,

(b) $\rho(\alpha, \beta) \le \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$.

$\dashv$

The following immediate fact will also be quite useful.

14.7 Lemma. *If α is a limit point of C_β, then $\rho(\xi, \alpha) = \rho(\xi, \beta)$ for all $\xi < \alpha$.*

$$\dashv$$

The following as well is an immediate consequence of the fact that the definition of ρ is closely tied to the notion of a minimal walk along the square sequence.

14.8 Lemma. $\rho(\alpha, \gamma) \geq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$ *whenever $\alpha \leq \beta \leq \gamma$ and β belongs to the trace of the walk from γ to α.* $\dashv$

Using Lemmas 14.7 and 14.8 one proves the following fact exactly as in the case of ρ_κ of Section 12 (the proof of Lemma 12.11).

14.9 Lemma. *If $0 < \beta \leq \gamma$ and if β is a limit ordinal, then there is $\bar{\beta} < \beta$ such that $\rho(\alpha, \gamma) \geq \rho(\alpha, \beta)$ for all α in the interval $[\bar{\beta}, \beta)$.* $\dashv$

The proof of the following fact is also completely analogous to the proof of the corresponding fact for the local version ρ_κ considered above in Section 12 (the proof of Lemma 12.12).

14.10 Lemma. $P_\nu(\gamma) = \{\beta < \gamma : \rho(\beta, \gamma) \leq \nu\}$ *is a closed subset of γ for all $\gamma < \kappa^+$ and $\nu < \kappa$.* $\dashv$

The discussion of $\rho : [\kappa^+]^2 \longrightarrow \kappa$ now splits naturally into two cases depending on whether κ is a regular or a singular cardinal (with the case $\mathrm{cf}(\kappa) = \omega$ of special importance). This is done in the following two sections.

15. Successors of regular cardinals

In this section κ is a fixed regular cardinal, C_α ($\alpha < \kappa^+$) a fixed $\square_\kappa$-sequence and $\rho : [\kappa^+]^2 \longrightarrow \kappa$ the corresponding ρ-function.

For $\nu < \kappa$ and $\alpha < \beta < \kappa^+$ set $\alpha <_\nu \beta$ if and only if $\rho(\alpha, \beta) \leq \nu$. $\hspace{2cm}$ (I.183)

The following is an immediate consequence of the analysis of ρ given in the previous section.

15.1 Lemma.

(a) $<_\nu$ is a closed tree ordering of κ^+ of height $\leq \kappa$ for all $\nu < \kappa$,

(b) $<_\nu \subseteq <_\mu$ whenever $\nu \leq \mu < \kappa$,

(c) $\in \restriction \kappa^+ = \bigcup_{\nu < \kappa} <_\nu$.

$$\dashv$$

15.2 Remark. Recall that by Lemma 14.3 for every regular cardinal $\lambda \leq \kappa$ the $\in$-relation of κ^+ can also be written as an increasing union of a λ-sequence $<_\nu$ $(\nu < \lambda)$ of closed tree orderings. When $\lambda < \kappa$, however, we can no longer insist that the trees $(\kappa^+, <_\nu)$ have heights $\leq \kappa$. Using an additional set-theoretic assumption (compatible with the existence of a $\square_\kappa$-sequence) one can strengthen (a) of Lemma 15.1 and have that the height of $(\kappa^+, <_\nu)$ is $< \kappa$ for all $\nu < \kappa$. Without additional set-theoretic assumptions, these trees, however, do have some properties of smallness not covered by Lemma 15.1.

15.3 Lemma. *If $\kappa > \omega$, then no tree $(\kappa^+, <_\nu)$ has a branch of size κ.*

Proof. Suppose towards a contradiction that some tree $(\kappa^+, <_\nu)$ does have a branch of size κ and let B be one such fixed branch (maximal chain). By Lemmas 14.5 and 14.10, if $\gamma = \sup(B)$ then B is a closed and unbounded subset of γ of order-type κ. Since κ is regular and uncountable, $C_\gamma \cap B$ is unbounded in C_γ, so in particular we can find $\alpha \in C_\gamma \cap B$ such that $\mathrm{tp}(C_\gamma \cap \alpha) > \nu$. Reading off the definition of $\rho(\alpha, \gamma)$ we conclude that $\rho(\alpha, \beta) = \mathrm{tp}(C_\gamma \cap \alpha) > \nu$. Similarly we can find a $\beta > \alpha$ belonging to the intersection of $\lim(C_\gamma)$ and B. Then $C_\beta = C_\gamma \cap \beta$ so $\alpha \in C_\beta$ and therefore $\rho(\alpha, \beta) = \mathrm{tp}(C_\beta \cap \alpha) > \nu$ contradicting the fact that $\alpha <_\nu \beta$. $\dashv$

15.4 Lemma. *If $\kappa > \omega$, then no tree $(\kappa^+, <_\nu)$ has a Souslin subtree of height κ.*

Proof. Forcing with a Souslin subtree of $(\kappa^+, <_\nu)$ of height κ would produce an ordinal γ of cofinality κ and a closed and unbounded subset B of C_γ forming a chain of the tree $(\kappa^+, <_\nu)$. It is well-known that in this case B would contain a ground model subset of size κ contradicting Lemma 15.3. $\dashv$

15.5 Lemma. *If $\kappa > \omega$ then for every $\nu < \kappa$ and every family A of κ pairwise disjoint finite subsets of κ^+ there exists $A_0 \subseteq A$ of size κ such that for all $a \neq b$ in A_0 and all $\alpha \in a$, $\beta \in b$ we have $\rho(\alpha, \beta) > \nu$.*

Proof. We may assume that for some n and all $a \in A$ we have $|a| = n$. Let $a(0), \ldots, a(n-1)$ enumerate a given element a of A increasingly. By Lemma 15.4, shrinking A, we may assume that $a(i)$ $(a \in A)$ is an antichain of $(\kappa^+, <_\nu)$ for all $i < n$. Going to a subfamily of A of equal size we may assume to have a well-ordering $<_w$ of A with the property that if $a <_w b$ then no node from a is above a node from b in the tree ordering $<_\nu$. Define $f : [A]^2 \longrightarrow \{0\} \cup (n \times n)$ by letting $f(a, b) = 0$ if $a \cup b$ is an $<_\nu$-antichain; otherwise, assuming $a <_w b$, let $f(a, b) = (i, j)$ where (i, j) is the minimal pair such that $a(i) <_\nu b(j)$. By the Dushnik-Miller partition theorem ([18]), either there exists $A_0 \subseteq A$ of size κ such that f is constantly equal to 0 on $[A_0]^2$ or there exist $(i, j) \in n \times n$ and an infinite $A_1 \subseteq A$ such that

f is constantly equal to (i, j) on $[A_1]^2$. The first alternative is what we want, so let us see that the second one is impossible. Otherwise, choose $a <_w b <_w c$ in A_1. Then $a(i)$ and $b(i)$ are both $<_\nu$-dominated by $c(j)$, so they must be $<_\nu$-comparable, contradicting our initial assumption about A. This completes the proof.   ⊣

The unboundedness property of Lemma 15.5 can be quite useful in designing forcing notions satisfying good chain conditions. Having such applications in mind, we shall now work on refining further this kind of unboundedness property of the ρ-function.

15.6 Lemma. *Suppose $\kappa > 0$, let $\gamma < \kappa^+$ and let $\{\alpha_\xi, \beta_\xi\}$ ($\xi < \kappa$) be a sequence of pairwise disjoint elements of $[\kappa^+]^{\leq 2}$. Then there is an unbounded set $\Gamma \subseteq \kappa$ such that $\rho\{\alpha_\xi, \beta_\eta\} \geq \min\{\rho\{\alpha_\xi, \gamma\}, \rho\{\beta_\eta, \gamma\}\}$ for all $\xi \neq \eta$ in Γ.*[33]

Proof. Clearly we may assume that the sequences α_ξ ($\xi < \kappa$) and β_ξ ($\xi < \kappa$) are strictly increasing. For definiteness we assume that $\alpha_\xi \leq \beta_\xi$ for all ξ. The case $\alpha_\xi > \beta_\xi$ for all ξ is considered similarly. Let

$$\alpha = \sup \alpha_\xi \text{ and } \beta = \sup \beta_\xi. \tag{I.184}$$

Then α and β are ordinals of cofinality κ, so the order-types of C_α and C_β are both equal to κ. It will be convenient to assume that the two sequences $\{\alpha_\xi\}$ and $\{\beta_\xi\}$ are actually indexed by C_α rather than κ. It is then clear we may assume that

$$\beta_\xi \geq \alpha_\xi \geq \xi \text{ for all } \xi \in C_\alpha. \tag{I.185}$$

Case 1: $\alpha = \beta$. By Lemma 14.9, for each limit point ν of C_α there is $f(\nu) < \nu$ in C_α such that

$$\rho(\xi, \alpha_\nu), \rho(\xi, \beta_\nu) \geq \rho(\xi, \nu) = \rho(\xi, \alpha) \text{ for all } \xi \in [f(\nu), \nu). \tag{I.186}$$

Fix a stationary $\Gamma \subseteq \lim(C_\alpha)$ such that f is constant on Γ. Then

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) \text{ for all } \xi < \eta \text{ in } \Gamma, \text{ and} \tag{I.187}$$

$$\rho(\beta_\eta, \alpha_\xi) \geq \rho(\beta_\eta, \alpha) \text{ for all } \eta < \xi \text{ in } \Gamma. \tag{I.188}$$

Subcase 1^a**:** $\gamma \geq \alpha$. Using the two subadditive properties of ρ in Lemma 14.6 one easily concludes that $\rho(\xi, \alpha) = \rho(\xi, \gamma)$ for any $\xi < \alpha$ such that $\rho(\xi, \alpha) > \rho(\alpha, \gamma)$. So, going to a tail of Γ we may assume that

$$\rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma) \text{ and } \rho(\beta_\xi, \alpha) = \rho(\beta_\xi, \gamma) \text{ for all } \xi \in \Gamma. \tag{I.189}$$

---

[33]Here, and everywhere else later in this Chapter, the convention is that, $\rho\{\alpha, \beta\}$ is meant to be equal to $\rho(\alpha, \beta)$ if $\alpha < \beta$, equal to $\rho(\beta, \alpha)$ if $\beta < \alpha$, and equal to 0 if $\alpha = \beta$.

Consider $\xi < \eta$ in Γ. Combining (I.187) and the first equality of (I.189) gives us

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma),$$

which is sufficient for the conclusion of Lemma 15.6. Suppose $\xi > \eta$ are chosen from Γ in this order. Combining (I.188) and the second equality of (I.189) gives us

$$\rho(\beta_\eta, \alpha_\xi) \geq \rho(\beta_\eta, \alpha) = \rho(\beta_\eta, \gamma),$$

which is also sufficient for the conclusion of Lemma 15.6.

Subcase 1^b: $\gamma < \alpha$. Using Lemma 14.5 and going to a tail of the set Γ we may assume that Γ lies above γ and that

$$\rho(\alpha_\xi, \alpha), \rho(\beta_\xi, \alpha) > \rho(\gamma, \alpha) \text{ for all } \xi \in \Gamma. \tag{I.190}$$

Applying the subadditive property 14.6(b) of ρ we get for all $\xi \in \Gamma$:

$$\rho(\gamma, \alpha_\xi) \leq \max\{\rho(\gamma, \alpha), \rho(\alpha_\xi, \alpha)\} = \rho(\alpha_\xi, \alpha), \text{ and} \tag{I.191}$$

$$\rho(\gamma, \beta_\xi) \leq \max\{\rho(\gamma, \alpha), \rho(\beta_\xi, \alpha)\} = \rho(\beta_\xi, \alpha). \tag{I.192}$$

If $\xi < \eta$ are chosen from Γ in this order then (I.187) and (I.191) give us

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) \geq \rho(\gamma, \alpha_\xi),$$

which is sufficient for the conclusion of Lemma 15.6. If $\xi > \eta$ are chosen from Γ in this order, then combining (I.188) and (I.192) give us

$$\rho(\beta_\eta, \alpha_\xi) \geq \rho(\beta_\eta, \alpha) \geq \rho(\gamma, \beta_\eta),$$

which is again sufficient for the conclusion of Lemma 15.6.

Case 2: $\alpha < \beta$. We may assume all $\beta_\xi > \alpha$ for all ξ and working as in the Case 1 we find a stationary set Γ of limit points of C_α such that

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) \text{ for all } \xi < \eta \text{ in } \Gamma. \tag{I.193}$$

By Lemma 14.5 for each $\eta \in \Gamma$ there is $g(\eta) \in C_\alpha$ such that $\rho(\xi, \alpha) > \rho(\alpha, \beta_\eta)$ for all ξ in the interval $[g(\eta), \alpha)$. Using the two subadditive properties of ρ from Lemma 14.6, one concludes that $\rho(\xi, \alpha) = \rho(\alpha, \beta_\eta)$ for all $\xi \in [g(\eta), \alpha)$. Intersecting Γ with the closed and unbounded subset of C_α of all ordinals that are closed under the mapping g we may assume

$$\rho(\alpha_\xi, \beta_\eta) = \rho(\alpha_\xi, \alpha) \text{ for all } \xi > \eta \text{ in } \Gamma. \tag{I.194}$$

Subcase 2^a: $\gamma < \alpha$. Applying Lemma 14.5 again and going to a tail of Γ we may assume that Γ lies above γ and

$$\rho(\alpha_\xi, \alpha) > \rho(\gamma, \alpha) \text{ for all } \xi \in \Gamma. \tag{I.195}$$

Applying the subadditive properties of ρ we get

$$\rho(\gamma, \alpha_\xi) \leq \max\{\rho(\gamma, \alpha), \rho(\alpha_\xi, \alpha)\} = \rho(\alpha_\xi, \alpha) \text{ for all } \xi \in \Gamma. \qquad (\text{I}.196)$$

If $\xi < \eta$ are chosen from Γ in this order, then combining the inequalities (I.193) and (I.196) we get

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) \geq \rho(\gamma, \alpha_\xi),$$

which is sufficient to give us the conclusion of Lemma 15.6. If $\xi > \eta$ are chosen from Γ in this order, then combining (I.194) and (I.196) we get

$$\rho(\alpha_\xi, \beta_\eta) = \rho(\alpha_\xi, \alpha) \geq \rho(\gamma, \alpha_\xi),$$

which is again sufficient for the conclusion of Lemma 15.6.

Subcase 2^b: $\gamma \geq \alpha$. Going to a tail of Γ we may assume that $\rho(\alpha_\xi, \alpha) > \rho(\alpha, \gamma)$, so as before the subadditive properties give us that

$$\rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma) \text{ for all } \xi \in \Gamma. \qquad (\text{I}.197)$$

If $\xi < \eta$ are chosen from Γ in this order then combining the inequality (I.193) and equality (I.197) we have

$$\rho(\alpha_\xi, \beta_\eta) \geq \rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma),$$

which is sufficient to give us the conclusion of Lemma 15.6. If $\xi > \eta$ are chosen from Γ in this order, then combining the equalities (I.194) and (I.197) we have

$$\rho(\alpha_\xi, \beta_\eta) = \rho(\alpha_\xi, \alpha) = \rho(\alpha_\xi, \gamma),$$

which is again sufficient to give us the conclusion of Lemma 15.6. This finishes the proof. $\dashv$

15.7 Lemma. *Suppose κ is λ-inaccessible* [34] *for some $\lambda < \kappa$ and that A is a family of size κ of subsets of κ^+, all of size $< \lambda$. Then for every ordinal $\nu < \kappa$ there is a subfamily B of A of size κ such that for all a and b in B:*

(a) *$\rho\{\alpha, \beta\} > \nu$ for all $\alpha \in a \setminus b$ and $\beta \in b \setminus a$.*

(b) *$\rho\{\alpha, \beta\} \geq \min\{\rho\{\alpha, \gamma\}, \rho\{\beta, \gamma\}\}$ for all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$.*

Proof. Consider first the case $\lambda = \omega$. Going to a subfamily, assume A forms a Δ-system with root r and that for some fixed integer $n \geq 1$ and all $a \in A$ we have $|a \setminus r| = n$. By Lemma 15.5, going to a subfamily we may assume that A satisfies (a). For $a \in A$ let $a(0), \ldots, a(n-1)$ be the increasing

_____
[34]I.e. $\nu^\tau < \kappa$ for all $\nu < \kappa$ and $\tau < \lambda$.

enumeration of $a \setminus r$. Going to a subfamily, we may assume that A can be enumerated as a_ξ ($\xi < \kappa$) in such a way that $a_\xi(i)$ ($\xi < \kappa$) is strictly increasing for all $i < n$. By $n^2 \cdot |r|$ successive applications of Lemma 15.6 we find a single unbounded set $\Gamma \subseteq \kappa$ such that for all $\gamma \in r$, $(i, j) \in n \times n$ and $\xi \neq \eta$ in Γ:

$$\rho\{a_\xi(i), a_\eta(j)\} \geq \min\{\rho\{a_\xi(i), \gamma\}, \rho\{a_\eta(j), \gamma\}\}. \qquad (I.198)$$

This gives us the conclusion (b) of the Lemma.

Let us now consider the general case. Using the λ-inaccessibility of κ we may assume that A forms a Δ-system with root r and that members of A have some fixed order-type, i.e. that for some fixed ordinal $\mu < \lambda$ we can enumerate each $a \setminus r$ for $a \in A$ in increasing order as $a(i)$ ($i < \mu$). Using λ-inaccessibility again and refining A we may assume that there is an enumeration a_ξ ($\xi < \kappa$) of A such that $a_\xi(i)$ ($\xi < \kappa$) is a strictly increasing sequence for all $i < \mu$. For $i < \mu$, set

$$\delta(i) = \sup_\xi a_\xi(i).$$

Then $\mathrm{cf}(\delta(i)) = \kappa$ and therefore tp $C_{\delta(i)} = \kappa$ for all $i < \mu$. Going to a smaller cardinal we may assume that λ is a regular cardinal $< \kappa$. Applying Lemma 14.9 simultaneously μ times gives us a regressive map $f : \kappa \longrightarrow \kappa$ such that for every $\eta < \kappa$ of cofinality λ, all $\xi \in [c_{\delta(i)}(f(\eta)), c_{\delta(i)}(\eta))$ and $i < \mu$:

$$\rho(\xi, a_\eta(i)) \geq \rho(\xi, c_{\delta(i)}(\eta)) = \rho(\xi, \delta(i)). \qquad (I.199)$$

Here $c_{\delta(i)}(\xi)$ ($\xi < \kappa$) refers to the increasing enumeration of the set $C_{\delta(i)}$ for each $i < \mu$. Pick a stationary set Γ of cofinality λ ordinals $< \kappa$ such that f is constant on Γ. It follows that for all $i, j < \mu$, if $\delta(i) = \delta(j)$ then

$$\rho\{a_\xi(i), a_\eta(j)\} \geq \rho(a_\xi(i), \delta(i)) \text{ for } \xi < \eta \text{ in } \Gamma, \qquad (I.200)$$

$$\rho\{a_\eta(j), a_\xi(i)\} \geq \rho(a_\eta(j), \delta(i)) \text{ for } \xi > \eta \text{ in } \Gamma. \qquad (I.201)$$

On the other hand, if $i, j < \mu$ such that $\delta(i) < \delta(j)$ we have

$$\rho\{a_\xi(i), a_\eta(j)\} \geq \rho(a_\xi(i), \delta(i)) \text{ for all } \xi < \eta \text{ in } \Gamma. \qquad (I.202)$$

Intersecting Γ with a closed and unbounded subset of κ we also assume that

$$a_\xi(i) \geq c_{\delta(i)}(\xi) \text{ for all } i < \mu \text{ and } \xi \in \Gamma. \qquad (I.203)$$

$$a_\xi(i) \leq a_\xi(j) \text{ for all } \xi \in \Gamma \text{ and } i, j < \mu \text{ such that } \delta(i) \leq \delta(j). \qquad (I.204)$$

Since $\rho(\alpha, \delta(i)) \geq \mathrm{tp}(C_{\delta(i)} \cap \alpha)$ for every $i < \mu$ and $\alpha < \delta(i)$, combining (I.200)-(I.204) we see that

$$\rho\{a_\xi(i), a_\eta(j)\} \geq \min\{\xi, \eta\} \text{ for all } i, j < \mu \text{ and } \xi \neq \eta \text{ in } \Gamma. \qquad (I.205)$$

So replacing Γ with $\Gamma \setminus (\nu + 1)$ gives us the conclusion (a) of Lemma 15.7.

Suppose now we are given two coordinates $i, j < \mu$ and an ordinal γ in the root r. Let

$$\alpha_\xi = a_\xi(i) \text{ and } \beta_\xi = a_\xi(j)$$

for $\xi \in \Gamma$. Note that (I.200) and (I.201) give us the conditions (I.187) and (I.188) of the Case 1 (i.e. $\delta(i) = \alpha = \beta = \delta(j)$) in the proof of Lemma 15.6 while (I.202) gives us the condition (I.193) of the Case 2 (i.e. $\delta(i) = \alpha < \beta = \delta(j)$) in the proof of Lemma 15.6. Observe that in this proof of Lemma 15.6, once the set Γ was obtained by a single application of the Pressing Down Lemma to finally give us (I.187) and (I.188) of Case 1 or (I.193) of Case 2, the rest of the refinements of Γ all lie in the restriction of the closed and unbounded filter to Γ. In other words, we can now proceed as in the case $\lambda = \omega$ and successively apply this refining procedure for each $\gamma \in r$ and $(i, j) \in \mu \times \mu$ obtaining a set $\Gamma_0 \subseteq \Gamma$ with the property that $\Gamma \setminus \Gamma_0$ is nonstationary and

$$\rho\{a_\xi(i), a_\eta(j)\} \geq \min\{\rho\{a_\xi(i), \gamma\}, \rho\{a_\eta(j), \gamma\}\} \text{ for all } \gamma \in r; \qquad \text{(I.206)}$$
$$i, j < \mu \text{ and } \xi \neq \eta \text{ in } \Gamma_0.$$

Since this is clearly equivalent to the conclusion (b) of Lemma 15.7, the proof is completed. $\dashv$

15.8 Definition. $D : [\kappa^+]^2 \longrightarrow [\kappa^+]^{<\kappa}$ is defined by

$$D(\alpha, \beta) = \{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}.$$

(Note that $D(\alpha, \beta) = \{\xi \leq \alpha : \rho(\xi, \beta) \leq \rho(\alpha, \beta)\}$, so we could take the formula

$$D\{\alpha, \beta\} = \{\xi \leq \min\{\alpha, \beta\} : \rho(\xi, \alpha) \leq \rho\{\alpha, \beta\}\}$$

as our definition of $D\{\alpha, \beta\}$ when there is no any implicit assumption about the ordering between α and β as there is whenever we write $D(\alpha, \beta)$.)

15.9 Lemma. *If κ is λ-inaccessible for some $\lambda < \kappa$, then for every family A of size κ of subsets of κ^+, all of size $< \lambda$, there exists $B \subseteq A$ of size κ such that for all a and b in B and all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$:*

(a) $\alpha, \beta > \gamma$ *implies* $D\{\alpha, \gamma\} \cup D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$,

(b) $\beta > \gamma$ *implies* $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$,

(c) $\alpha > \gamma$ *implies* $D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$,

(d) $\gamma > \alpha, \beta$ *implies* $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$ *or* $D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$.

Proof. Choose $B \subseteq A$ of size κ satisfying the conclusion (b) of Lemma 15.7. Pick $a \neq b$ in B and consider $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$. By the conclusion of 15.7(b), we have

$$\rho\{\alpha, \beta\} \geq \min\{\rho\{\alpha, \gamma\}, \rho\{\beta, \gamma\}\}. \qquad \text{(I.207)}$$

a. Suppose $\alpha, \beta > \gamma$. Note that in this case a single inequality $\rho(\gamma, \alpha) \leq \rho\{\alpha, \beta\}$ or $\rho(\gamma, \beta) \leq \rho\{\alpha, \beta\}$ given to us by (I.207) implies that we actually have both inequalities simultaneously holding. The subadditivity of ρ gives us $\rho(\xi, \alpha) \leq \rho\{\alpha, \beta\}$, or equivalently $\rho(\xi, \beta) \leq \rho\{\alpha, \beta\}$ for any $\xi \leq \gamma$ with $\rho(\xi, \gamma) \leq \rho(\gamma, \alpha)$ or $\rho(\xi, \gamma) \leq \rho(\gamma, \beta)$. This is exactly the conclusion of 15.9(a).

b. Suppose that $\beta > \gamma > \alpha$. Using the subadditivity of ρ we see that in both cases given to us by (I.207) we have that $\rho(\alpha, \gamma) \leq \rho\{\alpha, \beta\}$. So the inclusion $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$ follows immediately.

c. Suppose that $\alpha > \gamma > \beta$. The conclusion $D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$ follows from the previous case by symmetry.

d. Suppose that $\gamma > \alpha, \beta$. Then $\rho(\alpha, \gamma) \leq \rho\{\alpha, \beta\}$ gives $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$ while $\rho(\beta, \gamma) \leq \rho\{\alpha, \beta\}$ gives us $D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$.

This completes the proof. $\dashv$

15.10 Remark. Note that $\min\{x, y\} \in D\{x, y\}$ for every $\{x, y\} \in [\kappa^+]^2$, so the conclusion (a) of Lemma 15.9 in particular means that $\gamma < \min\{\alpha, \beta\}$ implies $\gamma \in D\{\alpha, \beta\}$. In applications, one usually needs this consequence of 15.9(a) rather than 15.9(a) itself.

