Makassar, 30 November 2020

STRUKTUR ALJABAR II

— Pertemuan XI —

(Catatan)

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Struktur Aljabar II/ Pertemuan ke-11/ Catatan Makassar, 13 November 2020

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Struktur Aljabar II : Catatan Pertemuan ke - 11

Teorema Dasar Isomorfisma Ring
(Teorema I Isomorfisma Ring)

Sebelumnya, diperkenalkan terlebih dahulu "Kernel dari Homomorfisma Ring".

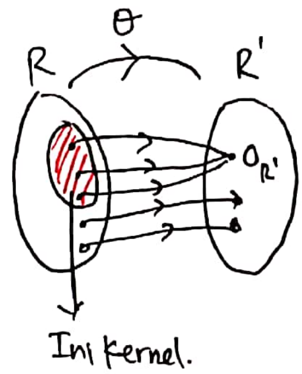
Kernel dari Homomorfisma Ring

<u>Definisi</u>

Misalkan $\theta : R \longrightarrow R'$ suatu homomorfisma ring, kernel / inti dari θ ditulis

$$\ker(\theta) = \text{Inti}(\theta) = \{a \in R \mid \theta(a) = 0_{R'}\}$$

$0_{R'}$ unsur nol di R'.



$\boxed{T_1}$

Jika $\theta : R \to R'$ suatu homomorfisma ring, maka $\ker(\theta)$ adalah Ideal dari R.

<u>Bukti</u>

(1) Adb. $\ker(\theta) \neq \phi$

Perhatikan bahwa $0_R \in R$, $0_{R'} \in R'$ dan $\theta(0_R) = 0_{R'}$. Jadi, $0_R \in \ker(\theta)$

$\therefore \ker(\theta) \neq \phi$

(2) $\ker(\theta) \subseteq R$ [Jelas dari definisi $\ker(\theta)$].

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Makassar, 13 November 2020

(3) Ambil $a, b \in \ker(\theta)$, $r \in R$ sebarang

$a \in \ker(\theta) \Rightarrow \theta(a) = 0_{R'}$

$b \in \ker(\theta) \Rightarrow \theta(b) = 0_{R'}$

Perhatikan bahwa

$$\begin{aligned}\theta(a-b) &= \theta(a+(-b)) \\ &= \theta(a) + \theta(-b) \\ &= \theta(a) - \theta(b) \\ &= 0_{R'} - 0_{R'} \\ &= 0_{R'}\end{aligned}$$

$\therefore a-b \in \ker(\theta)$

I Ideal

$\forall a, b \in I \Rightarrow a-b \in I$

$\forall a \in I, r \in R$

Berlaku: $ra \in I$

$ar \in I$.

$\theta(-b) = -\theta(b)$

$\forall b \in R$

dilain pihak

$$\begin{aligned}\theta(ra) &= \theta(r) \cdot \theta(a) \\ &= \theta(r) \cdot 0_{R'} \\ &= 0_{R'}\end{aligned}$$

$$\begin{aligned}\theta(ar) &= \theta(a) \cdot \theta(r) \\ &= 0_{R'} \cdot \theta(r) \\ &= 0_{R'}\end{aligned}$$

$\therefore ra \in \ker(\theta)$, $\therefore ar \in \ker(\theta)$

$\therefore \ker(\theta)$ adalah Ideal di R.

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$\boxed{T_2}$ $\boxed{P_1}$

Jika $\theta : R \longrightarrow R'$ suatu homomorfisma ring
Maka θ monomorfisma jika dan hanya jika

$$Ker(\theta) = \{0_R\}$$

Bukti :

$\Rightarrow$ Akan ditunjukkan : Jika θ satu-satu, maka $Ker(\theta) = \{0_R\}$

Ambil $a, b \in R$ sebarang

Karena θ satu-satu, maka jelas berlaku :

$$\theta(a) = \theta(b) \Rightarrow a = b$$

Karena $Ker(\theta) \subseteq R$, maka jelas berlaku :

$\forall\, c, d \in Ker(\theta)$ berlaku $\theta(c) = \theta(d) \Rightarrow c = d$ .....(*)

dilain pihak, menurut definisi kernel :

$$\theta(c) = 0_{R'} \quad \text{dan} \quad \theta(d) = 0_{R'}$$

maka $\theta(c) = \theta(d) = 0_{R'}$.

Dan dari (*), diperoleh $c = d$, atau dengan kata lain anggota di $Ker(\theta)$ adalah tunggal.

Dan menurut Sifat-sifat Homomorfisma Gelanggang $(T_1\ (1))$ diketahui bahwa : $\theta(0_R) = 0_{R'}$. Ini mengindikasikan bahwa $Ker(\theta) = \{0_{R'}\}$.

$\therefore$ θ satu-satu $\Rightarrow Ker(\theta) = \{0_R\}$

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- Akan ditunjukkan: Jika $Ker(\theta) = \{0_R\}$ maka θ satu-satu.

