École polytechnique

Third-year Research Project

Multi-currency arbitrage and calibration of a Wishart model

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1 Model-free bounds on option prices using improved Fréchet bounds

1.1 Introduction

Foreign exchange markets exhibit, for various reasons, very interesting properties compared to traditional markets. In this study, we will focus on a particular feature, that may allow arbitrage opportunities to arise. Using empirical market data, we aim to find ways to identify such arbitrage opportunities, first using model-free approaches, which will constitute the core of the present report and presentation.

First, let's quickly describe the kind of arbitrage we are interested in. Let's take three currencies A,B and C. Basically, we expect the prices of those three currencies to be related. Theoretically, we expect the following relation:

$$P_{AtoB} * P_{BtoC} = P_{AtoC}$$

Otherwise, an obvious arbitrage exists for this set of currencies.

What we'll try to investigate in this project, is the existence of similar arbitrage opportunity in the call option market, within the broader currency market, using empirical data. The data we will use consists of an array gathering the implied volatility of call options related to 4 currencies (USD, EUR, JPY, GBP), for 14 maturities and 18 strikes.

USD/EUR								Strikes en % du fwd de la maturité en question				
Pdate	8-févr-19							USD/EUR: l'USD en Euros				
Spot	0,883											
Fwds	**ATMF vol**	**Smile**	**50%**	**60%**	**70%**	**80%**	**90%**	**95%**	**100%**	**105%**	**110%**	**120%**
0,883	4,58	**11-févr-19**	15,61	13,59	11,62	9,58	7,34	6,03	4,58	7,57	9,58	12,50
0,882	5,76	**15-févr-19**	13,68	12,09	10,56	9,03	7,43	6,56	5,76	7,65	9,10	11,32
0,882	5,74	**22-févr-19**	11,93	10,63	9,40	8,18	6,94	6,28	5,74	7,24	8,44	10,31
0,881	5,89	**1-mars-19**	11,01	9,90	8,85	7,82	6,80	6,27	5,89	7,11	8,13	9,75
0,881	6,15	**8-mars-19**	10,75	9,72	8,76	7,84	6,92	6,46	6,15	7,25	8,19	9,70
0,879	6,43	**10-avr-19**	10,45	9,52	8,66	7,83	7,03	6,63	6,43	7,29	8,06	9,34
0,876	6,48	**9-mai-19**	9,68	8,90	8,19	7,52	6,88	6,56	6,48	7,30	8,05	9,29
0,870	6,86	**8-août-19**	10,95	9,97	9,07	8,20	7,35	6,93	6,86	7,57	8,30	9,52
0,863	7,16	**8-nov-19**	11,43	10,39	9,43	8,51	7,61	7,19	7,16	7,77	8,46	9,63
0,856	7,29	**10-févr-20**	11,42	10,39	9,44	8,53	7,63	7,25	7,29	7,85	8,52	9,68
0,844	7,57	**10-août-20**	11,13	10,20	9,34	8,52	7,74	7,49	7,57	7,98	8,49	9,46
0,832	7,71	**10-févr-21**	10,98	10,10	9,29	8,52	7,81	7,62	7,71	8,04	8,49	9,36
0,811	8,14	**10-févr-22**	11,17	10,31	9,52	8,77	8,16	8,06	8,14	8,38	8,71	9,43
0,774	8,57	**8-févr-24**	11,06	10,26	9,54	8,88	8,50	8,48	8,57	8,73	8,95	9,51

Figure 1: Sample of the work data

Our aim is to search for arbitrage opportunities on all sets of 3 of thoses currencies, for all those maturities and strikes. For clarity purposes, this report will focus on the currency exchange rates triangle USD/EUR, JPY/EUR and JPY/USD.

2 Defining the problem

In the previous figure, we plot the implied volatility of the set of options as well as the spot value of the asset in the top left corner of the figure, and the different forwards for different maturities i the first column.

Let's denote S_t an exchange rate, for example USD/EUR.
The pay-off of a European Call option on this exchange rate, with maturity T and strike K is $(S_T - K)_+$.
Using the Black-Scholes model, the price of an option is defined by the formula :

$$C(T, K) = S_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$$

where :

$$d_1 = \frac{\ln S_0/K + (r_d - r_f + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Where :

- S_0 is the current spot rate
- K is the strike price
- $N(x)$ is the cumulative normal distribution function
- r_d is the domestic risk-free simple interest rate
- r_f is the foreign risk-free simple interest rate
- T is the time to maturity
- σ is the volatility of the FX rate

2.1 Estimation of the interest rates

Using the available data, we can directly identify all the variables above except for r_d and r_f. In the pricing formula of a Call option, we only need the value of the difference $r_d - r_f$, so we will try to estimate it using the information we know.

In the first column are gathered the forward values, based on the spot value of the exchange rate that is in the top left corner of the previous figure. Using this, we can estimate the difference between the domestic and foreign interest rates.

The forward F_T for maturity T is given by :

$$F_T = S_0 e^{(r_d - r_f)T}$$

Therefore : $\ln F_T = \ln S_0 + (r_d - r_f)T$ and $\ln F_T$ is thus a linear function of the difference $r_d - r_f$. By fitting a simple linear regression to this variable with regards to the maturities, we can estimate its slope $r_d - r_f$. We'll from now on use this value in our model.

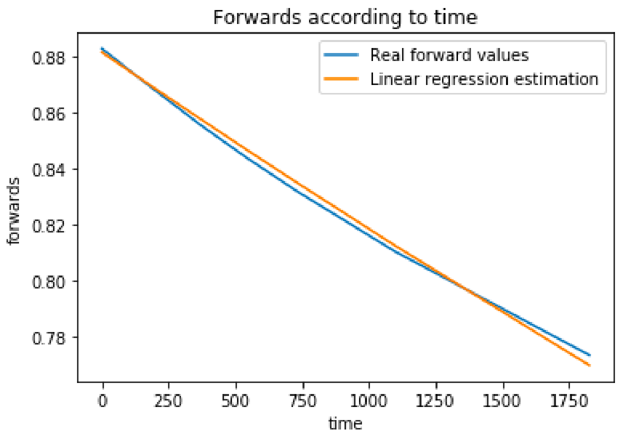


Figure 2: Linear regression of the logarithm of the forwards

This method gives the estimation : $r_d - r_f = -0.0001$. We'll use this value from now on in our model.