15.11 Definition. The Δ-*function* of some family $\mathcal{F}$ of subsets of some ordinal κ (respectively, a family of functions with domain κ) is the function $\Delta : [\mathcal{F}]^2 \longrightarrow \kappa$ defined by

$$\Delta(f, g) = \min(f \triangle g),$$

(respectively, $\Delta(f, g) = \min\{\xi : f(\xi) \neq g(\xi)\}$).

Note the following property of Δ:

15.12 Lemma. $\Delta(f, g) \geq \min\{\Delta(f, h), \Delta(g, h)\}$ *for all* $\{f, g, h\} \in [\mathcal{F}]^3$.
$\dashv$

15.13 Remark. This property can be very useful when transferring objects that live on κ to objects on $\mathcal{F}$. This is especially interesting when $\mathcal{F}$ is of size larger than κ while all of its restrictions $\mathcal{F} \restriction \nu = \{f \cap \nu : f \in \mathcal{F}\}$ $(\nu < \kappa)$ have size $< \kappa$, i.e. when $\mathcal{F}$ is a *Kurepa family* (see for example [12]). We shall now see that it is possible to have a Kurepa family $\mathcal{F} = \{f_\alpha : \alpha < \kappa^+\}$ whose Δ-function is dominated by ρ, i.e. $\Delta(f_\alpha, f_\beta) \leq \rho(\alpha, \beta)$ for all $\alpha < \beta < \kappa^+$.

15.14 Theorem. *If* $\square_\kappa$ *holds and* κ *is* λ-*inaccessible then there is a* λ-*closed* κ-*cc forcing notion* $\mathcal{P}$ *that introduces a Kurepa family on* κ.

Proof. Going to a larger cardinal, we may assume that λ is regular. Put p in $\mathcal{P}$, if p is a one-to-one function from a subset of κ^+ of size $< \lambda$ into the family of all subsets of κ of size $< \lambda$ such that for all α and β in $\text{dom}(p)$:

$$p(\alpha) \cap p(\beta) \text{ is an initial part of } p(\alpha) \text{ and of } p(\beta), \tag{I.208}$$

$$\Delta(p(\alpha), p(\beta)) \leq \rho(\alpha, \beta) \text{ provided that } \alpha \neq \beta. \tag{I.209}$$

Let $p \leq q$ whenever $\operatorname{dom}(p) \supseteq \operatorname{dom}(q)$ and $p(\alpha) \supseteq q(\alpha)$ for all $\alpha \in \operatorname{dom}(q)$. Clearly $\mathcal{P}$ is a λ-closed forcing notion. To see that it satisfies the κ-chain condition, let A be a given subset of $\mathcal{P}$ of size κ. By the assumption that κ is λ-inaccessible, going to a subfamily of A of size κ, we may assume that A forms a Δ-system, or more precisely, that $\operatorname{dom}(p)$ $(p \in A)$ forms a Δ-system with root d, that $\bigcup \operatorname{rng}(p)$ $(p \in A)$ forms a Δ-system with root c and that any two members of A generate isomorphic structures via the natural isomorphism that fixes the roots. By Lemma 15.7 there exists $B \subseteq A$ of size κ such that for all $p, q \in B$, all $\alpha \in \operatorname{dom}(p) \setminus \operatorname{dom}(q)$, all $\beta \in \operatorname{dom}(q) \setminus \operatorname{dom}(p)$ and all $\gamma \in \operatorname{dom}(p) \cap \operatorname{dom}(q) (= d)$:

$$\rho\{\alpha, \beta\} > \max(c), \tag{I.210}$$

$$\rho\{\alpha, \beta\} \geq \min\{\rho\{\alpha, \gamma\}, \rho\{\beta, \gamma\}\}. \tag{I.211}$$

Suppose $p, q \in B$ and that $a = (\bigcup \operatorname{rng}(p)) \setminus c$ lies entirely below $b = (\bigcup \operatorname{rng}(q)) \setminus c$. Define $r : \operatorname{dom}(p) \cup \operatorname{dom}(q) \longrightarrow [\kappa]^{<\lambda}$ as follows. For $\gamma \in d$, let $r(\gamma) = p(\gamma) \cup q(\gamma)$. If $\beta \in \operatorname{dom}(q) \setminus \operatorname{dom}(p)$ and if there is $\gamma \in d$ such that $\Delta(q(\beta), q(\gamma)) \in b$, the γ must be unique. Put $r(\beta) = p(\gamma) \cup q(\beta)$ in this case. If such γ does not exist, put $r(\beta) = q(\beta) \cup \{\max(c) + 1\}$. For $\alpha \in \operatorname{dom}(p) \setminus \operatorname{dom}(q)$, put $r(\alpha) = p(\alpha)$. Note that this choice of r does not change the behaviour of Δ on pairs coming from $[\operatorname{dom}(p)]^2$ or $[\operatorname{dom}(q)]^2$, so in checking that r satisfies (I.208) it suffices to assume $\alpha \in \operatorname{dom}(p) \setminus \operatorname{dom}(q)$ and $\beta \in \operatorname{dom}(q) \setminus \operatorname{dom}(p)$. If $r(\beta)$ was defined as $p(\gamma) \cup q(\beta)$ for some $\gamma \in d$ we have that $\Delta(r(\alpha), r(\beta)) = \Delta(p(\alpha), p(\gamma))$ and that $\Delta(q(\beta), q(\gamma)) \in b$. It follows that

$$
\begin{aligned}
\Delta(r(\alpha), r(\beta)) &= \min\{\Delta(p(\alpha), p(\gamma)), \Delta(q(\beta), q(\gamma))\} \\
&\leq \min\{\rho\{\alpha, \gamma\}, \rho\{\beta, \gamma\}\}.
\end{aligned}
\tag{I.212}
$$

Applying (I.211) we get the desired inequality $\Delta(r(\alpha), r(\beta)) \leq \rho\{\alpha, \beta\}$. If $r(\beta)$ was chosen to be $q(\beta) \cup \{\max(c) + 1\}$. Then $\Delta(r(\alpha), r(\beta)) \leq \max(c) + 1$ and this is $\leq \rho\{\alpha, \beta\}$ by (I.210).

Let $\dot{G}$ be the generic filter of $\mathcal{P}$ and let $\dot{\mathcal{F}} = \{\dot{f}_\alpha : \alpha < \kappa^+\}$, where $\dot{f}_\alpha$ is the union of $p(\alpha)$ $(p \in \dot{G})$. A simple genericity argument shows that each $\dot{f}_\alpha$ intersects every interval if the form $[\nu, \nu + \lambda)$ $(\nu < \kappa)$. It follows (see (I.208)) that $\dot{\mathcal{F}} \restriction \nu$ has size at most λ for all $\nu < \kappa$. This finishes the proof. $\dashv$

15.15 Corollary. *If $\square_{\omega_1}$ holds, and so in particular if ω_2 is not a Mahlo cardinal in the constructible universe, then there is a property K poset, forcing the Kurepa hypothesis.* $\dashv$

15.16 Remark. This is a variation on a result of Jensen who proved that under $\square_{\omega_1}$ there is a ccc poset forcing the Kurepa hypothesis. It should be noted that Jensen also proved (see [36]) that in the Levy collapse of a Mahlo cardinal to ω_2 there is no ccc poset forcing the Kurepa hypothesis. It should also be noted that Veličković [91] was the first to use ρ to reprove Jensen's result. He has also shown (see [91]) that ρ easily yields an example of a function with the 'Δ-property' in the sense of Baumgartner and Shelah [5]. Recall that a function with the Δ-property in the sense of [5] is a function that satisfies some of the properties of the function D of Lemma 15.9 and a function that forms a ground for the well-known forcing construction of a locally compact scattered topology on ω_2 all of whose Cantor-Bendixson ranks are countable. We now use ρ to improve the chain-condition of the Baumgartner-Shelah forcing.

15.17 Theorem. *If $\square_{\omega_1}$ holds then there is a property K forcing notion that introduces a locally compact scattered topology on ω_2 all of whose Cantor-Bendixson ranks are countable.*

Proof. Let $\mathcal{P}$ be the set of all $p = \langle D_p, \leq_p, M_p \rangle$ where D_p is a finite subset of ω_2, where $\leq_p$ is a partial ordering of D_p compatible with the well-ordering and $M_p : [D_p]^2 \longrightarrow [\omega_2]^{<\omega}$ has the following properties:

$$M_p\{\alpha, \beta\} \subseteq D\{\alpha, \beta\} \cap D_p, \tag{I.213}$$

$$M_p\{\alpha, \beta\} = \{\alpha\} \text{ if } \alpha \leq_p \beta \text{ and } M_p\{\alpha, \beta\} = \{\beta\} \text{ if } \beta \leq_p \alpha, \tag{I.214}$$

$$\gamma \leq_p \alpha, \beta \text{ for all } \gamma \in M_p\{\alpha, \beta\}, \tag{I.215}$$

$$\text{for every } \delta \leq_p \alpha, \beta \text{ there is } \gamma \in M_p\{\alpha, \beta\} \text{ such that } \delta \leq_p \gamma. \tag{I.216}$$

We let $p \leq q$ if and only if $D_p \supseteq D_q$, $\leq_p \restriction D_q = \leq_q$ and $M_p \restriction [D_q]^2 = M_q$. To verify that $\mathcal{P}$ satisfies Knaster's chain condition, let A be a given uncountable subset of $\mathcal{P}$. Shrinking A we may assume that A forms a Δ-system with root R and that the conditions of A generate isomorphic structures over R. By Lemma 15.9 there is an uncountable subset B of A such that for all p and q in B and all $\alpha \in D_p \setminus D_q$, $\beta \in D_q \setminus D_p$ and $\gamma \in R = D_p \cap D_q$:

$$\alpha, \beta > \gamma \quad \text{implies} \quad D\{\alpha, \gamma\} \cup D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}, \tag{I.217}$$

$$\beta > \gamma \quad \text{implies} \quad D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}, \tag{I.218}$$

$$\alpha > \gamma \quad \text{implies} \quad D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}. \tag{I.219}$$

Consider two conditions p and q from B. Let $r = \langle D_r, \leq_r, M_r \rangle$ be defined as follows: $D_r = D_p \cup D_q$ and $\alpha \leq_r \beta$ if and only if $\alpha \leq_p \beta$, or $\alpha \leq_q \beta$, or there is $\gamma \in R$ such that $\alpha \leq_p \gamma \leq_q \beta$ or $\alpha \leq_q \gamma \leq_p \beta$. It is easily checked that $\leq_r$ is a partial ordering on D_r compatible with the well-ordering such

that $\leq_r \restriction D_p = \leq_p$ and $\leq_r \restriction D_q = \leq_q$. Define $M_r : [D_r]^2 \longrightarrow [\omega_2]^{<\omega}$ by letting $M_r \restriction [D_p]^2 = M_p$, $M_r \restriction [D_q]^2 = M_q$, $M_r\{\alpha, \beta\} = \{\alpha\}$ if $\alpha \leq_r \beta$, $M_r\{\alpha, \beta\} = \{\beta\}$ if $\beta \leq_r \alpha$ and if $\alpha \in D_p \setminus R$ and $\beta \in D_q \setminus R$ are $\leq_r$-incomparable, set

$$M_r\{\alpha, \beta\} = \{\gamma \in D\{\alpha, \beta\} \cap D_r : \gamma \leq_r \alpha, \beta\}. \qquad (\text{I}.220)$$

Only nontrivial to check is that a so defined r satisfies (I.216). So let $\delta \leq_r \alpha, \beta$ be given. If $\alpha, \beta \in D_p \setminus D_q$ and $\delta \in D_q \setminus D_p$ then there exist $\gamma_\alpha, \gamma_\beta \in R$ such that $\delta \leq_q \gamma_\alpha \leq_p \alpha$ and $\delta \leq_q \gamma_\beta \leq_p \beta$. By (I.216) for q there exists $\gamma \in M_q\{\gamma_\alpha, \gamma_\beta\}$ such that $\delta \leq_q \gamma$. Since $M_q\{\gamma_\alpha, \gamma_\beta\} \subseteq R$ we conclude that $\gamma \in R$. Thus $\gamma \leq_p \alpha, \beta$, so by (I.216) there is $\bar{\gamma} \in M_p\{\alpha, \beta\}$ such that $\gamma \leq_p \bar{\gamma}$ and therefore $\delta \leq_r \bar{\gamma}$. The case $\alpha \in R$, $\beta \in D_p \setminus D$ and $\delta \in D_q \setminus D_p$ is quite similar.

Consider now the case when $\alpha \in D_p \setminus D_q$, $\beta \in D_q \setminus D_p$ and they are $\leq_r$-incomparable, i.e. $M_r\{\alpha, \beta\}$ is given by (I.220). For symmetry we may consider only the subcase $\delta \leq_r \alpha, \beta$ with $\delta \in D_p$. Then $\delta \leq_p \alpha$ and $\delta \leq_p \gamma \leq_q \beta$ for some $\gamma \in R$. By (I.216) for p there is $\bar{\gamma} \in M_p\{\alpha, \gamma\}$ such that $\delta \leq_p \bar{\gamma}$. It remains to show that $\bar{\gamma}$ was put in $M_r\{\alpha, \beta\}$ in (I.220), i.e. that $\bar{\gamma}$ belongs to $D\{\alpha, \beta\}$. Note that since $\leq_r$ agrees with the well-ordering, we have that $\beta > \gamma$, so by (I.218) we have that $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$. By (I.213) for p we have that $\bar{\gamma} \in M_p\{\alpha, \gamma\} \subseteq D\{\alpha, \gamma\}$ so we can conclude that $\bar{\gamma} \in D\{\alpha, \beta\}$. The rest of the cases are symmetric to the ones above.

Let $\dot{G}$ be a generic filter of $\mathcal{P}$ and let $\leq_{\dot{G}}$ be the union of $\leq_p$ $(p \in \dot{G})$. For $\alpha < \omega_2$ set $\dot{B}_\alpha = \{\xi : \xi \leq_{\dot{G}} \alpha\}$. Let τ be the topology on ω_2 generated by $\dot{B}_\alpha$ $(\alpha < \omega_2)$ as a clopen subbasis. It is easily checked that τ is a Hausdorff locally compact scattered topology on ω_2. A simple density argument and induction on α shows that the interval $[\omega\alpha, \omega\alpha + \omega)$ is the αth Cantor-Bendixson rank of the space (ω_2, τ). $\qquad\dashv$

15.18 Definition. A function $f : [\omega_2]^2 \longrightarrow [\omega_2]^{\leq\omega}$ has *property* Δ if for every uncountable set A of finite subsets of ω_2 there exist a and b in A such that for all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$:

$$
\begin{array}{llll}
(a) & \alpha, \beta > \gamma & \text{implies} & \gamma \in f\{\alpha, \beta\}, \\
(b) & \beta > \gamma & \text{implies} & f\{\alpha, \gamma\} \subseteq f\{\alpha, \beta\}, \\
(c) & \alpha > \gamma & \text{implies} & f\{\beta, \gamma\} \subseteq f\{\alpha, \beta\}.
\end{array}
$$

15.19 Remark. This definition is due to Baumgartner and Shelah [5] who have used it in their well-known forcing construction. It should be noted that they were also able to force a function with the property Δ using a σ-closed ω_2-cc poset. As shown above (a fact first checked in [5, p.129]), the function

$$D\{\alpha, \beta\} = \{\xi \leq \min\{\alpha, \beta\} : \rho(\xi, \alpha) \leq \rho\{\alpha, \beta\}\}$$

has property Δ. However, Lemma 15.9 shows that the function D has many other properties that are of independent interest and that are likely to be needed in other forcing constructions of this sort. The reader is referred to papers of Koszmider [41] and Rabus [58] for further work in this area.

16. Successors of singular cardinals

In the previous section we have witnessed the fact that the function $\rho : [\kappa^+]^2 \longrightarrow \kappa$ defined from a $\square_\kappa$-sequence C_α $(\alpha < \kappa^+)$ can be a quite useful tool in stepping-up objects from κ to κ^+. In this section we analyse the stepping-up power of ρ under the assumption that κ is a singular cardinal of cofinality ω. So let κ_n $(n < \omega)$ be a strictly increasing sequence of regular cardinals converging to κ fixed from now on. This immediately gives rise to a rather striking tree decomposition $<_n$ $(n < \omega)$ of the $\in$-relation on κ^+:

$$\alpha <_n \beta \text{ if and only if } \rho(\alpha, \beta) \leq \kappa_n. \qquad (\text{I}.221)$$

16.1 Lemma.

(1) $\in \restriction \kappa^+ = \bigcup_{n<\omega} <_n$,

(2) $<_n \subseteq <_{n+1}$,

(3) $(\kappa^+, <_n)$ *is a tree of height* $\leq \kappa_n^+$.

$\dashv$

16.2 Definition. For $\alpha < \kappa^+$ and $n < \omega$ set

$$F_n(\alpha) = \{\xi \leq \alpha : \rho(\xi, \alpha) \leq \kappa_n\},$$

$$f_\alpha(n) = \text{tp}(F_n(\alpha)).$$

Let $L = \{f_\alpha : \alpha < \kappa^+\}$, considered as a linearly ordered set with the lexicographical ordering. Since L is a subset of $^\omega\kappa$, it has an order-dense subset of size κ, so in particular it contains no well-ordered subset of size κ^+. The following result shows that, however, every subset of L of smaller size is the union of countably many well-ordered subsets.

16.3 Lemma. *For each $\beta < \kappa^+$, $L_\beta = \{f_\alpha : \alpha < \beta\}$ can be decomposed into countably many well-ordered subsets.*

Proof. Let $L_{\beta n} = \{f_\alpha : \alpha \in F_n(\beta)\}$ for $n < \omega$. Note that the projection $f \longmapsto f \restriction (n+1)$ is one-to-one on $L_{\beta n}$ so each $L_{\beta n}$ is lexicographically well-ordered. $\dashv$

16.4 Remark. Note that $\mathcal{K} = \{\{(n, f_\alpha(n)) : n < \omega\} : \alpha < \kappa^+\}$ is a family of countable subsets of $\omega \times \kappa$ which has the property that $\mathcal{K} \upharpoonright X = \{K \cap X : K \in \mathcal{K}\}$ has size $\leq |X| + \aleph_0$ for every $X \subseteq \omega \times \kappa$ of size $< \kappa$. We now give an application of the existence of such a family of countable sets, due to Fremlin [25]. This requires a few standard notions from measure theory and real analysis.

16.5 Definition. A *Radon measure space* is a Hausdorff topological space X with a σ-finite measure μ defined on a field of subsets of X containing open sets with the property that the measure of every set is equal to the supremum of measures of its compact subsets. When $\mu(X) = 1$ then (X, μ) is called *Radon probability space.* Finally recall that the *Maharam type* of (X, μ) is the least cardinality of a subset of the measure algebra of μ which completely generates the algebra.

16.6 Definition. A directed set D is *Tukey reducible* to a directed set E, written as $D \leq E$, if there is $f : D \longrightarrow E$ which maps unbounded sets to unbounded sets. In case D and E are ideals of subsets, this is equivalent to saying that there is $q : E \longrightarrow D$ which is monotone and has cofinal range. We say that D and E are *Tukey equivalent,* $D \equiv E$, whenever $D \leq E$ and $E \leq D$.

16.7 Theorem. *If there is a locally countable subset of $[\kappa]^\omega$ of size $\mathrm{cf}[\kappa]^\omega$ then the null ideal of every atomless Radon probability space of Maharam type κ is Tukey equivalent to the product $\mathcal{N} \times [\kappa]^\omega$ where $\mathcal{N}$ is the ideal of measure zero subsets of the unit interval.*

Proof. We shall give an argument for the case $X = \{0, 1\}^\kappa$ and μ the Haar measure of X, referring the reader to the general result from [25] that the null ideal of any atomless probability space of Maharam type κ is Tukey equivalent to the null ideal $\mathcal{N}_\kappa$ of $\{0, 1\}^\kappa$. The only nontrivial direction is to produce a Tukey map from $\mathcal{N}_\kappa$ into $\mathcal{N}_\omega \times [\kappa]^\omega$. Let $\mathcal{K} \subseteq [\kappa]^\omega$ of size $\mathrm{cf}[\kappa]^\omega$ with the property that $\mathcal{K} \upharpoonright \Gamma = \{a \cap \Gamma : a \in \mathcal{K}\}$ is countable for all countable $\Gamma \subseteq \kappa$. Let $h : \mathcal{K} \longrightarrow [\kappa]^\omega$ be a mapping whose range is cofinal in $[\kappa]^\omega$ when considered as a directed set ordered by the inclusion. Let $\mathcal{C} = \{a \cup h(a) : a \in \mathcal{K}\}$. Then $\mathcal{C}$ is a cofinal subset of $[\kappa]^\omega$ with the property that $\mathcal{C}(b) = \{a \in \mathcal{C} : a \subseteq b\}$ is countable for every $b \in [\kappa]^\omega$. For $c \in \mathcal{C}$ fix a bijection $e_c : c \longrightarrow \omega$. This gives us a way to define a function π_c mapping cylinders of $\{0, 1\}^\kappa$ with supports included in c into 2^ω in the natural way. Let $\mathcal{N}_\kappa^{cl}$ be the family of all null subsets of $\{0, 1\}^\kappa$ that are countably supported cylinders of $\{0, 1\}^\kappa$. Clearly, $\mathcal{N}_\kappa^{cl}$ is cofinal in $\mathcal{N}_\kappa$, so it suffices to produce a Tukey map $f : \mathcal{N}_\kappa^{cl} \longrightarrow \mathcal{N}_\omega \times [\kappa]^\omega$. Given $N \in \mathcal{N}_\kappa^{cl}$, find $c(N) \in \mathcal{C}$ such that $\mathrm{supp}(N) \subseteq c(N)$ and put

$$f(N) = (\pi_{c(N)}{}''N, c(N)).$$

It is straightforward to check that f maps unbounded subsets of $\mathcal{N}_\kappa^{cl}$ to unbounded subsets of the product $\mathcal{N}_\omega \times [\kappa]^\omega$. ⊣

16.8 Remark. Note that the hypothesis of Fremlin's theorem is true when $[\kappa]^\omega$ has cofinality κ and so it is always satisfied for $\kappa < \aleph_\omega$. By remark 16.4 the hypothesis of Theorem 16.7 is satisfied for all κ if one assumes $\square_\kappa$ and $\mathrm{cf}[\kappa]^\omega = \kappa^+$ for every κ of uncountable cofinality. Note that the transfer from a locally countable $\mathcal{K}$ to a cofinal $\mathcal{C} = \{a \cup h(a) : a \in \mathcal{K}\}$ in the above proof does not preserve local countability. To obtain a locally countable cofinal $\mathcal{C} \subseteq [\kappa]^\omega$ one needs to work a bit harder along the same lines.

16.9 Definition. If a family $\mathcal{K} \subseteq [S]^\omega$ is at the same time locally countable and cofinal in $[S]^\omega$ then we call it a *cofinal Kurepa family* (cofinal K-family in short). Two cofinal K-families $\mathcal{H}$ and $\mathcal{K}$ are *compatible* if $H \cap K \in \mathcal{H} \cap \mathcal{K}$ for all $H \in \mathcal{H}$ and $K \in \mathcal{K}$. We say that $\mathcal{K}$ *extends* $\mathcal{H}$ if they are compatible and if $\mathcal{H} \subseteq \mathcal{K}$.

16.10 Remark. Note that the size of any cofinal K-family $\mathcal{K}$ on a set S is equal to the cofinality of $[S]^\omega$. Note also that for every $X \subseteq S$ there is $Y \supseteq X$ of size $\mathrm{cf}[X]^\omega$ such that $K \cap Y \in \mathcal{K}$ for all $K \in \mathcal{K}$.

16.11 Definition. Define $\mathrm{CK}(\theta)$ to be the statement that every sequence $\mathcal{K}_n$ ($n < \omega$) of comparable cofinal K-families with domains included in θ which are closed under $\cup$, $\cap$ and $\setminus$ can be extended to a single cofinal K-family on θ, which is also closed under these three operations.

16.12 Lemma. $\mathrm{CK}(\omega_1)$ *is true and if* $\mathrm{CK}(\theta)$ *is true for some* θ *such that* $\mathrm{cf}[\theta]^\omega = \theta$ *then* $\mathrm{CK}(\theta^+)$ *is also true.*

Proof. The easy proof of $\mathrm{CK}(\omega_1)$ is left to the reader.

