Extensive Form Games I

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Introduction

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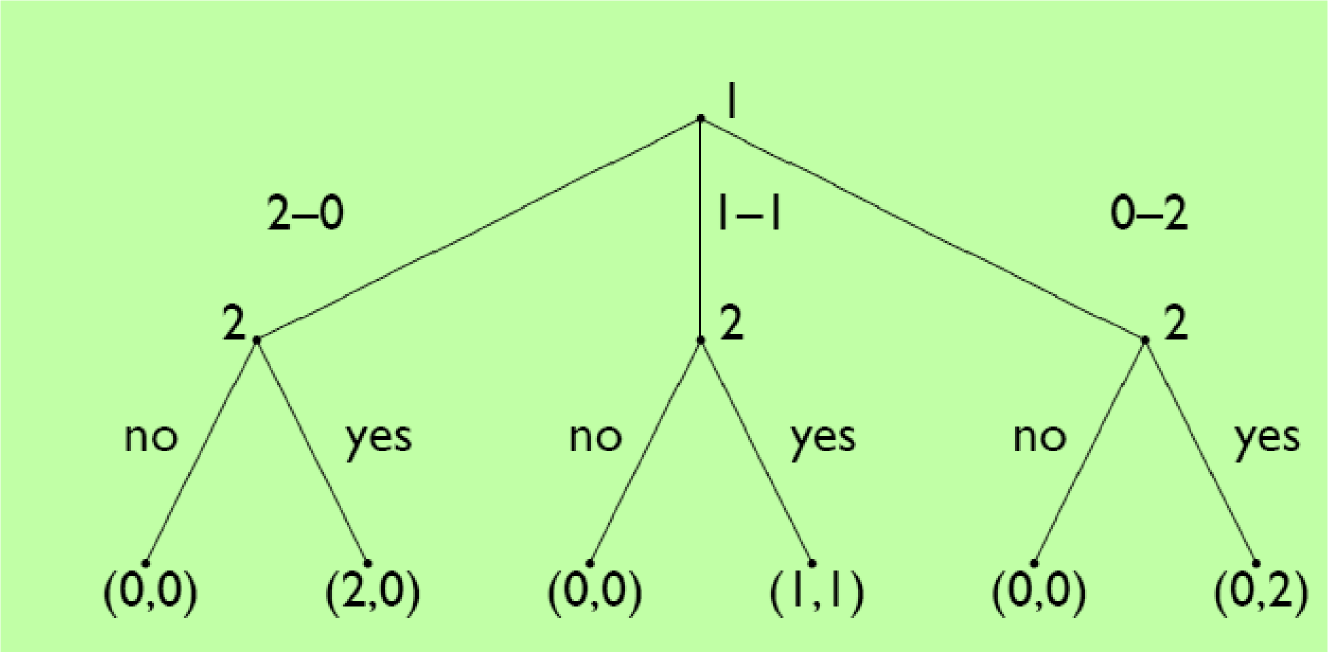
- The extensive form is an alternative representation that makes the **Sequential** structure explicit.
- Two variants:
  - perfect information extensive-form games
  - imperfect-information extensive-form games

Example: Sharing game

- Suppose players 1 and 2 are two children
- Someone offers them two cookies, but only if they can agree how to share them
- player 1 chooses one of the following options:
  - player 1 gets 2 cookies: (2,0)
  - They each get 1 cookie: (1,1)
  - player 1 gets 0 cookies: (0,2)
- player 2 chooses to accept or reject the sharing strategy:
  - Accept => they each get their cookies
  - Reject => neither gets any

Example: Sharing game

- Tree representation (extensive form)

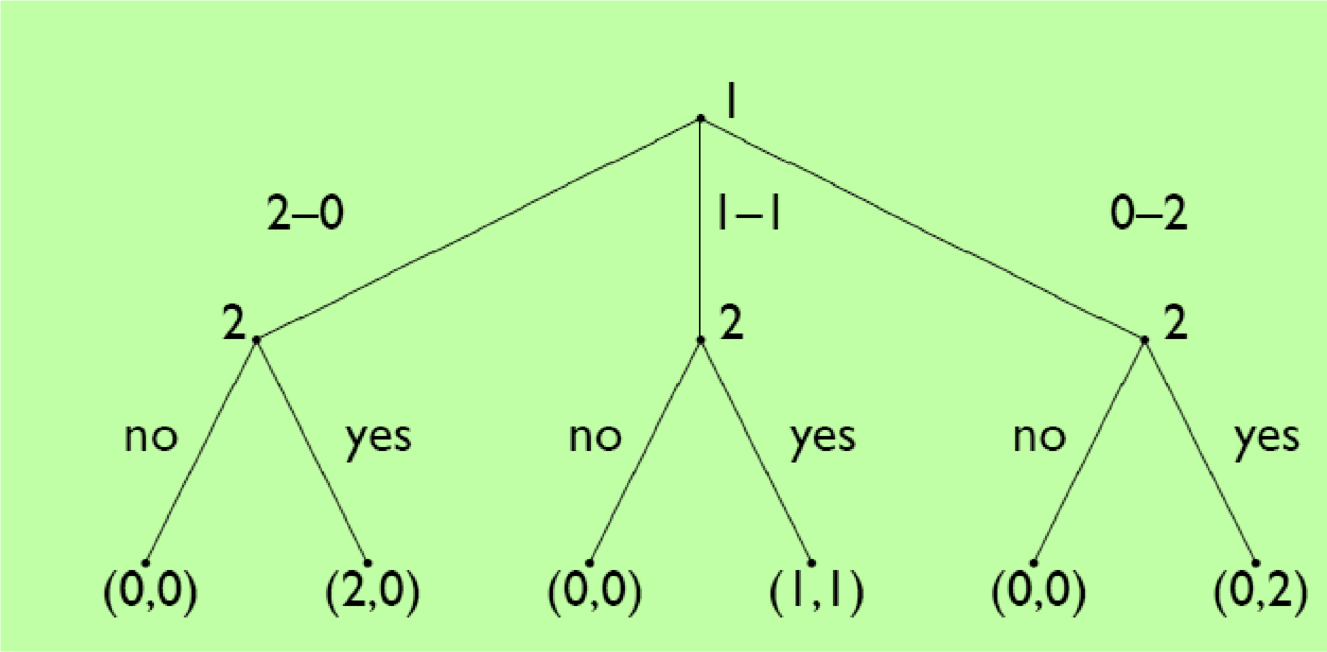


Example: Sharing game

- How many pure strategies does each player have?

