Repeated Games

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Introduction

- Many (most?) interactions occur more than once.
- Firms in a marketplace
- Political alliances
- Friends (favor exchange...)
- Workers (team production...)

Example

- A repeated Prisoner's Dilemma game (Cooperative Behavior)

	C	D
C	3,3	0,5
D	5,0	1,1

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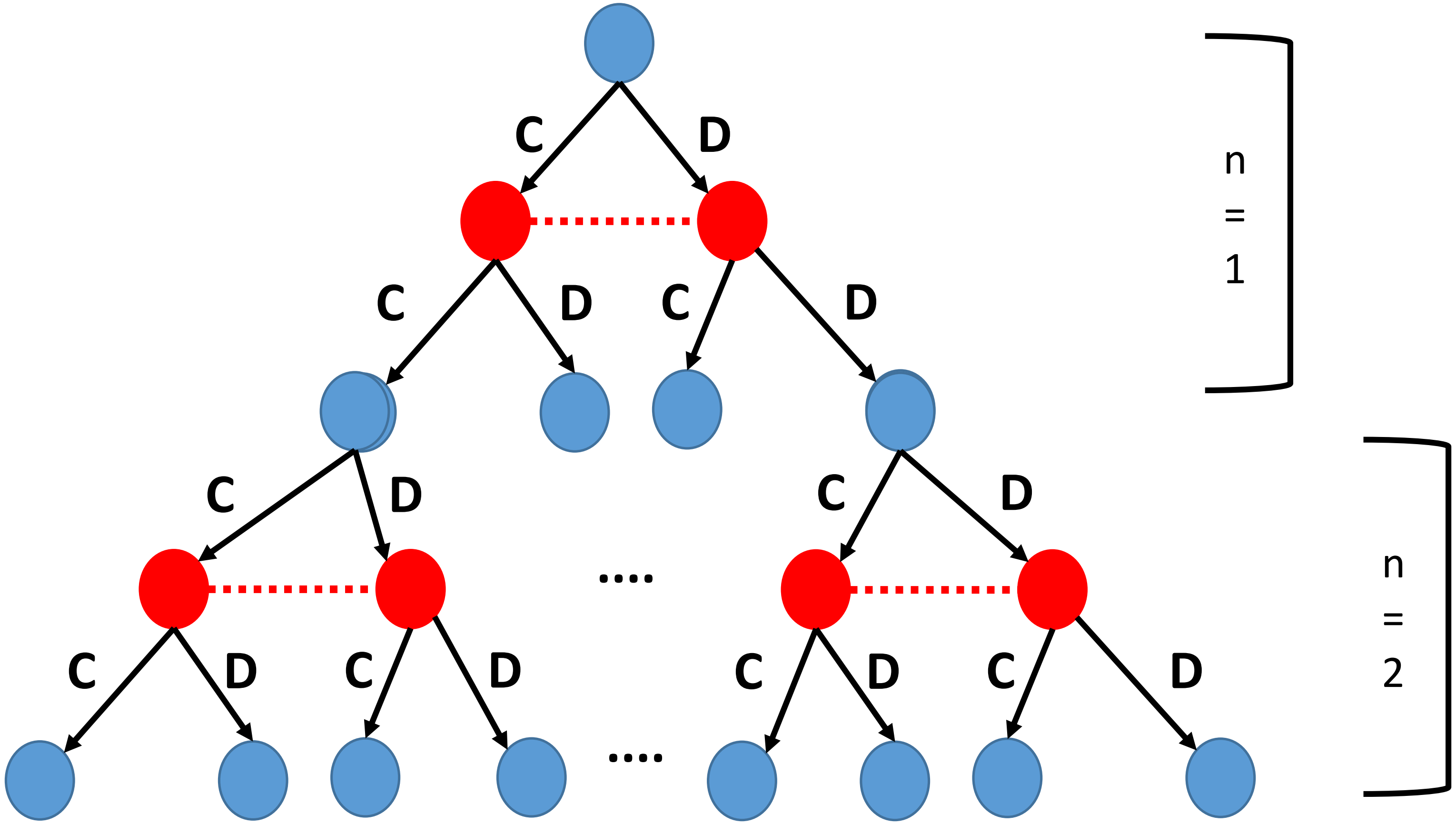
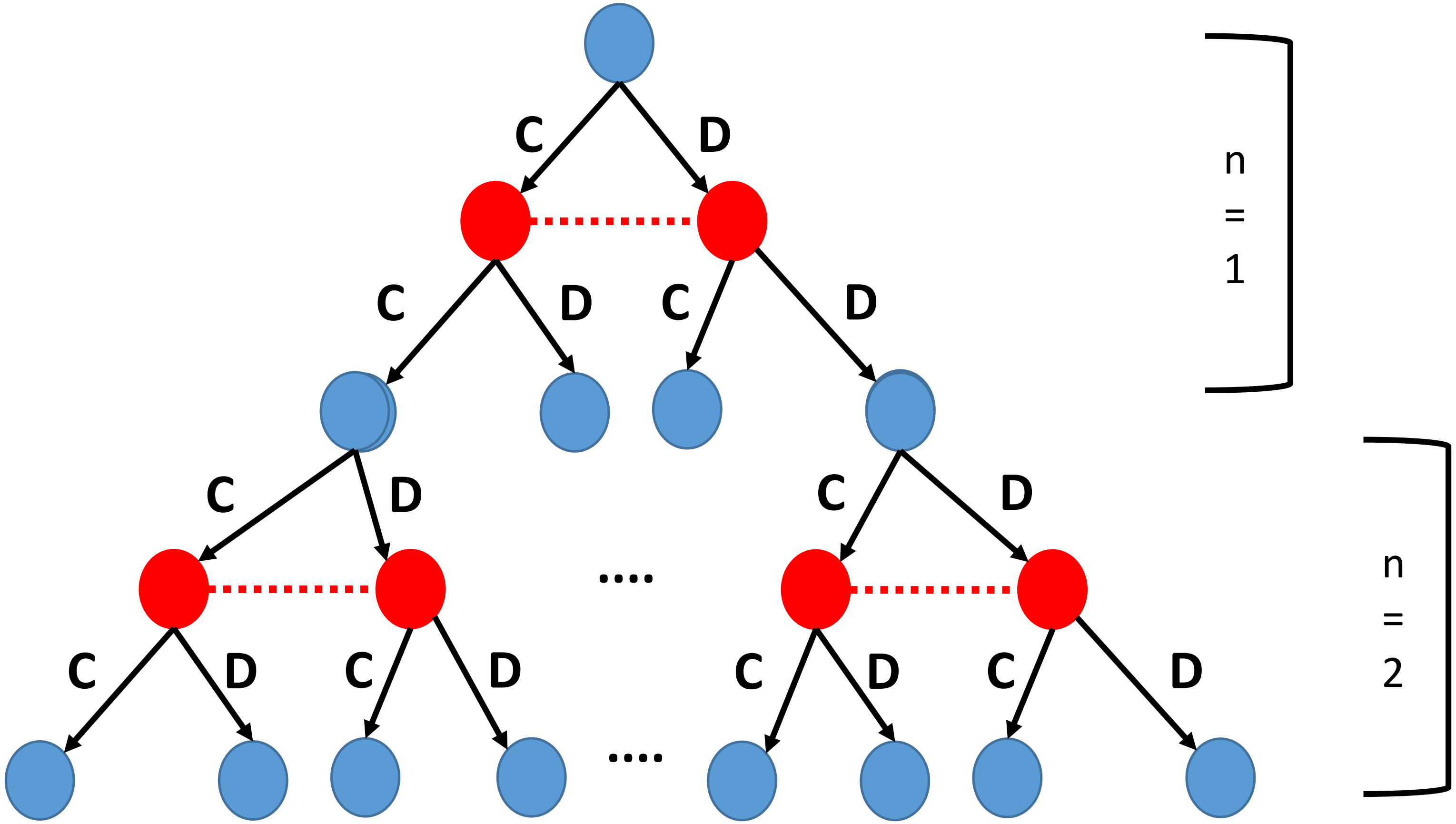
- Finitely repeated (n)

- Infinitely repeated (n -> ∞)

Finitely repeated PD game



Finitely repeated PD game



- By backward induction, NE is also (All D, All D).

