The Bargaining Set

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Introduction to Bargaining Set

- If agents desire the kind of stability offered by the core, they will be unable to reach an agreement.
- They have no choice but to relax their stability requirements.
- Need a solution that allows agents to reach an agreement, but maintain some stability.
- Then, we will consider the bargaining set, which relaxes the requirements of the core

A solution concept:

The bargaining set

Let (N, v, S) be a game with coalition structure and x an imputation.

The bargaining set models stability in the following sense:

Any argument from an agent i against a payoff distribution x is of the following form:

> I get too little in the imputation x, and agent j gets too much! I can form a coalition that excludes j in which some members benefit and all members are at least as well off as in x.

The argument is ineffective for the bargaining set if agent j can answer the following:

> I can form a coalition that excludes agent i in which all agents are at least as well off as in x, and as well off as in the payoff proposed by i for those who were offered to join i in the argument.

Definition (Objection)

Let (N, v, S) be a game with coalition structure, $x \in X_{(N,v,S)}$ (the set of all feasible payoff vectors for (N, v, S)),

$e \in S$ be a coalition, and i and j two distinct members of e ($(i, j) \in e^2$, $i \neq j$).

An objection of i against j is a pair (P, y) where

- $P \subseteq N$ is a coalition such that $i \in P$ and $j /\in P$.
- $y \in R^p$ where p is the size of P
- $y(P)$ 6 $v(P)$ (y is a feasible payoff distribution for the agents in P)
- $\forall k \in P$, $y_k > x_k$ and $y_i > x_i$ (agent i strictly benefits from y, and the other members of P do not do worse in y than in x.)

An objection (P, y) of i against j is a potential threat by coalition P, which contains i but not j, to deviate from x.

The goal is not to change S, but to obtain a side payment from j to i, i.e., to modify x within $X_{(N,v,S)}$.

Definition (Stability)

Let (N, v, S) a game with coalition structure. A vector $x \in X(N,v,S)$ is stable iff for each objection at x there is a counter-objection.

Definition (Pre-bargaining set)

The pre-bargaining set (preBS) is the set of all stable members of $X_{(N,v,S)}$.

Lemma

Let (N, v, S) a game with coalition structure, we have $Core(N, v, S) \subseteq preBS(N, v, S)$.

This is true since, if $x \in Core(N, v, S)$, no agent i has any objection against any other agent j.

Example

Let (N, v) be a 7-player simple majority game, i.e.

$v(C) = \{ 1$ if $|C| > 4$

$\{0$ otherwise

Let us consider $x = \langle - 1/5 , 1/5 , . . . , 1/5 \rangle$. It is clear that $x(N) = 1$.

Let us prove that x is in the pre-bargaining set of the game (N, v, {N}).

Objections within members of {2, 3, 4, 5, 6, 7} will have a counter objection by symmetry.

Let us consider the objections (P, y) of 1 against another member of {2, 3, 4, 5, 6, 7}.

Since the players {2, . . . , 7} play symmetric roles, we consider an objection (P, y) of 1 against 7 using successively as P {1, 2, 3, 4, 5, 6}, {1, 2, 3, 4, 5}, {1, 2, 3, 4}, {1, 2, 3}, {1, 2} and {1}q. We will look for a counter-objection of player 7 using (Q, z).

- $P = \{1, 2, 3, 4, 5, 6\}$. We need to find the payoff vector $y \in R6$ so that (P, y) is an objection.

  $y = \langle \alpha, 1/5 + \alpha_2, 1/5 + \alpha_3, \ldots, 1/5 + \alpha_6 \rangle$.

  The conditions for (P, y) to be an objection are the following:

  - each agent is as well off as in x: $\alpha > -1/5$, $\alpha_i \geq 0$
  - y is feasible for coalition P: $\Sigma^6_{i=2}(\alpha_i + 1/5) + \alpha \leq 1$.

  w.l.o.g $0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6$.

  Then $\sum^6_{i=2}(1/5 + \alpha_i) + \alpha = 5/5 + \sum^6_{i=2} \alpha_i + \alpha = 1 + \sum^6_{i=2} \alpha_i + \alpha \leq 1$.

  Then $\sum^6_{i=2} \alpha_i \leq -\alpha < 1/5$.

  We need to find a counter-objection for (P, y).

  claim: we can choose $Q = \{2, 3, 4, 7\}$ and $z = \langle 1/5 + \alpha_2, 1/5 + \alpha_3, 1/5 + \alpha_4, 1/5 + \alpha_5 \rangle$

  $z(Q) = 1/5 + \alpha_2 + 1/5 + \alpha_3 + 1/5 + \alpha_4 + 1/5 + \alpha_5 = 4/5 + \sum^5_{i=2} \alpha_i \leq 1$ since

  $\sum^5_{i=2} \alpha_i \leq \sum^6_{i=2} \alpha_i < 1/5$ so z is feasible.

  It is clear that $\forall i \in Q, z_i > x_i$ and that $\forall i \in Q \cap P, z_i > y_i$

  Hence, (Q, z) is a counter-objection.

