Bayesian Games

By Marzie Nilipour

Spring 2023

Contents

- Incomplete Information Games
- Some Examples
- Bayesian Nash Equilibrium (BNE)

Introduction

- Perfect Information games
- Imperfect Information games
  - Nature (Chance Node)
- Complete Information games
- Incomplete Information games

Complete vs. Incomplete Information Games

- In **complete information** games, everyone knows:
  - The number of players
  - The actions available to each player
  - The payoff associated with each action vector

Complete vs. Incomplete Information Games

- In **complete information** games, everyone knows:
  - The number of players
  - The actions available to each player
  - The payoff associated with each action vector

- Note: These are still valid assumptions for imperfect information games

Complete vs. Incomplete Information Games

- In **incomplete information (Bayesian)** games
  - We represent players' uncertainties about the every game being played,
  - This uncertainty is represented as a probability distribution over a set of possible games

Complete vs. Incomplete Information Games

- In **incomplete information (Bayesian)** games
  - We represent players' uncertainties about the every game being played,
  - This uncertainty is represented as a probability distribution over a set of possible games

- Some assumptions: All possible games have
  - the same number of agents,
  - the same strategy space for each agent,
  - they differ only in their payoffs.

Contents

- Incomplete Information Games
- Some Examples
- Bayesian Nash Equilibrium (BNE)