- P1:

- P2:



Formal Definition

A (finite) perfect-information game (in extensive form) is defined by the tuple $(N, A, H, Z, \chi, \rho, \sigma, u)$, where:

- Players: N is a set of n players
- Actions: A is a (single) set of actions
- Choice nodes and labels for these nodes:
    - Choice nodes: H is a set of non-terminal choice nodes
    - Action function: $\chi : H \to 2^A$ assigns to each choice node a set of possible actions
    - Player function: $\rho : H \to N$ assigns to each non-terminal node h a player $i \in N$ who chooses an action at h
- Terminal nodes: Z is a set of terminal nodes, disjoint from H

Formal Definition

- Successor function: $\sigma : H \times A \rightarrow H \cup Z$ maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$
  - Choice nodes form a tree: nodes encode history
- Utility function: $u = (u_1, \ldots, u_n)$; $u_i : Z \rightarrow \mathbb{R}$ is a utility function for player i on the terminal nodes Z

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  - Choice nodes: H
  - Action function: $\chi : H \to 2^A$
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- Terminal nodes: Z
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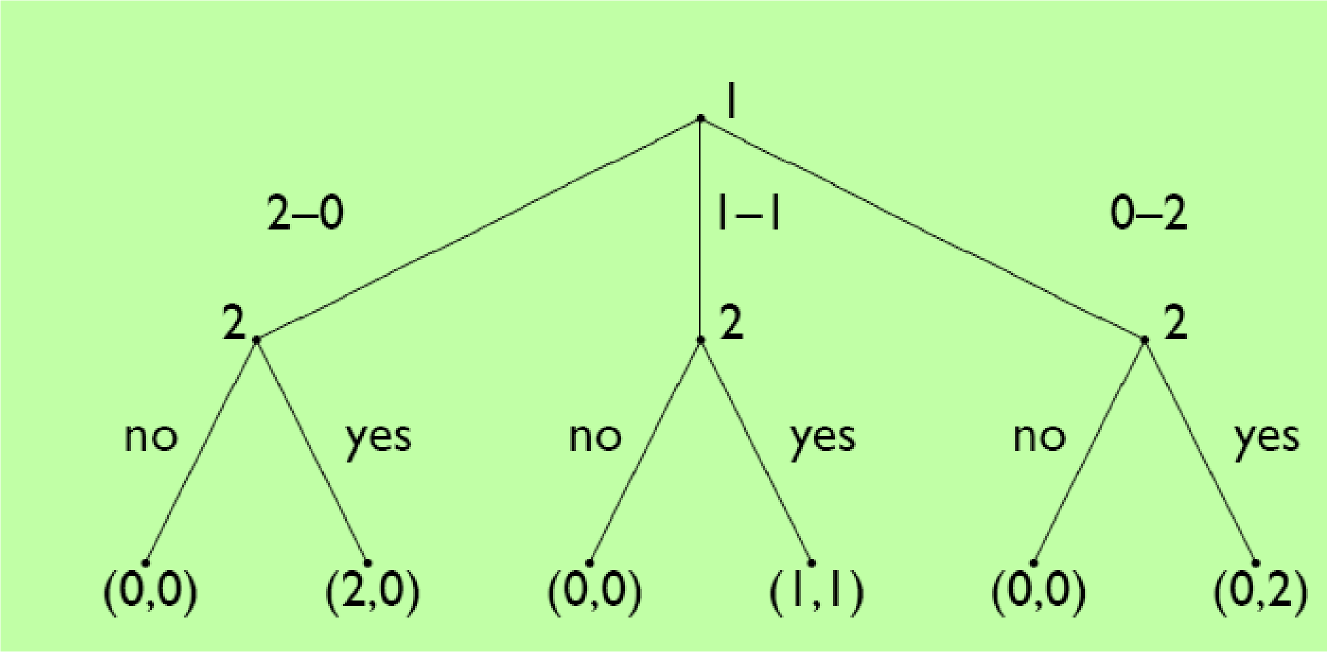
Example: Sharing game

- How many pure strategies does each player have?

- P1: 3

- P2: 8

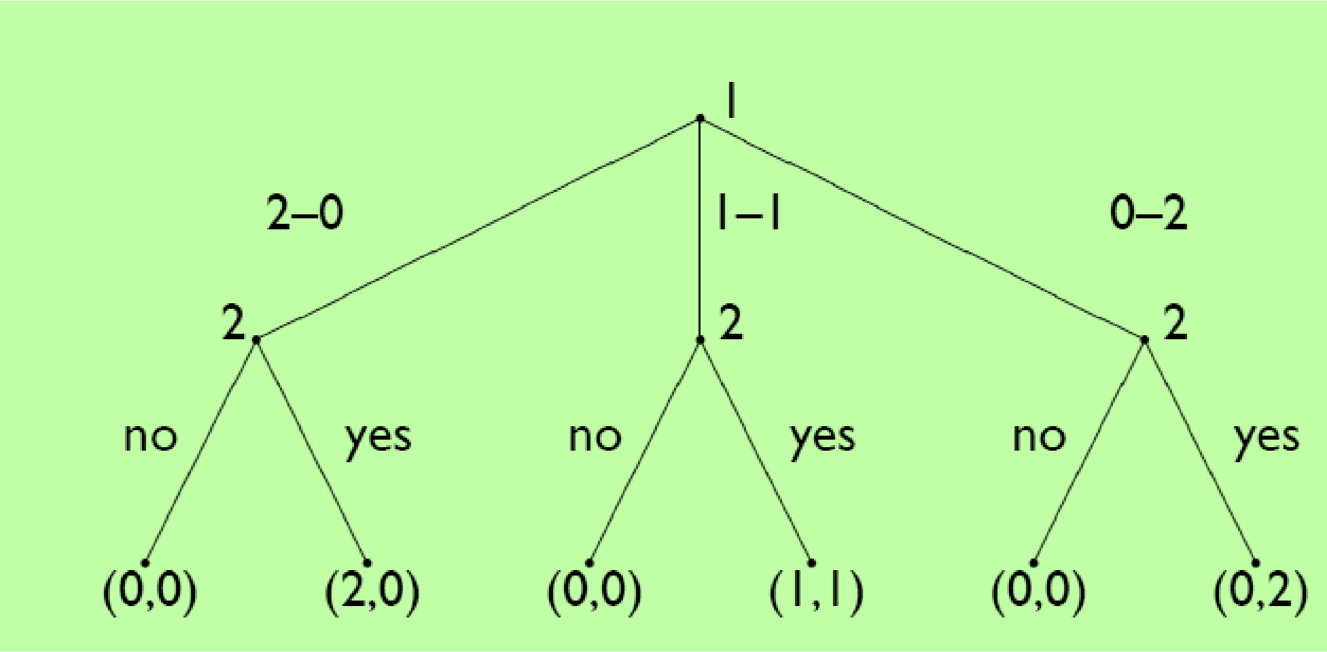


Example: Sharing game

- What is pure strategies?