2.2 Volatility smile

The plot of the implied volatilities with regards to strikes for the different maturities shows a clear smile structure for the small maturities, and a smirk structure for the largest maturities.

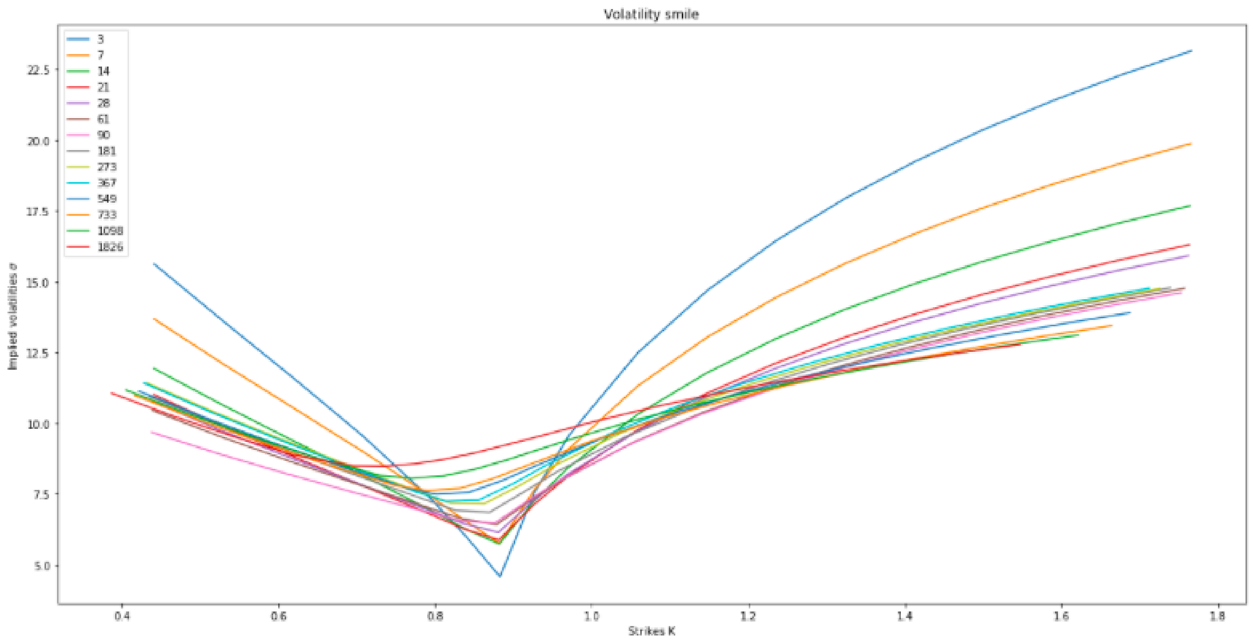


Figure 3: Volatility smile for the USD/EUR exchange rate, with maturities in days

2.3 European Call option prices

Using the formula above for the pricing of European Call options on exchange rates, and given the estimated value of the difference of foreign and domestic interest rates, we can compute the option price for all the maturities.

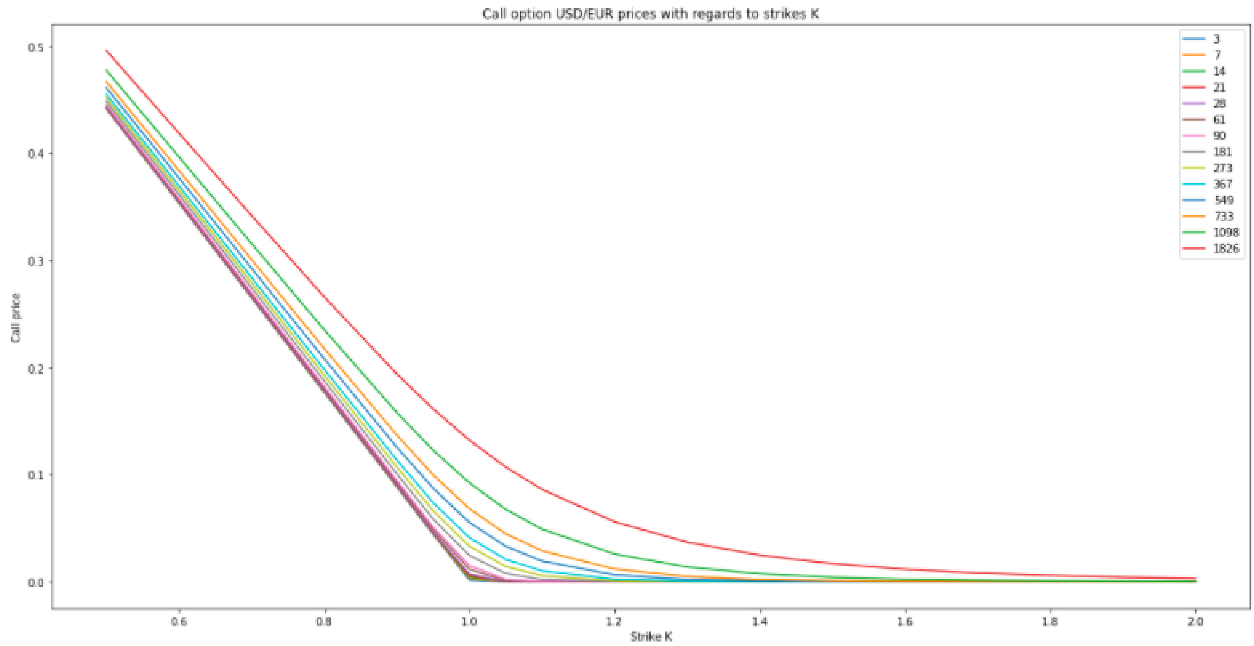


Figure 4: European Call option prices for the USD/EUR exchange rate for different maturities in days

