Makassar, 2 Juni 2024

Analisis Kompleks

Pertemuan ke - 10

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Analisis Kompleks / Pertemuan ke-10 / Catatan

Malassar, 18 Mei 2024

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Fungsi Kompleks

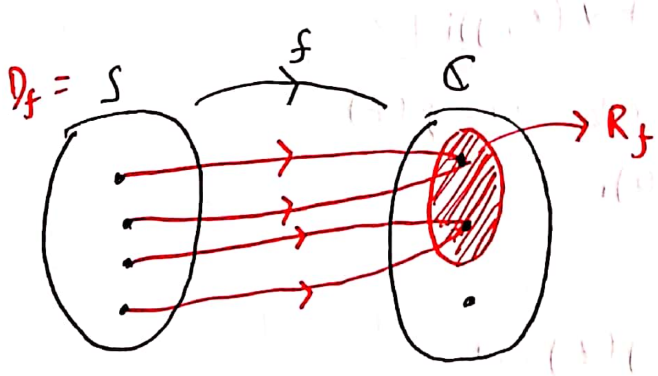
$\boxed{D}$

Fungsi $f : S \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ kompleks adalah suatu aturan yang mengaitkan setiap $z \in S$ dengan tepat satu $w \in \mathbb{C}$

$D_f = S$

$R_f = \{ w \in \mathbb{C} \mid w = f(z) \text{ u/suatu } z \in D_f \}$

$= f(S)$



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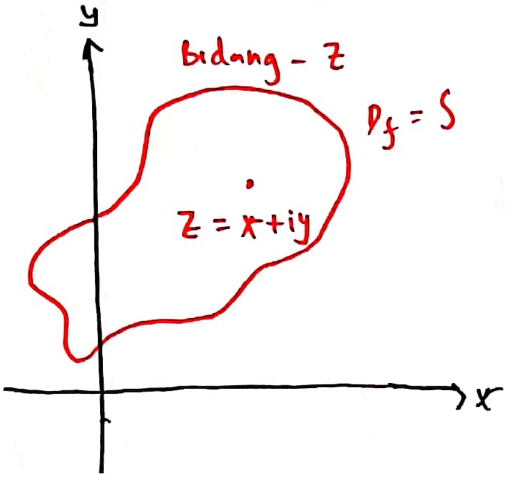
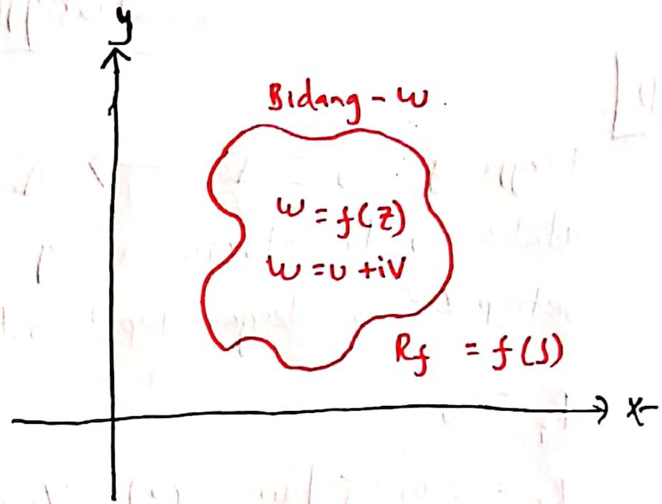
$$z = x + iy = (x, y)$$

$$\underbrace{x}_{Re(z)} \qquad \underbrace{y}_{Im(z)}$$





(1) Jika $z = x + iy$ maka $w = f(z)$

Berbentuk $f(z) = U(x, y) + V(x, y)i$

(2) Jika $z = r \text{ cis } \theta = re^{i\theta}$ maka $w = f(z)$

$w = f(z) = U(r, \theta) + V(r, \theta)i$

$\boxed{N}$

Jika $V(x, y) = 0$ maka $f(z) = w$

Fungsi kompleks bernilai real.

$\boxed{E}$ $f(z) = z^2$

(1) Misal $z = x + yi$, maka

$$f(z) = f(x + yi)$$
$$= (x + yi)^2$$
$$= \underbrace{(x^2 - y^2)}_{U(x,y)} + \underbrace{(2xy)}_{V(x,y)}i$$

