PERTEMUAN I

Analisis Kompleks

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Makassar, 11 Februari 2021

Complex Variables
and Applications

Bilangan Kompleks

- Real
  - Irrasional
  - Rasional
    - Bulat
      - Bulat negatif
      - Cacah
        - Nol
        - Asli
          - Satu
          - Prima
          - Komposit
    - Pecahan
- Imaginer

Bilangan kompleks

$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$

didefinisikan $\sqrt{-1} = i$, $i^2 = -1$

$3 \in \mathbb{C} \rightarrow 3 = 3 + 0i$

$0 \in \mathbb{C} \rightarrow 0 = 0 + 0i$

$z = a + bi \in \mathbb{C}$ → $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$

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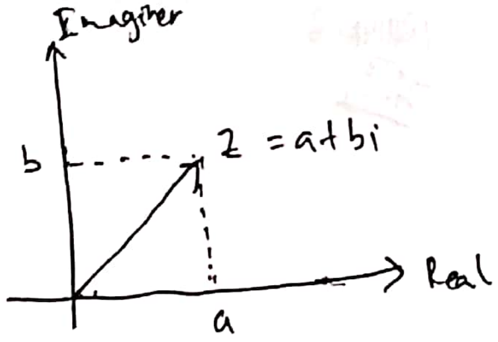

$z = a + bi$

$\hookrightarrow z = (a, b)$

komp. real

komp. imaginer

Operasi Bilangan Kompleks

Misal $z_1 = a + bi \in \mathbb{C} \xrightarrow{\text{dapat ditulis}} z_1 = (a, b)$ → disebut pasangan terurut

$z_2 = c + di \in \mathbb{C} \xrightarrow{\text{dapat ditulis}} z_2 = (c, d)$ → disebut pasangan terurut

(1) Penjumlahan / pengurangan

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

saat dipandang sebagai pasangan terurut

$$z_1 + z_2 = (a, b) + (c, d) = (a + c, b + d)$$

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

$$z_1 - z_2 = (a, b) - (c, d) = (a - c, b - d)$$

$\boxed{E}$

$z_1 = 3 + 2i$, $z_2 = 5 + 7i$ → $z_1 + z_2 = 8 + 9i$, $z_1 - z_2 = -2 - 5i$

(2) Perkalian

$$z_1 z_2 = (a + bi) \cdot (c + di) = ac + (ad)i + (bc)i - bd = (ac - bd) + (ad + bc)i$$

$$z_1 z_2 = (a, b)(c, d) = (ac - bd, ad + bc)$$

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Mantap

(3.) Pembagian

$z_1 = a + bi$

$z_2 = c + di \neq 0$

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$

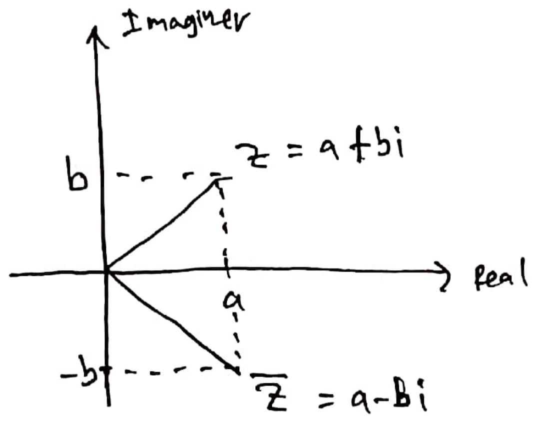
$$= \frac{ac - (ad)i + (bc)i + bd}{c^2 + d^2}$$

$$= \frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

Konjugat / Sekawan

konjugat bilangan kompleks $z = a + bi$ ditulis $\bar{z} = a - bi$



PERTEMUAN II

Analisis Kompleks

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Analisis Kompleks / Syam - ok / Pertemuan II Imanuel AS 1811141008 Makassar, 18 Februari 2021

Pendahuluan

Bilangan kompleks dan satuan imajinernya seperti apa yaa... mm

$$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$$

Ingat bahwa bilangan kompleks ini, bisa kita pandang sebagai pasangan terurut. Seperti berikut ini,

$$z = a + bi \in \mathbb{C} \xrightarrow{\text{pasangan terurut}} z = (a, b)$$

(part) real (part) Imaginer

Perhatikan, misalkan kita punya

$$z = (a, b) = \underbrace{(a,0)}_{a} + \underbrace{(b,0)}_{b} \underbrace{(0,1)}_{i}$$

$$i = (0,1) \quad , \quad i = \sqrt{-1} \longrightarrow i^2 = -1$$

$$\begin{aligned} i^2 = i \cdot i &= (0,1)(0,1) \\ &= (0.0 - 1.1 \,,\, 0.1 + 1.0) \qquad \text{[Definisi Perkalian Bil. komp]} \\ &= (-1, 0) \\ &= -1 \end{aligned}$$

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Sifat - Sifat Pada Bilangan Kompleks

Bil. Real — berlaku
- sifat Aljabar
- sifat Urutan
- sifat Kelengkapan

Di bilangan kompleks, ternyata sifat aljabar juga berlaku.

Sifat Aljabar Bilangan Kompleks

- Penjumlahan (+)

(1) Tertutup

$$\forall z_1, z_2 \in \mathbb{C} \Rightarrow z_1 + z_2 \in \mathbb{C}$$

(2) Asosiatif

$$\forall z_1, z_2, z_3 \in \mathbb{C} \Rightarrow z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

(3) Unsur Identitas / Unsur nol

Terdapat $0 \in \mathbb{C}$ sehingga untuk setiap $z \in \mathbb{C}$ berlaku

$$0 + z = z + 0 = z$$

(4) Invers

Untuk setiap $z \in \mathbb{C}$ terdapat $-z \in \mathbb{C}$ sehingga

$$z + (-z) = (-z) + z = 0$$

catatan

$0 = 0 + 0i$, $z = a + bi$, $a, b \in \mathbb{R}$

(5) Komutatif

$\forall z_1, z_2 \in \mathbb{C}$ berlaku

$$z_1 + z_2 = z_2 + z_1$$

catatan

$0 + 0i$, $z = a + bi$, $a, b \in \mathbb{R}$

lebih jauh, kita punya fakta bahwa $(\mathbb{C}, +)$ adalah Grup Abelian.

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4## Imanuel

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- Perkalian (x)

(1) Tertutup

$\forall z_1, z_2 \in \mathbb{C} \Rightarrow z_1 . z_2 \in \mathbb{C}$

(2) Assosiatif

$\forall z_1, z_2, z_3 \in \mathbb{C}$ berlaku $z_1(z_2 z_3) = (z_1 z_2) z_3$

(3) Unsur Identitas / kesatuan

Terdapat $1 \in \mathbb{C}$ sehingga untuk setiap $z \in \mathbb{C}$ berlaku

$$z . 1 = 1 . z = z$$

Catatan : $1 = 1 + 0i$

(4) Unsur Invers

$\forall z \in \mathbb{C}, z \neq 0$, terdapat $\frac{1}{z} \in \mathbb{C}$ sehingga

$$z\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right) z = 1$$

Catatan :
$$z^{-1} = \frac{1}{z} = \frac{1}{a+bi}$$
$$= \frac{1}{a+bi} \times \frac{a-bi}{a-bi}$$
$$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} . i$$

(5) Komutatif

$\forall z_1, z_2 \in \mathbb{C} \Rightarrow z_1 z_2 = z_2 z_1$

lebih jauh, $(\mathbb{C} \setminus \{0\}, \times) \to$ Grup Abel

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$\boxed{N}$

$z_1, z_2 \in \mathbb{C}$

(1) $z_1 - z_2 = z_1 + (-z_2)$

(2) $\frac{z_1}{z_2} = z_1 \cdot \left(\frac{1}{z_2}\right), \; z_2 \neq 0$

dimana $1 = 1 + 0i$

dan $0 = 0 + 0i$

$\boxed{E}$

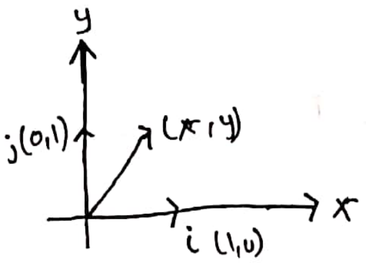
$z = 2 + 3i \rightarrow z^{-1} = \dots$

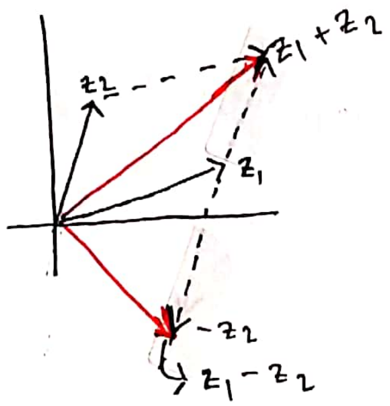
$$z^{-1} = \frac{2}{(2)^2 + (3)^2} - \frac{3}{(2)^2 + (3)^2} \cdot i$$

$$= \frac{4}{13} - \frac{3}{13} i$$

$z \cdot z^{-1} = 1$

Tapsiran Geometri





$\left.\begin{aligned} z &= a + bi \\ z &= x + yi \end{aligned}\right\}$ Basis $\{1, i\}$

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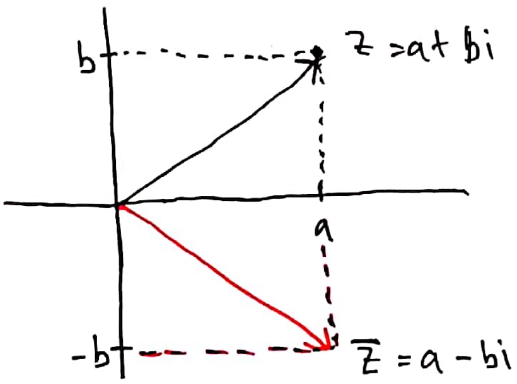
Konjugat Bilangan Kompleks

Misal $z = a+bi \in \mathbb{C}$, konjugat dari z ditulis $\bar{z} = a - bi$



Sifat konjugat bilangan kompleks

$\forall z_1, z_2 \in \mathbb{C}$ berlaku

(1) $\bar{\bar{z}} = z$

(2) $z + \bar{z} = 2 \, Re(z)$

(3) $z - \bar{z} = 2i \, Im(z)$

(4) $z\bar{z} = (Re(z))^2 + (Im(z))^2$

(5) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(6) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(7) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(8) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

(9) $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2 \, Re(z_1 \bar{z}_2)$

$\boxed{N}$ $z = a+bi \rightarrow Re(z) = a$ dan $Im(z) = b$

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BUKTI

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① Adb. $\overline{\overline{z}} = z$

Ambil $z \in \mathbb{C}$ sebarang, tulis

$z = a + bi$ untuk suatu $a, b \in \mathbb{R}$

$\overline{z} = a - bi$. Jadi

$\overline{\overline{z}} = \overline{a - bi} = a - (-b)i$

$= a + bi = z$ //

③ Adb. $z - \overline{z} = 2i \ \mathrm{Im}(z)$

Ambil $z \in \mathbb{C}$ sebarang,

Tulis, $z = a + bi$ u/ suatu $a, b \in \mathbb{R}$

Perhatikan bahwa,

$z - \overline{z} = (a + bi) - \overline{(a + bi)}$

$= a + bi - (a - bi)$

$= 2bi$

$= 2i \ \mathrm{Im}(z)$ //

PR 1 Buktikan sifat konjugat nomor 1/ sd 9.

