Graph Labeling Problems

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Preliminaries

Definition

A simple graph G is an ordered pair of a non empty set V of objects called vertices together with a binary relation, which is irreflexive and symmetric.

Definition

Let G be a simple graph of order p and size q. By labeling of a graph, we mean assign some identification to the vertices or edges (or both) of the graph.

Preliminaries

There are basically two types of labelings of graph, namely

- Quantitative Labelings (Assignment of some numbers to the elements of graph).
- Qualitative Labelings (Assignment of qualitative nature of the elements of graph).

Type of labelings

Quantitative Labelings: These labeling have inspired research by vide variety of applications in radio-astronomy, development of missile guidance codes, spectral characterization of materials using X-ray crystallography etc. Probably the most popular of them all, called graceful labeling. let $\mathcal{R}$ be the set of real numbers and let A be a subset of $\mathcal{R}$.

$$* : \mathcal{R} \times \mathcal{R} \to \mathcal{R}.$$

Quantitative Labeling

$$f : V(G) \to A$$

induces

$$f^* : E(G) \to \mathcal{R}$$

where

$$f * (e = uv) = *(f(u), f(v)) = f(u) * f(v).$$

Type of Quantitative Labelings

- If $f : V \rightarrow \{0, 1, \dots, q\}$ is an injective mapping and $f^*(uv) = |f(u) - f(v)|$, $\forall uv \in E$. If $f^*(E) = \{1, 2, \dots, q\}$, then f is called a graceful labeling of G.
- If $f : V \rightarrow \mathcal{N}$ is an injective mapping and $f^*(uv) = (f(u) + f(v))$, $\forall uv \in E$. If f and f^* are both injective, then f is called an additive labeling of G.
- If $f : V \rightarrow \mathcal{Z}_q$ is mapping and $f^*(uv) = (f(u) + f(v))(\mathrm{mod}\, q)$, $\forall uv \in E$. If f^* are both injective, then f is called a hormonious labeling of G.

Type of Quantitative Labelings

- If $f : V \to \mathcal{Z}_q$ is mapping and $f^*(uv) = (f(u) + f(v))(\mathrm{mod}\, q)$, $\forall uv \in E$ such that f^* are all distinct, then f is called a felicitous labeling of G.
- If $f : V \to \{0, 1\}$ and $f^*(uv) = |f(u) - f(v)|$. If $|f_n(0) - f_n(1)| \leq 1$ and $|f_n^*(0) - f_n^*(1)| \leq 1$, then f is called a cordial labeling of G.
- If $f : V \to \{1, 2, \ldots, p\}$ is a bijection such that $\sum_{v \in N(u)} f(v)$ is constant. Then f is called a sigma labeling of G.

Type of Quantitative Labelings

- If $f : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ is a bijection and $f^*(uv) = f(u) + f(v) + f(uv)$ is constant. Then f is called a magic labeling of graph.
- If $f : V \rightarrow \{1, 2, \ldots, p\}$ is a bijection such that $f^*(uv) = |f(u) - f(v)|$, $\forall uv \in E$. If $f^*(E) = \{1, 2, \ldots, q\}$, then f is called a Skolem graceful labeling of G.

Type of Labelings

Qualitative Labelings: This labelings have inspired research in unrelated areas of human enquiry such as conflict resolutions in social psychology, electrical circuit theory, energy crises etc. Consider a social group. We can associate a graph as a mathematical model in which vertices represents the individuals and edge represents the relation between two individuals. When we study the attitudinal behavior of the the people in the social group. We can assign each individual by some sign as:

Qualitative Labeling

- If $\sigma : V(G) \rightarrow \{+, -\}$, then we have signed graph as mathematical model(point-signed graph) and is denoted by $S = (G, \sigma)$.
- If $\rho : E(G) \rightarrow \{+, -\}$ then we have signed graph as mathematical model(line-signed graph) and is denoted by $S = (G, \rho)$.

Motivation

Since every graph can be numbered (labeled) in infinitely many ways, one is interested to classes of numberings which are determined by imposing additional constraints of either mathematical or physical significance.
Thus utilization of numbered graph models requires imposing of additional constraints which characterize the problem being investigated.

