# More about Normal-Form Games

By Marzie Nilipour Spring 2023

# Outline

- Correlated Equilibrium
- ε-Nash Equilibrium

• We already know a lot about this game.

	А	В
Α	2, 1	0, 0
В	0, 0	1, 2

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	А	В
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- Coordination game
  - without communication, it is possible that the two players might fail to coordinate

- Pure-strategy equilibria
  - NE = (A,A) and (B,B)
  - Payoff profiles (2,1) and (1,2)
  - Unfair, but pareto optimal

	А	В
Α	2, 1	0, 0
В	0, 0	1, 2

Mixed-strategy equilibrium

	А	В	
А	2, 1	0, 0	р
В	0, 0	1, 2	<b>1</b> -p
	q	1-q	

• Player1's expected utility?

$$E[U_{1}(A,(q,1-q))] = 2q + 0(1-q)$$

$$E[U_{1}(B,(q,1-q))] = 0q + 1(1-q)$$

$$2q = (1-q) \Rightarrow q = \frac{1}{3}$$

Player2's expected utility?

$$E[U_{2}((p,1-p),A)] = 1p + 0(1-p)$$

$$E[U_{2}((p,1-p),B)] = 0p + 2(1-p)$$

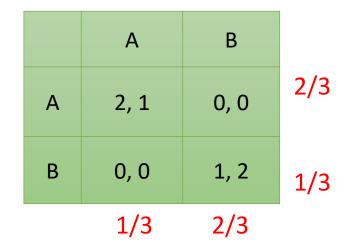
$$1p = 2(1-p) \Rightarrow p = \frac{2}{3}$$

	А	В	
А	2, 1	0, 0	р
В	0, 0	1, 2	1-p
	q	1-q	

Mixed-strategy equilibrium

Player 1 Player 2 
$$\left[ \left( \frac{2}{3}, \frac{1}{3} \right), \left( \frac{1}{3}, \frac{2}{3} \right) \right]_{p \quad 1-p \quad q \quad 1-q}$$

	А	В	
А	2, 1	0, 0	2/3
В	0, 0	1, 2	1/3
	1/3	2/3	

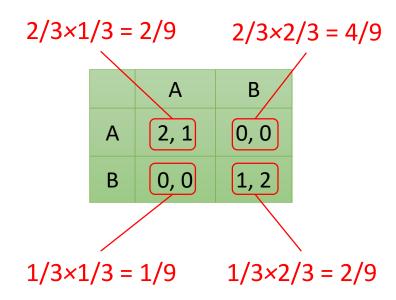


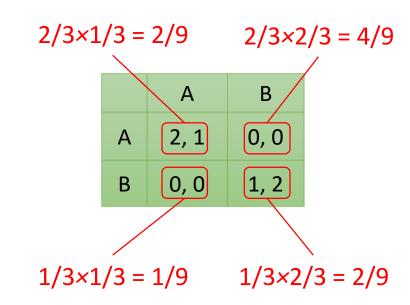
• What is the probability for the two players meet or not to meet?

	Α	В	
А	2, 1	0, 0	2/3
В	0, 0	1, 2	1/3
	1/3	2/3	

- What is the probability for the two players meet or not to meet?
- $\rightarrow$  Prob(meet) = 2/3\*1/3+1/3\*2/3=4/9
- → I- Prob(meet) = 5/9 !!!

• Payoff profile?





- Payoff profile?
  - Each player's payoff at this profile =  $\frac{2}{9}$ \*2 + 0 + 0 +  $\frac{2}{9}$ \*1 =  $\frac{2}{3}$
  - $(^2/_3, ^2/_3)$  is Fair

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  - $(^{2}/_{3}, ^{2}/_{3})$  is Fair
- Is this profile pareto optimal?

	А	В	
А	2, 1	0, 0	2/3
В	0, 0	1, 2	1/3
	1/3	2/3	1

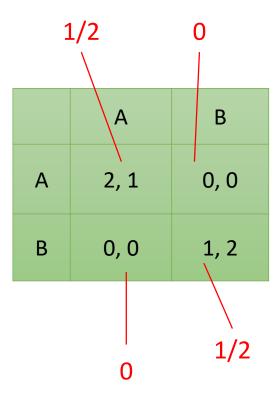
- Payoff profile?
  - Each player's payoff at this profile =  $\frac{2}{9}$ \*2 + 0 + 0 +  $\frac{2}{9}$ \*1 =  $\frac{2}{3}$
  - $(\frac{2}{3}, \frac{2}{3})$  is Fair
- Is this profile pareto optimal?
  - No, this profile is Pareto dominated by (A,A) and (B,B)

• How we can obtain both pareto optimality and fairness in NE payoffs?

	Α	В
А	2, 1	0, 0
В	0, 0	1, 2

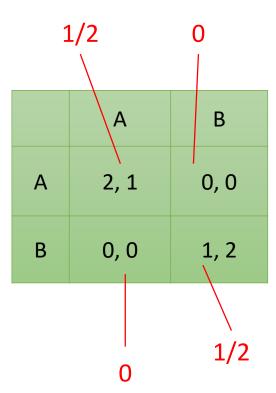
How we can obtain both pareto optimality and fairness in NE payoffs?

