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Pangkat dan Bilangan Kompleks

$z = a + bi \quad , a,b \in \mathbb{R}$

$\hookrightarrow r = \sqrt{a^2+b^2}$

$\hookrightarrow z = r(\cos\theta + i \cdot \sin\theta) = r \text{ cis } \theta$
$= re^{i\theta}$

$z = re^{i\theta} \longrightarrow z^n = (re^{i\theta})^n = r^n \cdot e^{i\theta n} \quad , n \in \mathbb{N}$ [Induksi]

$\boxed{z^n = r^n e^{in\theta}} \quad n \in \mathbb{N}$

- Pangkat Bilangan Bulat

Misal $n \in \mathbb{Z}$, $\boxed{z^n = r^n e^{in\theta}}$

untuk $n = 0 \longrightarrow z^0 = 1$

untuk $n > 0 \longrightarrow z^n = r^n e^{i\theta n}$. [Induksi]

untuk $n < 0 \longrightarrow$ Tulis $n = -m$, $m \in \mathbb{N}$

akibatnya , $z^n = z^{-m} = (z^{-1})^m$

$$= \left(\frac{1}{r} e^{-i\theta}\right)^m$$
$$= r^{-m} \cdot e^{-m(i\theta)}$$
$$= r^n \cdot e^{i\theta n}$$

$$\boxed{z^n = r^n e^{in\theta}}$$

Kesimpulannya :

Jika $z = re^{i\theta}$, $z \neq 0$ maka $z^n = r^n \cdot e^{in\theta}$, $n \in \mathbb{Z}$

$\boxed{E}$ Tentukan nilai dari z^7 jika $z = \sqrt{3} + i$

Solusi :

$z = \sqrt{3} + i < \begin{matrix} a = \sqrt{3} \\ b = 1 \end{matrix} \Big\} r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \text{arc tan}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$

$z = re^{i\theta} = 2e^{(\pi/6) \cdot i}$

Jadi, diperoleh $z^7 = (2e^{(\pi/6)i})^7 = 2^7 \cdot e^{\frac{7\pi}{6} \cdot i} = (2^6 e^{i\pi}) \cdot (2e^{\pi/6 \cdot i})$

$= (64(-1)) \cdot (z)$

$= (-64) \cdot (\sqrt{3} + i)$

$= -64\sqrt{3} - 64i$ //

$e^{i\pi} = \cos\pi + i \cdot \sin\pi$
$= -1 + 0$
$= -1$

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Teorema de' Moivre

Jika $z \in \mathbb{C}$ dengan $z = r(\cos\theta + i \cdot \sin\theta)$

maka, $z^n = r^n(\cos n\theta + i \cdot \sin n\theta)$; $\forall n \in \mathbb{Z}$

Bukti

(1) Kasus $n=0$

$$z^0 = r^0(\cos 0 + i \cdot \sin 0)$$
$$= 1(1 + 0)$$
$$= \underline{1} \in \mathbb{C}$$

(2) Kasus $n \in \mathbb{Z}^+$

Gunakan Induksi Matematika

(*) Untuk $n=1$ maka $z = r(\cos\theta + i \cdot \sin\theta) \in \mathbb{C}$

(*) Misal untuk $n=k$ benar, yaitu

$$z^k = r^k(\cos(k\theta) + i \cdot \sin(k\theta))$$

akan ditunjukkan benar untuk $n=k+1$

yaitu :

$$z^{k+1} = r^{k+1}(\cos(k+1)\theta + i \cdot \sin(k+1)\theta)$$

Perhatikan bahwa

$$z^{k+1} = z^k \cdot z$$
$$= r^k(\cos(k\theta) + i \cdot \sin(k\theta))(r(\cos\theta + i \cdot \sin\theta))$$
$$= r^k \operatorname{cis}(k\theta) \cdot r \operatorname{cis}(\theta)$$
$$= r^{k+1} \cdot \operatorname{cis}(k\theta + \theta)$$
$$= r^{k+1} \cdot \operatorname{cis}(k+1)\theta$$
$$= r^{k+1}(\cos(k+1)\theta + i \sin(k+1)\theta)$$

