Randomization and Mixed Strategies

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Mixed strategies

- So far, we have been discussing how to achieve NE by players selecting their pure strategies
- In principle, players can also randomize over their pure strategies
- Let’s seen an example!

Example

- Rock, paper, scissors game
- Pure strategies = {R, S, P}

- Any dominated strategies?

- Pure NE?

	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

Example

- Rock, paper, scissors game
- Pure strategies = {R, S, P}

	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

- Any dominated strategies? No

- Pure NE? No

- Claim: there is a NE if player choose with probability 1/3 each of his pure strategies

Reminder!

- A payoff matrix only shows the payoffs for pure-strategy profiles

- For mixed strategies, use expected value (utility)
  - X: a random variable
  - $E[X]$: Expected value of X
  - A weighted average of different values of X

$$E[X] = \sum x.p(x)$$

Expected utilities

$$E\left[U_1\left(R,\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)\right]=\frac{1}{3}0+\frac{1}{3}1+\frac{1}{3}(-1)=0$$

$$E\left[U_1\left(S,\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)\right]=\frac{1}{3}(-1)+\frac{1}{3}0+\frac{1}{3}1=0$$

$$E\left[U_1\left(P,\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)\right]=\frac{1}{3}1+\frac{1}{3}(-1)+\frac{1}{3}0=0$$

$$\Rightarrow E\left|U_1\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)\right|=\frac{1}{3}0+\frac{1}{3}0+\frac{1}{3}0=0$$

Definition 1

Definition: Mixed strategies

A mixed strategy p_i is a randomization over i's pure strategies

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A mixed strategy p_i is a randomization over i's pure strategies

- $p_i(s_i)$ is the probability that p_i assigns to pure strategy s_i
- $p_i(s_i)$ could be zero → in RSP: (1/2, 1/2, 0)
- $p_i(s_i)$ could be one → in RSP: 'P' a pure strategy if $p_i(P) = 1$

Definition 2

Definition: Expected Payoffs

The expected payoff of the mixed strategy p_i
is the weighted average of
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The expected payoff of the mixed strategy p_i is the weighted average of the expected payoffs of each of the <u>pure strategies</u> in <u>the mix of -i</u>

- Every player is mixing, hence you have to take the joint probabilities for a strategy profile to occur

Another example

- Battle of the sexes game

- Player 1: p = (1/5, 4/5)
- Player 2: q = (1/2, 1/2)

- What's the players expected payoff by using *p and q*?

		Player 2		
		M	N	
Player 1	M	2,1	0,0	1/5
	N	0,0	1,2	4/5
		½	½	

Another example

- Battle of the sexes game

- Player 1's expected payoff by using p?

		Player 2 M	Player 2 N	
Player 1	M	2,1	0,0	1/5
	N	0,0	1,2	4/5
		½	½	

$$E\left[U_1\left(M,\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{2}2+\frac{1}{2}0=1$$

$$E\left[U_1\left(N,\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{2}0+\frac{1}{2}1=\frac{1}{2}$$

Another example

- Battle of the sexes game

- Player 2's expected payoff by using q?

		Player 2 M	Player 2 N	
Player 1	M	2,1	0,0	1/5
	N	0,0	1,2	4/5
		½	½	

$$E\left[U_2\left(\left(\frac{1}{5},\frac{4}{5}\right),M\right)\right]=\frac{1}{5}1+\frac{4}{5}0=\frac{1}{5}$$

$$E\left[U_2\left(\left(\frac{1}{5},\frac{4}{5}\right),N\right)\right]=\frac{1}{5}0+\frac{4}{5}2=\frac{8}{5}$$

Another example

- Battle of the sexes game

			Player 2	
		M	N	
Player 1	M	2,1	0,0	1/5
	N	0,0	1,2	4/5
		½	½	

- What's the players expected payoff totally?

$$E\left[U_1\left(\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{5}1+\frac{4}{5}\frac{1}{2}=\frac{3}{5}$$

$$E\left[U_2\left(\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)\right]=\frac{1}{2}\frac{1}{5}+\frac{1}{2}\frac{8}{5}=\frac{9}{10}$$

Definition 3

Definition: Mixed Best Response

if $p_i^*(s_i) > 0 \Rightarrow s_i^*$ is also a BR to p_{-i}^*

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To compute mixed strategies NE, $E[u_i(s_i, p_{-i})]$ must be the same for all pure strategy s_i such that $p_{-i}(s_i) > 0$.