- P1:

- P2:

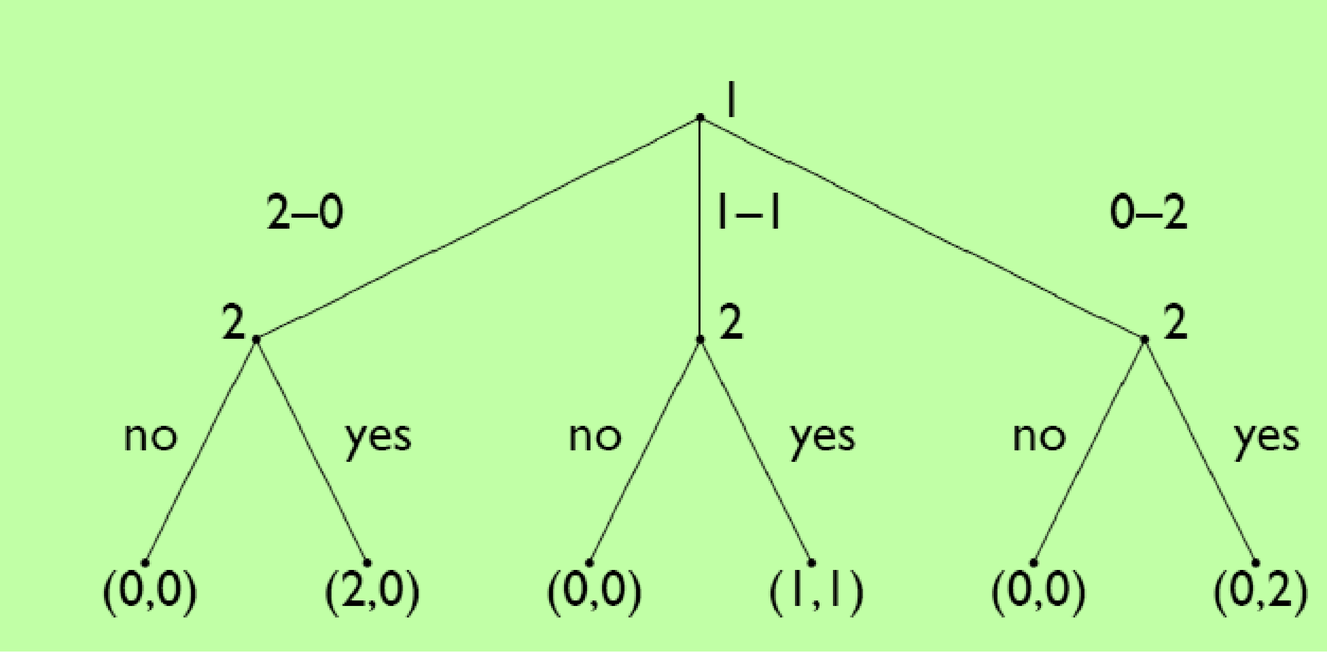


Example: Sharing game

- What is pure strategies?

- P1:
- S_1 = {2-0, 1-1, 0-2}

- P2:
- S_2 = {(yes, yes, yes), (yes, yes, no),
  (yes, no, yes), (yes, no, no),
  (no, yes, yes), (no, yes, no),
  (no, no, yes), (no, no, no)}



Informal Definition

A ***pure strategy*** for a player in a game of perfect information is a ***complete plan*** of actions to take at each node belonging to that player.

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- Each pure strategy must specify an action at every node where it's the agent's move

Formal Definition

Definition (pure strategies)

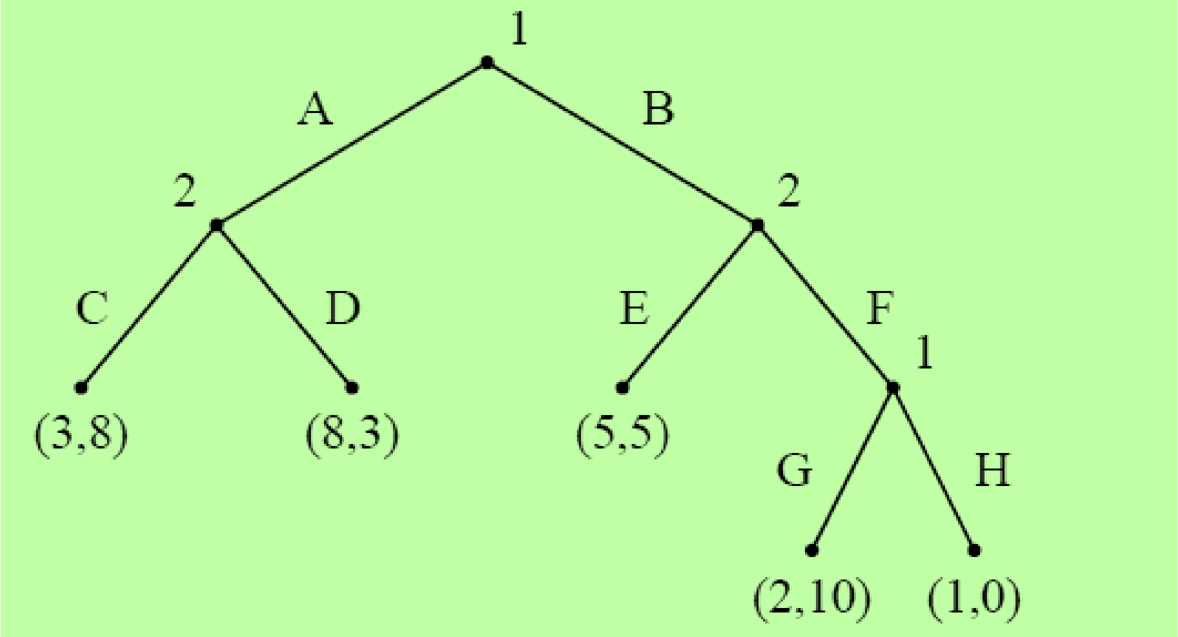
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect-information extensive-form game. Then the pure strategies of player i consist of the cross product