- Neither pure nor mixed NE
- Flip a coin
  - Heads → both choose A
  - Tails → both choose B



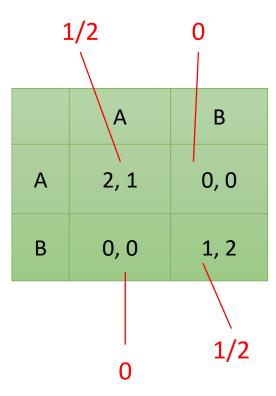
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- Neither pure nor mixed NE
- Flip a coin
  - Heads → both choose A
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  - Payoff profile?
    - Each player's payoff at this profile = 1/2\*2 + 0 + 0 + 1/2\*1 = 3/2
    - (1.5, 1.5) is Fair and Pareto optimal



# Correlated Equilibrium

 A new randomization on each strategy of a game which expected utility is strictly higher than those of NE

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 A new randomization on each strategy of a game which expected utility is strictly higher than those of NE

**Theorem** For every Nash equilibrium  $\sigma^*$  there exists a corresponding correlated equilibrium  $\sigma$ .

# Correlated Equilibrium

• No agent i can benefit by deviating from  $\sigma_i$ , so  $\sigma$  is a correlated equilibrium

- There also are correlated equilibria that aren't equivalent to NE
  - e.g., Battle of the Sexes

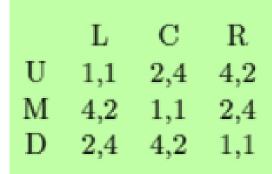
• Find a correlated equilibrium in which the sum of the players' payoff is higher than NE.

	L	$^{\mathrm{C}}$	R
U	1,1	2,4	4,2
M	4,2	1,1	2,4
D	2,4	$^{4,2}$	1,1

 Find a correlated equilibrium in which the sum of the players' payoff is higher than any NE.

Pure strategy?

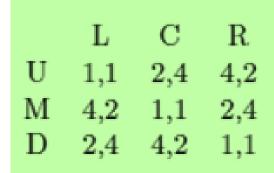
Dominated strategy?



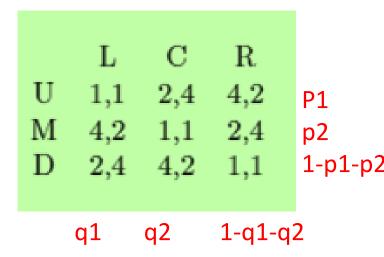
 Find a correlated equilibrium in which the sum of the players' payoff is higher than any NE.

Pure strategy? No

Dominated strategy? No

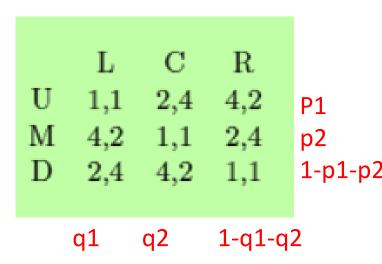


Mixed strategy?



#### Mixed strategy?

```
\begin{split} & \text{Expected Utility}[Player1 \; (\text{U}, (q_1 \, , q_2 \, , 1 \, \cdot (q_1 \, + \, q_2))] = q_1 \, + \, 2q_2 \, + \, 4 \; (1 \, \cdot q_1 \, - \, q_2 \, ) = \\ & 3q_1 - 2q_2 \, + \, 4 \\ & \text{Expected Utility}[Player1 \; (\text{M}, \; (q_1 \, , q_2 \, , \; 1 \, \cdot \, (q_1 \, + \, q_2))] = \, 4q_1 \, + \, q_2 \, + \, 2 \; (1 \, \cdot q_1 \, - \, q_2 \, ) = \\ & 2q_1 - q_2 \, + \, 2 \\ & \text{Expected Utility}[Player1 \; (\text{D}, \; (q_1 \, , q_2 \, , \; 1 \, \cdot \, (q_1 \, + \, q_2))] = \, 2q_1 \, + \, 4q_2 \, + \, (1 \, \cdot q_1 \, - \, q_2 \, ) = \\ & q_1 \, + \, 3q_2 \, + \, 1 \end{split}
```



#### Mixed strategy?

$$\begin{aligned} &\text{Expected Utility}[Player1~(\text{U},(q_1\,,q_2\,,1\,\text{-}\,(q_1+\,q_2))] = q_1 + 2q_2 + 4~(1\text{-}q_1-q_2\,) = \\ &-3q_1 - 2q_2 + 4 \\ &\text{Expected Utility}[Player1~(\text{M},~(q_1\,,q_2\,\,,\,1\,\text{-}\,(q_1+\,q_2))] = 4q_1 + q_2 + 2~(1\text{-}q_1-q_2\,\,) = \\ &2q_1 - q_2 + 2 \\ &\text{Expected Utility}[Player1~(\text{D},~(q_1\,,q_2\,\,,\,1\,\text{-}\,(q_1+\,q_2))] = 2q_1 + 4q_2 + (1\text{-}q_1-q_2\,\,) = \\ &q_1 + 3q_2 + 1 \end{aligned}$$