Infinitely repeated games

- When n -> ꝏ or
- When end of the game is not determined

Infinitely repeated games

- When n -> ∞ or
- When end of the game is not determined

- What is a player's utility for playing an infinitely repeated game?
  - Average reward
  - Discounted reward

Average Reward

Definition

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player i, the average reward of i is

$$\lim_{k \to \infty} \sum_{j=1}^{k} \frac{r_j}{k}.$$

Discounted Reward

Definition

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player i and discount factor β with $0 < \beta < 1$, i's future discounted reward is

$$\sum_{j=1}^{\infty} \beta^j r_j.$$

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- Two equivalent interpretations of the discount factor (β) :
  - Currency depreciation over time
  - Probabilistic perspective:
    - β (continuing the game), 1-β (ending the game)

Strategy Space

- Some famous strategies (repeated PD):
  - Tit-for-tat: Start out cooperating. If the opponent defected, defect in the next round. Then go back to cooperation.
  - Trigger: Start out cooperating. If the opponent ever defects, defect forever.

Discounted Repeated Game

- Stage game: (N, A, u)
- Discount factors: $\beta_1, \ldots, \beta_n$, $\beta_i \in [0, 1]$
- Assume a common discount factor for now: $\beta_i = \beta$ for all i
- Payoff from a play of actions $a^1, \ldots, a^t, \ldots$:

$$\sum_t \beta_i^t u_i(a^t)$$

History

- Histories of length t: $H^t = \{h^t : h^t = (a^1, \ldots, a^t) \in A^t\}$

- All finite histories: $H = \cup_t H^t$

- A strategy: $s_i : H \rightarrow \quad A_i$

Prisoner’s Dilemma

- $A_i = \{C, D\}$
- A history for three periods: $(C, C), (C, D), (D, D)$
- A strategy for period 4 would specify what a player would do after seeing $(C, C), (C, D), (D, D)$ played in the first three periods...

Prisoner's Dilemma

- Both players defect forever after if anyone ever deviates: Grim Trigger
- Payoffs?

	C	D
C	3,3	0,5
D	5,0	1,1

Prisoner's Dilemma

- Cooperate: $3 + \beta 3 + \beta^2 3 + \beta^3 3 \ldots = \frac{3}{1-\beta}$
- Defect: $5 + \beta 1 + \beta^2 1 + \beta^3 1 \ldots = 5 + \beta \frac{1}{1-\beta}$
- Difference: $-2 + \beta 2 + \beta^2 2 + \beta^3 2 \ldots = \beta \frac{2}{1-\beta} - 2$

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- Difference is nonnegative if $\beta \frac{2}{1-\beta} - 2 \geq 0$ or $\beta \geq (1-\beta)$, so $\beta \geq 1/2$

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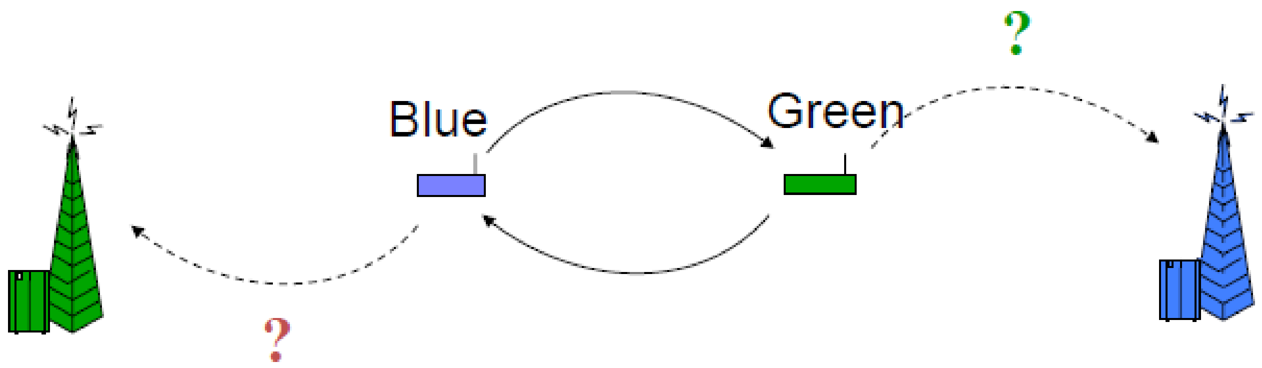
Prisoner's Dilemma

