Analisis Kompleks / Catatan pertemuan ke-3

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Modulus Bilangan Kompleks

Misal $z = a + bi \in \mathbb{C}$, modulus dari z ditulis $|z| = \sqrt{a^2 + b^2}$

$$z = a + bi \begin{cases} \text{Re}(z) = a \\ \text{Im}(z) = b \end{cases}$$

$$\downarrow$$

$$a, b \in \mathbb{R}$$

Misal $z_1, z_2 \in \mathbb{C}$, maka jarak z_1 ke z_2 adalah $|z_1 - z_2|$.

$\rightarrow |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Real $\xrightarrow{\text{sifat}}$ Aljabar
Real $\xrightarrow{\text{sifat}}$ Urutan
Real $\xrightarrow{\text{sifat}}$ Kelengkapan

Kompleks $\xrightarrow{\text{sifat}}$ Aljabar ✓
Kompleks $\xrightarrow{\text{sifat}}$ Urutan ✗ (Tidak berlaku)

$3, 2 \in \mathbb{R} \rightarrow 2 < 3$

$2i, 3i \in \mathbb{C} \rightarrow 2i < 3i$ ✗

$3i < 2i$ ✗

Sifat Modulus

$\forall z, z_1, z_2 \in \mathbb{C}$ berlaku

(1) $|z| = |\bar{z}| = |-z|$

✓ (2) $|z|^2 = z\bar{z} = (\text{Re}(z))^2 + (\text{Im}(z))^2$

(3) $\text{Re}(z) \leq |\text{Re}(z)| \leq |z|$

(4) $\text{Im}(z) \leq |\text{Im}(z)| \leq |z|$

✓ (5) $|z^{-1}| = \dfrac{1}{|z|}$, $z \neq 0$

(6) $|\text{Re}(z)| + |\text{Im}(z)| \leq |z|\sqrt{2}$

(7) $|z_1 z_2| = |z_1| \cdot |z_2|$

(8) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$

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Bukti

(2) Adb. $|z|^2 = z\bar{z} = (Re(z))^2 + (Im(z))^2$

Ambil sebarang $z \in \mathbb{C}$.

Tulis, $z = a+bi$ u/ suatu $a, b \in \mathbb{R}$

$Re(z) = a$, $Im(z) = b$

Perhatikan bahwa,

- $|z|^2 = (\sqrt{a^2+b^2})^2 = a^2 + b^2$ .... (1)
- $z\bar{z} = (a+bi)(\overline{a+bi})$
  $= (a+bi)(a-bi)$
  $= a^2 + b^2$ .... (2)
- $(Re(z))^2 + (Im(z))^2 = a^2 + b^2$ .... (3)

Dari (1), (2) dan (3) diperoleh

$$|z|^2 = z\bar{z} = (Re(z))^2 + (Im(z))^2 \quad \blacksquare$$

(5) Adb. $|z^{-1}| = \frac{1}{|z|}$, $z \neq 0$

Ambil $z \in \mathbb{C}$ sebarang, $z \neq 0$

Tulis, $z = a+bi$ u/ suatu $a, b \in \mathbb{R}$

Perhatikan bahwa

$$|z^{-1}| = \left|\frac{1}{z}\right|$$

$$= \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(-\frac{b}{a^2+b^2}\right)^2}$$

$$= \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}}$$

$$= \sqrt{\frac{\cancel{(a^2+b^2)}}{(a^2+b^2)^{\cancel{2}}}}$$

$$= \frac{1}{\sqrt{a^2+b^2}}$$

$$= \frac{1}{|z|} \quad \blacksquare$$

$$\frac{1}{z} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi}$$

$$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

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Sifat Modulus

(1.) Adb. $\forall z \in \mathbb{C}$ berlaku $|z| = |\bar{z}| = |-z|$

Ambil sebarang $z \in \mathbb{C}$

Tulis, $z = a + bi$; u/suatu $a, b \in \mathbb{R}$

Note that,

$$
\begin{aligned}
|\bar{z}| &= |a - bi| \\
&= |a + (-bi)| \\
&= \sqrt{a^2 + (-b)^2} \\
&= \sqrt{a^2 + b^2} \\
&= |z| \quad \ldots\ldots (*)
\end{aligned}
$$

dan

$$
\begin{aligned}
|-z| &= |-(a+bi)| \\
&= |-a - bi| \\
&= \sqrt{(-a)^2 + (-b)^2} \\
&= \sqrt{a^2 + b^2} \\
&= |z| \quad \ldots\ldots (**)
\end{aligned}
$$