② Adb. $z + \overline{z} = 2 \ \mathrm{Re}(z)$

Ambil $z \in \mathbb{C}$ sebarang,

Tulis, $z = a + bi$ u/suatu $a, b \in \mathbb{R}$

Note that,

$z + \overline{z} = (a + bi) + \overline{(a + bi)}$

$= (a + bi) + (a - bi)$

$= (a + a) + (b - b)i$

$= 2a + (0)i$

$= 2a$

$= 2 \ \mathrm{Re}(z)$

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(4) Adb. $z\bar{z} = (Re(z))^2 + (Im(z))^2$

Ambil sebarang $z \in \mathbb{C}$

Tulis, $z = a+bi$ u/ suatu $a,b \in \mathbb{R}$

Note that,

$$
\begin{aligned}
z\bar{z} &= (a+bi) \cdot \overline{(a+bi)} \\
&= (a+bi) \cdot (a-bi) \\
&= (a.a) - (ab)i + (ba)i + b^2 \\
&= a^2 + 0 + b^2 \\
&= (Re(z))^2 + (Im(z))^2 \quad _\#
\end{aligned}
$$

(5) Adb. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Tulis, $z_1 = a+bi$ u/ suatu $a,b \in \mathbb{R}$ dan $z_2 = c+di$ u/ suatu $c,d \in \mathbb{R}$

Note that,

$$
\begin{aligned}
\overline{z_1 + z_2} &= \overline{(a+bi) + (c+di)} \\
&= \overline{(a+c) + (b+d)i} \\
&= (a+c) - (b+d)i \\
&= (a-bi) + (c-di) \\
&= \overline{z_1} + \overline{z_2}
\end{aligned}
$$

(6) Adb. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Tulis, $z_1 = a+bi$ u/ suatu $a,b \in \mathbb{R}$

dan $z_2 = c+di$ u/ suatu $c,d \in \mathbb{R}$

Note that

$$
\begin{aligned}
\overline{z_1 - z_2} &= \overline{(a+bi) - (c+di)} \\
&= \overline{(a-c) + (b-d)i} \\
&= (a-c) - (b-d)i \\
&= (a-bi) - (c-di) \\
&= \overline{z_1} - \overline{z_2}
\end{aligned}
$$

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(7) Adb. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Tulis, $z_1 = a + bi$ u/ suatu $a, b \in \mathbb{R}$

dan $z_2 = c + di$ u/ suatu $c, d \in \mathbb{R}$

Note that

$$\begin{aligned}
\overline{z_1 \cdot z_2} &= \overline{(a+bi)\cdot(c+di)} \\
&= \overline{ac + (ad)i + (bc)i - bd} \\
&= \overline{(ac-bd) + (ad+bc)i} \\
&= (ac-bd) - (ad+bc)i \\
&= ac - (ad)i - (bc)i - bd \\
&= (a-bi)\cdot(c-di) \\
&= \overline{z_1} \cdot \overline{z_2} \quad //
\end{aligned}$$

(8) Adb. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$

Ambil $z_1, z_2 \in \mathbb{C}$ sebarang, dengan $z_2 \neq 0$

Tulis, $z_1 = a + bi$ u/ suatu $a, b \in \mathbb{R}$

dan $z_2 = c + di$ u/ suatu $c, d \in \mathbb{R}$

Note that,

$$\begin{aligned}
\overline{\left(\frac{z_1}{z_2}\right)} &= \overline{\left(\frac{a+bi}{c+di}\right)} \\
&= \overline{\left(\frac{a+bi}{c+di} \times \frac{c-di}{c-di}\right)} \\
&= \overline{\left(\frac{ac - (ad)i + (bc)i + bd}{c^2 + d^2}\right)} \\
&= \overline{\left(\left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i\right)} \\
&= \left(\frac{ac+bd}{c^2+d^2}\right) - \left(\frac{bc-ad}{c^2+d^2}\right)i
\end{aligned}$$

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$$\hookrightarrow = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{ad-bc}{c^2+d^2}\right)i$$

$$= \frac{ac+(ad)i-(bc)i+bd}{c^2+d^2}$$

$$= \frac{a-bi}{c-di} \times \frac{c+di}{c+di}$$

$$= \frac{a-bi}{c-di}$$

$$= \frac{\overline{z_1}}{\overline{z_2}} \quad //$$

(9) Adb. $z_1 \cdot \overline{z_2} + \overline{z_1} \cdot z_2 = 2\,\mathrm{Re}(z_1 \overline{z_2})$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Tulis, $z_1 = a+bi$ u/ suatu $a,b \in \mathbb{R}$

dan $z_2 = c+di$ u/ suatu $c,d \in \mathbb{R}$

Note that,

$$\begin{aligned}
z_1 \cdot \overline{z_2} + \overline{z_1} \cdot z_2 &= (a+bi \cdot \overline{c+di}) + (\overline{a+bi} \cdot c+di) \\
&= (a+bi \cdot c-di) + (a-bi \cdot c+di) \\
&= (ac-(ad)i+(bc)i+bd) + (ac+(ad)i-(bc)i+bd) \\
&= [(ac+bd)-(ad-bc)i] + [(ac+bd)+(ad-bc)i] \\
&= ((ac+bd)+(ac+bd)) - ((ad-bc)-(ad-bc))i \\
&= 2(ac+bd) - (0)i \\
&= 2(ac+bd) \\
&= 2\,\mathrm{Re}(z_1 \cdot \overline{z_2}) \quad //
\end{aligned}$$

PERTEMUAN III

Analisis Kompleks

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Analisis Kompleks / Catatan pertemuan ke-3 Imanuel AS 1811141008 Makassar, 26 Februari 2021

Modulus Bilangan Kompleks

Misal $z = a + bi \in \mathbb{C}$, modulus dari z ditulis $|z| = \sqrt{a^2 + b^2}$

$$z = a + bi \begin{cases} Re(z) = a \\ Im(z) = b \end{cases}$$

$$\downarrow$$

$$a, b \in \mathbb{R}$$

$\Rightarrow |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Misal $z_1, z_2 \in \mathbb{C}$, maka jarak z_1 ke z_2 adalah $|z_1 - z_2|$.

Real $\xrightarrow{\text{sifat}}$ Aljabar

Real $\xrightarrow{\text{sifat}}$ Urutan

Real $\xrightarrow{\text{sifat}}$ Kelengkapan

Kompleks $\xrightarrow{\text{sifat}}$ Aljabar ✓

Kompleks $\xrightarrow{\text{sifat}}$ Urutan ✗ (Tidak berlaku)

$3, 2 \in \mathbb{R} \rightarrow 2 < 3$

$2i, 3i \in \mathbb{C} \rightarrow 2i < 3i$ ✗

$3i < 2i$ ✗

Sifat Modulus

$\forall z, z_1, z_2 \in \mathbb{C}$ berlaku

(1) $|z| = |\bar{z}| = |-z|$

✓ (2) $|z|^2 = z\bar{z} = (Re(z))^2 + (Im(z))^2$

(3) $Re(z) \leq |Re(z)| \leq |z|$

(4) $Im(z) \leq |Im(z)| \leq |z|$

✓ (5) $|z^{-1}| = \frac{1}{|z|}$, $z \neq 0$

(6) $|Re(z)| + |Im(z)| \leq |z|\sqrt{2}$

(7) $|z_1 z_2| = |z_1| \cdot |z_2|$

(8) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

Bukti

(2) Adb. $|z|^2 = z\bar{z} = (Re(z))^2 + (Im(z))^2$

Ambil sebarang $z \in \mathbb{C}$.

Tulis, $z = a+bi$ u/ suatu $a, b \in \mathbb{R}$

$Re(z) = a$, $Im(z) = b$

Perhatikan bahwa,

- $|z|^2 = (\sqrt{a^2+b^2})^2 = a^2 + b^2$ .... (1)
- $z\bar{z} = (a+bi)(\overline{a+bi})$

$= (a+bi)(a-bi)$

$= a^2 + b^2$ .... (2)

- $(Re(z))^2 + (Im(z))^2 = a^2 + b^2$ .... (3)

Dari (1), (2) dan (3) diperoleh

$$|z|^2 = z\bar{z} = (Re(z))^2 + (Im(z))^2 \quad \blacksquare$$

(5) Adb. $|z^{-1}| = \frac{1}{|z|}$, $z \neq 0$

Ambil $z \in \mathbb{C}$ sebarang, $z \neq 0$

Tulis, $z = a+bi$ u/ suatu $a, b \in \mathbb{R}$

Perhatikan bahwa

$$|z^{-1}| = \left|\frac{1}{z}\right|$$

$$= \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(-\frac{b}{a^2+b^2}\right)^2}$$

$$= \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}}$$

$$= \sqrt{\frac{\cancel{(a^2+b^2)}}{(a^2+b^2)^{\cancel{2}}}}$$

$$= \frac{1}{\sqrt{a^2+b^2}}$$

$$= \frac{1}{|z|} \quad \blacksquare$$

$$\frac{1}{z} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi}$$

$$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i$$

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Imanuel

Sifat Modulus

(1.) Adb. $\forall z \in \mathbb{C}$ berlaku $|z| = |\bar{z}| = |-z|$

Ambil sebarang $z \in \mathbb{C}$

Tulis, $z = a + bi$; u/suatu $a, b \in \mathbb{R}$

Note that,

$$|\bar{z}| = |a - bi| = |a + (-bi)| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z| \quad \ldots\ldots (*)$$

dan

$$|-z| = |-(a+bi)| = |-a - bi| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z| \quad \ldots\ldots (**)$$

$\therefore$ Dari persamaan $(*)$ dan $(**)$ diperoleh bahwa

$$|z| = |\bar{z}| = |-\bar{z}|$$

$\blacksquare$

(3) Adb. $\forall z \in \mathbb{C}$ berlaku $Re(z) \leq |Re(z)| \leq |z|$

Ambil sebarang $z \in \mathbb{C}$, Tulis $z = a + bi$, u/suatu $a, b \in \mathbb{R}$

$\Rightarrow Re(z) = a$

$\Rightarrow |Re(z)| = |a|$

$\Rightarrow |z| = |a + bi| = \sqrt{a^2 + b^2}$

(i) Adit. $Re(z) \leq |Re(z)|$

Perhatikan bahwa

$a \leq |a|$ karena sesuai definisi nilai mutlak:

$a = |a|$ bila $a \geq 0$

$a < |a|$ bila $a < 0$

$\Rightarrow a \leq |a|$ atau

$Re(z) \leq |Re(z)|$

(ii) Adit. $|Re(z)| \leq |z|$

Perhatikan bahwa

$|a| \leq |a + bi|$

atau

$|a| \leq \sqrt{a^2 + b^2}$

karena sesuai definisi nilai mutlak:

$|a| = \sqrt{a^2 + b^2}$ u/ $b = 0$

$|a| < \sqrt{a^2 + b^2}$ u/ $b \neq 0$

Maka

$|a| \leq \sqrt{a^2 + b^2}$

atau

$|Re(z)| \leq |z|$

$\therefore$ Dari (i) dan (ii) diperoleh

$$Re(z) \leq |Re(z)| \leq |z|$$

$\blacksquare$

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(4) Adb. $\forall z \in \mathbb{C}$ berlaku $Im(z) \leq |Im(z)| \leq |z|$.

