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Computer Simulation

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Chapter Three: Statistical Models

Purpose and Overview (1)

- What is the goal of studying statistical models?
  - Many phenomenon in real-world are rather **probabilistic** rather than being **deterministic**
  - Therefore, it is necessary to **search for good statistical models** to describe the **alterations** of the intended probabilistic phenomenon **correctly** with **good precision**
- Based on sampling the intended phenomena and the results, an appropriate model can be developed by:
  - Select a known distribution through educated **guesses**
  - Make **estimate of the parameter(s)** for the guessed distribution function(s)
  - Test for goodness of fit to select the best candidate

Purpose and Overview (2)

- Note: It is possible to have multiple distribution functions describing a single phenomena
  - It is our job to compare the outcomes of the functions to the real-world data
    - This is what we call as **reconciliation**
  - Afterwards, the most suited distribution function will be selected
- In this chapter, we are going to study:
  - Different distribution functions
  - Their specifications
  - Applications that they could be applied to

Random Variables (1)

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- A random variable is a parameter of a specific phenomena, which is **measurable** but its **value is random**
- Discrete random variables
  - If X could be assigned with only a set of countable values
  - Example: tossing a coin, status of a production line, which is working or is failed
- Continuous random variables
  - If X could be assigned with any values within a finite or infinite range
  - Example: Received Signal Strength Index (RSSI) in wireless communications (in dB), or strength of wind in power systems

Random Variables (2)

- Now consider the number of jobs arriving at a job shop
  - Let X be the number of jobs arriving each week at a the shop
  - $\mathrm{R_x}$ is all of the possible values of X, which could be $\in \{0,1,2,\ldots\}$
    - $\mathrm{R_x}$ is called as **support** or **sample space**
    - It is possible to have no requests, while we could have more
  - The probability that the random variable X equals to $\mathrm{x_i}$ is denoted by $\mathrm{P(x_i) = P(X = x_i)}$
  - Note: Probability of having event $\mathrm{x_i}$ for any i is greater or equal to zero:

$$P(x_i) \geq 0, for\ any\ i$$

  - Note: Sum of all probability values for all the events $\mathrm{x_i}$ (any i) would be equal to 1:

$$\sum_{i=1}^{\infty} P(x_i) = 1$$

Random Variables (3)

- Couples of $[x_i, P(x_i)]$ for all of the i values, is called the **probability distribution** of random variable X
  - $P(x_i)$ is known as Probability Mass Function (PMF) of random variable X
    - PMF gives us the probability that X is equal to x_i
- Reminding some important statistical knowledge
- Random experiment
  - A function which is **replicable**, but gives us **different random outcomes** every time we iterate it
  - Notify that all the outcomes are known priorly, but we don't know which one would come up after the experiment
    - Example: tossing a coin, throwing a dice, distant to the target in archery

Random Variables (4)

- Sample space:
  - Set of all the possible outcome values for a random experiment
  - It is indicated with Ω
    - Example: In tossing a coin we have $\Omega = \{H, T\}$, or in archery, the distance to the red target has $\Omega = [0, r]$
      - Since in the later, x_i can get any numbers and we have an uncountable set, the random variable is **continuous**
- Event:
  - Consider a countable sample space
  - Any **subset of this sample space**, creates an event
    - Example: In tossing a coin for two times, $A = \{TH, TT\}$ is an event representing an observation of tails in the first round of throwing
  - If $A \subseteq \Omega \rightarrow \acute{A}(Complement\ Event) \subseteq \Omega$
    - $\Omega\ (Definite\ Event) \subseteq \Omega \rightarrow \emptyset\ (Impossible\ Event) \subseteq \Omega$

Random Variables (5)

- Event space:
  - A set of all the subsets of our sample space (Ω)
  - Denoted with F
    - $\emptyset \in F$, and $\Omega \in F$
    - If A is member of F, then its complement ($\acute{A}$) is also member of F
    - If $A_1, A_2, \ldots$ are **sequence of independent events** ($\in F$), then their unity will be also member of F

$$\bigcup_{i=1}^{\infty} A_i \in F$$

- Inconsistent events:
  - If A, and B are two events of random variable X, they are called inconsistent if their subscription is null

$$\bigcap_{i=1}^{\infty} A_i = \emptyset$$

Random Variables (6)

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- Inconsistent events (Cont.):
  - In this case, only one event (A, or B) could be occurred
  - Sequence of $A_1, A_2, \ldots$ are called **mutually exclusive** if only one of them could occur at a time instance:

$$A_i \cap A_j = \emptyset$$

  - If the sequence of events $A_1, A_2, \ldots$ have the following property, then we call them **monotone**:

$$A_1 \subset A_2 \subset A_3 \ldots$$

- Probability function (formal definition):
  - Any function who makes a correspondence between **event space** (F) to **real numbers** (R)

Random Variables (7)

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- Probability function (Cont.):
  - Any probability function must guarantee the three following conditions:
    - Probability value for every event is a non-negative number
      - $P(A) \geq 0$
    - The probability of the sample space equals to 1
      - $P(\Omega) = 1$
    - Based on the definition of mutual exclusive events, we have the following relation:
      - $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Important Statistical Theorems

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- Probability of event ∅ is zero: $P(\emptyset) = 0$
- Probability of uniting two **mutual exclusive** events is equal to adding their probability individually:

$$P(A \cup B) = P(A) + P(B)$$

- Probability of complement of A, equals to subtracting probability of A from 1: $P(\acute{A}) = 1 - P(A)$
- If event A is subset of event B, probability value of A is less than probability value of B: $If\ A \subseteq B\ then\ P(A) \leq P(B)$
- If C is an event created by uniting A, and B, then probability of C equals to:

$$P(C) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Final Marks About Random Variable (1)

- With the help of **random variables**, we can assign a real number for every event of the sample space
- Since events occur randomly, it is obvious that we are facing random values for the intended phenomena
  - This is why we call it a random variable
- Accordingly, we could say that random variables are somehow a **function**
  - Example: In case of throwing a symmetric coin for two times, the sample space would be $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$
    - If we consider our random variable X, as the number of heads:

$$X(\{HT\}) = X(\{TH\}) = 1, X(\{TT\}) = 0, X(\{HH\}) = 2$$

  - As you can see, we have assigned a real number for every member of the sample space

Final Marks About Random Variable (2)

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- Now, let's determine the probabilities for different values of our random variable
  - We want to calculate $P(X = x_i)$
  - Recall: x_i is member of sample space, and it could get those values we discussed in the last slide $\in \{0,1,2\}$

$$P(X = 1) = P(\{HT, TH\}) = \frac{2}{4} = 0.5$$

$$P(X = 2) = P(\{HH\}) = \frac{1}{4} = 0.25$$

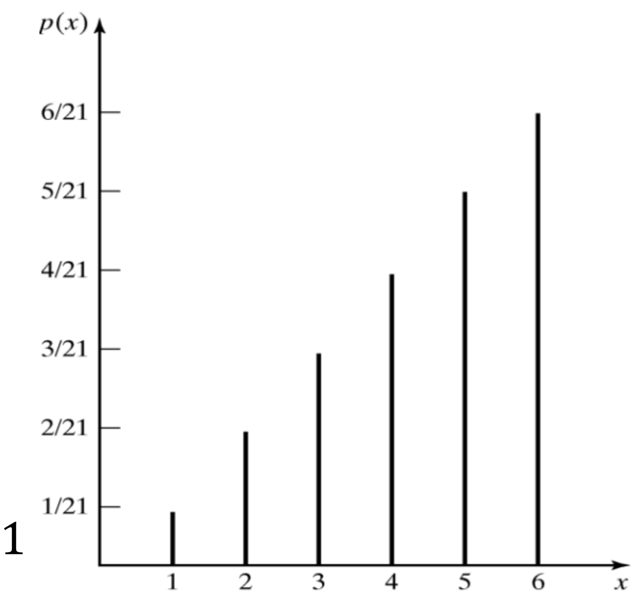
$$P(X = 4) = P(X < 0) = P(1 < X < 2) = P(\emptyset) = 0$$

- Formal definition of a random variable X:
  - Consider a random experiment with Ω as its sample space, F as its event space, and A as its simple events
  - Random variable is a **function**, which assigns a **real value** to every event, and is denoted with $X(A) \in R$

Example

- Assume a dice is loaded so that the probability of having a given face landed up is proportional to the **number of spots showing**
  - What is our random variable? The face that lands up
    - So, every face will be assigned to $\Omega \in \{1,2,\dots,6\}$
- In the following chart and table, the probability values are depicted for different values of X

x_i	1	2	3	4	5	6
$P(x_i)$	1/21	2/21	3/21	4/21	5/21	6/21



  - As you can see:

$$P(X = x_i) \geq 0 \;\; for\ all\ i \quad \text{and} \quad \sum_{i=1}^{6} P(x_i) = 1$$

Continuous Random Variables (1)

- First we discuss a little bit more about support (R_x)
  - Support (or **range space**) is a set of values which a random variable could get with a non-negative probability
  - The probability function is only indicated for values that are supported by R_x
    - For values out of R_x, probability function gives zero
- If X could be assigned with **any** values within a finite or infinite range, it is a continuous random variable
  - The range space R_x of a continuous variable is an **interval** or a **collection of intervals**
- Lifetime of a lamp bulb is an instance of continuous random variables

Continuous Random Variables (2)

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- Probability of X having values in [a,b] interval is calculated with the following:

$$P(a \leq X \leq b) = \int_{a}^{b} \boldsymbol{f(x)d_x}$$

  - In this equation, $f(x)$ is the **probability density function (PDF)** for continuous random variable X
- Is it possible to calculate the probability in a specific point (y) (not an interval)?
  - First, we need to get familiar with another concept
- In statistics, F_X is known as the **cumulative distribution function (CDF)**, and it indicates $F_X(y) = P(X \leq y)$

Continuous Random Variables (3)

- CDF is a continuous function:

$$F_X(y) = F_X(y^+) = F_X(y^-)$$

- In order to calculate $P(X = y)$:

$$P(X = y) = F_x(y^+) - F_x(y^-) = 0$$

- Therefore, the probability value for continuous random variables in a single point of the support set would be **zero**
  - **Any other option?** This is where $f(x)$ comes in mind, which represents the derivative of the CDF
    - If X is a random variable with strict continuous CDF (F_x), a function f exists:

$$F_X(y) = \int_{-\infty}^{y} f(t)d_t \rightarrow f_X(y) = F_x\acute{}(y)$$

Specifications of PDF (1)

- Also, based on what we have already discussed in **slide 16**, the probability of a continuous random variable in a specific point is zero:

$$P(X = x_0) = P(x_0 \leq X \leq x_0) = \int_{x_0}^{x_0} f(x) d_x = 0$$

- For all of the values of R_x, the PDF is equal or greater than zero: $f_X(x_i) \geq 0, for\ all\ x_i \in R_x$
- Integration of $f(x)$ for values over the support is 1:

$$\int_{R_x} f(x) d_x = 1$$

- Value of $f(x)$ for all of other values out of R_x is zero:

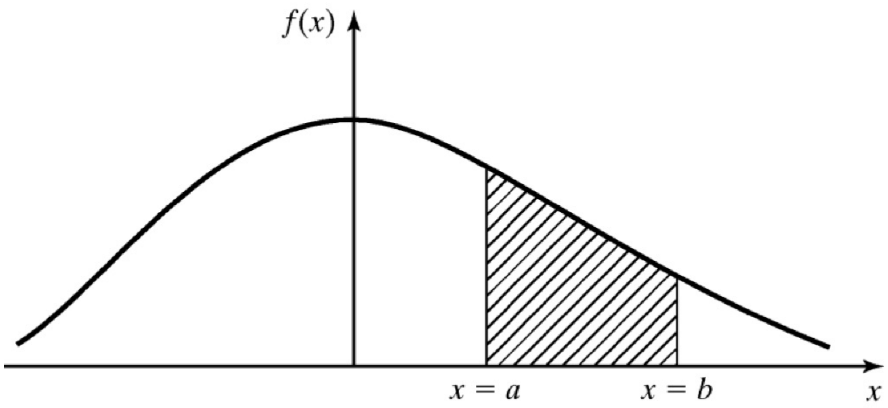
$$f_X(x_i) = 0, if\ x_i is\ not\ in\ R_x$$

Specifications of PDF (2)

- Note: probability of X between a, and b does not depend on openness or closeness of the boundaries

$$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$$

- According to the principles of integration
  - In order to obtain the probability of continuous variable X in [a,b], we can use the area under the PDF curve

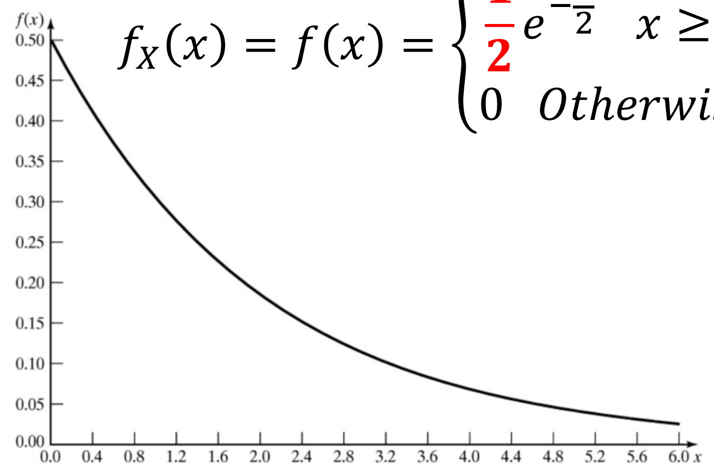


Example

- Life of an inspection device is given by X, a continuous random variable with a PDF depicted in the following:

- As we will discuss later, this random variable has an exponential distribution
  - Based on its PDF, it has a mean lifetime of 2 years
    - This number could be obtained by reversing the value of λ
      - We will discuss about the specifications of the distribution functions later in this chapter

$$f_X(x) = f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & Otherwise \end{cases}$$



- Calculate the probability of a lifetime between 2 to 3 years?