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$f(z) = |z|^2$

Misal $z = x + iy$

$f(z) = f(x+iy) = |x+iy|^2$

$= (x^2+y^2) + 0i$

① Fungsi Polinom (Suku banyak) di Bil. Kompleks

$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$

$a_n \neq 0$, $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$

$f(z) = z^2 + 1$ ada pembuat nol

$z^2 + 1 = 0$

$z = \pm\sqrt{-1} \begin{cases} i \\ -i \end{cases}$

secara umum

$f(z) = z^n + 1$

$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$

$f(z) = 0$

Banyaknya solusi

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② Fungsi rasional di Bil. kompleks

$$f(z) = \frac{P(z)}{Q(z)}$$, P, Q suku banyak

$Q(z) \neq 0$

$\boxed{E}$ $$f(z) = \frac{z^2+1}{2z}$$

$f(x) = x^{\frac{1}{2}}$

$z \in \mathbb{C} \setminus \{0\}$, $(z = re^{i\theta})$

$f(z) = z^{\frac{1}{2}}$ mempunyai 2 nilai

akar 1 : $z^{\frac{1}{2}} = \sqrt{r} \cdot e^{\frac{i\theta}{2}}$ $\quad r = |z|$

akar 2 : $z^{\frac{1}{2}} = -\sqrt{r} \cdot e^{\frac{i\theta}{2}}$ $\quad -\pi < \theta < \pi$

$f(z) = \sqrt{r} \cdot e^{\frac{i\theta}{2}}$ adalah fungsi

$\boxed{D}$ Misal $A \subseteq \mathbb{C}$, $B \subseteq \mathbb{C}$, f, g fungsi yang didefinisikan

$w = f(z)$, $z \in A$

$t = g(z)$, $z \in B$

Operasi f dan g pada $D = A \cap B$ didefinisikan

(1) $(f+g)(z) = f(z) + g(z)$

(2) $(f-g)(z) = f(z) - g(z)$

(3) $(fg)(z) = f(z) \cdot g(z)$

(4) $(cf)(z) = c \cdot f(z)$, $c \in \mathbb{R}$

(5) $\left(\frac{f}{g}\right)(z) = \frac{f(z)}{g(z)}$, $g(z) \neq 0$

(6) $(f^n)(z) = (f(z))^n$, $n \in \mathbb{N}$

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$\boxed{D}$ Misal $f : D_f \to R_f$ $g : D_g \to R_g$

adalah fungsi kompleks.

Jika $R_f \cap D_g \neq \phi$ maka terdapat suatu fungsi kompleks $h : E \to R_g$ dengan $E \subseteq D_f$ disebut fungsi komposisi f dan g ditulis $g \circ f$ yaitu

$$h(z) = (g \circ f)(z) = g(f(z))$$

D_f dari $g \circ f$ adalah prapeta $R_f \cap D_g$ terhadap fungsi f

$$D_{g \circ f} = f^{-1}(R_f \cap D_g) = \{ z \in D_f \mid f(z) \in R_f \cap D_g \}$$

R_f dari $g \circ f$ adalah peta $R_f \cap D_g$ terhadap fungsi g

$$R_{g \circ f} = g(R_f \cap D_g) = \{ g(z) \mid z \in R_f \cap D_g \}$$

$\boxed{E}$ Misal $f(z) = 3z + 1$ dan $g(z) = z^2 + z + 1 - i$

(1) Tentukan $(f+g)(z)$

(2) Selidiki apakah fungsi $g \circ f$ terdefinisi? Jika ya, tentukan $(g \circ f)(z)$

Solusi:

(1) $D_f = \mathbb{C}$, $D_g = \mathbb{C}$, $D_f \cap D_g \neq \phi$

$f + g$ terdefinisi pada $D_f \cap D_g \neq \phi$

sehingga

$$\begin{aligned}(f+g)(z) &= f(z) + g(z) \\ &= (3z+1) + (z^2 + z + 1 - i) \\ &= z^2 + 4z + 1 \ //\end{aligned}$$

next

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b) $R_f = \mathbb{C}$; $D_g = \mathbb{C}$, karena

$R_f \cap D_g = \mathbb{C} \neq \phi$, akibatnya

$g \circ f$ terdefinisi dengan

$$(g \circ f)(z) = g(f(z))$$
$$= g(3z+i)$$
$$= (3z+i)^2 + (3z+i) + 1 - i$$
$$= 9z^2 + 6zi - 1 + (3z+i) + 1 - i$$
$$= 9z^2 + (6i+3)z$$
$$= 9z^2 + (3+6i)z$$

Kalau diperhatikan, fungsi kompleks mirip dengan fungsi dua variabel rel. Kenapa? Karena z itu bisa dipandang sebagai $z = x+iy$.