Ambil sebarang $z \in \mathbb{C}$

Tulis, $z = a + bi$; untuk suatu $a, b \in \mathbb{R}$

- $Im(z) = b$
- $|Im(z)| = |b|$
- $|z| = |a+bi| = \sqrt{a^2+b^2}$

(i) Adit. $Im(z) \leq |Im(z)|$

Perhatikan bahwa,

$b \leq |b|$

karena sesuai dengan definisi nilai mutlak bahwa,

$b = |b|$ jika $b \geq 0$

$b < |b|$ jika $b < 0$

$\Rightarrow b \leq |b|$ atau $Im(z) \leq |Im(z)|$

(ii) Adit. $|Im(z)| \leq |z|$

Perhatikan bahwa,

$|b| \leq |a+bi|$

atau

$|b| \leq \sqrt{a^2+b^2}$

karena sesuai dengan definisi nilai mutlak bahwa,

$b = \sqrt{a^2+b^2}$ jika $a = 0$

$b < \sqrt{a^2+b^2}$ jika $a \neq 0$

$\Rightarrow |b| \leq \sqrt{a^2+b^2}$ atau $|Im(z)| \leq |z|$

$\therefore$ Dari (i) dan (ii) diperoleh bahwa

$Im(z) \leq |Im(z)| \leq |z|$ ▢

Dipindai dengan CamScanner

Imanuel AS / 1811141008 Makassar, 3 April 2021

(6) Adb. $|Re(z)| + |Im(z)| \leq |z|\sqrt{2}$

Ambil sebarang $z \in \mathbb{C}$

Tulis, $z = a + bi$; untuk suatu $a, b \in \mathbb{R}$

- $|Re(z)| = |a|$
- $|Im(z)| = |b|$
- $|z| = \sqrt{a^2 + b^2}$

Perhatikan bahwa untuk setiap $a, b \in \mathbb{R}$ berlaku:

$$(|a| - |b|)^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2|a||b|$$

$$(|a| + |b|)^2 = |a|^2 + |b|^2 + 2|a||b| \leq |a|^2 + |b|^2 + |a|^2 + |b|^2$$
$$= 2(|a|^2 + |b|^2)$$
$$= 2(a^2 + b^2)$$
$$= 2|z|^2$$

$$\boxed{\begin{aligned} |z|^2 &= |z| \cdot |z| \\ &= |\sqrt{a^2+b^2}| \cdot |\sqrt{a^2+b^2}| \\ &= |a^2 + b^2| \end{aligned}}$$

Perhatikan bahwa, hal berikut berlaku:

$$|a|^2 + |b|^2 \leq |a|^2 + |b|^2 + 2|a| \cdot |b| \leq 2|z|^2$$

sehingga diperoleh

$$(|a| + |b|)^2 \leq 2|z|^2$$
$$|a| + |b| \leq \sqrt{2|z|^2}$$
$$|a| + |b| \leq \sqrt{2} \cdot |z|$$
$$|Re(z)| + |Im(z)| \leq |z| \cdot \sqrt{2}$$

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(7) Adb. $|z_1 . z_2| = |z_1| . |z_2|$; $\forall z_1, z_2 \in \mathbb{C}$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Tulis, $z_1 = a_1 + b_1 i$ untuk suatu $a_1, b_1 \in \mathbb{R}$

$z_2 = a_2 + b_2 i$ untuk suatu $a_2, b_2 \in \mathbb{R}$

Note that

$$|z_1 . z_2|^2 = (z_1 . z_2) . (\overline{z_1 . z_2})$$
$$= (z_1 . z_2)(\overline{z_1} . \overline{z_2})$$
$$= (z_1 . \overline{z_1})(z_2 . \overline{z_2})$$
$$= |z_1|^2 . |z_2|^2$$

Jadi, $|z_1 z_2| = |z_1| . |z_2|$ ∎

(8) Adb. $\forall z_1, z_2$, $z_2 \neq 0$ berlaku $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Tulis, $z_1 = a_1 + b_1 i$ untuk suatu $a_1, b_1 \in \mathbb{R}$

$z_2 = a_2 + b_2 i$ untuk suatu $a_2, b_2 \in \mathbb{R}$

Note that

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right) . \overline{\left(\frac{z_1}{z_2}\right)}$$
$$= \left(\frac{z_1}{z_2}\right) . \frac{\overline{z_1}}{\overline{z_2}}$$
$$= \frac{z_1 . \overline{z_1}}{z_2 . \overline{z_2}}$$
$$= \frac{|z_1|^2}{|z_2|^2}$$

Jadi, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ ∎

Imanuel AS / 1811141008 Makassar, 28 Februari 2021

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Ketaksamaan Segitiga

$\forall z_1, z_2 \in \mathbb{C}$ berlaku

✓ (1) $|z_1 + z_2| \leq |z_1| + |z_2|$

✓ (2) $|z_1 + z_2| \geq |z_1| - |z_2|$

(3) $|z_1 + z_2| \geq ||z_1| - |z_2||$ banyak di buku hanya $|z_1 + z_2| \geq -(|z_1| - |z_2|)$

(4) $|z_1 - z_2| \leq |z_1| + |z_2|$

✓ (5) $|z_1 - z_2| \geq |z_1| - |z_2|$ banyak dibuku hanya $|z_1 - z_2| \leq |z_1| + |z_2|$

(6) $|z_1 - z_2| \geq ||z_1| - |z_2||$

Bukti:

(1) Adb. $|z_1 + z_2| \leq |z_1| + |z_2|$

Perhatikan bahwa

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) \qquad [z^2 = z \cdot \bar{z}]$$

$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= z_1 \cdot \bar{z}_1 + z_1 \cdot \bar{z}_2 + z_2 \cdot \bar{z}_1 + z_2 \cdot \bar{z}_2$$

$$= |z_1|^2 + z_1 \cdot \bar{z}_2 + z_2 \cdot \bar{z}_1 + |z_2|^2 \qquad [|z|^2 = z \cdot \bar{z}]$$

maka diperoleh

$$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$= (|z_1| + |z_2|)^2 \qquad [(a+b)^2 = a^2 + 2ab + b^2]$$

$\therefore$ Berdasarkan teorema di analisis real, $a^2 \leq b^2$, $a, b \geq 0 \Rightarrow a < b$
maka diperoleh $|z_1 + z_2|^2 \leq |z_1| + |z_2|$

Perhatikan

$$z_1\bar{z}_2 + z_2\bar{z}_1 = z_1\bar{z}_2 + \overline{\bar{z}_2 z_1}$$

$$= z_1\bar{z}_2 + \overline{z_1\bar{z}_2}$$

$$= 2\,\mathrm{Re}(z_1\bar{z}_2)$$

$$\leq 2|z_1 \cdot \bar{z}_2|$$

$$= 2|z_1||\bar{z}_2|$$

$$= 2|z_1||z_2|$$

$$\overline{\bar{z}_2 \cdot z_1} = \bar{\bar{z}}_2 \cdot \bar{z}_1 = z_2 \cdot \bar{z}_1$$

tulis kembali dari sifat ini

$\boxed{z + \bar{z} = 2\,\mathrm{Re}(z)}$

$z = z_1\bar{z}_2$ ganti menjadi

$\boxed{z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\,\mathrm{Re}(z_1\bar{z}_2)}$

sifat modulus No 3
$\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z|$

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Makassar, 26 Februari 2021.

(2) Adb. $|z_1 + z_2| \geq |z_1| - |z_2|$

Bukti :

Ambil $z_1, z_2 \in \mathbb{C}$ sebarang.

Perhatikan bahwa

$$
\begin{aligned}
|z_1| &= |z_1 + z_2 - z_2| \\
&= |(z_1 + z_2) + (-z_2)| \\
&\leq |z_1 + z_2| + |-z_2| \\
&= |z_1 + z_2| + |z_2|
\end{aligned}
$$

[Sifat Modulus No. 1 ; $|z| = |-z|$

diperoleh

$$|z_1| \leq |z_1 + z_2| + |z_2|$$

$$\Rightarrow |z_1 + z_2| \geq |z_1| - |z_2| \quad \boxtimes$$

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(3) Adb. $|z_1+z_2| \geq ||z_1|-|z_2||$

Ambil $z_1, z_2 \in \mathbb{C}$ sebarang

$$|z_2| = |z_2+z_1-z_1|$$
$$= |(z_2+z_1)+(-z_1)|$$
$$\leq |z_2+z_1| + |-z_1| \quad \text{[Ketaksamaan segitiga (1)]}$$

diperoleh,

$$|z_2| \leq |z_2+z_1| + |z_1|$$
$$|z_2|-|z_1| \leq |z_1+z_2| \quad \text{[Kedua ruas ditambah } -|z_1|\text{]}$$
$$-|z_2|+|z_1| \geq -|z_1+z_2| \quad \text{[Kedua ruas dikali } -1\text{]}$$
$$|z_1|-|z_2| \geq -|z_1+z_2| \quad \ldots\ldots (*)$$

Selanjutnya, perhatikan bahwa

$$|z_1| = |z_1+z_2-z_2|$$
$$= |(z_1+z_2)+(-z_2)|$$
$$\leq |z_1+z_2| + |-z_2|$$

diperoleh,

$$|z_1| \leq |z_1+z_2| + |z_2|$$
$$|z_1|-|z_2| \leq |z_1+z_2| \quad \text{[Kedua ruas ditambah } -|z_2|\text{]}$$
$$-|z_1|+|z_2| \geq -|z_1+z_2| \quad \text{[Kedua ruas dikali } -1\text{]}$$
$$|z_2|-|z_1| \geq -|z_1+z_2| \quad \ldots\ldots (**)$$

∴ Dari persamaan (*) dan (**) diperoleh

$$-|z_1+z_2| \leq |z_1|-|z_2| \leq |z_1+z_2|$$

Maka,

$$||z_1|-|z_2|| \leq ||z_1+z_2||$$
$$||z_1|-|z_2|| \leq |z_1+z_2|$$

atau

$$|z_1+z_2| \geq ||z_1|-|z_2|| \quad \blacksquare$$

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Makassar, 29 Maret 2021

(4) Adb. $|z_1 - z_2| \leq |z_1| + |z_2|$

Ambil $z_1, z_2 \in \mathbb{C}$ sebarang

Perhatikan bahwa,

$$|z_1| = |z_1 - z_2 + z_2|$$
$$= |(z_1 - z_2) + z_2|$$
$$\geq |z_1 - z_2| - |z_2| \quad \text{[Ketaksamaan segitiga (2)]}$$

diperoleh,

$$|z_1| \geq |z_1 - z_2| - |z_2|$$
$$|z_1| + |z_2| \geq |z_1 - z_2| \quad \text{[kedua ruas ditambah } |z_2|\text{]}$$
$$|z_1 - z_2| \leq |z_1 + z_2|$$

$$\therefore |z_1 - z_2| \leq |z_1| + |z_2|$$

(5) Adb. $|z_1 - z_2| \geq |z_1| - |z_2|$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$ sebarang