Suppose $\mathrm{CK}(\theta)$ and let $\mathcal{K}_n$ ($n < \omega$) be a given sequence of compatible cofinal K-families as in the hypothesis of $\mathrm{CK}(\theta^+)$. By Remark 16.10 there is a strictly increasing sequence δ_ξ ($\xi < \theta^+$) of ordinals $< \theta$ such that $\mathcal{K}_n \restriction \delta_\xi \subseteq \mathcal{K}_n$ for all $\xi < \theta^+$ and $n < \omega$. Recursively on $\xi < \theta^+$ we construct a chain $\mathcal{H}_\xi$ ($\xi < \theta^+$) of cofinal K-families as follows. If $\xi = 0$ or $\xi = \eta + 1$ for some η, using $\mathrm{CK}(\delta_\xi)$ we can find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\mathcal{H}_{\xi-1}$ and $\mathcal{K}_n \restriction \delta_\xi$ ($n < \omega$). If ξ has uncountable cofinality then the union of $\bar{\mathcal{H}}_\xi = \bigcup_{\eta < \xi} \mathcal{H}_\eta$ is a cofinal K-family with domain included in δ_ξ, so using $\mathrm{CK}(\delta_\xi)$ we can find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\bar{\mathcal{H}}_\xi$ and $\mathcal{K}_n \restriction \delta_\xi$ ($n < \omega$). If ξ has countable cofinality, pick a sequence $\{\xi_n\}$ converging to ξ and use $\mathrm{CK}(\delta_\xi)$ to find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\mathcal{H}_{\xi_n}$ ($n < \omega$) and $\mathcal{K}_n$ ($n < \omega$). When the recursion is done, set $\mathcal{H} = \bigcup_{\xi < \theta^+} \mathcal{H}_\xi$. Then $\mathcal{H}$ is a cofinal K-family on θ^+ extending $\mathcal{K}_n$ ($n < \omega$). ⊣

16.13 Corollary. *For each* $n < \omega$ *there is a cofinal Kurepa family on* ω_n.
⊣

16.14 Definition. Let κ be a cardinal of cofinality ω. A *Jensen matrix* on κ^+ is a matrix $\mathrm{J}_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ of subsets of κ^+ with the following properties, where κ_n $(n < \omega)$ is some fixed increasing sequence of cardinals converging to κ:

(1) $|\mathrm{J}_{\alpha n}| \leq \kappa_n$ for all $\alpha < \kappa^+$ and $n < \omega$,

(2) for all $\alpha < \beta$ and $n < \omega$ there is $m < \omega$ such that $\mathrm{J}_{\alpha n} \subseteq \mathrm{J}_{\beta m}$,

(3) $\bigcup_{n<\omega}[\mathrm{J}_{\beta n}]^\omega = \bigcup_{\alpha<\beta}\bigcup_{n<\omega}[\mathrm{J}_{\alpha n}]^\omega$ whenever $\mathrm{cf}(\beta) > \omega$,

(4) $[\kappa^+]^\omega = \bigcup_{\alpha<\kappa^+}\bigcup_{n<\omega}[\mathrm{J}_{\alpha n}]^\omega$.

16.15 Remark. The notion of a Jensen matrix is the combinatorial essence behind Silver's proof of Jensen's model-theoretic two-cardinal transfer theorem in the constructible universe (see [35, appendix]), so the matrix could equally well be called 'Silver matrix'. It has been implicitly or explicitly used on several places in the literature. The reader is referred to the paper of Foreman and Magidor [24] which gives a quite complete discussion of this notion and its occurrence in the literature.

16.16 Lemma. *Suppose some cardinal κ of countable cofinality carries a Jensen matrix $\mathrm{J}_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ relative to some sequence of cardinals κ_n $(n < \omega)$ that converge to κ. If $\mathrm{CK}(\kappa_n)$ holds for all $n < \omega$ then $\mathrm{CK}(\kappa^+)$ is also true.*

Proof. Let $\mathcal{K}_n$ $(n < \omega)$ be a given sequence of compatible cofinal K-families with domains included in κ^+. Given $\mathrm{J}_{\alpha n}$, there is a natural continuous chain $\mathrm{J}_{\alpha n}^\xi$ $(\xi < \omega_1)$ of subsets of κ^+ of size $\leq \kappa_n$ such that $\mathrm{J}_{\alpha n}^0 = \mathrm{J}_{\alpha n}$ and $\mathrm{J}_{\alpha n}^{\xi+1}$ equal to the union of all $K \in \bigcup_{n<\omega}\mathcal{K}_n$ which intersect $\mathrm{J}_{\alpha n}^\xi$. Let $\mathrm{J}_{\alpha n}^* = \bigcup_{\xi<\omega_1}\mathrm{J}_{\alpha n}^\xi$. It is easily seen that $\mathrm{J}_{\alpha n}^*$ $(\alpha < \kappa^+, n < \omega)$ is also a Jensen matrix. By recursion on α and n we define a sequence $\mathcal{H}_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ of compatible cofinal K-families as follows. If $\alpha = \beta + 1$ or $\alpha = 0$ and $n < \omega$ using $\mathrm{CK}(\kappa_n)$ we can find a cofinal K-family $\mathcal{H}_{\alpha n}$ with domain $\mathrm{J}_{\alpha n}^*$ compatible with $\mathcal{H}_{\alpha m}$ $(m < n)$, $\mathcal{H}_{(\alpha-1)m}$ $(m < \omega)$ and $\mathcal{K}_m \restriction \mathrm{J}_{\alpha n}^*$ $(m < \omega)$. If $\mathrm{cf}(\alpha) = \omega$ let α_n $(n < \omega)$ be an increasing sequence of ordinals converging to α. Using $\mathrm{CK}(\kappa_n)$ we can find a cofinal K-family $\mathcal{H}_{\alpha n}$ which extends $\mathcal{H}_{\alpha m}$ $(m < n)$, $\mathcal{K}_m \restriction \mathrm{J}_{\alpha n}^*$ $(m < \omega)$ and each of the families $\mathcal{H}_{\alpha_i k}$ $(i < \omega, k < \omega$ and $\mathrm{J}_{\alpha_i k}^* \subseteq \mathrm{J}_{\alpha n}^*)$. Finally suppose that $\mathrm{cf}(\alpha) > \omega$. For $n < \omega$, set

$$\mathcal{H}_{\alpha n} = [\mathrm{J}_{\alpha n}^*]^\omega \cap \left(\bigcup_{\xi<\alpha}\bigcup_{m<\omega}\mathcal{H}_{\xi m}\right).$$

Using the properties of the Jensen matrix (especially (3)) as well as the compatibility of $\mathcal{H}_{\xi m}$ $(\xi < \alpha, m < \omega)$ one easily checks that $\mathcal{H}_{\alpha n}$ is a cofinal K-family with domain $\mathrm{J}_{\alpha n}^*$ which extends each member of $\mathcal{H}_{\alpha m}$ $(m < n)$

and $\mathcal{K}_m \restriction \mathrm{J}^*_{\alpha n}$ ($m < \omega$) and which is compatible with all of the previously constructed families $\mathcal{H}_{\xi m}$ ($\xi < \alpha, m < \omega$). When the recursion is done we set

$$\mathcal{H} = \bigcup_{\alpha < \kappa^+} \bigcup_{n < \omega} \mathcal{H}_{\alpha n}.$$

Using the property (4) of $\mathrm{J}^*_{\alpha n}$ ($\alpha < \kappa^+, n < \omega$), it follows easily that $\mathcal{H}$ is a cofinal K-family on κ^+ extending $\mathcal{K}_n$ ($n < \omega$). $\dashv$

16.17 Theorem. *If a Jensen matrix exists on any successor of a cardinal of cofinality ω, then a cofinal Kurepa family exists on any domain.* $\dashv$

The ρ-function $\rho : [\kappa^+]^2 \longrightarrow \kappa$ associated with a $\square_\kappa$-sequence C_α ($\alpha < \kappa^+$) for some singular cardinal κ of cofinality ω leads to the matrix

$$F_n(\alpha) = \{\xi < \alpha : \rho(\xi, \alpha) \leq n\}(\alpha < \kappa^+, n < \omega) \qquad (\text{I.222})$$

which has the properties (1)-(3) of 16.14 as well as some other properties not captured by the definition of a Jensen matrix. If one additionally has a sequence a_α ($\alpha < \kappa^+$) of countable subsets of κ^+ that is cofinal in $[\kappa^+]^\omega$ one can extend the matrix (I.222) as follows:

$$M_{\beta n} = \bigcup_{\alpha <_n \beta} (a_\alpha \cup \{\alpha\}) \ (\beta < \kappa^+, n < \omega). \qquad (\text{I.223})$$

(Recall that $<_n$ is the tree ordering on κ^+ defined by the formula $\alpha <_n \beta$ iff $\rho(\alpha, \beta) \leq \kappa_n$ where κ_n is a fixed increasing sequence of cardinals converging to κ.) The matrix $M_{\beta n}$ ($\beta < \kappa^+, n < \omega$) has properties not captured by Definition 16.14 that are of independent interest.

16.18 Lemma.

(1) $\alpha <_n \beta$ implies $M_{\alpha n} \subseteq M_{\beta n}$,

(2) $M_{\alpha m} \subseteq M_{\alpha n}$ whenever $m < n$,

(3) if $\beta = \sup\{\alpha : \alpha <_n \beta\}$ then $M_{\beta n} = \bigcup_{\alpha <_n \beta} M_{\alpha n}$,

(4) every countable subset of κ^+ is covered by some $M_{\beta n}$.

(5) $\mathcal{M} = \{M_{\beta n} : \beta < \kappa^+, n < \omega\}$ is a locally countable family if we have started with a locally countable $\mathcal{K} = \{a_\alpha : \alpha < \kappa^+\}$.

$\dashv$

16.19 Remark. One can think of the matrix $\mathcal{M} = \{M_{\beta n} : \beta < \kappa^+, n < \omega\}$ as a version of a 'morass' for the singular cardinal κ (see [92]). It would be interesting to see how far one can go in this analogy. We give a few applications just to illustrate the usefulness of the families we have constructed so far.

16.20 Definition. A *Bernstein decomposition* of a topological space X is a function $f : X \longrightarrow 2^{\mathbb{N}}$ with the property that f takes all the values from $2^{\mathbb{N}}$ on any subset of X homeomorphic to the Cantor set.

16.21 Remark. The classical construction of Bernstein [8] can be interpreted by saying that every space of size at most continuum admits a Bernstein decomposition. For larger spaces one must assume Hausdorff's separation axiom, a result of Nešetril and Rődl (see [55]). In this context Malykhin was able to extend Bernstein's result to all spaces of size $< \mathfrak{c}^{+\omega}$ (see [50]). To extend this to all Hausdorff spaces, some use of square sequences seems natural. In fact, the first Bernstein decompositions of an arbitrary Hausdorff space have been constructed using $\square_\kappa$ and $\kappa^\omega = \kappa^+$ for every $\kappa > \mathfrak{c}$ of cofinality ω by Weiss [96] and Wolfsdorf [97]. We shall now see that cofinal K-families are quite natural tools in constructions of Bernstein decompositions.

16.22 Theorem. *Suppose every regular $\theta > \mathfrak{c}$ supports a cofinal Kurepa family of size θ. Then every Hausdorff space admits a Bernstein decomposition.*

Proof. Let us say that a subset S of a topological space X is *sequentially closed in X* if it contains all limit points of sequences $\{x_n\} \subseteq S$ that converge in X. We say S is *σ-sequentially closed in X* if it can be decomposed into countably many sequentially closed sets. For a given cardinal θ, let $B(\theta)$ denote the statement that for every Hausdorff space X of size $\leq \theta$, every σ-sequentially closed subset S of X and every Bernstein decomposition $f : S \longrightarrow 2^{\mathbb{N}}$ there is a Bernstein decomposition $g : X \longrightarrow 2^{\mathbb{N}}$ extending f. Using the fact that a Hausdorff space of size $\leq \mathfrak{c}$ has at most $\mathfrak{c}$ copies of the Cantor set and following the idea of Bernstein's original proof, one sees that $B(\mathfrak{c})$ holds. Note also that by our assumption $\theta^{\aleph_0} = \theta$ for every cardinal $\theta \geq \mathfrak{c}$ of uncountable cofinality. So if we have $B(\theta)$ for such θ one proves $B(\theta^+)$ by constructing a chain f_ξ ($\xi < \theta^+$) of partial Bernstein decompositions on any Hausdorff topology τ on θ^+ using the fact that the set C of all $\delta < \theta^+$ which are σ-sequentially closed in (θ^+, τ) is closed and unbounded in θ^+. Thus if δ_ξ ($\xi < \theta^+$) enumerates C increasingly we will have $g_\xi : \delta_\xi \longrightarrow 2^{\mathbb{N}}$ and g_ξ compatible with the given partial Bernstein decomposition f of a given σ-sequentially closed set $S \subseteq \theta^+$. Note that $S \cap \delta_\xi$ will be σ-sequentially closed, so the use of $B(\delta_\xi)$ to get f_ξ will be possible. This inductive procedure encounters a crucial difficulty only when $\theta = \kappa^+$ for some $\kappa > \mathfrak{c}$ of cofinality ω. This is where a cofinal K-family $\mathcal{K}$ on κ^+ is useful. Let τ be a given Hausdorff topology on θ^+ and for $K \in \mathcal{K}$ let K' be the collection of all limits of τ-convergent sequences of elements of K. Note that if a set $\Gamma \subseteq \kappa^+$ is $\mathcal{K}$-closed in the sense that $\mathcal{K} \restriction \Gamma \subseteq \mathcal{K}$ (i.e. $K \cap \Gamma \in \mathcal{K}$ for all $K \in \mathcal{K}$) then Γ' will be equal to the union of all K' ($K \in \mathcal{K} \restriction \Gamma$). So if $|\Gamma| > \mathfrak{c}$ is a cardinal of uncountable cofinality then

$|\Gamma'| = |\Gamma|$. It follows that every $\delta < \theta^+$ with the property that $\delta = \lim \delta_n$ for some sequence δ_n $(n < \omega)$ of ordinals $\leq \delta$ with the property that $\mathcal{K} \restriction \delta_n \subseteq \mathcal{K}$ for all n can be decomposed into countably many sets $\Gamma_{\delta n}$ $(n < \omega)$ such that for each n, the set $\Gamma_{\delta n}$ is σ-sequentially closed in τ and $|\Gamma_{\delta n}|$ is a cardinal $> \mathfrak{c}$ of uncountable cofinality. However, note that the set C of all such $\delta < \theta^+$ is closed and unbounded in θ^+ so as above we use our inductive hypothesis $B(\theta)$ $(\theta < \kappa)$ to construct a chain $g_\xi : \delta_\xi \longrightarrow 2^{\mathbb{N}}$ $(\xi < \theta^+)$ of Bernstein decompositions. This completes the proof. $\dashv$

It is interesting that various less pathological classes of spaces admit a local version of Theorem 16.22.

16.23 Theorem. *A sigma-product* [35] *of the unit interval admits a Bernstein decomposition if its index set carries a cofinal Kurepa family. So in particular, any metric space that carries a cofinal Kurepa family admits a Bernstein decomposition.*

Proof. So let $\mathcal{K}$ be a fixed well-founded cofinal K-family on Γ and let $<_w$ be a well-ordering of $\mathcal{K}$ compatible with $\subseteq$. For $K \in \mathcal{K}$ fix a Bernstein decomposition $f_K : X_K \longrightarrow 2^{\mathbb{N}}$ where $X_K = \{x \in X : \operatorname{supp}(x) \subseteq K\}$. Define $f : X \longrightarrow 2^{\mathbb{N}}$ by letting $f(x) = f_K(x)$ where K is the $<_w$-minimal $K \in \mathcal{K}$ such that $\operatorname{supp}(x) \subseteq K$. It is easily checked that f is a Bernstein decomposition of the sigma-product. $\dashv$

16.24 Definition. Recall the notion of a *coherent family* of partial functions indexed by some ideal $\mathfrak{I}$, a family of the form $f_a : a \longrightarrow \omega$ $(a \in \mathfrak{I})$ with the property that $\{x \in a \cap b : f_a(x) \neq f_b(x)\}$ is finite for all $a, b \in \mathfrak{I}$.

It can be seen (see [89]) that the P-ideal dichotomy (see 4.45) has a strong influence on such families provided $\mathfrak{I}$ is a P-ideal of countable subsets of some set Γ.

16.25 Theorem. *Assuming the P-ideal dichotomy, for every coherent family of functions $f_a : a \longrightarrow \omega$ $(a \in \mathfrak{I})$ indexed by some P-ideal $\mathfrak{I}$ of countable subsets of some set Γ, either*

    1. *There is an uncountable $\Delta \subseteq \Gamma$ such that $f_a \restriction \Delta$ is finite-to-one for all $a \in \mathfrak{I}$, or*

    2. *There is $g : \Gamma \longrightarrow \omega$ such that $g \restriction a =^* f_a$ for all $a \in \mathfrak{I}$.*

Proof. Let $\mathfrak{L}$ be the family of all countable subsets b of Γ for which one can find an a in $\mathfrak{I}$ such that $b \setminus a$ is finite and f_a is finite-to-one on b. To see that $\mathfrak{L}$ is a P-ideal, let $\{b_n\}$ be a given sequence of members of $\mathfrak{L}$ and for

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[35] Recall that a *sigma-product* is the subspace of some Tychonoff cube $[0,1]^\Gamma$ consisting of all mappings x such that $\operatorname{supp}(x) = \{\xi \in \Gamma : x(\xi) \neq 0\}$ is countable. The set Γ is its *index-set* .

each n fix a member a_n of $\mathfrak{I}$ such that f_{a_n} is finite-to-one on b_n. Since $\mathfrak{I}$ is a P-ideal, we can find $a \in \mathfrak{I}$ such that $a_n \setminus a$ is finite for all n. Note that for all n, $b_n \setminus a$ is finite and that f_a is finite-to-one on b_n. For $n < \omega$, let

$$b_n^* = \{\xi \in b_n \cap a : f_a(\xi) > n\}.$$

Then b_n^* is a cofinite subset of b_n for each n, so if we set b to be equal to the union of the b_n^*'s, we get a subset of a which almost includes each b_n and on which f_a is finite-to-one. It follows that b belongs to $\mathfrak{L}$. This finishes the proof that $\mathfrak{L}$ is a P-ideal. Applying the P-ideal dichotomy to $\mathfrak{L}$, we get the two alternatives that translate into the alternatives 1 and 2 of Theorem 16.25.   ⊣

This leads to the natural question whether for any set Γ one can construct a family $\{f_a : a \longrightarrow \omega\}$ of finite-to-one mappings indexed by $[\Gamma]^\omega$. This question was answered by Koszmider [40] using the notion of a Jensen matrix discussed above. We shall present Koszmider's result using the notion of a cofinal Kurepa family instead.

16.26 Theorem. *If Γ carries a cofinal Kurepa family then there is a coherent family $f_a : a \longrightarrow \omega$ ($a \in [\Gamma]^\omega$) of finite-to-one mappings.*

Proof. Let $\mathcal{K}$ be a fixed well-founded cofinal K-family on Γ and let $<_w$ be a well-ordering of $\mathcal{K}$ compatible with $\subseteq$. It suffices to produce a coherent family of finite-to-one mappings indexed by $\mathcal{K}$. This is done by induction on $<_w$. Suppose $K \in \mathcal{K}$ and $f_H : H \longrightarrow \omega$ is determined for all $H \in \mathcal{K}$ with $H <_w K$. Let H_n ($n < \omega$) be a sequence of elements of $\mathcal{K}$ that are $<_w K$ and have the property that for every $H \in \mathcal{K}$ with $H <_w K$ there is $n < \omega$ such that $H \cap K =^* H_n \cap K$. So it suffices to construct a finite-to-one $f_K : K \longrightarrow \omega$ which coheres with each f_{H_n} ($n < \omega$), a straightforward task.   ⊣

16.27 Corollary. *For every nonnegative integer n there is a coherent family $f_a : a \longrightarrow \omega$ ($a \in [\omega_n]^\omega$) of finite-to-one mappings.*   ⊣

16.28 Remark. It is interesting that 'finite-to-one' cannot be replaced by 'one-to-one' in these results. For example, there is no coherent family of one-to-one mappings $f_a : a \longrightarrow \omega$ ($a \in [\mathfrak{c}^+]^\omega$). We finish this section with a typical application of coherent families of finite-to-one mappings discovered by Scheepers [61].

16.29 Theorem. *If there is a coherent family $f_a : a \longrightarrow \omega$ ($a \in [\Gamma]^\omega$) of finite-to-one mappings, then there is $F : [[\Gamma]^\omega]^2 \longrightarrow [\Gamma]^{<\omega}$ with the property that for every strictly $\subseteq$-increasing sequence a_n ($n < \omega$) of countable subsets of Γ, the union of $F(a_n, a_{n+1})$ ($n < \omega$) covers the union of a_n ($n < \omega$).*

Proof. For $a \in [\Gamma]^\omega$ let $x_a : \omega \longrightarrow \omega$ be defined by letting $x_a(n) = |\{\xi \in a : f_a(\xi) \leq n\}|$. Note that x_a is eventually dominated by x_b whenever a is a proper subset of b. Choose $\Phi : \omega^\omega \longrightarrow \omega^\omega$ with the property that $x <^* y$ implies $\Phi(y) <^* \Phi(x)$, where $<^*$ is the ordering of eventual dominance on ω^ω (i.e. $x <^* y$ if $x(n) < y(n)$ for all but finitely many n's). Define another family of functions $g_a : a \longrightarrow \omega$ $(a \in [\Gamma]^\omega)$ by letting

$$g_a(\xi) = \Phi(x_a)(f_a(\xi)). \tag{I.224}$$

Note the following interesting property of g_a $(a \in [\Gamma]^\omega)$:

$$F(a, b) = \{\xi \in a : g_b(\xi) \geq g_a(\xi)\} \text{ is finite for all } a \subsetneq b \text{ in } [\Gamma]^\omega. \tag{I.225}$$

So if a_n $(n < \omega)$ is a strictly $\subseteq$-increasing sequence of countable subsets of Γ and $\bar{\xi}$ belongs to some $a_{\bar{n}}$ then the sequence of integers $g_{a_n}(\bar{\xi})$ $(\bar{n} \leq n < \omega)$ must have some place $n \geq \bar{n}$ with the property that $g_{a_n}(\bar{\xi}) < g_{a_{n+1}}(\bar{\xi})$, i.e. a place $n \geq \bar{n}$ such that $\xi \in F(a_n, a_{n+1})$. $\dashv$

16.30 Remark. Note that if κ is a singular cardinal of cofinality ω with the property that $\mathrm{cf}[\theta]^\omega < \kappa$ for all $\theta < \kappa$, then the existence of a cofinal Kurepa family on κ^+ implies the existence of a Jensen matrix on κ^+. So these two notions appear to be quite close to each other. The three basic properties of the function $\rho : [\kappa^+] \longrightarrow \kappa$ (14.5 and 14.6(a),(b)) seem much stronger in view of the fact that the linear ordering as in 16.3 cannot exist for κ above a supercompact cardinal and the fact that Foreman and Magidor [24] have produced a model with a supercompact cardinal that carries a Jensen matrix on any successor of a singular cardinal of cofinality ω. Note that Chang's conjecture of the form $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega)$ for some singular κ of cofinality ω implies that every locally countable family $\mathcal{K} \subseteq [\kappa]^\omega$ must have size $\leq \kappa$. So, one of the models of set theory that has no cofinal K-family on, say $\aleph_{\omega+1}$, is the model of Levinski, Magidor and Shelah [49]. It seems still unknown whether the conclusion of Theorem 16.29 can be proved without additional set-theoretic assumptions.

17. The oscillation mapping

In what follows, θ will be a fixed regular infinite cardinal.

$$\mathrm{osc} : \mathcal{P}(\theta)^2 \longrightarrow \mathrm{Card} \tag{I.226}$$

is defined by

$$\mathrm{osc}(x, y) = |x \setminus (\sup(x \cap y) + 1)/ \sim |,$$

where $\sim$ is the equivalence relation on $x \setminus (\sup(x \cap y) + 1)$ defined by letting $\alpha \sim \beta$ iff the closed interval determined by α and β contains no point from

y. So, if x and y are disjoint, $\mathrm{osc}(x,y)$ is simply the number of convex pieces the set x is split by the set y. The oscillation mapping has proven to be a useful device in various schemes for coding information. It usefulness in a given context depend very much of the corresponding 'oscillation theory', a set of definitions and lemmas that disclose when it is possible to achieve a given number as oscillation between two sets x and y in a given family $\mathcal{X}$. The following definition reveals the notion of largeness relevant to the oscillation theory that we develop in this section.

17.1 Definition. A family $\mathcal{X} \subseteq \mathcal{P}(\theta)$ is *unbounded* if for every closed and unbounded subset C of θ there exist $x \in \mathcal{X}$ and an increasing sequence $\{\delta_n : n < \omega\} \subseteq C$ such that $\sup(x \cap \delta_n) < \delta_n$ and $[\delta_n, \delta_{n+1}) \cap x \neq \emptyset$ for all $n < \omega$.

This notion of unboundedness has proven to be the key behind a number of results asserting the complex behaviour of the oscillation mapping on $\mathcal{X}^2$. The case $\theta = \omega$ seems to contain the deeper part of the oscillation theory known so far (see [79],[80, §1] and [88]), though in this section we shall only consider the case $\theta > \omega$. We shall also restrict ourselves to the family $\mathcal{K}(\theta)$ of all closed bounded subsets of θ rather than the whole power-set of θ. The following is the basic result about the behaviour of the oscillation mapping in this context.

