Makassar, 5 Oktober 2020

Teori Modul / Pertemuan ke-7 / Topik Diskusi

Nama : Imanuel AS

NIM : 181141008

D1

1. $M_3(F) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \middle| a,b,c,d \in F \right\} \leftarrow$ Ring

$F^3 = \{ (x \;\; y \;\; z) \mid x, y, z \in F \} \leftarrow M$

didefinisikan

$F^3 \times M_3(F) \longrightarrow F^3$

$\left[(x \;\; y \;\; z), \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right] \longmapsto \big((x.a + y.d + z.g) \;\; (x.b + y.e + z.h) \;\; (x.c + y.f + z.i) \big)$

Buktikan F^3 modul kanan atas $M_3(F)$

Penyelesaian :

Ambil $A, B, C \in F^3$, $\alpha \in M_3(F)$ sebarang

Tulis

$A = (x_1 \;\; y_1 \;\; z_1)$ u/suatu $x_1, y_1, z_1 \in F$

$B = (x_2 \;\; y_2 \;\; z_2)$ u/suatu $x_2, y_2, z_2 \in F$

$C = (x_3 \;\; y_3 \;\; z_3)$ u/suatu $x_3, y_3, z_3 \in F$

$\alpha = \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix}$ u/suatu $a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, i_1 \in F$

$\beta = \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix}$ u/suatu $a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2, i_2 \in F$

Imanuel AS / 1811141008 [signature]

Makassar, 5 oktober 2020

(1) Adb. $(F^3, +)$ Grup Abelian

(a) Adb. $\forall A, B \in F^3 \Rightarrow A+B \in F^3$

Note that,

$$A+B = (x_1\ y_1\ z_1) + (x_2\ y_2\ z_2)$$
$$= (x_1+x_2\ \ y_1+y_2\ \ z_1+z_2) \in F^3$$

(b) Adb. $\forall A, B, C \in F^3 \Rightarrow A+(B+C) = (A+B)+C$

Note that,

$$A+(B+C) = (x_1\ y_1\ z_1) + [(x_2\ y_2\ z_2) + (x_3\ y_3\ z_3)]$$
$$= (x_1\ y_1\ z_1) + (x_2+x_3\ \ y_2+y_3\ \ z_2+z_3)$$
$$= (x_1+(x_2+x_3)\ \ y_1+(y_2+y_3)\ \ z_1+(z_2+z_3))$$
$$= ((x_1+x_2)+x_3\ \ (y_1+y_2)+y_3\ \ (z_1+z_2)+z_3)$$
$$= (x_1+x_2\ \ y_1+y_2\ \ z_1+z_2) + (x_3\ y_3\ z_3)$$
$$= [(x_1\ y_1\ z_1) + (x_2\ y_2\ z_2)] + (x_3\ y_3\ z_3)$$
$$= (A+B) + C$$

Dipindai dengan CamScanner

Imanuel AS/1811141008

Makassar, 5 Oktober

(c) Adb. $\exists\, 0_{F^3} \in F^3,\ \forall A \in F^3 \ni 0_{F^3} + A = A + 0_{F^3} = A$

Terdapat $0_{F^3} = (0_F \ \ 0_F \ \ 0_F) \in F^3$ sehingga untuk setiap

$A = (x_1 \ \ y_1 \ \ z_1) \in F^3$ u/suatu $x_1, y_1, z_1 \in F$ berlaku

$$\begin{aligned} 0_{F^3} + A &= (0_F \ \ 0_F \ \ 0_F) + (x_1 \ \ y_1 \ \ z_1) \\ &= (0_F + x_1 \ \ 0_F + y_1 \ \ 0_F + z_1) \\ &= (x_1 \ \ y_1 \ \ z_1) \\ &= A \text{ .... } (*) \end{aligned}$$

dan

$$\begin{aligned} A + 0_{F^3} &= (x_1 \ \ y_1 \ \ z_1) + (0_F \ \ 0_F \ \ 0_F) \\ &= (x_1 + 0_F \ \ y_1 + 0_F \ \ z_1 + 0_F) \\ &= (x_1 \ \ y_1 \ \ z_1) \\ &= A \text{ .... } (**) \end{aligned}$$

Karena $(*) = (**)$ maka $0_{F^3} + A = A + 0_{F^3} = A$

Imanuel AS / 1811141008 ~~Matematika~~ Makassar, 5 Oktober 2020

(d) Adb. $\forall A \in F^3$, $\exists -A \in F^3 \ni A+(-A) = (-A)+A = 0_{F^3}$

untuk setiap $A = (x_1 \quad y_1 \quad z_1) \in F^3$

Pilih $-A = (-x_1 \quad -y_1 \quad -z_1) \in F^3$

sehingga

$$\begin{aligned} A+(-A) &= (x_1 \quad y_1 \quad z_1) + (-x_1 \quad -y_1 \quad -z_1) \\ &= (x_1+(-x_1) \quad y_1+(-y_1) \quad z_1+(-z_1)) \\ &= (0_F \quad 0_F \quad 0_F) \\ &= 0_{F^3} \quad \ldots\ldots\ldots\ldots\ldots (*) \end{aligned}$$

$$\begin{aligned} (-A)+A &= (-x_1 \quad -y_1 \quad -z_1) + (x_1 \quad y_1 \quad z_1) \\ &= (-x_1+x_1 \quad -y_1+y_1 \quad -z_1+z_1) \\ &= (0_F \quad 0_F \quad 0_F) \\ &= 0_{F^3} \quad \ldots\ldots\ldots\ldots\ldots (**) \end{aligned}$$

