Makassar, 30 November 2020

STRUKTUR ALJABAR II

— Pertemuan XII —

(catatan)

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Struktur Aljabar II / Pertemuan ke-12 / catatan Makassar, 13 November 2020

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Struktur Aljabar II : Catatan Pertemuan ke-12.

Teorema Isomorfisma II
dan
Teorema Isomorfisma III
dari Ring

Teorema Isomorfisma II

Jika S, T ideal dari ring R dan $S \subseteq T$, maka

$$R/T \cong \frac{R/S}{T/S}$$

(baca: isomorp)

<u>Bukti</u>:

(1) Adb. T/S terdefinisi

Perhatikan $S \subseteq T \subseteq R$

(a) $\forall a, b \in S \Rightarrow a - b \in S$

(b) $\forall a \in S, t \in T \Rightarrow ta \in S, at \in S$

Jadi, S ideal T

$\therefore T/S$ terdefinisi

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(2) Adb. $\dfrac{R/S}{T/S}$ terdefinisi.

(a) Ambil $\bar{a}, \bar{b} \in T/S$, $\bar{r} \in R/S$ sebarang.

(untuk kuliah kita, kita sepakati menggunakan kurung karena saya)
- Pak Sahlan -

Tulis, $\bar{a} = S + t_1$ u/suatu $t_1 \in T$

$\bar{b} = S + t_2$ u/suatu $t_2 \in T$

$\bar{r} = S + r$ u/suatu $r \in R$

Perhatikan bahwa,

$$\bar{a} - \bar{b} = (S + t_1) - (S + t_2)$$

$$= S + (t_1 - t_2) \in T/S$$

$$\bar{r}\,\bar{a} = (S + r)(S + t_1) = S + (r t_1) \in T/S$$

$$\bar{a}\,\bar{r} = (S + t_1)(S + r) = S + (t_1 r) \in T/S$$

Jadi, T/S ideal dari R/S

$\therefore \dfrac{R/S}{T/S}$ terdefinisi

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Definisikan pengaitan

$$\theta : R/S \longrightarrow R/T \qquad , \theta(S+r) = T+r$$
$$S+r \longmapsto T+r$$

(3) Adb. θ pemetaan

Ambil $\overline{r_1}, \overline{r_2} \in R/S$ sebarang.

Tulis, $\overline{r_1} = S + r_1$ u/suatu $r_1 \in R$

$\overline{r_2} = S + r_2$ u/suatu $r_2 \in R$

dengan $\overline{r_1} = \overline{r_2}$.

Adb. $\theta(\overline{r_1}) = \theta(\overline{r_2})$

Perhatikan bahwa,

$$\overline{r_1} = \overline{r_2}$$
$$\Rightarrow S+r_1 = S+r_2$$
$$\Rightarrow r_1 - r_2 \in S \subseteq T \qquad [\, S+a = S+b \Leftrightarrow a-b \in S \,]$$
$$\Rightarrow T+r_1 = T+r_2$$
$$\Rightarrow \theta(S+r_1) = \theta(S+r_2)$$
$$\Rightarrow \theta(\overline{r_1}) = \theta(\overline{r_2})$$

$\therefore$ θ pemetaan

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(4) Adb. θ homomorfisma

Ambil $\bar{r}_1, \bar{r}_2 \in R/S$ sebarang.

Tulis $\bar{r}_1 = S + r_1$ u/suatu $r_1 \in R$

$\bar{r}_2 = S + r_2$ u/suatu $r_2 \in R$

$$
\begin{aligned}
\theta(\bar{r}_1 + \bar{r}_2) &= \theta\big((S+r_1) + (S+r_2)\big) \\
&= \theta\big(S + (r_1+r_2)\big) \\
&= T + (r_1+r_2) \\
&= (T+r_1) + (T+r_2) \\
&= \theta(S+r_1) + \theta(S+r_2) \\
&= \theta(\bar{r}_1) + \theta(\bar{r}_2)
\end{aligned}
$$

$$
\begin{aligned}
\theta(\bar{r}_1 \bar{r}_2) &= \theta\big((S+r_1)(S+r_2)\big) \\
&= \theta\big(S + (r_1 r_2)\big) \\
&= T + (r_1 r_2) \\
&= (T+r_1)(T+r_2) \\
&= \theta(S+r_1) \cdot \theta(S+r_2) \\
&= \theta(\bar{r}_1) \cdot \theta(\bar{r}_2)
\end{aligned}
$$

$\therefore$ θ homomorfisma ring.