17.2 Lemma. *If $\mathcal{X}$ is an unbounded subfamily of $\mathcal{K}(\theta)$ then for every positive integer n there exist x and y in $\mathcal{X}$ such that $\mathrm{osc}(x,y) = n$.*

Proof. Choose an $\in$-chain $\mathcal{M}$ of length θ of elementary submodels M of some large enough structure of the form H_λ such that $\mathcal{X} \in M$ and $M \cap \theta \in \theta$. Applying the fact that $\mathcal{X}$ is unbounded with respect to the closed and unbounded set $\{M \cap \theta : M \in \mathcal{M}\}$ we can find $y \in \mathcal{X}$ such that:

$$\delta_i = M_i \cap \theta \notin y \text{ for all } i < n, \qquad (I.227)$$

$$\delta_0 \cap y \neq \emptyset \text{ and } y \setminus \delta_{n-1} \neq \emptyset, \qquad (I.228)$$

$$[\delta_{i-1}, \delta_i) \cap y \neq \emptyset \text{ for all } 0 < i < n. \qquad (I.229)$$

Let

$$J_0 = [0, \sup(y \cap \delta_0)], \qquad (I.230)$$

$$J_i = [\delta_{i-1}, \max(y \cap \delta_i)] \text{ for } 0 < i < n, \qquad (I.231)$$

$$J_n = [\delta_{n-1}, \max(y)]. \qquad (I.232)$$

Let $\mathcal{F}_n$ be the collection of all increasing sequences $I_0 < I_1 < \ldots < I_n$ of closed intervals of θ such that $I_0 = J_0$ and such that there is $x = x_I$ in $\mathcal{X}$ such that

$$x \subseteq I_0 \cup I_1 \cup \ldots \cup I_n, \qquad (I.233)$$

$$\max(I_i) \in x \text{ for all } i \leq n. \tag{I.234}$$

Clearly, $\mathcal{F}_n \in M_0$ and $\langle J_0, J_1, \ldots, J_n \rangle \in \mathcal{F}_n$. Let $\mathcal{F}_{n-1}$ be the collection of all $\langle I_0, \ldots, I_{n-1} \rangle$ such that for every $\alpha < \theta$ there exists an interval I of θ above α such that $\langle I_0, \ldots, I_{n-1}, I \rangle \in \mathcal{F}_n$. Note that $\mathcal{F}_{n-1} \in M_0$ and that using the elementarity of M_{n-1} one proves that $\langle J_0, \ldots, J_{n-1} \rangle \in \mathcal{F}_{n-1}$. Let $\mathcal{F}_{n-2}$ be the collection of all $\langle I_0, \ldots, I_{n-2} \rangle$ such that for every $\alpha < \theta$ there exists interval I above α such that $\langle I_0, \ldots, I_{n-2}, I \rangle \in \mathcal{F}_{n-1}$. Then again $\mathcal{F}_{n-2} \in M_0$ and using the elementarity of M_{n-2} one shows that the restriction $\langle J_0, \ldots, J_{n-2} \rangle$ belongs to $\mathcal{F}_{n-2}$. Continuing in this way we construct families $\mathcal{F}_n, \mathcal{F}_{n-1}, \ldots, \mathcal{F}_0$ such that:

$$\mathcal{F}_i \in M_0 \text{ for all } i \leq n, \tag{I.235}$$

$$\mathcal{F}_0 = \{J_0\}. \tag{I.236}$$

$$\begin{array}{l} \langle I_0, \ldots, I_{n-i} \rangle \in \mathcal{F}_{n-i} \text{ implies that for all } \alpha < \theta \text{ there exists} \\ \text{an interval } I \text{ above } \alpha \text{ such that } \langle I_0, \ldots, I_{n-i}, I \rangle \in \mathcal{F}_{n-i+1}. \end{array} \tag{I.237}$$

Recursively in $i \leq n$ we choose intervals $I_0 < I_1 < \ldots < I_n$ such that:

$$\langle I_0, \ldots, I_i \rangle \in \mathcal{F}_i \text{ for all } i \leq n, \tag{I.238}$$

$$J_{i-1} < I_i \in M_{i-1} \text{ for } 0 < i \leq n. \tag{I.239}$$

Clearly, there is no problem in choosing the sequence using the elementarity of the submodels M_{i-1} and the condition (I.237). By the definition $\langle I_0, \ldots I_n \rangle \in \mathcal{F}_n$ means that there is $x \in \mathcal{X}$ satisfying (I.233) and (I.234). It follows that $\max(x \cap y) = \max(I_0)$ and that

$$x \cap I_i \neq \emptyset \quad (0 < i \leq n) \tag{I.240}$$

are the convex pieces of $x \setminus (\max(x \cap y) + 1)$ to which this set is split by y. It follows that $\operatorname{osc}(x, y) = n$. This finishes the proof. $\dashv$

Lemma 17.2 also has a rectangular form.

17.3 Lemma. *If $\mathcal{X}$ and $\mathcal{Y}$ are two unbounded subfamilies of $\mathcal{K}(\theta)$ then for all but finitely many positive integers n there exist $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $\operatorname{osc}(x, y) = n$.* $\dashv$

Recall the notion of a nontrivial C-sequence C_α $(\alpha < \theta)$ on θ from Section 11, a C-sequence with the property that for every closed and unbounded subset C of θ there is a limit point δ of C such that $C \cap \delta \nsubseteq C_\alpha$ for all $\alpha < \theta$.

17.4 Definition. For a subset D of θ let $\lim(D)$ denote the set of all $\alpha < \theta$ with the property that $\alpha = \sup(D \cap \alpha)$. A subsequence C_α $(\alpha \in \Gamma)$ of some C-sequence C_α $(\alpha < \theta)$ is *stationary* if the union of all $\lim(C_\alpha)$ $(\alpha \in \Gamma)$ is a stationary subset of θ.

17.5 Lemma. *A stationary subsequence of a nontrivial C-sequence on θ is an unbounded family of subsets of θ.*

Proof. Let C_α $(\alpha \in \Gamma)$ be a given stationary subsequence of a nontrivial C-sequence on θ. Let C be a given closed and unbounded subset of θ. Let Δ be the union of all $\lim(C_\alpha)$ $(\alpha \in \Gamma)$. Then Δ is a stationary subset of θ. For $\xi \in \Delta$ choose $\alpha_\xi \in \Gamma$ such that $\xi \in \lim(C_{\alpha_\xi})$. Applying the assumption that C_α $(\alpha \in \Gamma)$ is a nontrivial C-subsequence, we can find $\xi \in \Delta \cap \lim(C)$ such that

$$C \cap [\eta, \xi) \nsubseteq C_{\alpha_\xi} \text{ for all } \eta < \xi. \qquad (I.241)$$

If such ξ cannot be found using the stationarity of the set $\Delta \cap \lim(C)$ we would be able to use the Pressing Down Lemma on the regressive mapping that would give us an $\eta < \xi$ violating (I.241) and get that a tail of C trivializes C_α $(\alpha \in \Gamma)$. Using (I.241) and the fact that $C_{\alpha_\xi} \cap \xi$ is unbounded in ξ we can find a strictly increasing sequence δ_n $(n < \omega)$ of elements of $(C \cap \xi) \setminus C_{\alpha_\xi}$ such that $[\delta_n, \delta_{n+1}) \cap C_{\alpha_\xi} \neq \emptyset$ for all n. So the set C_{α_ξ} satisfies the conclusion of Definition 17.1 for the given closed and unbounded set C. $\dashv$

Recall that $\mathbb{Q}_\theta$ denotes the set of all finite sequences of ordinals $< \theta$ and that we consider it ordered by the right lexicographical ordering. We need the following two further orderings on $\mathbb{Q}_\theta$:

$$s \sqsubseteq t \text{ if and only if } s \text{ is an initial segment of } t. \qquad (I.242)$$

$$s \sqsubset t \text{ if and only if } s \text{ is a proper initial part of } t. \qquad (I.243)$$

17.6 Definition. Given a C-sequence C_α $(\alpha < \theta)$ we can define an *action* $(\alpha, t) \longmapsto \alpha_t$ of $\mathbb{Q}_\theta$ on θ recursively on the ordering $\sqsubseteq$ of $\mathbb{Q}_\theta$ as follows:

$$\alpha_\emptyset = \alpha, \qquad (I.244)$$

$$\alpha_{\langle \xi \rangle} = \text{ the } \xi\text{th member of } C_\alpha \text{ if } \xi < \mathrm{tp}(C_\alpha); \text{ otherwise } \alpha_{\langle \xi \rangle} = \alpha, \quad (I.245)$$

$$\alpha_{t ^\frown \langle \xi \rangle} = (\alpha_t)_{\langle \xi \rangle}. \qquad (I.246)$$

17.7 Remark. Note that if $\rho_0(\alpha, \beta) = t$ for some $\alpha < \beta < \theta$ then $\beta_t = \alpha$. In fact, if $\beta = \beta_0 > \ldots > \beta_n = \alpha$ is the walk from β to α along the C-sequence, each member of the trace $\mathrm{Tr}(\alpha, \beta) = \{\beta_0, \beta_1, \ldots, \beta_n\}$ has the form β_s where s is the uniquely determined initial part of t. Note, however, that in general $\beta_t = \alpha$ does not imply that $\rho_0(\alpha, \beta) = t$.

17.8 Notation. Given a C-sequence C_α $(\alpha < \theta)$ on θ we shall use $\mathrm{osc}(\alpha, \beta)$ to denote $\mathrm{osc}(C_\alpha, C_\beta)$.

17.9 Theorem. *If C_α $(\alpha < \theta)$ is a nontrivial C-sequence on θ, then for every unbounded set $\Gamma \subseteq \theta$ and positive integer n there exist $\alpha < \beta$ in Γ and $t \sqsubseteq \rho_0(\alpha, \beta)$ such that*

(a) $\mathrm{osc}(\alpha_t, \beta_t) = n$,

(b) $\mathrm{osc}(\alpha_s, \beta_s) = 1$ *for all* $s \sqsubset t$.

Proof. Choose an elementary submodel M of some large enough structure of the form H_λ such that $\bar{\delta} = M \cap \theta \in \theta$ and M contains all the relevant objects. Choose $\bar{\beta} \in \Gamma$ above $\bar{\delta}$ and let

$$\bar{\beta} = \bar{\beta}_0 > \bar{\beta}_1 > \ldots > \bar{\beta}_k = \bar{\delta}$$

be the minimal walk from $\bar{\beta}$ to $\bar{\delta}$ along the C-sequence. Let $\bar{k} = k - 1$ if $C_{\bar{\beta}_{k-1}} \cap \bar{\delta}$ is unbounded; otherwise $\bar{k} = k$. Recall our implicit assumption about essentially all C-sequences that we consider here: if γ is a limit ordinal then a $\xi \in C_\gamma$ occupying a successor place in C_γ must be a successor ordinal. In this case, since δ is a limit ordinal and since $\bar{\delta} \in C_{\bar{\beta}_{k-1}}$, either it must be that $\bar{\beta}_{k-1}$ is a limit ordinal and $\bar{\delta} = \sup(C_{\bar{\beta}_{k-1}} \cap \bar{\delta})$ or $\bar{\beta}_{k-1} = \bar{\delta} + 1$. It follows that $\bar{\beta}_{\bar{k}}$ is a limit ordinal. Set $t = \rho_0(\bar{\beta}_{\bar{k}}, \bar{\beta})$. Let

$$\emptyset = t \upharpoonright 0, t \upharpoonright 1, \ldots, t \upharpoonright \bar{k} = t$$

be our notation for all the restrictions of the sequence t. Note that $\bar{\beta}_i = \bar{\beta}_{t \upharpoonright i}$ for $i \leq k$. Let Γ_0 be the set of all $\beta \in \Gamma$ such that:

$$\beta = \beta_\emptyset > \beta_{t \upharpoonright 1} > \ldots > \beta_t, \tag{I.247}$$

$$\sup(C_{\beta_{t \upharpoonright i}} \cap \beta_t) = \sup(C_{\bar{\beta}_i} \cap \bar{\beta}_{\bar{k}}) \text{ for all } i < \bar{k}. \tag{I.248}$$

Note that $\Gamma_0 \in M$ and $\bar{\beta} \in \Gamma_0$. Let $\Delta = \{\beta_t : \beta \in \Gamma_0\}$. Then $\Delta \in M$ and $\bar{\beta}_{\bar{k}} \in \Delta$. By the very choice of $\bar{k}$, $\bar{\delta}$ belongs to $\lim(C_{\bar{\beta}_{\bar{k}}})$, so using the elementarity of M one concludes that C_δ $(\delta \in \Delta)$ is a stationary subsequence of our original sequence. Shrinking Δ a bit if necessary, we may assume that for every $\gamma \in \Delta$ an $\alpha \in \Gamma_0$ with $\alpha_t = \gamma$ can be found below the next member of Δ above γ. By Lemma 17.5, C_δ $(\delta \in \Delta)$ is an unbounded family of subsets of θ and so we can apply the basic oscillation Lemma 17.2 and find $\gamma < \delta$ such that $\mathrm{osc}(\alpha_t, \beta_t) = n$. Using the fact $\alpha < \delta$ and the fact that α and β satisfy (I.248) we conclude that $\mathrm{osc}(\alpha_{t \upharpoonright i}, \beta_{t \upharpoonright i}) = 1$ for all $i < \bar{k}$. This finishes the proof. $\dashv$

17.10 Corollary. *Suppose a regular uncountable cardinal θ carries a non-trivial C-sequence. Then there is $f : [\theta]^2 \longrightarrow \omega$ which takes all the values from ω on any set of the form $[\Gamma]^2$ for an unbounded subset of θ.*

Proof. Given $\alpha < \beta < \theta$, if there is $t \sqsubseteq \rho_0(\alpha, \beta)$ satisfying 17.9(a),(b) put

$$f(\alpha, \beta) = \mathrm{osc}(\alpha_t, \beta_t) - 2 \tag{I.249}$$

otherwise put $f(\alpha, \beta) = 0$. $\dashv$

17.11 Remark. The class of all regular cardinals θ that carry a nontrivial C-sequence is quite extensive. It includes not only all successor cardinals but also some inaccessible as well as hyperinaccessible cardinals such as for example, first inaccessible or first Mahlo cardinals. In view of the well-known Ramsey-theoretic characterization of weak compactness, Corollary 17.10 leads us to the following natural question.

17.12 Question. Can the weak compactness of a strong limit regular uncountable cardinal be characterized by the fact that for every $f : [\theta]^2 \longrightarrow \omega$ there exists an unbounded set $\Gamma \subseteq \theta$ such that $f''[\Gamma]^2 \neq \omega$? This is true when ω is replaced by 2, but can any other number beside 2 be used in this characterization?

18. The square-bracket operation

In this Section we show that the basic idea of the square-bracket operation on ω_1 introduced in Definition 7.3 extends to a general setting on an arbitrary uncountable regular cardinal θ that carries a nontrivial C-sequence C_α $(\alpha < \theta)$. The basic idea is based on the oscillation map defined in the previous section and, in particular, on the property of this map exposed in Theorem 17.9: for $\alpha < \beta < \theta$ we set

> $[\alpha\beta] = \beta_t$, where $t \sqsubseteq \rho_0(\alpha, \beta)$ is such that $\mathrm{osc}(\alpha_t, \beta_t) \geq 2$
> but $\mathrm{osc}(\alpha_s, \beta_s) = 1$ for all $s \sqsubset t$; if such t does not exist, we   (I.250)
> let $[\alpha\beta] = \alpha$.

Thus, $[\alpha\beta]$ is the first place visited by β on its walk to α where a nontrivial oscillation with the corresponding step of α occurs. What Theorem 17.9 is telling us is that the nontrivial oscillation indeed happens most of the time. Results that would say that the set of values $\{[\alpha\beta] : \{\alpha, \beta\} \in [\Gamma]^2\}$ is in some sense large no matter how small the unbounded set $\Gamma \subseteq \theta$ is, would correspond to the results of Lemmas 7.4-7.5 about the square-bracket operation on ω_1. It turns out that this is indeed possible and to describe it we need the following definition.

18.1 Definition. A C-sequence C_α $(\alpha < \theta)$ on θ *avoids* a given subset Δ of θ if $C_\alpha \cap \Delta = \emptyset$ for all limit ordinals $\alpha < \theta$.

18.2 Lemma. *Suppose C_α $(\alpha < \theta)$ is a given C-sequence on θ that avoids a set $\Delta \subseteq \theta$. Then for every unbounded set $\Gamma \subseteq \theta$, the set of elements of Δ not of the form $[\alpha\beta]$ for some $\alpha < \beta$ in Γ is nonstationary in θ.*

Proof. Let Ω be a given stationary subset of Δ. We need to find $\alpha < \beta$ in Γ such that $[\alpha\beta] \in \Omega$. For a limit $\delta \in \Omega$ fix a $\beta = \beta(\delta) \in \Gamma$ above δ and let

$$\beta = \beta_0 > \beta_1 > \ldots > \beta_{k(\delta)} = \delta$$

be the walk from β to δ along the sequence C_α ($\alpha < \theta$). Since $\delta \notin C_\alpha$ for any limit α and since $C_{\alpha+1} = \{\alpha\}$ for all α, we must have that $\beta_{k(\delta)-1} = \delta + 1$. In particular,

$$C_{\beta_i} \cap \delta \text{ is bounded in } \delta \text{ for all } i < k(\delta). \tag{I.251}$$

Applying the Pressing Down Lemma gives us a stationary set $\Omega_0 \subseteq \Omega$, an integer k, ordinals ξ_i ($i < k$), and a sequence $t \in \mathbb{Q}_\theta$ such that for all $\delta \in \Omega_0$:

$$k(\delta) = k \text{ and } \rho_0(\delta, \beta(\delta)) = t, \tag{I.252}$$

$$\max(C_{\beta(\delta)_i} \cap \delta) = \xi_i \text{ for all } i < k. \tag{I.253}$$

Intersecting Ω_0 with a closed and unbounded subset of θ, we may assume for every $\delta \in \Omega_0$ that $\beta(\delta)$ is smaller than the next member of Ω_0 above δ. Note that any C-sequence on a regular uncountable cardinal that avoids a stationary subset of the cardinal, in particular, must be nontrivial. Thus C_δ ($\delta \in \Omega_0$) is a stationary subsequence of a nontrivial C-sequence, and so by Lemma 17.5, unbounded as a family of subsets of θ. By the basic oscillation Lemma 17.2, we can find $\gamma < \delta$ in Ω_0 such that $\mathrm{osc}(C_\gamma, C_\delta) = 2$. Then by the choice of Ω, if $\alpha = \beta(\gamma)$ and $\beta = \beta(\delta)$ then $\gamma = \alpha_t$, $\delta = \beta_t$ and $\alpha < \delta$. Comparing this with (I.252) and (I.253) we conclude that $[\alpha\beta] = \beta_t = \delta \in \Omega$. This finishes the proof. $\dashv$

It is clear that the argument gives the following more general result.

18.3 Lemma. *Suppose C_α ($\alpha < \theta$) avoids $\Delta \subseteq \theta$ and let A be a family of size θ consisting of pairwise disjoint finite sets, all of some fixed size n. Then the set of all elements of Δ that are not of the form $[a(1)b(1)] = [a(2)b(2)] = \ldots = [a(n)b(n)]$ for some $a \neq b$ in A is nonstationary in θ.* $\dashv$

Since $[\alpha\beta]$ belongs to the trace $\mathrm{Tr}(\alpha, \beta)$ of the walk from β to α it is not surprising that $[\cdot\cdot]$ strongly depends on the behaviour of Tr. For example the following should be clear from the proof of Lemma 18.2.

18.4 Lemma. *The following are equivalent for a given C-sequence and a set Ω:*

(a) $\Omega \setminus \{[\alpha\beta] : \{\alpha, \beta\} \in [\Gamma]^2\}$ is nonstationary in θ,

(b) $\Omega \setminus \bigcup\{\mathrm{Tr}(\alpha, \beta) : \{\alpha, \beta\} \in [\Gamma]^2\}$ is nonstationary in θ.

$\dashv$

This fact suggests the following definition.

18.5 Definition. The *trace filter* of a given C-sequence C_α ($\alpha < \theta$) is the normal filter on θ generated by sets of the form $\bigcup\{\mathrm{Tr}(\alpha, \beta) : \{\alpha, \beta\} \in [\Gamma]^2\}$ where Γ is an unbounded subset of θ.

18.6 Remark. Having a proper (i.e. $\neq \mathcal{P}(\theta)$) trace filter is a strengthening of the nontriviality requirement on a given C-sequence C_α ($\alpha < \theta$). For example, if a C-sequence avoids a stationary set $\Omega \subseteq \theta$, then its trace filter is nontrivial and in fact no stationary subset of Γ is a member of it. Note the following analogue of Lemma 18.4: the trace filter of a given C-sequence is the normal filter generated by sets of the form $\{[\alpha\beta] : \{\alpha, \beta\} \in [\Gamma]^2\}$ where Γ is an unbounded subset of θ. So to obtain the analogues of the results of Section 7 about the square-bracket operation on ω_1 one needs a C-sequence C_α ($\alpha < \theta$) on θ whose trace filter is not only nontrivial but also not θ-saturated, i.e. it allows a family of θ pairwise disjoint positive sets. It turns out that the hypothesis of Lemma 18.2 is sufficient for both of these conclusions.

18.7 Lemma. *If a C-sequence on θ avoids a stationary subset of θ, then there exist θ pairwise disjoint subsets of θ that are positive with respect to its trace filter.* [36]

Proof. This follows from the well-known fact (see [37]) that if there is a normal, nontrivial and θ-saturated filter on θ, then for every stationary $\Omega \subseteq \theta$ there exists $\lambda < \theta$ such that $\Omega \cap \lambda$ is stationary in λ (and the fact that the stationary set which is avoided by the C-sequence does not reflect in this way). $\dashv$

18.8 Corollary. *If a regular cardinal θ admits a nonreflecting stationary subset then there is $c : [\theta]^2 \longrightarrow \theta$ which takes all the values from θ on any set of the form $[\Gamma]^2$ for some unbounded set $\Gamma \subseteq \theta$.* $\dashv$

To get such a c, one composes the square-bracket operation of some C-sequence, that avoids a stationary subset of θ, with a mapping $* : \theta \longrightarrow \theta$ with the property that the $*$-preimage of each point from θ is positive with respect to the trace filter of the square sequence. In other words c is equal to the composition of $[\cdot\cdot]$ and $*$, i.e. $c(\alpha, \beta) = [\alpha\beta]^*$. Note that, as in Section 7, the property of the square-bracket operation from Lemma 18.3 leads to the following rigidity result which corresponds to Lemma 7.10.

18.9 Lemma. *The algebraic structure $(\theta, [\cdot\cdot], *)$ has no nontrivial automorphisms.* $\dashv$

18.10 Remark. Note that every θ which is a successor of a regular cardinal κ admits a nonreflecting stationary set. For example $\Omega = \{\delta < \theta : \mathrm{cf}\,\delta = \kappa\}$ is such a set. Thus any C-sequence on θ that avoids Ω leads to a square bracket operation which allows analogues of all the results from Section 7 about the square-bracket operation on ω_1. The reader is urged to examine

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[36]A subset A of the domain of some filter $\mathcal{F}$ is *positive* with respect to $\mathcal{F}$ if $A \cap F \neq \emptyset$ for every $F \in \mathcal{F}$.

these analogues. We finish this section by giving a projection of the square-bracket operation (the analogue of 7.14) which has been used by Shelah and Steprans [69] in order to construct a family of nonisomorphic extraspecial p-groups that contain no large abelian subgroups.

Suppose now that $\theta = \kappa^+$ for some regular cardinal κ and fix a C-sequence C_α ($\alpha < \kappa^+$) on κ^+ such that $\mathrm{tp}(C_\alpha) \leq \kappa$ for all α, or equivalently, such that C_α ($\alpha < \kappa^+$) avoids the set $\Omega_\kappa = \{\delta < \kappa^+ : \mathrm{cf}\ \delta = \kappa\}$. Let $[\cdot\cdot]$ be the corresponding square-bracket operation. Let λ be the minimal cardinal such that $2^\lambda \geq \kappa^+$. Choose a sequence r_ξ ($\xi < \kappa^+$) of distinct subsets of λ. Let $\mathcal{H}$ be the collection of all maps $h : \mathcal{P}(D(h)) \longrightarrow \kappa^+$ where $D(h)$ is a finite subset of λ. Let $\pi : \kappa^+ \longrightarrow \mathcal{H}$ be a map with the property that $\pi^{-1}(h) \cap \Omega_\kappa$ is stationary for all $h \in \mathcal{H}$. Finally, define an operation $[\![\cdot\cdot]\!]$ on κ^+ as follows:

$$[\![\alpha\beta]\!] = \pi([\alpha\beta])(r_\alpha \cap D(\pi([\alpha\beta]))). \tag{I.254}$$

The following is a simple consequence of the property 18.3 of the square-bracket operation.

18.11 Lemma. *For every family A of size κ^+ consisting of pairwise disjoint finite subsets of κ^+ all of some fixed size n and every sequence $\xi_0, \ldots, \xi_{n-1}$ of ordinals $< \kappa^+$ there exist $a \neq b$ in A such that $[\![a(i)b(i)]\!] = \xi_i$ for all $i < n$.* $\dashv$

18.12 Remark. Shelah and Steprans consider the function $\Phi : [\kappa^+]^2 \longrightarrow 2$ defined by $\Phi(\alpha, \beta) = 0$ iff $[\![\alpha\beta]\!] = 0$ in [69] which they then incorporate into a general construction, due to Ehrenfeucht and Faber, to obtain an extraspecial p-group G_Γ associated to essentially every unbounded subset Γ of κ^+. They show that G_Γ contains no abelian subgroup of size κ^+ using not only Lemma 18.11 but also the behaviour of $[\![a(i)b(j)]\!]$ for $i \neq j < n$, $a \neq b \in A$ that follows from the behaviour of the square-bracket operation, exposed above in Lemma 7.7.

For sufficiently large cardinals θ we have the following variation on the theme first encountered above in Theorem 11.2.

