Zero-Sum Games

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2 person zero-sum games

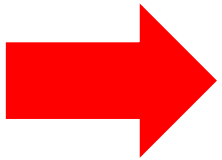
- A kind of symmetric games
- Player 1 maximizes matrix entry, while player 2 minimizes
- Game matrix has single entry (payoff of player 1)

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

- Game matrix has single entry (payoff of player 1)

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$



	Heads	Tails
Heads	1	-1
Tails	-1	1

Example

		Player 2			
		A	B	C	D
Player 1	A	12	-1	1	0
	B	3	1	3	-18
	C	5	2	4	3
	D	-16	1	2	-1

- Is there any strictly dominated strategies?
  - For player 1:
  - For player 2:

Example

		Player 2			
		A	B	C	D
Player 1	A	12	-1	1	0
	B	3	1	3	-18
	C	5	2	4	3
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- Is there any strictly dominated strategies?
  - For player 1: B is strictly dominated by C
  - For player 2: C is strictly dominated by B

Analyzing zero-sum games

- How player1 play?
- a conservative agent maximizing worst-case payoff

- How player2 play?
- to punish the other agent as much as possible

Example

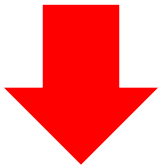
Player 1 \ Player 2	A	B	C	D
A	12	-1	1	0
B	3	1	3	-18
C	5	2	4	3
D	-16	1	2	-1

- choose maximum entry in each column
- choose the minimum among these
- this is the minimax value

- choose minimum entry in each row
- choose the maximum among these
- this is maximin value

Example

		Player 2				
		A	B	C	D	
Player 1	A	12	-1	1	0	-1
	B	3	1	3	-18	-18
	C	5	2	4	3	2
	D	-16	1	2	-1	-16
		12	2	4	3	

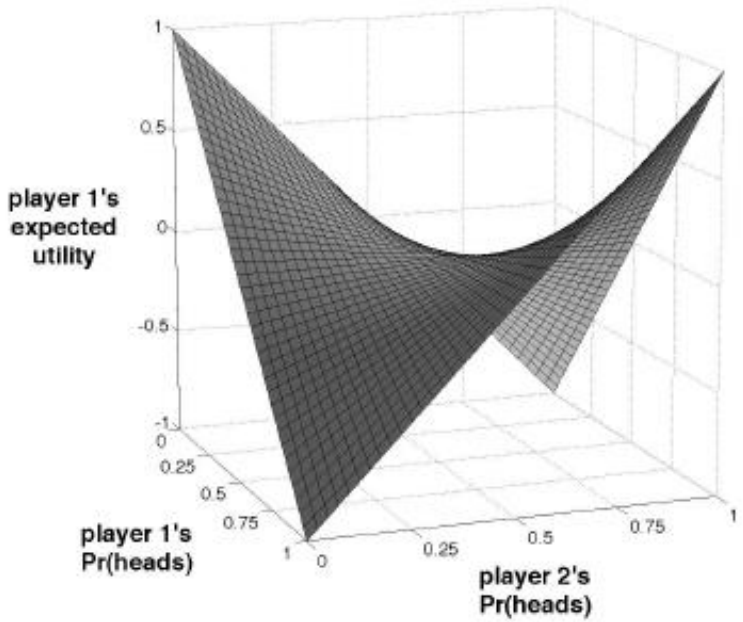


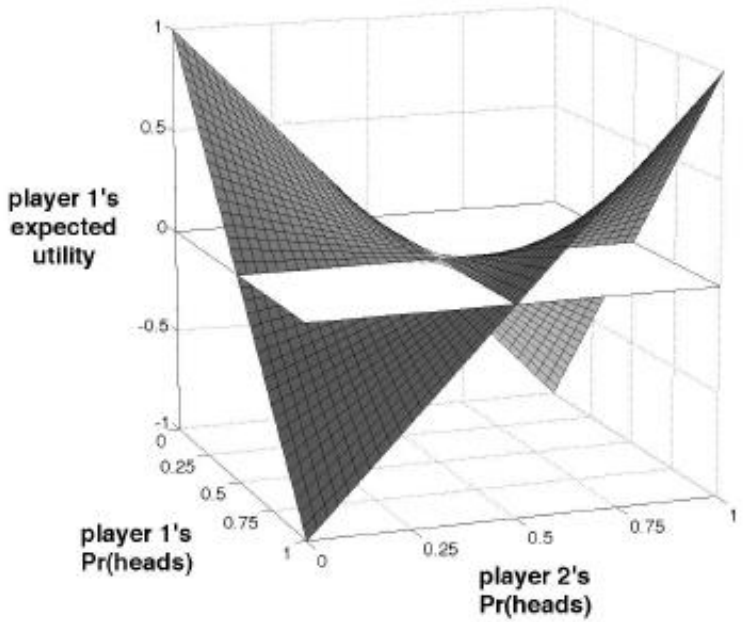
(Maxmin value, Minmax value) = (2,2)

(Maxmin strategy, Minmax strategy) = (C,B)

Saddle point

- if minimax == maximin, then this is the saddle point of game





Maxmin

- Player *i*'s maxmin strategy is a strategy that maximizes *i*'s worst-case payoff, in the situation where all the other players (S_i) happen to play the strategies which cause the greatest harm to *i*.

- The maxmin value (or safety level) of the game for player *i* is that minimum payoff guaranteed by a maxmin strategy.

Maxmin

- Formal definition

Definition (Maxmin)

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$, and the maxmin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$.

Minmax

- Player *i*'s minmax strategy against the other player in a 2-player game is a strategy that minimizes $S\__i$'s best-case payoff

- The minmax value for *i* against others is his payoff in minmax strategy.

Minmax

- Formal definition

Definition (Minmax, 2-player)

In a two-player game, the minmax strategy for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

About von Neumann

- John von Neumann (1903, 1957)
- von Neumann theorem at 1928

Minimax Theorem

Theorem (Minimax theorem (von Neumann, 1928))

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

Minimax Theorem

- Each player's maxmin value is equal to his minmax value.
- The maxmin value for player 1 is called the value of the game.

- For both players, the set of maxmin strategies coincides with the set of minmax strategies.

- Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria.

2*2 zero sum games

- Minmax or maxmin produces the same result as method for finding NE in general 2 * 2 games;