Best Response, Nash Equilibrium

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Example

	X	Y	Z
A	2, 1	0, 1	1, 0
B	0, 1	2, 1	1, 0
C	1, 1	1, 0	0, 0
D	1, 0	0, 1	0, 0

- Weakly Dominated strategy for player1?
- Weakly Dominated strategy for player2?

Example

	X	Y	Z
A	2, 1	0, 1	1, 0
B	0, 1	2, 1	1, 0
C	1, 1	1, 0	0, 0
D	1, 0	0, 1	0, 0

- Weakly Dominated strategy for player1? D is Weakly Dominated by A.
- Weakly Dominated strategy for player2? Z is Weakly Dominated by X or Y.

Example

- After elimination D,Z:

	X	Y
A	$2, 1$	$0, 1$
B	$0, 1$	$2, 1$
C	$1, 1$	$1, 0$

- Now, Weakly Dominated strategy for player1?
- Weakly Dominated strategy for player2?

Example

- After elimination D,Z:

	X	Y
A	$2, 1$	$0, 1$
B	$0, 1$	$2, 1$
C	$1, 1$	$1, 0$

- Now, Weakly Dominated strategy for player1? No
- Weakly Dominated strategy for player2? Y is W.D by X

Example

- After elimination Y:

	X
A	2, 1
B	0, 1
C	1, 1

- Now, Weakly Dominated strategy for player1?
- Weakly Dominated strategy for player2?

Example

- After elimination Y:

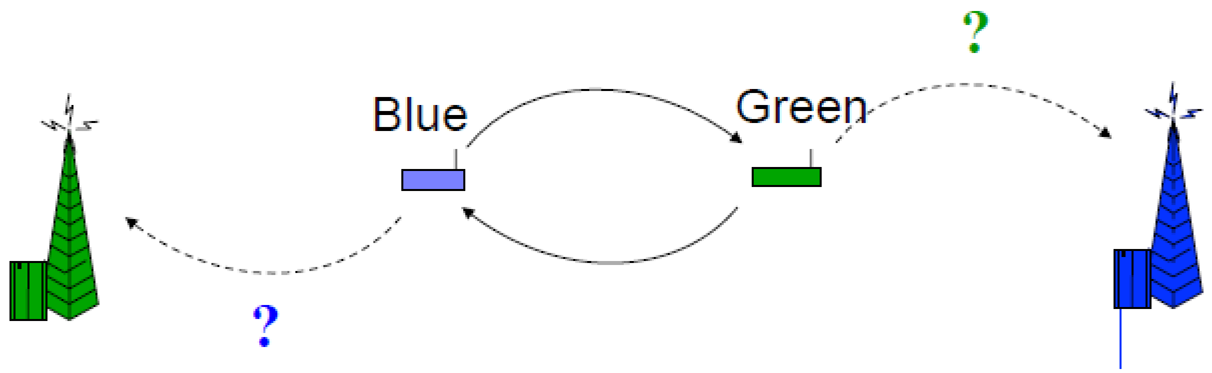
	X
A	$2, 1$
B	$0, 1$
C	$1, 1$

- Now, Weakly Dominated strategy for player1? B is W.D by A, then C is W.D by A.
- Weakly Dominated strategy for player2? No
- Finally, the reduced game is

	X
A	$2, 1$

Forwarder Game

- Reward for packet delivering to destination: 1
- Cost of packet forwarding: c
- $0 < c \ll 1$



- Model this situation in normal form game.