Karena $(*) = (**)$ maka $A+(-A) = (-A)+A = 0_{F^3}$

Imanuel AS / 1811141008 OFFC MABS

Makassar, 5 oktober 20..

(e) Adb. $\forall A, B \in F^3 \Rightarrow A+B = B+A$

Note that,

$$A+B = (x_1 \quad y_1 \quad z_1) + (x_2 \quad y_2 \quad z_2)$$
$$= (x_1+x_2 \quad y_1+y_2 \quad z_1+z_2)$$
$$= (x_2+x_1 \quad y_2+y_1 \quad z_2+z_1)$$
$$= (x_2 \quad y_2 \quad z_2) + (x_1 \quad y_1 \quad z_1)$$
$$= B + A$$

$\therefore (F^3, +)$ Grup Abelian

(2) Terhadap operasi penggandaan skalar $\bullet$ memenuhi keempat aksioma, yakni :

(a) Adb. $A \bullet \alpha \in F^3$; $\forall A \in F^3, \alpha \in M_3(F)$

Note that,

$$A \bullet \alpha = (x_1 \quad y_1 \quad z_1) \bullet \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix}$$
$$= (x_1a_1 + y_1d_1 + z_1g_1 \quad x_1b_1 + y_1e_1 + z_1h_1 \quad x_1c_1 + y_1f_1 + z_1i_1) \in F^3$$

(b) Adb. $(A+B)\alpha = (A \cdot \alpha) + (B \cdot \alpha)$; $\forall A, B \in F^3, \alpha \in M_3(F)$

Note that,

$$(A+B)\alpha = \left[(x_1 \ y_1 \ z_1) + (x_2 \ y_2 \ z_2)\right] \bullet \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix}$$
$$= (x_1+x_2 \quad y_1+y_2 \quad z_1+z_2) \bullet \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix}$$
$$= \big((x_1+x_2)a_1 + (y_1+y_2)d_1 + (z_1+z_2)g_1 \quad (x_1+x_2)b_1 + (y_1+y_2)e_1 + (z_1+z_2)h_1 \quad (x_1+x_2)c_1 + (y_1+y_2)f_1 + (z_1+z_2)\ldots$$
$$= \big(x_1a_1 + x_2a_1 + y_1d_1 + y_2d_1 + z_1g_1 + z_2g_1 \quad x_1b_1 + x_2b_1 + y_1e_1 + y_2e_1 + z_1h_1 + z_2h_1 \quad x_1c_1 + x_2c_1 + y_1f_1 + y_2f_1 + z_1i_1 + \ldots$$
$$= (x_1a_1 + y_1d_1 + z_1g_1 \quad x_1b_1 + y_1e_1 + z_1h_1 \quad x_1c_1 + y_1f_1 + z_1i_1) +$$
$$(x_2a_1 + y_2d_1 + z_2g_1 \quad x_2b_1 + y_2e_1 + z_2h_1 \quad x_2c_1 + y_2f_1 + z_2i_1)$$
$$= (x_1 \ y_1 \ z_1) \bullet \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} + (x_2 \ y_2 \ z_2) \bullet \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix}$$
$$= A \bullet \alpha + B \bullet \alpha$$

Irwansyah AS / 1811141008

Makassar, 5 Oktober 2020

(c). $A\cdot(\alpha+\beta) = (A\cdot\alpha) + (A\cdot\beta)$; $\forall A \in F^3, \alpha, \beta \in M_3(F)$

Note that,

$$A\cdot(\alpha+\beta) = (x_1 \; y_1 \; z_1)\cdot\left[\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix}\right]$$

$$= (x_1 \; y_1 \; z_1)\cdot\begin{pmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & e_1+e_2 & f_1+f_2 \\ g_1+g_2 & h_1+h_2 & i_1+i_2 \end{pmatrix}$$

$$= \big(x_1\cdot(a_1+a_2)+y_1(d_1+d_2)+z_1(g_1+g_2) \quad x_1\cdot(b_1+b_2)+y_1(e_1+e_2)+z_1(h_1+h_2) \quad x_1(c_1+c_2)+y_1(f_1+f_2)+z_1(i_1+i_2)\big)$$

$$= \big(x_1a_1+x_1a_2+y_1d_1+y_1d_2+z_1g_1+z_1g_2 \quad x_1b_1+x_1b_2+y_1e_1+y_1e_2+z_1h_1+z_1h_2 \quad x_1c_1+x_1c_2+y_1f_1+y_1f_2+z_1i_1+z_1i_2\big)$$