- $P = \{1, 2, 3, 4, 5\}$. The vector $y = \langle \alpha, 1/5 + \alpha_2, 1/5 + \alpha_3, 1/5 + \alpha_4, 1/5 + \alpha_5 \rangle$

  is an objection when

  $\alpha > -1/5$, $\alpha_i \geq 0$, $\sum^5_{i=2}(1/5 + \alpha_i) + \alpha \leq 1$

  This time, we have $\sum^5_{i=2} (1/5 + \alpha_i) + \alpha = 4/5 + \sum^5_{i=2} \alpha_i + \alpha \leq 1$

  Then $\sum^5_{i=2} \alpha_i \leq 1 - 4/5 - \alpha = 1/5 - \alpha$ and finally $\sum^5_{i=2} \alpha_i \leq 1/5 - \alpha < 2/5$.

  We need to find a counter-objection to (P, y)

  claim: we can choose $Q = \{2, 3, 6, 7\}$, $z = \langle 1/5 + \alpha_2, 1/5 + \alpha_3, 1/5 , 1/5 \rangle$

  It is clear that $\forall i \in Q, z_i > x_i$ and $\forall i \in P \cap Q\ z_i > y_i$ (for agent 2 and 3).

  $z(Q) = 1/5 + \alpha_2 + 1/5 + \alpha_3 + 1/5 + 1/5 = 4/5 + \alpha_2 + \alpha_3$. We have $\alpha_2 + \alpha_3 < 1/5$, otherwise, we would have $\alpha_2 + \alpha_3 \geq 1/5$ and since the α_i are ordered, we would then have $\sum^5_{i=2}\alpha_i \geq 2/5$, which is not possible. Hence $z(Q) \leq 1$ which proves z is feasible

Using similar arguments, we find a counter-objection for each other objections (you might want to fill in the details at home).

- $P = \{1, 2, 3, 4\}$, $y = \langle \alpha, 1/5 + \alpha_1, 1/5 + \alpha_2, 1/5 + \alpha_3 \rangle$, $\alpha > -1/5$, $\alpha_i \geq 0$, $\sum^4_{i=2} \alpha_i + \alpha \leq 2/5 \Rightarrow \sum^4_{i=2} \alpha_i \leq 2/5 - \alpha < 3/5$.

$\rightarrow Q = \{2, 5, 6, 7\}$, $z = \langle 1/5 + \alpha_2, 1/5 , 1/5 , 1/5 \rangle$ since $\alpha_2 \leq 1/5$

- $|P| \leq 3$ $P = \{1, 2, 3\}$, $v(P) = 0$, $y = \langle \alpha, \alpha_1, \alpha_2 \rangle$, $\alpha > -1/5$, $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 \leq -\alpha < 1/5$

$\rightarrow Q = \{4, 5, 6, 7\}$, $z = \langle 1/5 , 1/5 , 1/5 , 1/5 \rangle$ will be a counter argument

(1 cannot provide more than 1/5 to any other agent).

- For each possible objection of 1, we found a counter-objection. Using similar arguments, we can find a counter-objection to any objection of player 7 against player 1.

$\rightarrow x \in preBS(N, v, S)$.

Bargaining set

In the example, agent 1 gets - 1/5 when $v(C) > 0$ for all coalition $C \subseteq N$! This shows that the pre-bargaining set may not be individually rational.

Let $I(N, v, S) = \{x \in X_{(N,v,S)} \mid x_i \geq v(\{i\})\ \forall i \in N\}$ be the set of individually rational payoff vector in $X_{(N,v,S)}$.

Lemma

If a game is weakly superadditive, $I(N, v, S) \neq \emptyset$.

Definition (Bargaining set)

Let (N, v, S) a game in coalition structure.

The bargaining set (BS) is defined by $BS(N, v, S) = I(N, v, S) \cap preBS(N, v, S)$.

Lemma

We have $Core(N, v, S) \subseteq BS(N, v, S)$.

Theorem

Let (N, v, S) a game with coalition structure. Assume that $I(N, v, S) \neq \emptyset$. Then the bargaining set $BS(N, v, S)$ 6 = $\emptyset$

Proof

It is possible to give a direct proof of this theorem (a bit long, (Section 4.2 in Introduction to the Theory of Cooperative Games)).

We will show this result in a different way in the lecture about the nucleolus next week.

Definition (weighted voting games)

A game $(N, w_i \in N, q, v)$ is a weighted voting game when v satisfies unanimity, monotonicity and the valuation function is defined as

$v(S) =$

$\{1$ when $\sum_{i \in S} w_i \geq q$

$\{0$ otherwise

We note such a game by $(q : w_1, \ldots, w_n)$

Let (N, v) be the game associated with the 6-player weighted majority game (3:1,1,1,1,1,0).

Agent 6 is a null/dummy player since its weight is 0.

Nevertheless $\langle 1/7, \ldots, 1/7, 2/7 \rangle \in BS(N, v)$

Agent 6 is a dummy, however, it receives a payoff of 2/7, which is larger than agents who are not dummy!

Summary

- We introduced the bargaining set, and looked at some examples.

pros: it is guaranteed to be non-empty, when the core is non-empty, it is contained in the bargaining set.

cons: it may not be reasonable from above.