$$\prod_{h \in H, \rho(h) = i} \chi(h)$$

Another Example

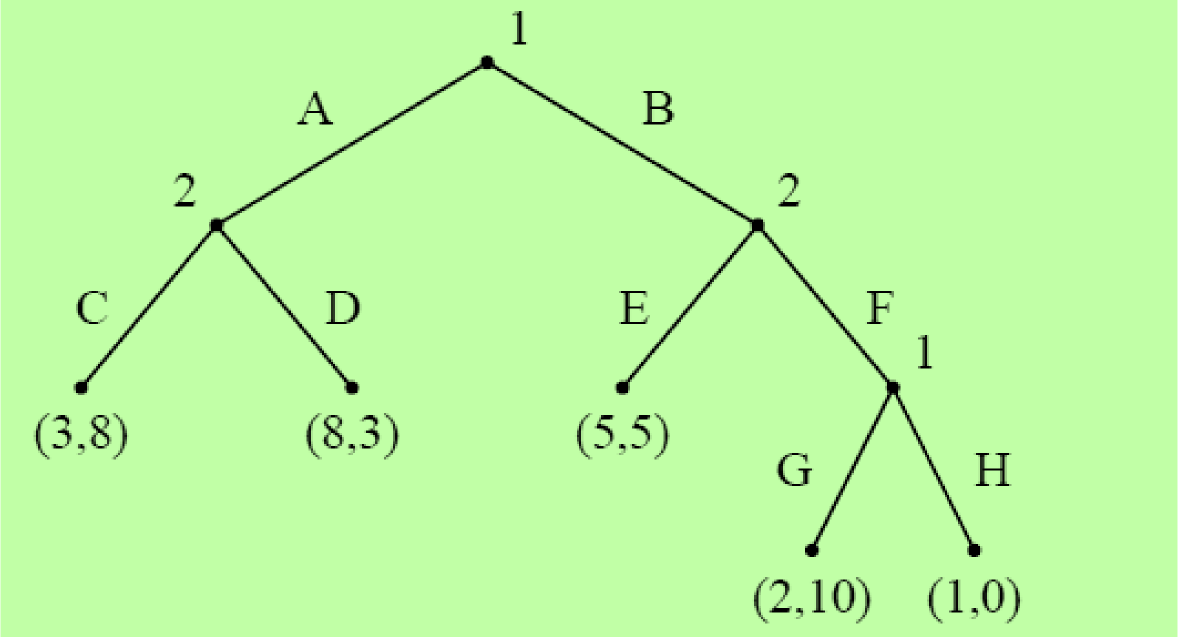
- pure strategies for player1:

- pure strategies for player2:



Another Example

- pure strategies for player2:
  - S_2 = {(C,E), (C,F), (D,E), (D,F)}
- pure strategies for player1:
  - S_1 = {(A,G), (A,H), (B,G), (B,H)}



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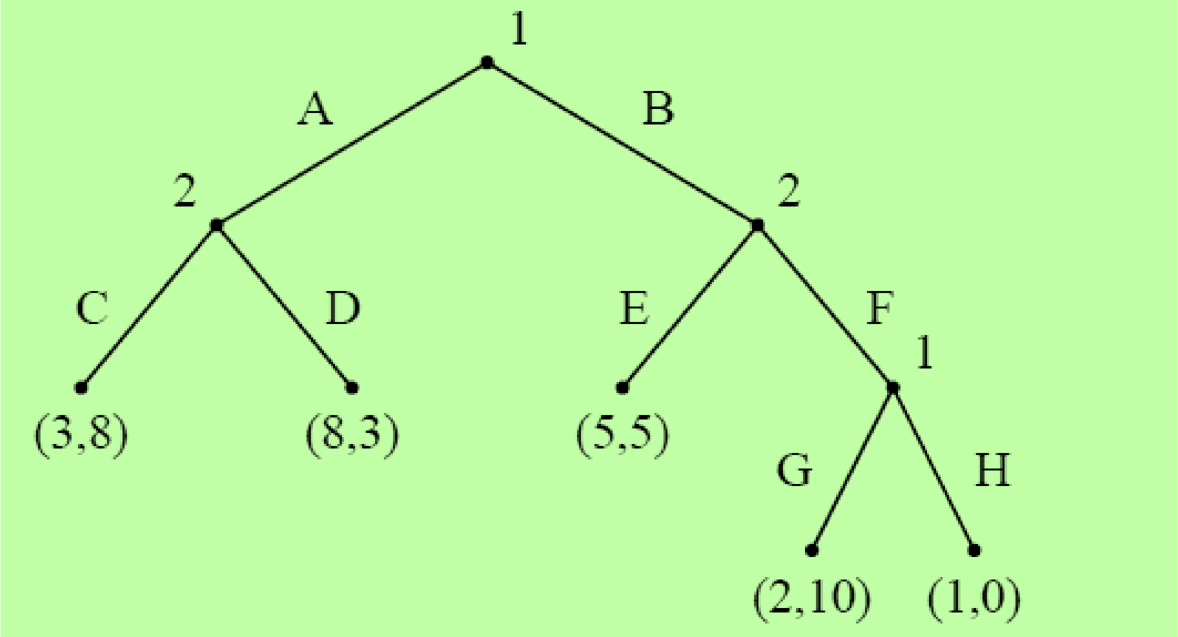
- pure strategies for player1:
  - S_1 = {(A,G), (A,H), (B,G), (B,H)}
  - Are (A,G) and (A,H) the same strategy?