Forwarder Game

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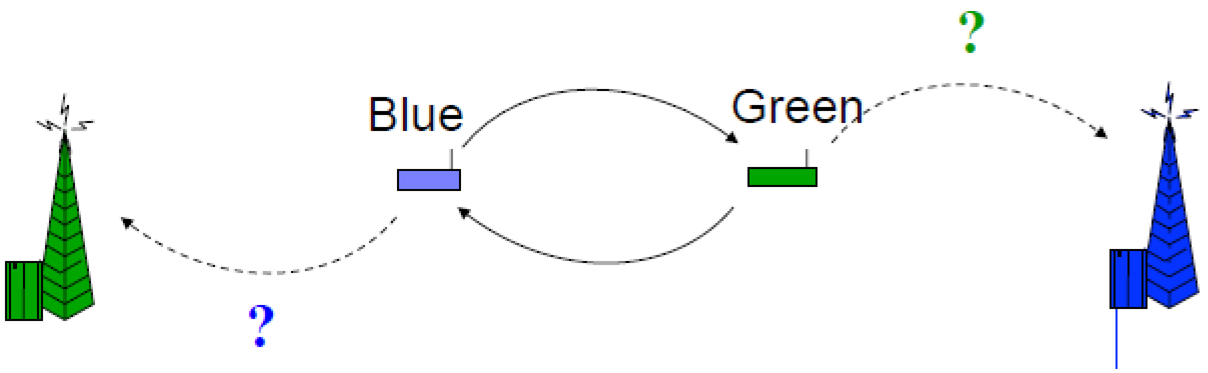

- Model this situation in normal form game.

Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 1)
Drop	(1, -c)	(0, 0)

Forwarder Game

- Is it a symmetric game?

Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 1)
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Forwarder Game

- Is it a symmetric game? Yes

Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 1)
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Forwarder Game

- Is it a symmetric game? Yes

- What is best responses?
  - For player1:

  - For player2:

Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 1)
Drop	(1, -c)	(0, 0)

Forwarder Game

- Is it a symmetric game? Yes

- What is best responses?
  - For player1: br_1(Forward) = Drop and br_1(Drop) = Drop
  - For player2: br_2(Forward) = Drop and br_2(Drop) = Drop

Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 1)
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Forwarder Game

- Is it a symmetric game? Yes
- What is best responses?
  - For player1: br_1(Forward) = Drop and br_1(Drop) = Drop
  - For player2: br_2(Forward) = Drop and br_2(Drop) = Drop
- Similar to witch previous games?

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Forwarder Game

- Is it a symmetric game? Yes
- What is best responses?
  - For player1: br_1(Forward) = Drop and br_1(Drop) = Drop
  - For player2: br_2(Forward) = Drop and br_2(Drop) = Drop
- Similar to witch previous games?
  - Prisoner's Dilemma

Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 1)
Drop	(1, -c)	(0, 0)

Best Response

Definition: Best Response

Player i's strategy $\hat{s}_i$ is a BR to strategy s_{-i} of other players if:

$$u_i(\hat{s}_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \text{ for all } s'_i \text{ in } S_i$$

or

$$\hat{s}_i \text{ solves } \max u_i(s_i, s_{-i})$$

- Question: This definition is similar to which of the previous definitions?

Reminder

Definition: Best Response

Player i's strategy $\hat{s}_i$ is a BR to strategy s_{-i} of other players if:

$$u_i(\hat{s}_i , s_{-i}) \geq u_i(s'_i , s_{-i}) \text{ for all } s'_i \text{ in } S_i$$

or

$$\hat{s}_i \text{ solves } max\ u_i(s_i , s_{-i})$$

Definition: Strict dominance

We say player i's strategy s_i' is strictly dominated by player i's strategy s_i if:

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \text{ for all } s_{-i}$$

Best response vs. Strict dominance

- S_{-i} = [$S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n$]
- S = [S_{-i}, S_i]