18.13 Theorem. *Suppose θ is bigger than the continuum and carries a C-sequence avoiding a stationary set Γ of cofinality $> \omega$ ordinals in θ. Let A be a family of θ pairwise disjoint finite subsets of θ, all of some fixed size n. Then for every stationary $\Gamma_0 \subseteq \Gamma$ there exist $s, t \in \omega^n$ and a positive integer k such that for every $l < \omega$ there exist $a < b$ [37] in A and $\delta_0 > \delta_1 > \ldots > \delta_l$ in $\Gamma_0 \cap (\max(a), \min(b))$ such that:*

(1) $\rho_2(\delta_{i+1}, \delta_i) = k$ for all $i < l$,

---

[37]Recall that if a and b are two sets of ordinals, then the notation $a < b$ means that $\max(a) < \min(b)$.

(2) $\rho_0(a(i), b(j)) = \rho_0(\delta_0, b(j))^\frown \rho_0(\delta_1, \delta_0)^\frown \ldots ^\frown \rho_0(\delta_l, \delta_{l-1})^\frown \rho_0(a(i), \delta_l)$
   for all $i, j < n$,

(3) $\rho_2(\delta_0, b(j)) = t_j$ and $\rho_2(a(i), \delta_l) = s_i$ for all $i, j < n$.

Proof. For $\Delta \subseteq \Gamma$ and $\delta < \theta$ set

$$S_\delta(A) = \bigcap_{\gamma < \delta} \{\langle \rho_2(a(i), \delta) : i < n \rangle : a \in A \upharpoonright [\gamma, \delta)\}, \qquad (\text{I.255})$$

$$T_\delta(\Delta) = \bigcap_{\gamma < \delta} \{\rho_2(\alpha, \delta) : \alpha \in \Delta \cap [\gamma, \delta)\}. \qquad (\text{I.256})$$

Note that $S_\delta(A)$ and $T_\delta(\Delta)$ range over subsets of ω^n and ω, respectively, so if δ has uncountable cofinality and if A and Δ are unbounded in δ then both of these sets, $S_\delta(A)$ and $T_\delta(\Delta)$, are nonempty. Let Γ_0 be a given stationary subset of Γ. Since θ has size bigger than the continuum, starting from Γ_0 one can find an infinite sequence $\Gamma_0 \supseteq \Gamma_1 \supseteq \ldots$ of subsets of Γ such that $\Gamma_\omega = \bigcap_{i < \omega} \Gamma_i$ is stationary in θ and

$$S_\delta(A) = S_\varepsilon(A) \neq \emptyset \text{ for all } \delta, \varepsilon \in \Gamma_1, \qquad (\text{I.257})$$

$$T_\delta(\Gamma_i) = T_\varepsilon(\Gamma_i) \neq \emptyset \text{ for all } \delta, \varepsilon \in \Gamma_{i+1} \text{ and } i < \omega. \qquad (\text{I.258})$$

Pick $\delta \in \Gamma_\omega$ such that $\Gamma_\omega \cap \delta$ is unbounded in δ and choose $b \in A$ above δ. Let $t_j = \rho_2(\delta, b_j)$ for $j < n$. Now let s be an arbitrary member of $S_\delta(A)$ and k an arbitrary member of $T_\delta(\Gamma_\omega)$. To check that these objects satisfy the conclusion of the Theorem, let $l < \omega$ be given. Let $\delta_0 = \delta$ and let $\gamma_0 < \delta_0$ be an upper bound for all sets of the form $C_\xi \cap \delta_0$ ($\xi \in \text{Tr}(\delta_0, b(j)), \xi \neq \delta_0, j < n$). Then the walk from any $b(j)$ to any $\alpha \in [\gamma_0, \delta_0)$ must pass through δ_0. Since $k \in T_\delta(\Gamma_\omega) \subseteq T_\delta(\Gamma_l)$ we can find $\delta_1 \in \Gamma_l \cap (\gamma_0, \delta_0)$ such that $\rho_2(\delta_1, \delta_0) = k$. Choose $\gamma_1 \in (\gamma_0, \delta_1)$ such that the walk from δ_0 to any $\alpha \in [\gamma_1, \delta_1)$ must pass through δ_1. Since $k \in T_\delta(\Gamma_\omega) \subseteq T_\delta(\Gamma_{l-1}) = T_{\delta_1}(\Gamma_{l-1})$ there exists $\delta_2 \in \Gamma_{l-1} \cap (\gamma_1, \delta_1)$ such that $\rho_2(\delta_2, \delta_1) = k$. Choose $\gamma_2 \in (\gamma_1, \delta_2)$ such that the walk from δ_1 to any $\alpha \in [\gamma_2, \delta_2)$ must pass through δ_2, and so on. Proceeding in this way, we find $\delta = \delta_0 > \delta_1 > \ldots > \delta_l > \gamma_l > \gamma_{l-1} > \ldots > \gamma_0$ such that:

$$\delta_i \in \Gamma_{l-i+1} \text{ for all } 0 < i \leq l, \qquad (\text{I.259})$$

$$\rho_2(\delta_{i+1}, \delta_i) = k \text{ for all } i < l, \qquad (\text{I.260})$$

for every $i < l$ the walk from δ_i to some $\alpha \in [\gamma_{i+1}, \delta_{i+1})$ passes through δ_{i+1}. $\qquad (\text{I.261})$

Since $s \in S_\delta(A) = S_{\delta_l}(A)$, we can find $a \in A \upharpoonright [\gamma_l, \delta_l)$ such that $\rho_2(a(i), \delta_l) = s(i)$ for all $i < n$. Now, given $i, j < n$, the walk from $b(j)$ to $a(i)$ first passes through all the δ_i ($i \leq l$), so $\rho_0(a(i), b(j))$ is simply equal to the concatenation of $\rho_0(\delta_0, b(j))$, $\rho_0(\delta_1, \delta_0)$, ..., $\rho_0(\delta_l, \delta_{l-1})$, $\rho_0(a(i), \delta_l)$. This together with (I.260) gives us the three conclusions of Theorem 18.13. $\dashv$

From now on θ is assumed to be a fixed cardinal satisfying the hypotheses of Theorem 18.13. It turns out that Theorem 18.13 gives us a way to define another square-bracket operation which has complex behavior not only on squares of unbounded subsets of θ but also on rectangles formed by two unbounded subsets of θ. To define this new operation we choose a mapping $h : \omega \longrightarrow \omega$ such that:

$$\text{for every } k, m, n, p < \omega \text{ and } s \in \omega^n \text{ there exist } l < \omega \text{ such} \\ \text{that } h(m + l \cdot k + s(i)) = m + p \text{ for all } i < n. \tag{I.262}$$

18.14 Definition. $[\cdots]_h : [\theta]^2 \longrightarrow \theta$ is defined by letting $[\alpha\beta]_h = \beta_t$ where $t = \rho_0(\alpha, \beta) \upharpoonright h(\rho_2(\alpha, \beta))$.

Thus, $[\alpha\beta]_h$ is the $h(\rho_2(\alpha, \beta))$th place that β visits on its walk to α. It is clear that Theorem 18.13 and the choice of h in (I.262) give us the following conclusion.

18.15 Lemma. *Let A be a family of θ pairwise disjoint finite subsets of θ, all of some fixed size n, and let Ω be an unbounded subset of θ. Then almost every* [38] *$\delta \in \Gamma$ has the form $[a(0)\beta]_h = [a(1)\beta]_h = \ldots = [a(n-1)\beta]_h$ for some $a \in A$, $\beta \in \Omega$, $a < \beta$.* $\dashv$

In fact, one can get a projection of this square-bracket operation with seemingly even more complex behavior. Keeping the notation of Theorem 18.13, pick a function $\xi \longmapsto \xi^*$ from θ to ω such that $\{\xi \in \Gamma : \xi^* = n\}$ is stationary for all n. This gives us a way to consider the following projection of the trace function $\mathrm{Tr}^* : [\theta]^2 \longrightarrow \omega^{<\omega}$:

$$\mathrm{Tr}^*(\alpha, \beta) = \langle \min(C_\beta \setminus \alpha)^* \rangle^\frown \mathrm{Tr}^*(\alpha, \min(C_\beta \setminus \alpha)), \tag{I.263}$$

where we stipulate that $\mathrm{Tr}^*(\gamma, \gamma) = \langle \gamma^* \rangle$ for all γ. It is clear that the proof of Theorem 18.13 allows us to add the following conclusions:

18.13* Theorem. *Under the hypothesis of Theorem 18.13, its conclusion can be extended by adding the following two new statements:*

(4) $\mathrm{Tr}^*(\delta_1, \delta_0) = \ldots = \mathrm{Tr}^*(\delta_l, \delta_{l-1})$,

(5) The maximal term of the sequence $\mathrm{Tr}^*(\delta_1, \delta_0) = \ldots = \mathrm{Tr}^*(\delta_l, \delta_{l-1})$ is bigger than the maximal term of any of the sequences $\mathrm{Tr}^*(\delta_0, b(j))$ or $\mathrm{Tr}^*(a(i), \delta_l)$ for $i, j < n$.

$\dashv$

18.16 Definition. For $\alpha < \beta < \theta$, let $[\alpha\beta]^* = \beta_t$ for t the minimal initial part of $\rho_0(\alpha, \beta)$ such that $\beta_t^* = \max(\mathrm{Tr}^*(\alpha, \beta))$.

---

[38] Here 'almost every' is to be interpreted by 'all except a nonstationary set'.

Thus $[\alpha\beta]^*$ is the first place in the walk from β to α where the function $*$ reaches its maximum among all other places visited during the walk. Note that combining the conclusions (1)-(5) of Theorem $18.13^{(*)}$ we get:

18.13^{**} Theorem. *Under the hypothesis of Theorem 18.13, its conclusion can be extended by adding the following:*

(6) $[a(i)b(j)]^* = [\delta_1\delta_0]^*$ for all $i, j < n$.

$\dashv$

Having in mind the property of $[\cdot\cdot]_h$ stated in Lemma 18.15, the following variation is now quite natural.

18.17 Definition. $[\alpha\beta]^*_h = [\alpha[\alpha\beta]^*]_h$ for $\alpha < \beta < \theta$.

Using Theorem $18.13^{(**)}(1)$-(6) one easily gets the following conclusion.

18.18 Lemma. *Let A be a family of θ pairwise disjoint finite subsets of θ, all of some fixed size n. Then for all but nonstationarily many $\delta \in \Gamma$ one can find $a < b$ in A such that $[a(i)b(j)]^*_h = \delta$ for all $i, j < n$.* $\dashv$

18.19 Remark. Composing $[\cdot\cdot]^*_h$ with a mapping $\pi : \theta \longrightarrow \theta$ with the property that $\pi^{-1}(\xi) \cap \Gamma$ is stationary for all $\xi < \theta$, one gets a projection of $[\cdot\cdot]^*_h$ for which the conclusion of Lemma 18.18 is true for *all* $\delta < \kappa$. Assuming that θ is moreover a successor of a regular cardinal κ (of size at least continuum), in which case Γ can be taken to be $\{\delta < \kappa^+ : \operatorname{cf} \delta = \kappa\}$, and proceeding as in 18.11 above we get a projection $[\![\cdot\cdot]\!]^*_h$ with the following property:

18.20 Lemma. *For every family A of pairwise disjoint finite subsets of κ^+ all of some fixed size n and for every $q : n \times n \longrightarrow \kappa^+$ there exist $a < b$ in A such that $[\![a(i)b(j)]\!]^*_h = q(i, j)$ for all $i, j < n$.* $\dashv$

18.21 Remark. The first example of a cardinal with such a complex binary operation was given by the author [79] using the oscillation mapping described above in Section 17. It was the cardinal $\mathfrak{b}$, the minimal cardinality of an unbounded subset of ω^ω under the ordering of eventual dominance. The oscillation mapping restricted to some well-ordered unbounded subset W of ω^ω is perhaps still the most interesting example of this kind due to the fact that its properties are preserved in forcing extensions that do not change the unboundedness of W (although they can collapse cardinals and therefore destroy the properties of the square-bracket operations on them). This absoluteness of osc is the key feature behind its applications in various coding procedures (see, e.g. [85]).

18.22 Theorem. *For every regular cardinal κ of size at least the continuum, the κ^+-chain condition is not productive, i.e., there exist two partially ordered sets $\mathcal{P}_0$ and $\mathcal{P}_1$ satisfying the κ^+-chain condition but their product $\mathcal{P}_0 \times \mathcal{P}_1$ fails to have this property.*

Proof. Fix two disjoint stationary subsets Γ_0 and Γ_1 of $\{\delta < \kappa^+ : \mathrm{cf}\,\delta = \kappa\}$. Let $\mathcal{P}_i$ be the collection of all finite subsets p of κ^+ with the property that $[\alpha\beta]_h^* \in \Gamma_i$ for all $\alpha < \beta$ in p. By Lemma 18.18, $\mathcal{P}_0$ and $\mathcal{P}_1$ are κ^+-cc posets. Their product $\mathcal{P}_0 \times \mathcal{P}_1$, however, contains a family $\langle\{\alpha\}, \{\alpha\}\rangle\,(\alpha < \kappa^+)$ of pairwise incomparable conditions. $\dashv$

18.23 Remark. Theorem 18.22 is due to Shelah [65] who proved it using similar methods. The first ZFC-examples of non-productiveness of the κ^+-chain condition were given by the author in [77] using what is today known under the name 'pcf theory'. After the full development of pcf theory it became apparent that the basic construction from [77] applies to every successor of a singular cardinal [66]. A quite different class of cardinals θ with θ-cc non-productive was given by the author in [76]. For example, $\theta = \mathrm{cf}\,\mathfrak{c}$ is one of these cardinals. For an overview of recent advances in this area, the reader is referred to [53]. The following problem seems still open:

18.24 Question. Suppose that θ is a regular strong limit cardinal and the θ-chain condition is productive. Is θ necessarily a weakly compact cardinal?

19. Unbounded functions on successors of regular cardinals

In this section κ is a regular cardinal and $C_\alpha\,(\alpha < \kappa^+)$ is a fixed sequence with $\mathrm{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$. Define $\rho^* : [\kappa^+]^2 \longrightarrow \kappa$ by

$$
\begin{aligned}
\rho^*(\alpha, \beta) \;=\; \sup\{&\mathrm{tp}(C_\beta \cap \alpha), \rho^*(\alpha, \min(C_\beta \setminus \alpha)), \\
&\rho^*(\xi, \alpha) : \xi \in C_\beta \cap \alpha\},
\end{aligned}
\tag{I.264}
$$

where we stipulate that $\rho^*(\gamma, \gamma) = 0$ for all $\gamma < \kappa^+$. Since $\rho^*(\alpha, \beta) \geq \rho_1(\alpha, \beta)$ for all $\alpha < \beta < \kappa^+$ by Lemma 10.1 we have the following:

19.1 Lemma. *For $\nu < \kappa$, $\alpha < \kappa^+$ the set $P_\nu(\alpha) = \{\xi \leq \alpha : \rho^*(\xi, \alpha) \leq \nu\}$ has size no more than $|\nu| + \aleph_0$.* $\dashv$

The proof of the following subadditivity properties of ρ^* is very similar to the proof of the corresponding fact for the function ρ from Section 14.

19.2 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho^*(\alpha, \gamma) \leq \max\{\rho^*(\alpha, \beta), \rho^*(\beta, \gamma)\}$,

(b) $\rho^*(\alpha, \beta) \leq \max\{\rho^*(\alpha, \gamma), \rho^*(\beta, \gamma)\}$.

$\dashv$

We mention a typical application of this function to the problem of exis-
tance of partial square sequences which, for example, have some applications
in the pcf theory (see [9]).

19.3 Theorem. *For every regular uncountable cardinal $\lambda < \kappa$ and station-
ary $\Gamma \subseteq \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \lambda\}$, there is a stationary set $\Sigma \subseteq \Gamma$ and a
sequence C_α ($\alpha \in \Sigma$) such that:*

(1) *C_α is a closed and unbounded subset of α,*

(2) *$C_\alpha \cap \xi = C_\beta \cap \xi$ for every $\xi \in C_\alpha \cap C_\beta$.*

Proof. For each $\delta \in \Gamma$, choose $\nu = \nu(\delta) < \kappa$ such that the set $P_{<\nu}(\delta) =
\{\xi < \delta : \rho^*(\xi, \delta) < \nu\}$ is unbounded in δ and closed under taking suprema
of sequences of size $< \lambda$. Then there is $\bar{\nu}, \bar{\mu} < \kappa$ and stationary $\Sigma \subseteq \Gamma$ such
that $\nu(\delta) = \bar{\nu}$ and $\mathrm{tp}(P_{<\bar{\nu}}(\delta)) = \bar{\mu}$ for all $\delta \in \Sigma$. Let C be a fixed closed
and unbounded subset of $\bar{\mu}$ of order-type λ. Finally, for $\delta \in \Gamma$ set

$$C_\delta = \{\alpha \in P_{<\bar{\nu}}(\delta) : \mathrm{tp}(P_{<\bar{\nu}}(\alpha)) \in C\}.$$

Using Lemma 19.2, one easily checks that C_α ($\alpha \in \Sigma$) satisfies the conditions
(1) and (2). ⊣

Another application concerns the fact exposed above in Section 16, that
the inequalities 19.2(a),(b) are particularly useful, when κ has cofinality
ω. Also consider the well-known phenomenon first discovered by K.Prikry
(see [37]), that in some cases, the cofinality of a regular cardinal κ can be
changed to ω, while preserving all cardinals.

19.4 Theorem. *In any cardinal-preserving extension of the universe, which
has no new bounded subsets of κ, but in which κ has a cofinal ω-sequence
diagonalizing the filter of closed and unbounded subsets of κ restricted to
the ordinals of cofinality $> \omega$, there is a sequence $C_{\alpha n}(\alpha \in \lim (\kappa^+), n < \omega)$
such that for all $\alpha < \beta$ in $\lim(\kappa^+)$:*

(1) *$C_{\alpha n}$ is a closed subset of α for all n,*

(2) *$C_{\alpha n} \subseteq C_{\alpha m}$, whenever $n \leq m$,*

(3) *$\alpha = \bigcup_{n < \omega} C_{\alpha n}$,*

(4) *$\alpha \in \lim(C_{\beta n})$ implies $C_{\alpha n} = C_{\beta n} \cap \alpha$.*

Proof. For $\alpha < \kappa^+$, let D_α be the collection of all $\nu < \kappa$ for which $P_{<\nu}(\alpha)$
is *σ-closed*, i.e., closed under supremums of all of its countable bounded
subsets. Clearly, D_α contains a closed unbounded subset of κ, restricted to
cofinality $> \omega$ ordinals. Note that

$$\nu \in D_\beta \text{ and } \rho^*(\alpha, \beta) < \nu \text{ imply that } \nu \in D_\alpha. \tag{I.265}$$

In the extended universe, pick a strictly increasing sequence ν_n $(n < \omega)$, which converges to κ and has the property that for each $\alpha < \kappa^+$, there is $n < \omega$ such that $\nu_m \in D_\alpha$ for all $m \geq n$. Let $n(\alpha)$ be the minimal such integer n.

Fix $\alpha \in \lim(\kappa^+)$ and $n < \omega$. If there is $\gamma \geq \alpha$ such that $n \geq n(\gamma)$ and

$$\alpha \in \lim(P_{<\nu_n}(\gamma)) \tag{I.266}$$

let $\gamma(\alpha)$ be the minimal such γ and let

$$C_{\alpha n} = \overline{P_{<\nu_n}(\gamma(\alpha))} \cap \alpha. \tag{I.267}$$

If such a $\gamma \geq \alpha$ does not exist, let $C_{\alpha n} = \emptyset$ for $n < n(\alpha)$ and let

$$C_{\alpha n} = \overline{P_{<\nu_n}(\alpha)} \cap \alpha. \tag{I.268}$$

for $n \geq n(\alpha)$. Clearly, this choice of $C_{\alpha n}$'s satisfies (1),(2) and (3). To verify (4), let α be a limit point of some $C_{\beta n}$.

Suppose first that $\mathrm{cf}(\alpha) = \omega$. If $C_{\beta n}$ was defined according to the case I.267 so is $C_{\alpha n}$. Since $P_{<\nu_n}(\gamma(\beta))$ is σ-closed we conclude that $\alpha \in P_{<\nu_n}(\gamma(\beta))$ which by 19.2(a),(b) means that $P_{<\nu}(\alpha) = P_{<\nu}(\gamma(\beta)) \cap \alpha$ for all $\nu \geq \nu_n$ and therefore $D_{\gamma(\beta)} \setminus \nu_n \subseteq D_\alpha$. It follows that $n(\alpha) \geq n$ and therefore that $\gamma(\alpha) = \alpha$. Hence, $C_{\alpha n} = C_{\beta n} \cap \alpha$. If $C_{\beta n}$ was defined according to the case I.268 then our assumption $\alpha \in \lim(C_{\beta n})$ and the fact that $n \geq n(\beta)$ means that $C_{\alpha n}$ was defined according to I.267. Similarly as above we conclude that actually $\alpha \in P_{<\nu_n}(\beta)$ which gives us $n \geq n(\alpha)$ and $\gamma(\alpha) = \alpha$. It follows that $C_{\alpha n} = C_{\beta n} \cap \alpha$ in this case as well.

Suppose now that $\mathrm{cf}(\alpha) > \omega$. If $C_{\beta n}$ was defined according to I.267 then so is $C_{\alpha n}$ and so $P_{\nu_n}(\gamma(\alpha)) \cap \alpha$ and $P_{\nu_n}(\gamma(\beta)) \cap \alpha$ are two σ-closed and unbounded subsets of α. So their intersection is unbounded in α which by 19.2(a),(b) gives us that $P_{\nu_n}(\gamma(\alpha)) \cap \alpha = P_{\nu_n}(\gamma(\beta)) \cap \alpha$. It follows that $C_{\alpha n} = C_{\beta n} \cap \alpha$ in this case. Suppose now that $C_{\beta n}$ was defined according to I.268. Then $\gamma = \beta$ satisfies $n \geq n(\gamma)$ and I.266. So $C_{\alpha n}$ was defined according to the case I.267. It follows that $P_{\nu_n}(\gamma(\alpha)) \cap \alpha$ and $P_{\nu_n}(\beta) \cap \alpha$ are two σ-closed and unbounded subsets of α so their intersection is unbounded in α. Applying 19.2(a),(b) we get that $P_{\nu_n}(\gamma(\alpha)) \cap \alpha = P_{\nu_n}(\beta) \cap \alpha$. It follows that $C_{\alpha n} = C_{\beta n} \cap \alpha$ in this case as well. $\dashv$

19.5 Remark. The combinatorial principle appearing in the statement of Theorem 19.4 is a member of a family of square principles that has been studied systematically by Schimmerling and others (see e.g. [62]). It is definitely a principle sufficient for all of the applications of $\square_\kappa$ appearing in Section 16 above.

19.6 Definition. A function $f : [\kappa^+]^2 \longrightarrow \kappa$ is *unbounded* if $f''[\Gamma]^2$ is unbounded in κ for every $\Gamma \subseteq \kappa^+$ of size κ. We shall say that such an f is

strongly unbounded if for every family A of size κ^+, consisting of pairwise disjoint finite subsets of κ^+, and every $\nu < \kappa$ there exists $A_0 \subseteq A$ of size κ such that $f(\alpha, \beta) > \nu$ for all $\alpha \in a$, $\beta \in b$ and $a \neq b$ in A_0.

19.7 Lemma. *If $f : [\kappa^+]^2 \longrightarrow \kappa$ is unbounded and subadditive (i.e. it satisfies the two inequalities 19.2(a),(b)), then f is strongly unbounded.*

Proof. For $\alpha < \beta < \kappa^+$, set $\alpha <_\nu \beta$ if and only if $f(\alpha, \beta) \leq \nu$. Then our assumption about f satisfying 19.2(a) and (b) reduces to the fact that each $<_\nu$ is a tree ordering on κ^+ compatible with the usual ordering on κ^+. Note that the unboundedness property of f is preserved by any forcing notion satisfying the κ-chain condition, so in particular no tree $(\kappa^+, <_\nu)$ can contain a Souslin subtree of height κ. In the proof of Lemma 15.5 above we have seen that this property of $(\kappa^+, <_\nu)$ alone is sufficient to conclude that every family A of κ many pairwise disjoint subsets of κ^+ contains a subfamily A_0 of size κ such that for every $a \neq b$ in A_0 every $\alpha \in a$ is $<_\nu$-incomparable to every $\beta \in b$, which is exactly the conclusion of f being strongly unbounded. ⊣

19.8 Lemma. *The following are equivalent:*

(1) *There is a structure $(\kappa^+, \kappa, <, R_n)_{n<\omega}$ with no elementary substructure B of size κ such that $B \cap \kappa$ is bounded in κ.*

(2) *There is an unbounded function $f : [\kappa^+]^2 \longrightarrow \kappa$,*

(3) *There is a strongly unbounded, subadditive function $f : [\kappa^+]^2 \longrightarrow \kappa$.*

Proof. Only the implication from (2) to (3) needs some argument. So let $f : [\kappa^+]^2 \longrightarrow \kappa$ be a given unbounded function. Increasing f we may assume that each of its sections $f_\alpha : \alpha \longrightarrow \kappa$ is one-to-one. For a set $\Gamma \subseteq \kappa^+$, let the f-closure of Γ, $\mathrm{CL}_f(\Gamma)$, be the minimal $\Delta \supseteq \Gamma$ with the property that $f''[\Delta]^2 \subseteq \Delta$ and $f_\alpha^{-1}(\xi) \in \Delta$, whenever $\alpha \in \Delta$ and $\xi < \sup(\Delta \cap \kappa)$. Define $e : [\kappa^+]^2 \longrightarrow \kappa$ by letting

$$e(\alpha, \beta) = \sup(\kappa \cap \mathrm{CL}_f(\rho^*(\alpha, \beta) \cup P_{\rho^*(\alpha,\beta)}(\alpha))). \qquad (\mathrm{I.269})$$

We show the two subadditive properties for e:

$$e(\alpha, \gamma) \leq \max\{e(\alpha, \beta), e(\beta, \gamma)\} \text{ whenever } \alpha \leq \beta \leq \gamma. \qquad (\mathrm{I.270})$$

By 19.2(a) either $\rho^*(\alpha, \gamma) \leq \rho^*(\alpha, \beta)$ in which case $P_{\rho^*(\alpha,\gamma)}(\alpha) \subseteq P_{\rho^*(\alpha,\beta)}(\alpha)$ holds, or $\rho^*(\alpha, \gamma) \leq \rho^*(\beta, \gamma)$ when we have $P_{\rho^*(\alpha,\gamma)}(\alpha) \subseteq P_{\rho^*(\beta,\gamma)}(\beta)$. By the definition of e, the first case gives us the inequality $e(\alpha, \gamma) \leq e(\alpha, \beta)$ and the second gives us $e(\alpha, \gamma) \leq e(\beta, \gamma)$.

$$e(\alpha, \beta) \leq \max\{e(\alpha, \gamma), e(\beta, \gamma)\} \text{ whenever } \alpha \leq \beta \leq \gamma. \qquad (\mathrm{I.271})$$

By 19.2(b) either $\rho^*(\alpha, \beta) \leq \rho^*(\alpha, \gamma)$ in which case $P_{\rho^*(\alpha,\beta)}(\alpha) \subseteq P_{\rho^*(\alpha,\gamma)}(\alpha)$ holds, or $\rho^*(\alpha, \beta) \leq \rho^*(\beta, \gamma)$ when we have $P_{\rho^*(\alpha,\beta)}(\alpha) \subseteq P_{\rho^*(\beta,\gamma)}(\beta)$. In the first case we get $e(\alpha, \beta) \leq e(\alpha, \gamma)$ and in the second $e(\alpha, \beta) \leq e(\beta, \gamma)$.