Example

- Lions and Antelopes

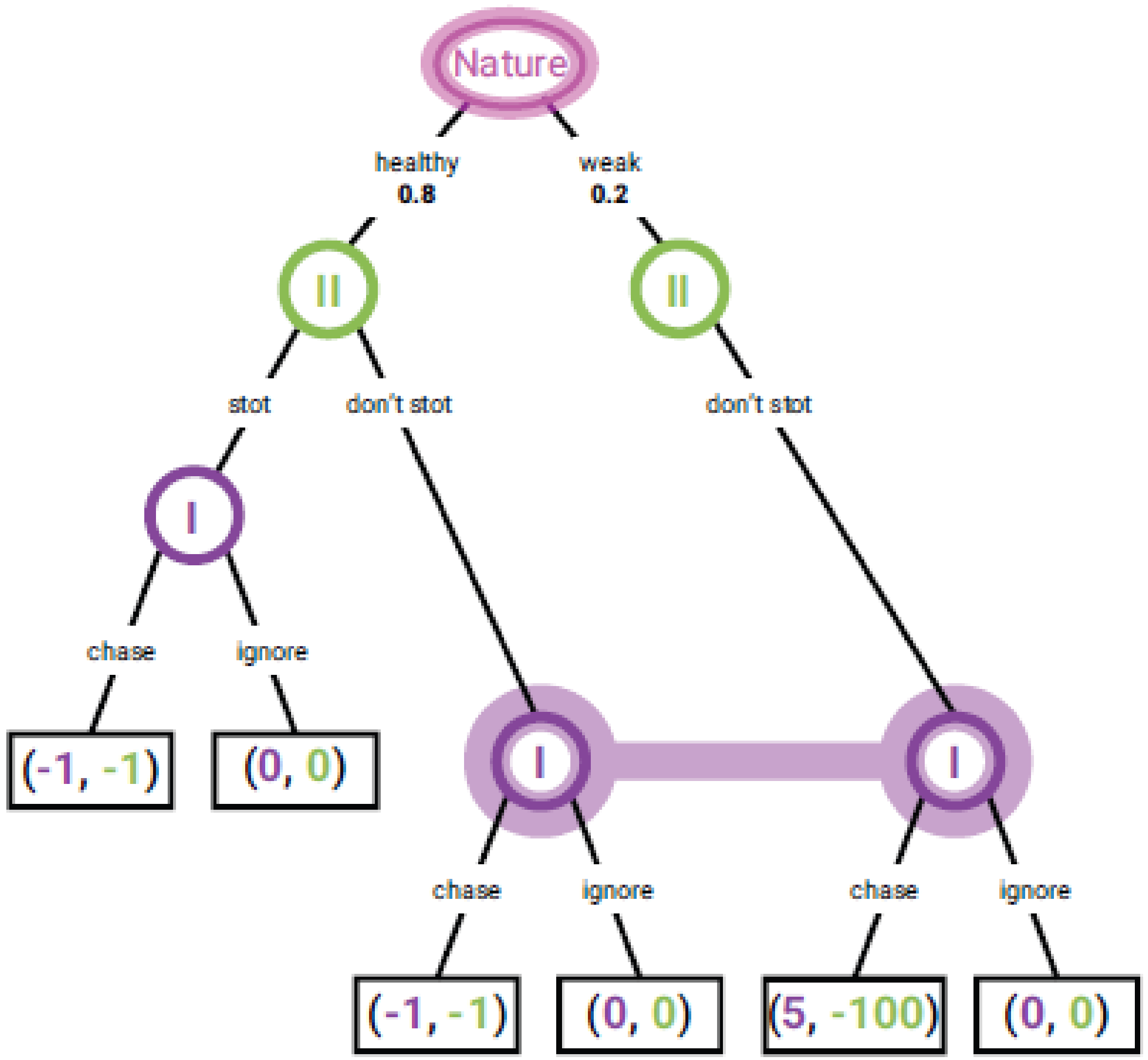
Example

- Lions and Antelopes

- Antelopes have been observed to jump energetically when they notice a lion.
- One theory is that the antelopes are signaling danger to others at some distance, in a community-spirited gesture.
- The currently accepted theory is that the signal is intended for the lion, to indicate that the antelope is in good health and is unlikely to be caught in a chase.
- This is the idea behind signaling.

Example

- Lions and Antelopes



Example

- This can be modeled as a combination of two simple games (A_H and A_W), depending on whether the antelope is healthy or weak,
- In each case the antelope has only one strategy (to run if chased), but the lioness has the choice of chasing (C) or ignoring (I):

$$A^H = \begin{array}{cc|c|} & & \text{antelope} \\ \cline{3-3} & & \text{run if chased} \\ \cline{2-3} \text{lioness} & \multicolumn{1}{|c|}{\text{chase}} & (-1, -1) \\ & \multicolumn{1}{|c|}{\text{ignore}} & (0, 0) \\ \cline{2-3} \end{array} \quad \text{and} \quad A^W = \begin{array}{cc|c|} & & \text{antelope} \\ \cline{3-3} & & \text{run if chased} \\ \cline{2-3} \text{lioness} & \multicolumn{1}{|c|}{\text{chase}} & (5, -100) \\ & \multicolumn{1}{|c|}{\text{ignore}} & (0, 0) \\ \cline{2-3} \end{array}$$

Example

- This can be modeled as a combination of two simple games (AH and AW), depending on whether the antelope is healthy or weak,
- In each case the antelope has only one strategy (to run if chased), but the lioness has the choice of chasing (C) or ignoring (I):

$A^H =$

lioness	antelope: run if chased
chase	$(-1, -1)$
ignore	$(0, 0)$

and $A^W =$

lioness	antelope: run if chased
chase	$(5, -100)$
ignore	$(0, 0)$

- The lioness does not know which game she is playing | and if 20% of the antelopes are weak, then the lioness can expect a payoff of (0.8)(-1)+(0.2)(5) = 0.2.

Informal Definition

Bayesian games. In many situations, however, the players have probabilistic prior information about which game is being played. Under this assumption, a game of incomplete information can be converted to a game of complete but imperfect information using **moves by nature** and information sets.

Informal Definition

Bayesian games. In many situations, however, the players have probabilistic prior information about which game is being played. Under this assumption, a game of incomplete information can be converted to a game of complete but imperfect information using **moves by nature** and information sets.

A **Bayesian game** is an extensive-form game of imperfect information, with a first move by nature and probabilities that are in common knowledge to all the players. Different players may have different information about the outcome of the move by nature. This is captured by their information sets.

Another Example

- Sheriff’s Dilemma
- A sheriff is faces an armed suspect and they each must (simultaneously) decide whether to shoot the other or not.