Perhatikan bahwa,

$$|z_1| = |z_1 - z_2 + z_2|$$
$$= |(z_1 - z_2) + z_2|$$
$$\leq |z_1 - z_2| + |z_2| \quad \text{[Ketaksamaan segitiga (1)]}$$

diperoleh,

$$|z_1| \leq |z_1 - z_2| + |z_2|$$
$$|z_1| - |z_2| \leq |z_1 - z_2| \quad \text{[kedua ruas ditambah } -|z_2|\text{]}$$
$$|z_1 - z_2| \geq |z_1| - |z_2|$$

$$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$$

$\blacksquare$

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Makassar, 29 Maret 2021

(6) Adb. $|z_1 - z_2| \geq ||z_1| - |z_2||$

Ambil sebarang $z_1, z_2 \in \mathbb{C}$

Perhatikan bahwa,

$$|z_2| = |z_2 - z_1 + z_1|$$
$$= |(z_2 - z_1) + z_1|$$
$$\leq |z_2 - z_1| + |z_1| \quad \text{[Ketaksamaan segitiga (1)]}$$
$$= |z_1 - z_2| + |z_1|$$

diperoleh,

$$|z_2| \leq |z_1 - z_2| + |z_1|$$
$$|z_2| - |z_1| \leq |z_1 - z_2| \quad \text{[kedua ruas ditambah } -|z_1|\text{]}$$
$$-|z_2| + |z_1| \geq -|z_1 - z_2| \quad \text{[kedua ruas dikali } -1\text{]}$$
$$-|z_1 - z_2| \leq -|z_2| + |z_1|$$
$$-|z_1 - z_2| \leq |z_1| - |z_2| \quad \ldots\ldots (*)$$

Selanjutnya, perhatikan bahwa

$$|z_1| = |z_1 - z_2 + z_2|$$
$$= |(z_1 - z_2) + z_2|$$
$$\leq |z_1 - z_2| + |z_2| \quad \text{[Ketaksamaan segitiga (1)]}$$

diperoleh

$$|z_1| \leq |z_1 - z_2| + |z_2|$$
$$|z_1| - |z_2| \leq |z_1 - z_2| \quad \text{[kedua ruas ditambah } -|z_2|\text{]}$$
$$|z_1 - z_2| \geq |z_1| - |z_2| \quad \ldots\ldots (**)$$

$\therefore$ Dari persamaan (*) dan (**) diperoleh

$$-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2|$$
$$|z_1| - |z_2| \leq |z_1 - z_2|$$
$$||z_1 - z_2|| \geq ||z_1| - |z_2||$$
$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

$\blacksquare$

PERTEMUAN IV

Analisis Kompleks

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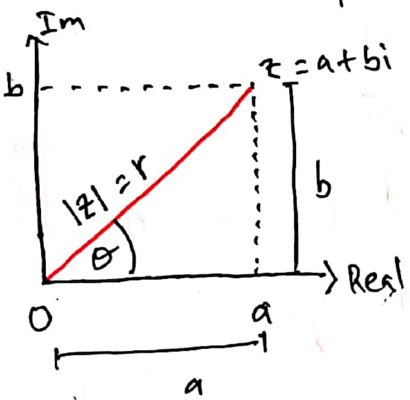
Analisis Kompleks / Pertemuan ke-4 / Catatan Imanuel AS 1811141008 Makassar, 7 Maret 2021

Bentuk Polar

Argumen bilangan kompleks



$|z| = \sqrt{a^2+b^2}$

Argumen dari $z \in \mathbb{C}$, $z \neq 0$ ditulis $\arg(z)$

Misal $z = a+bi \neq 0$, $\theta = \arg(z)$

$r = |z| = \sqrt{a^2+b^2}$

$\boxed{N}$

$\arg(z)$ tidak tunggal, $\theta + 2n\pi$, $n \in \mathbb{Z}$

nilai tunggal dari $\arg(z)$ ditulis

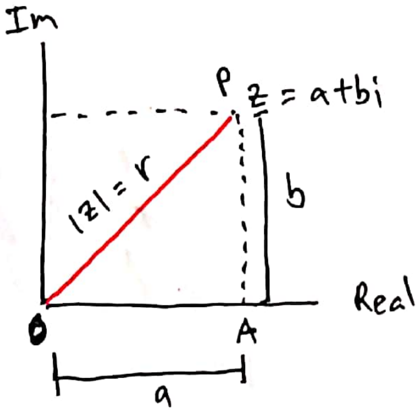
$\mathrm{Arg}(z)$ dengan $-\pi < \mathrm{Arg} < \pi$

Imanuel AS / 181141068 Makassar, 9 Maret 2021.

Imanuel

hubungan arg(z) dengan Arg(z) adalah

$$\boxed{\arg(z) = \text{Arg}(z) + 2n\pi \quad , n \in \mathbb{Z}}$$



$$|z| = r = \sqrt{a^2+b^2}$$

$$\tan\theta = \frac{b}{a} \Rightarrow \theta = \text{arc}\left(\tan\frac{b}{a}\right)$$

$$\sin\theta = \frac{b}{r} \Rightarrow b = r\sin\theta$$

$$\cos\theta = \frac{a}{r} \Rightarrow a = r\cos\theta$$

$$\begin{aligned} z = a+bi &= r\cos\theta + i(r\sin\theta) \\ &= r(\cos\theta + i.\sin\theta) \\ &= r.\text{cis}\,\theta \end{aligned}$$

$$\text{cis}\,\theta = \cos\theta + i.\sin\theta$$

Imanuel AS / 1811141008 Manuu Makassar, 7 Maret 2021

$z = a + bi$

$\Downarrow$

$z = r \text{ cis } \theta = r(\cos\theta + i \cdot \sin\theta)$

(Bentuk Polar)

$\boxed{E}$

Nyatakan dalam bentuk polar

(1) $z = 1 + i$

Jawab:

$z = 1 + i \begin{cases} a = 1 \\ b = 1 \end{cases}$ $r = |z| = \sqrt{a^2 + b^2} = \sqrt{2}$

$\theta = \text{Arg}(z) = \text{arc}\left(\tan\left(\frac{1}{1}\right)\right) = \pi/4$

diperoleh

$z = r \text{ cis } \theta = \sqrt{2} \cdot \text{cis}(\pi/4) = \sqrt{2} \cdot (\cos(\pi/4) + i \cdot \sin(\pi/4))$

(2) Nyatakan bentuk polar berikut

$z = \sqrt{8} \text{ cis}(\pi/4)$ dalam bentuk $z = a + bi$.

Jawab:

$z = \sqrt{8} \cos(\pi/4)$

$= \sqrt{8}\left(\cos(\pi/4) + i \cdot \sin(\pi/4)\right)$

$= \sqrt{8}\left(\frac{1}{2}\sqrt{2} + i \cdot \left(\frac{1}{2}\sqrt{2}\right)\right)$

$= 2 + 2i$

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Misal

$z_1 = r_1 \operatorname{cis}(\theta_1)$

$z_2 = r_2 \operatorname{cis}(\theta_2)$

$$\boxed{\begin{array}{l} z_1 = a_1 + b_1 i \quad , \quad z_2 = a_2 + b_2 i \\ z_1 = z_2 \iff a_1 = a_2 \ , \ b_1 = b_2 \end{array}}$$

$$z_1 = z_2 \iff r_1 \cdot \operatorname{cis}(\theta_1) = r_2 \cdot \operatorname{cis}(\theta_2)$$

$$\iff r_1 = r_2 \ , \ \theta_1 = \theta_2 + 2n\pi \ , \ n \in \mathbb{Z}$$

Konjugat $z = r \cdot \operatorname{cis} \theta$ adalah

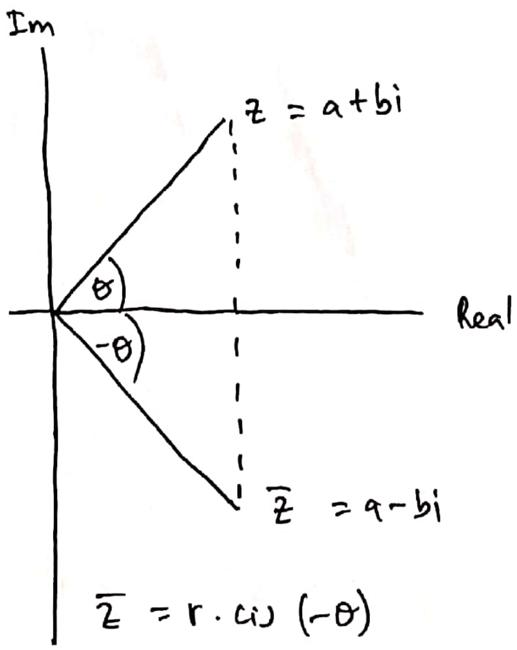
$$\boxed{\bar{z} = r \operatorname{cis}(-\theta)}$$

(1) $z = r \cdot \operatorname{cis}(\theta) \Rightarrow \bar{z} = r \operatorname{cis}(-\theta)$

<u>Bukti</u>

Misal $z = a + bi \Rightarrow \bar{z} = a - bi$

$r = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$



$\bar{z} = r \cdot \operatorname{cis}(-\theta)$

Imanuel AS / 1811141008 Imanuel Makassar, 8 Maret 2021

(2) Jika $z_1 = r_1 \cdot cis(\theta_1)$ dan $z_2 = r_2 \cdot cis(\theta_2)$

Maka

(a) $z_1 \cdot z_2 = r_1 . r_2 \cdot cis(\theta_1 + \theta_2)$

[KUIS] (b) $\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot cis(\theta_1 - \theta_2)$, $z_2 \neq 0$

Bukti

(a) $z_1 \cdot z_2 = [r_1 \cdot cis(\theta_1)] \cdot [r_2 \cdot cis(\theta_2)]$

$= [r_1 \cdot (\cos\theta_1 + i \cdot \sin\theta_1)][r_2 (\cos\theta_2 + i \cdot \sin\theta_2)]$

$= r_1 . r_2 (\cos\theta_1 + i \cdot \sin\theta_1)(\cos\theta_2 + i \cdot \sin\theta_2)$

$= r_1 . r_2 [\cos\theta_1 \cdot \cos\theta_2 + i \cdot \cos\theta_1 \cdot \sin\theta_2 + i \sin\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2]$

$= r_1 . r_2 \cdot [(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2) + i(\cos\theta_1 \cdot \sin\theta_2 + \sin\theta_1 \cdot \cos\theta_2)]$

$= r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i \cdot \sin(\theta_1 + \theta_2)]$

$= r_1 . r_2 . cis(\theta_1 + \theta_2)$

(b) (Jawaban ada di lembar kuis)

Imanuel AS / 1811141008 Malassang 7 Maret 2021

Bentuk Eksponen

$z = a + bi \rightarrow z = r \cdot cis\,\theta$

$r = \sqrt{a^2 + b^2}$

$\theta = \arctan\left(\frac{b}{a}\right)$

$z = r \cdot cis\,\theta = r(\cos\theta + i \cdot \sin\theta)$

$= re^{i\theta}\ [e^{i\theta} = cis\,\theta]$

Bentuk eksponen $z \neq 0$ adalah

$z = r \cdot cis\,\theta = r(\cos\theta + i \cdot \sin\theta) = re^{i\theta}$

$\boxed{N}$

(1) Bentuk eksponen z tidak tunggal

$z = re^{i\theta} = re^{i(\theta + 2n\pi)}$, $n \in \mathbb{Z}$

(2) $z_1 = r_1 e^{i \cdot \theta_1}$, $z_2 = r_2 e^{i \cdot \theta_2}$

$z_1 = z_2 \Leftrightarrow r_1 = r_2$ dan

$\theta_1 = \theta_2 + 2n\pi$, $n \in \mathbb{Z}$

(3) Jika $z_1 = r_1 e^{i \cdot \theta_1}$ dan $z_2 = r_2 \cdot e^{i\theta_2}$

maka :