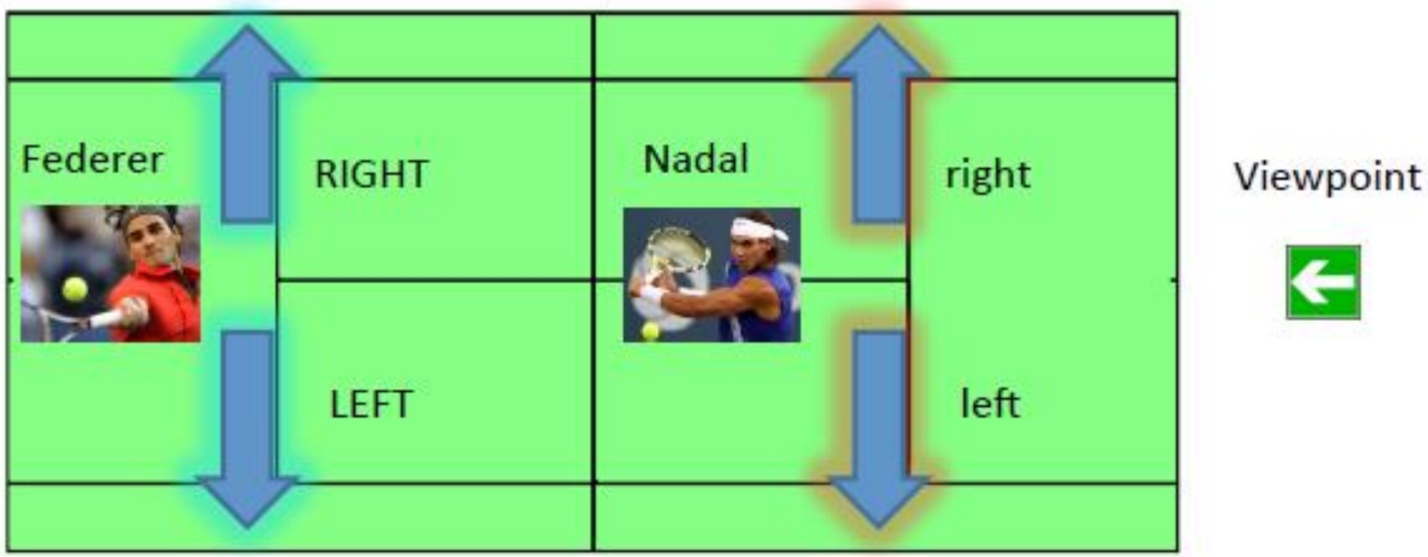
Definition 4

Definition: Mixed Strategies Nash Equilibrium

A mixed strategy profile $\left(p_1^*, p_2^*, \ldots, p_N^*\right)$ is a mixed strategy NE if for each player *i*:
p_i^* is a BR to p_{-i}^*

- Let’s go to some examples for finding p,q values

Tennis game



Tennis game

		Nadal	
		l	*r*
Federer	*L*	50,50	80,20
	R	90,10	20,80

- Is there any dominated strategies?
- Is there any pure strategy NE profile?

Tennis game

		Nadal *l*	Nadal *r*
Federer	*L*	50,50	80,20
Federer	*R*	90,10	20,80

- Is there any dominated strategies? No
- Is there any pure strategy NE profile? No

Tennis game

- How to find mixed strategy NE?
  - Each player's randomization is the best response to the other player's randomization

		Nadal		
		l	*r*	
Federer	*L*	50,50	80,20	p
	R	90,10	20,80	1-p
		q	1-q	

Tennis game

- Federer's expected utility?

$$E\left[U_{Federer}\left(L,(q,1-q)\right)\right]=50q+80(1-q)$$
$$E\left[U_{Federer}\left(R,(q,1-q)\right)\right]=90q+20(1-q)$$

		Nadal *l*	Nadal *r*	
Federer	*L*	50,50	80,20	p
	R	90,10	20,80	1-p
		q	1-q	

Tennis game

- Federer's expected utility?

$$E\left[U_{Federer}\left(L,(q,1-q)\right)\right]=50q+80(1-q)$$

$$E\left[U_{Federer}\left(R,(q,1-q)\right)\right]=90q+20(1-q)$$

Definition 3

$$\Rightarrow 50q+80(1-q)=90q+20(1-q)$$

$$\Rightarrow 40q=60(1-q)$$

$$\Rightarrow q=0.6$$

		Nadal		
		l	*r*	
Federer	*L*	50,50	80,20	p
	R	90,10	20,80	1-p
		q	1-q	

Tennis game

- Similarly, Nadal's expected utility?

		Nadal		
		l	*r*	
Federer	*L*	50,50	80,20	p
	R	90,10	20,80	1-p
		q	1-q	

$$E\left[U_{Nadal}\left((p,1-p),l\right)\right] = 50p + 10(1-p)$$

$$E\left[U_{Nadal}\left((p,1-p),r\right)\right] = 20p + 80(1-p)$$

$$\Rightarrow 50p + 10(1-p) = 20p + 80(1-p)$$

$$\Rightarrow 30p = 70(1-p)$$

$$\Rightarrow p = 0.7$$

Tennis game

		Nadal		
		l	*r*	
Federer	*L*	50,50	80,20	p
	R	90,10	20,80	1-p
		q	1-q	

- Mixed strategy NE = $\{(p\ , 1-p), (q, 1-q)\}$

$$= \{(0.7\ , 0.3), (0.6{,}0.4)\}$$

L R l r

Tennis game

- What would happen if Nadal jumped to the left more often than 0.6?
  - Federer would be better of playing the pure strategy **'R'!**
- What if he jumped less often than 0.6?
  - Federer would be shooting to the **'L'** all time**!**

		Nadal *l*	Nadal *r*
Federer	*L*	50,50	80,20
Federer	*R*	90,10	20,80

$$\{(\underset{L}{0.7}\,,\underset{R}{0.3}),(\underset{l}{0.6},\underset{r}{0.4})\}$$

NASH THEOREM

Every finite game has
a mixed strategy Nash equilibrium

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- Why this is important?
- Armed with this theorem, we also know that every finite game has an equilibrium, and thus we can simply try to locate the equilibria.