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(3) Kasus $n \in \mathbb{Z}^-$

Misal, $n = -m$, $m \in \mathbb{Z}^+$

Perhatikan bahwa

$$
\begin{aligned}
z^n &= z^{-m} \\
&= \frac{1}{z^m} \\
&= \frac{1}{r^m \operatorname{cis}(m\theta)} \\
&= \frac{1}{r^m(\cos m\theta + i \cdot \sin \theta)} \\
&= \frac{1}{r^m(\cos(m\theta) + i \cdot \sin(m\theta))} \times \frac{(\cos(m\theta) - i \cdot \sin(m\theta))}{(\cos(m\theta) - i \cdot \sin(m\theta))} \\
&= \frac{\cos(m\theta) - i \cdot \sin(m\theta)}{r^m(\cos^2(m\theta) + \sin^2(m\theta))} \\
&= r^{-m}(\cos(m\theta) - i \cdot \sin(m\theta)) \\
&= r^{-m}(\cos(-m\theta) - i \cdot \sin(m\theta)) \\
&= r^n(\cos n\theta + i \cdot \sin n\theta)
\end{aligned}
$$

$\therefore$ Teorema de' Moivre

$$
\begin{aligned}
z^n &= r^n(\cos n\theta + i \cdot \sin n\theta) \quad \forall n \in \mathbb{Z} \\
&= r^n \operatorname{cis}(n\theta) \quad \forall n \in \mathbb{Z} \\
&= r^n \cdot e^{in\theta} \quad \forall n \in \mathbb{Z}
\end{aligned}
$$

E

(1) Misal $a+bi = (i-1)^{49}\left(\cos(\pi/40) + i\cdot\sin(\pi/40)\right)^{10}$.

Tentukan $a+b = \ldots$?

Solusi

Perhatikan bahwa

$$(i-1) < \begin{matrix} a=1 \\ b=-1 \end{matrix} \Big\} \; r = \sqrt{a^2+b^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$\theta = \arc\tan\left(\frac{b}{a}\right) = \arc\tan(-1) = 3\pi/4 \quad (\text{Kuadran II})$$

akibatnya berdasarkan Teorema de'Moivre diperoleh,

$$\begin{aligned}(i-1)^{49} &= (\sqrt{2})^{49}\cdot\left(\cos(3\pi/4) + i\cdot\sin(3\pi/4)\right)^{49} \\ &= (\sqrt{2})^{49}\cdot\left(e^{\frac{3}{4}\pi i}\right)^{49}\end{aligned}$$

Sehingga diperoleh

$$\begin{aligned}&(i-1)^{49}\cdot\left(\cos(\pi/40) + i\cdot\sin(\pi/40)\right)^{10} \\ &= (\sqrt{2})^{49}\cdot\left(e^{(\frac{3}{4}\pi)i}\right)^{49}\left(e^{\frac{\pi}{40}i}\right)^{10} \\ &= (\sqrt{2})^{49}\cdot\left(e^{(\frac{147}{4}\pi)i}\right)\left(e^{(\frac{\pi}{4})i}\right) \\ &= (\sqrt{2})^{49}\cdot e^{(\frac{148}{4}\pi)i} \\ &= (\sqrt{2})^{49}\cdot e^{(37\pi)i} \\ &= (\sqrt{2})^{49}\cdot e^{\pi i} \\ &= (\sqrt{2})^{49}\cdot(\cos\pi + i\cdot\sin\pi) \\ &= (\sqrt{2})^{49}\cdot(-1+0) \\ &= -(\sqrt{2})^{49}\end{aligned}$$

$\therefore$ Dengan demikian diperoleh

$$-(\sqrt{2})^{49} = a+bi < \begin{matrix} a = -(\sqrt{2})^{49} \\ b = 0 \end{matrix}$$