(a) $z_1 \cdot z_2 = r_1 r_2 \cdot e^{i(\theta_1 + \theta_2)}$

(b) $\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$

Imanuel AS / 1811141008 [illegible] Makassar, 9 Maret 2021

Bukti

(3) (a) $z_1 = r_1 \cdot cis\ \theta_1$

$z_2 = r_2 \cdot cis\ \theta_2$

$z_1 \cdot z_2 = (r_1 \cdot cis\ \theta_1)(r_2 \cdot cis\ \theta_2)$

$= r_1 . r_2\ (cis\ (\theta_1 + \theta_2))$

$= r_1 \cdot r_2 \cdot e^{i(\theta_1 + \theta_2)}$ //

(b) $z_1 = r_1 \cdot cis\ (\theta_1)$

$z_2 = r_2 \cdot cis\ (\theta_2)$, $z \neq 0$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} . cis\ (\theta_1 - \theta_2)$

$= \frac{r_1}{r_2} . e^{i(\theta_1 - \theta_2)}$

//

(2) $z_1 = z_2 \Leftrightarrow r_1 . cis(\theta_1) = r_2 \cdot cis(\theta_2)$

$\Leftrightarrow r_1 \cdot e^{i\theta_1} = r_2 \cdot e^{i\theta_2}$

$\Leftrightarrow r_1 = r_2$, $\theta_1 = \theta_2 + 2n\pi$, $n \in \mathbb{Z}$

(1) $\theta = \arg(z)$, dan θ tidak tunggal karena $\arg(z)$ tidak tunggal dimana $\theta + 2n\pi$, $n \in \mathbb{Z}$.

Jadi, $z = re^{i\theta} = re^{i(\theta + 2n\pi)}$, $n \in \mathbb{Z}$

Analisis Kompleks / Pertemuan ke -4 / Kuis

Imanuel AS Makassar, 4 Maret 2021
1811141008 Manuel

(2) Jika $z_1 = r_1 \cdot cis(\theta_1)$ dan $z_2 = r_2 \cdot cis(\theta_2)$ maka, tunjukkan bahwa

(b.) $\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot cis(\theta_1 - \theta_2)$, $z_2 \neq 0$

Penyelesaian:

$$\frac{z_1}{z_2} = \frac{r_1 \cdot cis(\theta_1)}{r_2 \cdot cis(\theta_2)}$$

$$= \frac{r_1 (\cos(\theta_1) + i \cdot \sin(\theta_1))}{r_2 (\cos(\theta_2) + i \cdot \sin(\theta_2))}$$

$$= \frac{r_1 (\cos(\theta_1) + i \cdot \sin(\theta_1))}{r_2 (\cos(\theta_2) + i \cdot \sin(\theta_2))} \cdot \frac{\cos(\theta_2) - i \cdot \sin(\theta_2)}{\cos(\theta_2) - i \cdot \sin(\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \left[\frac{\cos(\theta_1) \cdot \cos(\theta_2) - \cos(\theta_1) \cdot i \cdot \sin(\theta_2) + i \cdot \sin(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2)}{\cos^2(\theta_2) + \sin^2(\theta_2)} \right]$$

$$= \frac{r_1}{r_2} \cdot \left[\frac{\cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2) + i(\sin(\theta_1) \cdot \cos(\theta_2) - \sin(\theta_2) \cdot \cos(\theta_1))}{1} \right]$$

$$= \frac{r_1}{r_2} \left[\frac{\cos(\theta_1 - \theta_2) + i(\sin(\theta_1 - \theta_2))}{1} \right]$$

$$= \frac{r_1}{r_2} \left[\cos(\theta_1 - \theta_2) + i(\sin(\theta_1 - \theta_2)) \right]$$

$$= \frac{r_1}{r_2} \cdot \left[cis(\theta_1 - \theta_2) \right] \quad // \quad \blacksquare$$

PERTEMUAN V

Analisis Kompleks

Imanuel A S

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Pangkat dan Bilangan Kompleks

$z = a + bi \quad , a,b \in \mathbb{R}$

$\hookrightarrow r = \sqrt{a^2 + b^2}$

$\hookrightarrow z = r(\cos\theta + i \cdot \sin\theta) = r \text{ cis } \theta$
$= r e^{i\theta}$

$z = re^{i\theta} \longrightarrow z^n = (re^{i\theta})^n = r^n \cdot e^{i\theta n} \quad , n \in \mathbb{N}$ [Induksi]

$\boxed{z^n = r^n e^{in\theta}} \quad n \in \mathbb{N}$

- Pangkat Bilangan Bulat

Misal $n \in \mathbb{Z}$, $\boxed{z^n = r^n e^{in\theta}}$

untuk $n = 0 \longrightarrow z^0 = 1$

untuk $n > 0 \longrightarrow z^n = r^n e^{i\theta n}$. [Induksi]

untuk $n < 0 \longrightarrow$ Tulis $n = -m$, $m \in \mathbb{N}$

akibatnya , $z^n = z^{-m} = (z^{-1})^m$

$= \left(\frac{1}{r} e^{-i\theta}\right)^m$

$= r^{-m} \cdot e^{-m(i\theta)}$

$= r^n \cdot e^{i\theta n}$

$\boxed{z^n = r^n e^{in\theta}}$

Kesimpulannya :

Jika $z = re^{i\theta}$, $z \neq 0$ maka $z^n = r^n \cdot e^{in\theta}$, $n \in \mathbb{Z}$

$\boxed{E}$ Tentukan nilai dari z^7 jika $z = \sqrt{3} + i$

Solusi :

$z = \sqrt{3} + i \begin{cases} a = \sqrt{3} \\ b = 1 \end{cases} \Big\} \; r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \text{arc tan}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$

$z = re^{i\theta} = 2e^{(\pi/6) i}$

Jadi, diperoleh $z^7 = \left(2e^{(\pi/6)i}\right)^7 = 2^7 \cdot e^{\frac{7\pi}{6} \cdot i}$

$= (2^6 e^{i\pi}) \cdot (2e^{\pi/6 \cdot i})$

$= (64(-1)) \cdot (z)$

$= (-64) \cdot (\sqrt{3} + i)$

$= -64\sqrt{3} - 64i$ //

$e^{i\pi} = \cos\pi + i \cdot \sin\pi$
$= -1 + 0$
$= -1$

Imanuel AS / 181114008 Makassar, 13 Maret 2021

Teorema de' Moivre

Jika $z \in \mathbb{C}$ dengan $z = r(\cos\theta + i \cdot \sin\theta)$

maka, $z^n = r^n (\cos n\theta + i \cdot \sin n\theta)$; $\forall n \in \mathbb{Z}$

Bukti

(1) Kasus $n = 0$

$z^0 = r^0 (\cos 0 + i \cdot \sin 0)$

$= 1(1 + 0)$

$= \underline{1} \in \mathbb{C}$

(2) Kasus $n \in \mathbb{Z}^+$

Gunakan Induksi Matematika

(*) Untuk $n = 1$ maka $z = r(\cos\theta + i \cdot \sin\theta) \in \mathbb{C}$

(*) Misal untuk $n = k$ benar, yaitu

$$z^k = r^k (\cos(k\theta) + i \cdot \sin(k\theta))$$

akan ditunjukkan benar untuk $n = k+1$

yaitu :

$$z^{k+1} = r^{k+1} (\cos(k+1)\theta + i \cdot \sin(k+1)\theta)$$

Perhatikan bahwa

$z^{k+1} = z^k \cdot z$

$= r^k (\cos(k\theta) + i \cdot \sin(k\theta))(r(\cos\theta + i \cdot \sin\theta))$

$= r^k \operatorname{cis}(k\theta) \cdot r \operatorname{cis}(\theta)$

$= r^{k+1} \cdot \operatorname{cis}(k\theta + \theta)$

$= r^{k+1} \cdot \operatorname{cis}(k+1)\theta$

$= r^{k+1} (\cos(k+1)\theta + i \sin(k+1)\theta)$

Imanuel AS / 1811141008 Makassar, 13 Maret 2021

(3) Kasus $n \in \mathbb{Z}^-$

Misal, $n = -m$, $m \in \mathbb{Z}^+$

Perhatikan bahwa

$$
\begin{aligned}
z^n &= z^{-m} \\
&= \frac{1}{z^m} \\
&= \frac{1}{r^m \operatorname{cis}(m\theta)} \\
&= \frac{1}{r^m(\cos m\theta + i \cdot \sin \theta)} \\
&= \frac{1}{r^m(\cos(m\theta) + i \cdot \sin(m\theta))} \times \frac{(\cos(m\theta) - i \cdot \sin(m\theta))}{(\cos(m\theta) - i \cdot \sin(m\theta))} \\
&= \frac{\cos(m\theta) - i \cdot \sin(m\theta)}{r^m(\cos^2(m\theta) + \sin^2(m\theta))} \\
&= r^{-m}(\cos(m\theta) - i \cdot \sin(m\theta)) \\
&= r^{-m}(\cos(-m\theta) - i \cdot \sin(m\theta)) \\
&= r^n(\cos n\theta + i \cdot \sin n\theta)
\end{aligned}
$$

$\therefore$ Teorema de' Moivre

$$
\begin{aligned}
z^n &= r^n(\cos n\theta + i \cdot \sin n\theta) \quad \forall n \in \mathbb{Z} \\
&= r^n \operatorname{cis}(n\theta) \quad \forall n \in \mathbb{Z} \\
&= r^n \cdot e^{in\theta} \quad \forall n \in \mathbb{Z}
\end{aligned}
$$

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E

(1) Misal $a+bi = (i-1)^{49} (\cos(\pi/40) + i \cdot \sin(\pi/40))^{10}$.

Tentukan $a+b = \ldots$?