Sheriff's Dilemma

- **Players:** a Sheriff and an armed suspect
- Sheriff must decide to shoot or not
- The suspect is either criminal with probability p or not with probability 1-p
- The sheriff would rather shoot if the suspect shoots, but not if the suspect does not
- The criminal would rather shoot even sheriff does not
- The innocent suspect would rather not shoot even if the sheriff shoots

Sheriff's Dilemma

Good: 1-p	Suspect \ Sheriff	Shoot	Not
	Shoot	-3, -1	-1, -2
	Not	-2, -1	0, 0

Bad: p	Suspect \ Sheriff	Shoot	Not
	Shoot	0, 0	2, -2
	Not	-2, -1	-1, 1

Sheriff's Dilemma

- There are **two strictly dominated** strategies in two cases
- We need to calculate the expected payoffs for each action

$Eu_s(Shoot) = -1(1-p)+0(p)$
$Eu_s(Not\text{-}Shoot) = 0(1-p)-2(p)$

If p>1/3 ➔ Sheriff shoots
If p<1/3 ➔ Sheriff does not shoot
If p=1/3 ➔ Any mixture

Good: 1-p

Suspect \ Sheriff	Shoot	Not
Shoot	-3, -1	-1, -2
Not	-2, -1	0, 0

Bad: p

Suspect \ Sheriff	Shoot	Not
Shoot	0, 0	2, -2
Not	-2, -1	-1, 1

Formal Definition 1

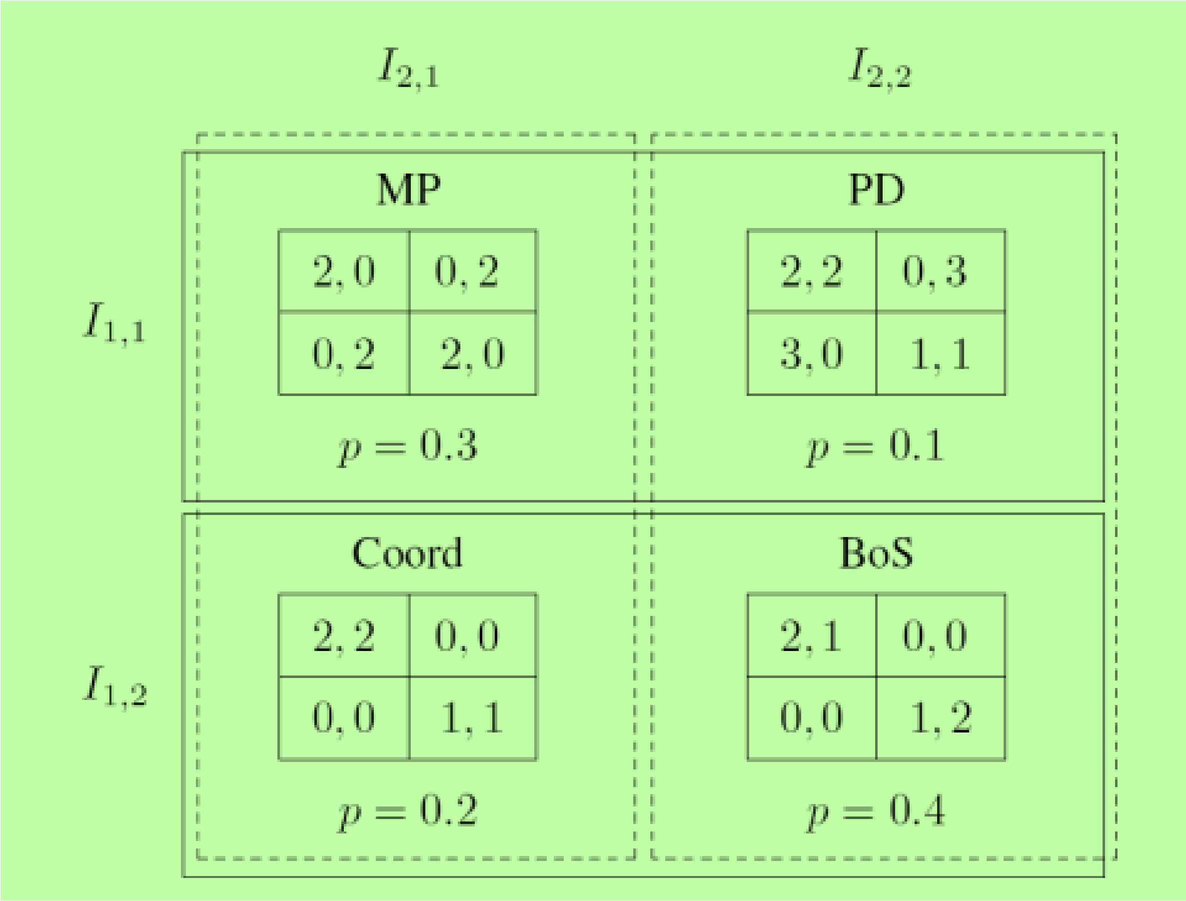
- **Bayesian game**: a set of games that differ only in their payoffs, a common prior defined over them, and a partition structure over the games for each agent.