Cumulative Distribution Function (1)

- Cumulative Distribution Function (CDF) is used whenever $P(X \leq x_i)$ is required
- As mentioned before, CDF of a random variable X is denoted by $F_X(x_i)$
- If X is discrete, then: $F_X(x) = \sum_{\substack{\text{all} \\ x_i \leq x}} P(x_i)$
- If X is continuous, then $F_X(x) = \int_{-\infty}^{x} f(t)dt$
- Most important specifications of CDF:
  - $F_X(x)$ is a strict ascending function: $a < b \rightarrow F_X(a) \leq F_X(b)$
    - Since $\mathrm{F_X(x_i)} = P(X \leq x_i)$, and it adds up all of the probabilities of values less then x_i, it is obvious that with increasing x_i, value of $F_X(x_i)$ gets bigger and gets closer to 1

Cumulative Distribution Function (2)

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- Boundries of $F_X(x_i)$
  - Assume that R_x could range between $-\infty$, and $+\infty$ (any other values could be considered)
  - $F_X(-\infty) = P(X \leq -\infty) = 0 \rightarrow \lim_{x \to -\infty} F_X(x) = 0$
    - Because X cannot get values less than the lower boundary of R_x
  - $F_X(+\infty) = P(X \leq +\infty) = 1 \rightarrow \lim_{x \to +\infty} F_X(x) = 1$
    - Based on the principles of probability
  - Therefore, F_X could range between [0,1]
- All probability questions about *X* can be answered in terms of the CDF
  - Example:

$$P(a < X \leq b) = F(b) - F(a), \text{ for all } a < b$$

Example (1)

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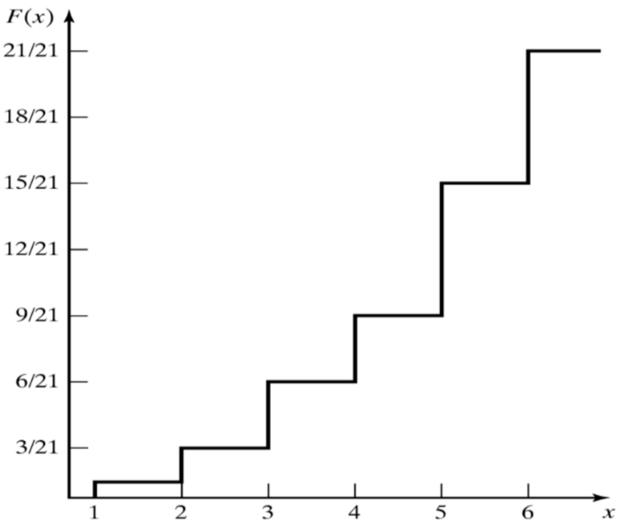
- Recall our dice-tossing example in previous slides
- As you can see, the table of values are different from what we have seen for PDF

x	(-∞,1)	[1,2)	[2,3)	[3,4)	[4,5)	[5,6)	[6,∞)
F(x)	0	1/21	3/21	6/21	10/21	15/21	21/21



- Notes:
  - Since values less than 1 are not part of R_x, value of F_X would be zero
  - On the other hand, for values greater then the maximum bound, F_X would be equal to 1
  - Similar to PDF, the data in the table could be represented as a chart

Example (2)

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- In our previously mentioned inspection device, the PDF of the lifetime was defined as an exponential distribution:

$$f_X(x) = f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x \geq 0 \\ 0 & Otherwise \end{cases}$$

- What is the probability that the device lasts for less than 2 years:
  - First obtain CDF based on our PDF

$$F_X(x) = \int_{-\infty}^{+\infty} f(x)d_x = \int_0^x \frac{1}{2}e^{-\frac{t}{2}} \ d_t = 1 - e^{-\frac{x}{2}}$$

$$\rightarrow P(0 \leq X \leq 2) = F_X(2) - F_X(0) = 1 - e^{-\frac{2}{2}} = 0.63$$

- The probability that it lasts between 2 and 3 years:

$$P(2 \leq X \leq 3) = F_X(3) - F_X(2) = \left(1 - e^{-\frac{3}{2}}\right) - \left(1 - e^{-\frac{2}{2}}\right) = 0.14$$

Statistical Index

- Statistical indexes are parameters or criterions, which are used to compare at least two statistical populations
- Indexes are generally grouped into three categories
  - Measures of central tendency
    - Mode, median, mean (also known as expected value), quantile (quartile, decile, percentile)
  - Measures of dispersion
    - Variance, and standard deviation
  - Measures of relative dispersion
    - Coefficient of variation, coefficient of skewness, and coefficient of Kurtosis

Measures of Central Tendency

- Any index, which determines the center of the population and is surrounded by other observations
- Expected value is one the most important central tendency measures
  - Assume random variable X
  - The expected value for X is denoted by E(X), and is calculated as follows:
    - If X is discrete: $E(x) = \sum_{\text{all } i} x_i p(x_i)$ **Probability Mass Function**
    - If X is continuous: $E(x) = \int_{-\infty}^{\infty} x f(x) dx$ **Probability Density Function**
- E(X) is only defined for values residing in R_x
  - $-\infty$, and $+\infty$ should be replaced with the boundaries if needed
    - Because, f(x) gives zero outside of R_x

Expected Value (1)

- E(X) talks about the values which may come in future
  - It may be considered identical to average (or mean) of the sampled data, which has happened until now
- E(X) could be also represented with the first order momentum
  - Momentum in statistics is a quantitative value or a measure for **determining the shape** of a probability function or the distribution of random variables
  - In physics, momentum or torque is also used to define the specifications of an entity (mass, center of gravity, …)
  - It is useful to exploit momentum to describe different aspects of a random variable
    - This is done with different orders of momentum over R_x

Expected Value (2)

- The more complicated the distribution function gets, more number of momentum orders are required for understanding its specifications
- The n^{th} order momentum for a real function f(x) around c is defined as follows:

$$\mu_n = \int_{-\infty}^{+\infty} (x - c)^n f(x) d_x$$

  - If $c = \bar{x}$: it is called as the **central momentum**
  - Typically, c is set as zero, when μ is called **raw momentum** (or the momentum around zero)
    - In this case, if n=1: $\mu_1 = \int_{-\infty}^{+\infty} x f(x) d_x = \mathrm{E(X)}$
      - This is why we say expected value is the 1st order momentum of the random variable X

Measures of Dispersion

- Based on the relation between E(X), and momentum
  - The expected value tries to determine the **focus** of the intended population around the intended point
- But, there is a more important statistical criterion
  - Variance, which is denoted by V(X), VAR(X), σ^2
- While E(X), and mean are important measures of tendency for evaluation of two populations around them, but we may face with two identical values for each population
  - In such cases, E(X) cannot be used for comparison
  - We need to use measures of dispersion
    - They are used to compare the **dispersion** of populations around the center, and select the desired set of data

Variance and Standard Deviation

- Variance
  - One of the most important measures of dispersion
    - Indicates the deviation of observations $(x_1, x_2, \dots)$ around the mean value

$$V(X) = \sigma^2 = \frac{\sum(x_i - mean)^2}{n} = E\left[\left(x - E(X)\right)^2\right]$$

$$V(X) = \sigma^2 = E(X^2) - E(X)^2$$

- Standard Deviation
  - Assume that the unit of observations is meters (m)
    - Hence, the unit of V(X) will be m^2
    - It is complicated to compare the dispersion between populations → to solve this issue, we use the square root of V(X)
      - It is called the standard deviation, which gives us a measure of dispersion with same unit as our data (and its mean)

Example (1)

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- The expected value (mean) of lifetime of the previous inspection device is calculated as:

$$\left.\begin{array}{l} f_X(x) = f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & Otherwise \end{cases} \\ \\ E(X) = \int_{-\infty}^{+\infty} x f(x) d_x \end{array}\right\} E(X) = \int_0^{+\infty} x \times \frac{1}{2} e^{-\frac{x}{2}} d_x = \frac{1}{2} \int_0^{+\infty} x e^{-\frac{x}{2}} d_x$$

Partial Integration $\Rightarrow$ $E(X) = -xe^{-\frac{x}{2}} [0, +\infty] + \int_0^{+\infty} e^{-\frac{x}{2}} d_x = \mathbf{2}$

- As you can remember, we mentioned that in exponential distribution, the mean value could be obtained by inversing the rate (λ)
  - Here, $\lambda = \frac{1}{2} \rightarrow E(X) = mean = \mathbf{2}$

Example (2)

- The variance, and standard deviation is calculated as follows:

$$E(X^2) = \frac{1}{2}\int_0^{+\infty} x^2 e^{-\frac{x}{2}} d_x = -x^2 e^{-\frac{x}{2}}[0, +\infty] + \int_0^{+\infty} e^{-\frac{x}{2}} d_x = 8$$

- Recall: V(X) can be calculated as $E(X^2) - E(X)^2$
  - $8 - 2^2 = 4$
- The standard deviation is the square root of V(X)
  - $\sigma = 2$

Useful Statistical Models

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- In the rest of this chapter, the well-known statistical distributions for discrete and continuous random variables are introduced
  - Their specifications will be discussed
- Prior to this, a number of statistical models appropriate to some **application areas** are discussed
- The areas include:
  - Queueing systems
  - Inventory and supply-chain systems
  - Reliability and maintainability
  - Limited data

Queueing Systems (1)

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- In a queueing system, **interarrival** and **service-time** patterns can be probabilistic
  - For more queueing examples, see Chapter 2
- Sample statistical models for interarrival or service time distribution:
  - Exponential distribution
    - If service times are completely random
    - In other words, it is unclear when the customers arrive or how long their service will take
  - Normal distribution
    - If there isn't much difference between the probability of different values
    - Fairly constant but with some random variability (either positive or negative)

Queueing Systems (2)

- Sample statistical models for interarrival or service time distribution (Cont.):
  - Truncated normal distribution
    - Similar to normal distribution but with **restricted boundaries**
      - The values of the random variable are bounded
    - In normal distribution, R_x could range from $-\infty$ to $+\infty$, but we may have an entity where its values are only positive (or negative)
  - Gamma and Weibull distributions
    - More general than exponential
      - Their mathematical representation has more parameters for **tuning** and **adjusting** to different variables
      - Different values of parameters results in different **distribution shapes**
    - They are more complicated, but gives us more freedom to apply to different applications

Inventory and Supply Chain (1)

- In realistic inventory and supply-chain systems, there are at least three random variables:
  - The number of units demanded per order or per time period
  - The time between demands
  - The lead time
- Sample statistical models for lead time distribution:
  - Gamma
- Sample statistical models for demand distribution:
  - Poisson:
    - Since it is a discrete distribution, it is used for modeling number of demands in a period, or number of units per order
    - It is a simple model with only a single parameter to be tuned with the sampled data

Inventory and Supply Chain (2)

- Sample statistical models for demand distribution (Cont.):
  - Poisson:
    - Its CDF is available in form of tables
      - Even without tables, due to simplicity of calculations with its equation, the probability values could be simply obtained
  - Negative binomial distribution:
    - Creates longer tail than Poisson
    - Useful for more large demands
  - Geometric:
    - A special case of negative binomial given **at least one demand has been received**
- Based on the specifications of the intended random variable, suitable distribution is selected