Solusi

Perhatikan bahwa

$(i-1) < \begin{matrix} a = 1 \\ b = -1 \end{matrix} \} r = \sqrt{a^2+b^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\theta = \text{arc} \tan\left(\frac{b}{a}\right) = \text{arc} \tan(-1) = 3\pi/4$ (Kuadran II)

akibatnya berdasarkan Teorema de' Moivre diperoleh,

$$(i-1)^{49} = (\sqrt{2})^{49} \cdot (\cos(3\pi/4) + i \cdot \sin(3\pi/4))^{49}$$
$$= (\sqrt{2})^{49} \cdot \left(e^{\frac{3}{4}\pi i}\right)^{49}$$

Sehingga diperoleh

$$(i-1)^{49} \cdot (\cos(\pi/40) + i \cdot \sin(\pi/40))^{10}$$
$$= (\sqrt{2})^{49} \cdot \left(e^{\left(\frac{3}{4}\pi\right)i}\right)^{49} \left(e^{\frac{\pi}{40} i}\right)^{10}$$
$$= (\sqrt{2})^{49} \cdot \left(e^{\left(\frac{147}{4}\pi\right)i}\right)\left(e^{\left(\frac{\pi}{4}\right)i}\right)$$
$$= (\sqrt{2})^{49} \cdot e^{\left(\frac{148}{4}\pi\right)i}$$
$$= (\sqrt{2})^{49} \cdot e^{(37\pi)i}$$
$$= (\sqrt{2})^{49} \cdot e^{\pi i}$$
$$= (\sqrt{2})^{49} \cdot (\cos\pi + i \cdot \sin\pi)$$
$$= (\sqrt{2})^{49} \cdot (-1+0)$$
$$= -(\sqrt{2})^{49}$$

$\therefore$ Dengan demikian diperoleh

$-(\sqrt{2})^{49} = a+bi < \begin{matrix} a = -(\sqrt{2})^{49} \\ b = 0 \end{matrix}$

Jadi $a+b = -(\sqrt{2})^{49} + 0 = -(\sqrt{2})^{49}$ //

TUGAS Imanuel AS/181114100 8 Makassar, 13 Maret 2021

Tentukan nilai dari $5\,Re(z) + 7\,Im(z)$

Jika $z = (3-3i)^{2010}$

Penyelesaian :

Perhatikan bahwa

$(3-3i)$: $a = 3$, $b = -3$ $\Big\}$ $r = \sqrt{a^2+b^2} = \sqrt{3^2+(-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$\theta = \arc\tan\left(\frac{b}{a}\right) = \arc\tan\left(\frac{-3}{3}\right) = \arc\tan(-1) = \frac{3\pi}{4}$ (Kuadran II)

akibatnya berdasarkan Teorema de' Moivre diperoleh,

$$(3-3i)^{2010} = (3\sqrt{2})^{2010} \cdot \left(\cos\left(\frac{3\pi}{4}\right) + i\cdot\sin\left(\frac{3\pi}{4}\right)\right)^{2010}$$

$$= (3\sqrt{2})^{2010} \cdot \left(e^{\frac{3\pi}{4}i}\right)^{2010}$$

$$= (3\sqrt{2})^{2010} \cdot \left(e^{\frac{6030}{4}\pi\cdot i}\right)$$

$$= (3\sqrt{2})^{2010} \cdot \left(\cos\left(\frac{6030}{4}\pi\right) + i\cdot\sin\left(\frac{6030}{4}\pi\right)\right)$$

$$= (3\sqrt{2})^{2010} \cdot \left(\cos\left(\frac{3015}{2}\pi\right) + i\cdot\sin\left(\frac{3015}{2}\pi\right)\right)$$

$$= (3\sqrt{2})^{2010} \cdot \left(0 + i\cdot(-1)\right)$$

$$= (3\sqrt{2})^{2010} \cdot (-i)$$

Maka diperoleh

$-(3\sqrt{2})^{2010}\,i = a+bi$: $a = 0$ atau $Re(z) = 0$; $b = -(3\sqrt{2})^{2010}$ atau $Im(z) = -(3\sqrt{2})^{2010}$

$$\therefore\ 5\,Re(z) + 7\,Im(z) = 5\cdot(0) + 7\cdot -(3\sqrt{2})^{2010}$$

$$= -7\cdot(3\sqrt{2})^{2010}$$

//

Imanuel AS / 1811141008   Makassar, 13 Maret 2021

$$\cos\left(\frac{3015}{2}\pi\right) = \cos\left(\frac{3\pi}{2} + \frac{3012\pi}{2}\right)$$

$$= \cos\left(\frac{3\pi}{2} + 2 \cdot \frac{3012}{2 \cdot 2} \cdot \pi\right)$$

$$= \cos\left(\frac{3\pi}{2} + 2 \cdot \frac{3012}{4} \cdot \pi\right)$$

$$= \cos\left(\frac{3\pi}{2} + 2 \cdot 753 \pi\right)$$

= Menggunakan $\cos(t + 2k\pi) = \cos(t)$, $k \in \mathbb{Z}$

diperoleh,

$$= \cos\left(\frac{3\pi}{2}\right)$$

$$= \cos(270^\circ)$$

$$= 0$$

$$\sin\left(\frac{3015}{2}\pi\right) = \sin\left(\frac{3\pi}{2} + \frac{3012\pi}{2}\right)$$

$$= \sin\left(\frac{3\pi}{2} + 2 \cdot \frac{3012}{2 \cdot 2} \cdot \pi\right)$$

$$= \sin\left(\frac{3\pi}{2} + 2 \cdot \frac{3012}{4} \cdot \pi\right)$$

$$= \sin\left(\frac{3\pi}{2} + 2 \cdot 753 \pi\right)$$

= Menggunakan $\sin(t + 2k\pi) = \sin(t)$, $k \in \mathbb{Z}$

diperoleh,

$$= \sin\left(\frac{3\pi}{2}\right)$$

$$= \sin(270^\circ)$$

$$= -1$$

PERTEMUAN VI

Analisis Kompleks

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Analisis Kompleks / Pertemuan ke-6 / Catatan

Makassar, 29 Maret 2021

Imanuel Agung Sembe / 1811141008

Akar dari Bilangan Kompleks

$z = a + bi \in \mathbb{C} \rightsquigarrow z^n = r^n . cis\,(n\theta)$

$= r^n \,(\cos n\theta + i \cdot \sin n\theta)$

$\downarrow$

$z = re^{i\theta} = r. cis\,\theta$

$r = \sqrt{a^2 + b^2}$ $\qquad \theta = \arctan\left(\frac{b}{a}\right)$

Definisi

Misal $z, w \in \mathbb{C}$, akar pangkat n dan w ditulis $w^{\frac{1}{n}}$ didefinisikan sebagai bilangan kompleks z ditulis

$$z^n = w \quad , n \in \mathbb{N} \quad , n \geq 2$$

$$z = \sqrt[n]{w} = w^{\frac{1}{n}}.$$

Imanuel Agung Sembe / 1811141008 Manday   Makassar, 31 Maret 2021

B

Tentukan semua nilai $z \in \mathbb{C}$ yang memenuhi $z = (i)^{\frac{1}{3}}$

Solusi

Cara I : Misal $z = i^{\frac{1}{3}} \Rightarrow z^3 = i \Rightarrow z^3 - i = 0$

diperoleh

$$z^3 - i = 0$$

$$\Rightarrow (z+i)(z^2 - iz - 1) = 0$$

$$\Rightarrow z + i = 0 \quad \vee \quad z^2 - iz - 1 = 0$$

$$z_1 = -i \qquad z_{2,3} = \frac{-(-i) \pm \sqrt{(-i)^2 - 4(1)(-1)}}{2(1)}$$

$$z_{2,3} = \frac{i \pm \sqrt{-1+4}}{2}$$

$$z_2 = \frac{i}{2} + \frac{1}{2}\sqrt{3} \text{ dan } z_3 = \frac{i}{2} - \frac{1}{2}\sqrt{3}$$

$\therefore Hp = \{ -i , \frac{1}{2}\sqrt{3} + \frac{i}{2} , -\frac{1}{2}\sqrt{3} + \frac{i}{2} \}$

Cara II : Misal $z = i^{\frac{1}{3}} \Rightarrow z^3 = i \Rightarrow z^3 - i = 0$

misal $z = a + bi$, $a, b \in \mathbb{R}$, akibatnya

$$z^3 - i = 0$$

$$\Rightarrow (a+bi)^3 - i = 0$$

$$\Rightarrow a^3 + 3a^2bi - 3ab^2 - ib^3 - i = 0$$

$$\Rightarrow (a^3 - 3ab^2) + (3a^2b - b^3 - 1)i = 0$$

$$\Rightarrow (a^3 - 3ab^2) + (3a^2b - b^3 - 1)i = 0 + 0i$$

Akibatnya, $\left.\begin{aligned} a^3 - 3ab^2 &= 0 \\ 3a^2b - b^3 - 1 &= 0 \end{aligned}\right\}$ cari solusi dari sistem persamaan ini

Diperoleh,

$a_1 = 0 \rightarrow b_1 = -1 \Rightarrow z_1 = a_1 + b_1 i = -i$

$a_2 = \frac{1}{2}\sqrt{3} \rightarrow b_2 = \frac{1}{2} \Rightarrow z_2 = a_2 + b_2 i = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$

$a_3 = -\frac{1}{2}\sqrt{3} \rightarrow b_3 = \frac{1}{2} \Rightarrow z_3 = a_3 + b_3 i = -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

$\therefore Hp = \{ -i , \frac{1}{2}\sqrt{3} + \frac{1}{2}i , -\frac{1}{2}\sqrt{3} + \frac{1}{2}i \}$

Imanuel AS / 1811141008 Makassar, 31 Maret 2021

Cara III : Misal $z = i^{\frac{1}{3}} \Rightarrow z^3 = i$

$z = r \text{ cis } \phi = r(\cos\phi + i \sin\phi)$

untuk $i \Rightarrow r = \sqrt{0^2+1^2} = 1$

$\phi = \frac{\pi}{2}$

Akibatnya,

$$z^3 = i$$

$$\Rightarrow (r(\cos\theta + i\cdot\sin\theta))^3 = 1(\cos\phi + i\cdot\sin\phi)$$

$$\Rightarrow r^3(\cos 3\theta + i \sin 3\theta) = 1(\cos(\pi/2) + i\cdot\sin(\pi/2))$$

Berdasarkan kesamaan bilangan kompleks diperoleh

$r^3 = 1 \Rightarrow r = 1$

$3\theta = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \Rightarrow \theta = \frac{\pi}{6} + \frac{2}{3}k\pi, k \in \mathbb{Z}$

diperoleh $z = r \text{ cis } \theta$

Perhatikan bahwa, karena θ periodik, maka nilainya akan berulang

Misalkan, perhatikan bahwa untuk $(k=-1) = (k=2)$

perhatikan bahwa untuk $(k=-2) = (k=1)$

perhatikan bahwa untuk $(k=-3) = (k=0)$

Maka diperoleh,

$k=0 \Rightarrow z_1 = r \text{ cis}(\theta)$

$= 1(\cos(\pi/6) + i\cdot\sin(\pi/6))$

$= \frac{1}{2}\sqrt{3} + \frac{1}{2}i$

$k=1 \Rightarrow z_2 = r \text{ cis}(\theta)$

$= 1(\cos(5\pi/6) + i\cdot\sin(5\pi/6))$

$= -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

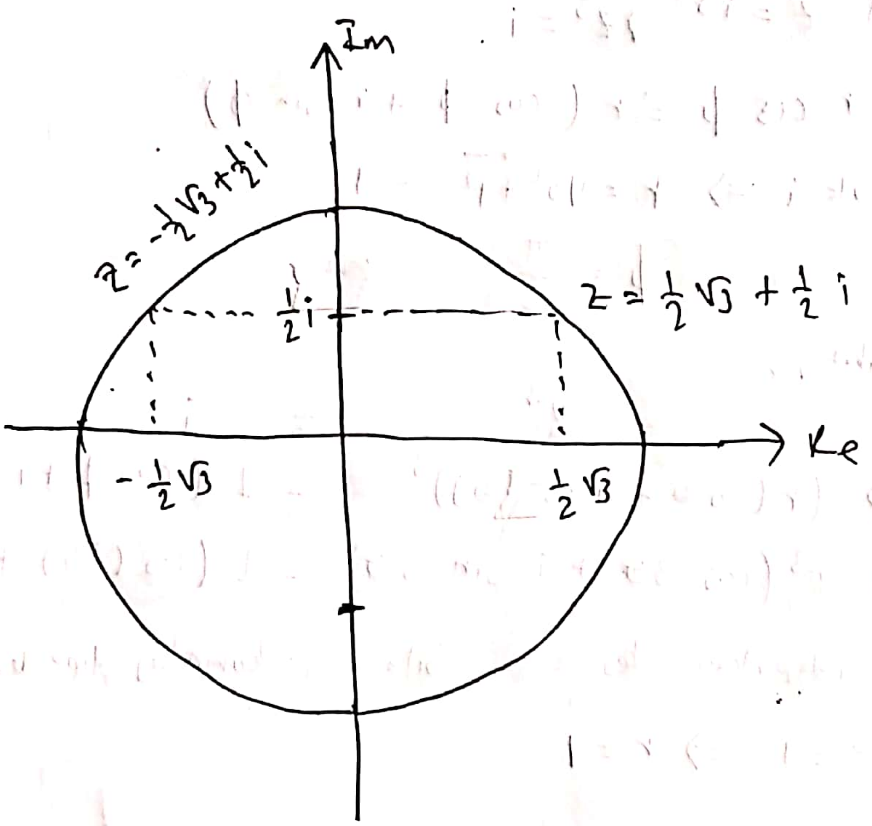
$k=2 \Rightarrow z_3 = r \text{ cis}(\theta)$