This plot noticeably satisfies known properties of the Call option prices function C with regards to the strikes K, such as the fact that C is a convex decreasing function. Additionally:

$$\lim_{K\to\infty} C(T,K) = 0$$

$$\lim_{K\to 0} C(T,K) = S_0$$

We can then relate the cumulative density function and the probability density function of S_T using the fact that :

$$C(T,K) = E^Q[e^{-rT}(S_T - K)_+]$$

$$\frac{\partial C}{\partial K} = -E^Q[1_{S_T \geq K}] = -P(S_T \geq K)$$

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT} p(K)$$

2.4 Multiple underlyings

Let's consider two Call options on two different exchange rates :
S_T^1 is the rate USD/EUR, $C^1(T,K)$ the price of the corresponding option and $p^1(K)$ its density.
S_T^2 is the rate JPY/EUR, $C^2(T,K)$ the price of the corresponding option and $p^2(K)$ its density.
S_T^3 is the third rate USD/JPY and can be given by : $S_T^3 = \frac{S_T^1}{S_T^2}$.
The payoff in euros of an option on the third exchange rate is $(S_T^3 - K)_+ S_T^2 = (S_T^1 - K S_T^2)_+$.
Given this payoff, this option can be seen as a spread option with a coefficient.
Therefore : $C^3(T,K) = E^Q[e^{-rT}(S_T^1 - K S_T^2)_+]$

The problem we are confronted with is that we know the distributions of S_T^1 and S_T^2 (p^1 and p^2), but not their joint distribution. All we can do is simply compute bounds for the prices of the third option.

2.5 Bounds

In the article [Improved Frechet bounds and model-free pricing of multi-asset options][1] by Dr. Peter Tankov, the price of the third option follows the formula :

$$\pi(C) = -f(0,0) + E[f(X,0)] + E[f(0,Y)] + \int_0^\infty \int_0^\infty \mu(dx*dy)(1 - F_X(x) - F_Y(y) + C(F_X(x), F_Y(y)))$$

where C is a copula and $f : (x, y) \mapsto (x - Ky)_+$ The term with the integral can be simplified into the following formula :

$$\int_0^\infty \int_0^\infty \mu(dx * dy)(1 - F_X(x) - F_Y(y) + C(F_X(x), F_Y(y))) = -\int_0^\infty G(z, \frac{z}{K})dz$$

where :

$$G(x, y) = 1 - F_X(x) - F_Y(y) + C(F_X(x), F_Y(y))$$

We use improved Fréchet bounds for the copula C :

$$C(u, v) \leq min(u, v)$$

$$max(u + v - 1, 0) \leq C(u, v)$$

These bounds immediately give us bounds on the function G and therefore on the price of the option $\pi(C)$:

$$G_{min}(x, y) = 1 - F_X(x) - F_Y(y) + min(F_X(x), F_Y(y))$$
$$G_{max}(x, y) = 1 - F_X(x) - F_Y(y) + max(F_X(x) + F_Y(y) - 1, 0)$$

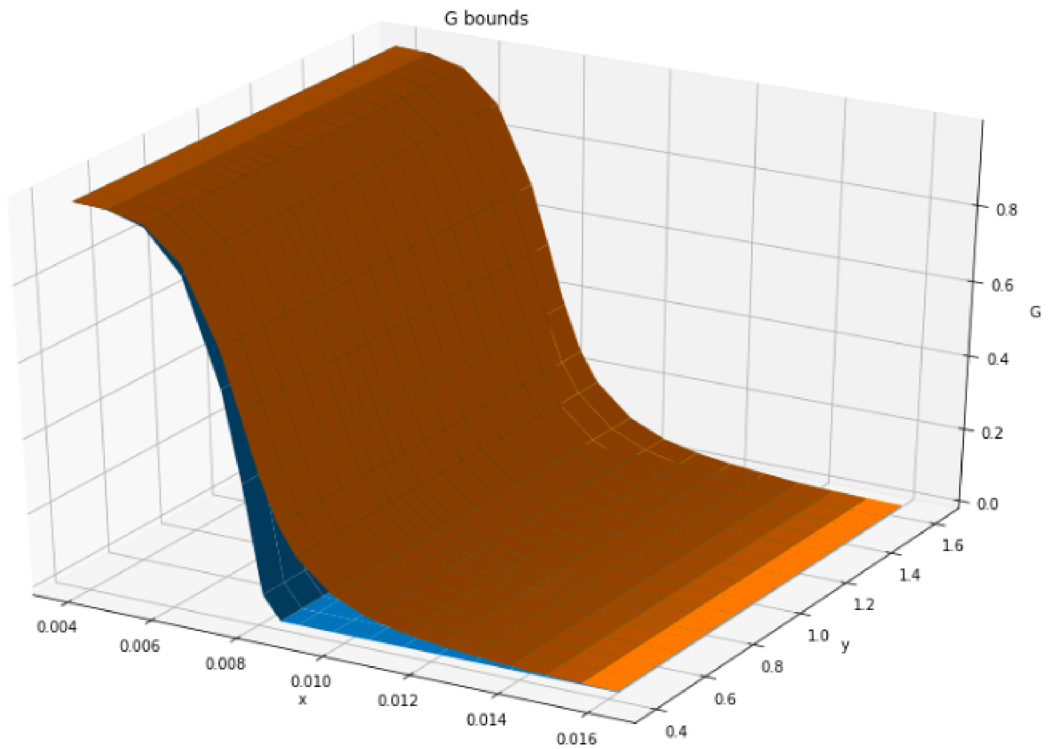


Figure 5: Two-dimensional plot of the bounds G_{min} and G_{max}

[1] arXiv:1004.4153v2 [q-fin.PR] 25 Mar 2011

Therefore, given that $f(0,0) = 0$ and $f(0,Y) = (0 - KY)_+ = 0$:

$$\pi_{min}(C) = E[f(X,0)] - \int_0^\infty G_{max}(z, \frac{z}{K})dz$$

$$\pi_{max}(C) = E[f(X,0)] - \int_0^\infty G_{min}(z, \frac{z}{K})dz$$

In order to compute these bounds, we need the cumulative distribution function F_X and F_Y, which we approximate by interpolating the option price function and then deriving it with respect to the strike.