Ambil $a, b \in R$ sebarang, dengan $\theta(a) = \theta(b)$

Perhatikan bahwa,

$$\theta(a) = \theta(b)$$

$\theta(a) + (-\theta(b)) = 0_{R'}$ [Kedua ruas $+(-\theta(b))$]

$\theta(a) + \theta(-b) = 0_{R'}$ [T_1 (2) Sifat-sifat Homomorfisma Gelanggang]

$\theta(a + (-b)) = 0_{R'}$ [θ homomorfisma ring]

$a + (-b) \in Ker(\theta)$ [Sesuai Definisi Kernel]

$a + (-b) = 0_R$ [$Ker\ \theta = \{0_R\}$]

$a = b$ [Kedua ruas $+b$]

$\therefore$ karena $\theta(a) = \theta(b) \Rightarrow a = b$, artinya θ adalah fungsi satu-satu.

$\therefore$ Karena θ satu-satu maka $Ker(\theta) = \{0_R\}$

dan $Ker(\theta) = \{0_R\}$ maka θ satu-satu,

Maka disimpulkan:

Jika $\theta : R \to R'$ homomorfisma ring

Maka θ monomorfisma jika dan hanya jika

$Ker(\theta) = \{0_R\}$.

Monomorfisma = homomorfisma + injektif (satu-satu)

$\blacksquare$

Dipindai dengan CamScanner

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Pemetaan Natural

$\boxed{T_3}$

Jika R ring dan I ideal dari R maka terdapat epimorfisma:

$$\theta : R \longrightarrow R/I$$
$$a \longmapsto I + a$$

Bukti

Misal R ring dan I ideal di R, didefinikan pengaitan: (Pengaitan bkn pemetaan) pengaitan belum tentu pemetaan.

$$\theta : R \longrightarrow R/I \quad , \theta(a) = I + a$$
$$a \longmapsto I + a$$

(1) Adb. θ pemetaan

Ambil $a, b \in R$ sebarang dengan $a = b$

$\theta(a) = I + a = I + b = \theta(b)$

$\therefore \theta$ pemetaan

(2) Adb. θ homomorfisma ring

Ambil $a, b \in R$ sebarang.

Perhatikan bahwa

$$\begin{aligned} \theta(a+b) &= I + (a+b) \\ &= (I+a) + (I+b) \\ &= \theta(a) + \theta(b) \end{aligned}$$

$$\begin{aligned} \theta(ab) &= I + (ab) \\ &= (I+a)(I+b) \\ &= \theta(a) \cdot \theta(b) \end{aligned}$$

$\therefore \theta$ homomorfisma ring

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(3) Adb. θ Surjektif.

Ambil $\bar{a} \in R/I$ sebarang.

Tulis, $\bar{a} = I + a$ u/suatu $a \in R$.

Pilih $a \in R$, sehingga

$$\theta(a) = I + a = \bar{a}$$

$\therefore$ θ Surjektif

$\therefore$ θ Epimorfisma Ring

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Teorema Dasar Isomorfisma Ring

Jika R ring, I ideal dari R dan $\phi : R \to R'$ dan $\phi : R \to R'$ yang epimorfisma ring dengan $\operatorname{Ker}(\phi) = I$ maka terdapat secara tunggal isomorfisma ring:

$$\theta : R/I \longrightarrow R'$$
$$(I + a) \longmapsto \phi(a)$$

sehingga diagram bentuk komutatif

$$\begin{tikzcd}
R \arrow[r, "\phi"] \arrow[d, "\tau"'] & R' \\
R/I \arrow[ur, "\theta"'] &
\end{tikzcd}$$

yaitu : $\phi = \theta \circ \tau$

Bukti:

Misalkan R ring, I ideal R dan $\phi : R \to R'$ epimorfisma ring dengan $\operatorname{Ker}(\phi) = I$, Definisikan pengaitan.

$$\theta : R/I \longrightarrow R'$$
$$I + a \longmapsto \phi(a)$$

Diketahui:

$$\tau : R \longrightarrow R/I$$
$$a \longmapsto I + a$$

Merupakan epimorfisma (Pemetaan Natural).

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(1) Adb. θ pemetaan

Ambil $\bar{a}, \bar{b} \in R/I$ sebarang.

Tulis, $\bar{a} = I + a$ u/suatu $a \in R$.

$\bar{b} = I + a$ u/suatu $b \in R$.

dengan $\bar{a} = \bar{b}$.

Adb. $\theta(\bar{a}) = \theta(\bar{b})$

Perhatikan bahwa:

$$\bar{a} = \bar{b}$$

$$\Rightarrow I + a = I + b$$

$$\Rightarrow a - b \in I = Ker(\phi) \quad [I+a = I+b \Leftrightarrow a-b \in I]$$

$$\Rightarrow \phi(a-b) = 0_{R'}$$

$$\Rightarrow \phi(a+(-b)) = 0_{R'}$$

$$\Rightarrow \phi(a) + \phi(-b) = 0_{R'}$$

$$\Rightarrow \phi(a) - \phi(b) = 0_{R'}$$

$$\Rightarrow \phi(a) = \phi(b)$$

$$\Rightarrow \theta(I+a) = \theta(I+b)$$

$$\Rightarrow \theta(\bar{a}) = \theta(I)$$

$\therefore$ θ Pemetaan

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(2) Adb. Diagram

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\tau"'] & R' \\ R/I \arrow[ur, "\theta"'] & \end{tikzcd}$$

komutatif, yaitu $\phi = \theta \circ \tau$

Ambil $a \in R$ sebarang.