$= 1(\cos(3\pi/2) + i\cdot\sin(3\pi/2))$

$= -i$

$\therefore HP = \{\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -i\}$

Imanuel Agung Sembe / 181114008 Makassar, 31 Maret 2021



Latihan

Tentukan akar-akar dari persamaan

1. $(-i)^{\frac{1}{2}}$
2. $(1+i)^{\frac{1}{2}}$
3. $\left(\frac{1-i\sqrt{3}}{2}\right)^{\frac{1}{2}}$
4. $z^3 + i = 0$
5. $(-i)^{1/4}$
6. $z^6 = \frac{1-i}{\sqrt{3}-i}$
7. $\left(\frac{2z-1}{3z-4}\right)^2 = 2 + 2\sqrt{3}\, i$
8. $\left(\frac{5z-2}{3z+1}\right)^2 = 2\sqrt{2} + 2\sqrt{2}\, i$

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Latihan :

Tentukan akar-akar dari persamaan:

(1.) $(i)^{\frac{1}{2}}$

Penyelesaian :

Misal $z = (i)^{\frac{1}{2}} \Rightarrow z^2 = -i$

$z = r \cdot cis\ \phi = r(\cos\phi + i \cdot \sin\phi)$

Untuk $-i \Rightarrow r = \sqrt{a^2+b^2} = \sqrt{0^2+(-1)^2} = \sqrt{1} = 1$

$\phi = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{-1}{0}\right) = \arctan(\text{tak terdefinisi}) = \frac{\pi}{2}$

Akibatnya,

$z^2 = -i$

$\Rightarrow (r(\cos\theta + i\cdot\sin\theta))^2 = (r(\cos\phi + i\cdot\sin\phi))$

$\Rightarrow r^2(\cos 2\theta + i\cdot\sin 2\theta) = (1(\cos(\frac{\pi}{2}) + i\cdot\sin(\frac{\pi}{2}))$

$\Rightarrow r^2(\cos 2\theta + i\cdot\sin 2\theta) = (\cos(\frac{\pi}{2}) + i\cdot\sin(\frac{\pi}{2}))$

Berdasarkan kesamaan bilangan kompleks, diperoleh

$r^2 = 1 \Rightarrow r\sqrt{1} = 1$

$2\theta = \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z} \Rightarrow \theta = \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$

diperoleh $z = r\ cis\ \theta$

$k = 0 \Rightarrow z_1 = r \cdot cis(\theta)$

$= (\cos(\frac{\pi}{4}) + i\cdot\sin(\frac{\pi}{4}))$

$= (\frac{1}{2}\sqrt{2} + i\cdot\frac{1}{2}\sqrt{2})$

$= \frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}$

$k = 1 \Rightarrow z_2 = r \cdot cis(\theta)$

$= (\cos(\frac{5\pi}{4}) + i\cdot\sin(\frac{5\pi}{4}))$

$= (-\frac{1}{2}\sqrt{2} + i\cdot -\frac{1}{2}\sqrt{2})$

$= -\frac{1}{2}\sqrt{2} - \frac{i}{2}\sqrt{2}$

$=$

$\therefore HP = \{\frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2} - \frac{i}{2}\sqrt{2}\}$

Imanuel Agung Sembe / 1811141008 Malegasar 31 Maret 2021

(2.) $(1+i)^{\frac{1}{2}}$

Penyelesaian:

Misal $z = (1+i)^{\frac{1}{2}} \Rightarrow z^2 = (1+i)$

$z = r \cdot cis\ \phi = r(\cos\phi + i \cdot \sin\phi)$

untuk $(1+i) \Rightarrow r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$\phi = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{1}{1}\right) = \arctan(1) = \frac{\pi}{4}$

Akibatnya,

$z^2 = 1+i$ → lihat cat pert. 4 hal 3

$\Rightarrow (r \cdot (\cos\theta + i \cdot \sin\theta))^2 = (r(\cos\phi + i \cdot \sin\phi))$

$\Rightarrow r^2 \cdot (\cos 2\theta + i \cdot \sin 2\theta) = \sqrt{2} \cdot \left(\cos\left(\frac{\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{4}\right)\right)$

Berdasarkan kesamaan bilangan kompleks diperoleh

$r^2 = \sqrt{2} \Rightarrow r = 2^{\frac{1}{4}}$

$2\theta = \frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z} \Rightarrow \theta = \frac{\pi}{8} + k\pi,\ k \in \mathbb{Z}$

diperoleh $z = r \cdot cis\ \theta$

Maka diperoleh

$k = 0 \Rightarrow z_1 = r \cdot cis\ \theta$

$= 2^{\frac{1}{4}}\left(\cos\left(\frac{\pi}{8}\right) + i \cdot \sin\left(\frac{\pi}{8}\right)\right)$

$= 2^{\frac{1}{4}}\left(\frac{\sqrt{2+\sqrt{2}}}{2} + i \cdot \frac{\sqrt{2-\sqrt{2}}}{2}\right)$

$= \frac{\sqrt{2+\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} + \frac{\sqrt{2-\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} \cdot i$

$= (1) \cdot 2^{\frac{1}{4}} + \frac{\sqrt{2-\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} \cdot i$

$= 2^{\frac{1}{4}} + \frac{\sqrt{2-\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} \cdot i$

$k = 1 \Rightarrow z_2 = r \cdot cis\ \theta$

$= 2^{\frac{1}{4}}\left(\cos\left(\frac{9\pi}{8}\right) + i \cdot \sin\left(\frac{9\pi}{8}\right)\right)$

$= 2^{\frac{1}{4}}\left(-\frac{\sqrt{2+\sqrt{2}}}{2} + i \cdot -\frac{\sqrt{2-\sqrt{2}}}{2}\right)$

$= -\frac{\sqrt{2+\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} - \frac{\sqrt{2-\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} \cdot i$

$\therefore HP = \{$ $= (-1) \cdot 2^{\frac{1}{4}} - \frac{\sqrt{2-\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} \cdot i$

$= -2^{\frac{1}{4}} - \frac{\sqrt{2-\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} \cdot i$

$\therefore HP = \left\{2^{\frac{1}{4}} + \frac{\sqrt{2+\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} \cdot i,\ -2^{\frac{1}{4}} - \frac{\sqrt{2+\sqrt{2}}}{2} \cdot 2^{\frac{1}{4}} \cdot i\right\}$

Imanuel Agung Sambo / 1811141008 Imanuel Makassar, 31 Maret 2021

(3.) $\left(\frac{1 - i\sqrt{3}}{2}\right)^{\frac{1}{2}}$

Penyelesaian.

Misal $z = \left(\frac{1 - i\sqrt{3}}{2}\right)^{\frac{1}{2}} \Rightarrow z^2 = \left(\frac{1 - i\sqrt{3}}{2}\right)$

$z = r \cdot \text{cis}\ \phi = r\left(\cos\phi + i \cdot \sin\phi\right)$

Untuk $\left(\frac{1 - i\sqrt{3}}{2}\right) \Rightarrow r = \sqrt{a^2 + b^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)} = 1$

$\phi = \text{arc tan}\left(\frac{b}{a}\right) = \text{arc tan}\left(\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}\right) = \text{arc tan}\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$

Akibatnya,

$$z^2 = \left(\frac{1 - i\sqrt{3}}{2}\right)$$

$$\Rightarrow \left(r\left(\cos\theta + i \cdot \sin\theta\right)\right)^2 = \left(r\left(\cos\phi + i \cdot \sin\phi\right)\right)$$

$$\Rightarrow r^2\left(\cos 2\theta + i \cdot \sin 2\theta\right) = 1\left(\cos\left(\frac{5\pi}{6}\right) + i \cdot \sin\left(\frac{5\pi}{6}\right)\right)$$

Berdasarkan kesamaan bilangan kompleks, diperoleh

$r^2 = 1 \Rightarrow r = 1$

$2\theta = \frac{5\pi}{6} + 2k\pi,\ k \in \mathbb{Z} \Rightarrow \theta = \frac{5\pi}{12} + k\pi,\ k \in \mathbb{Z}$

diperoleh $z = r \cdot \text{cis}\ \theta$

Maka diperoleh

$$k = 0 \Rightarrow z_1 = r \cdot \text{cis}\ \theta$$
$$= 1\left(\cos\left(\frac{5\pi}{12}\right) + i \cdot \sin\left(\frac{5\pi}{12}\right)\right)$$
$$= 1\left(\frac{\sqrt{6} - \sqrt{2}}{4} + i \cdot \frac{\sqrt{6} + \sqrt{2}}{4}\right)$$
$$= \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right) + \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right) \cdot i$$

$$k = 1 \Rightarrow z_2 = r \cdot \text{cis}\ \theta$$
$$= 1\left(\cos\left(\frac{17\pi}{12}\right) + i \cdot \sin\left(\frac{17\pi}{12}\right)\right)$$
$$= 1 \cdot \left(\frac{-\sqrt{6} + \sqrt{2}}{4} + i \cdot -\left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)\right)$$
$$= \left(\frac{-\sqrt{6} + \sqrt{2}}{4}\right) - \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right) i$$

$\therefore HP = \left\{\frac{\sqrt{6} - \sqrt{2}}{4} + \frac{\sqrt{6} + \sqrt{2}}{4} \cdot (i),\ \frac{-\sqrt{6} + \sqrt{2}}{4} - \frac{\sqrt{6} + \sqrt{2}}{4} \cdot (i)\right\}$

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4. $z^3 + i = 0$

Penyelesaian

Misal $z^3 + i = 0 \Rightarrow z^3 = -i$

$z = r \cdot cis\,\phi = r \cdot (\cos\phi + i \cdot \sin\phi)$

Untuk $-i \Rightarrow r = \sqrt{a^2+b^2} = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$

$\phi = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{-1}{0}\right) = \arctan(\text{tak terdefinisi}) = \frac{\pi}{2}$

Akibatnya,

$z^3 = -i$

$\Rightarrow (r(\cos\theta + i \cdot \sin\theta))^3 = (r(\cos\phi + i \cdot \sin\phi))$

$\Rightarrow r^3(\cos 3\theta + i \cdot \sin 3\theta) = (1(\cos(\frac{\pi}{2}) + i \cdot \sin(\frac{\pi}{2})))$