$$= \big(x_1a_1+y_1d_1+z_1g_1 \quad x_1b_1+y_1e_1+z_1h_1 \quad x_1c_1+y_1f_1+z_1i_1\big) +$$
$$\big(x_1a_2+y_1d_2+z_1g_2 \quad x_1b_2+y_1e_2+z_1h_2 \quad x_1c_2+y_1f_2+z_1i_2\big)$$

$$= (x_1 \; y_1 \; z_1)\cdot\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} + (x_1 \; y_1 \; z_1)\cdot\begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix}$$

$$= A\cdot\alpha + A\cdot\beta$$

Imanuel AS/ 181114008

Makassar, 5 oktober 20

(d) Adb. $A \cdot (\alpha \cdot \beta) = (A \cdot \alpha) \cdot \beta \quad ; \forall A \in F^3, \alpha, \beta \in M_3(F)$

Note that,

$$A \cdot (\alpha \cdot \beta) = (x_1 \;\; y_1 \;\; z_1) \cdot \left[\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} \right]$$

$$= (x_1 \;\; y_1 \;\; z_1) \cdot \begin{pmatrix} a_1a_2 + b_1d_2 + c_1g_2 & a_1b_2 + b_1e_2 + c_1h_2 & a_1c_2 + b_1f_2 + c_1i_2 \\ d_1a_2 + e_1d_2 + f_1g_2 & d_1b_2 + e_1e_2 + f_1h_2 & d_1c_2 + e_1f_2 + f_1i_2 \\ g_1a_2 + h_1d_2 + i_1g_2 & g_1b_2 + h_1e_2 + i_1h_2 & g_1c_2 + h_1f_2 + i_1i_2 \end{pmatrix}$$

$$= \Big([x_1(a_1a_2 + b_1d_2 + c_1g_2) + y_1(d_1a_2 + e_1d_2 + f_1g_2) + z_1(g_1a_2 + h_1d_2 + i_1g_2)] \;\; [x_1(a_1b_2 + b_1e_2 + c_1h_2) + y_1(d_1b_2 + e_1e_2 + f_1h_2) + z_1(g_1b_2 + h_1e_2 + i_1h_2)] \;\; [x_1(a_1c_2 + b_1f_2 + c_1i_2) + y_1(d_1c_2 + e_1f_2 + f_1i_2) + z_1(g_1c_2 + h_1f_2 + i_1i_2)]$$

$$= \Big[\big(x_1(a_1a_2) + y_1(d_1a_2) + z_1(g_1a_2)\big) + \big(x_1(b_1d_2) + y_1(e_1d_2) + z_1(h_1d_2)\big) + \big(x_1(c_1g_2) + y_1(f_1g_2) + z_1(i_1g_2)\big) \Big] +$$

$$\Big[\big(x_1(a_1b_2) + y_1(d_1b_2) + z_1(g_1b_2)\big) + \big(x_1(b_1e_2) + y_1(e_1e_2) + z_1(h_1e_2)\big) + \big(x_1(c_1h_2) + y_1(f_1h_2) + z_1(i_1h_2)\big) \Big] +$$

$$\Big[\big(x_1(a_1c_2) + y_1(d_1c_2) + z_1(g_1c_2)\big) + \big(x_1(b_1f_2) + y_1(e_1f_2) + z_1(h_1f_2)\big) + \big(x_1(c_1i_2) + y_1(f_1i_2) + z_1(i_1i_2)\big) \Big]$$

$$= \Big([x_1a_1 + y_1d_1 + z_1g_1] \;\; [x_1b_1 + y_1e_1 + z_1h_1] \;\; [x_1c_1 + y_1f_1 + z_1i_1] \Big) \cdot \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix}$$

$$= \left[(x_1 \;\; y_1 \;\; z_1) \cdot \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} \right] \cdot \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} = (A \cdot \alpha) \cdot \beta$$

Imanuel AS / 1811141008 Ayu Maull

Makassar, 5 Oktober 2020

Kelupaan :

Adb. $F^3 \neq \phi$

Misal $A = (x_1 \quad y_1 \quad z_1)$ untuk suatu $x_1, y_1, z_1 \in F$

Jelas bahwa $A \in F^3$ tidak kosong, karena terdapat $x_1, y_1, z_1 \in F$ anggota A.

$\therefore$ Karena $F^3 \neq \phi$, dan $M_3(F)$ Ring dengan operasi penggandaan skalar $\cdot$ didefinisikan, memenuhi:

(1) $(F^3, +)$ Grup Abelian

(2) terhadap operasi penggandaan skalar $\cdot$ memenuhi keempat aksioma.

Maka $F^3 = \{(x \quad y \quad z) \mid x, y, z \in F\}$ disebut modul kanan atas $M_3(F)$ $\blacksquare$ (Terbukti)