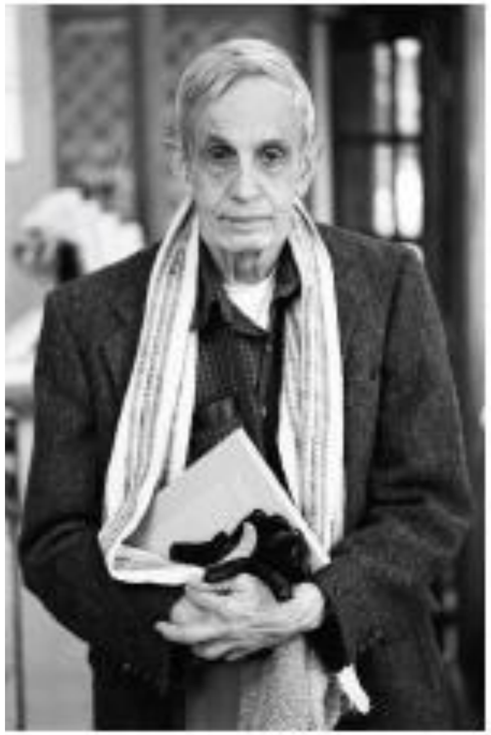
- S.D: for all other players strategies (S_{-i})
- BR: for all player i's strategies (S'_i in S_i)

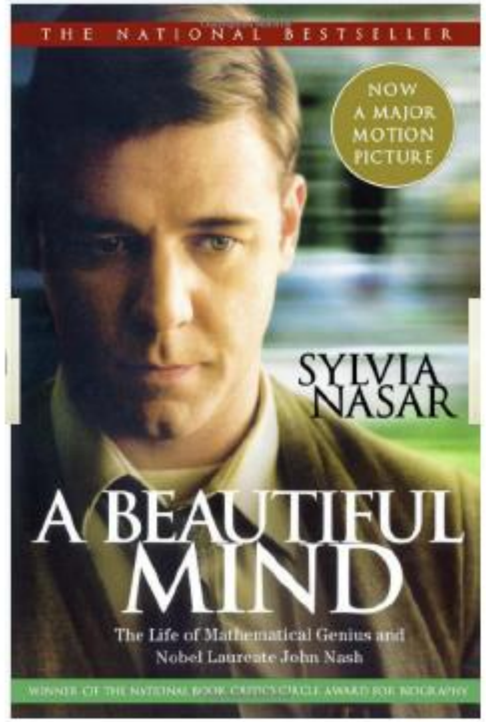
Main Lesson

Rational players don't choose a strategy that is never a Best Response!

About Nash

- John Nash (1928, 2015)
- Princeton Mathematics Department
- Economic Nobel prize at 1994
- Abel Prize at 2015





Nash Equilibrium

- All players simultaneously play best response to others

Definition (1): Nash Equilibrium

A strategy profile (s_1*, s_2*,…, s_N*) is a **Nash Equilibrium (NE)** if, for each i, her choice s_i* is a best response to the other players' choices s_{-i}*

Nash Equilibrium = Mutual best responses

Nash Equilibrium

Definition (2): Nash Equilibrium

At **Nash Equilibrium** no player can increase its payoff by deviating unilaterally.

No regret for every player!

Nash Equilibrium

Definition (3): **Nash Equilibrium**

Strategy profile s^* constitutes a **Nash Equilibrium** if, for each player *i*,

Where: $$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \forall s_i \in S_i$$

$u_i \in U$ utility function of player i

$s_i \in S_i$ strategy of player i

Challenges

- Does any game have a Nash equilibrium?

- Is there a game with more than one Nash equilibrium?

Finding Nash Equilibrium

		Player 2		
		l	c	r
	U	0,4	4,0	5,3
Player 1	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

- NE= ?

Finding Nash Equilibrium

		Player 2		
		l	c	r
	U	0,4	4,0	5,3
Player 1	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

- Definition1: What is best responses?

✧ $BR_1(l) =$ $BR_2(U) =$
✧ $BR_1(c) =$ $BR_2(M) =$
✧ $BR_1(r) =$ $BR_2(D) =$

Finding Nash Equilibrium

		Player 2		
		l	c	r
Player 1	U	0,4	4,0	5,3
	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

- Definition1: What is best responses?

