More about Normal-Form Games

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Outline

- Correlated Equilibrium
- ε-Nash Equilibrium

Battle of sexes

- We already know a lot about this game.

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

Battle of sexes

- We already know a lot about this game.

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

- Coordination game
  - without communication, it is possible that the two players might fail to coordinate

Battle of sexes

- Pure-strategy equilibria
  - NE = (A,A) and (B,B)
  - Payoff profiles (2,1) and (1,2)
  - Unfair, but pareto optimal

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

Battle of sexes

- Mixed-strategy equilibrium

	A	B	
A	2, 1	0, 0	p
B	0, 0	1, 2	1-p
	q	1-q	

Battle of sexes

- Player1's expected utility?

$$\left.\begin{aligned} E\left[U_1\left(\text{A},(q,1-q)\right)\right] &= 2q+0(1-q) \\ E\left[U_1\left(\text{B},(q,1-q)\right)\right] &= 0q+1(1-q) \end{aligned}\right\} 2q=(1-q) \Rightarrow q=\frac{1}{3}$$

- Player2's expected utility?

$$\left.\begin{aligned} E\left[U_2\left((p,1-p),\text{A}\right)\right] &= 1p+0(1-p) \\ E\left[U_2\left((p,1-p),\text{B}\right)\right] &= 0p+2(1-p) \end{aligned}\right\} 1p=2(1-p) \Rightarrow p=\frac{2}{3}$$

	A	B	
A	2, 1	0, 0	p
B	0, 0	1, 2	1-p
	q	1-q	

Battle of sexes

- Mixed-strategy equilibrium

$$\left[\overset{\text{Player 1}}{\left(\underset{p}{\frac{2}{3}}, \underset{1-p}{\frac{1}{3}}\right)}, \overset{\text{Player 2}}{\left(\underset{q}{\frac{1}{3}}, \underset{1-q}{\frac{2}{3}}\right)}\right]$$

	A	B	
A	2, 1	0, 0	2/3
B	0, 0	1, 2	1/3
	1/3	2/3	

Battle of sexes

	A	B	
A	2, 1	0, 0	2/3
B	0, 0	1, 2	1/3
	1/3	2/3	

- What is the probability for the two players meet or not to meet?

Battle of sexes

	A	B	
A	2, 1	0, 0	2/3
B	0, 0	1, 2	1/3
	1/3	2/3	

- What is the probability for the two players meet or not to meet?

➔ Prob(meet) = 2/3*1/3+1/3*2/3=4/9

➔ 1- Prob(meet) = 5/9 !!!

Battle of sexes

2/3×1/3 = 2/9   2/3×2/3 = 4/9

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

1/3×1/3 = 1/9   1/3×2/3 = 2/9

- Payoff profile?

Battle of sexes

2/3×1/3 = 2/9 2/3×2/3 = 4/9

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

1/3×1/3 = 1/9 1/3×2/3 = 2/9

- Payoff profile?
  - Each player's payoff at this profile = ${}^{2}/_{9}$*2 + 0 + 0 + ${}^{2}/_{9}$*1 = ${}^{2}/_{3}$
  - $({}^{2}/_{3}, {}^{2}/_{3})$ is Fair

Battle of sexes

	A	B	
A	2, 1	0, 0	2/3
B	0, 0	1, 2	1/3
	1/3	2/3	

- Payoff profile?
  - Each player's payoff at this profile = $^2/_9$*2 + 0 + 0 + $^2/_9$*1 = $^2/_3$
  - ($^2/_3$, $^2/_3$) is Fair
- Is this profile pareto optimal?

Battle of sexes

	A	B	
A	2, 1	0, 0	2/3
B	0, 0	1, 2	1/3
	1/3	2/3	

- Payoff profile?
  - Each player's payoff at this profile = ${}^{2}/_{9}$*2 + 0 + 0 + ${}^{2}/_{9}$*1 = ${}^{2}/_{3}$
  - (${}^{2}/_{3}$, ${}^{2}/_{3}$) is Fair
- Is this profile pareto optimal?
  - No, this profile is Pareto dominated by (A,A) and (B,B)

Battle of sexes

- How we can obtain both pareto optimality and fairness in NE payoffs?