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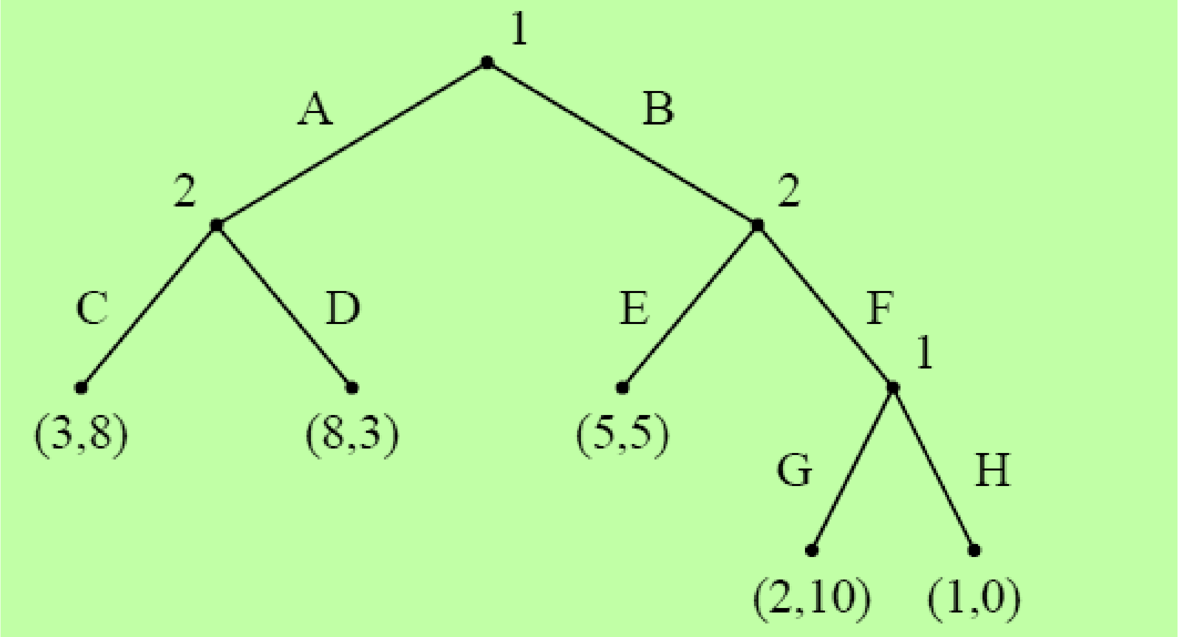
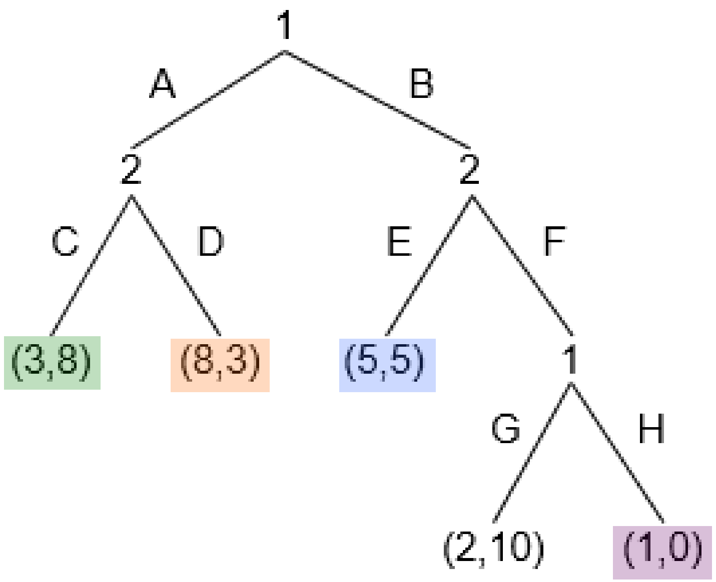


- pure strategies for player1:
  - S_1 = {(A,G), (A,H), (B,G), (B,H)}
  - Are (A,G) and (A,H) the same strategy?
    - No

Converting an extensive-form game into normal form

- Every game tree corresponds to an equivalent normal-form game



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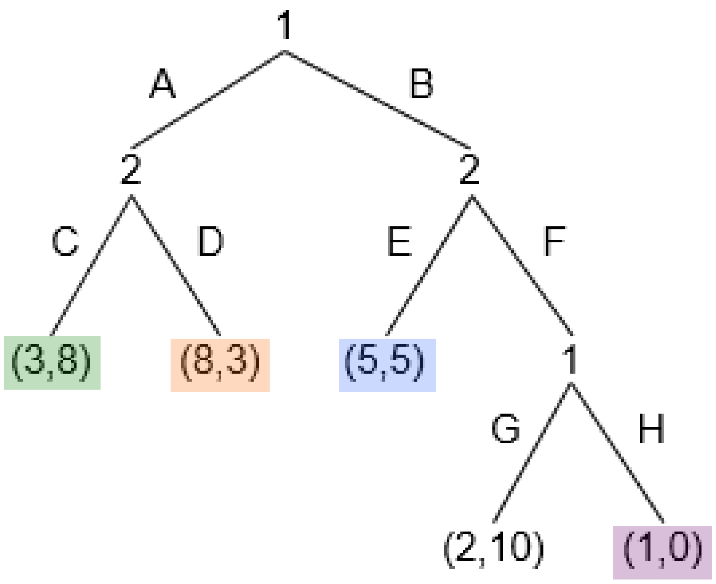

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3,8	3,8	8,3	8,3
(A,H)	3,8	3,8	8,3	8,3
(B,G)	5,5	2,10	5,5	2,10
(B,H)	5,5	1,0	5,5	1,0

Converting an extensive-form game into normal form

- Each terminal node may occur several times in the payoff matrix
  - Can cause exponential blowup
    - 5 outcomes in the game tree
    - 16 outcomes in the payoff matrix

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3,8	$\boxed{3,8}$	8,3	8,3
(A,H)	3,8	$\boxed{3,8}$	8,3	8,3
(B,G)	5,5	2,10	5,5	2,10
(B,H)	$\boxed{5,5}$	1,0	5,5	1,0

- Extensive form games have more compact representation

A point

Converting a normal form game into perfect information game

- we can't always perform the reverse transformation
  - e.g., matching pennies cannot be written as a perfect-information extensive form game

Extensive form vs. Normal form

- What are the pure Nash equilibria?

	CE	CF	DE	DF
AG	$3,8$	$3,8$	$8,3$	$8,3$
AH	$3,8$	$3,8$	$8,3$	$8,3$
BG	$5,5$	$2,10$	$5,5$	$2,10$
BH	$5,5$	$1,0$	$5,5$	$1,0$

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  - ((A,H), (C,F))
  - ((B,H), (C,E))

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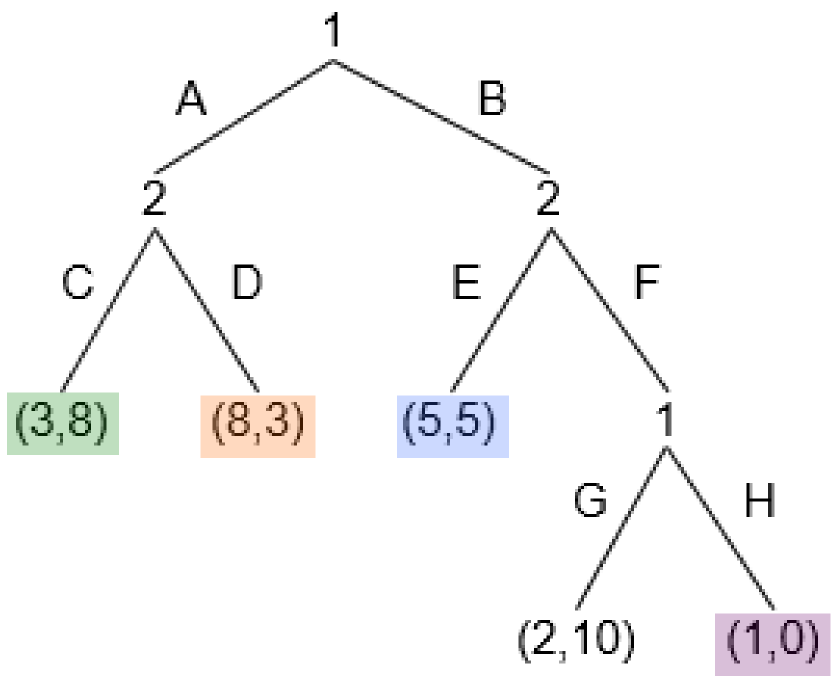