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(5) Adb. θ Surjektif

Ambil $\bar{r} \in R/T$ sebarang.

Tulis $\bar{r} = T + r$ u/suatu $r \in R$.

Pilih $\bar{r_1} = S + r \in R/S$ sehingga

$\theta(\bar{r_1}) = \theta(S+r) = T + r = \bar{r}$

Jadi, θ Surjektif.

(6) Adb. $Ker(\theta) = T/S$ — $Ker(\theta) \subseteq T/S$; $T/S \subseteq Ker(\theta)$

$$\theta : R/S \longrightarrow R/T$$
$$S + r \longmapsto T + r \qquad \theta(S+r) = T + r$$

Perhatikan bahwa,

$$\begin{aligned} Ker(\theta) &= \{\bar{r} = S + r \in R/S \mid \theta(\bar{r}) = 0_{R/T}\} \\ &= \{\bar{r} = S + r \in R/S \mid \theta(S+r) = T\} \\ &= \{\bar{r} = S + r \in R/S \mid T + r = T + 0_R\} \\ &= \{\bar{r} = S + r \in R/S \mid r \in T\} \\ &= T/S \end{aligned}$$

Berdasarkan Teorema Dasar Isomorfisma, $R/T \cong \dfrac{R/S}{T/S}$

Imanuel AS / 1811141008 ~~[illegible]~~ Manua   Makassar, 14 Novem 2020

Teorema Isomorfisma III
(Teorema Diamond)

$$
\begin{tikzcd}
 & S+T \arrow[dl, no head] \arrow[dr, no head] & \\
S \arrow[dr, no head] & & T \arrow[dl, no head] \\
 & S \cap T &
\end{tikzcd}
$$

$S + T = \{a + b \mid a \in S, b \in T\}$

$$\frac{S+T}{T} \cong \frac{S}{S \cap T}$$

$$\frac{S+T}{S} \cong \frac{T}{S \cap T}$$

Versi 1

Jika S ideal dari ring R dan T subring R, maka

$$\frac{S+T}{S} \cong \frac{T}{S \cap T}$$

Versi 2

Jika T ideal dari ring R dan S subring R, maka

$$\frac{S+T}{T} \cong \frac{S}{S \cap T}$$

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Bukti Teorema Isomorfisma III (Teorema Diamond)

Buktikan bahwa jika S ideal dari ring R dan T subring dari R, maka

$$\frac{S+T}{S} \cong \frac{T}{S \cap T}$$

Penyelesaian :

(1.) Adb. $\frac{T}{S \cap T}$ terdefinisi

Akan ditunjukkan $S \cap T$ ideal dari T

$S \cap T \neq \phi$ karena $\exists\ 0_R \in S \cap T$.

Perhatikan bahwa $S \cap T \subseteq T \subseteq R$

Ambil $a, b \in S \cap T$, $t \in T$ sebarang

$a, b \in S \cap T \Rightarrow a, b \in S$ dan $a, b \in T$.

a) $a, b \in S \Rightarrow a - b \in S$ [Karena S ideal R]

$a, b \in T \Rightarrow a - b \in T$ [Karena T subring R]

Maka $a - b \in S \cap T$

b) $a \in S$, $t \in T \subseteq R \Rightarrow at \in S$ dan $ta \in S$ [Karena S ideal R]

$a \in T$, $t \in T \Rightarrow at \in T$ dan $ta \in T$ [Karena T subring R]

$at \in S$ dan $at \in T$ maka $at \in S \cap T$

$ta \in S$ dan $ta \in T$ maka $ta \in S \cap T$

Jadi, $S \cap T$ ideal T.