Definition (Bayesian Game: Information Sets)

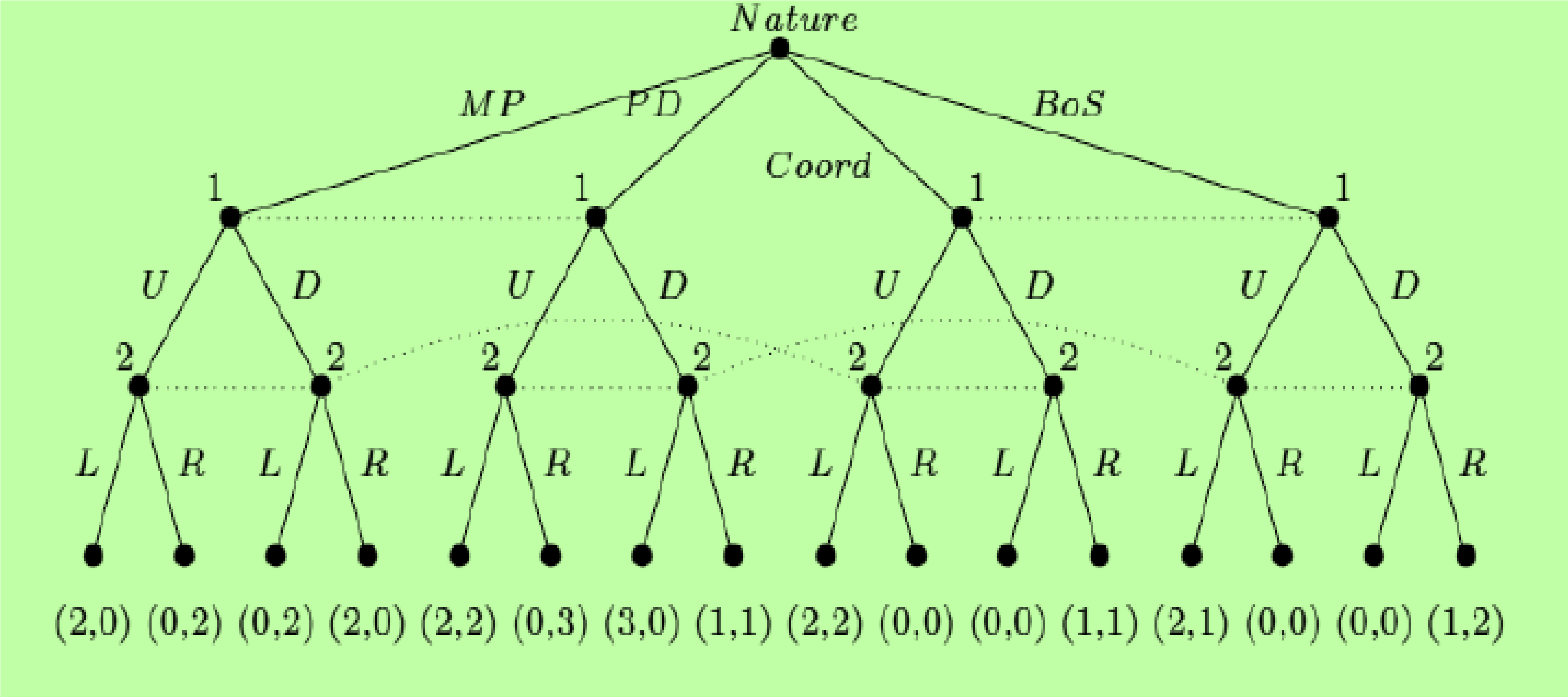
A Bayesian game is a tuple (N, G, P, I) where