Perhatikan bahwa,

$$z = x+iy \quad , x, y \in \mathbb{R}$$
$$z = r \operatorname{cis} \theta = re^{i\theta} \quad , r = |z| = \sqrt{x^2+y^2}$$
$$\theta = \operatorname{Arg}(z)$$

Kalau dimisalkan $f(z) = w$, $w = u + iv$

Maka diperoleh $f(x+iy) = u + iv$ ← $u = u(x,y)$, $v = v(x,y)$

bisa juga $f(r \operatorname{cis} \theta) = u + iv$ ← $u = u(r,\theta)$, $v = v(r,\theta)$

$$w = f(z) = u(x,y) + iv(x,y)$$
$$w = f(z) = u(r,\theta) + iv(r,\theta)$$

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E

Misal $f(z) = z^2 + z - 3$,

[Coba sendiri]

Nyatakan f dalam bentuk

$f(z) = u(x,y) + iv(x,y)$ dan

$f(z) = u(r,\theta) + iv(r,\theta)$

Penyelesaian:

- Misal $z = x + yi$; u/ suatu $x, y \in \mathbb{R}$ maka :

$$
\begin{aligned}
f(z) &= (x+yi)^2 + (x+yi) - 3 \\
&= (x^2 + 2xyi - y^2) + (x+yi) - 3 \\
&= \underbrace{(x^2 - y^2 + x - 3)}_{u(x,y)} + \underbrace{(2xy + y)i}_{v(x,y)}
\end{aligned}
$$

$u(x,y) + v(x,y)$

- Misal $z = r.\text{cis}\,\theta = r.e^{i\theta}$, maka

$$
\begin{aligned}
f(z) &= f(r.\text{cis}\,\theta) \\
&= (r.\text{cis}\,\theta)^2 + (r.\text{cis}\,\theta) - 3 \\
&= (r^2.\text{cis}\,2\theta) + (r.\text{cis}\,\theta) - 3 \\
&= [r^2(\cos 2\theta + i.\sin 2\theta)] + [r(\cos\theta + i.\sin\theta)] - 3 \\
&= [r^2.\cos 2\theta + (r^2.\sin 2\theta)i] + [r.\cos\theta + (r.\sin\theta)i] - 3 \\
&= \underbrace{(r^2.\cos 2\theta + r.\cos\theta - 3)}_{u(r,\theta)} + \underbrace{(r^2.\sin 2\theta + r.\sin\theta)i}_{v(r,\theta)}
\end{aligned}
$$

$u(r,\theta) + v(r,\theta)$

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1. Tentukan

(a) $f(2i)$ jika $f(z) = z^2 - 2z - 1$

Penyelesaian:

Misal $z = a + bi$; u/suatu $a, b \in \mathbb{R}$

Maka $f(z) = f(a + bi)$

$= (a + bi)^2 - 2(a + bi) - 1$

Sehingga untuk $a = 0$ dan $b = 2$, diperoleh

$$
\begin{aligned}
f(2i) &= f(0 + 2i) \\
&= (0 + 2i)^2 - 2(0 + 2i) - 1 \\
&= (2i)^2 - 2(2i) - 1 \\
&= 4(i)^2 - 4i - 1 \\
&= -4 - 4i - 1 \\
&= -4i - 5 \\
&= -5 - 4i \; //
\end{aligned}
$$

(b.) $f(2 - i)$ jika $f(z) = 3z^2 - i\bar{z}$

Penyelesaian:

$$
\begin{aligned}
f(2 - i) &= 3(2 - i)^2 - i\overline{(2 - i)} \\
&= 3(2 - i)^2 - i(2 + i) \\
&= 3(-4i + 5) - (2i + i^2) \\
&= -12i + 15 - (2i + (-1)) \\
&= -12i + 15 - 2i + 1 \\
&= -14i + 16 \\
&= 16 - 14i \; //
\end{aligned}
$$

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(c) $f(-i)$ jika $f(z) = \frac{z+1}{z-1}$

Penyelesaian :

$$f(-i) = \frac{(-i)+1}{(-i)-1} = \frac{(-i)+1}{(-i)-1} \cdot \frac{(-i)+1}{(-i)+1} = \frac{(i)^2+(-i)+(-i)+1}{(i)^2-i+i-1} = -i$$ //

(d) $f(-4-4i)$ jika $f(z) = |z|^2 - [Re(z)]^2$

Penyelesaian :

$$f(-4-4i) = |-4-4i|^2 - [-4]^2$$

$$= \left(\sqrt{(-4)^2+(-4)^2}\right)^2 - [16]$$

$$= 16+16 - 16$$

$$= 16$$ //

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(2.) Nyatakan f kedalam bentuk

$u(x,y) + iv(x,y)$ dan $u(r,\theta) + iv(r,\theta)$

(a.) $f(z) = z^2 + 3z^3$

Penyelesaian :

•) Misal $z = x + yi$ maka

$$f(z) = (x+yi)^2 + 3(x+yi)^3$$

$$= (x^2 + 2xyi - y^2) + 3(x^3 - 3xy^2 + (3x^2y - y^3)i)$$

$$= x^2 + 2xyi - y^2 + 3x^3 - 9xy^2 + (9x^2y - 3y^3)i$$

$$= \underbrace{(x^2 - y^2 + 3x^3 - 9xy^2)}_{u(x,y)} + \underbrace{(2xy + 9x^2y - 3y^3)}_{v(x,y)}i$$