  - $BR_1(l) = M$ $BR_2(U) = l$
  - $BR_1(c) = U$ $BR_2(M) = c$
  - $BR_1(r) = D$ $BR_2(D) = r$

- NE = (D,r)

Finding Nash Equilibrium

			Player 2	
		l	c	r
	U	0,4	4,0	5,3
Player 1	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

- Definition2: deviation unilaterally?
  - If $s^* = (U, l)$, then deviation is profitable?
  - If $s^* = (U, c)$, then deviation is profitable?
  - If $s^* = (U, r)$, then deviation is profitable?
  - ….
  - If $s^* = (D, c)$, then deviation is profitable?
  - If $s^* = (D, r)$, then deviation is profitable?

Finding Nash Equilibrium

		Player 2		
		l	c	r
Player 1	U	0,4	4,0	5,3
	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

- Definition2: deviation unilaterally?
  - If $s^* = (U, l)$, then deviation is profitable? Yes
  - If $s^* = (U, c)$, then deviation is profitable? Yes
  - If $s^* = (U, r)$, then deviation is profitable? Yes
  - ….
  - If $s^* = (D, c)$, then deviation is profitable? Yes
  - If $s^* = (D, r)$, then deviation is profitable? No for each player $\Rightarrow$ NE

Finding Nash Equilibrium

- NE for Prisoner's Dilemma game?

- NE for Driving game (coordination game)?

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

	Left	Right
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

Examples

- NE for Prisoner's Dilemma game?
  - NE = (D,D)
- NE for Driving game (coordination game)?
  - NE = (Left, Left) and (Right, Right)

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$\boxed{-3, -3}$

	Left	Right
Left	$\boxed{1, 1}$	$0, 0$
Right	$0, 0$	$\boxed{1, 1}$

Examples

- Pick a number game
- What is NE for n players?
  - NE = (1,1,…,1)

Examples

- Matching Penny game
  - One player wants to match, other player wants to mismatch
- What is NE?

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Examples

- Matching Penny game
- What is NE?
    - No pure NE

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Examples

- Battle of the sexes (BS) game
- What is NE?

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Examples

- Battle of the sexes (BS) game
- What is NE?
  - NE = (B, B) and (F, F)

	B	F
B	$\boxed{2, 1}$	$0, 0$
F	$0, 0$	$\boxed{1, 2}$

First Theorem

- Iterated Elimination of Strictly Dominated Strategies (IESDS)

If G is a finite game and solved by IESDS, then the resulting outcome is unique NE of G. (order independent)

Second Theorem

- Iterated Elimination of Weakly Dominated Strategies (IEWDS)

If G is a finite game and solved by IEWDS, then the resulting outcome is a Nash equilibria of G.

This outcome does not need to be unique. (order dependent)

Third Theorem

- Iterated Elimination of Never Best Responses (IENBR)

If G is a finite game and solved by IENBR, then the resulting outcome is unique NE. (order independent)

Analyzing Games

- From the point of view of an outside observer, can some strategy profiles of a game be said to be better than others?

- Are there ways to prefer one profile to another?

Pareto Optimality

- Informal Definition

Definition

A strategy profile is pareto optimal if it is not possible to increase the payoff of any player without decreasing the payoff of another player.

Example

- Driving game
- Is there any pareto optimal?

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Example

- Driving game
- Is there any pareto optimal?
  - PO = (left,left) and (right,right)

	Left	Right
Left	$\boxed{1, 1}$	0, 0
Right	0, 0	$\boxed{1, 1}$

Example

- PD game
- Is there any pareto optimal?

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Example

- PD game
- Is there any pareto optimal?
  - PO = (C,C) and (C,D) and (D,D)

	C	D
C	$\boxed{-1,-1}$	$\boxed{-4,0}$
D	$\boxed{0,-4}$	$-3,-3$

Pareto Dominance

- Sometimes, one strategy profile s is at least as good for every agent as another profile s', and there is some agent who strictly prefers s to s'.

- In this case, it seems reasonable to say that s is better than s' we say that s Pareto-dominates s'.

Pareto Optimality

- Formal Definition

Definition (Pareto Optimality)

A profile s^* Pareto-optimal if there is no other profile that Pareto-dominates it.