Motivation

The necessary constraints arise naturally in studying the wide variety of seemingly unrelated practical applications for which numbered graphs provide underlying mathematical models. Some embodiments of this theory are as follows:

Motivation

- The design of certain important classes of good non periodic codes for pulse radar and missile guidance is equivalent to numbering of complete graph in such a way that all edge numbers are distinct. The vertex numbers then determine the time positions at which pulses are transmitted. Corresponding radar pulse and missile-guidance code problems have been the subject of investigation for several years.

Motivation

- Determination of crystal structures from X-ray diffraction data has long been a concern of crystallographers. The ambiguities inherent in this procedure are now beginning to be understood. In some cases the same diffraction information may correspond to more than one structure. This problem is mathematically equivalent to determining of all numbering of the appropriate graphs which produce produce a pre-specified set of edge numbers.

Motivation

- Non natural method s of encoding the integers from 0 to $b^n - 1$ using n digit vectors from the b-symbol alphabet have been devised to minimize the seriousness of errors occurring in a single digit. These encodings have been the subject of an extensive literature. The corresponding graph problem involves numbering the vertices of the square lattice grid, b on a side in n dimensions with integers from 0 to $b^n - 1$, in a way that optimizes some statistical function(typically the mean or the variance) of the edge numbers.

Graceful Graphs

Definition

If $f : V \rightarrow \{0, 1, \ldots, q\}$ is an injective mapping and $f^*(uv) = |f(u) - f(v)|$, $\forall uv \in E$. If $f^*(E) = \{1, 2, \ldots, q\}$, then f is called a graceful labeling(numbering) of G and the graph which admits such numbering is called graceful graph and non graceful otherwise.

Graceful Graphs

Remark

If each vertex u of G is assigned a value $f(u)$(usually an integer) and each edge uv is assigned a value determined by induced edge function f^*. For economy, we wish to optimize the value of largest integer assigned to any vertex of G.

Observation

If f is an optimal numbering of G then there exists a vertex u in G such that $f(u) = 0$.

Graceful Graphs

Observation

If f is graceful numbering of a graph G then $f^c = M(f) - f(u), \forall u \in V(G)$ is also a graceful numbering(called a complement of graceful numbering) of G, where $M(f)$ is the maximum number assign to a vertex of G under f.

Graceful Graphs

Theorem

A necessary condition for a graph to be graceful is that it is possible to partition $V(G)$ into V_o and V_e such that the number of edges across V_o and V_e is $\lfloor \frac{q+1}{2} \rfloor$.

Theorem

The complete graph K_p is graceful if and only if $p \leq 4$.

Graceful Graphs

Problem

What should be least number of edges deleted from K_p such that resulting graph is graceful?

Problem

What should be the minimum number of maximum number assigned to the vertices of K_p such that all edge labels are distinct?

Notching of Metal Bar

There is a classical combinatorial problem came from mechanical engineering, the notching of metal bar of length k at integer points in such a way that all distances between two notches or between a notch and an end points are distinct.

- If there are $(n-2)$ notches and two end points, then there are $\binom{n}{2}$ length which must be distinct.
- This problem is equivalent to the numbering of K_n. The smallest k for which notch problem has solution is $\theta(K_n)$.

Notching of Metal Bar

- This is also called as (n, k)-ruler of length k with n marks on it in such a way that $\binom{n}{2}$ distances between two marking are all distinct. This ruler is also called non redundant ruler.
- If $\theta(K_n) = k$, then it is called as Golomb Ruler.

Golomb Ruler

Definition

It is a ruler with n marks placed on it end-to-end so that all $\binom{n}{2}$ distances that can be measured by such a ruler are distinct. If the maximum distances measured by such a ruler is least possible, then that ruler is called Golomb Ruler.

X-ray Crystallography

Position of atom in a crystal structures are made by X-ray diffraction patterns. Measurements indicate the set of inter atomic distances in crystal lattice. But in general, do not necessarily, unambiguously specify the absolute position of atoms.

Mathematically, one can find the finite set of integers $R = \{0 = a_1 < a_2 < \cdots < a_n\}$ corresponds to one atom position and $S = \{0 = b_1 < b_2 < \cdots < b_n\}$ corresponds to another atom position.

Application

They may have the same difference set $D(R) = D(S) = \{|a_i - a_j| : i > j\}$. Since the diffraction pattern determine the set of differences. It is difficult to say that which homeometric set R and S produce it.