$\therefore \frac{T}{S \cap T}$ terdefinisi.

Dipindai dengan CamScanner

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(2) Adb. $\frac{S+T}{S}$ terdefinisi

Akan ditunjukkan : S ideal S+T.

$S+T = \{a+b \mid a \in S, b \in T\} \neq \emptyset$ karena $S \neq \emptyset$ dan $T \neq \emptyset$.

Perhatikan : $S \subseteq S+T \subseteq R$; $T \subseteq R$

Ambil $m, n \in S$, $p \in S+T$ sebarang,

Tulis, $p = a+b$ untuk suatu $a \in S, b \in T$

a.) $m, n \in S \Rightarrow m-n \in S$ [Karena S ideal R]

b.) $m \in S$, $p \in S+T$

$$m \cdot p = m(a+b) = ma + mb \in S$$

[Karena S ideal R dan $S \subseteq R$ maka $ma \in S$.
Karena S ideal R dan $T \subseteq R$ maka $mb \in S$.
Karena S ideal ring R maka $ma+mb \in S$.]

$$pm = (a+b)m = am + bm \in S.$$

[Karena S ideal R dan $S \subseteq R$ maka $am \in S$.
Karena S ideal R dan $T \subseteq R$ maka $bm \in S$.
Karena S ideal ring R maka $am+bm \in S$.]

Jadi, S ideal S+T.

$\therefore \frac{S+T}{S}$ terdefinisi.

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Definisi Pengaitan

$$\theta : T \longrightarrow \frac{S+T}{S} \qquad \theta(a) = S + a$$
$$a \longmapsto S + a$$

(3) Adb. θ pemetaan

Ambil $a_1, a_2 \in T \subseteq R$ sebarang

dengan $a_1 = a_2$

Adb. $\theta(a_1) = \theta(a_2)$

Perhatikan bahwa

$$a_1 = a_2$$
$$\Leftrightarrow S + a_1 = S + a_2 \qquad \text{[Karena S ideal R, koset kanan]}$$
$$\Leftrightarrow \theta(a_1) = \theta(a_2) \qquad \text{[Sesuai definisi } \theta\text{]}$$

$\therefore$ θ pemetaan

(4) Adb. θ homomorfisma

Ambil $a_1, a_2 \in T$ sebarang

Perhatikan

$$\begin{aligned} \theta(a_1 + a_2) &= S + (a_1 + a_2) \\ &= (S + a_1) + (S + a_2) \\ &= \theta(a_1) + \theta(a_2) \end{aligned}$$

$$\begin{aligned} \theta(a_1 a_2) &= S + (a_1 a_2) \\ &= (S + a_1) \cdot (S + a_2) \\ &= \theta(a_1) \cdot \theta(a_2) \end{aligned}$$

$\therefore$ θ homomorfisma ring

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(5) Adb. θ Surjektif

Ambil $\bar{a} \in \frac{S+T}{S}$ sebarang.

Tulis, $\bar{a} = S + a$ u/ suatu $a \in T$.

Pilih $a \in T$.

Sehingga, $\theta(a) = S + a = \bar{a}$

Jadi, θ Surjektif.

$\therefore$ θ epimorfisma.

(6) Adb. $\text{Ker}(\theta) = S \cap T$

$$\theta : T \longrightarrow \frac{S+T}{S} \qquad \theta(a) = S + a$$
$$a \longmapsto S + a$$

Perhatikan bahwa :

$$\begin{aligned}
\text{Ker}(\theta) &= \{a \in T \mid \theta(a) = 0_{\frac{S+T}{S}}\} \\
&= \{a \in T \mid \theta(a) = S\} \\
&= \{a \in T \mid S + a = S + 0_T\} \\
&= \{a \in T \mid a - 0_T = a \in S\} \\
&= S \cap T.
\end{aligned}$$

$\therefore$ Berdasarkan Teorema Dasar Isomorfisma, Terbukti :

$$\frac{S+T}{S} \cong \frac{T}{S \cap T}.$$