By Lemma 19.7, to get the full conclusion of (3) it suffices to show that e is unbounded. So let Γ be a given subset of κ^+ of order-type κ and let $\nu < \kappa$ be a given ordinal. Since $e(\alpha, \beta) \geq \rho^*(\alpha, \beta)$ for all $\alpha < \beta < \kappa^+$ we may assume that ρ^* is bounded on Γ, or more precisely, that for some $\bar{\nu} \leq \nu$ and all $\alpha < \beta$ in Γ, $\rho^*(\alpha, \beta) \leq \bar{\nu}$. For $\alpha \in \Gamma$, let $\beta(\alpha)$ be the minimal point of Γ above α. Then there is $\mu \leq \bar{\nu}$ and $\Delta \subseteq \Gamma$ of size κ such that $\rho^*(\alpha, \beta(\alpha)) = \mu$ for all $\alpha \in \Delta$. Note that the sequence of sets $P_\mu(\alpha)$ $(\alpha \in \Delta)$ forms a chain under end-extension. Let Γ^* be the union of this sequence of sets. Since f is unbounded, there exist $\xi < \eta$ in Γ^* such that $f(\xi, \eta) > \nu$. Pick $\alpha \in \Delta$ above η. Then $\xi, \eta \in P_\mu(\alpha)$ and therefore $e(\alpha, \beta(\alpha)) \geq f(\xi, \eta) > \nu$. This finishes the proof. $\dashv$

19.9 Remark. Recall that *Chang's conjecture* is the model-theoretic transfer principle asserting that every structure of the form $(\omega_2, \omega_1, <, \ldots)$ with a countable signature has an uncountable elementary submodel B with the property that $B \cap \omega_1$ is countable. This principle shows up in many considerations including the first two uncountable cardinals ω_1 and ω_2. For example, it is known that it is preserved by ccc forcing extensions, that it holds in the Silver collapse of an ω_1-Erdős cardinal, and that it in turn implies that ω_2 is an ω_1-Erdős cardinal in the core model of Dodd and Jensen (see, e.g. [13],[37]).

19.10 Corollary. *The negation of Chang's conjecture is equivalent to the statement that there exists $e : [\omega_2]^2 \longrightarrow \omega_1$ such that:*

(a) *$e(\alpha, \gamma) \leq \max\{e(\alpha, \beta), e(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$,*

(b) *$e(\alpha, \beta) \leq \max\{e(\alpha, \gamma), e(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$,*

(c) *For every uncountable family A of pairwise disjoint finite subsets of ω_2 and every $\nu < \omega_1$ there exists an uncountable $A_0 \subseteq A$ such that $e(\alpha, \beta) > \nu$ whenever $\alpha \in a$ and $\beta \in b$ for every $a \neq b \in A_0$.*

$\dashv$

19.11 Remark. Note that if a mapping $e : [\omega_2]^2 \longrightarrow \omega_1$ has properties (a),(b) and (c) of Corollary 19.10, then $D_e : [\omega_2]^2 \longrightarrow [\omega_2]^{\aleph_0}$ defined by $D_e\{\alpha, \beta\} = \{\xi \leq \min\{\alpha, \beta\} : e(\xi, \alpha) \leq e\{\alpha, \beta\}\}$ satisfies the weak form of Definition 15.18, where only the conclusion (a) is kept. The full form of Definition 15.18, however, cannot be achieved assuming only the negation of Chang's conjecture. This shows that the function ρ, based on a $\square_{\omega_1}$-sequence, is a considerably deeper object than an $e : [\omega_2]^2 \longrightarrow \omega_1$ satisfying 19.10(a),(b),(c).

Recall that the successor of the continuum is characterized as the minimal cardinal θ with the property that every $f : [\theta]^2 \longrightarrow \omega$ is constant on the square of some infinite set. We shall now see that in slightly weakening the partition property by replacing squares by rectangles one gets a characterization of a quite different sort. To see this, let us use the arrow notation

$$\begin{pmatrix} \theta \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} \omega \\ \omega \end{pmatrix}_\omega^{1,1}$$

to succinctly express the statement that for every map $f : \theta \times \theta \longrightarrow \omega$, there exist infinite sets $A, B \subseteq \theta$ such that f is constant on their product. Let θ_2 be the minimal θ which fails to satisfy this property. Note that $\omega_1 < \theta_2 \leq \mathfrak{c}^+$. The following result shows that θ_2 can have the minimal possible value ω_2, as well as that θ_2 can be considerably smaller than the continuum.

19.12 Theorem. *Chang's conjecture is equivalent to the statement that* $\begin{pmatrix} \omega_2 \\ \omega_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \omega \\ \omega \end{pmatrix}_\omega^{1,1}$ *holds in every ccc forcing extension.*

Proof. We have already noted that Chang's conjecture is preserved by ccc forcing extensions, so in order to prove the direct implication, it suffices to show that Chang's conjecture implies that every $f : \omega_2 \times \omega_2 \longrightarrow \omega$ is constant on the product of two infinite subsets of ω_2. Given such an f, using Chang's conjecture it is possible to find an elementary submodel M of H_{ω_3} such that $f \in M$, $\delta = M \cap \omega_1 \in \omega_1$ and $\Gamma = M \cap \omega_2$ is uncountable. Choose $i \in \omega$ and uncountable $\Delta \subseteq \Gamma$ such that $f(\delta, \beta) = i$ for all $\beta \in \Delta$. Choose $\alpha_0 < \delta$ such that $\Delta_0 = \{\beta \in \Delta : f(\alpha_0, \beta) = i\}$ is uncountable. Such α_0 exists by elementarity of M. Pick $\beta_0 \in \Delta_0$ above ω_1. Note that by the elementarity of M, for every $\beta \in \Delta_0$ there exist unboundedly many $\alpha < \delta$ such that $f(\alpha, \beta_0) = f(\alpha, \beta) = i$. So we can choose $\alpha_1 < \delta$ above α_0 such that the set $\Delta_1 = \{\beta \in \Delta_0 : f(\alpha_1, \beta) = i\}$ is uncountable. Then again for every $\beta \in \Delta_1$ there exist unboundedly many $\alpha < \delta$ such that $f(\alpha, \beta_0) = f(\alpha, \beta_1) = f(\alpha, \beta) = i$, and so on. Continuing in this way, we build two increasing sequences $\{\alpha_n\} \subseteq \delta$ and $\{\beta_n\} \subseteq \Gamma \setminus \omega_1$ such that $f(\alpha_n, \beta_m) = i$ for all $n, m < \omega$.

Assume now that Chang's conjecture fails and fix $e : [\omega_2]^2 \longrightarrow \omega$ as in Corollary 19.10. We shall describe a ccc forcing notion $\mathcal{P}$ which forces an $f : \omega_2 \times \omega_2 \longrightarrow \omega$ which is not constant on the product of any two infinite subsets of ω_2. The conditions of $\mathcal{P}$ are simply maps of the form $p : [D_p]^2 \longrightarrow \omega$ where D_p is some finite subset of ω_2 such that

> $p(\xi, \alpha) \neq p(\xi, \beta)$ whenever $\xi < \alpha < \beta$ belong to D_p and have the property that $e(\xi, \alpha) > e(\alpha, \beta)$.   (I.272)

We order $\mathcal{P}$ by letting p extend q if and only if p extends q as a function and

$$p(\xi, \alpha) \neq p(\xi, \beta) \text{ whenever } \alpha < \beta \text{ belong to } D_q \text{ and } \xi \text{ belongs to } D_p \setminus D_q. \tag{I.273}$$

Let A be a given uncountable subset of $\mathcal{P}$. Refining A, we may assume that D_p $(p \in A)$ forms a Δ-system with root D and that $F_p = e''[D_p]^2$ $(p \in A)$ form an increasing Δ-system on ω_1 with root F. Moreover, we may assume that for every p and q in A, the finite structures they generate together with e are isomorphic via the isomorphism that fixes D and F. By 19.10(c) there is uncountable $A_0 \subseteq A$ such that for all $p, q \in A_0$,

$$e(\alpha, \beta) > \max(F) \text{ whenever } \alpha \in D_p \setminus D \text{ and } \beta \in D_q \setminus D. \tag{I.274}$$

We claim that every two conditions p and q from A_0 are compatible. To see this, extend $p \cup q$ to a mapping $r : [D_p \cup D_q]^2 \longrightarrow \omega$ which is one-to-one on new pairs, avoiding the old values. To see that r is indeed a member of $\mathcal{P}$ we need to check (I.272) for r. So let $\xi < \alpha < \beta$ be three given ordinals from D_r such that $e(\xi, \alpha) > e(\alpha, \beta)$. We have to prove that $r(\xi, \alpha) \neq r(\xi, \beta)$. By the choice of r we may assume that $\{\xi, \alpha\}$ and $\{\xi, \beta\}$ are old pairs, i.e. that they belong to $[D_p]^2 \cup [D_q]^2$. Since p and q both satisfy (I.272), we may consider only the case when (say) $\alpha \in D_p \setminus D_q$, $\beta \in D_q \setminus D_p$ and $\xi \in D$. Recall that $e(\xi, \alpha) > e(\alpha, \beta)$ and the subadditive properties of e (see 19.10(a),(b)) give us the equality $e(\xi, \alpha) = e(\xi, \beta)$, i.e. $e(\xi, \alpha)$ belongs to F contradicting (I.274). To see that r extends p and q, by symmetry it suffices to check that r extends, say, p. So suppose $\xi < \alpha < \beta$ are given such that $\alpha < \beta$ belong to D_p and ξ belongs to $D_q \setminus D_p$. We need to show that $r(\xi, \alpha) \neq r(\xi, \beta)$. By the choice on new pairs, this is automatic if one of the two pairs is new. So we may consider the case when both α and β are in D. Since $r(\xi, \alpha) = q(\xi, \alpha)$ and $r(\xi, \beta) = q(\xi, \beta)$ in this case the inequality $r(\xi, \alpha) \neq r(\xi, \beta)$ would follow from (I.272) for q if we show that $e(\xi, \alpha) > e(\alpha, \beta)$. Otherwise $e(\xi, \alpha)$ would belong to the root F and by the fact that all conditions are isomorphic, $e(\xi(s), \alpha) \in F$ holds for all $s \in A_0$, where $\xi(s) \in D_s$ $(s \in A_0)$ are the copies of ξ. This of course contradicts the property 19.10(c) of e since $\xi(s) \neq \xi(t)$ when $s \neq t$.

Forcing with $\mathcal{P}$ gives us a mapping $g : [\omega_2]^2 \longrightarrow \omega$ with the property that $\{\xi < \alpha : g(\xi, \alpha) = g(\xi, \beta)\}$ is finite for all $\alpha < \beta < \omega_2$. Define $f : \omega_2 \times \omega_2 \longrightarrow \omega$ by letting $f(\alpha, \beta) = 2g\{\alpha, \beta\} + 2$ when $\alpha < \beta$, $f(\alpha, \beta) = 2g\{\alpha, \beta\} + 1$ when $\alpha > \beta$, and $f(\alpha, \beta) = 0$ when $\alpha = \beta$. Then f is not constant on any product of two infinite sets. ⊣

19.13 Remark. The relative size of θ_2 (or its higher-dimensional analogues $\theta_3, \theta_4, \ldots$) in comparison to the sequence of cardinals $\omega_2, \omega_3, \omega_4, \ldots$ is of considerable interest, both in Set Theory and Model Theory (see e.g. [63],

[82], [87]). On the other hand, even the following most simple questions, left open by Theorem 19.12, are still unanswered.

19.14 Question. Can one prove any of the bounds like $\theta_2 \leq \omega_3$, $\theta_3 \leq \omega_4$, $\theta_4 \leq \omega_5$, etc. without appealing to additional axioms?

Note that by Corollary 19.10, Chang's conjecture is equivalent to the statement that within every decomposition of the usual ordering on ω_2 as an increasing chain of tree orderings, one of the trees has an uncountable chain. Is it possible to have decompositions of $\in \restriction \omega_2$ into an increasing ω_1-chain of tree orderings of countable heights? It turns out that the answer to this question is equivalent to a different well-known combinatorial statement about ω_2 rather than Chang's conjecture itself. Recall that $f : [\kappa^+]^2 \longrightarrow \kappa$ is *transitive* if

$$f(\alpha, \gamma) \leq \max\{f(\alpha, \beta), f(\beta, \gamma)\} \text{ whenever } \alpha \leq \beta \leq \gamma. \qquad (\text{I.275})$$

Given a transitive map $f : [\kappa^+]^2 \longrightarrow \kappa$, one defines $\rho_f : [\kappa^+]^2 \longrightarrow \kappa$ recursively on $\alpha \leq \beta < \kappa$ as follows:

$\rho_f(\alpha, \beta)$ is equal to the supremum of:

1. $f(\min(C_\beta \setminus \alpha), \beta)$,
2. $\text{tp}(C_\beta \cap \alpha)$,   (I.276)
3. $\rho_f(\alpha, \min(C_\beta \setminus \alpha))$,
4. $\rho_f(\xi, \alpha)$ $(\xi \in C_\beta \cap \alpha)$.

19.15 Lemma. *For every transitive map $f : [\kappa^+]^2 \longrightarrow \kappa$ the corresponding $\rho_f : [\kappa^+]^2 \longrightarrow \kappa$ has the following properties:*

(a) $\rho_f(\alpha, \gamma) \leq \max\{\rho_f(\alpha, \beta), \rho_f(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$,

(b) $\rho_f(\alpha, \beta) \leq \max\{\rho_f(\alpha, \gamma), \rho_f(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$,

(c) $|\{\xi \leq \alpha : \rho_f(\xi, \alpha) \leq \nu\}| \leq |\nu| + \aleph_0$ for $\nu < \kappa$ and $\alpha < \kappa^+$,

(d) $\rho_f(\alpha, \beta) \geq f(\alpha, \beta)$ for all $\alpha < \beta < \kappa^+$.

Proof. Proofs of (a)-(c) are almost identical to the corresponding proofs for the function ρ itself and so we avoid repetition. The inequality (d) is deduced from (I.276) by using the transitivity of f as follows:

$$\rho_f(\alpha, \beta) \geq \max_{i < n} f(\beta_{i+1}, \beta_i) \geq f(\alpha, \beta),$$

where $\beta = \beta_0 > \beta_1 > \ldots > \beta_{n-1} > \beta_n = \beta$ is the walk from β to α along the given C-sequence. $\dashv$

19.16 Remark. Transitive maps are frequently used combinatorial objects, especially when one works with quotient structures. Adding the extra sub-additivity condition 19.15(b), one obtains a considerably more subtle object which is much less understood.

19.17 Definition. Let $f_\alpha : \kappa \longrightarrow \kappa$ $(\alpha < \kappa^+)$ be a given sequence of functions such that $f_\alpha <^* f_\beta$ whenever $\alpha < \beta$ (i.e. $\{\nu < \kappa : f_\alpha(\nu) \geq f_\beta(\nu)\}$ is bounded in κ whenever $\alpha < \beta$). Then the *corresponding transitive map* $f : [\kappa^+]^2 \longrightarrow \kappa$ is defined by

$$f(\alpha, \beta) = \min\{\mu < \kappa : f_\alpha(\nu) < f_\beta(\nu) \text{ for all } \nu \geq \mu\}. \tag{I.277}$$

Let ρ_f be the corresponding ρ-function that dominates this particular f and for $\nu < \kappa$ let $<_\nu^f$ be the corresponding tree ordering of κ^+, i.e.

$$\alpha <_\nu^f \beta \text{ if and only if } \rho_f(\alpha, \beta) \leq \nu. \tag{I.278}$$

19.18 Lemma. *Suppose $f_\alpha \leq g$ for all $\alpha < \kappa^+$ where $\leq$ is the ordering of everywhere dominance. Then for every $\nu < \kappa$ the tree $(\kappa^+, <_\nu^f)$ has height $\leq g(\nu)$.*

Proof. Let P be a maximal chain of $(\kappa^+, <_\nu^f)$. $f(\alpha, \beta) \leq \rho_f(\alpha, \beta) \leq \nu$ for every $\alpha < \beta$ in P. It follows that $f_\alpha(\nu) < f_\beta(\nu) \leq g(\nu)$ for all $\alpha < \beta$ in P. So P has order-type $\leq g(\nu)$. $\dashv$

Note that if we have a function $g : \kappa \longrightarrow \kappa$ which bounds the sequence f_α $(\alpha < \kappa^+)$ in the ordering $<^*$ of eventual dominance, then the new sequence $\bar{f}_\alpha = \min\{f_\alpha, g\}$ $(\alpha < \kappa^+)$ is still strictly $<^*$-increasing but now bounded by g even in the ordering of everywhere dominance. So this proves the following result of Galvin (see [34],[59]).

19.19 Corollary. *The following two conditions are equivalent for every regular cardinal κ.*

(1) *There is a sequence $f_\alpha : \kappa \longrightarrow \kappa$ $(\alpha < \kappa^+)$ which is strictly increasing and bounded in the ordering of eventual dominance.*

(2) *The usual order-relation of κ^+ can be decomposed into an increasing κ-sequence of tree orderings of heights $< \kappa$.*

            $\dashv$

19.20 Remark. The assertion that every strictly $<^*$-increasing κ^+-sequence of functions from κ to κ is $<^*$-unbounded is strictly weaker than Chang's conjecture and in the literature it is usually referred to as *weak Chang's conjecture*. This statement still has considerable large cardinal strength (see [14]). Also note the following consequence of Corollary 19.19 which can be deduced from Lemmas 19.1 and 19.2 above as well.

19.21 Corollary. *If κ is a regular limit cardinal (e.g. $\kappa = \omega$), then the usual order-relation of κ^+ can be decomposed into an increasing κ-sequence of tree orderings of heights $< \kappa$.* $\dashv$

20. Higher dimensions

The reader must have noticed already that in this Chapter so far, we have only considered functions of the form $f : [\theta]^2 \longrightarrow I$ or equivalently sequences $f_\alpha : \alpha \longrightarrow I$ ($\alpha < \theta$) of one-place functions. To obtain analogous results about functions defined on higher-dimensional cubes $[\theta]^n$ one usually develops some form of *stepping-up procedure* that lifts a function of the form $f : [\theta]^n \longrightarrow I$ to a function of the form $g : [\theta^+]^{n+1} \longrightarrow I$. The basic idea seems quite simple. One starts with a coherent sequence $e_\alpha : \alpha \longrightarrow \theta$ ($\alpha < \theta^+$) of one-to-one mappings and wishes to define $g : [\theta^+]^{n+1} \longrightarrow I$ as follows:

$$g(\alpha_0, \alpha_1, \ldots, \alpha_n) = f(e(\alpha_0, \alpha_n), \ldots, e(\alpha_{n-1}, \alpha_n)). \qquad (\text{I.279})$$

In other words, we use e_{α_n} to send $\{\alpha_0, \ldots, \alpha_{n-1}\}$ to the domain of f and then apply f to the resulting n-tuple. The problem with such a simple-minded definition is that for a typical subset Γ of θ^+, the sequence of restrictions $e_\delta \restriction (\Gamma \cap \delta)$ ($\delta \in \Gamma$) may not cohere, so we cannot produce a subset of θ that would correspond to Γ and on which we would like to apply some property of f. It turns out that the definition (I.279) is basically correct except that we need to replace e_{α_n} by $e_{\tau(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)}$, where $\tau : [\theta^+]^3 \longrightarrow \theta^+$ is defined as follows (see Definition 17.6):

$$\tau(\alpha, \beta, \gamma) = \gamma_t, \text{ where } t = \rho_0(\alpha, \gamma) \cap \rho_0(\beta, \gamma). \qquad (\text{I.280})$$

The function ρ_0 to which (I.280) refers is of course based on some C-sequence C_α ($\alpha < \theta^+$) on θ^+. The following result shows that if the C-sequence is carefully chosen, the function τ will serve as a stepping-up tool.

20.1 Lemma. *Suppose ρ_0 and τ are based on some $\square_\theta$-sequence C_α ($\alpha < \theta^+$) and let κ be a regular uncountable cardinal $\leq \theta$. Then every set $\Gamma \subseteq \theta^+$ of order-type κ contains a cofinal subset Δ such that, if $\varepsilon = \sup(\Gamma) = \sup(\Delta)$, then $\rho_0(\xi, \varepsilon) = \rho_0(\xi, \tau(\alpha, \beta, \gamma))$ for all $\xi < \alpha < \beta < \gamma$ in Δ.*

Proof. For $\delta \in \lim(C_\varepsilon)$, let ξ_δ be the minimal element of Γ above δ. Let ε_δ be the maximal point of the trace $\text{Tr}(\delta, \xi_\delta)$ of the walk from ξ_δ to δ along the $\square_\theta$-sequence C_α ($\alpha < \theta^+$). Thus $\varepsilon_\delta = \delta$ or $\varepsilon_\delta = \min(\text{Tr}(\delta, \xi_\delta) \setminus (\delta + 1))$ and $C_\xi \cap \delta$ is bounded in δ for all $\xi \in \text{Tr}(\delta, \xi_\delta)$ strictly above ε_δ. Let $f(\delta)$ be a member of $C_\varepsilon \cap \delta$ which bounds all these sets. By the Pressing Down Lemma there is a stationary set $\Sigma \subseteq \lim(C_\varepsilon)$ and $\bar{\delta} \in C_\varepsilon$ such that $f(\delta) \leq \bar{\delta}$ for all $\delta \in \Sigma$. Shrinking Σ we may assume that $\xi_\gamma < \delta$ whenever $\gamma < \delta$ are members of Σ. Note that

$$\varepsilon_\gamma = \min(\text{Tr}(\xi_\alpha, \xi_\gamma) \cap \text{Tr}(\xi_\beta, \xi_\gamma)) \text{ for every } \alpha < \beta < \gamma \text{ in } \Sigma. \qquad (\text{I.281})$$

In other words, $\tau(\xi_\alpha, \xi_\beta, \xi_\gamma) = \varepsilon_\gamma$ for all $\alpha < \beta < \gamma$ in Σ. By the definition of ε_δ for $\delta \in \lim(C_\varepsilon)$, δ belongs to C_{ε_δ} and so by the implicit assumption

that nonlimit points of $C_{\varepsilon\delta}$ are successor ordinals, we conclude that δ is a limit point of $C_{\varepsilon\delta}$. Thus the section $(\rho_0)_\delta$ of the function ρ_0 is an initial part of both the section $(\rho_0)_\varepsilon$ and $(\rho_0)_{\varepsilon\delta}$, which is exactly what is needed for the conclusion of Lemma 20.1. $\dashv$

Recall that for a given C-sequence C_α $(\alpha < \theta^+)$ such that $\mathrm{tp}(C_\alpha) \le \theta$ for all $\alpha < \theta^+$, the range of ρ_0 is the collection of all finite sequences of ordinals $< \theta$. It will be convenient to identify $\mathbb{Q}_\theta$ with θ via, for example, Gödel's coding of sequences of ordinals by ordinals. This identification gives us a well-ordering $<_w$ of $\mathbb{Q}_\theta$ of length θ . We use this identification to lift-up of an arbitrary map $f : [\theta]^n \longrightarrow I$ (really, $f : [\mathbb{Q}_\theta]^n \longrightarrow I$) to a map $g : [\theta^+]^{n+1} \longrightarrow I$ by the following formula:

$$g(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n) = f(\rho_0(\alpha_0, \varepsilon), \ldots, \rho_0(\alpha_{n-1}, \varepsilon)), \qquad \text{(I.282)}$$

where $\varepsilon = \tau(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$.