- N is a set of agents,
- G is a set of games with N agents each such that if $g, g' \in G$ then for each agent $i \in N$ the strategy space in g is identical to the strategy space in g',
- $P \in \Pi(G)$ is a common prior over games, where $\Pi(G)$ is the set of all probability distributions over G, and
- $I = (I_1, \ldots, I_N)$ is a set of partitions of G, one for each agent.

Formal Definition 1: Example



Formal Definition 1: Example



Formal Definition 2

Directly represent uncertainty over utility function using the notion of **epistemic type**.

Definition

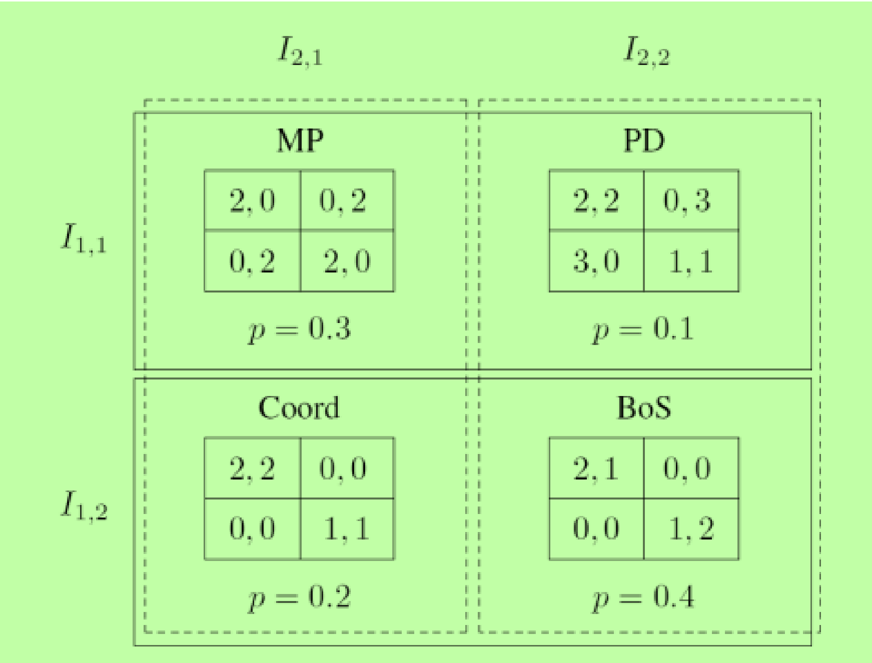
A Bayesian game is a tuple (N, A, Θ, p, u) where

- N is a set of agents,
- $A = (A_1, \dots, A_n)$, where A_i is the set of actions available to player i,
- $\Theta = (\Theta_1, \dots, \Theta_n)$, where Θ_i is the type space of player i,
- $p : \Theta \to [0, 1]$ is the common prior over types,
- $u = (u_1, \dots, u_n)$, where $u_i : A \times \Theta \to \mathbb{R}$ is the utility function for player i.

A Few Words About Type

- The type of agent encapsulates all the information possessed by the agent that is not common knowledge, e.g.,
- The agent’s knowledge of his private payoff function.
- His beliefs about other agents’ payoffs
- Their beliefs about his own payoff
- Any other higher-order beliefs …

Formal Definition 2: Example



a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Contents

- Incomplete Information Games
- Some Examples
- Bayesian Nash Equilibrium (BNE)