Reliability and Maintainability

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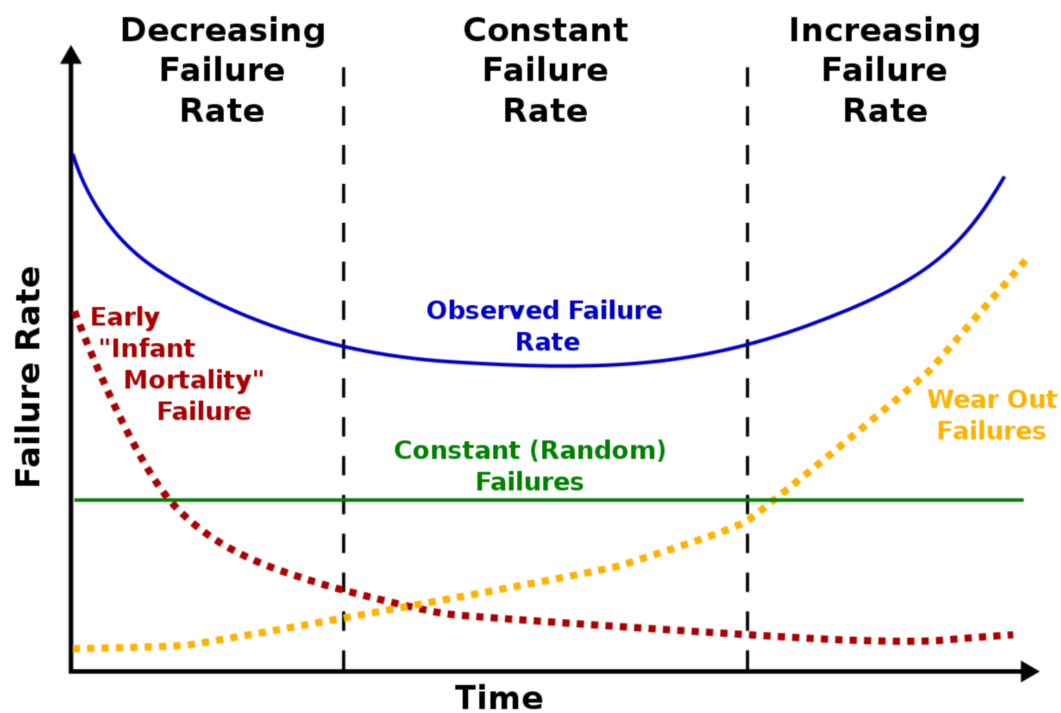
- Reliability is the conditional probability of correct functioning of a device at t, while it was correctly working at the beginning of our analysis $(t = 0)$
  - Denoted with $R(t)$, and reported for (0,t] interval
- Time to failure (TTF) is one of the most important parameters for reliability analysis
  - Indicates the time between two consecutive failures
- If failures occur randomly, TTF is modeled with exponential distribution
- If standby redundancy is considered for different parts of the system with **each** having exponential TTF
  - TTF of the **entire system** is modeled with Gamma

Reliability and Maintainability

- If failure is due to a large number of defects in a system of components
  - The Weibull distribution could be used
- If failures occur due to wear
  - The normal distribution will be used



The Bathtub Curve

Other Areas

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- For cases with limited data, some useful distributions are:
  - Uniform, triangular and beta
- Other distributions include Bernoulli, binomial and hyper-exponential
- The important issue is to select the **best-fit** distribution based on the collected data
  - To model the intended variable with high precision
  - In case we realized the distribution does not act well, change it or try to tune its parameters
    - More information in Chapter 8
  - This skill could be improved with more practice and using your previous experience

Discrete Distributions

- Discrete RV are used to describe random phenomena in which countable number of values can occur
- In this section, we will learn about:
  - Bernoulli trials and Bernoulli distribution
  - Binomial distribution
  - Geometric and negative binomial distribution
  - Poisson distribution
- Note: Many distributions are constructed based on a basic model
  - Example: Tossing a coin and observing H/T is a random variable with Bernoulli distribution
    - If we iterate this experiment and define a variable which indicates number of H/Ts in n iterations → **a variable with binomial distribution**

Bernoulli Trials and Bernoulli Distribution (1)

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- Bernoulli Trial
  - An experiment with only 2 outcomes: success or failure
- If we define X as our Bernoulli random variable (outcome of the Bernoulli trial):
  - Let X = 1 if the experiment is a success
  - and X = 0 if the experiment is a failure
  - The Bernoulli distribution:
    - Probabilities in j-th trial: $P(x_j) = \begin{cases} p, & x_j = 1, j = 1,2,3,\dots,n \\ 1-p = q, & x_j = 0, j = 1,2,3,\dots,n \\ 0, & Otherwise \end{cases}$
- Example:
  - Tossing a homogenous coin, observing H → p=1/2, and q=1/2
    - We may have an unusual coin with other probability values, but we still have Bernoulli trial
  - Throwing a dice 2 times, observing same values → p=1/6, and q=5/6

Bernoulli Trials and Bernoulli Distribution (2)

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- Expected value for a random variable with Bernoulli distribution:
  - $E(X) = \sum_{i=0,1} x_i \times P(x_i) = 1 \times p + 0 \times (1-p) = p$
- Variance value for a random variable with Bernoulli distribution:
  - $V(X) = E(X^2) - E(X)^2 = p - p^2 = p \times (1-p) = p \times q$
- Bernoulli process:
  - If we **repeat n Bernoulli trials**, where trials are **independent**, we call it a Bernoulli process
  - To obtain the probability of a sequence of Bernoulli events:
    - $P(x_1, x_2, \ldots, x_n) = P(x_1) \times P(x_2) \times \cdots \times P(x_n)$

Binomial Distribution (1)

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- Consider a **single** Bernoulli trial with a constant success probability p
  - If we assume the **number of successes** in **n trials** as a random variable X, it will follow the Binomial distribution
  - Note: The number of trials n is constant
- The probability function of X is defined as the following:

$$P(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0, 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

The number of outcomes having the required number of successes or failures

Probability that there are x successes and (n-x) failures

Binomial Distribution (2)

- In the PMF equation, it is obvious that X cannot get values more than n
  - Because we have repeated the trials for n times
  - So, for $x > n$, the value of P(x)=0
- Expected value for random variable X is obtained according to the following:

$$E(X) = \sum_{for\ all\ i} x_i P(x_i) = np$$

- Variance could be also calculated based on the previous equations:

$$V(X) = n \times pq$$

Binomial Distribution Example (1)

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- Assume that in the process of manufacturing ICs, there is a 2% probability of having a defected chip
  - Every day, a group of 50 ICs are sampled randomly
  - If they observe more than 2 broken ICs, the production process will be shut down
- Calculate the probability of shut down in this manufacturing plant?
  - First we need to determine the random variable
    - Since we want to count the **number of defected ICs**, we are facing a **discrete** variable
    - Since every IC could be produced correctly or with defect, each instance of production is a Bernoulli trial
    - Since the **number of failed productions** is important to us (a number of failed Bernoulli trials), we are facing with a binomial distribution

Binomial Distribution Example (2)

- Therefore, we have 50 Bernoulli trials in this example
  - Assume that p=0.02 is indicating the probability of failure, and q=0.98 is representing the correct IC
  - To calculate the probability of x failures in 50 trials:

$$P(X=x)=\begin{cases}\binom{50}{x}\,0.02^{x}0.98^{50-x}, & x=0,1,2,\ldots,50\\ 0, & \text{otherwise}\end{cases}$$

- We want to obtain probability of shutdown (having more than 2 IC defects)

$$\mathrm{P}(\mathrm{X}>2)=1-\mathrm{P}(\mathrm{X}\le 2)=1-\sum_{\mathrm{x}=0}^{2}\binom{50}{\mathrm{x}}(0.02)^{\mathrm{x}}(0.98)^{50-\mathrm{x}}=$$

$$1\text{-}[(0.98)^{50}+50(0.02)(0.98)^{49}+1225(0.02)^{2}(0.98)^{48}]=1-0.92=\mathbf{8}\%$$

Geometric Distribution (1)

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- In binomial distribution, the random variable is the **number of successes** in **n repetitions** of Bernoulli trials with a constant success probability p
- Now assume we modify this a little bit:
  - **How many trials must be taken to reach the first success?**
    - The number of trials will be our random variable with a geometric distribution
    - The **probability of success/failure** and **n** are still constant and known
- Anytime we need to model the number of trials to reach the first success (or failure) we use this distribution
  - Example: In a computer network, the probability of having the first collision after a number of packet receptions in a hub is modeled with geometric distribution

Geometric Distribution (2)

- The probability function of geometric distribution is represented as the following:

$$P(x) = \begin{cases} q^{x-1}p, x = 1,2,3,\dots \\ 0, \qquad otherwise \end{cases}$$

  - p is indicating the first success occurring on the xth trial, and after x-1 trials, which were failed
  - q is the fail probability
- In order to obtain E(X), and V(X), use the equations below:

$$\mathrm{E(X) = 1/p}$$

$$\mathrm{V(X) = q/p^2}$$

Negative Binomial Distribution (1)

- It is used to model the number of Bernoulli trials required (X) to reach k^{th} success (or failure)
  - In other words, how many times we should repeat the trials to have k successes (or failures)
  - Probability function for random variable X with parameters p, and k is defined as follows:

$$P(x) = \begin{cases} \binom{x-1}{k-1} q^{x-k} p^k, & \mathrm{x} = k, k+1, k+2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- Values of expected value, and variance for random variable X is calculated with:

$$E(X) = \frac{k}{p} \qquad V(X) = kq/p^2$$

Example

- 1) It has been reported that 40% of the ink-jet printers are returned after production in the quality-check section
  - In what chance, having the first approved printer is the 3rd printer?
  - Answer: we want to have the first success after 2 failures
    - **Geometric distribution** with p=0.6, and q=0.4

$$P(X = 3) = (0.4)^2 \times 0.6 = 0.096 = \sim 10\%$$

    - So, based on the parameters, there is a ~10% chance of approving a printer on the 3rd inspection
- 2) If we desire to calculate the probability of approving the 2nd printer on our 3rd inspection
  - **Negative binomial distribution** with p=0.4, and k=2

$$P(X = 3) = \binom{3-1}{2-1} \times (0.4)^{3-2} (0.6)^2 = \binom{2}{1} \times 0.4 \times (0.6)^2 = 0.288 = \sim 30\%$$

Poisson Distribution (1)

- Similar to binomial distribution, the **number of successes** is important in the Poisson distribution
  - **But, in a temporal or spatial unit**
- In other words, if in random experiments, the events are dependent on temporal or spatial interval, the variable could be modeled with Poisson
- Examples:
  - Number of customers entering a market during 8:00-9:00 A.M.
  - Number of pizza orders during a working day
  - Number of packet drops in a router gateway in a week
  - Number of vehicles passing a specific 1Km of a highway

Poisson Distribution (2)

- A random variable with Poisson distribution must meet the three following conditions:
  - Events occurring in different intervals are independent
    - This is also known as **memoryless** aspect of Poisson distribution
      - When an event occurs, it does not have any effect on the probability of the next event
  - Probability of having more than one event at the same time is zero
  - Probability of an event in an interval is proportional to the size of the interval
- When to use Poisson?
  - Modeling events, which their **average interarrivals are known**, but the **exact instance of occurrence is unknown**, and is **random**
    - So, the events will be stochastic

Poisson Distribution Function (1)

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- Recall: Poisson is similar to binomial distribution, where the number of successes is proportional to the number of trials n
  - Therefore, the probability of successes in n trials (P_n) will be also dependent on n (n could be assumed to be our interval ☺)
  - What is the relation between P_n and n? $n \times P_n = \alpha$
    - Where α is indicating the number of events occurred in an interval
    - Note: If P_n is constant → number of success has **binomial distribution**
    - Accordingly, let's obtain the Poisson distribution function
  - Consider X to be the number of successes
    - Based on binomial distribution, we have:

$$P(X = x) = \underbrace{\boxed{\binom{n}{x}\left(\frac{\alpha}{n}\right)^{x}}}_{A}\ \underbrace{\boxed{\left(1 - \frac{\alpha}{n}\right)^{n-x}}}_{B} \qquad \mathrm{I}$$

Poisson Distribution Function (2)

- Consider part **A** of the equation
  - If n moves towards ∞:

$$\lim_{n\to\infty}\frac{n!}{x!\,(n-x)!}\times\left(\frac{\alpha}{n}\right)^{x}=\frac{\alpha^{x}}{x!}\lim_{n\to\infty}\cancelto{1}{\frac{n!}{(n-x)!\times n^{x}}}=\frac{\alpha^{x}}{x!}\qquad \mathrm{II}$$