Perhatikan bahwa,

$$\begin{aligned} (\theta \circ \tau)(a) &= \theta(\tau(a)) \\ &= \theta(I+a) \\ &= \phi(a) \end{aligned}$$

Jadi, $\phi = \theta \circ \tau$ (komutatif)

(3) Adb. θ homomorfisma

Ambil $\bar{a}, \bar{b} \in R/I$ sebarang

Tulis, $\bar{a} = I+a$ u/suatu $a \in I$

$\bar{b} = I+b$ u/suatu $b \in I$

Perhatikan bahwa

$$\begin{aligned} \theta(\bar{a}+\bar{b}) &= \theta((I+a)(I+b)) \\ &= \theta(I+(a+b)) \\ &= \phi(a+b) \\ &= \phi(a)+\phi(b) \\ &= \theta(I+a)+\theta(I+b) \\ &= \theta(\bar{a})+\theta(\bar{b}) \end{aligned}$$

$$\begin{aligned} \theta(\bar{a}\bar{b}) &= \theta((I+a)(I+b)) \\ &= \theta(I+(ab)) \\ &= \phi(ab) \\ &= \phi(a)\cdot\phi(b) \\ &= \theta(I+a)\cdot\theta(I+b) \\ &= \theta(\bar{a})\cdot\theta(\bar{b}) \end{aligned}$$

$\therefore$ θ homomorfisma ring.

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(4) Adb. θ surjektif

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\tau"'] & R' \\ R/I \arrow[ur, "\theta"'] & \end{tikzcd}$$

ϕ Epimorfisma

τ Epimorfisma

$\theta \circ \tau = \phi$

Ambil $a' \in R'$ sebarang.

Karena ϕ epimorfisma, akibatnya terdapat $a \in R$ sehingga $\phi(a) = a'$ dan τ epimorfisma.

Jadi, $\tau(a) = I + a = \bar{a} \in R/I$

Jadi, terdapat $\bar{a} \in R/I$, $\bar{a} = I + a$, $a \in R$

sehingga,

$$\begin{aligned} \theta(\bar{a}) &= \theta(I + a) \\ &= \theta(\tau(a)) \\ &= (\theta \circ \tau)(a) \\ &= \phi(a) \\ &= a' \end{aligned}$$

Jadi, θ surjektif.

.Imanuel AS / 1811141008 [signature]

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(5) Adb. θ Injektif

Ambil $\bar{a}, \bar{b} \in R/I$ sebarang

Tulis $\bar{a} = I + a$ u/suatu $a \in R$

$\bar{b} = I + b$ u/suatu $b \in R$

dengan $\theta(a) = \theta(b)$.

Adb. $\bar{a} = \bar{b}$.

Perhatikan bahwa,

$$\theta(\bar{a}) = \theta(\bar{b})$$

$$\Rightarrow \theta(I+a) = \theta(I+b)$$

$$\Rightarrow \phi(a) = \phi(b)$$

$$\Rightarrow \phi(a) + (-\phi(b)) = 0_{R'}$$

$$\Rightarrow \phi(a) + \phi(-b) = 0_{R'}$$

$$\Rightarrow \phi(a + (-b)) = 0_{R'}$$

Jadi, $a + (-b) \in \ker(\phi) = I$

Akibatnya, $a - b \in I$.

$$\Rightarrow I + a = I + b$$

$$\Rightarrow \bar{a} = \bar{b}$$

$\therefore$ θ Injektif.

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(6) Adb. θ tunggal

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\tau"'] & R' \\ R/I \arrow[ur, shift left, "\theta"] \arrow[ur, shift right, "\theta'"'] & \end{tikzcd}$$

Misal terdapat

$$\theta' : R/I \longrightarrow R'$$
$$I + a \longmapsto \phi(a)$$

Adb. $\theta = \theta'$

Perhatikan bahwa,

$\phi = \theta \circ \tau$ dan

$\phi = \theta' \circ \tau$

Ambil $\bar{a} \in R/I$ sebarang,

Tulis, $\bar{a} = I + a$ u/suatu $a \in R$, diperoleh

$$\begin{aligned} \theta(\bar{a}) &= \theta(I+a) \\ &= \theta(\tau(a)) \\ &= (\theta \circ \tau)(a) \\ &= \phi(a) \\ &= (\theta' \circ \tau)(a) \\ &= \theta'(I+a) \\ &= \theta'(I+a) \\ &= \theta'(\bar{a}) \end{aligned}$$

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\tau(a) \downarrow (I+a)"'] & R' \\ R/I \arrow[ur, shift left, "\theta"] \arrow[ur, shift right, "\theta'"'] & \end{tikzcd}$$

$\downarrow$

$\phi = \theta \circ \tau$

$\phi = \theta' \circ \tau$

Jadi, $\theta = \theta'$

Q.E.D //