$\therefore$ Dari persamaan $(*)$ dan $(**)$ diperoleh bahwa

$$|z| = |\bar{z}| = |-\bar{z}| \quad \blacksquare$$

(3) Adb. $\forall z \in \mathbb{C}$ berlaku $Re(z) \leq |Re(z)| \leq |z|$

Ambil sebarang $z \in \mathbb{C}$, Tulis $z = a + bi$, u/suatu $a, b \in \mathbb{R}$

$\Rightarrow Re(z) = a$

$\Rightarrow |Re(z)| = |a|$

$\Rightarrow |z| = |a + bi| = \sqrt{a^2 + b^2}$

(i) Adit. $Re(z) \leq |Re(z)|$

Perhatikan bahwa

$a \leq |a|$ karena sesuai definisi nilai mutlak:

$a = |a|$ bila $a \geq 0$

$a < |a|$ bila $a < 0$

$\Rightarrow a \leq |a|$ atau

$Re(z) \leq |Re(z)|$

(ii) Adit. $|Re(z)| \leq |z|$

Perhatikan bahwa

$|a| \leq |a + bi|$

atau

$|a| \leq \sqrt{a^2 + b^2}$

karena sesuai definisi nilai mutlak:

$|a| = \sqrt{a^2 + b^2}$ u/ $b = 0$

$|a| < \sqrt{a^2 + b^2}$ u/ $b \neq 0$

Maka

$|a| \leq \sqrt{a^2 + b^2}$

atau

$|Re(z)| \leq |z|$

$\therefore$ Dari (i) dan (ii) diperoleh

$$Re(z) \leq |Re(z)| \leq |z| \quad \blacksquare$$

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(4) Adb. $\forall z \in \mathbb{C}$ berlaku $Im(z) \leq |Im(z)| \leq |z|$.

Ambil sebarang $z \in \mathbb{C}$

Tulis, $z = a + bi$; untuk suatu $a, b \in \mathbb{R}$

- $Im(z) = b$
- $|Im(z)| = |b|$
- $|z| = |a+bi| = \sqrt{a^2+b^2}$

(i) Adit. $Im(z) \leq |Im(z)|$

Perhatikan bahwa,

$b \leq |b|$

karena sesuai dengan definisi nilai mutlak bahwa,

$b = |b|$ jika $b \geq 0$

$b < |b|$ jika $b < 0$

$\Rightarrow b \leq |b|$ atau $Im(z) \leq |Im(z)|$

(ii) Adit. $|Im(z)| \leq |z|$

Perhatikan bahwa,

$|b| \leq |a+bi|$

atau

$|b| \leq \sqrt{a^2+b^2}$

karena sesuai dengan definisi nilai mutlak bahwa,

$b = \sqrt{a^2+b^2}$ jika $a = 0$

$b < \sqrt{a^2+b^2}$ jika $a \neq 0$

$\Rightarrow |b| \leq \sqrt{a^2+b^2}$ atau $|Im(z)| \leq |z|$

$\therefore$ Dari (i) dan (ii) diperoleh bahwa

$Im(z) \leq |Im(z)| \leq |z|$ $\boxed{//}$

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(6) Adb. $|Re(z)| + |Im(z)| \leq |z|\sqrt{2}$

