تمرین ۳ آمار

سؤال ۱)

$$U = X$$
$$V = \frac{X}{Y} \Rightarrow Y = \frac{U}{V}$$

الف) x و y مستقل $\leftarrow$

$$f_{U,V}(u,v) = \left|\text{Jacobian}(u,v)\right| f_{x,y}(x,y)$$

$$= \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right| f_x(x)\, f_y(y) = \left|\begin{matrix} 1 & 0 \\ \frac{1}{v} & \frac{-u}{v^2} \end{matrix}\right| \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}$$

$$= \left|\frac{u}{v^2}\right| \frac{1}{2\pi} e^{-\frac{u^2}{2}} \times e^{-\frac{u^2}{2v^2}}$$

ب)

$$f_V(v) = \int_{-\infty}^{+\infty} f_{U,V}(u,v)\,du = \int_{-\infty}^{+\infty} \frac{|u|}{2\pi v^2} e^{\frac{-u^2}{2}\left(1+\frac{1}{v^2}\right)} du$$

$$= 2\int_{0}^{+\infty} \frac{u}{2\pi v^2} e^{\frac{-u^2}{2}\left(1+\frac{1}{v^2}\right)} du = 2\int_{0}^{+\infty} \frac{1}{4\pi v^2} e^{-\frac{t}{2}\left(1+\frac{1}{v^2}\right)} dt$$

$$= \frac{-1}{\pi(v^2+1)} e^{\frac{-t}{2}\left(1+\frac{1}{v^2}\right)} \Big|_0^{+\infty} = \frac{-1}{\pi(v^2+1)}(0-1) = \frac{1}{\pi(v^2+1)}$$

از توزیع کوشی با متغیرهای $\gamma=1$ و $x_0=0$ پیروی می کند

سوال ۳)

الف -

$$Y_1 \sim Exp(\lambda_1) \qquad X = \frac{Y_1}{Y_2}$$
$$Y_2 \sim Exp(\lambda_2)$$

$$F_x(x) = \iint_{\frac{y_1}{y_2} \leq x} f_{y_1, y_2}(y_1, y_2)\, dy_1\, dy_2 \Rightarrow \quad x \leq 0 \Rightarrow F_x(x) = 0$$

y_1 و y_2 مستقل $\Leftarrow$

$$F_x(x) = \iint_{\frac{y_1}{y_2} \leq x} f_{y_1}(y_1)\, f_{y_2}(y_2)\, dy_1\, dy_2 = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{x y_2} f_{y_1}(y_1)\, dy_1 \right) f_{y_2}(y_2)\, dy_2$$

$$= \int_{-\infty}^{+\infty} F_{y_1}(x y_2)\, f_{y_2}(y_2)\, dy_2 = \int_{0}^{+\infty} \lambda_2 (1 - e^{-\lambda_1 (x y_2)}) e^{-\lambda_2 y_2}\, dy_2$$

$$= \int_0^{+\infty} \lambda_2 e^{-\lambda_2 y_2}\, dy_2 - \int_0^{+\infty} \lambda_2 e^{-y_2(\lambda_1 x + \lambda_2)}\, dy_2$$

$$= 1 + \frac{\lambda_2}{\lambda_1 x + \lambda_2} e^{-y_2(\lambda_1 x + \lambda_2)} \Big|_0^{+\infty} = 1 - \frac{\lambda_2}{\lambda_1 x + \lambda_2} \Rightarrow F_x(x) = \begin{cases} 0 & 0 \geq x \\ 1 - \frac{\lambda_2}{\lambda_1 x + \lambda_2} & x > 0 \end{cases}$$

$$x \geq 0 \Rightarrow f_x(x) = \frac{d}{dx} F_x(x) = -\lambda_2 \frac{d}{dx} \left(\frac{1}{\lambda_1 x + \lambda_2} \right) = \frac{\lambda_1 \lambda_2}{(\lambda_1 x + \lambda_2)^2} \Rightarrow f_x(x) = \begin{cases} 0 & x < 0 \\ \frac{\lambda_1 \lambda_2}{(\lambda_1 x + \lambda_2)^2} & x \geq 0 \end{cases}$$

ب -

$$y_2 > 1 \Rightarrow x = \frac{y_1}{y_2} < 1$$

$$\Rightarrow F_x(1) = 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

(سؤال ۴)

$$Z = \frac{\sum_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}}$$

$$M_x(t) = E[e^{tZ}]$$

$$B = E\left[e^{t\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}}}\right] = E\left[\exp\left(t\frac{x_1}{\sqrt{np(1-p)}}\right)\exp\left(t\frac{x_2}{\sqrt{np(1-p)}}\right)\dots\exp\left(t\frac{x_n}{\sqrt{np(1-p)}}\right)\cdot\exp\left(\frac{-tnp}{\sqrt{np(1-p)}}\right)\right]$$

$$B = E\left[\exp\left(t\frac{x_1}{\sqrt{np(1-p)}}\right)\right]\cdot E\left[\exp\left(t\frac{x_2}{\sqrt{np(1-p)}}\right)\right]\dots \Leftarrow$$ ها مستقل X_i