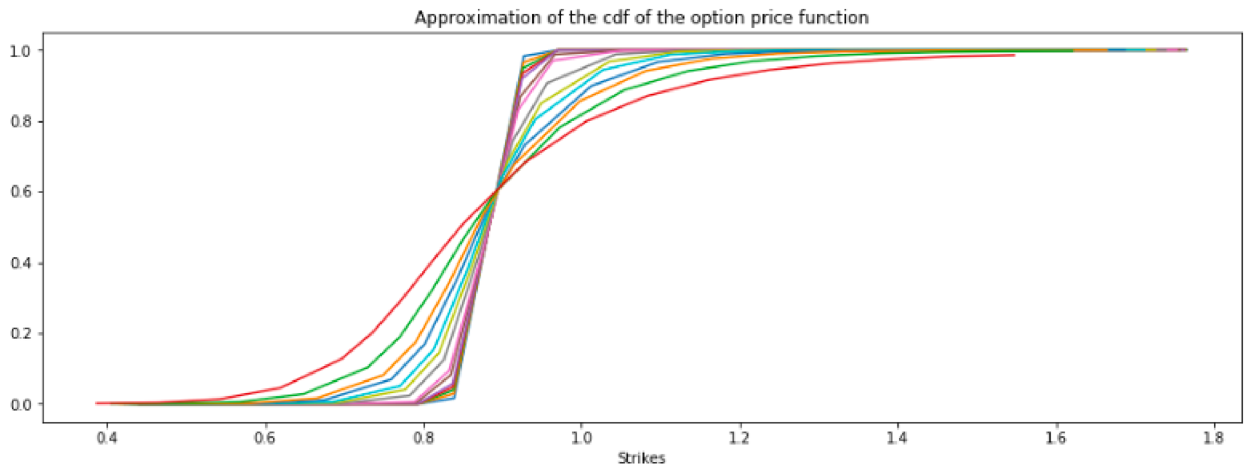


Figure 6: Cumulative distribution function of the option prices with regards to strikes K, for maturities in days

2.6 Results

We computed the previous boundaries for options with regards to strikes and for different maturities. We can also note another one, the trivial bound for the price of a European Call option :

$$(S_0 - Ke^{-rT})_+ \leq C(T,K)$$

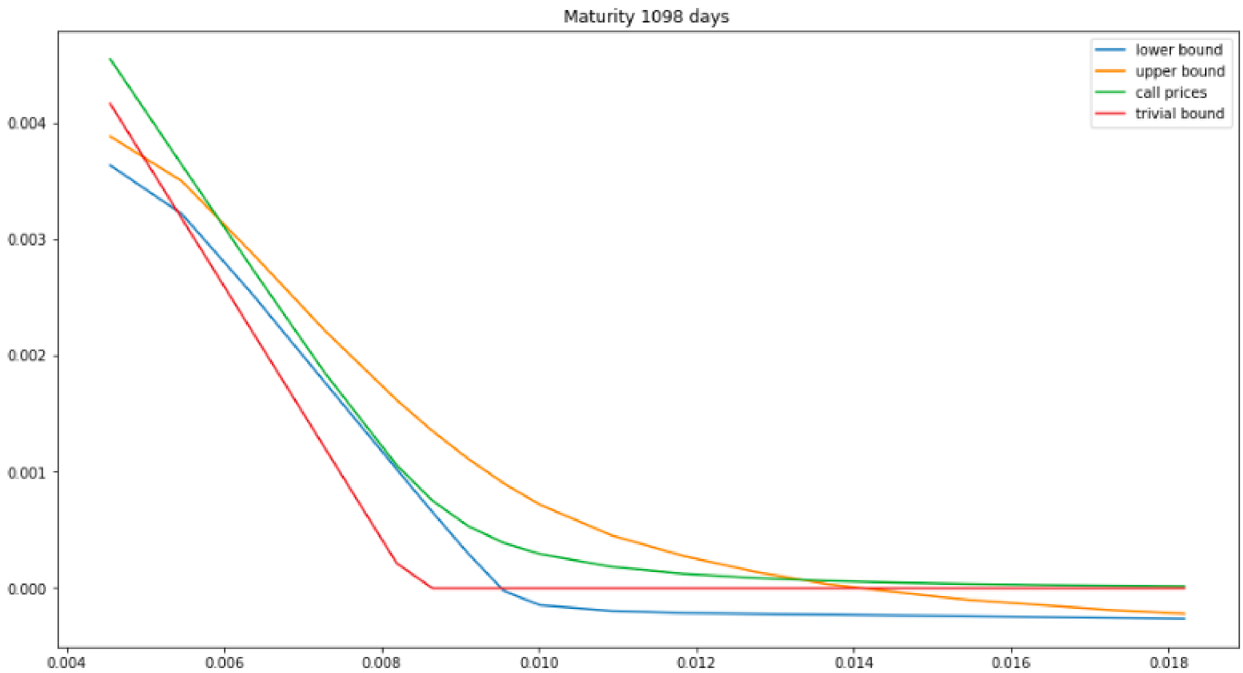


Figure 7: Bounds on the prices of the third Call option

Firstly, the figure above shows that the green line representing the actual option prices is well above the trivial bound represented by the red line. Secondly, it seems that for most values of K, the actual option prices are within the upper and lower bounds (orange and blue lines respectively) computed using the previous method. However, for small values of K, or very large values of K, it seems that the option prices get outside of these bounds, from which we can conclude that arbitrage opportunities exists for these maturities and strikes.