Ambil sebarang $z \in \mathbb{C}$

Tulis, $z = a + bi$; untuk suatu $a, b \in \mathbb{R}$

- $|Re(z)| = |a|$
- $|Im(z)| = |b|$
- $|z| = \sqrt{a^2+b^2}$

Perhatikan bahwa untuk setiap $a, b \in \mathbb{R}$ berlaku:

$$(|a| - |b|)^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2\,|a||b|$$

$$(|a| + |b|)^2 = |a|^2 + |b|^2 + 2|a||b| \leq |a|^2 + |b|^2 + |a|^2 + |b|^2$$
$$= 2(|a|^2 + |b|^2)$$
$$= 2(a^2 + b^2)$$
$$= 2|z|^2$$

$$\boxed{\begin{aligned} |z|^2 &= |z| \cdot |z| \\ &= |\sqrt{a^2+b^2}| \cdot |\sqrt{a^2+b^2}| \\ &= |a^2+b^2| \end{aligned}}$$

Perhatikan bahwa, hal berikut berlaku :

$$|a|^2 + |b|^2 \leq |a|^2 + |b|^2 + 2|a| \cdot |b| \leq 2|z|^2$$

sehingga diperoleh

$$(|a| + |b|)^2 \leq 2|z|^2$$
$$|a| + |b| \leq \sqrt{2|z|^2}$$
$$|a| + |b| \leq \sqrt{2} \cdot |z|$$
$$|Re(z)| + |Im(z)| \leq |z| \cdot \sqrt{2}$$

$\blacksquare$

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(7) Adb. $|z_1 . z_2| = |z_1| . |z_2|$; $\forall z_1, z_2 \in \mathbb{C}$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Tulis, $z_1 = a_1 + b_1 i$ untuk suatu $a_1, b_1 \in \mathbb{R}$

$z_2 = a_2 + b_2 i$ untuk suatu $a_2, b_2 \in \mathbb{R}$

Note that

$$|z_1 . z_2|^2 = (z_1 . z_2) \cdot (\overline{z_1 . z_2})$$
$$= (z_1 . z_2)(\overline{z_1} . \overline{z_2})$$
$$= (z_1 . \overline{z_1})(z_2 . \overline{z_2})$$
$$= |z_1|^2 . |z_2|^2$$

Jadi, $|z_1 z_2| = |z_1| . |z_2|$ ∎

(8) Adb. $\forall z_1, z_2$, $z_2 \neq 0$ berlaku $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Tulis, $z_1 = a_1 + b_1 i$ untuk suatu $a_1, b_1 \in \mathbb{R}$

$z_2 = a_2 + b_2 i$ untuk suatu $a_2, b_2 \in \mathbb{R}$

Note that

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right) \cdot \overline{\left(\frac{z_1}{z_2}\right)}$$
$$= \left(\frac{z_1}{z_2}\right) \cdot \frac{\overline{z_1}}{\overline{z_2}}$$
$$= \frac{z_1 . \overline{z_1}}{z_2 . \overline{z_2}}$$
$$= \frac{|z_1|^2}{|z_2|^2}$$

Jadi, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ ∎

Ketaksamaan Segitiga

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Makassar, 28 Februari 2021

$\forall\, z_1, z_2 \in \mathbb{C}$ berlaku

(1) $|z_1 + z_2| \leq |z_1| + |z_2|$

(2) $|z_1 + z_2| \geq |z_1| - |z_2|$

(3) $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$ bkanya di bulu hrsnya $|z_1 + z_2| \geq -(|z_1| - |z_2|)$

(4) $|z_1 - z_2| \leq |z_1| + |z_2|$

(5) $|z_1 - z_2| \geq |z_1| - |z_2|$ bkannya dibuku hrsnya $|z_1 - z_2| \leq |z_1| + |z_2|$

(6) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

Bukti :

(1) Adb. $|z_1 + z_2| \leq |z_1| + |z_2|$

Perhatikan bahwa

$$
\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} && [\, z^2 = z \cdot \bar{z} \,] \\
&= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\
&= z_1 \cdot \bar{z}_1 + z_1 \cdot \bar{z}_2 + z_2 \cdot \bar{z}_1 + z_2 \cdot \bar{z}_2 \\
&= |z_1|^2 + z_1 \cdot \bar{z}_2 + z_2 \cdot \bar{z}_1 + |z_2|^2 && [\, |z|^2 = z \cdot \bar{z} \,] \\
&\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2 && [\, (a+b)^2 = a^2 + 2ab + b^2 \,]
\end{aligned}
$$

maka diperoleh

$\therefore$ Berdasarkan teorema di analisis real, $a^2 \leq b^2,\ a, b \geq 0 \Rightarrow a < b$ maka diperoleh $|z_1 + z_2|^2 \leq |z_1| + |z_2|$