$$E\left[\exp\left(t\frac{X_i}{\sqrt{np(1-p)}}\right)\right] = \sum f_x(x)\, e^{t\frac{x}{\sqrt{np(1-p)}}} = p\, e^{t\frac{1}{\sqrt{np(1-p)}}} + (1-p)\, e^{t\frac{0}{\sqrt{np(1-p)}}}$$

$$= p\, e^{\frac{t}{\sqrt{np(1-p)}}} + 1-p \Rightarrow B = E\left[\exp\left(t\frac{x_i}{\sqrt{np(1-p)}}\right)\right]^n E\left[\exp\left(\frac{-tnp}{\sqrt{np(1-p)}}\right)\right]$$

$$\Rightarrow B = \left(1-p+pe^{\frac{t}{\sqrt{np(1-p)}}}\right)^n e^{\frac{-tnp}{\sqrt{np(1-p)}}}$$

سؤال ۵)

$$E[E[X|Y]] = E[X] \quad , \quad E[X\,E[Y|X]] = E[XY]$$

$$\mathrm{cov}(x,y) = \cancel{E} \; E(yx) - E(x)\,E(y)$$

$$\Rightarrow \mathrm{cov}(X+A, Y+B) - \mathrm{cov}(A,B) = E\big((X+A)(Y+B)\big) - E(X+A)\,E(Y+B) - E(AB) + E(A)\,E(B)$$

$$= E(AB) + E(XY) + E(AY) + E(XB) - E(X)E(Y) - E(A)E(B) - E(X)E(B) - E(Y)E(A) - E(AB) + E(A)E(B)$$

$$\Rightarrow ۲E(XY) \;\cancel{E}\; - ۲E(x)E(y) = ۲\,\mathrm{cov}(X,Y)$$

$$\begin{cases} E(A) = E\big(E(X|Y)\big) = E(X), \quad E(B) = E(Y) \\ E(AY) = E\big(Y\,E(X|Y)\big) = E(XY), \quad E(XB) = E(XY) \end{cases}$$

سوال ۶) الف-

$0 \le u, v \le 1$

$$f_{xy}(x,y) = \begin{cases} \frac{1}{2} & 0 \le u+v < 1 \\ \frac{1}{2} & 1 \le u+v < 2 \\ 0 & \text{other} \end{cases} \qquad f_X(u) = \int f_{XY}(u,v)\,dv$$

$$0 \le u+v < 1 \Rightarrow 0 \le v \le 1-u \quad , \quad 1 \le u+v < 2 \Rightarrow 1-u \le v < 1$$

$$f_X(x) = \int f_{XY}(x,y)\,dy \Rightarrow f_X(u) = \int_0^{1-u} \frac{1}{2} + \int_{1-u}^{1} \frac{3}{2} = \frac{1}{2}+u$$

ب-

$$P\left(X+Y \le \frac{3}{2}\right) = P\left(Y \le \frac{3}{2}-X\right) = \int_0^1 \int_0^{\min(\frac{3}{2}-x,1)} f_{XY}(x,y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}\,dy\,dx + \int_0^1 \int_{1-x}^{\min(\frac{3}{2}-x,1)} \frac{3}{2}\,dy\,dx = \frac{1}{4} + \frac{3}{8} + \frac{3}{16} = \frac{13}{16}$$

$$1 - P\{x^2+y^2 \le 1\} = P\{x^2+y^2 \ge 1\}$$

$$\Rightarrow \left(\frac{1}{2}\times 1 \times 1\right)\times\frac{1}{2} + \left(\frac{1}{4}\pi - \frac{1}{2}\times 1\times 1\right)\times\frac{3}{2} = \frac{3\pi}{8} - \frac{1}{2} \Rightarrow P\{x^2+y^2 \ge 1\} = 1 - \left(\frac{3\pi}{8} - \frac{1}{2}\right)$$

$$= \frac{3}{2} - \frac{3\pi}{8}$$

سؤال ۷) الف-

$$Var(X_1+\dots+X_n) = \underbrace{Var(X_1)}_{\sigma^2} + Var(X_2+\dots+X_n) + 2\,cov(X_1, X_2+\dots X_n)$$

$$\Rightarrow Var(X_1+\dots+X_n) = n\sigma^2 + 2\sum_{j>i} cov(x_i, x_j) = n\sigma^2 + \alpha\,(n)(n-1)\times\eta = n\sigma^2 + n(n-1)\eta$$

ب-

$$cov(Y_m, Y_{m+j}) = E[(X_m+X_{m+1}+X_{m+2})(X_{m+j}+X_{m+j+1}+X_{m+j+2})] - E[X_m+X_{m+1}+X_{m+2}]\cdot E[X_{m+j}+X_{m+j+1}+X_{m+j+2}]$$

اگر $j>2$ باشد، Y_m و Y_{m+j} مستقل هستند و حتی I و J مستقل هستند، $cov(I,J)=0$ است

$$\Rightarrow cov(Y_m, Y_{m+j}) = \begin{cases} 9\eta & j>2 \\ 8\eta+\sigma^2 & j=2 \\ 7\eta+2\sigma^2 & j=1 \\ 6\eta+3\sigma^2 & j=0 \end{cases}$$