

Computer Problem Set 7

Monte Carlo approximation of the Greeks

In the context of the Black-Scholes model with constant interest rate r , constant volatility parameter $\sigma > 0$, and Brownian motion B under the risk-neutral measure:

$$dS_t := S_t(r dt + \sigma dB_t),$$

we are interested in the numerical approximation of the no arbitrage price of a binary option defined by the payoff $\mathbf{1}_{\{S_T \leq K\}}$ at the maturity date T , for some $K > 0$, together with the corresponding optimal hedging strategy

$$C_0 := e^{-rT} \mathbb{E}[\mathbf{1}_{\{S_T \leq K\}}] \quad \text{and} \quad \Delta_0 := e^{-rT} \frac{\partial}{\partial S_0} \mathbb{E}[\mathbf{1}_{\{S_T \leq K\}}].$$

In terms of the cdf of the $\mathcal{N}(0, 1)$ distribution, direct calculation leads to

$$C_0 := e^{-rT} \mathbf{N}(-d_-(X_0, \sigma^2 T)), \quad \Delta_0 := \frac{-e^{-rT}}{S_0 \sqrt{\sigma^2 T}} \mathbf{N}'(-d_-(X_0, \sigma^2 T)), \quad \text{with } X_0 := \frac{S_0}{K e^{-rT}},$$

$$\text{and } d_-(x, v) := \frac{\ln(x)}{\sqrt{v}} - \frac{1}{2} \sqrt{v}.$$

1. We first focus on the Monte Carlo approximation of C_0 .
 - (a) Build a program which returns C_0 and Δ_0 for given values of r, σ, S_0, T, K .
 - (b) Build a program which returns a Monte Carlo approximation C_0^N of C_0 based on N copies of B_T .
 - (c) Discuss the numerical results using the parameters values $r = 0.02$, $\sigma_0 = 0.4$, $S_0 = 100$, $T = 0.9$, $K = 80 + i$, $i = 0, \dots, 40$.
2. We denote $C_0^N(S_0)$ to emphasize the dependence of this function on S_0 .
 - (a) Build a program which returns the centered finite-differences approximation of Δ_0 and the corresponding 5% confidence interval:

$$\Delta_0^{N, \varepsilon} := \frac{C_0^N(S_0 + \varepsilon) - C_0^N(S_0 - \varepsilon)}{2\varepsilon}.$$
 - (b) Using the parameters values of Question 1c, discuss numerically the choice of the parameters ε and N .
3. By writing the price C_0 as an integral with respect to the distribution of S_T , we obtain the following representation:

$$\Delta_0 = e^{-rT} \mathbb{E}[\mathbf{1}_{\{S_T \leq K\}} \frac{B_T}{S_0 \sigma T}]$$

- (a) Build a program which returns a Monte Carlo approximation $\hat{\Delta}_0^N$ of Δ_0 based on the last representation.
- (b) Using the parameters values of Question 1c, compare the performances of the approximation Δ_0^N and $\hat{\Delta}_0^N$.