Let us examine how this stepping-up procedure works on a particular example.

20.2 Theorem. *Suppose θ is an arbitrary cardinal for which $\square_\theta$ holds. Suppose further that for some regular $\kappa > \omega$ and integer $n \ge 2$ there is a map $f : [\theta]^n \longrightarrow [[\theta]^{<\kappa}]^{<\kappa}$ such that:*

(1) $A \subseteq \min(a)$ for all $a \in [\theta]^n$ and $A \in f(a)$.

(2) For all $\nu < \kappa$ and $\Gamma \subseteq \theta$ of size κ there exist $a \in [\Gamma]^n$ and $A \in f(a)$ such that $\mathrm{tp}(A) \ge \nu$ and $A \subseteq \Gamma$.

Then θ^+ and κ satisfy the same combinatorial property, but with $n + 1$ in place of n.

Proof. We assume that the domain of f is actually equal to the set $[\mathbb{Q}_\theta]^n$ rather than $[\theta]^n$ and define g by

$$g(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n) = (\rho_0)_\varepsilon^{-1}(f(\rho_0(\alpha_0, \varepsilon), \ldots, \rho_0(\alpha_{n-1}, \varepsilon))), \qquad \text{(I.283)}$$

where $\varepsilon = \tau(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$, and where ρ_0 and τ are based on some fixed $\square_\theta$-sequence. Note that the transformation does not necessarily preserve (1), so we intersect each member of a given $g(a)$ with $\min(a)$ in order to satisfy this condition. To check (2), let $\Gamma \subseteq \theta$ be a given set of size κ. By Lemma 20.1, shrinking Γ we may assume that Γ has order-type κ and that if $\varepsilon = \sup(\Gamma)$, then

$$\rho_0(\xi, \varepsilon) = \rho_0(\xi, \tau(\alpha, \beta, \gamma)) \text{ for all } \alpha < \beta < \gamma \text{ in } \Gamma. \qquad \text{(I.284)}$$

It follows that g restricted to $[\Gamma]^{n+1}$ satisfies the formula

$$g(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n) = (\rho_0)_\varepsilon^{-1}(f(\rho_0(\alpha_0, \varepsilon), \ldots, \rho_0(\alpha_{n-1}, \varepsilon))). \qquad \text{(I.285)}$$

Shrinking Γ further we assume that the mapping $(\rho_0)_\varepsilon$, as a map from the ordered set $(\theta^+, \in)$ into the ordered set $(\mathbb{Q}_\theta, <_w)$ is strictly increasing when restricted to Γ. Given an ordinal $\nu < \kappa$, we apply (2) for f to the set $\Delta = \{\rho_0(\alpha, \varepsilon) : \alpha \in \Gamma\}$ and find $a \in [\Delta]^n$ and $A \in f(a)$ such that $\text{tp}(A) \geq \nu$ and $A \subseteq \Delta$. Let $\{\alpha_0, \dots, \alpha_{n-1}\}$ be the increasing enumeration of the preimage $(\rho_0)_\varepsilon^{-1}(a)$ and pick $\alpha_n \in \Gamma$ above α_{n-1}. Let B be the preimage $(\rho_0)_\varepsilon^{-1}(A)$. Then $B \in g(\alpha_0, \dots, \alpha_{n-1}, \alpha_n)$, $\text{tp}(B) \geq \nu$ and $B \subseteq \Gamma$. This finishes the proof. $\dashv$

20.3 Remark. It should be noted that Velleman [93] was the first to attempt to step-up the combinatorial property appearing in Theorem 20.2 using his version of the gap-2 morass. Unfortunately the stepping up procedure of [93] worked only up to the value $n = 3$ so it remains unclear if the higher-gap morass could be used to reach all the other values of n.

If we apply this stepping-up procedure to the projection $[\![\cdot\cdot]\!]$ of the square-bracket operation defined in 7.14, one obtains analogues of families $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ of Example 7.16 for ω_2 instead of ω_1. To make this precise, we first fix a $\square_{\omega_1}$-sequence C_α $(\alpha < \omega_2)$ and consider the corresponding maps $\rho_0 : [\omega_2]^2 \longrightarrow \omega_1^{<\omega}$ and $\rho : [\omega_2]^2 \longrightarrow \omega_1$. Let $\bar{\rho} : [\omega_2]^2 \longrightarrow \omega_1$ be defined by

$$\bar{\rho}(\alpha, \beta) = 2^{\rho(\alpha,\beta)} \cdot (2 \cdot tp\{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\} + 1).$$

Then for $\alpha < \omega_2$, the mapping $\bar{\rho}_\alpha : \alpha \longrightarrow \omega_1$ that sends ξ to $\bar{\rho}(\xi, \alpha)$ is 1-1 and we have the following coherence property:

$$\bar{\rho}_\alpha = \bar{\rho}_\beta \upharpoonright \alpha \text{ whenever } \alpha \in \lim(C_\beta). \tag{I.286}$$

20.4 Definition. The ternary operation $[\![\cdot\cdot]\!] : [\omega_2]^3 \longrightarrow \omega_2$ is defined as follows:
$$[\![\alpha\beta\gamma]\!] = (\bar{\rho}_{\tau(\alpha,\beta,\gamma)})^{-1}[\![\bar{\rho}_{\tau(\alpha,\beta,\gamma)}(\alpha)\,\bar{\rho}_{\tau(\alpha,\beta,\gamma)}(\beta)]\!].$$

It should be clear that the stepping-up method lifts the property 7.15 of $[\![\cdot\cdot]\!]$ to the following property of the ternary operation $[\![\cdot\cdot]\!]$:

20.5 Lemma. *Let A be a given uncountable family of pairwise disjoint subsets of ω_2, all of some fixed size n. Let $\xi_1, \dots, \xi_n$ be a sequence of ordinals $< \omega_2$ such that for all $i = 1, \dots, n$ and all but countably many $a \in A$ we have that $\xi_i < a(i)$. Then there exist a, b and c in A such that $[\![a(i)b(i)c(i)]\!] = \xi_i$ for $i = 1, \dots, n$.* $\dashv$

As in Example 7.16, let us identify ω_2 and $3 \times [\omega_2]^{<\omega}$ and view $[\![\cdot\cdot]\!]$ as having the range $3 \times [\omega_2]^{<\omega}$ rather than ω_2. Let $[\![\cdot\cdot]\!]_0$ and $[\![\cdot\cdot]\!]_1$ be the two projections of $[\![\cdot\cdot]\!]$. Set

$$\mathcal{G} = \{G \in [\omega_2]^{<\omega} : [\![\alpha\beta\gamma]\!]_0 = 0 \text{ for all } \{\alpha, \beta, \gamma\} \in [G]^3\},$$

$$\mathcal{H} = \{H \in [\omega_2]^{<\omega} : [\![\alpha\beta\gamma]\!]_0 = 1 \text{ for all } \{\alpha, \beta, \gamma\} \in [H]^3\}.$$

Note the following immediate consequence of Lemma 20.5.

20.6 Lemma. *For every uncountable family A of disjoint 2-element subsets of ω_2 there is an arbitrarily large finite subset B of A with $\{b(0) : b \in B\} \in \mathcal{G}$ and $\{b(1) : b \in B\} \in \mathcal{H}$.* ⊣

Recall that the parameter in all our definitions so far is a fixed $\square_{\omega_1}$-sequence, so we also have the corresponding ρ-function at our disposal. We use it to define an analogue of the family $\mathcal{K}$ of Example 7.16 as follows. In this case, let $\mathcal{K}$ be the collection of all finite sets $\{a_i : i < n\}$ of elements of $[\omega_2]^2$ such that for all $i < j < k < n$:

$$a_i < a_j < a_k, \rho(\alpha, \beta) \leq \rho(\beta, \gamma) \text{ holds if } \alpha \in a_i, \beta \in a_j \text{ and } \gamma \in a_k, \quad \text{(I.287)}$$

$$[\![a_i(0)a_j(0)a_k(0)]\!]_0 = [\![a_i(1)a_j(1)a_k(1)]\!]_0 = 2, \quad \text{(I.288)}$$

$$[\![a_i(0)a_j(0)a_k(0)]\!]_1 = [\![a_i(1)a_j(1)a_k(1)]\!]_1 = \bigcup_{l<i} a_l. \quad \text{(I.289)}$$

The following property of $\mathcal{K}$ is a consequence of Lemma 20.5 and the subadditivity of ρ.

20.7 Lemma. *For every sequence a_ξ ($\xi < \omega_1$) of 2-element subsets of ω_2 such that $a_\xi < a_\eta$ whenever $\xi < \eta$ and for every positive integer n there exist $\xi_0 < \xi_1 < \ldots < \xi_{n-1} < \omega_1$ such that $\{a_{\xi_i} : i < n\} \in \mathcal{K}$.* ⊣

The particular definition of $\mathcal{K}$ is made in order to have the following property of $\mathcal{K}$ which is analogous to (I.79) of Section 7:

> If K and L are two distinct members of $\mathcal{K}$, then there are no more than 9 ordinals α such that $\{\alpha, \beta\} \in K$ and $\{\alpha, \gamma\} \in L$ for some $\beta \neq \gamma$. $\quad$ (I.290)

The definitions of $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ immediately lead to the following analogues of (I.77) and (I.78):

> $\mathcal{G}$ and $\mathcal{H}$ contain all singletons, are closed under subsets and are 3-orthogonal to each other. $\quad$ (I.291)

> $\mathcal{G}$ and $\mathcal{H}$ are both 5-orthogonal to the family of all unions of members of $\mathcal{K}$. $\quad$ (I.292)

So we can proceed as in Example 7.16 and for a function x from ω_2 into $\mathbb{R}$, define

$$\|x\|_{\mathcal{H},2} = \sup\{(\sum_{\alpha \in \mathcal{H}} x(\alpha)^2)^{\frac{1}{2}} : H \in \mathcal{H}\},$$

$$\|x\|_{\mathcal{K},2} = \sup\{(\sum_{\{\alpha,\beta\} \in \mathcal{K}} (x(\alpha) - x(\beta))^2)^{\frac{1}{2}} : K \in \mathcal{K}\}.$$

Let $\|\cdot\| = \max\{\|\cdot\|_\infty, \|\cdot\|_{\mathcal{H},2}, \|\cdot\|_{\mathcal{K},2}\}$, let $\bar{E}_2 = \{x : \|x\| < \infty\}$, and let E_2 be the closure of the linear span of $\{1_\alpha : \alpha \in \omega_2\}$ inside $(\bar{E}_2, \|\cdot\|)$. Then as in Example 7.16 one shows, using the properties of $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ listed above in Lemmas 20.6 and 20.7 and in (I.287-I.292), that E_2 is a Banach space with $\{1_\alpha : \alpha \in \omega_2\}$ as its transitive basis and with the property that every bounded linear operator $T : E_2 \longrightarrow E_2$ can be written as $S + D$, where S is an operator with separable range and D is a diagonal operator relative to the basis $\{1_\alpha : \alpha \in \omega_2\}$ with only countably many changes of the constants. The new feature here is the use of (I.287) in checking that the projections on countable sets of the form $\{\alpha < \beta : \rho(\alpha, \beta) < \nu\}$ $(\alpha < \omega_2, \nu < \omega_1)$ are (uniformly) bounded. This is needed among other things in showing, using Lemma 20.6, that T can be written as a sum of one such projection and a diagonal operator relative to $\{1_\alpha : \alpha \in \omega_2\}$.

Using the interpolation method of [11], one can turn E_2 into a reflexive example. Thus, we have the following:

20.8 Example. Assuming $\square_{\omega_1}$, there is a reflexive Banach space E with a transitive basis of type ω_2 with the property that every bounded operator $T : E \longrightarrow E$ can be written as a sum of an operator with a separable range and a diagonal operator (relative to the basis) with only countably many changes of constants. ⊣

20.9 Remark. In [42], P.Koszmider has shown that such a space cannot be constructed on the basis of the usual axioms of set theory. We refer the reader to that paper for more details about these kinds of examples of Banach spaces.

For the rest of this section we shall examine the stepping-up method with less restrictions on the given C-sequence C_α $(\alpha < \theta^+)$ on which it is based.

20.10 Theorem. *The following are equivalent for a regular cardinal θ such that* $\log \theta^+ = \theta$. [39]

(1) *There is a substructure of the form $(\theta^{++}, \theta^+, <, \ldots)$ with no substructure B of size θ^+ with $B \cap \theta^+$ of size θ.*

(2) *There is $f : [\theta^{++}]^3 \longrightarrow \theta^+$ which takes all the possible values on the cube of any subset Γ of θ^{++} of size θ^+.*

---

[39]$\log \kappa = \min\{\lambda : 2^\lambda \geq \kappa\}$.

Proof. To prove the nontrivial direction from (1) to (2), we use Lemma 19.8 and choose a strongly unbounded and subadditive $e : [\theta^{++}]^2 \longrightarrow \theta^+$. We also choose a C-sequence C_α $(\alpha < \theta^+)$ such that $\mathrm{tp}(C_\alpha) \leq \theta$ for all $\alpha < \theta^+$ and consider the corresponding function $\rho^* : [\theta^+]^2 \longrightarrow \theta$ defined above in I.264. Finally we choose a one-to-one sequence r_α $(\alpha < \theta^{++})$ of elements of $\{0,1\}^{\theta^+}$ and consider the corresponding function $\Delta : [\theta^{++}]^2 \longrightarrow \theta^+$:

$$\Delta(\alpha, \beta) = \Delta(r_\alpha, r_\beta) = \min\{\nu : r_\alpha(\nu) \neq r_\beta(\nu)\}. \qquad \text{(I.293)}$$

The definition of $f : [\theta^{++}]^3 \longrightarrow \theta$ is given according to the following two rules applied to a given triple $x = \{\alpha, \beta, \gamma\} \in [\theta^{++}]^3$ $(\alpha < \beta < \gamma)$:

Rule 1: If $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{lex} r_\beta <_{lex} r_\gamma$ or $r_\alpha >_{lex} r_\beta >_{lex} r_\gamma$, let

$$f(\alpha, \beta, \gamma) = \min(P_\nu(\Delta(\beta, \gamma)) \setminus \Delta(\alpha, \beta)),$$

where $\nu = \rho^*(\min\{\xi \leq \Delta(\alpha, \beta) : \rho^*(\xi, \Delta(\alpha, \beta)) \neq \rho^*(\xi, \Delta(\beta, \gamma))\}, \Delta(\beta, \gamma))$.

Rule 2: If $\alpha \in x$ is such that r_α is lexicographically between the other two r_ξ's for $\xi \in x$, if $\beta \in x \setminus \{\alpha\}$ is such that $\Delta(r_\alpha, r_\beta) > \Delta(r_\alpha, r_\gamma)$, where γ is the remaining element of x and if x does not fall under Rule 1, let

$$f(\alpha, \beta, \gamma) = \min(P_\nu(e(\beta, \gamma)) \setminus e(\alpha, \beta)),$$

where $\nu = \rho^*\{\Delta(\alpha, \beta), e(\beta, \gamma)\}$.

The proof of the Theorem is finished once we show the following: for every stationary set Σ of cofinality θ ordinals $< \theta^+$ and every $\Gamma \subseteq \theta^{++}$ of size θ^+ there exist $\alpha < \beta < \gamma$ in Γ such that $f(\alpha, \beta, \gamma) \in \Sigma$. Clearly we may assume that Γ has order-type θ^+. Let $R_\Gamma = \{r_\alpha : \alpha \in \Gamma\}$.

Case 1: R_Γ contains a subset of size θ^+ which is lexicographically well-ordered or conversely well-ordered. Going to a subset of Γ, we may assume that

$$\Delta(r_\alpha, r_\beta) = \Delta(r_\alpha, r_\gamma) \text{ for all } \alpha < \beta < \gamma \text{ in } \Gamma. \qquad \text{(I.294)}$$

It follows that Rule 1 applies in the definition of $f(\alpha, \beta, \gamma)$ for any triple $\alpha < \beta < \gamma$ of elements of Γ. Let $\Delta = \{\Delta(r_\alpha, r_\beta) : \{\alpha, \beta\} \in [\Gamma]^2\}$. Then Δ is a subset of θ^+ of size θ^+ and the range of f on $[\Gamma]^3$ includes the range of $[\cdot\cdot]$ on $[\Delta]^2$, where $[\cdot\cdot]$ is the square-bracket operation on θ^+ defined as follows:

$$[\alpha\beta] = \min(P_\nu(\beta) \setminus \alpha), \qquad \text{(I.295)}$$

where $\nu = \rho^*(\min\{\xi \leq \alpha : \rho^*(\xi, \alpha) \neq \rho^*(\xi, \beta)\}, \beta)$. Note that this is the exact analogue of the square-bracket operation that has been analysed above in Section 7. So exactly as in Section 7, one establishes the following fact.

For $\Delta \subseteq \theta^+$ unbounded, the set $\{[\alpha\beta] : \{\alpha, \beta\} \in [\Delta]^2\}$ contains almost all ordinals $< \theta^+$ of cofinality θ. $\qquad \text{(I.296)}$

Case 2: R_Γ contains no subsets of size θ^+ which are lexicographically well-ordered or conversely well-ordered. For $t \in \{0,1\}^{<\theta^+}$, let $\Gamma_t = \{\alpha \in \Gamma : t \subseteq r_\alpha\}$. Let T be the set of all $t \in \{0,1\}^{<\theta^+}$ for which Γ_t has size θ^+. Shrinking Γ we may assume that T is either a downwards closed subtree of $\{0,1\}^{<\theta^+}$ of size θ and height $\lambda < \theta^+$, or a downwards closed Aronszajn subtree of $\{0,1\}^{<\theta^+}$ of height θ^+. For $\beta \in \Gamma$, let

$$e_\beta''\Gamma^+ = \{e\{\beta,\gamma\} : \gamma \in \Gamma, r_\beta <_{lex} r_\gamma\}. \tag{I.297}$$

Case 2.1: There exist θ^+ many $\beta \in \Gamma$ such that the set $e_\beta''\Gamma^+$ is bounded for all $\beta \in \Gamma$. Choose an elementary submodel M of some large enough structure of the form H_κ such that $\delta = M \cap \theta^+ \in \Sigma$ and M contains all the relevant objects. By the elementarity of M, strong unboundedness of e and the fact that T contains no θ^+-chain, there exist β and γ in $\Gamma \setminus M$ such that $r_\gamma <_{lex} r_\beta$, $\varepsilon = e\{\beta,\gamma\} \geq \delta$ and $\Delta(r_\beta, r_\gamma) < \delta$. Let $\bar{\nu} = \rho^*(\delta, \varepsilon)$. Note that there exist θ many $\xi > \Delta(r_\beta, r_\gamma)$ such that $(r_\beta \restriction \xi)^\frown 0$ belongs to M and $r_\beta(\xi) = 1$. This is clear in the case when T is an Aronszajn tree, by the elementarity of M and the fact that $\text{cf}(\delta) = \theta$. However, this is also true in case T has height $\lambda < \theta^+$ and has size θ, since from our assumption $\log \theta^+ = \theta$ the ordinal λ must have cofinality exactly θ. So by the property 19.1 of ρ^* there is one such ξ such that $\nu = \rho^*(\xi, \varepsilon) \geq \bar{\nu}$. Let $t = (r_\beta \restriction \xi)^\frown 0$. Then $t \in T \cap M$ and so Γ_t is a subset of Γ of size θ^+ which belongs to M. Using the strong unboundedness of e and the elementarity of M, there must be an α in $\Gamma_t \cap M$ such that $e(\alpha, \beta) > \sup(P_\nu(\varepsilon) \cap \delta)$. Since $e(\alpha, \beta)$ belongs to $e_\alpha''\Gamma^+$, a set which, by the assumption of Case 2.1, is bounded in θ^+, we conclude that $e(\alpha, \beta) < \delta$. It follows that $f(\alpha, \beta, \gamma)$ is defined according to Rule 2, and therefore

$$f(\alpha, \beta, \gamma) = \min(P_\nu(\varepsilon) \setminus e(\alpha, \beta)) = \delta. \tag{I.298}$$

Case 2.2: For every β in some tail of Γ the set $e_\beta''\Gamma^+$ is unbounded in θ^+. Going to a tail, we assume that this is true for all $\beta \in \Gamma$. Let M and $\delta = M \cap \theta^+ \in \Sigma$ be chosen as before. If T is Aronszajn, using elementarity of M, there must be $\beta \in \Gamma \cap M$ and $\bar{\delta} < \delta$ of cofinality θ such that there exist unboundedly many ξ below $\bar{\delta}$ such that the set

$$\Omega_\xi = \{e\{\beta,\chi\} : \chi \in \Gamma, r_\beta <_{lex} r_\chi, \Delta(r_\beta, r_\chi) = \xi\} \tag{I.299}$$

is unbounded in θ^+. If T is a tree of height $\lambda < \theta^+$, $\bar{\delta} = \lambda$ will satisfy this for all but θ many $\beta \in \Gamma$, so in particular, we can pick one such $\beta \in \Gamma \cap M$. Choose $\gamma \in \Gamma$ such that $r_\beta <_{lex} r_\gamma$, $\varepsilon = e(\beta, \gamma) \geq \delta$ and $\Delta(r_\beta, r_\gamma) < \bar{\delta}$. Let $\bar{\nu} = \rho^*(\delta, \varepsilon)$. By the property 19.1 of ρ^*, there is $\bar{\mu} < \bar{\delta}$ such that $\rho^*(\xi, \varepsilon) \geq \bar{\nu}$ for all $\xi \in [\bar{\mu}, \bar{\delta})$. Choose $\xi \in [\bar{\mu}, \bar{\delta})$ above $\Delta(r_\beta, r_\gamma)$ such that the set Ω_ξ is unbounded in θ^+. Let $\nu = \rho^*(\xi, \varepsilon)$. Then $\nu \geq \bar{\nu}$. The set $P_\nu(\varepsilon) \cap \delta$ being of size $< \theta$ is bounded in δ. Since $\Omega_\xi \in M$, there is $\alpha \in \Gamma \cap M$ above β

such that $\Delta(r_\beta, r_\alpha) = \xi$, $r_\beta <_{lex} r_\alpha$ and $e(\beta, \alpha) > \sup(P_\nu(\varepsilon) \cap \delta)$. Note that $r_\alpha <_{lex} r_\gamma$, so the definition of $f\{\alpha, \beta, \gamma\}$ obeys Rule 2. Applying this rule, we get

$$f\{\alpha, \beta, \gamma\} = \min(P_\nu(\varepsilon) \setminus e(\beta, \alpha)) = \delta. \tag{I.300}$$

This finishes the proof of Theorem 20.10. $\dashv$

20.11 Theorem. *If θ is a regular strong limit cardinal carrying a nonreflecting stationary set, then there is $f : [\theta^+]^3 \longrightarrow \theta$ which takes all the values from θ on the cube of any subset of θ^+ of size θ.*

Proof. This is really a corollary of the proof of Theorem 20.10, so let us only indicate the adjustments. By Corollary 19.21 and Lemma 19.7, we can choose a strongly unbounded subadditive map $e : [\theta^+]^2 \longrightarrow \theta$. By the assumption about θ we can choose a C-sequence C_α $(\alpha < \theta)$ avoiding a stationary set $\Sigma \subseteq \theta$ and consider the corresponding notion of a walk, trace, ρ_0-function and the square-bracket operation $[\cdot\cdot]$ as defined in (I.250) above. As in the proof of Theorem 20.10, we choose a one-to-one sequence r_α $(\alpha < \theta^+)$ of elements of $\{0, 1\}^\theta$ and consider the corresponding function $\Delta : [\theta^+]^2 \longrightarrow \theta$:

$$\Delta(\alpha, \beta) = \Delta(r_\alpha, r_\beta) = \min\{\nu < \theta : r_\alpha(\nu) \neq r_\beta(\nu)\}. \tag{I.301}$$

The definition of $f : [\theta^+]^3 \longrightarrow \theta$ is given according to the following rules, applied to a given $x \in [\theta^+]^3$.

Rule 1: If $x = \{\alpha < \beta < \gamma\}$, $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{lex} r_\beta <_{lex} r_\gamma$, or $r_\alpha >_{lex} r_\beta >_{lex} r_\gamma$, let

$$f\{\alpha, \beta, \gamma\} = [\Delta(\alpha, \beta)\Delta(\beta, \gamma)].$$

Rule 2: If $\alpha \in x$ is such that r_α is lexicographically between the other two r_ξ's for $\xi \in x$, if $\beta \in x \setminus \{\alpha\}$ is such that $\Delta(r_\alpha, r_\beta) > \Delta(r_\alpha, r_\gamma)$, where γ is the remaining element of x, and they do not satisfy the conditions of Rule 1, set

$$f\{\alpha, \beta, \gamma\} = \min(\mathrm{Tr}(\Delta(\alpha, \beta), e\{\beta, \gamma\}) \setminus e\{\alpha, \beta\}),$$

i.e. $f\{\alpha, \beta, \gamma\}$ is the minimal point on the trace of the walk from $e\{\beta, \gamma\}$ to $\Delta(\alpha, \beta)$ above the ordinal $e\{\alpha, \beta\}$; if such a point does not exist, set $f\{\alpha, \beta, \gamma\} = 0$.