Perhatikan

$$
\begin{aligned}
z_1 \bar{z}_2 + z_2 \bar{z}_1 &= z_1 \bar{z}_2 + \overline{\bar{z}_2 z_1} \\
&= z_1 \bar{z}_2 + \overline{z_1 \bar{z}_2} \\
&= 2\,\mathrm{Re}(z_1 \bar{z}_2) \\
&\leq 2\,|z_1 \cdot \bar{z}_2| \\
&= 2\,|z_1||\bar{z}_2| \\
&= 2\,|z_1|\,|z_2|
\end{aligned}
$$

$\overline{\bar{z}_2 \cdot z_1} = \bar{\bar{z}}_2 \cdot \bar{z}_1 = z_2 \cdot \bar{z}_1$

Tulis kembali dari sifat ini

$\boxed{z + \bar{z} = 2\,\mathrm{Re}(z)}$

$z = z_1 \bar{z}_2$ ganti menjadi

$\boxed{z_1 + \bar{z}_1 = 2\,\mathrm{Re}(z_1 \bar{z}_2)}$

sifat modulus No 3

$\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z|$

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(2) Adb. $|z_1 + z_2| \geq |z_1| - |z_2|$

Bukti :

Ambil $z_1, z_2 \in \mathbb{C}$ sebarang.

Perhatikan bahwa

$$
\begin{aligned}
|z_1| &= |z_1 + z_2 - z_2| \\
&= |(z_1 + z_2) + (-z_2)| \\
&\leq |z_1 + z_2| + |-z_2| \\
&= |z_1 + z_2| + |z_2| \quad [\text{Sifat Modulus No. 1} ; |z| = |-z|]
\end{aligned}
$$

diperoleh

$$|z_1| \leq |z_1 + z_2| + |z_2|$$

$$\Rightarrow |z_1 + z_2| \geq |z_1| - |z_2| \quad \blacksquare$$

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(3) Adb. $|z_1+z_2| \geq ||z_1|-|z_2||$

Ambil $z_1, z_2 \in \mathbb{C}$ sebarang

$$|z_2| = |z_2+z_1-z_1|$$
$$= |(z_2+z_1)+(-z_1)|$$
$$\leq |z_2+z_1| + |-z_1| \quad \text{[Ketaksamaan segitiga (1)]}$$

diperoleh,

$$|z_2| \leq |z_2+z_1| + |z_1|$$
$$|z_2|-|z_1| \leq |z_1+z_2| \quad \text{[Kedua ruas ditambah } -|z_1|\text{]}$$
$$-|z_2|+|z_1| \geq -|z_1+z_2| \quad \text{[Kedua ruas dikali } -1\text{]}$$
$$|z_1|-|z_2| \geq -|z_1+z_2| \quad \ldots\ldots (*)$$

Selanjutnya, perhatikan bahwa

$$|z_1| = |z_1+z_2-z_2|$$
$$= |(z_1+z_2)+(-z_2)|$$
$$\leq |z_1+z_2| + |-z_2|$$

diperoleh,

$$|z_1| \leq |z_1+z_2| + |z_2|$$
$$|z_1|-|z_2| \leq |z_1+z_2| \quad \text{[Kedua ruas ditambah } -|z_2|\text{]}$$
$$-|z_1|+|z_2| \geq -|z_1+z_2| \quad \text{[Kedua ruas dikali } -1\text{]}$$
$$|z_2|-|z_1| \geq -|z_1+z_2| \quad \ldots\ldots (**)$$

$\therefore$ Dari persamaan (*) dan (**) diperoleh

$$-|z_1+z_2| \leq |z_1|-|z_2| \leq |z_1+z_2|$$

Maka,

$$||z_1|-|z_2|| \leq ||z_1+z_2||$$
$$||z_1|-|z_2|| \leq |z_1+z_2|$$

atau

$$|z_1+z_2| \geq ||z_1|-|z_2|| \quad \blacksquare$$

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(4) Adb. $|z_1 - z_2| \leq |z_1| + |z_2|$