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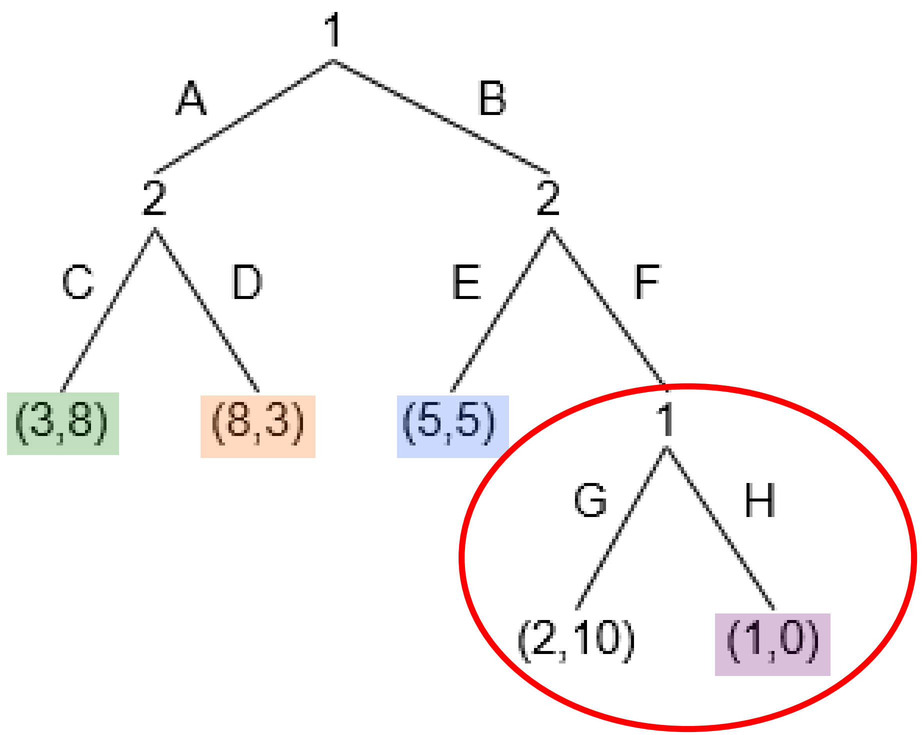

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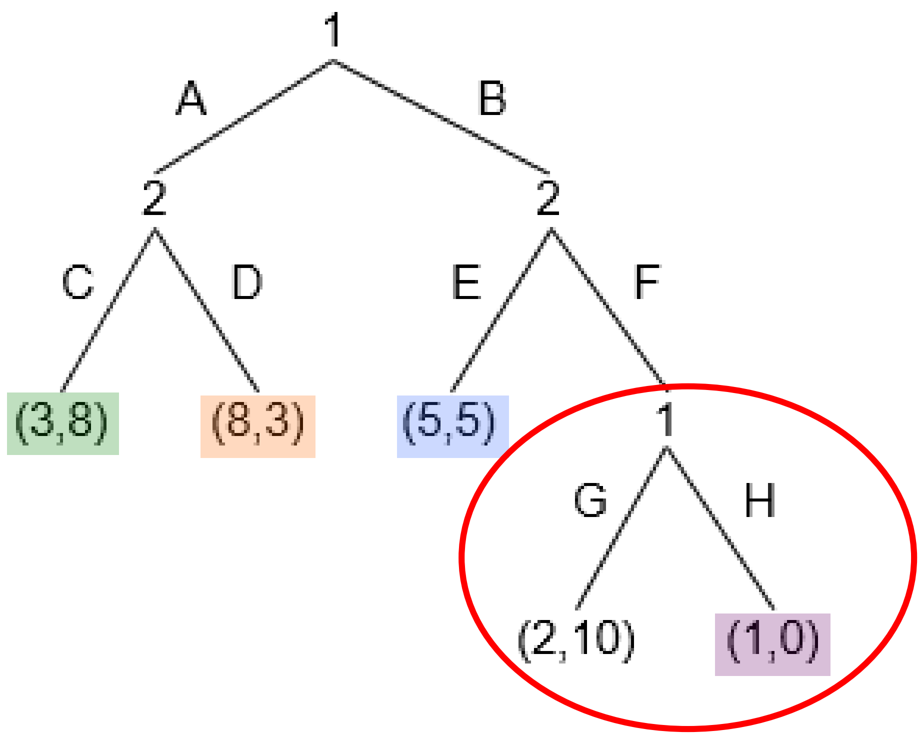

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  - G dominates H for player 1.
  - H is “off-path”, if you are rational!

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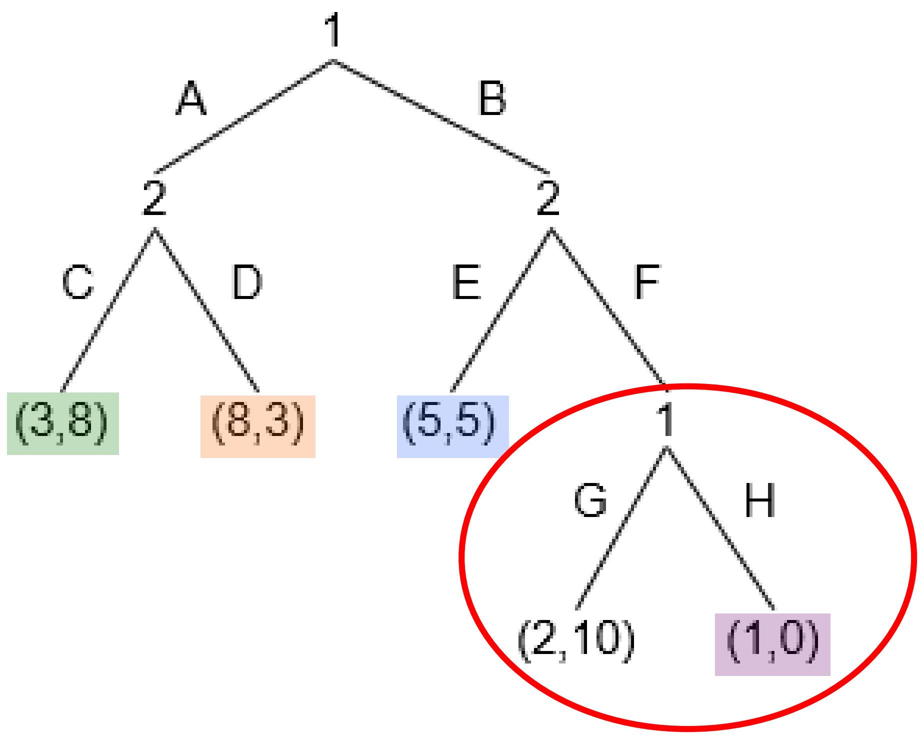