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

Battle of sexes

- How we can obtain both pareto optimality and fairness in NE payoffs?

- Neither pure nor mixed NE
- Flip a coin
  - Heads → both choose A
  - Tails → both choose B

1/2 0

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

0 1/2

Battle of sexes

- How we can obtain both pareto optimality and fairness in NE payoffs?

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  - Payoff profile?

1/2 0

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Battle of sexes

- How we can obtain both pareto optimality and fairness in NE payoffs?

1/2 0

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

0 1/2

- Neither pure nor mixed NE
- Flip a coin
  - Heads → both choose A
  - Tails → both choose B
  - Payoff profile?
    - Each player’s payoff at this profile = 1/2*2 + 0 + 0 + 1/2*1 = 3/2
    - (1.5, 1.5) is Fair and Pareto optimal

Correlated Equilibrium

- A new randomization on each strategy of a game which expected utility is strictly higher than those of NE

Correlated Equilibrium

- A new randomization on each strategy of a game which expected utility is strictly higher than those of NE

Theorem *For every Nash equilibrium σ^* there exists a corresponding correlated equilibrium σ.*

Correlated Equilibrium

- No agent *i* can benefit by deviating from σ_i, so $\boldsymbol{\sigma}$ is a correlated equilibrium

- There also are correlated equilibria that aren't equivalent to NE
  - e.g., Battle of the Sexes

Question

- Find a correlated equilibrium in which the sum of the players' payoff is higher than NE.

	L	C	R
U	1,1	2,4	4,2
M	4,2	1,1	2,4
D	2,4	4,2	1,1

Question

- Find a correlated equilibrium in which the sum of the players' payoff is higher than any NE.

- Pure strategy?

- Dominated strategy?

	L	C	R
U	1,1	2,4	4,2
M	4,2	1,1	2,4
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Question

- Find a correlated equilibrium in which the sum of the players' payoff is higher than any NE.

- Pure strategy? No

- Dominated strategy? No

	L	C	R
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D	2,4	4,2	1,1

Question

- Mixed strategy?

	L	C	R	
U	1,1	2,4	4,2	P1
M	4,2	1,1	2,4	p2
D	2,4	4,2	1,1	1-p1-p2
	q1	q2	1-q1-q2	

Question

- Mixed strategy?

Expected Utility[$Player1$ (U, $(q_1, q_2, 1 - (q_1 + q_2))$] = $q_1 + 2q_2 + 4\,(1 - q_1 - q_2) = -3q_1 - 2q_2 + 4$

Expected Utility[$Player1$ (M, $(q_1, q_2, 1 - (q_1 + q_2))$] = $4q_1 + q_2 + 2\,(1 - q_1 - q_2) = 2q_1 - q_2 + 2$

Expected Utility[$Player1$ (D, $(q_1, q_2, 1 - (q_1 + q_2))$] = $2q_1 + 4q_2 + (1 - q_1 - q_2) = q_1 + 3q_2 + 1$

	L	C	R	
U	1,1	2,4	4,2	P1
M	4,2	1,1	2,4	p2
D	2,4	4,2	1,1	1-p1-p2
	q1	q2	1-q1-q2	

Question

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Expected Utility[$Player1$ (U, $(q_1, q_2, 1 - (q_1 + q_2))$] $= q_1 + 2q_2 + 4(1 - q_1 - q_2) =$ $-3q_1 - 2q_2 + 4$

Expected Utility[$Player1$ (M, $(q_1, q_2, 1 - (q_1 + q_2))$] $= 4q_1 + q_2 + 2(1 - q_1 - q_2) =$ $2q_1 - q_2 + 2$