U M D	L 1,1 4,2 2,4	C 2,4 1,1 4,2	R 4,2 2,4 1,1	P1 p2 1-p1-p2
	q1	q2	1-q1-q2	2

$$\begin{cases}
-3q_1 - 2q_2 + 4 = 2q_1 - q_2 + 2 \\
-3q_1 - 2q_2 + 4 = q_1 + 3q_2 + 1
\end{cases}
q_1 = q_2 = \frac{1}{3}$$



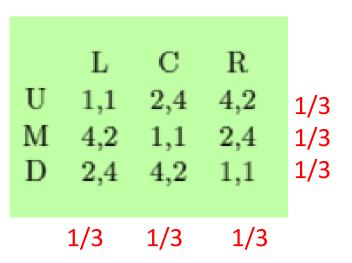
$$q_1 = q_2 = \frac{1}{3}$$

Mixed Strategy NE = 
$$\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$$

U M D	L 1,1 4,2 2,4	C 2,4 1,1 4,2	R 4,2 2,4 1,1	1/3 1/3 1/3
	1/3	1/3	1/3	

Mixed Strategy NE = 
$$\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$$

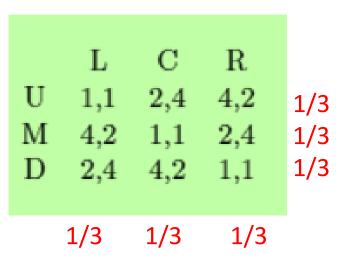
Expected payoff for each player?



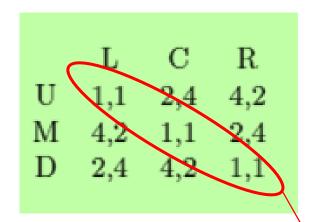
Mixed Strategy NE = 
$$\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$$

Expected payoff for each player?

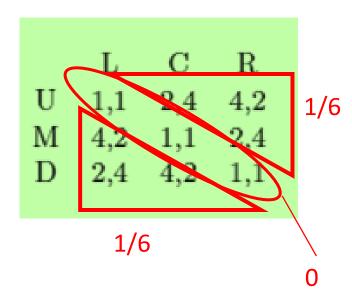
$$\frac{1}{3} * \frac{1}{3} (1*3 + 2*3 + 4*3) = \frac{7}{3}$$



• What is correlated equilibrium?

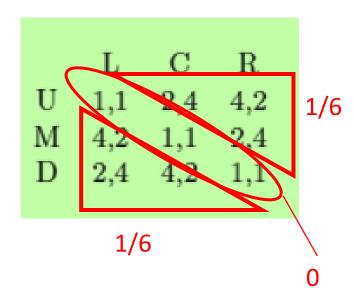


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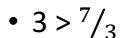
Expected payoff for each player?

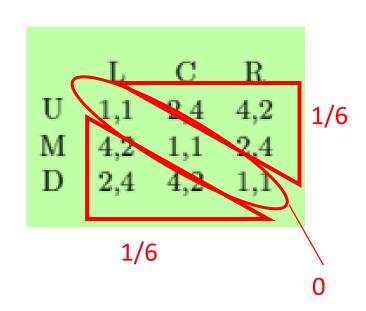


What is correlated equilibrium?

Expected payoff for each player?

$$\frac{1}{6}$$
\* (3\*4 + 3\*2) = 3





#### Definition (3): Nash Equilibrium

Strategy profile s\* constitutes a **Nash Equilibrium** if, for each player *i*,

Where:  $u_i(s_i^*, s_{-i}^*) \ge u_i(s_i, s_{-i}^*), \forall s_i \in S_i$ 

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**Definition**  $(\epsilon$ -Nash) Fix  $\epsilon > 0$ . A strategy profile  $s = (s_1, \ldots, s_n)$  is an  $\epsilon$ -Nash equilibrium if, for all agents i and for all strategies  $s'_i \neq s_i$ ,  $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) - \epsilon$ .

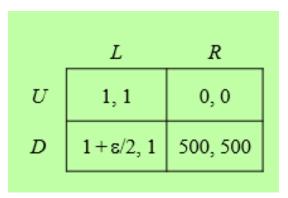
- $\varepsilon$ -Nash equilibria exist for every  $\varepsilon > 0$ 
  - Every NE is surrounded by a region of  $\epsilon$ -Nash equilibria

- Computationally useful
  - Finding NE algorithms can stop when they get close sufficiently

• Example

• NE?

• ε-Nash equilibrium?



Example

• NE? (*D*, *R*)

- ε-Nash equilibrium? (*U, L*)
  - Neither agent can gain more than  $\epsilon$  by deviating