The proof of the Theorem is finished once we show that for every stationary $\Omega \subseteq \Sigma$ and every $\Gamma \subseteq \theta^+$ of size θ, there exist $x \in [\Gamma]^3$ such that $f(x) \in \Omega$. This is done, as in the proof of Theorem 20.10, by considering the two cases. Case 1 is reduced to the property 18.2 of the square-bracket operation. The treatment of Case 2 is similar and a bit simpler than in the proof of Theorem 20.10, since the case that T is of height $< \theta$ is impossible due to our assumption that θ is a strong limit cardinal. This finishes the proof. $\dashv$

Since $\log \omega_1 = \omega$, the following is an immediate consequence of Theorem 20.10.

20.12 Theorem. *Chang's conjecture is equivalent to the statement that for every $f : [\omega_2]^3 \longrightarrow \omega_1$ there is uncountable $\Gamma \subseteq \omega_2$ such that $f''[\Gamma]^3 \neq \omega_1$.*

$$\dashv$$

20.13 Remark. Since this same statement is stronger for functions from higher dimensional cubes $[\omega_2]^n$ into ω_1 the Theorem 20.12 shows that they are all equivalent to Chang's conjecture. Note also that $n = 3$ is the minimal dimension for which this equivalence holds, since the case $n = 2$ follows from the Continuum Hypothesis, which has no relationship to Chang's conjecture.

For the rest of this section we examine the stepping-up procedure without the assumption that some form of Chang's conjecture is false. So let θ be a given regular uncountable cardinal and let C_α $(\alpha < \theta^+)$ be a fixed C-sequence such that $\mathrm{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \theta^+$. Let $\rho^* : [\theta^+]^2 \longrightarrow \theta$ be the ρ^*-function defined above in (I.264). Recall that, in case C_α $(\alpha < \theta^+)$ is a $\square_\theta$-sequence, the key to our stepping-up procedure was the function $\tau : [\theta^+]^3 \longrightarrow \theta^+$ defined by the formula (I.280). Without the assumption of C_α $(\alpha < \theta^+)$ being a $\square_\theta$-sequence, the following related function turns out to be a good substitute: $\chi : [\theta^+]^3 \longrightarrow \omega$ defined by

$$\chi(\alpha, \beta, \gamma) = |\rho_0(\alpha, \gamma) \cap \rho_0(\beta, \gamma)|. \tag{I.302}$$

Thus $\chi(\alpha, \beta, \gamma)$ is equal to the length of the common part of the walks $\gamma \to \alpha$ and $\gamma \to \beta$.

20.14 Definition. A subset Γ of θ^+ is *stable* if χ is bounded on $[\Gamma]^3$.

20.15 Lemma. *Suppose that Γ is a stable subset of θ^+ of size θ. Then $\{\rho^*(\alpha, \beta) : \{\alpha, \beta\} \in [\Omega]^2\}$ is unbounded in θ for every $\Omega \subseteq \Gamma$ of size θ.*

Proof. Suppose ρ^* is bounded by ν on $[\Omega]^2$ for some $\Omega \subseteq \theta^+$ of size θ. We need to show that χ is unbounded on $[\Omega]^3$. Therefore we may assume that Ω is of order-type θ and let $\lambda = \sup(\Omega)$. Construct a sequence $\Omega = \Omega_0 \supseteq \Omega_1 \supseteq \dots$ of subsets of Ω of order-type θ such that:

$$\chi(\alpha, \beta, \gamma) \geq n \text{ for all } \{\alpha, \beta, \gamma\} \in [\Omega_n]^3. \tag{I.303}$$

Given Ω_n, choose an $\in$-chain $\mathcal{M}$ of length θ of elementary submodels M of some large enough structure of the form H_κ such that $M \cap \theta \in \theta$ and M contains all the relevant objects. Now choose another elementary submodel N of H_κ which contains $\mathcal{M}$ as well as all the other relevant objects such that $N \cap \theta \in \theta$. Let $\lambda_N = \sup(N \cap \lambda)$. Choose $\gamma \in \Omega_n$ above λ_N. Let $\gamma = \gamma_0 > \dots > \gamma_k = \lambda_N$ be the walk from γ to λ_N. Let $\bar\gamma$ be the γ_i with

the smallest index such that $C_{\gamma_i} \cap \lambda_N$ is unbounded in λ_N. Then $\bar{\gamma}$ is either equal to λ_N or γ_{k-1}. Choose $\lambda_0 \in N \cap \lambda$ such that:

$$\lambda_0 > \max(C_{\gamma_i} \cap \lambda_N) \text{ for all } i < k \text{ for which } C_{\gamma_i} \cap \lambda_N \text{ is bounded in } \lambda_N. \tag{I.304}$$

Then $\{\gamma_i : i < k\} \setminus \bar{\gamma} \subseteq \mathrm{Tr}(\alpha, \gamma)$ for all $\alpha \in N \cap \Omega_n$ above λ_0. So if we choose $\alpha < \beta$ in $N \cap \Omega_0$ such that $C_{\bar{\gamma}} \cap [\alpha, \beta) \neq \emptyset$, then

$$\mathrm{Tr}(\alpha, \gamma) \cap \mathrm{Tr}(\beta, \gamma) = \{\gamma_i : i \leq k\} \setminus \bar{\gamma}. \tag{I.305}$$

From this and our assumption $\chi(\alpha, \beta, \gamma) \geq n$ we conclude that this set has size at least n. For $M \in \mathcal{M}$, let $\lambda_M = \sup(M \cap \lambda)$ and $C = \{\lambda_M : M \in \mathcal{M}\}$. Then C is a closed and unbounded subset of λ of order-type θ. By our assumption that ρ^* is bounded by ν on $[\Omega]^2$, we get:

$$
\begin{aligned}
\mathrm{tp}(C_{\bar{\gamma}} \cap \lambda_N) &\leq \sup\{\mathrm{tp}(C_{\bar{\gamma}} \cap \alpha) : \alpha \in \Omega \cap [\lambda_0, \lambda_N)\} \\
&\leq \sup\{\rho^*(\alpha, \gamma) : \alpha \in \Omega \cap [\lambda_0, \lambda_N)\} \\
&\leq \nu.
\end{aligned}
\tag{I.306}
$$

Since $C \cap [\lambda_0, \lambda_N)$ has order-type $> \nu$ there must be $M \in \mathcal{M} \cap N$ such that $\lambda_M \notin C_{\bar{\gamma}}$ and $\lambda_M > \lambda_0$. Thus, we have an elementary submodel M of H_κ which contains all the relevant objects and a $\gamma \in \Omega_n$ above λ_M such that the walk $\gamma \to \lambda_M$ contains all the points from the set $\{\gamma_i : i < k\} \setminus \bar{\gamma}$, but includes at least the point $\min(C_{\bar{\gamma}} \setminus \lambda_M)$ which is strictly above λ_M. Hence, if $\gamma = \gamma_0 > \gamma_1 > \ldots > \gamma_l = \lambda_M$ is the trace of $\gamma \to \lambda_M$ and if $\hat{\gamma}$ is the γ_i $(i \leq l)$ with the smallest index, subject to the requirement that $C_{\hat{\gamma}} \cap \lambda_M$ is unbounded in λ_M, then $\{\gamma_i : i < l\} \setminus \hat{\gamma}$ has size at least $n + 1$. Let $\hat{\lambda}_0 \in \lambda \cap M$ be such that:

$$\hat{\lambda}_0 > \max(C_{\gamma_i} \cap \lambda_M) \text{ for all } i < l \text{ for which } C_{\gamma_i} \cap \lambda_M \text{ is bounded in } \lambda_M. \tag{I.307}$$

Then $\{\gamma_i : i < l\} \setminus \bar{\gamma} \subseteq \mathrm{Tr}(\alpha, \gamma)$ for all $\alpha \in M \cap \Omega_n$ above $\hat{\lambda}_0$. Using the elementarity of M we conclude that for every $\delta < \lambda$ there exist $\varepsilon \in \Omega_n \setminus \delta$ and a set $a_\varepsilon \subseteq [\delta, \varepsilon)$ of size at least $n + 1$ such that $a_\varepsilon \subseteq \mathrm{Tr}(\alpha, \varepsilon)$ for all $\alpha \in \Omega_n \cap [\hat{\lambda}_0, \delta)$. Hence we can choose a closed and unbounded set D of $[\hat{\lambda}_0, \lambda)$ of order-type θ, for each $\delta \in D$ an $\varepsilon(\delta) \in \Omega_n \setminus \delta$ and an $a_{\varepsilon(\delta)} \subseteq [\delta, \varepsilon(\delta))$ such that for every $\delta \in D$:

$$a_{\varepsilon(\delta)} \subseteq \mathrm{Tr}(\alpha, \varepsilon(\delta)) \text{ for } \alpha \in [\hat{\lambda}_0, \delta), \tag{I.308}$$

$$\varepsilon(\delta) < \min(D \setminus (\delta + 1)). \tag{I.309}$$

Finally let $\Omega_{n+1} = \{\varepsilon(\delta) : \delta \in D\}$. Then

$$a_{\varepsilon(\gamma)} \subseteq \operatorname{Tr}(\varepsilon(\alpha), \varepsilon(\gamma)) \cap \operatorname{Tr}(\varepsilon(\beta), \varepsilon(\gamma)) \text{ for all } \alpha < \beta < \gamma \text{ in } \Delta. \qquad \text{(I.310)}$$

This gives us that $\chi(\alpha, \beta, \gamma) \geq n + 1$ for every triple $\alpha < \beta < \gamma$ in Ω_{n+1}, finishing the inductive step and therefore the proof of Lemma 20.15. $\dashv$

20.16 Definition. The 3-*dimensional version of the oscillation mapping*, osc : $[\theta^+]^3 \longrightarrow \omega$, is defined on the basis of the 2-dimensional version of Section 17 as follows

$$\operatorname{osc}(\alpha, \beta, \gamma) = \operatorname{osc}(C_{\beta_s} \setminus \alpha, C_{\gamma_t} \setminus \alpha),$$

where $s = \rho_0(\alpha, \beta) \restriction \chi(\alpha, \beta, \gamma)$ and $t = \rho_0(\alpha, \gamma) \restriction \chi(\alpha, \beta, \gamma)$.

In other words, we let n be the length of the common part of the two walks $\gamma \to \alpha$ and $\gamma \to \beta$. Then we consider the walks $\gamma = \gamma_0 > \ldots > \gamma_k = \alpha$ and $\beta = \beta_0 > \ldots > \beta_l = \alpha$ from γ to α and β to α respectively; if both k and l are bigger than n (and note that $k \geq n$), i.e. if γ_n and β_n are both defined, we let $\operatorname{osc}(\alpha, \beta, \gamma)$ be equal to the oscillation of the two sets $C_{\beta_n} \setminus \alpha$ and $C_{\gamma_n} \setminus \alpha$. If not, we let $\operatorname{osc}(\alpha, \beta, \gamma) = 0$.

20.17 Lemma. *Suppose that Γ is a subset of θ^+ of size κ, a regular uncountable cardinal, and that every subset of Γ of size κ is unstable. Then for every positive integer n, there exist $\alpha < \beta < \gamma$ in Γ such that $\operatorname{osc}(\alpha, \beta, \gamma) = n$.*

Proof. We may assume that Γ is actually of order-type κ and let $\lambda = \sup(\Gamma)$. Let $\mathcal{M}$ be an $\in$-chain of length κ of elementary submodels M of $H_{\theta^{++}}$ such that $M \cap \kappa \in \kappa$ and M contains all the relevant objects. For $M \in \mathcal{M}$, let $\lambda_M = \sup(\lambda \cap M)$. Now choose another elementary submodel N of $H_{\theta^{++}}$ containing $\mathcal{M}$ as well as all the other relevant objects with $N \cap \kappa \in \kappa$. Let $\lambda_N = \sup(\lambda \cap N)$ and choose $\gamma \in \Gamma$ above λ_N and let $\gamma = \gamma_0 > \ldots > \gamma_k = \lambda_N$ be the walk from γ to λ_N. Let $\bar{\gamma}$ and λ_0 be chosen as above in (I.304). Then $\bar{\gamma} = \gamma_k$ or $\bar{\gamma} = \gamma_{k-1}$ and

$$\operatorname{Tr}(\alpha, \gamma) \cap \operatorname{Tr}(\beta, \gamma) = \{\gamma_i : i \leq k\} \setminus \bar{\gamma} \text{ for all } \alpha < \beta \text{ in } \Gamma \cap [\lambda_0, \lambda_N) \text{ such that } C_{\bar{\gamma}} \cap [\alpha, \beta) \neq \emptyset. \qquad \text{(I.311)}$$

So if a tail of the set $\{\lambda_M : M \in \mathcal{M} \cap N\}$ is included in $C_{\bar{\gamma}}$, then using the elementarity of N, we would be able to construct a cofinal subset $\Gamma_0 \subseteq \Gamma$ such that χ is constantly equal to $\bar{k}$ on $[\Gamma_0]^3$, where $\bar{k}$ is the place of $\bar{\gamma}$ in the sequence $\gamma_0, \ldots, \gamma_k$. This, of course, would contradict our assumption that Γ contains no large stable subsets. It follows that the set of all λ_N for $M \in \mathcal{M} \cap N$ which do not belong to $C_{\bar{\gamma}}$ is cofinal in λ_N. So we can pick $M_1 \in M_2 \in \ldots \in M_{n+1}$ in $\mathcal{M} \cap N$ such that, if $\lambda_i = \lambda_{M_i}$ $(1 \leq i \leq n+1)$, then:

$$\lambda_i \in [\lambda_0, \lambda) \setminus C_{\bar{\gamma}} \text{ for all } 1 \leq i \leq n+1, \qquad \text{(I.312)}$$

$$[\lambda_i, \lambda_{i+1}) \cap C_{\bar{\gamma}} \neq \emptyset \text{ for all } 1 \leq i \leq n. \tag{I.313}$$

Fix $\alpha \in \Gamma \cap M_1$ above $\max(C_{\bar{\gamma}} \cap \lambda_1)$. For $1 \leq i \leq n$, pick $\bar{\lambda}_i \in [\lambda_i, \lambda_{i+1}) \cap M_{i+1}$ such that $\bar{\lambda}_i \geq \max(C_{\bar{\gamma}} \cap \lambda_{i+1})$. For $1 \leq i \leq n$, let $\bar{I}_i = [\lambda_i, \bar{\lambda}_i]$, and let $\bar{I}_{n+1} = [\lambda_n, \gamma]$. Now set $\mathcal{F}$ to be the collection of all increasing sequences $I_1 < I_2 < \ldots < I_{n+1}$ of closed intervals of ordinals $< \lambda$ with the property that there is $\beta = \beta(\vec{I})$ in $\Gamma \cap I_{n+1}$ such that, if $\beta = \beta_0 > \ldots > \beta_l = \alpha$ is the walk from β to α, then:

$$l > \bar{k} \text{ and } \beta_{\bar{k}} = \min(\mathrm{Tr}(\alpha, \beta) \cap I_{n+1}), \tag{I.314}$$

$$C_{\beta_{\bar{k}}} \setminus \alpha \subseteq \bigcup_{i=1}^{n+1} I_i, \tag{I.315}$$

$$C_{\beta_{\bar{k}}} \cap I_i \neq \emptyset \text{ for all } 1 \leq n \leq n+1. \tag{I.316}$$

Clearly, $\mathcal{F} \in M_1$ and $\langle \bar{I}_1, \ldots, \bar{I}_{n+1} \rangle \in \mathcal{F}$, as γ is a witness of this. So working as in the proof of Lemma 17.2, we can find $\mathcal{F}_0 \subseteq \mathcal{F}$ in M_1 such that, if some sequence $\vec{J}$ of length $\leq n$ (including the case $\vec{J} = \emptyset$) end-extends to a member of $\mathcal{F}_0$, then for every $\xi < \lambda$ there is a closed interval K included in $[\xi, \lambda)$ such that the concatenation $\vec{J}^\frown K$ also end-extends to a sequence in $\mathcal{F}_0$. Working inductively on $1 \leq i \leq n+1$ we can select a sequence $I_1, I_2, \ldots, I_n$ such that:

$$I_1, \ldots, I_i \in M_{i+1} \text{ for } 1 \leq i \leq n, \tag{I.317}$$

$$I_i \subseteq [\bar{\lambda}_i, \lambda_{i+1}) \text{ for } 1 \leq i \leq n. \tag{I.318}$$

$$\langle I_1, \ldots, I_i \rangle \text{ end-extends to a member of } \mathcal{F}_0. \tag{I.319}$$

Clearly, there is no problem in choosing these objects, using the elementarity of the models M_i ($1 \leq i \leq n+1$) and the splitting property of $\mathcal{F}_0$. Working in M_{n+1}, we find an interval I_{n+1} such that $\langle I_0, \ldots, I_n, I_{n+1} \rangle$ belongs to $\mathcal{F}_0$. Let $\beta = \beta(\langle I_0, \ldots, I_{n+1} \rangle)$. Since $C_{\bar{\gamma}}$ intersects the interval $[\lambda_1, \bar{\lambda}_1)$ it separates α from β, so by (I.311) we get that $\tau(\alpha, \beta, \gamma) = \bar{k}$, hence by definition

$$\mathrm{osc}(\alpha, \beta, \gamma) = \mathrm{osc}(C_{\beta_{\bar{k}}} \setminus \alpha, C_{\gamma_{\bar{k}}} \setminus \alpha). \tag{I.320}$$

Recall that $\gamma_{\bar{k}} = \bar{\gamma}$, so referring to (I.313),(I.315) and (I.316) we conclude that $I_1 \cap C_{\beta_{\bar{k}}}, \ldots, I_{n-1} \cap C_{\beta_{\bar{k}}}$ and $(I_n \cup I_{n+1}) \cap C_{\beta_{\bar{k}}}$ are the n convex pieces in which the set $C_{\bar{\gamma}} \setminus \alpha$ splits the set $C_{\beta_{\bar{k}}} \setminus \alpha$. Hence $\mathrm{osc}(C_{\beta_{\bar{k}}} \setminus \alpha, C_{\gamma_{\bar{k}}} \setminus \alpha) = n$. This completes the proof of Lemma 20.17. $\dashv$

Applying the last two Lemmas to the subsets of θ^+ of size θ, we get an interesting dichotomy:

20.18 Lemma. *Every $\Gamma \subseteq \theta^+$ of size θ can be refined to a subset Ω of size θ such that either:*

(1) ρ^* is unbounded and therefore strongly unbounded on Ω, or

(2) the oscillation mapping takes all its possible values on the cube of Ω.

$$\dashv$$

We finish the section with a typical application of this dichotomy.

20.19 Theorem. *Suppose θ is a regular cardinal such that $\log \theta^+ = \theta$. Then there is $f : [\theta^{++}]^3 \longrightarrow \omega$ which takes all the values from ω on the cube of any subset of θ^{++} of size θ^+.*

Proof. We choose two C-sequences C_α $(\alpha < \theta^+)$ and C_α^+ $(\alpha < \theta^{++})$ on θ^+ and θ^{++} respectively, such that $\mathrm{tp}(C_\alpha) \leq \theta$ for all $\alpha < \theta^+$ and $\mathrm{tp}(C_\alpha^+) \leq \theta^+$ for all $\alpha < \theta^{++}$. Let $\rho^* : [\theta^+]^2 \longrightarrow \theta$ and $\rho^{*+} : [\theta^{++}]^2 \longrightarrow \theta^+$ be the corresponding ρ^*-functions defined above in I.264. Also choose a one-to-one sequence r_α $(\alpha < \theta^{++})$ of elements of $\{0,1\}^{\theta^+}$ and consider the corresponding function $\Delta : [\theta^{++}]^2 \longrightarrow \theta^+$ defined in (I.293). We define $f : [\theta^{++}]^3 \longrightarrow \theta^+$ according to the following two cases for a given triple $\alpha < \beta < \gamma$ of elements of θ^{++}.

Case 1: $(C_{\beta_s} \cap C_{\gamma_t}) \setminus \alpha \neq \emptyset$, where $s = \rho_0(\alpha, \beta) \restriction \chi(\alpha, \beta, \gamma)$ and $t = \rho_0(\alpha, \gamma) \restriction \chi(\alpha, \beta, \gamma)$ of course assuming that $\rho_0(\alpha, \beta)$ has length at least $\chi(\alpha, \beta, \gamma)$.

Rule 1: If $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{lex} r_\beta <_{lex} r_\gamma$ or $r_\alpha >_{lex} r_\beta >_{lex} r_\gamma$, set

$$f(\alpha, \beta, \gamma) = \min(P_\nu(\Delta(\beta, \gamma)) \setminus \Delta(\alpha, \beta)),$$

where $\nu = \rho^*(\min\{\xi \leq \Delta(\alpha, \beta) : \rho^*(\xi, \Delta(\alpha, \beta)) \neq \rho^*(\xi, \Delta(\beta, \gamma))\}, \Delta(\beta, \gamma))$.

Rule 2: If $\bar\alpha \in \{\alpha, \beta, \gamma\}$ is such that $r_{\bar\alpha}$ is lexicographically between the other two r_ξ's for $\xi \in \{\alpha, \beta, \gamma\}$, if $\bar\beta \in \{\alpha, \beta, \gamma\} \setminus \{\bar\alpha\}$ is such that $\Delta(r_{\bar\alpha}, r_{\bar\beta}) > \Delta(r_{\bar\alpha}, r_{\bar\gamma})$, where $\bar\gamma$ is the remaining member of $\{\alpha, \beta, \gamma\}$, and if $\{\alpha, \beta, \gamma\}$ does not fall under Rule 1, let

$$f(\alpha, \beta, \gamma) = \min(P_\nu(\rho^{*+}\{\bar\beta, \bar\gamma\}) \setminus \rho^{*+}(\alpha, \beta)),$$

where $\nu = \rho^*\{\Delta(\alpha, \beta), \rho^{*+}(\beta, \gamma)\}$.

Case 2: $(C_{\beta_s} \cap C_{\gamma_t}) \setminus \alpha = \emptyset$, where $s = \rho_0(\alpha, \beta) \restriction \chi(\alpha, \beta, \gamma)$ and $t = \rho_0(\alpha, \gamma) \restriction \chi(\alpha, \beta, \gamma)$ of course assuming that $\rho_0(\alpha, \beta)$ has length at least $\chi(\alpha, \beta, \gamma)$. Let
$$f(\alpha, \beta, \gamma) = \mathrm{osc}(\alpha, \beta, \gamma).$$

If a given triple $\alpha < \beta < \gamma$ does not fall into one of these two cases, let $f(\alpha, \beta, \gamma) = 0$.

The proof of Theorem 20.19 is finished if we show that for every $\Gamma \subseteq \theta^{++}$ of size θ^+, the image $f''[\Gamma]^3$ either contains all positive integers or almost all ordinals $< \theta^+$ of cofinality θ.

So let Γ be a given subset of θ^{++} of order-type θ^+ and let $\lambda = \sup(\Gamma)$. By the proof of Lemma 20.15, if Γ is stable, then it must contain a (stable) subset Ω such that for some $n < \omega$ and all $\alpha < \beta < \gamma$ in Ω:

$$\chi(\alpha, \beta, \gamma) = n, \tag{I.321}$$

$$(C_{\beta_s} \cap C_{\gamma_t}) \setminus \alpha \neq \emptyset \text{ where } s = \rho_0(\alpha, \beta) \restriction n \text{ and } t = \rho_0(\alpha, \gamma) \restriction n. \tag{I.322}$$

In other words, Case 1 of the definition of $f(\alpha, \beta, \gamma)$ applies to any triple $\alpha < \beta < \gamma$ from Ω. Note that in this case, the definition follows exactly the definition of the analogous function in the proof of Theorem 20.10 with ρ^{*+} in the role of the strongly unbounded subadditive function $e : [\theta^{++}]^2 \longrightarrow \theta^+$. By Lemma 20.15, ρ^{*+} is strongly unbounded on subsets of Ω, so by the proof of Theorem 20.10 we conclude that in this case the image $f''[\Omega]^3$ contains almost all ordinals $< \theta^+$ of cofinality θ.

Suppose now that no subset of Γ of size θ^+ is stable. By Lemma 20.17 (in fact, its proof) for every integer $n \geq 1$ there exist $\alpha < \beta < \gamma$ in Γ such that for $s = \rho_0(\alpha, \beta) \restriction \chi(\alpha, \beta, \gamma)$ and $t = \rho_0(\alpha, \gamma) \restriction \chi(\alpha, \beta, \gamma)$, where $\rho_0(\alpha, \beta)$ has length $> \chi(\alpha, \beta, \gamma)$:

$$\mathrm{osc}(\alpha, \beta, \gamma) = n, \tag{I.323}$$

$$(C_{\beta_s} \cap C_{\gamma_t}) \setminus \alpha = \emptyset. \tag{I.324}$$

This finishes the proof. $\dashv$

20.20 Corollary. *There is $f : [\omega_2]^3 \longrightarrow \omega$ which takes all the values on the cube of any uncountable subset of ω_2.* $\dashv$

20.21 Remark. Note that the dimension 3 in this Corollary cannot be lowered to 2 as long as one does not use some additional axioms to construct such f. Note also that the range ω cannot be replaced by a set of bigger size, as this would contradict Chang's conjecture. We have seen above that Chang's conjecture is equivalent to the statement that for every $f : [\omega_2]^3 \longrightarrow \omega_1$ there is an uncountable set $\Gamma \subseteq \omega_2$ such that $f''[\Gamma]^3 \neq \omega_1$. Is there a similar reformulation of the Continuum Hypothesis? More precisely, one can ask the following question.

20.22 Question. Is CH equivalent to the statement that for every $f : [\omega_2]^2 \longrightarrow \omega$ there exists an uncountable $\Gamma \subseteq \omega_1$ with $f''[\Gamma]^2 \neq \omega$?

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