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Analisis Kompleks

Pertemuan ke - 13

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Analisis Kompleks / Pertemuan ke - 13 Makassar, 28 Mei 2021

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Teorema Limit

(1.) Misalkan $f(z) = u(x,y) + iv(x,y)$

$z = x + iy$ dan $z_0 = x_0 + iy_0$

$w_0 = u_0 + iv_0$ maka

(*) $\lim_{z \to z_0} f(z) = w_0$ jika dan hanya jika

(**) $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ dan $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$

Proof :

$\Leftarrow$ Ambil $\varepsilon > 0$ sebarang karena

$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ akibatnya terdapat $\delta_1 > 0$ sehingga untuk

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1$ berlaku $|u - u_0| < \frac{\varepsilon}{2}$ ....(1)

dilain pihak

$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$ akibatnya terdapat $\delta_2 > 0$ sehingga untuk

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2$ berlaku $|v - v_0| < \frac{\varepsilon}{2}$) .... (2)

Pilih $\delta = \min\{\delta_1, \delta_2\}$ sehingga

untuk $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ berlaku

$$|f(z) - w_0| = |(u+iv) - (u_0 + iv_0)|$$
$$= |(u-u_0) + i(v-v_0)|$$
$$\leq |u-u_0| + |v-v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \lim_{z \to z_0} f(z) = w_0$

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$\Rightarrow$ Diketahui $(*)$ berlaku.

Kita tahu bahwa u/ setiap bilangan positif ε sebarang, selalu ada bilangan positif δ sedemikian sehingga

$(***)$ $$|(u+iv)-(u_0+iv_0)|<\varepsilon$$

dimana

$(****)$ $$0<|(x+iy)-(x_0+iy_0)|<\delta$$

Tapi

$$|u-u_0|\leq|(u-u_0)+i(v-v_0)|=|(u+iv)-(u_0+iv_0)|,$$
$$|v-v_0|\leq|(u-u_0)+i(v-v_0)|=|(u+iv)-(u_0+iv_0)|$$

dan

$$|(x+iy)-(x_0+iy_0)|=|(x-x_0)+i(y-y_0)|=\sqrt{(x-x_0)^2+(y-y_0)^2}$$

Oleh karena itu, dari $(***)$ dan $(****)$ diperoleh

$$|u-u_0|<\varepsilon \text{ dan } |v-v_0|<\varepsilon$$

dimana

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$$

Ini menunjukkan $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y)=u_0$ dan $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y)=v_0$.

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(2.) Misalkan $\lim_{z \to z_0} f(z) = w_0$ dan $\lim_{z \to z_0} t(z) = V_0$ maka

(a) $\lim_{z \to z_0} \left(f(z) + t(z) \right) = w_0 + V_0$

Bukti :

Analisis Pendahuluan

Jika $\varepsilon > 0$, maka $\frac{\varepsilon}{2} > 0$ pasti.

Karena diketahui $\lim_{z \to z_0} f(z) = w_0$, maka terdapat $\delta_1 > 0$ sedemikian sehingga

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - w_0| < \frac{\varepsilon}{2}$$

Karena diketahui $\lim_{z \to z_0} t(z) = V_0$, maka terdapat $\delta_2 > 0$ sedemikian sehingga

$$0 < |z - z_0| < \delta_2 \Rightarrow |t(z) - V_0| < \frac{\varepsilon}{2}$$

Pilih $\delta = \min\{\delta_1, \delta_2\}$

Bukti Formal

Diberikan $\varepsilon > 0$ sebarang,

Pilih $\delta = \min\{\delta_1, \delta_2\}$

Sehingga untuk $|z - z_0| < \delta$ maka

$$\left| \left(f(z) + t(z) \right) - \left(w_0 + V_0 \right) \right| = \left| \left(f(z) - w_0 \right) + \left(t(z) - V_0 \right) \right|$$

$$\leq |f(z) - w_0| + |t(z) - V_0| \quad [\text{Ketaksamaan } \triangle]$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \lim_{z \to z_0} \left(f(z) + t(z) \right) = w_0 + V_0$. //