Ambil $z_1, z_2 \in \mathbb{C}$ sebarang

Perhatikan bahwa,

$$
\begin{aligned}
|z_1| &= |z_1 - z_2 + z_2| \\
&= |(z_1 - z_2) + z_2| \\
&\geq |z_1 - z_2| - |z_2| \quad \text{[Ketaksamaan segitiga (2)]}
\end{aligned}
$$

diperoleh,

$$
\begin{aligned}
|z_1| &\geq |z_1 - z_2| - |z_2| \\
|z_1| + |z_2| &\geq |z_1 - z_2| \quad \text{[Kedua ruas ditambah } |z_2|\text{]} \\
|z_1 - z_2| &\leq |z_1 + z_2|
\end{aligned}
$$

$\therefore |z_1 - z_2| \leq |z_1| + |z_2|$

(5) Adb. $|z_1 - z_2| \geq |z_1| - |z_2|$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$ sebarang

Perhatikan bahwa,

$$
\begin{aligned}
|z_1| &= |z_1 - z_2 + z_2| \\
&= |(z_1 - z_2) + z_2| \\
&\leq |z_1 - z_2| + |z_2| \quad \text{[Ketaksamaan segitiga (1)]}
\end{aligned}
$$

diperoleh,

$$
\begin{aligned}
|z_1| &\leq |z_1 - z_2| + |z_2| \\
|z_1| - |z_2| &\leq |z_1 - z_2| \quad \text{[Kedua ruas ditambah } -|z_2|\text{]} \\
|z_1 - z_2| &\geq |z_1| - |z_2|
\end{aligned}
$$

$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$

$\blacksquare$

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(6) Adb. $|z_1 - z_2| \geq ||z_1| - |z_2||$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Perhatikan bahwa,

$$
\begin{aligned}
|z_2| &= |z_2 - z_1 + z_1| \\
&= |(z_2 - z_1) + z_1| \\
&\leq |z_2 - z_1| + |z_1| \qquad \text{[Ketaksamaan segitiga (1)]} \\
&= |z_1 - z_2| + |z_1|
\end{aligned}
$$

diperoleh,

$$
\begin{aligned}
|z_2| &\leq |z_1 - z_2| + |z_1| \\
|z_2| - |z_1| &\leq |z_1 - z_2| \qquad \text{[Kedua ruas ditambah } -|z_1|\text{]} \\
-|z_2| + |z_1| &\geq -|z_1 - z_2| \qquad \text{[Kedua ruas dikali } -1\text{]} \\
-|z_1 - z_2| &\leq -|z_2| + |z_1| \\
-|z_1 - z_2| &\leq |z_1| - |z_2| \quad \ldots\ldots (*)
\end{aligned}
$$

Selanjutnya, perhatikan bahwa

$$
\begin{aligned}
|z_1| &= |z_1 - z_2 + z_2| \\
&= |(z_1 - z_2) + z_2| \\
&\leq |z_1 - z_2| + |z_2| \qquad \text{[Ketaksamaan segitiga (1)]}
\end{aligned}
$$

diperoleh

$$
\begin{aligned}
|z_1| &\leq |z_1 - z_2| + |z_2| \\
|z_1| - |z_2| &\leq |z_1 - z_2| \qquad \text{[Kedua ruas ditambah } -|z_2|\text{]} \\
|z_1 - z_2| &\geq |z_1| - |z_2| \quad \ldots\ldots (**)
\end{aligned}
$$

$\therefore$ Dari persamaan $(*)$ dan $(**)$ diperoleh

$$
\begin{aligned}
-|z_1 - z_2| \leq |z_1| &- |z_2| \leq |z_1 - z_2| \\
||z_1| - |z_2|| &\leq |z_1 - z_2| \\
||z_1 - z_2|| &\geq ||z_1| - |z_2|| \\
|z_1 - z_2| &\geq ||z_1| - |z_2||
\end{aligned}
$$

$\blacksquare$