- Consider part **B** of the equation
  - If n moves towards ∞:

$$\lim_{n\to\infty}\left(1-\frac{\alpha}{n}\right)^{n-x}=e^{-\alpha}\qquad \mathrm{III}$$

- According to I, II, and III:

$$P(X=x)=e^{-\alpha}\frac{\alpha^{x}}{x!}\qquad x=0,1,2,\ldots$$

Poison Distribution Function (3)

- PDF of the Poisson distribution is defined as follows:

$$P(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^{x}}{x!}, & x = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

  - Where X=x indicates the number of successes (observations) in a Poisson experiment
  - It is called that X has a Poisson distribution with parameter α
  - α could be obtained by $\alpha = \lambda t$
    - Where λ is representing the **rate of occurrence** for the Poisson events

$$\lambda = \frac{Number\ of\ Events}{Time(Space)\ Unit} \rightarrow \alpha = \frac{Number\ of\ Events}{Time(Space)\ Unit} \times \boldsymbol{t}$$

Time period which you want to analyze or observe the random variable

Example

- An insurance company signs 3 contracts per week
  - What is the probability of signing at least one contract in one week?
  - Answer: Our random variable X is the number of contracts in an interval
    - So, Poisson is one candidate
    - But, does X support three conditions of Poisson distribution?
      - Only one contracts could be signed at a time ☺
      - Signing a contract is independent from other contracts ☺
      - More time indicates more contracts to be signed ☺

      Yes
  - First determine the $\alpha = \lambda t = \frac{3\ contracts}{1\ week} \times 1\ week = 3$
  - $P(X > 0) = 1 - P(X = 0) = 1 - \frac{e^{-3}(3)^0}{0!} = 95\%$

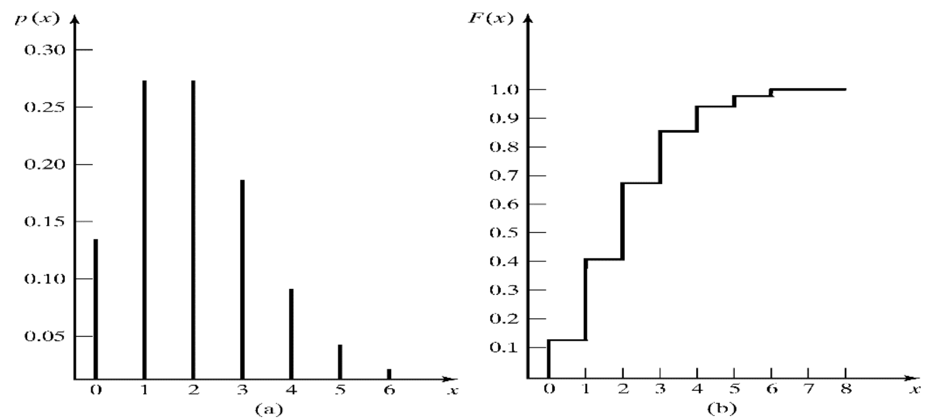
Poisson Distribution Function (4)

- The expected value, and variance of a random variable with Poisson distribution is obtained by the following:

$$E(X) = V(X) = \lambda t = \alpha$$

- The CDF of this distribution is calculated by: $F(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^{i}}{i!}$

- For $\alpha = 2$, the PDF, and CDF graphs for different success values (i in the above) are depicted



Continuous Distributions

- Continuous random variables can be used to describe random phenomena in which the variable can take **any values in an interval**
  - The distributions that are used to model such variables are continuous distributions
- In this section, the distributions that will be studied are:
  - Uniform
  - Exponential
  - Normal
  - Gamma
  - Erlang
  - Weibull
  - Lognormal
  - Triangular
  - Beta

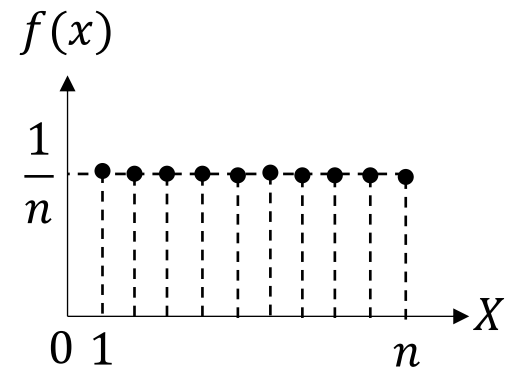
Uniform Distribution (1)

- This distribution could be used for both discrete and continuous variables
  - This depends on the support and the type of our intended variable
  - While discrete uniform variables can take values $S = \{1,2,3, \ldots, n\}$, continuous uniform variables can take any values in (a,b) interval
- Why they are called uniform?
  - Because their probability mass/density function gives identical probabilities for any value in the support interval
    - Recall: we use mass function for discrete variables, and density function for continuous variables

Uniform Distribution (2)

- If X is a discrete uniform variable with $S = \{1,2,3,\dots,n\}$, the probability function of this variable is denoted with:

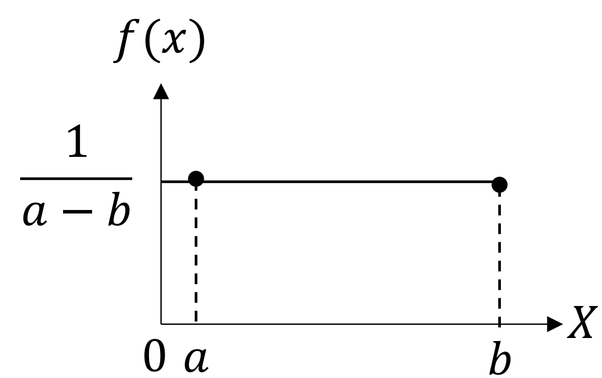
$$f_X(x) = P(X = x) = \frac{1}{n}, \qquad x \in S$$



  - As you can see, all the probabilities are 1/n

- If X is a continuous uniform variable with $S = (a, b)$, the PDF of this variable is denoted with:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$



Uniform Distribution (3)

- Value of E(X) in the uniform distribution is an instance between the two end points of the support: $E(X) = \frac{a+b}{2}$
- The variance is obtained by: $V(X) = \frac{(b-a)^2}{12}$
- Since the variable is continuous, we use integration to obtain CDF from $f_X(x)$:

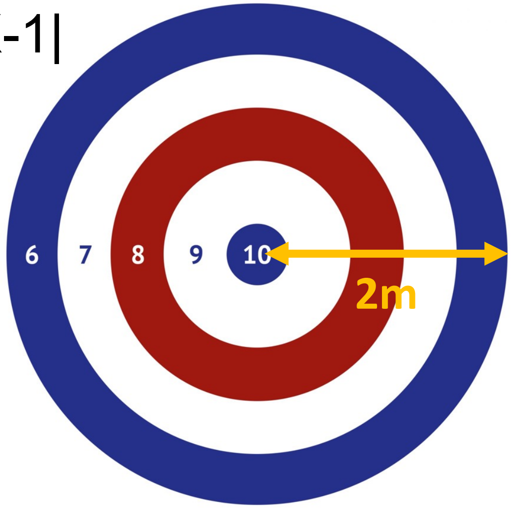
$$\mathrm{F_X(x)} = \mathrm{P}(\mathrm{X} \le x) = \int_a^x f_X(x) d_x = \int_a^x \frac{1}{b-a} d_x = \frac{x-a}{b-a}, a \le x \le b$$

  - □ It is obvious that for values less than a, the CDF gives zero, and for values greater than b, it gives 1
    - So, the uniform CDF is defined as:

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

Example

- Assume an athlete, who tries to shoot a target with 2 meter radius
  - What is the probability of having a distance less than 0.2m to the 1 meter point of this target?
- Answer:
  - If X is the distance to the center with $R_x = [0,2]$
  - Distance to the **center of the line** would be |X-1|
    - We can assume Y=X-1 as our RV
      - It has a uniform distribution in [0,1]



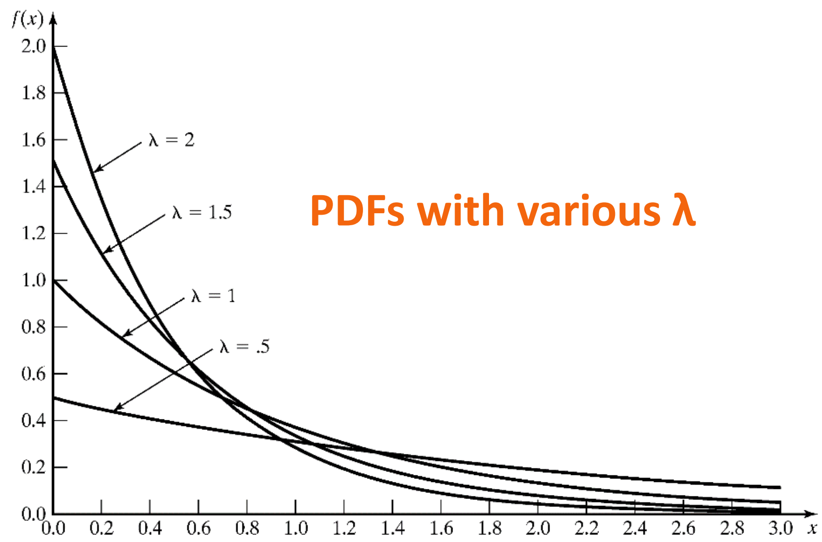
$$P(|Y| \le 0.2) = P(-0.2 \le Y \le 0.2) = F_Y(0.2) - F_Y(-0.2)$$

$$F_Y(y) = \frac{y-a}{b-a}, a = 0, and\ b = 1$$

$$\rightarrow P = \frac{0.2-0}{1-0} - 0 = 0.2$$

Exponential Distribution (1)

- If X is a continuous random variable with two conditions, it follows the exponential distribution:
  - It has a **non-negative** real support
  - Its probability density function is: $$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$
  - We say X has an exponential distribution with parameter λ
- Used to model interarrival times when arrivals are completely **random**, and to model service times that are **highly variable**



PDFs with various λ

Exponential Distribution (2)

- In Poisson experiments, the number of events (success or failure) in a specific interval is important
  - But, if we define X as the **time to the occurrence of events** (success or failure), then it follows the exponential distribution

  ⬇

  - In Poisson, λ indicates the number of events in a time unit
  - Therefore, the time between two events could be obtained by $1/\lambda$
    - Indicated with θ
- The PDF of exponential distribution could be also defined as:

$$f_X(x) = \frac{1}{\theta} e^{-\frac{1}{\theta}x}, x \geq 0$$

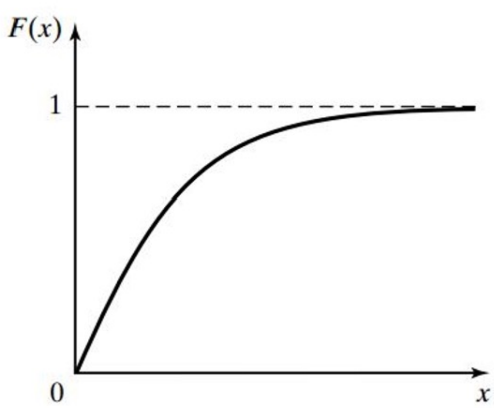
Exponential Distribution (3)

- Examples for this distribution:
  - The time a customers waits in a bank to make a deposit
  - The length of a fabric to be inspected for finding a tear
  - Life-time of ICs (or anything else)
- Exponential distribution has also the **memoryless** property
- Expected value, and the variance in exponential distribution are calculate with the following equations:

$E(X) = 1/\lambda \qquad V(X) = \theta^2$

- The CDF of this distribution:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$



Example

- On average, there is 1 crack in every 2 meters of a 6m grinder
  - If we define X as the length of the grinder, which does not include cracks, what is the probability that he grinder does not have any cracks?
- Answer:
  - First, we are talking about an **interval**, the first candidates for X would be **Poisson**, and **exponential**
  - If we consider that the first crack stays on **zero point** of the grinder, and the second occurs at a **length longer than 6 meters** → There is an interval between events → exponential ☺

$$\left.\begin{array}{c} P(X \geq 6) = 1 - P(X < 6) = 1 - F_X(6) = 1 - \left(1 - e^{-\lambda \times 6}\right) \\ \lambda = \frac{1}{2} = 0.5 \end{array}\right\} \begin{array}{l} P(X \geq 6) \\ = e^{-3} \\ = 0.0498 \end{array}$$

Gamma Distribution (1)

- Gamma distribution has two parameters in its structure
  - Therefore, is one of the important distributions
- The random variable X has a Gamma distribution with parameters β, and θ and the following density function:

$$f_X(x) = \begin{cases} \dfrac{\beta\theta}{\Gamma(\beta)}(\beta\theta \mathrm{x})^{\beta-1}\mathrm{e}^{-\beta\theta x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