Berdasarkan kesamaan bilangan kompleks, diperoleh

$r^3 = 1 \Rightarrow r = 1$

$3\theta = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \Rightarrow \theta = \frac{\pi}{6} + \frac{2}{3}k\pi, k \in \mathbb{Z}$

diperoleh $z = r \cdot cis\,\theta$

$k = 0 \Rightarrow z_1 = r \cdot cis\,\theta$
$= 1(\cos(\frac{\pi}{6}) + i \cdot \sin(\frac{\pi}{6}))$
$= 1(\frac{1}{2}\sqrt{3} + i \cdot \frac{1}{2})$
$= \frac{1}{2}\sqrt{3} + \frac{1}{2} \cdot i$

$k = 1 \Rightarrow z_2 = r \cdot cis\,\theta$
$= 1(\cos(\frac{5\pi}{6}) + i \cdot \sin(\frac{5\pi}{6}))$
$= 1(-\frac{1}{2}\sqrt{3} + i \cdot \frac{1}{2})$
$= -\frac{1}{2}\sqrt{3} + \frac{1}{2} i$

$k = 2 \Rightarrow z_3 = r \cdot cis\,\theta$
$= 1(\cos(\frac{9\pi}{6}) + i \cdot \sin(\frac{9\pi}{6}))$
$= 1(0 + i \cdot (-1))$
$= -i$

$\therefore HP = \{\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -i$

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(5.) $(-i)^{1/4}$

Penyelesaian :

Misal $z = (i)^{\frac{1}{4}} \Rightarrow z^4 = i$

$z = r \cdot cis\ \phi = r(\cos\phi + i \cdot \sin\phi)$

untuk $i \Rightarrow r = \sqrt{a^2+b^2} = \sqrt{0^2+1^2} = 1$

$\phi = arc\tan\left(\frac{b}{a}\right) = arc\tan\left(\frac{1}{0}\right) = arc\tan\ (\text{tak terdefinisi}) = \frac{\pi}{2}$

Akibatnya,

$$z^4 = i$$

$$\Rightarrow (r(\cos\theta + i \cdot \sin\theta))^4 = (r(\cos\phi + i \cdot \sin\phi))$$

$$\Rightarrow r^4(\cos 4\theta + i \cdot \sin 4\theta) = \left(1\left(\cos\left(\frac{\pi}{2}\right) + i \cdot \sin\left(\frac{\pi}{2}\right)\right)\right)$$

Berdasarkan kesamaan bilangan kompleks, diperoleh

$r^4 = 1 \Rightarrow r = 1$

$4\theta = \frac{\pi}{2} + 2k\pi \quad , k \in \mathbb{Z} \Rightarrow \theta = \frac{\pi}{8} + \frac{2}{4}k\pi \quad , k \in \mathbb{Z}$

diperoleh $z = r \cdot cis\ \theta$

$$k=0 \Rightarrow z_1 = r \cdot cis\ \theta$$
$$= 1\left(\cos\left(\frac{\pi}{8}\right) + i \cdot \sin\left(\frac{\pi}{8}\right)\right)$$
$$= \left(\frac{\sqrt{2+\sqrt{2}}}{2} + i \cdot \frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

$$k=1 \Rightarrow z_2 = r \cdot cis\ \theta$$
$$= 1\left(\cos\left(\frac{5\pi}{8}\right) + i \cdot \sin\left(\frac{5\pi}{8}\right)\right)$$
$$= \left(-\frac{\sqrt{2-\sqrt{2}}}{2} + i \cdot \frac{\sqrt{2+\sqrt{2}}}{2}\right)$$

$$k=2 \Rightarrow z_3 = r \cdot cis\ \theta$$
$$= 1\left(\cos\left(\frac{9\pi}{8}\right) + i \cdot \sin\left(\frac{9\pi}{8}\right)\right)$$
$$= \left(-\frac{\sqrt{2+\sqrt{2}}}{2} + i \cdot -\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

$$k=3 \Rightarrow z_4 = r \cdot cis\ \theta$$
$$= 1\left(\cos\left(\frac{13\pi}{8}\right) + i \cdot \sin\left(\frac{13\pi}{8}\right)\right)$$
$$= \left(\frac{\sqrt{2-\sqrt{2}}}{2} + i \cdot -\frac{\sqrt{2+\sqrt{2}}}{2}\right)$$

$$\therefore HP = \left\{\frac{\sqrt{2+\sqrt{2}}}{2} + i \cdot \frac{\sqrt{2-\sqrt{2}}}{2}, -\frac{\sqrt{2-\sqrt{2}}}{2} + i \cdot \frac{\sqrt{2+\sqrt{2}}}{2}, -\frac{\sqrt{2-\sqrt{2}}}{2} - i \cdot \frac{\sqrt{2-\sqrt{2}}}{2}, \frac{\sqrt{2-\sqrt{2}}}{2} - i \cdot \frac{\sqrt{2+\sqrt{2}}}{2}\right\}$$

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5. $z^6 = \dfrac{1-i}{\sqrt{3}-i}$

Penyelesaian :

$$z^6 = \frac{1-i}{\sqrt{3}-i} = \frac{1-i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i} = \frac{\sqrt{3}+1}{4} + \frac{(1-\sqrt{3})}{4} \cdot i$$

$$z = r \cdot cis\ \phi = r \cdot (\cos\phi + i \cdot \sin\phi)$$

untuk $\dfrac{\sqrt{3}+1}{4} + \dfrac{(1-\sqrt{3})}{4} \cdot i \Rightarrow r = \sqrt{a^2+b^2} = \sqrt{\left(\dfrac{\sqrt{3}+1}{4}\right)^2 + \left(\dfrac{(1-\sqrt{3})}{4}\right)^2}$

$$r = \frac{\sqrt{2}}{2}$$

$$\phi = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{\frac{1-\sqrt{3}}{4}}{\frac{\sqrt{3}+1}{4}}\right)$$

$$= \arctan\left(\frac{1-\sqrt{3}}{\sqrt{3}+1}\right)$$

$$= \arctan\left(\sqrt{3}-2\right)$$

$$= 345^\circ$$

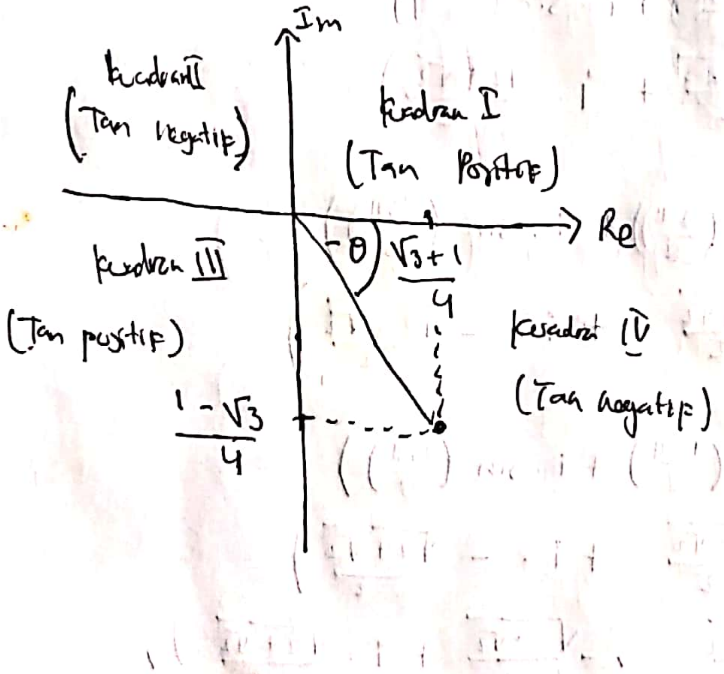
$$\phi = \frac{23\pi}{12}$$

Soal mengapa memilih 345°, bukan 165°

Dik: $a = \dfrac{\sqrt{3}+1}{4}$ (Positif)

$b = \dfrac{(1-\sqrt{3})}{4}$ (Negatif)

Jadi di dapat grafik sbb



next

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Akibatnya,

$$z^6 = \frac{\sqrt{3}+1}{4} + \frac{(1-\sqrt{3})}{4} \cdot i$$

$$\Rightarrow (r(\cos\theta + i.\sin\theta))^6 = (r(\cos\phi + i.\sin\phi))$$

$$\Rightarrow r^6(\cos 6\theta + i.\sin 6\theta) = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{11\pi}{12}\right) + i.\sin\left(\frac{11\pi}{12}\right)\right)$$

Berdasarkan kesamaan bilangan kompleks, diperoleh

$$r^6 = \frac{\sqrt{2}}{2} \Rightarrow r = \sqrt[6]{\frac{\sqrt{2}}{2}} = 0{,}94 = 1$$

$$6\theta = \frac{23\pi}{12} + 2k\pi, k \in \mathbb{Z} \Rightarrow \theta = \frac{23\pi}{72} + \frac{2}{6} \cdot k\pi, k \in \mathbb{Z}$$

diperoleh $z = r.\text{cis}\,\theta$

$$k = 0 \Rightarrow z_1 = r.\text{cis}\,\theta$$
$$= 1 \cdot \left(\cos\left(\frac{23\pi}{72}\right) + i.\sin\left(\frac{23\pi}{72}\right)\right)$$
$$= 1\,(0{,}5 + i.(0{,}8))$$
$$= 0{,}5 + (0{,}8)i$$

$$k = 1 \Rightarrow z_2 = r.\text{cis}\,\theta$$
$$= 1\left(\cos\left(\frac{47\pi}{72}\right) + i.\sin\left(\frac{47\pi}{72}\right)\right)$$
$$= 1\,(-0{,}4 + i \cdot 0{,}9)$$
$$= (-0{,}4) + (0{,}9)i$$

$$k = 2 \Rightarrow z_3 = r.\text{cis}\,\theta$$
$$= 1\left(\cos\left(\frac{71\pi}{72}\right) + i.\sin\left(\frac{71\pi}{72}\right)\right)$$
$$= 1\,(-1 + i \cdot 0)$$
$$= -1$$

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$$k=3 \Rightarrow z_4 = r \cdot cis\ \theta$$
$$= 1 \cdot \left(\cos\left(\frac{95\pi}{72}\right) + i \cdot \sin\left(\frac{95\pi}{72}\right)\right)$$
$$= 1 \cdot (-0,5 + i \cdot (-0,8))$$
$$= -0,5 - (0,8)i$$

$$k=4 \Rightarrow z_5 = r \cdot cis\ \theta$$
$$= 1 \cdot \left(\cos\left(\frac{119\pi}{72}\right) + i \cdot \sin\left(\frac{119\pi}{72}\right)\right)$$
$$= 1 \cdot (0,4 + i \cdot (-0,9))$$
$$= 0,4 - (0,9)i$$

$$k=5 \Rightarrow z_6 = r \cdot cis\ \theta$$
$$= 1 \cdot \left(\cos\left(\frac{143\pi}{72}\right) + i \cdot \sin\left(\frac{143\pi}{72}\right)\right)$$
$$= 1 \cdot (1 + i \cdot (0))$$
$$= 1$$

$$\therefore HP = \left\{ \begin{matrix} 0,5+(0,8)i, & -0,4+(0,9)i, & -1 \\ -0,5-(0,8)i, & 0,4-(0,9)i, & 1 \end{matrix} \right\}$$