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(b) $\lim_{z \to z_0} (f(z) - t(z)) = w_0 - v_0$

Bukti :

Karena bagian (a) sudah terbukti, maka hal berikut berlaku :

$$\lim_{z \to z_0} (f(z) - t(z)) = \lim_{z \to z_0} (f(z) + (-1) \cdot t(z))$$

$$= \lim_{z \to z_0} f(z) + \lim_{z \to z_0} (-1) \cdot t(z)$$

$$= \lim_{z \to z_0} f(z) + (-1) \cdot \lim_{z \to z_0} t(z)$$

$$= \lim_{z \to z_0} f(z) - \lim_{z \to z_0} t(z)$$

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(c) $\lim_{z \to z_0} (f(z) \cdot t(z)) = (w_0) \cdot (v_0)$

<u>Bukti</u> :

Dengan menggunakan teorema (a) dan (b), diperoleh:

$$\lim_{z \to z_0} [f(z) - w_0] = \lim_{z \to z_0} f(z) - \lim_{z \to z_0} w_0 = w_0 - w_0 = 0$$

$$\lim_{z \to z_0} [t(z) - v_0] = \lim_{z \to z_0} t(z) - \lim_{z \to z_0} v_0 = v_0 - v_0 = 0$$

Diambil $\varepsilon > 0$ sebarang, maka terdapat δ_1 ; $\delta_2 > 0$, sehingga:

Jika $0 < |z - z_0| < \delta_1$ maka $|(f(z) - w_0) - 0| < \sqrt{\varepsilon}$ dan

Jika $0 < |z - z_0| < \delta_2$ maka $|(t(z) - v_0) - 0| < \sqrt{\varepsilon}$

Pilih $\delta = \min\{\delta_1, \delta_2\}$

Untuk $0 < |z - z_0| < \delta$ diperoleh

$$|[f(z) - w_0][t(z) - v_0] - 0| = |f(z) - w_0| \cdot |t(z) - v_0|$$

$$< \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$$

$\therefore \lim_{z \to z_0} [f(z) - w_0][t(z) - v_0] = 0$

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Selanjutnya perhatikan bahwa,

$$[f(z) - w_0][t(z) - V_0] = f(z) \cdot t(z) - V_0 \cdot f(z) - w_0 \cdot t(z) + w_0 \cdot V_0$$

Atau,

$$f(z) \cdot t(z) = [f(z) - w_0][t(z) - V_0] + V_0 \cdot f(z) + w_0 \cdot t(z) - w_0 \cdot V_0$$

Kedua ruas dilimitkan, menjadi :

$$\lim_{z \to z_0} [f(z) \cdot t(z)] = \lim_{z \to z_0} \left([f(z) - w_0][t(z) - V_0] + V_0 \cdot f(z) + w_0 \cdot t(z) - w_0 \cdot V_0 \right)$$

$$= \lim_{z \to z_0} [f(z) - w_0][t(z) - V_0] + \lim_{z \to z_0} V_0 \cdot f(z) + \lim_{z \to z_0} w_0 \cdot t(z) - \lim_{z \to z_0} w_0 \cdot V_0$$

$$= 0 + V_0 \cdot \lim_{z \to z_0} f(z) + w_0 \cdot \lim_{z \to z_0} t(z) - \lim_{z \to z_0} w_0 \cdot V_0$$

$$= V_0 \cdot w_0 + w_0 \cdot V_0 - w_0 \cdot V_0$$

$$= V_0 \cdot w_0$$

$$\lim_{z \to z_0} [f(z) \cdot t(z)] = (w_0) \cdot (V_0)$$

// $\boxed{\text{Ib}}$

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(d) $\lim_{z \to z_0} \frac{f(z)}{t(z)} = \frac{w_0}{v_0}$, $V_0 \neq 0$

Bukti :

Pertama akan ditunjukkan :

$$\lim_{z \to z_0} \frac{1}{t(z)} = \frac{1}{v_0}$$

Diambil $\varepsilon > 0$ sebarang.