  - Where β is the shape parameter (due to its effect on the shape of the PDF graph), and θ is the scaling parameter
    - With tuning these two parameters, we could better adjust the model with the sampled data
  - **What is Γ(β)?**

Gamma Distribution (2)

- The Gamma function for all positive values of β is defined as the following:

$$\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} d_x$$

  - If we use partial integration: $\Gamma(\beta) = (\beta - 1)\Gamma(\beta - 1)$
- For integer values of β>0, the value of the Gamma function could be obtained with $\Gamma(\beta) = (\beta - 1)!$
- The value of expected value, and variance are calculated with:

$$E(X) = 1/\theta \qquad V(X) = 1/\beta\theta^2$$

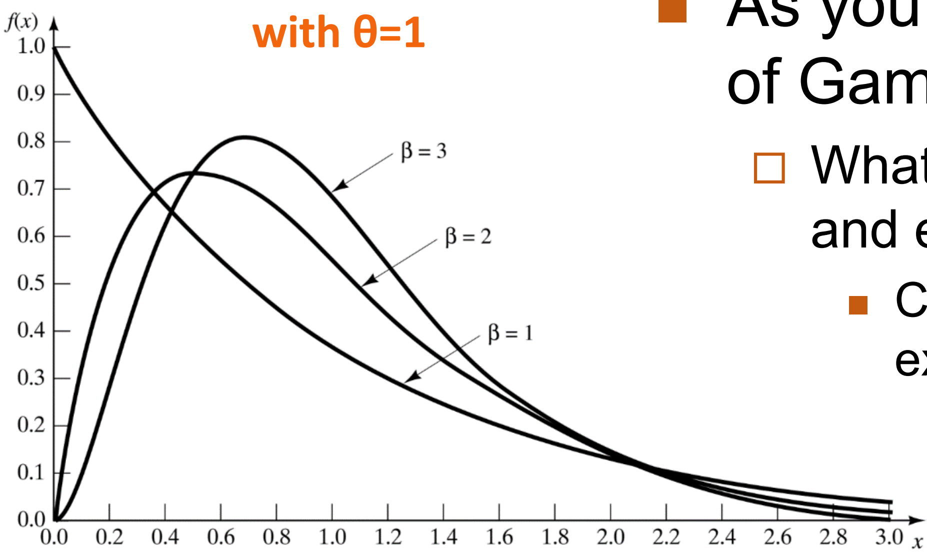
  - In future chapters, we will discuss that for estimating proper values of parameters in Gamma distribution, we could estimate θ based on E(X), and β based on V(X)

Gamma Distribution (3)

- The CDF for this distribution is determined as follows:

$$F_X(x) = \begin{cases} 1 - \int_x^{\infty} \frac{\beta\theta}{\Gamma(\beta)} (\beta\theta t)^{\beta-1} e^{-\beta\theta t} dt, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

PDFs for several Gamma distributions with θ=1



- As you can see, for $\beta = 1$, the PDF of Gamma is similar to exponential
  - What is the relation between Gamma and exponential?
    - Consider X as the sum of β independent exponential variables with parameter $\beta\theta$

$$X = X_1 + X_2 + \cdots + X_\beta$$

  - X will support Gamma with θ, and β parametes

Erlang Distribution (1)

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- The PDF of Gamma distribution is often referred to as the Erlang distribution of order (or number of phases) k, when β=k, is an integer
  - Therefore, Erlang is a more specific form of Gamma
- Erlang was a Danish scientist, who was an expert in telecommunications
- Erlang distribution and its relation with exponential distribution could be explained as follows:
  - A customer wants to get service from a telecommunication company
  - His/her request must pass through k stations
  - The next customer cannot apply for service until all k stations have served for the previous customer
    - Each station has an exponentially distributed service time (parameter kθ)

Erlang Distribution (2)

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- The CDF of the Erlang distribution is obtained by:

$$F_X(x) = \begin{cases} 1 - \sum_{i=0}^{k-1} \frac{e^{-k\theta x}(k\theta x)^i}{i!}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

- Expected value, and variance of random variable X with Erlang distribution is defined as follows:

$$E(X) = 1/\theta \qquad V(X) = 1/k\theta^2$$

  - We have k exponentially distributed variables with $\lambda = k\theta$
  - And expected value is a linear function:

$$E(X+Y) = E(X) + E(Y)$$
$$E(\alpha X) = \alpha E(X)$$

$$\Rightarrow E\left(\sum_{i=0}^{k-1} X_i\right) = k \times \left(\frac{1}{k\theta}\right)$$

Example (1)

- A college professor of electrical engineering is leaving home for the summer, but would like to have a light burning at all times to discourage burglars
  - The professor installs a device that will hold two light bulbs
  - The device will switch the current to the second bulb if the first bulb fails
  - The box in which the light bulbs are packaged says:
    - "Average life 1000 hours, exponentially distributed"
  - The professor will be gone 90 days (2160 hours)
- What is the probability that a light will be burning when the summer is over and the professor returns?

Example (2)

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- Answer:
  - The probability of a device working for more than x hours could be obtained by $P(X > x) = R(X > x) = 1 - F(X \leq x)$
    - R(X) is indicating the reliability **somehow**
    - F(X) is representing the probability of **failure**, which the device is working less than x hours
      - Easier to calculate based on CDF $F_X(X)$
  - In this example, the total operating time of the system is calculated by adding the operation time of two independent light bulbs ($X_S = X_1 + X_2$)
    - So, k=2
  - According to the problem, $\lambda = \frac{1}{1000} \rightarrow k\theta = \frac{1}{1000} \rightarrow \theta = \frac{1}{2000}$

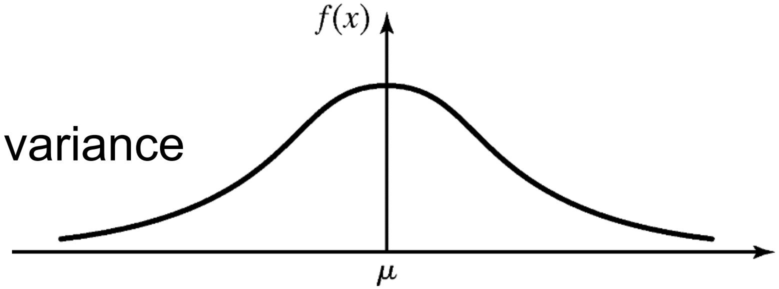
$$F(X \leq 2160) = F_X(2160) = 1 - e^{-2.16} \sum_{i=0}^{1} \frac{(2.16)^i}{i!} = 0.636 = \sim 64\% \rightarrow \boldsymbol{R(X) = \sim 36\%}$$

Normal Distribution (1)

- It is also known as the natural distribution, bell graph Gaussian distribution, and Gauss-Laplace distribution
- Normal distribution is one the most important distributions available in statistics
  - Why? Because, many of the fluctuations in values given by natural, and physical events surrounding us in the environment are centered around a specific value
    - These values are highly adjusted with the outcomes of this model
- Many applications are supported by Normal distribution:
  - Height of people
  - Dimension of a specific production manufactured in a factory
  - Errors in measurements
  - Blood pressure

Normal Distribution (2)

- It is indicated with $N(\mu, \sigma^2)$, and has two parameters:
  - The mean value μ
  - The standard deviation σ
    - Note: For presentation, we use the variance



- PDF of the normal distribution:

$$f_X(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < +\infty$$

- Special properties:
  - $\lim_{x \to -\infty} f_X(x) = \lim_{x \to +\infty} f_X(x) = 0$
  - PDF is symmetric about μ: $f_X(\mu - x) = f_X(\mu + x)$
  - The maximum value of the pdf occurs at $x = \mu$
    - The mean and mode are equal

Normal Distribution (3)

- Evaluation of CDF is complex
- There are two major approaches for this issue:
  - Use numerical methods (no closed form)
  - Independent from μ and σ (μ=0, σ=1), use the **standard normal distribution:** $Z \sim N(0,1)$
    - Transformation of variables

Normal Distribution (4)

- Transforming the variables
  - For random variable $X \sim N(\mu, \sigma^2)$, we recommend using a modified version of CDF, which transforms X to Z

$$F_X(x) = P(X < x) = P\left(\frac{X-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) = P\left(Z < \overset{Z}{\frac{x-\mu}{\sigma}}\right) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

  - The PDF, and CDF of this distribution are defined as follows:

$$\varphi_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < +\infty$$

$$\Phi_Z(z) = P(Z < z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} d_t$$

  - To simplify the calculation of probability values for the standard normal distribution, **look-up tables** are used:

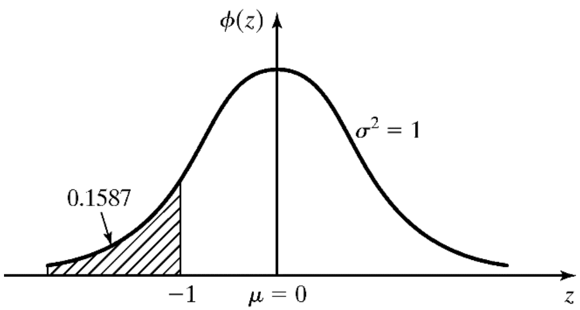
Z	0.00	0.01	0.02	…	0.05	…	0.09
0.0							
0.1							
…							
1.5					0.9394 $\Phi(Z < 1.55)$		
…							
2.9							

Example (1)

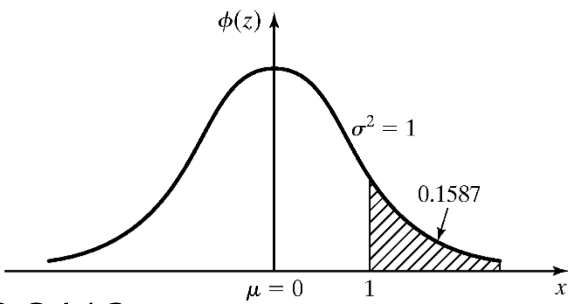
- The time required to load an oceangoing vessel, X, is distributed as ~N(12,4)
  - Calculate the probability that the vessel is loaded in less than 10 hours?
- Answer:
  - $\mu = 12$, and $\sigma = 2$, we are seeking $F_X(10)$
  - Transforming X to Z: $z = \frac{10-12}{2} = -1$
    - We must calculate $\Phi_Z(-1) = P(Z < -1)$
      - Negative values are not indicated in the table
  - Using the symmetry property, $\Phi_Z(-1)$ is the complement of $\Phi_Z(1)$

$$P(Z < -1) = P(Z > 1) = 1 - P(Z < 1) = \mathbf{0.1587}$$

    - From table (Page 540): $P(Z < 1) = \Phi_Z(1) = 0.8413$



(b)



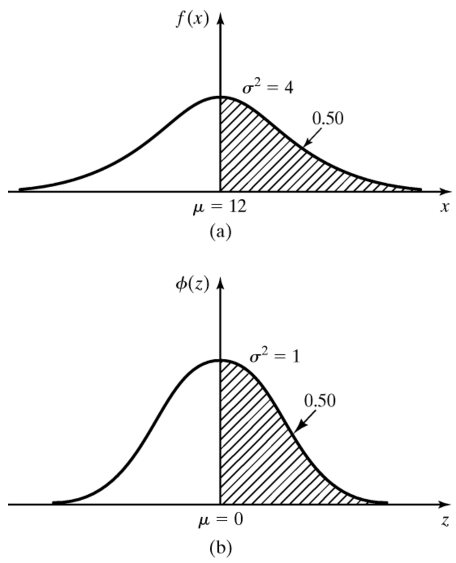
(a)

Example (2)

