# Sémantique des Langages de Programmation (SemLP)
# TD n$^o$ 3 : Unification & Termination

**Exercice 1 : (Ex. 52 in the course notes)**

Apply the unification algorithm viewed in the course to the following systems of equations :

**1.** $\{\ f(x, f(x, y)) = f(g(y), f(g(a), z))\ \}$

**2.** $\{\ f(x, g(y)) = f(y, g(g(x)))\ \}$

**Exercice 2 : *** (Ex. 55 in the course notes)**

**1.** Propose a method to transform a unification problem of the shape :

$$E = \{\ t_1 = s_1,\ \dots,\ t_n = s_n\ \}$$

over the signature $\Sigma = \{g_1, g_n\}$ with $n, m \geqslant 1$ into a unification problem $E'$ with the following properties :

1. $E'$ contains exactly one equation,
2. the terms in $E'$ range over the signature $\Sigma' = \{f\}$, where $f$ is binary,
3. $E$ has a solution if and only if $E'$ has a solution, and

**2.** Apply the method to the system below, where $x$, $y$ and $z$ are variables.

$$E = \{\ x = h(y),\ g(c, x, y) = g(y, z, z)\ \}$$

**Exercice 3 : (Ex. 56 in the course notes)**

Let $t, s, \dots$ be terms over the signature $\Sigma$. We say that $t$ is a *filter* for $s$ if there exists a substitution $S$ with $S\,t = s$. We denote this fact as $t \leqslant s$ [1]. Prove or disprove the following assertions :

**1.** If $t \leqslant s$, then $t$ and $s$ are unifiable.

**2.** If $t$ and $s$ are unifiable, then $t \leqslant s$ and $s \leqslant t$.

**3.** it $t \leqslant s$ and $s \leqslant t$, then $s$ and $t$ are unifiable.

**4.** For all $t$, $s$ there exists an $r$ with $r \leqslant t$, and $r \leqslant s$.

**5.** For all $t$, $s$, there exists an $r$ with $t \leqslant r$ and $s \leqslant r$.

1. Not to be confused with the notation $S \leqslant S'$ over substitutions.

# Termination

## Interpretation method

**Exercice 4 : (Ex. 63 in the course notes)**
Find a polynomial interpretation to show termination of the following TRS :

$$f(f(x,y),z) \to f(x,f(y,z)), \qquad f(x,f(y,z)) \to f(y,y)$$

**Exercice 5 : (Ex. 64 in the course notes)**

**1.** Find a polynomial interpretation for the following TRS :

$$\begin{aligned} x + \mathsf{0} &\to x, & x + \mathsf{s}(y) &\to \mathsf{s}(x+y), && \text{(addition)}\\ d(\mathsf{0}) &\to \mathsf{0}, & d(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{s}(d(x))), && \text{(double)}\\ q(\mathsf{0}) &\to \mathsf{0}, & q(\mathsf{s}(x)) &\to q(x) + \mathsf{s}(d(x)), && \text{(square)} \end{aligned}$$

**2.** Consider the term $t = q^{n+1}(s^2\mathsf{0})$ whose size is linear in $n$. Show that there is a reduction as shown below with length doubly exponential in $n$.

$$t \to^* q^{n+1}(s^{2^{2^n}})$$

## Recursive path ordering

**Exercice 6 : (Ex. 65 in the course notes)**
Consider the TRS :

$$(x+y)+z \to x+(y+z) \qquad x * \mathsf{s}(y) \to x + (y * x)$$

Find a status function $(\tau)$ such the following inequalities hold :

$$(x+y)+z >_r x+(y+z) \qquad x * \mathsf{s}(y) >_r x + (y * x)$$

**Exercice 7 : (Ex. 69 in the course notes)**
Consider the TRS encoding the *Ackermann function* and prove its termination.

$$\begin{array}{rclrcl} ack(\mathsf{z}, n) & \to & \mathsf{s}(\mathsf{z}) & ack(\mathsf{s}(\mathsf{z})) & \to & \mathsf{s}^2(\mathsf{z}) \\ ack(\mathsf{s}^2(m), \mathsf{z}) & \to & \mathsf{s}^2(m) & ack(\mathsf{s}(m), \mathsf{s}(n)) & \to & ack(ack(m, \mathsf{s}(n)), n) \end{array}$$

**Exercice 8 : (Ex. 70 in the course notes)**
Consider the TRS

$$b(c) \to r(\mathsf{s}(x)) \qquad r(\mathsf{s}(\mathsf{s}(x))) \to b(x)$$

**1.** Show that the TRS terminates by polynomial interpretation.

**2.** Show that there is no RPO on $\Sigma$ that can prove its termination.

**3.** RPO is a particular type of simplification order. Is there a simplification order that shows termination of the TRS above ?

**Exercice 9 : (Ex. 71 in the course notes)**
Consider the TRS :

$$f(f(x)) \to f(g(f(x)))$$

**1.** Show that it is terminating.

**2.** Show that there is no simplification order $>_r$ that contains $\to$.

**Exercice 10 : (Ex. 80 in the course notes)**
Consider the product order $\leqslant$ on $\mathbb{N}^k$ (vectors of natural numbers) :

$$(n_1, \dots, n_k) \leqslant (m_1, \dots, m_k) \text{ if } n_i \leqslant m_i, \text{ for } i \in [1, k]$$

**1.** Show that $<$ (the strict part of $\leqslant$) is well-founded.

**2.** Show by induction on $k$, that from every sequence $\{v_n\}_{n\in\mathbb{N}}$ in $\mathbb{N}^k$ we can extract a *growing subsequence*. Namely, that there is an injective function $\sigma : \mathbb{N} \to \mathbb{N}$ such that :

$$\forall n,\ v_{\sigma(n)} \leqslant v_{\sigma(n+1)}$$

**3.** Show that every set of incomparable elements in $\mathbb{N}^k$ (an anti-chain) is finite.