Karena $\lim_{z \to z_0} t(z) = V_0$ maka terdapat $\delta_1 > 0$ sedemikian sehingga

$$|t(z) - V_0| < \frac{|V_0|}{2} \quad \text{dimana } 0 < |z - z_0| < \delta_1$$

Untuk $0 < |z - z_0| < \delta_1$ diperoleh :

$$\begin{aligned} v_0 &= |V_0 - t(z) + t(z)| \\ &\leq |V_0 - t(z)| + |t(z)| \quad [\text{Ketaksamaan } \Delta] \\ &= |t(z) - V_0| + |t(z)| \\ &< \frac{|V_0|}{2} + |t(z)| \end{aligned}$$

Hal ini menunjukkan

$$|V_0| < \frac{|V_0|}{2} + |t(z)| \Rightarrow \frac{|V_0|}{2} < |t(z)| \Rightarrow \frac{1}{t(z)} < \frac{2}{|V_0|}$$

→ next

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Selanjutnya, terdapat juga $\delta_2 > 0$ sedemikian sehingga

$$|t(z) - V_0| < \frac{|V_0|^2}{2} \cdot \varepsilon \quad \text{dimana} \quad 0 < |z - z_0| < \delta_2$$

Pilih $\delta = \min\{\delta_1, \delta_2\}$. Jika $0 < |z - z_0| < \delta$, diperoleh:

$$\left|\frac{1}{t(z)} - \frac{1}{V_0}\right| = \left|\frac{V_0 - t(z)}{V_0 \cdot t(z)}\right|$$

$$= \frac{1}{|V_0 \cdot t(z)|} \cdot |V_0 - t(z)|$$

$$= \frac{1}{|V_0|} \cdot \frac{1}{|t(z)|} \cdot |t(z) - V_0|$$

$$< \frac{1}{|V_0|} \cdot \frac{2}{|V_0|} \cdot |t(z) - V_0|$$

$$< \frac{2}{|V_0|^2} \cdot \frac{|V_0|^2}{2} \cdot \varepsilon \quad = \varepsilon$$

$$\therefore \lim_{z \to z_0} \frac{1}{t(z)} = \frac{1}{V_0}$$

Selanjutnya,

$$\lim_{z \to z_0} \left[\frac{f(z)}{t(z)}\right] = \lim_{z \to z_0} \left[f(z) \cdot \frac{1}{t(z)}\right]$$

$$= \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} \frac{1}{t(z)} \quad [\text{Bagian (c)}]$$

$$= W_0 \cdot \frac{1}{V_0}$$

$$\lim_{z \to z_0} \left[\frac{f(z)}{t(z)}\right] = \frac{W_0}{V_0} \quad // \quad \blacksquare$$

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(3.) Jika $\lim_{z \to z_0} f(z) = w_0$ maka $\lim_{z \to z_0} |f(z)| = |w_0|$

(Kebalikan teorema ini tidak berlaku)

(4) $\lim_{z \to z_0} f(z) = 0 \Leftrightarrow \lim_{z \to z_0} |f(z)| = 0$

$\boxed{E}$

Hitung $\lim_{z \to 0} \frac{(\bar{z})^3}{|z|^2}$

Solusi :

Perhatikan bahwa

$$\lim_{z \to 0} \left| \frac{(\bar{z})^3}{|z|^2} \right| = \lim_{z \to 0} \frac{|\bar{z}|^3}{|z|^2} = \lim_{z \to 0} |z|$$

$$= \left| \lim_{z \to 0} z \right|$$

$$= 0$$

$$\therefore \lim_{z \to 0} \frac{(\bar{z})^3}{|z|^2} = 0$$