3 Cross-entropy methods

3.1 Introduction

Using the previous method based on improved Frechet bounds, we have shown how some arbitrage opportunities can be detected. However, we have no reason to believe that we have detected all of them, and that other opportunities have not gone undetected by the previous method. Therefore we may want to try other ways to fulfill the same goal.

The next two methods aim to detect arbitrage opportunities by using known conditions on option prices to try to build a fitting probability distribution. To do this, we'll use the entropy function between 2 probability distributions. This entropy can be written as:

$$H(p,q) = -\operatorname{E}_p[\log q]$$

In the next paragraphs, we'll look at the 3 currencies, and study the same options as before, for different strikes and maturities.

3.2 First method : Detection of arbitrage

In this method, we take a reference probability distribution P, and try to find the probability distribution Q that minimizes $H(P,Q)$, under the 3 known conditions on option prices :

$$C^3(T,K) = E^Q[e^{-rT}(S_T^1 - KS_T^2)_+]$$

with $C(T,K)$ the known market price, for all three couples of currencies. This means that we have one condition per maturity, strike, and couple of currencies.

Using this, we will compute:

$$\text{min H(p, q) - } \sum(\lambda_i E^Q[(S_T^1 - KS_T^2)_+] - C^3(T,K)) - \sum(\mu_i E^Q[(S_T^2 - KS_T^3)_+] - C^1(T,K)) \\ \text{- } \sum(\nu_i E^Q[(S_T^1 - KS_T^3)_+] - C^2(T,K)$$

which can finally be rewritten:

$$\text{min -E}[Z\log Z] - \sum(\lambda_i E^Q[Z(S_T^1 - KS_T^2)_+] - C^3(T,K)) - \\ \sum(\mu_i E^Q[Z(S_T^2 - KS_T^3)_+] - C^1(T,K)) - \sum(\nu_i E^Q[(S_T^1 - KS_T^3)_+] - C^2(T,K))$$

with $Z = \frac{dP}{dQ}$.

Finally, this means that we are trying to find the coefficients λ_i, μ_i and ν_i that minimize :

$$Z = \frac{exp(-\sum(\lambda_i E^Q[Z(S_T^1-KS_T^2)_+]) - \sum(\mu_i E^Q[Z(S_T^2-KS_T^3)_+])) - \sum(\nu_i E^Q[(S_T^1-KS_T^3)_+] - C^2(T,K)}{\operatorname{E}[exp(-\sum(\lambda_i E^Q[Z(S_T^1-KS_T^2)_+]) - \sum(\mu_i E^Q[Z(S_T^2-KS_T^3)_+]) - \sum(\nu_i E^Q[(S_T^1-KS_T^3)_+] - C^2(T,K))]} \tag{1}$$

Those coefficients will then be identified using numerical optimization. Knowing them, we can compute Z, and therefore Q. We can also use this information to fulfill our goal, and detect a potential arbitrage opportunity on the options related to these 3 currencies. Indeed, if there is no finite set of coefficients λ_i, μ_i and ν_i that minimize Z, we can deduce that no probability distribution Q satisfies the known conditions. Therefore, we have detected an arbitrage opportunity in this triplet of currencies.

One choice that may be important when using this method is the probability distribution P we use as a reference. For now, we'll use the two dimensional normal distribution with mean value 0 and standard deviation 1. We'll discuss later the potential benefit of using other distributions.

3.3 Second method : finding boundaries on option prices

In this second method, we'll search for the probability distribution Q that minimizes :

$$minC^2(K,T)-\alpha\,\mathrm{E}[Z\log Z]-\sum(\lambda_i E^Q[Z(S_T^1-KS_T^2)_+]-C^3(T,K))-\sum(\mu_i E^Q[Z(S_T^2-KS_T^3)_+]-C^1(T,K))$$

With C the price of the call option we investigate. To put it simply, the idea is to only take into account the known conditions on 2 couples of prices (3 in the previous method), and minimize the prize of the third couple, penalized by entropy. Therefore, for each real value of α, we can find the coefficients λ, μ (i.e. Z) that minimize this function, and from that the best probability distribution Q that satisfies the conditions. We can therefore obtain boundaries on the arbitrage-free values of the option price. They correspond to the limit of the formula above when α goes to 0. The next step for us is to compare the performance of those 3 methods (Fréchet bounds, cross-entropy with 3 conditions, cross-entropy with 2 conditions) in detecting arbitrage opportunities, and from that conclude our study of the first part of the problem.

3.4 Implementation problems

A few interesting technical points are worth discussing for these methods: namely, the computation of Z while trying to minimize it, and the influence of the distribution of probability P we use as a reference.

During each computation of Z, we have to estimate the expected value of a linear combination of option payoffs:

$$E[exp(-\sum(\lambda_i E^Q[Z(S_T^1-KS_T^2)_+])-\sum(\mu_i E^Q[Z(S_T^2-KS_T^3)_+])-\sum(\nu_i E^Q[(S_T^1-KS_T^3)_+]-C^2(T,K))]$$

A first approach we tried was a Monte-Carlo method, i.e. using a simulation of random variable following distribution P to estimate this value. However, doing this, we run into a few problems: long computation time, low precision of the estimate... More importantly, since our goal is to minimize Z numerically, these mistakes impacted the computation of Z's gradient, greatly increasing the number of iterations needed and the imprecision on Z's final estimated coefficients.

Noticing this, we proposed a second, better method. We now obtain this estimate of the expected value using a numerical integration. We use the optimal quantization grid of the 2-dimensional normal distribution.