Expected Utility[$Player1$ (D, $(q_1, q_2, 1 - (q_1 + q_2))$] $= 2q_1 + 4q_2 + (1 - q_1 - q_2) =$ $q_1 + 3q_2 + 1$

	L	C	R	
U	1,1	2,4	4,2	P1
M	4,2	1,1	2,4	p2
D	2,4	4,2	1,1	1-p1-p2
	q1	q2	1-q1-q2	

$$\begin{cases} -3q_1 - 2q_2 + 4 = 2q_1 - q_2 + 2 \\ -3q_1 - 2q_2 + 4 = q_1 + 3q_2 + 1 \end{cases} \Rightarrow q_1 = q_2 = {}^{1}/_{3}$$

Question

Mixed Strategy NE = $\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$

	L	C	R	
U	1,1	2,4	4,2	1/3
M	4,2	1,1	2,4	1/3
D	2,4	4,2	1,1	1/3
	1/3	1/3	1/3	

Question

$$\text{Mixed Strategy NE} = \{(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})\}$$

	L	C	R	
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D	2,4	4,2	1,1	1/3
	1/3	1/3	1/3	

- Expected payoff for each player?

Question

Mixed Strategy NE = $\{(\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$

	L	C	R	
U	1,1	2,4	4,2	1/3
M	4,2	1,1	2,4	1/3
D	2,4	4,2	1,1	1/3
	1/3	1/3	1/3	

- Expected payoff for each player?

$$\frac{1}{3}*\frac{1}{3}(1*3+2*3+4*3)=\frac{7}{3}$$

Question

- What is correlated equilibrium?

	L	C	R
U	1,1	2,4	4,2
M	4,2	1,1	2,4
D	2,4	4,2	1,1

0

Question

- What is correlated equilibrium?

	L	C	R
U	1,1	2,4	4,2
M	4,2	1,1	2,4
D	2,4	4,2	1,1

1/6

1/6

0

Question

- What is correlated equilibrium?

- Expected payoff for each player?

	L	C	R
U	1,1	2,4	4,2
M	4,2	1,1	2,4
D	2,4	4,2	1,1

1/6

1/6

0

Question

- What is correlated equilibrium?

	L	C	R
U	1,1	2,4	4,2
M	4,2	1,1	2,4
D	2,4	4,2	1,1

1/6

1/6

0

- Expected payoff for each player?

$$\frac{1}{6} * (3*4 + 3*2) = 3$$

- $3 > {}^{7}/_{3}$

ε-Nash Equilibrium

Definition (3): **Nash Equilibrium**

Strategy profile s^* constitutes a **Nash Equilibrium** if, for each player *i*,

Where: $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \forall s_i \in S_i$

ε-Nash Equilibrium

Definition (3): Nash Equilibrium

Strategy profile s^* constitutes a **Nash Equilibrium** if, for each player *i*,

Where: $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \forall s_i \in S_i$

Definition (**ϵ-Nash**) *Fix* $\epsilon > 0$. *A strategy profile* $s = (s_1, \ldots, s_n)$ *is an* ϵ-*Nash equilibrium if, for all agents* i *and for all strategies* $s_i' \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) - \epsilon$.

ε-Nash Equilibrium

- ε-Nash equilibria exist for every $\varepsilon > 0$
    - Every NE is surrounded by a region of ε-Nash equilibria

- Computationally useful
    - Finding NE algorithms can stop when they get close sufficiently

ε-Nash Equilibrium

- Example

	L	R
U	$1, 1$	$0, 0$
D	$1+\varepsilon/2, 1$	$500, 500$

- NE?
- ε-Nash equilibrium?

ε-Nash Equilibrium

- Example

	L	R
U	1, 1	0, 0
D	1+ε/2, 1	500, 500

- NE? *(D, R)*

- ε-Nash equilibrium? *(U, L)*
  - Neither agent can gain more than ε by deviating