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Need a new solution concept

- Modified version of NE

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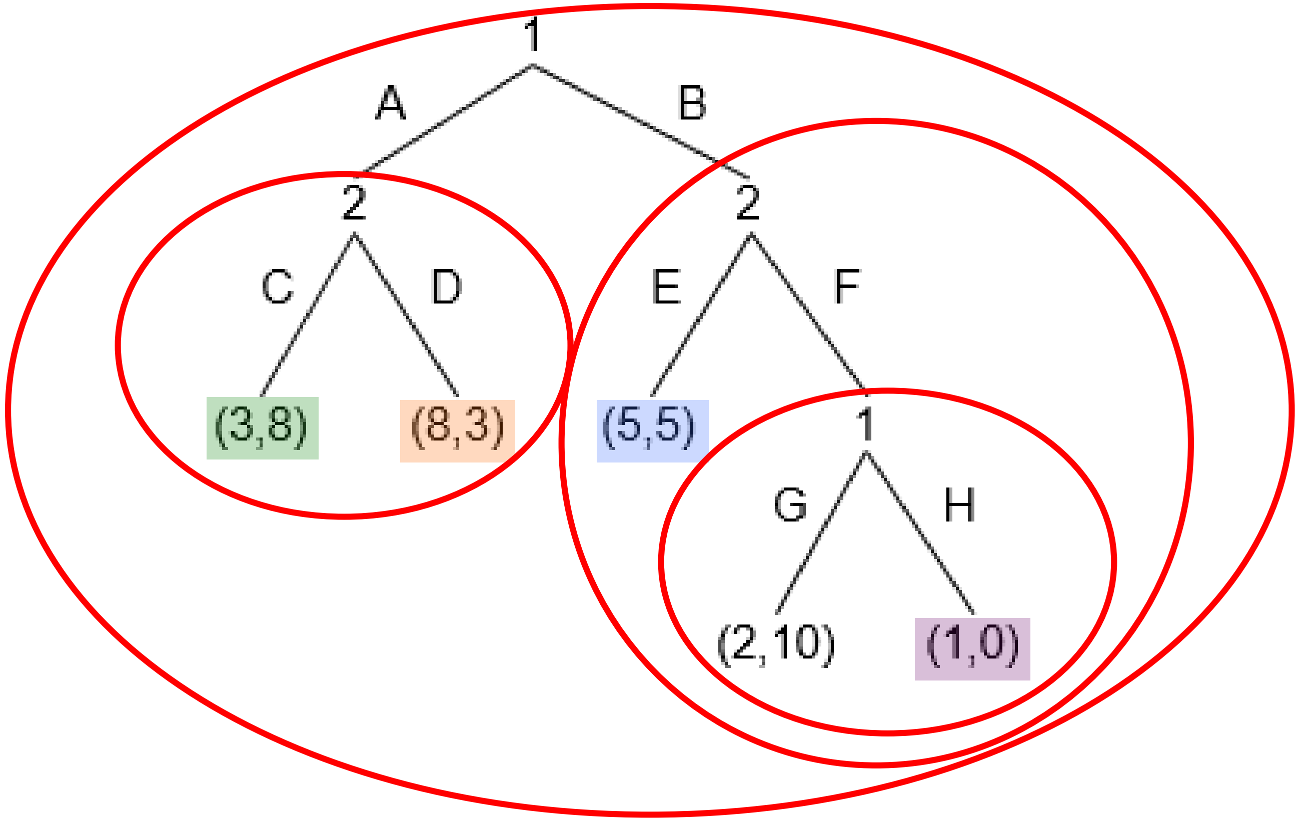
- Subgame Perfect Nash Equilibrium (SPNE)

Informal Definition

- A **sub-game** is a part of the game that looks like a game within the tree. It satisfies the two following properties:
  1. It starts from a **single node**
  2. It **comprises all successors** to that node

Subgames

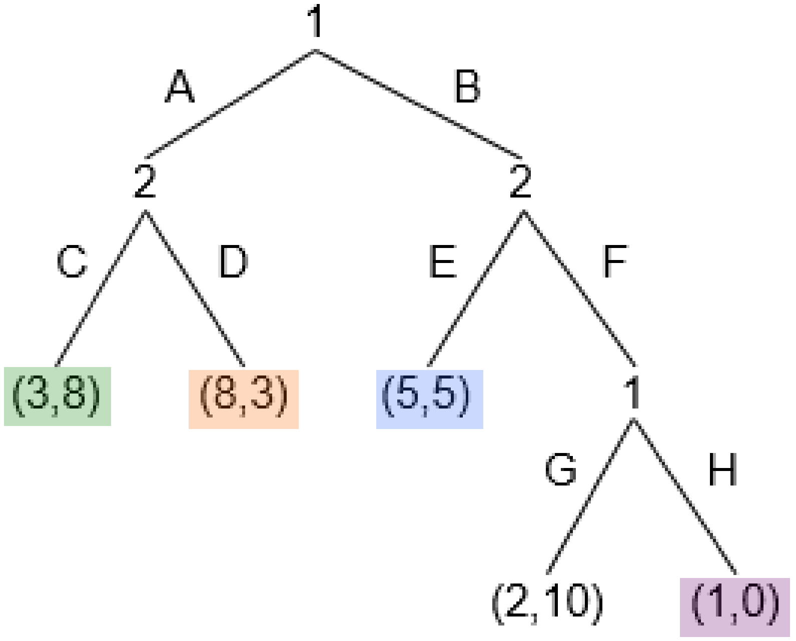


Subgame Perfect Equilibrium (SPE)

s is a subgame perfect equilibrium of G iff for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'

Again

- What are the pure Nash equilibria?
  - ((A,G), (C,F)) is subgame perfect
  - ((A,H), (C,F)) is not subgame perfect
  - ((B,H), (C,E)) is not subgame perfect