- What is the probability that the vessel is loaded in 12 hours or more?
- Answer:
  - $\mu = 12$, and $\sigma = 2$, we are seeking $P(X \geq 12) = 1 - \overbrace{\boxed{P(X < 12)}}^{F_X(12)}$
  - Transforming X to Z: $z = \frac{12-12}{2} = 0$
    - According to the table, $F_X(12) = \Phi_Z(0) = 0.5$

$$P(X \geq 12) = 1 - \Phi_Z(0) = \mathbf{0.5}$$

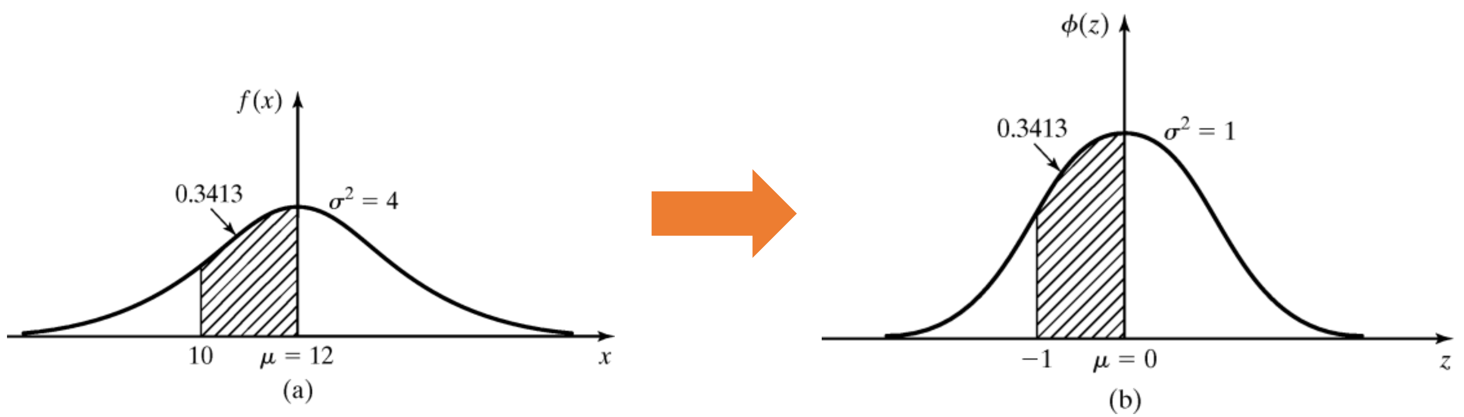
  - This could be obtained without calculation:
    - The mean value was 12
    - We also wanted to evaluate values $X \geq 12$
    - Due to the symmetric nature of normal graph, probability of having values higher or lower than the mean are the same = 50%



Example (3)

- What is the probability $P(10 \leq X \leq 12)$?
- Answer:
  - $\mu = 12$, and $\sigma = 2$, we are seeking $P(10 \leq X \leq 12) = F_X(12) - F_X(10)$
  - Transforming X to Z: $z = \frac{12-12}{2} = 0$, and $z = \frac{10-12}{2} = -1$
    - Based on what we calculated previously

$$P(10 \leq X \leq 12) = \Phi_Z(0) - \Phi_Z(-1) = 0.3413$$

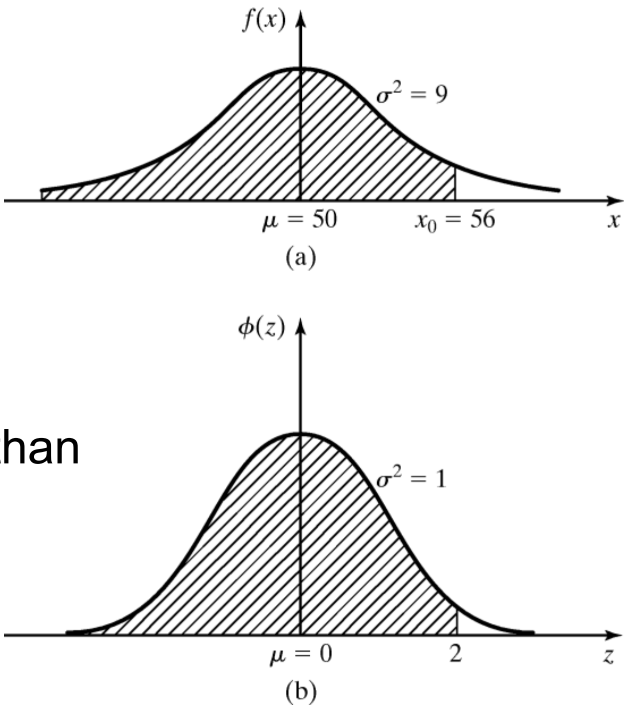


Example (4)

- Suppose a random variable X~N(50,9)
  - Calculate $F_X(56)$
- Answer:
  - $\mu = 50$, and $\sigma = 3$
  - Transforming X to Z: $z = \frac{56-50}{3} = 2$

$$P(Z < 2) = \Phi_Z(2) = \mathbf{0.9772}$$

  - With 98% chance, X gets values less than 56
  - As you can see, the area under $\Phi_Z(2)$ has been nearly fully scratched

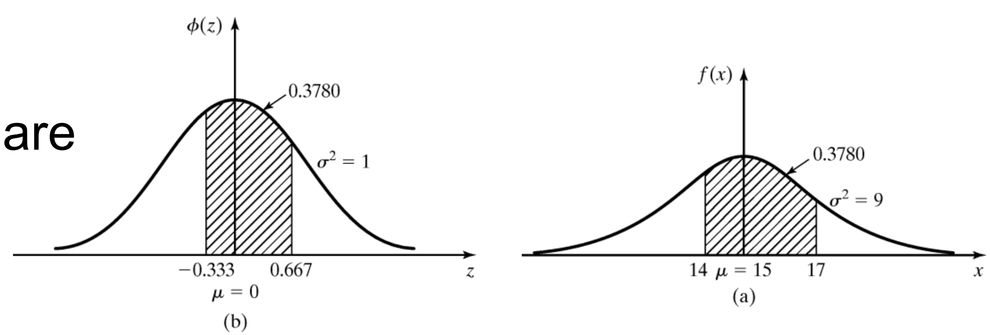


Example (5)

- The time required to pass a queue is modeled with N(15,9)
  - Calculate the probability that a customer spends 14 to 17 minutes in this queue
  - X~N(15,9) → $\mu = 15$, and $\sigma = 3$
  - Transforming X to Z: $z = \frac{17-15}{3} = \frac{2}{3}$, and $z = \frac{14-15}{3} = -\frac{1}{3}$

$$P(14 \leq X \leq 17) = \Phi_Z(0.667) - \Phi_Z(-0.333) =$$
$$\Phi_Z(0.667) - 1 + \Phi_Z(0.333) = 0.7476 - 0.3696 = \mathbf{0.3780}$$

- What is the point?
  - Obtaining the values, which are not explicitly indicated in the table, such as $\Phi_Z(0.667)$



Estimating Values Not Indicated in the Table

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- There is no value corresponding to $\Phi_Z(0.667)$
  - But we have values for $\Phi_Z(0.66)$, and $\Phi_Z(0.67)$ in the table
- In order to obtain $\Phi_Z(0.667)$ we use a simple interpolation:

$$\Phi_Z(0.667) = \boldsymbol{\Phi_Z(0.66)} + 7\times\left(\frac{\Phi_Z(0.67) - \boldsymbol{\Phi_Z(0.66)}}{\mathbf{10}}\right) = \mathbf{0.7476}$$

  - We have a 2 decimal precision in the table, but 3 decimal is required
    - The gap between values should be divided by 10 units

z_α	0.05	0.06	0.07	0.08	0.09	z_α
0.0	0.519 94	0.523 92	0.527 90	0.531 88	0.535 86	**0.0**
0.1	0.559 62	0.563 56	0.567 49	0.571 42	0.575 34	**0.1**
0.2	0.598 71	0.602 57	0.606 42	0.610 26	0.614 09	**0.2**
0.3	0.636 83	0.640 58	0.644 31	0.648 03	0.651 73	**0.3**
0.4	0.673 64	0.677 24	0.680 82	0.684 38	0.687 93	**0.4**
0.5	0.708 84	0.712 26	0.715 66	0.719 04	0.722 40	**0.5**
0.6	0.742 15	0.745 37	0.748 57	0.751 75	0.754 90	**0.6**
0.7	0.773 37	0.776 37	0.779 35	0.782 30	0.785 23	**0.7**
0.8	0.802 34	0.805 10	0.807 85	0.810 57	0.813 27	**0.8**
0.9	0.824 94	0.831 47	0.833 97	0.836 46	0.838 91	**0.9**
1.0	0.853 14	0.855 43	0.857 69	0.859 93	0.862 14	**1.0**
1.1	0.874 93	0.876 97	0.879 00	0.881 00	0.882 97	**1.1**
1.2	0.894 35	0.896 16	0.897 96	0.899 73	0.901 47	**1.2**
1.3	0.911 49	0.913 08	0.914 65	0.916 21	0.917 73	**1.3**
1.4	0.926 47	0.927 85	0.929 22	0.930 56	0.931 89	**1.4**
1.5	0.939 43	0.940 62	0.941 79	0.942 95	0.944 08	**1.5**
1.6	0.950 53	0.951 54	0.952 54	0.953 52	0.954 48	**1.6**
1.7	0.959 94	0.960 80	0.961 64	0.962 46	0.963 27	**1.7**
1.8	0.967 84	0.968 56	0.969 26	0.969 95	0.970 62	**1.8**
1.9	0.974 41	0.975 00	0.975 58	0.976 15	0.976 70	**1.9**
2.0	0.979 82	0.980 30	0.980 77	0.981 24	0.981 69	**2.0**
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	**2.1**
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	**2.2**
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	**2.3**
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	**2.4**
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	**2.5**
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	**2.6**
2.7	0.997 02	0.997 11	0.997 20	0.997 28	0.997 36	**2.7**
2.8	0.997 81	0.997 88	0.997 95	0.998 01	0.998 07	**2.8**
2.9	0.998 41	0.998 46	0.998 51	0.998 56	0.998 61	**2.9**
3.0	0.998 86	0.998 89	0.998 93	0.998 97	0.999 00	**3.0**
3.1	0.999 18	0.999 21	0.999 24	0.999 26	0.999 29	**3.1**
3.2	0.999 42	0.999 44	0.999 46	0.999 48	0.999 50	**3.2**
3.3	0.999 60	0.999 61	0.999 62	0.999 64	0.999 65	**3.3**
3.4	0.999 72	0.999 73	0.999 74	0.999 75	0.999 76	**3.4**
3.5	0.999 81	0.999 81	0.999 82	0.999 83	0.999 83	**3.5**
3.6	0.999 87	0.999 87	0.999 88	0.999 88	0.999 89	**3.6**
3.7	0.999 91	0.999 92	0.999 92	0.999 92	0.999 92	**3.7**
3.8	0.999 94	0.999 94	0.999 95	0.999 95	0.999 95	**3.8**
3.9	0.999 96	0.999 96	0.999 96	0.999 97	0.999 97	**3.9**

Example (6)

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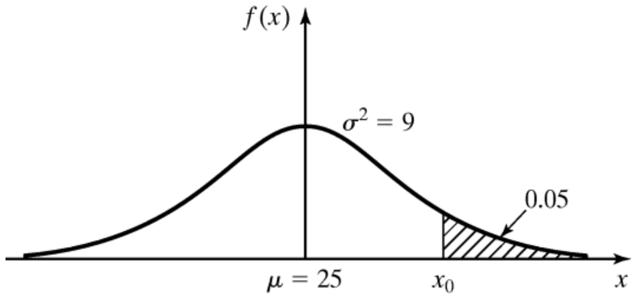
- Assume that the lead time for serving a request is following N(25,9)
  - Compute the lead time $(X = x)$ that will be exceeded only in 5% of the time?
    - Note: In this example, we have the probability, but x is unknown

$$P(X \geq x) = 1 - P(X < x) = 1 - F_X(x) = 1 - \Phi\left(\frac{x - 25}{3}\right) = \mathbf{0.05}$$

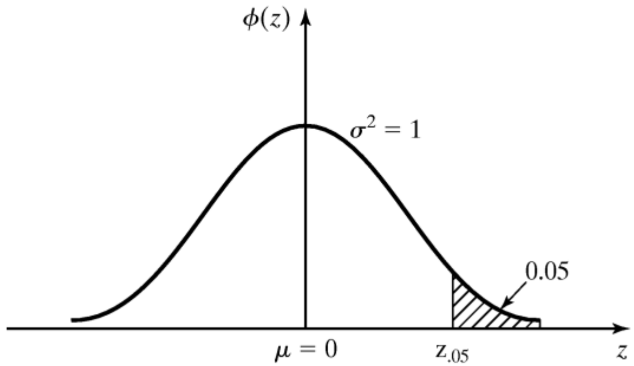
  - Now we refer to the table and search for the z that gives Φ(z)=0.95
    - Similar to previous example, you cannot find any z which supports above
    - Find two z_1, and z_2 who cover 0.95
      - $\Phi(z_1 = 1.64) = 0.94950$, and $\Phi(z_2 = 1.65) = 0.95053$
    - Use **mean** value to obtain $\Phi(z) = 0.95$ → $z = \frac{x-25}{3} = \frac{1.64+1.65}{2} = 1.645$
      - According to the above, $x = 29.935$ → In 5% of the time, customers wait more than 29.935 minutes

Example (7)

- Checkout the indicated values in the graphs
  - Both, the standard deviation, and the mean
  - Worthy to emphasis that the area under the curve is identical in both graphs
    - So, the probability will be the same



(a)



(b)

Weibull Distribution (1)