In 1939, Picard has shown that there can not be two non-equivalent ruler which measure s set of all distinct distances. G.S. Bloom has shown that the proof is not correct. He has disproved the same by a counter example as: $R = \{0, 1, 8, 11, 13, 17\}$ and $S = \{0, 1, 4, 10, 12, 17\}$.

History

Definition

The representation of first n natural numbers as differences of pairs of terms of an integer sequence of shortest possible length is called difference basis.

The actual story of representation of difference basis is appeared in 1938 in the work of J. Singer and A. Brauer from pure combinatorial number theoretic consideration.

History

In 1963, Ringel conjectured that K_{2n+1} could be decomposed into $2n+1$ subgraphs isomorphic to any given tree with n edges.

In 1966, A. Rosa has reported Kotzig suggestion that a stronger constraint would not falsify Ringel conjecture

Conjecture

Ringel-Kotzig Conjecture: K_{2n+1} can be cyclically decomposed into $2n+1$ subgraphs isomorphic to a given tree with n edges.

Decomposition

Intuitively, such a cyclic decomposition is accomplished by

- arbitrary choosing tree T_n with n edges;
- identifying the edges of T_n with a suitable set of edges in K_{2n+1}
- rotating each vertex and edge of T_n $2n$ times form it original position.

β-Valuation

In 1966, A. Rosa has defined the various types of labeling of vertices of graphs

Definition

An injection $f : V \to \{0, 1, \ldots, q\}$ and induced edge function $f^*(uv) = |f(u) - f(v)|, \forall uv \in E$ such that $f^*(E) = \{1, 2, \ldots, q\}$ is called a β-valuation of G.

In 1972, S.W. Golomb has defined the same graceful numbering of G. Now this is the popular term.

β-Valuation

Remark

If all trees are graceful, the Ringel-Kotzig Conjecture holds.

Conjecture

All trees are graceful.

α-Valuation

Rosa has defined an α-valuation to be graceful numbering with the additional property that there exists an integer k so that for each edge uv either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$.
In literature this numbering is also known as *balance or interlaced numbering*.

ρ-Valuation

Definition

An injection $f : V \rightarrow \{01, 2, \ldots, 2q\}$ and edge induced function is defined as $f^*(uv) = (2n + 1) - |f(u) - f(v)|$, $\forall uv \in E$ such that $f^*(E) = \{1, 2, \ldots, q\}$ is called ρ-valuation.

Remark

If all trees follows ρ-valuation then the Ringel-Kotzig Conjecture would be proved.

σ-Valuation

Definition

An injection $f: V \rightarrow \{01, 2, \ldots, 2q\}$ and edge induced function is defined as $f^*(uv) = |f(u) - f(v)|$, $\forall uv \in E$ such that $f^*(E) = \{1, 2, \ldots, q\}$ is called ρ-valuation.

Signed Graphs

Definition

A *signed graph* (for short, “sigraph") is a graph where edges are assigned *positive* or *negative* sign.

Remark

The edge which receives positive sign is called *positive edge* and the edge which receives negative sign is called *negative edge*. If all edges are either positive or negative then the sigraph is called as *homogeneous sigraph* and heterogeneous otherwise.

Graceful Signed Graphs

Definition

A sigraph S is called a (p, m, n)-sigraph if it has p vertices, m positive edges and n negative edges.

Definition

An injective function $f : V(S) \rightarrow \{0, 1, \ldots, q = m + n\}$ such that when each edge $uv \in E(S)$ is assigned $g_f(uv) = \sigma(uv)|f(u) - f(v)|$ the positive edges receive distinct labels from the set $\{1, 2, \ldots, m\}$ and the negative edges receive distinct labels from the set $\{-1, -2, \ldots, -n\}$ is called a graceful numbering of S.

Graceful Sigraphs

The sigraph S is called a *graceful sigraph* if it admits a graceful numbering and *nongraceful* otherwise.

Remark

If $n = 0$ then graceful numbering of sigraph coincides the notion of graceful graphs in Rosa Golomb sense.

Negation of Sigraph

Definition

For a sigraph S, $\eta(S)$ is a sigraph obtained from S by changing the sign of each edge of S to its opposite and is called the *negation* of S.

Observation

If f is a graceful numbering of a sigraph S then it is also a graceful numbering of the sigraph $\eta(S)$.

Thank You!!