Again

- What are the pure Nash equilibria?
    - ((A,G), (C,F)) is subgame perfect
    - ((A,H), (C,F)) is not subgame perfect
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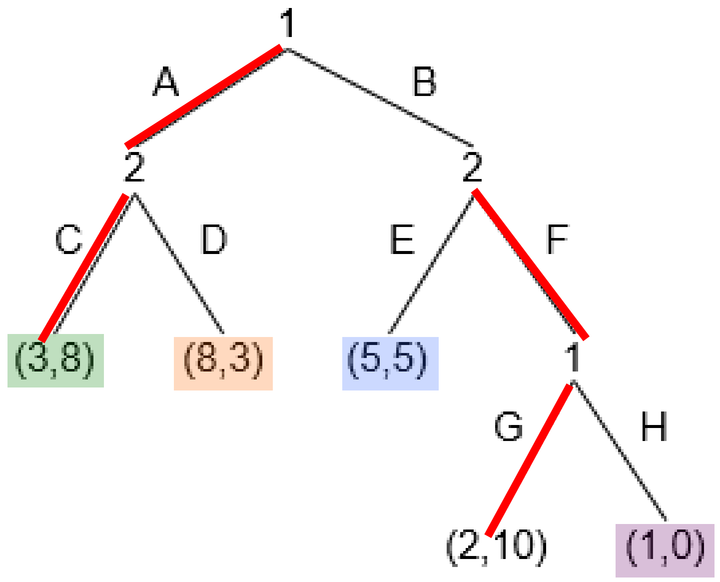
    - SPNE = ((A,G), (C,F))



Theorem

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Every perfect information game in extensive form has a ***SPNE***

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Every perfect information game in extensive form has a ***SPNE***

- Every SPNE is a NE,
- But vice versa in not correct definitely

Backward Induction

- Start with the last player and chose the strategies yielding higher payoff
- This simplifies the tree
- Continue with the before-last player and do the same thing
- Repeat until you get to the root

<u>This is a fundamental concept in game theory</u>

Computing SPNE

Idea: Identify the equilibria in the bottom-most trees, and adopt these as one moves up the tree

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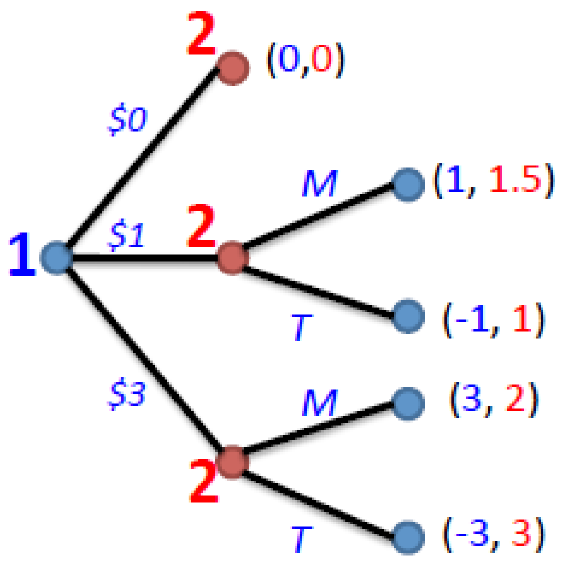
function BACKWARDINDUCTION (node h) **returns** $u(h)$
if $h \in Z$ **then**
  return $u(h)$
$best\_util \leftarrow -\infty$
forall $a \in \chi(h)$ **do**
  $util\_at\_child \leftarrow$ BACKWARDINDUCTION$(\sigma(h, a))$
  if $util\_at\_child_{\rho(h)} > best\_util_{\rho(h)}$ **then**
    $best\_util \leftarrow util\_at\_child$
return $best\_util$

$util\_at\_child$ is a vector denoting the utility for each player

Example 1



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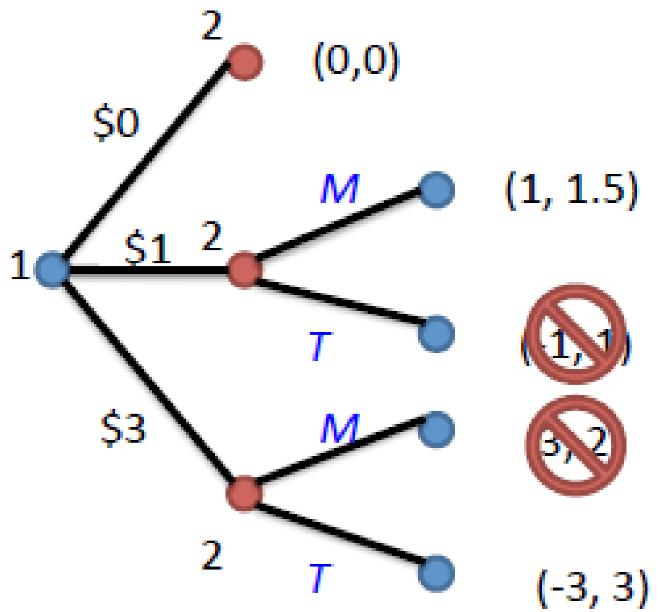

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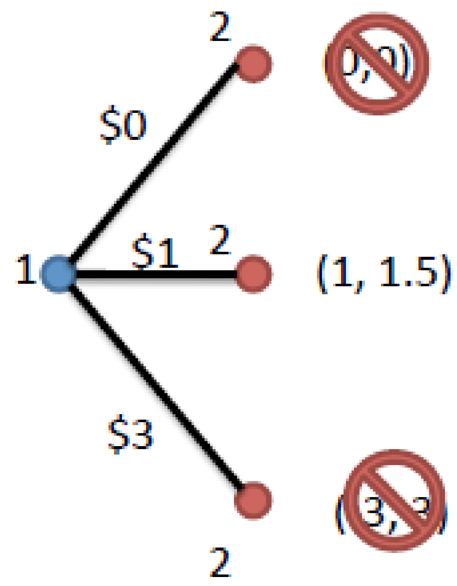

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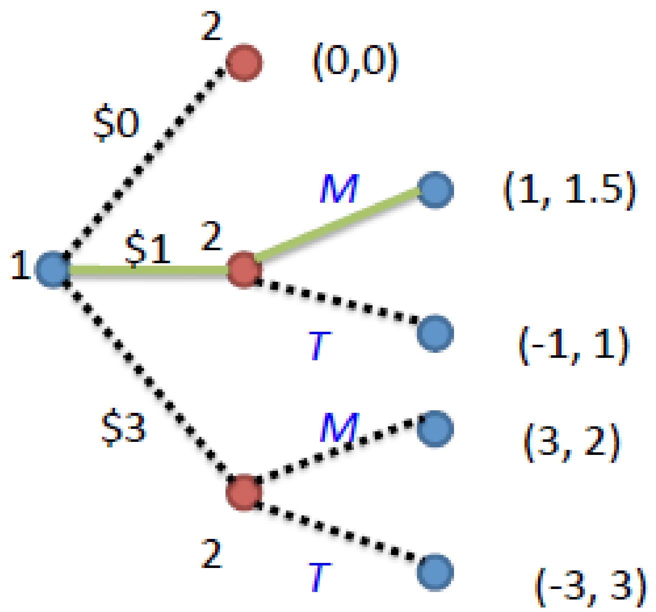

Example 2