- This distribution plays an important role in many applications, especially in analyzing reliable, and fault tolerant techniques
- If random variable X with **continuous**, and **non-negative** support, follows the indicated PDF, it could be modeled with Weibull distribution:

$$f_X(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right], & x \geq \nu \\ 0, & \text{otherwise} \end{cases}$$

- It has 3 parameters:
  - Location parameter: ν
  - Shape parameter: β , $(\beta > 0)$
  - Scale parameter: α, $(\alpha > 0)$
- It could be well-tuned with different data sets with higher degree of freedom

Weibull Distribution (2)

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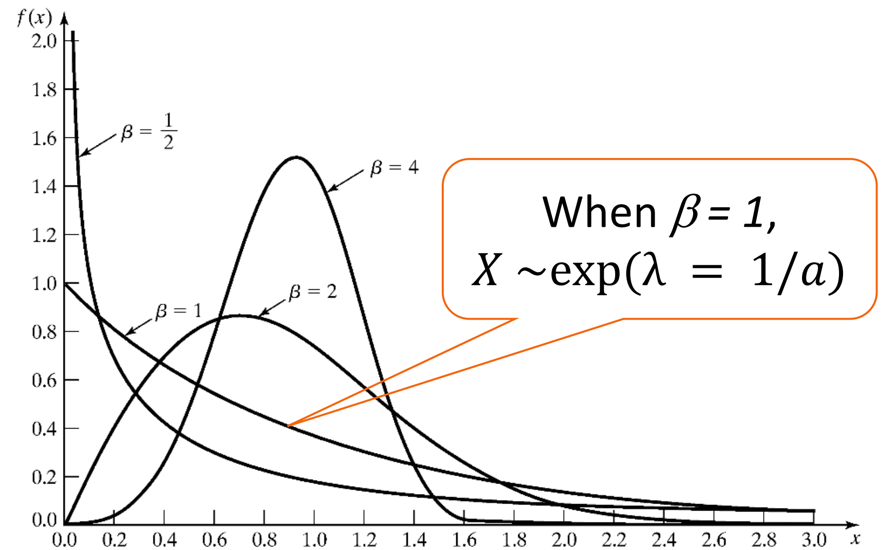
- Example:
  - Illustrated graph has been depicted with $\nu = 0$, and $\alpha = 1$
  - As you can see, with tuning parameters of the PDF, different graphs with completely different features could be obtained
    - Highly flexible

- While the Weibull distribution could be adjusted with various sampled data, it is **computationally complicated**



Relation Between Weibull and Other Distributions

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- As indicated in the previous slide, if $\nu = 0$, and $\beta = 1$, Weibull distribution turns into exponential:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- If $\alpha = 2$, the random variable X follows the Reyleigh distribution with the following PDF:

$$f_X(x) = \begin{cases} \frac{\beta}{2}\left(\frac{x-\nu}{2}\right)^{\beta-1} \exp[-\left(\frac{x-\nu}{2}\right)^{\beta}], & x \geq \nu \\ 0, & otherwise \end{cases}$$

  - This version of the distribution is widely used in modeling multipath fading in wireless communications, electromagnetic radiation, wind speed, etc.

Weibull Distribution (3)

- Expected value and variance of the Weibull distribution are calculated via the following equations:

$$E(X) = \nu + \alpha\Gamma(\frac{1}{\beta} + 1)$$

$$V(X) = \alpha^2[\Gamma\left(\frac{2}{\beta} + 1\right) - \Gamma\left(\frac{1}{\beta} + 1\right)^2]$$

- Recall: $\Gamma(x)$ is the Gamma function that we have previously discussed
- The CDF is calculated as follows:

$$F_X(x) = \begin{cases} 1 - e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta}, & x \geq \nu \\ 0, & otherwise \end{cases}$$

Example

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- Assume the TTF of a device has a Weibull distribution with $\nu = 0$, $\alpha = 200H$ and $\beta = 1/3$
  - Calculate the average TTF

$$\left.\begin{array}{c} E(X) = \cancelto{0}{\nu} + \alpha\Gamma\left(\frac{1}{\beta} + 1\right) = 200\times\Gamma(3+1) = 200\Gamma(4) \\ \Gamma(4) = (4-1)! = 6 \end{array}\right\} E(X) = 1200H$$

  - What is the probability of failure before 2000H?

$$\left.\begin{array}{c} P(X < 2000) = F_X(X = x) \\ x = 2000H \geq \nu = 0 \end{array}\right\} P(X < 2000) = 1 - e^{-\left(\frac{2000-0}{200}\right)^{\frac{1}{3}}=1-e^{-\frac{2}{15}}=0.884}$$

  - **With 89% chance, the device will be failed in less than 2000 hours**

Log-Normal Distribution (1)

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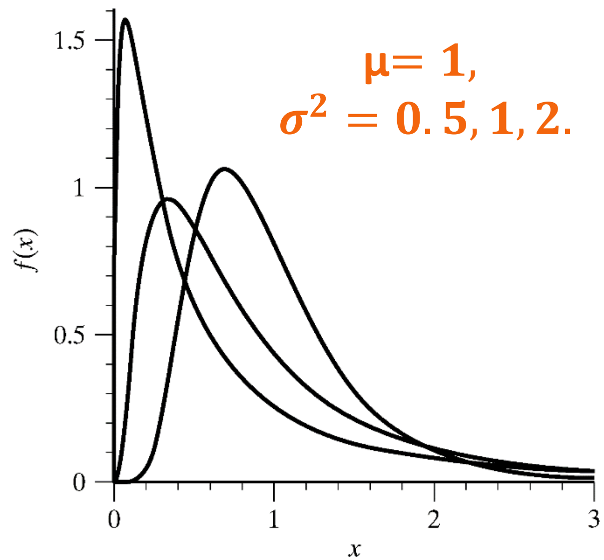
- It is used to model variables, which are changing in a cumulative fashion
  - Number of reviews regarding a gadget in Amazon
  - Amount of rainfall
  - Checkers play time
- There is a tight relation between log-normal and normal distributions
  - If there exists a random variable $Y \sim N(\mu, \sigma^2)$, a random variable X is also available, which $X = e^Y$
    - Then, X is called to support log-normal distribution with parameters μ, σ, and is denoted as $X \sim Lognormal\ (\mu, \sigma^2)$
  - Note: Parameters μ, σ^2 are not indicating the mean and variance of X, but they correspond to Y, and **only used for presentation**

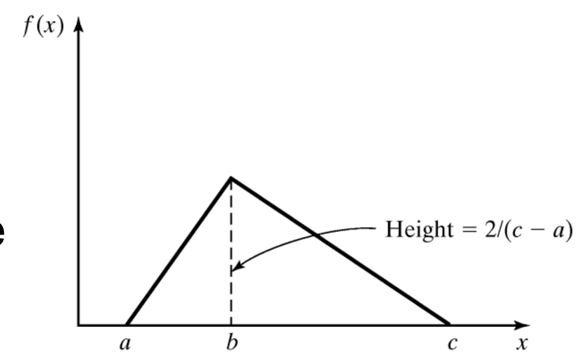
Log-Normal Distribution (2)

- The PDF of lognormal: $f_X(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right], & x > 0 \\ 0, & \text{otherwise} \end{cases}$
- Expected value and variance: $E(X) = e^{\mu + \frac{\sigma^2}{2}}$ $\quad V(X) = e^{2\mu + \frac{\sigma^2}{2}}(e^{\sigma^2} - 1)$
- With tuning the parameters of this distribution, we could fit the model with various data samples
  - This is in contrast with the normal distribution, which the structure of the PDF diagram does not change
    - It only gets shifted to sides **(constant σ, and variable μ)** or expansion/contraction **(variable σ, and constant μ)**



Triangular Distribution (1)

- A distribution used to model **fuzzy variables**
  - In contrast with binary, variables can get more than two values
  - These values are not necessarily numbers
    - They can be descriptive
    - Example: Energy level of a harvesting battery could be high, medium, or low
  - With Fuzzy logic, the system decisions could be made in a similar way to human decision makings
- It has three parameters: $a \leq b \leq c$
  - Lower bound **a**
  - The mode **b**
    - The event with most frequency of occurrence
  - Upper bound **c**



Triangular Distribution (2)

- PDF of the random variable X:

Ascending linear function

$$f_X(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x \le b \\ \dfrac{2(c-x)}{(c-b)(c-a)}, & b < x \le c \\ 0, & otherwise \end{cases}$$

Descending linear function

- To calculate the expected value for variable X, use the following equation:

$$E(X) = \frac{a+b+c}{3}$$

- Mode has a more pivotal role than E(X) in triangular distribution
  - It is calculated as: $Mode = b = 3E(X) - (a+c)$

Triangular Distribution (3)

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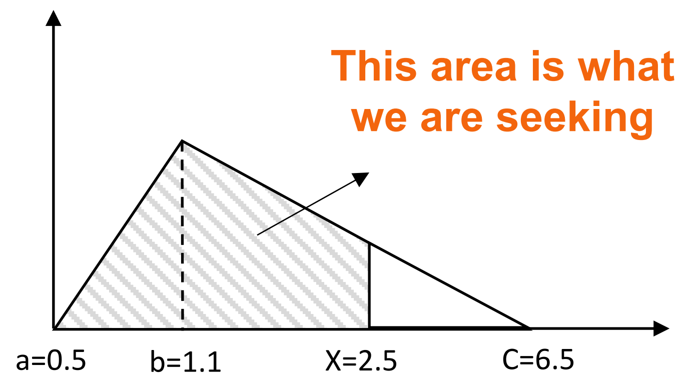
- CDF for the triangular distribution:

$$F_X(x) = \begin{cases} 0, & x \le a \\ \dfrac{(x-a)^2}{(b-a)(c-a)}, & a < x \le b \\ 1 - \dfrac{(c-x)^2}{(c-b)(c-a)}, & b < x \le c \\ 1, & x > c \end{cases}$$

- Example: An embedded processor executes the tasks with a triangularly distributed make-span ~T(0.5ms,1.1ms,6.5ms)
  - What is the probability of executing a task in less than 2.5ms?



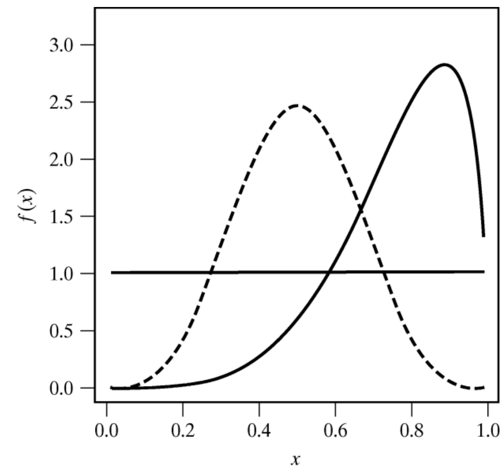
$$P(X < 2.5) = 1 - \frac{(6.5-2.5)^2}{(6.5-1.1)(6.5-0.5)} = 0.506$$

$b \le x \le c$

Beta Distribution (1)

- In cases, where random variable X ranges between 0, and 1, it **could be** modeled with Beta distribution
  - This is not a rule
    - Depends on the application and type of data
    - For instance, if the probability of all values are identical, uniform is preferred
- It has two shape parameters $0 < \beta_1, \beta_2$
- PDF of a Beta distributed variable X is defined as:

$$f_X(x) = \begin{cases} \dfrac{x^{\beta_1 - 1}(1-x)^{\beta_2 - 1}}{B(\beta_1, \beta_2)}, 0 < x < 1 \\ 0, \qquad\qquad otherwise \end{cases}$$



- B is the Beta function: $B(\beta_1, \beta_2) = \dfrac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1 + \beta_2)}$

Beta Distribution (2)

- What if the support is $(\mathrm{a} \neq 0, \mathrm{b} \neq 1)$?
  - We define a transformed random variable Y:

$$\mathrm{Y} = \mathrm{a} + (\mathrm{b} - \mathrm{a})\mathrm{X}$$
$$\mathrm{R_X} = (0,1)$$
$$\mathrm{R_Y} = (\mathrm{a}, \mathrm{b})$$

- In order to obtain E(X), and V(X) we use the following:

$$E(X) = \frac{\beta_1}{\beta_1 + \beta_2} \qquad V(X) = \frac{\beta_1 \beta_2}{(\beta_1 + \beta_2)^2(\beta_1 + \beta_2 + 1)}$$

- Obtain the expected value and variance equations for random variable Y

Poisson Process (1)

- In many applications, a counting function N(t) could be considered, which represents the number of events occurred in [0,t]
  - N(t) is the number of observations $\in \{0,1,2,\dots\}$
  - t=0 indicates the instance, where our observations has started
- A counting process $\{N(t), t \geq 0\}$ is a **Poisson process** with mean rate λ if:
  - In every instance of time, only one arrival could occur
  - The counting function has **stationary increments**
    - Meaning that the number of arrivals in [t,t+Δt] is only dependent on the length of interval (Δt) not the start of the interval (t)
  - The counting function has **independent increments**
    - Number of arrivals in two non-overlapping intervals are independent

Poisson Process (2)

- In a Poisson process, the probability of having n arrivals in [0,t] interval is calculated as:

$$P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, t \geq 0, n = 0,1,2, \ldots$$

- Recall: We could have used α instead of λt
  - λ indicates the rate of arrivals in a time unit
  - α indicates the number of arrivals in a time interval
- E(X), and V(X) are identical in Poisson process:

$$E(N(t)) = V\big(N(t)\big) = \alpha = \lambda t$$

- The above equations are valid for cases where we want to analyze the process in an interval starting from t=0
  - What if we want to analyze a Poisson process in $[t, t + \Delta t], t \neq 0$?