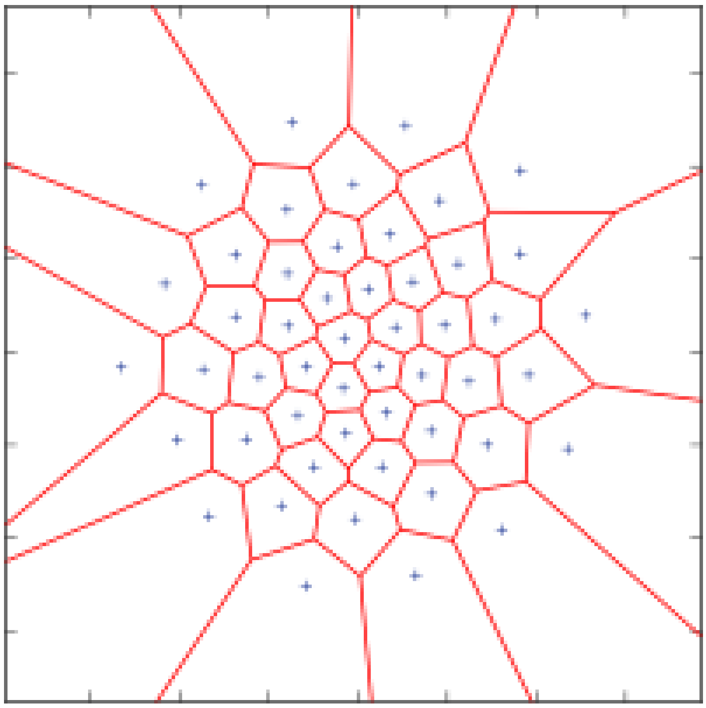


Figure 8: Optimal quantization grid for the 2-d normal distribution, N=50

This allows use to compute Z, find the eventual minimal value and associated coefficients, and hence detect arbitrages / establish boundaries.

Ajouter résultats, pas de convergence, passage au suivant

4 Explicit arbitrage finding

4.1 Introduction

Using the previous two broad categories of methods to detect arbitrages, we have encounter two practical problems that have impeded our ability to detect arbitrages. The first one is that, previously, arbitrage detection relied on running an optimization method and not finding a solution: However, such a detection criterion depends heavily on the chosen optimization algorithm and stopping criterion. Therefore, this method is unreliable to detect arbitrages, with a noticeable propensity to create false positives (i.e. the algorithm hasn't yet found an extremum, leading us to deduce falsely the presence of an arbitrage opportunity).

Therefore, we are now going to look into another, and last, model-free method. We are going to explicitly look for an arbitrage portfolio, by optimizing portfolio weights under certain constraints, and from there determine whether the set of assets we selected allows for arbitrage.

4.2 Best portfolio building method

Let's start by detailing the function we'll try to maximize. A preliminary idea would be to maximize the expected portfolio value at maturity, hoping that such a portfolio would also give very low probability of negative values.

$$\begin{aligned} \max_{\lambda_i} \quad & \textstyle\sum_1^N E^P(\lambda_i e^{-rT}(S_T^1 - KS_T^2)_+) \\ \text{under constraint} \quad & \textstyle\sum_1^N \lambda_i P_i = 0 \end{aligned} \tag{2}$$

However, to achieve our goal, it would be preferable for us to penalize more strongly negative values in this function, keep positive values bounded. Therefore, we'll maximise the expected utility of the portfolio at maturity, using as a utility function :

$$F(X, Y) = (1 - e^{-\alpha X})$$

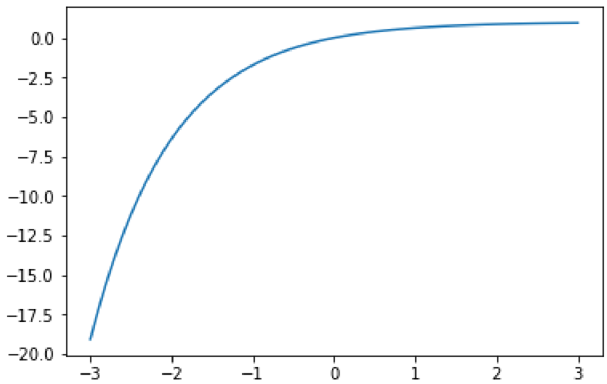
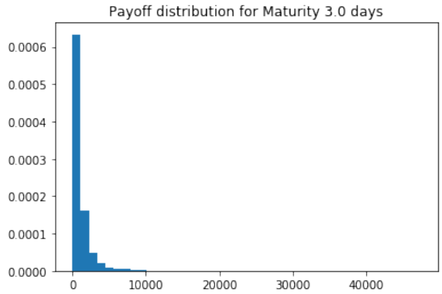


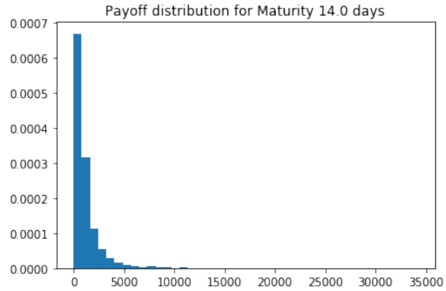
Figure 9: Chosen utility function, alpha=1.

4.3 The result

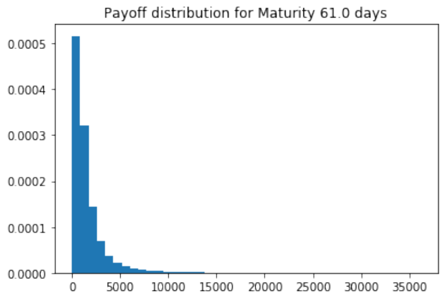
We can now run this algorithm, for a subset of options of constant maturity. For our probability law of reference, we chose a bidimensional normal distribution, with mean the mean of the strikes in the subset. From there, we obtain the following payoffs for the solution portfolio (the following plots show portfolio in the USD / EUR / GBP market) :



(a) Maturity 3 days ; expected payoff=2.8



(b) Maturity 14 days ; expected payoff=3.8



(c) Maturity 61 days ; expected payoff=5.3

Figure 10: Payoff empirical distribution for various maturities

Knowing the portfolio for each subset, an arbitrage portfolio would have a zero probability of having a negative payoff. Using a Monte-Carlo method, we can estimate this probability. Underneath is a plot of these estimates, for each best portfolio of constant maturity in the market.