Jadi $a+b = -(\sqrt{2})^{49} + 0 = -(\sqrt{2})^{49}$ //

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Tentukan nilai dari $5\,Re(z) + 7\,Im(z)$

Jika $z = (3-3i)^{2010}$

Penyelesaian :

Perhatikan bahwa

$(3-3i)$: $a = 3$, $b = -3$ $\Big\}$ $r = \sqrt{a^2+b^2} = \sqrt{3^2+(-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$\theta = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{-3}{3}\right) = \arctan(-1) = \frac{3\pi}{4}$ (Kuadran II)

akibatnya berdasarkan Teorema de' Moivre diperoleh,

$$(3-3i)^{2010} = (3\sqrt{2})^{2010} \cdot \left(\cos\left(\tfrac{3\pi}{4}\right) + i \cdot \sin\left(\tfrac{3\pi}{4}\right)\right)^{2010}$$

$$= (3\sqrt{2})^{2010} \cdot \left(e^{\frac{3\pi}{4}i}\right)^{2010}$$

$$= (3\sqrt{2})^{2010} \cdot \left(e^{\frac{6030}{4}\pi \cdot i}\right)$$

$$= (3\sqrt{2})^{2010} \cdot \left(\cos\left(\tfrac{6030}{4}\pi\right) + i \cdot \sin\left(\tfrac{6030}{4}\pi\right)\right)$$

$$= (3\sqrt{2})^{2010} \cdot \left(\cos\left(\tfrac{3015}{2}\pi\right) + i \cdot \sin\left(\tfrac{3015}{2}\pi\right)\right)$$

$$= (3\sqrt{2})^{2010} \cdot \left(0 + i \cdot (-1)\right)$$

$$= (3\sqrt{2})^{2010} \cdot (-i)$$

Maka diperoleh

$-(3\sqrt{2})^{2010} i = a + bi$: $a = 0$ atau $Re(z) = 0$; $b = -(3\sqrt{2})^{2010}$ atau $Im(z) = -(3\sqrt{2})^{2010}$

$$\therefore 5\,Re(z) + 7\,Im(z) = 5 \cdot (0) + 7 \cdot -(3\sqrt{2})^{2010}$$

$$= -7 \cdot (3\sqrt{2})^{2010} \quad //$$

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$$\cos\left(\frac{3015}{2}\pi\right) = \cos\left(\frac{3\pi}{2} + \frac{3012\pi}{2}\right)$$

$$= \cos\left(\frac{3\pi}{2} + 2\cdot\frac{3012}{2\cdot 2}\cdot\pi\right)$$

$$= \cos\left(\frac{3\pi}{2} + 2\cdot\frac{3012}{4}\cdot\pi\right)$$

$$= \cos\left(\frac{3\pi}{2} + 2\cdot 753\,\pi\right)$$

= Menggunakan $\cos(t + 2k\pi) = \cos(t)$, $k \in \mathbb{Z}$
diperoleh,

$$= \cos\left(\frac{3\pi}{2}\right)$$

$$= \cos(270^\circ)$$

$$= 0$$

$$\sin\left(\frac{3015}{2}\pi\right) = \sin\left(\frac{3\pi}{2} + \frac{3012\pi}{2}\right)$$

$$= \sin\left(\frac{3\pi}{2} + 2\cdot\frac{3012}{2\cdot 2}\cdot\pi\right)$$

$$= \sin\left(\frac{3\pi}{2} + 2\cdot\frac{3012}{4}\cdot\pi\right)$$

$$= \sin\left(\frac{3\pi}{2} + 2\cdot 753\,\pi\right)$$

= Menggunakan $\sin(t + 2k\pi) = \sin(t)$, $k \in \mathbb{Z}$
diperoleh,

$$= \sin\left(\frac{3\pi}{2}\right)$$

$$= \sin(270^\circ)$$

$$= -1$$