- What if we make defection more attractive:

- Cooperate: $3 + \beta 3 + \beta^2 3 + \beta^3 3 \ldots = \frac{3}{1-\beta}$
- Defect: $10 + \beta 1 + \beta^2 1 + \beta^3 1 \ldots = 10 + \beta \frac{1}{1-\beta}$
- Difference: $-7 + \beta 2 + \beta^2 2 + \beta^3 2 \ldots = \beta \frac{2}{1-\beta} - 7$
- Difference is nonnegative if $\beta \frac{2}{1-\beta} - 7 \geq 0$ or $2\beta \geq 7(1-\beta)$, so $\beta \geq 7/9$

	C	D
C	3,3	0,10
D	10,0	1,1

Repeated Forwarder's Game



Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 1)
Drop	(1, -c)	(0, 0)

stage payoff

NE in Finite Repeated Game

In the finite-horizon Repeated Forwarder's Dilemma, the strategy profile (All-D, All-D) is a Nash equilibrium.

Repeated Forwarder’s Game

Example strategies in the Forwarder’s Dilemma:

Blue (t)	initial move	F	D	strategy name
Green (t+1)	F	F	F	AllC
	F	F	D	Tit-For-Tat (TFT)
	D	D	D	AllD
	F	D	F	Anti-TFT

Analysis of Repeated Forwarder's Game

Infinite game with discounting: $\overline{u}_i = \sum_{t=0}^{\infty} u_i(t) \cdot \delta^t$

Blue strategy	Green strategy	Blue utility	Green utility
AllD	AllD	0	0
AllD	TFT	1	-c
AllD	AllC	1/(1-δ)	-c/(1-δ)
AllC	AllC	(1-c)/(1-δ)	(1-c)/(1-δ)
AllC	TFT	(1-c)/(1-δ)	(1-c)/(1-δ)
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TFT is the best strategy if δ is high enough!

NE in Infinite Repeated Game

In the Repeated Forwarder's Dilemma, if both players play AllD, it is a Nash equilibrium.

NE in Infinite Repeated Game

In the Repeated Forwarder's Dilemma, both players playing TFT is a Nash equilibrium (if $\delta > c$).

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Sketch of Proof:

If one deviate in stage t, then its payoff is:

$(1-\delta)\,[(1+\delta+\delta^2 \ldots + \delta^{t-1})(1-c) + \delta^t] =$

$1-c+\delta^t(c-\delta) \Rightarrow$

Hence if "$\delta > c$" there is no temptation to deviate

Or (i.e., other approach):

$1-(1-c) \leq \delta$ (u(C,C) forever – u (D,D) forever)

$c \leq \delta\,((1-c)/(1-\delta) - 0) \Rightarrow \delta > c$

Nash Equilibria

- With an infinite number of pure strategies, what can we say about Nash equilibria?
  - we won't be able to construct an induced normal form and then appeal to Nash's theorem to say that an equilibrium exists
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- We can characterize a set of payoffs that are achievable under equilibrium, without having to enumerate the equilibria.

Definition

- Consider any n-player game $G = (N, A, u)$ and any payoff vector $r = (r_1, r_2, \ldots, r_n)$.
- Let $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$.
  - i's minmax value: the amount of utility i can get when $-i$ play a minmax strategy against him

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Definition

A payoff profile r is feasible if there exist rational, non-negative values α_a such that for all i, we can express r_i as $\sum_{a \in A} \alpha_a u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$.

Folk Theorem

Theorem (Folk Theorem)

Consider any n-player game G and any payoff vector $(r_1, r_2, \ldots, r_n)$.

1. *If r is the payoff in any Nash equilibrium of the infinitely repeated G with average rewards, then for each player i, r_i is enforceable.*
2. *If r is both feasible and enforceable, then r is the payoff in some Nash equilibrium of the infinitely repeated G with average rewards.*