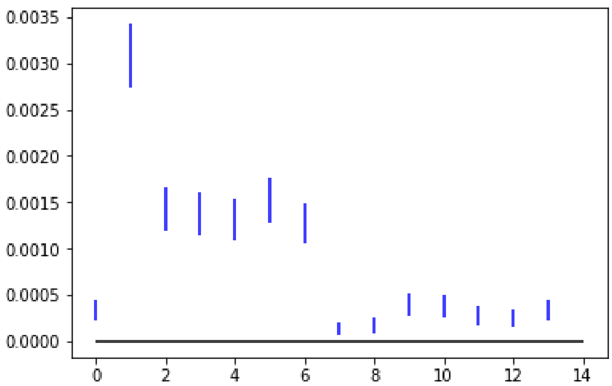


Figure 11: Monte Carlo estimate of negative payoff for each maturity subset, N=10^5

Using this, we can conclude that there is no arbitrage in this option market for the constant maturity subsets.

4.4 Adding a LASSO penalty

To expand on this method, if an arbitrage portfolio was to be found in a market, a good way to identify the smaller arbitrage subset would be to add a LASSO penalty to the function we aim to maximize.

$$
\begin{array}{lr}
\max\limits_{\lambda_i} & \sum_1^N E^P(\lambda_i e^{-rT}(S_T^1 - K S_T^2)_+) - \beta \sum_1^N |\lambda_i| \\
\text{under constraint} & \sum_1^N \lambda_i P_i = 0
\end{array}
\tag{3}
$$

What could then be done would be increasing beta until the solution portfolio ceases to allow for arbitrage, hence lending the smaller arbitrage portfolio within the set. Having shown the absence of arbitrage in our data, this method wasn't useful for us.

4.5 Conclusion : Model-free methods recap

We have proposed a few methods to identify arbitrage opportunities in the foreign exchange option market. First, we have used Frechet bounds to identify over priced/under priced options. Afterwards, we have proposed a cross entropy method, thus building a probability distribution we can use to propose bounds to the option price. However, this method was impractical, and did not lend reliable results. Finally, we have explicitly looked for arbitrage portfolio. The conclusion, across these methods, seems to be that arbitrage opportunities are absent in the studied market.

However, the fact that we did not find arbitrage doesn't definitely prove that they are absent. We can therefore try to find them through other ways, ending the model free approach and tackling the calibration of a Wishart model to fulfill this aim.

5 Calibration of a Wishart model

5.1 Introduction

In this section, we assume that S_t^1 and S_t^2 follow a conditionally Gaussian evolution, while the stochastic covariance matrix follow a Wishart process:

$$dS_t = diag[S_t](r1dt + \sqrt{\Sigma_t}dZ_t)$$

where $1 = (1, ..., 1)^T$ and $Z_t \in R^2$ is a vector Brownian motion. The covariance matrix follows the following evolution:

$$d\Sigma_t = (\omega\Omega^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t}dW_t Q + Q^T(dW_t)^T\sqrt{\Sigma_t}$$

with $\Omega, M, Q \in M_n$, Ω invertible and W_t is a matrix Brownian motion. In order to grant the strict positivity and the typical mean reverting feature of the volatility, the matrix M is assumed to be negative semi-definite, while Ω satisfies :

$$\Omega\Omega^T = \beta\Omega^T\Omega, \beta > n - 1$$

5.2 Pricing the option

Let's now consider the problem of pricing an option whose payoff is $F(S_T, T)$. From the risk-neutral argument, the price P_0 of such option can be written as risk neutral expected value :

$$P_0 = e^{-rT}E^Q[F(S_T, T)]$$

In our case of a basket option, the payoff can be written as :

$$F(X, Y) = (e^X - Ke^Y)_+$$

The price of the option can be expressed in terms of the inverse Fourier transform of the payoff function and the characteristic function of $X = S_T^1$ and $Y = S_T^2$:

$$\begin{aligned} P_0 &= e^{-rT}E^Q[F(X,Y)] \\ &= e^{-rT}\frac{1}{(2\pi)^2}E^Q\int_{R^2}\hat{F}(\xi, z)e^{-i\xi x - izy}d\xi dz \end{aligned}$$

where:

$$\begin{aligned} \hat{F}(\xi, z) &= \int_{R^2} e^{i\xi x + izy}(e^x - Ke^y)_+ dxdy \\ &= \int dxe^{i\xi x}\int dye^{izy}(e^x - Ke^y)1_{x>y+\ln K} \\ &= \int dxe^{i\xi x}\int_{-\infty}^{x-\ln k}\left(e^{izy}e^x - Ke^{(1+iz)y}\right)dy \\ &= \int_R dxe^{i\xi x}\Big(\frac{e^{x+iz(x-\ln K)}}{iz} - K\frac{e^{(1+iz)(x-\ln K)}}{1+iz}\Big) \end{aligned}$$

with the condition $Im(z) < -1$.

Here we are confronted with the problem of the integrability of the function. In order to bypass this problem, we can write the following:

$$\begin{aligned} P_0 &= e^{-rT}\int_{R^2}F(x,y)p(x,y)dxdy \\ &= e^{-rT}\int_{R^2}F(x,y)e^{-\alpha_1 x - \alpha_2 y}e^{\alpha_1 x + \alpha_2 y}p(x,y)dxdy \end{aligned}$$

where p is the density of (X,Y), and α_1, α_2 are chosen such that the function $F(x,y)e^{-\alpha_1 x-\alpha_2 y}$ admits a Fourier transform. This leads to the following conditions on α_1 and α_2 :

$$\alpha_2 > 1$$

$$\alpha_1 > \alpha_2 + 1$$

Let's denote $G(x,y) := F(x,y)e^{-\alpha_1 x-\alpha_2 y}$.

