

## Computer Problem Set 3

### Stochastic integration

*The present problem set is attached to Chapters 5 and 6 of the lectures notes. All implementations should be run with the value  $T = 2$ . For a positive integer  $n$ , we denote  $\Delta T := \frac{T}{n}$ ,  $t_i^n := i \Delta T$ ,  $i = 0, \dots, n$ . We consider a Brownian motion  $W$ , and we denote  $\Delta W_{t_i^n} := W_{t_i^n} - W_{t_{i-1}^n}$ ,  $i = 1, \dots, n$ .*

1. In view of the approximation of  $\int_0^T W_s dW_s$ , we consider the three following quantities:

$$I_n := \sum_{i=1}^n W_{t_{i-1}^n} \Delta W_{t_i^n}, \quad J_n := \sum_{i=1}^n W_{t_i^n} \Delta W_{t_i^n}, \quad K_n := \sum_{i=1}^n \frac{W_{t_i^n} + W_{t_{i-1}^n}}{2} \Delta W_{t_i^n}.$$

- (a) Simulate a sample of  $N = 1000$  copies of the randoms variables  $\frac{1}{2}W_T^2 - I_n$ ,  $\frac{1}{2}W_T^2 - J_n$ , and  $\frac{1}{2}W_T^2 - K_n$ .
  - (b) Compute the corresponding sample means, and comment on the results.
  - (c) Vary the value of  $n$  from 10 to 20, and provide a graph of the resulting sample means, together with the corresponding confidence intervals.
2. Address the previous questions with the random variables

$$A_n := \sum_{i=1}^n e^{t_{i-1}^n} \Delta W_{t_i^n}, \quad B_n := \sum_{i=1}^n e^{t_i^n} \Delta W_{t_i^n}, \quad \text{and} \quad C_n := \sum_{i=1}^n e^{\frac{t_i^n + t_{i-1}^n}{2}} \Delta W_{t_i^n}.$$

3. We now consider the random variables

$$A_n := \sin(W_T) + \frac{1}{2n} \sum_{i=1}^n \sin(W_{t_{i-1}^n}).$$

- (a) Simulate a sample of  $M = 1000$  copies of  $A_n$ , and plot the corresponding sample mean, with the appropriate confidence interval, as a function of  $n \in \{10, \dots, 200\}$ .
- (b) Comment the graph with appropriate justification.