Poisson Process (3)

- If it is desired to consider the number of arrivals in an interval $[t, t+\Delta t], t \neq 0$
  - According to the stationary incremental property of Poisson process, in order to obtain $P[N(t+\Delta t)-N(t)]$, we must use $\lambda\Delta t$:

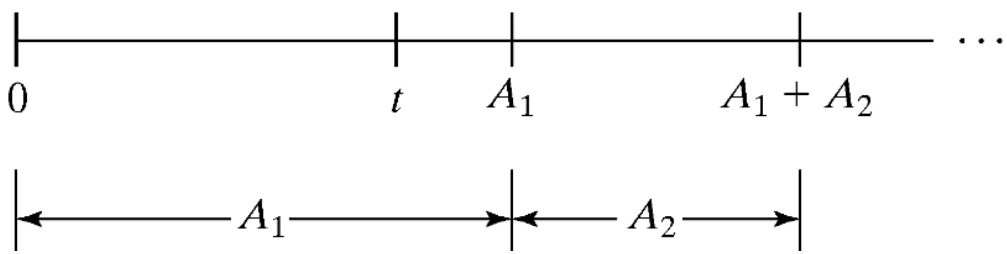
$$P[N(t+\Delta t)-N(t)] = \frac{e^{-\lambda\Delta t}(\lambda\Delta t)^n}{n!}, t \geq 0, n = 0,1,2,\ldots$$

  - Similarly, the E(X), and V(X), are calculated as:

$$E(N(t+\Delta t)-N(t)) = V(N(t+\Delta t)-N(t)) = \lambda\Delta t$$

Poisson Process (4)

- Now consider a Poisson process with a sequence of arrivals occurring on a temporal basis
  - 1st arrival happens on A_1, 2nd on $A_1 + A_2$, 3rd on $A_1 + A_2 + A_3$, etc.
    - Arrivals are depicted below:



    - Where A_i is the elapsed time between arrival i-1 and arrival i
      - We are talking about **interarrival times** here
- What should we do to calculate the probability that the 1st arrival occurs before t → $P(A_1 \leq t)$

Poisson Process (5)

- Assume that no arrivals occur between 0, and t
  - First arrival occurs after t → $A_1 > t$, and $N(t) = 0$

$$P(A_1 > t) = P(N(t) = 0) = P(n = 0) = \frac{e^{-\lambda t}\cancelto{1}{(\lambda t)^n}}{\cancelto{1}{n!}}$$

$$\Rightarrow P(A_1 > t) = e^{-\lambda t}$$

- In order to calculate the probability that the 1st arrival occurs in [0,t], the following equation is used:

$$P(A_1 \leq t) = 1 - P(A_1 > t) = \mathbf{1 - e^{-\lambda t}}$$

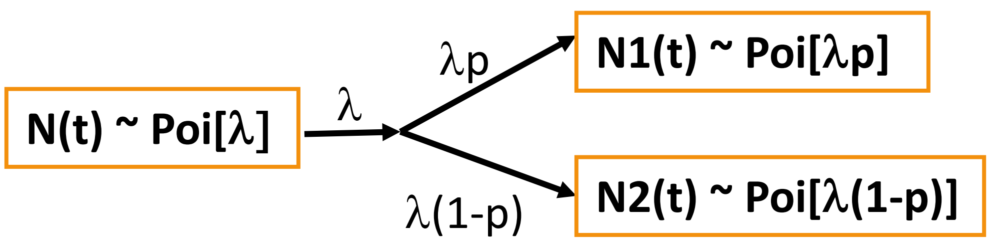
- This equation indicates that the interarrival times in a Poisson process are exponential with parameter λ

Arrival counts ~ Poi(λ) ⟺ **Interarrival time ~ Exp(1/λ)**

Stationary & Independent — Memoryless

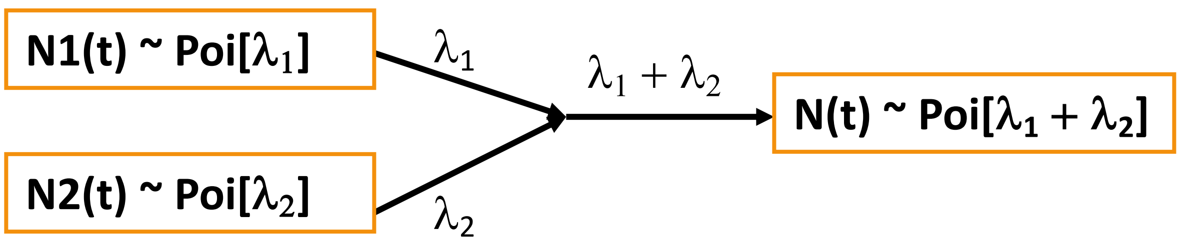
Splitting and Pooling (1)

- A Poisson process could be generated with **merging** two or more smaller Poisson process, or it could be **divided** into two or more processes
- Assume a Poisson process $\{N(t), t \geq 0\}$ with rate λ
- Splitting:
  - Suppose every event of this process can be classified as Type I, with probability p and Type II, with probability 1-p
  - During [0,t], N1(t) and N2(t) indicate the number of events corresponding to both of these types, where N(t) = N1(t) + N2(t)
  - It could be proven N1(t), and N2(t) are both Poisson processes with rates λp, and $\lambda(1-p)$



Splitting and Pooling (2)

- Pooling:
  - Suppose two Poisson processes N1(t), and N2(t) with parameters λ_1, and λ_2 are pooled together
  - With merging theses two processes, a bigger Poisson process N(t) is created, where N1(t) + N2(t) = N(t)
  - The rate of the new process is $\lambda_1 + \lambda_2$



Splitting and Pooling (3)

- Example 1:
  - Jobs arriving at a shop N(t) are in accordance with a Poisson process having rate λ
  - Suppose every arrival is marked **"high priority"** (type I) with probability 1/3 and **"low priority"** (type II) with probability 2/3
  - N1(t) and N2(t) are indicating the number of jobs, which both follow the Poisson process, with rates $\lambda/3$ and $2\lambda/3$, respectively
- Example 2:
  - A Poisson arrival stream with $\lambda_1 = 10$ arrivals per hour is combined (or pooled) with a Poisson arrival stream with $\lambda_2 = 17$ arrivals per hour
  - The combined process is a Poisson process with $\lambda = 27$ arrivals per hour

Nonstationary Poisson Process (NSPP)

- Recall: Poison processes have 3 conditions
- A Poisson process without the stationary increments (condition 2) is called as NSPP
  - NSPP are used for systems where the rate of arrival changes with time
    - So, NSPP are characterized by $\lambda(t)$, the arrival rate at time t
- Depending on the observation interval, λ would be different
- A number of use-cases for NSPP
  - Number of calls during a work-day
  - Number of pizza deliveries

Nonstationary Poisson Process (NSPP)

- One of the most important performance measurement parameters in NSPP is the expected number of arrivals by time t
- In order to achieve this, the mean rate must be determined as follows:
  - Denoted with $\Lambda(t)$:

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

- To be useful as an arrival-rate function, λ(t) must be nonnegative and integrable
- For a SPP with rate λ, we have $\Lambda(t) = \lambda$ as expected

Empirical Distributions (1)

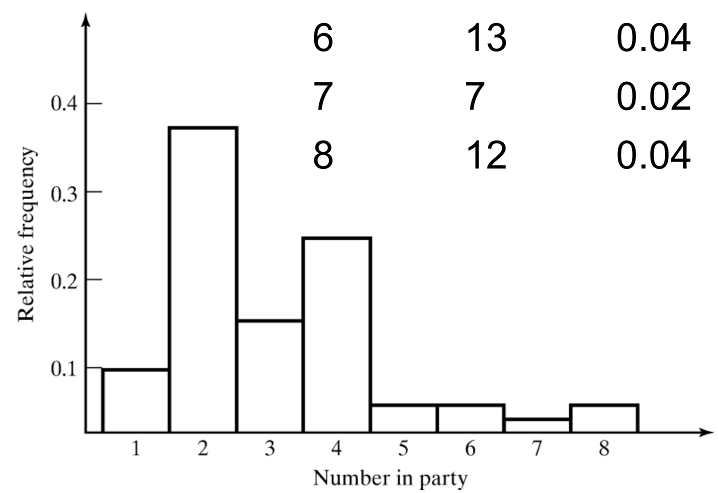
- In many situations, the gathered and sampled data could not be described with known distributions
  - They cannot be expressed with previously mentioned mathematical models
  - What should we do?
    - Use **graphics** and **tables** to represent the data as it is
    - These are called empirical distribution
- May be used when it is **impossible** or **unnecessary** to conclude that a random variable has any particular parametric distribution
  - Advantage: No assumption beyond the observed values in the sample
  - Disadvantage: Sample might not cover the entire range of possible values

Empirical Distributions (2)

- Example
  - Assume that the students are entering the dining area in groups of 1-8 members
  - We have monitored 300 groups, and counted the number of members in every group
  - Results are depicted in the histogram

Arrivals per Party	*Frequency*	*Relative Frequency*	*Cumulative Relative Frequency*
1	30	0.10	0.10
2	110	0.37	0.47
3	45	0.15	0.62
4	71	0.24	0.86
5	12	0.04	0.90
6	13	0.04	0.94
7	7	0.02	0.96
8	12	0.04	1.00

- This chart is somehow the PDF for random variable X
  - Where X is the number of students in a group



Empirical Distributions (3)

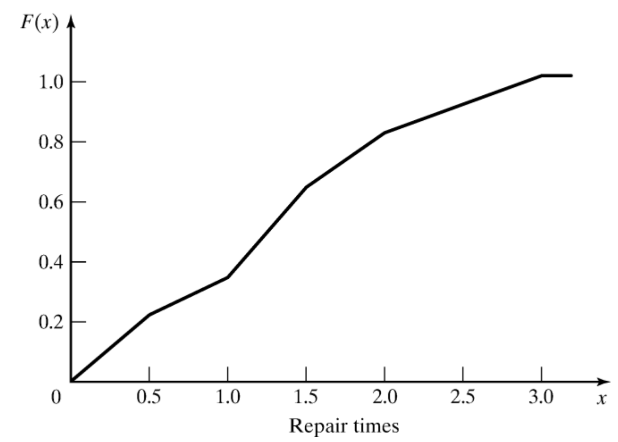
- Based on the histogram, there isn't any corresponding distributions matching the data
  - We have many upward and downward fluctuations
  - This amount of alterations cannot be described with known mathematical distributions
  - There may exists few distributions with **partial similarities**
    - We can use them as an estimate to simplify our modeling and calculations
    - But they will not produce an appropriate match to the data
- Shall we introduce a new distribution?
  - It is not rational
    - Because, there may not exist any other similar phenomena to be modeled with the same distribution
    - It is time consuming and complex

Stick with the table ☺

Empirical Distributions (4)

- Let's talk about another example:
  - The time required to repair a system, which has suffered a failure has been collected for the last 100 instances
  - The empirical CDF for this random variable is depicted in the following

Intervals (Hours)	Frequency	Relative Frequency	Cumulative Frequency
0<x<0.5	21	0.21	0.21
0.5<x<1.0	12	0.12	0.33
1.0<x<1.5	29	0.29	0.62
1.5<x<2.0	19	0.19	0.81
2.0<x<2.5	8	0.08	0.89
2.5<x<3.0	11	0.11	1.00



Summary

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- The world that the simulation analyst sees is probabilistic, not deterministic
- In this chapter:
  - Reviewed several important probability distributions
  - Showed applications of the probability distributions in a simulation context
- Important task in simulation modeling is the collection and analysis of the input data
  - Example: Hypothesize a distributional form for the input data
- Reader should know:
  - Difference between discrete, continuous, and empirical distributions
  - Poisson process and its properties