$$\begin{aligned}
P_0 &= e^{-rT}\int_{R^2} F(x,y)p(x,y)dxdy \\
&= e^{-rT}\int_{R^2} F(x,y)e^{-\alpha_1 x-\alpha_2 y}e^{\alpha_1 x+\alpha_2 y}p(x,y)dxdy \\
&= e^{-rT}E^Q[G(X,Y)e^{\alpha_1 x+\alpha_2 y}] \\
&= e^{-rT}\frac{1}{(2\pi)^2}E^Q\int_{R^2}\hat{G}(\xi,z)e^{-i\xi x-izy}e^{\alpha_1 x+\alpha_2 y}d\xi dz \\
&= e^{-rT}\frac{1}{(2\pi)^2}\int_{R^2}E^Q[e^{(\alpha_1-i\xi)x+(\alpha_2-iz)y}]\hat{G}(\xi,z)d\xi dz \\
&= e^{-rT}\frac{1}{(2\pi)^2}\int_{R^2}\psi(\alpha_1-i\xi,\alpha_2-iz)\hat{G}(\xi,z)d\xi dz
\end{aligned}$$

where:

$$\psi(\xi,z) = E^Q[e^{\xi x+zy}]$$

And: $\hat{G}(\xi,z)$ is the Fourier transform of G. The Laplace transform of S_T is given by:

$$\begin{aligned}
\psi(\xi,z) &= E^Q[e^{\xi S_T^1+zS_T^2}] \\
&= e^{Tr[A(\tau)\Sigma_T]+\gamma^T S_T+c(\tau)}
\end{aligned}$$

where $\gamma = (\xi,z)$ and $A(t)\in M_2$ is a deterministic function and is defined as follows:

$$A(\tau) = A_2^2(\tau)^{-1}A_1^2(\tau)$$

$$\begin{pmatrix} A_1^1(\tau) & A_2^1(\tau) \\ A_1^2(\tau) & A_2^2(\tau)\end{pmatrix} = exp\tau\begin{pmatrix} M+Q^T\rho\gamma^T & -2Q^TQ \\ \frac{1}{2}(\gamma\gamma^T-\gamma_1 e_{11}-\gamma_2 e_{22}) & -(M^T+\gamma\rho^T Q)\end{pmatrix}$$

And:

$$c(\tau) = -\frac{\beta}{2}Tr[\log(A_2^2(\tau))+\tau M^T+\tau\gamma(\rho^T Q)]+\tau Tr[r1\gamma^T]$$

and ρ represents a correlation term, that we choose to ignore in a first approach. We also suppose that the covariance matrix Σ is diagonal in this approach. We also set $r=0$. In this case:

$$\beta = 1$$

$$\Sigma = \begin{pmatrix}\sigma_1 & 0 \\ 0 & \sigma_2\end{pmatrix}$$

$$d\sigma_i = (\omega_i^2+2m_i\sigma_i)dt+2\sqrt{\sigma_i}q_i dW_T, i=1,2$$

where: $m_i, q_i \in R$ and $\omega_i \in R^*$.

$$\begin{pmatrix} A_1^1(\tau) & A_2^1(\tau) \\ A_1^2(\tau) & A_2^2(\tau)\end{pmatrix} = exp\tau\begin{pmatrix} M & -2(q_1^2+q_2^2) \\ \frac{1}{2}(x^2+y^2-x-y) & -M^T\end{pmatrix}$$

$$c(\tau) = -\frac{1}{2}Tr[\log(A_2^2(\tau)) + \tau M^T]$$

The price function P_0 is, in total, a function of parameters ω_i, m_i, q_i, τ. The idea is that we compute this price function with these 7 parameters and calibrate them with the real market prices. The optimization problem becomes the following:

$$\begin{array}{cc} \min\limits_{\omega_i, m_i, q_i, \tau} & \sum_{T,K} |P_0(\omega_i, m_i, q_i, \tau) - P^{\text{market}}|^2 \\ \text{s.t.} & \omega_i \neq 0 \end{array} \tag{4}$$

For the implementation, we start with random values of the parameters, then run an optimization scheme to get closer to the minimum of the objective function.

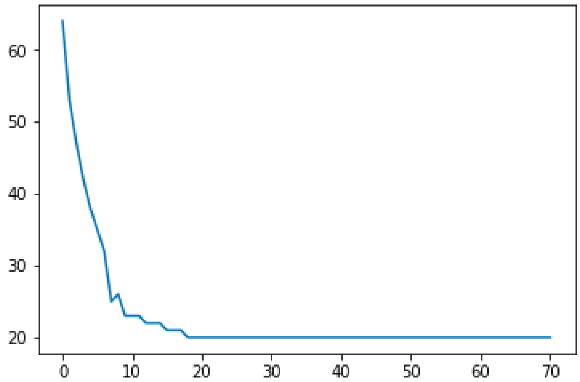


Figure 12: Evolution of the objective function with every iteration of the optimization scheme

The end of this optimization scheme gives us values of our parameters, but we are unsure of whether these parameters are the real values of the optimal ones, because the optimization scheme could stop before finding the optimum due to the specific stopping rules we set. Furthermore, this method doesn't seem very stable with regards to the parameters. Indeed, by slightly changing the initial value of one parameter, ω_1 for example, the optimal value varies widely.

6 Conclusion

Using the empirical data, we have devised a first, working system to detect arbitrage opportunities on call option prices in the foreign exchange market. This first method is model-free, and uses Frechet bounds to characterize arbitrage-free prices, and therefore pinpoint some of the prices on the market that do not satisfy this condition. Two other model-free approaches have been presented in this report.

Then we explored an approach based on the Wishart model, and the calibration of this model to try and replicate the market prices of the option.

Both approaches propose bounds on prices, or price estimates, that seem to indicate the absence of arbitrage opportunity in this market based on the available data.

To end this report, we would like to thank Dr. Peter Tankov for supervising this project. His disponibility and advice have been very helpful in understanding and devising the methods we presented above. Moreover, the data he gave us, and his article, are at the core of this project and of its eventual solution.