•) Misal $z = r \operatorname{cis} \theta = re^{i\theta}$, maka

$$f(z) = f(r \operatorname{cis} \theta) =$$

$$= (r \operatorname{cis} \theta)^2 + 3(r \operatorname{cis} \theta)^3$$

$$= (r^2 \cdot \operatorname{cis} 2\theta) + 3(r^3 \operatorname{cis} 3\theta)$$

$$= [r^2(\cos 2\theta + i \cdot \sin 2\theta)] + [3 \cdot (r^3(\cos 3\theta + i \cdot \sin 3\theta))]$$

$$= [r^2 \cdot \cos 2\theta + (r^2 \cdot \sin 2\theta)i] + [3r^3 \cdot \cos 3\theta + (3r^3 \cdot \sin 3\theta)i]$$

$$= \underbrace{(r^2 \cdot \cos 2\theta + 3r^3 \cdot \cos 3\theta)}_{u(r,\theta)} + \underbrace{(r^2 \cdot \sin 2\theta + 3r^3 \cdot \sin 3\theta)}_{v(r,\theta)}i$$

$$u(r,\theta) + v(r,\theta)$$

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(b) $f(z) = i\bar{z} + \text{Im}\left(\frac{i}{z}\right)$

Penyelesaian :

$\Rightarrow$ Misal $z = x + yi$, maka

$$f(z) = i\,\overline{(x+yi)} + \text{Im}\left(\frac{i}{x+yi}\right)$$

$$= i(x-yi) + \text{Im}\left(\frac{i}{x+yi}\cdot\frac{x-yi}{x-yi}\right)$$

$$= i(x-yi) + \text{Im}\left(\frac{i(x-yi)}{x^2+y^2}\right)$$

$$= (x-yi)i + \text{Im}\left(\frac{xi - y(i)^2}{x^2+y^2}\right)$$

$$= ix - y(i)^2 + \text{Im}\left(\frac{y + xi}{x^2+y^2}\right)$$

$$= y + xi + \text{Im}\left(\frac{y}{x^2+y^2} + \left(\frac{x}{x^2+y^2}\right)\cdot i\right)$$

$$= y + xi + \frac{x}{x^2+y^2}$$

$$= \underbrace{\left(y + \frac{x}{x^2+y^2}\right)}_{U(x,y)} + \underbrace{x}_{V(x,y)}\, i$$

$\Rightarrow$ Misal $z = r\ \text{cis}\ \theta$, maka

$$f(z) = i\,\overline{(r\cdot\text{cis}\,\theta)} + \text{Im}\left(\frac{i}{r\cdot\text{cis}\,\theta}\right)$$

$$= i\left[\overline{r\cdot\cos\theta + (r\cdot\sin\theta)i}\right] + \text{Im}\left(\frac{i}{r\cdot\cos\theta + (r\cdot\sin\theta)i}\cdot\frac{r\cdot\cos\theta - (r\cdot\sin\theta)i}{r\cdot\cos\theta - (r\cdot\sin\theta)i}\right)$$

$\overline{r\cdot\cos\theta + (r\cdot\sin\theta)i} = r\cdot\cos\theta - (r\cdot\sin\theta)i$ //

$$= (r\cdot\cos\theta)i - (r\cdot\sin\theta)(i)^2 + \text{Im}\left(\frac{(r\cdot\cos\theta)i - (r\cdot\sin\theta)\cdot(i)^2}{(r\cdot\cos\theta)^2 - (r\cdot\sin\theta)^2\cdot(i)^2}\right)$$

$$= (r\cdot\cos\theta)i - (r\cdot\sin\theta)(-1) + \text{Im}\left(\frac{(r\cdot\cos\theta)i - (r\cdot\sin\theta)(-1)}{(r\cdot\cos\theta)^2 - (r\cdot\sin\theta)^2(-1)}\right)$$

$$= r\cdot\sin\theta + (r\cdot\cos\theta)i + \text{Im}\left(\frac{r\cdot\sin\theta + (r\cdot\cos\theta)i}{(r\cdot\cos\theta)^2 + (r\cdot\sin\theta)^2}\right)$$

$$= r\cdot\sin\theta + (r\cdot\cos\theta)i + \frac{r\cdot\cos\theta}{(r\cdot\cos\theta)^2 + (r\cdot\sin\theta)^2}$$

$$= \underbrace{\left(r\cdot\sin\theta + \frac{r\cdot\cos\theta}{(r\cdot\cos\theta)^2 + (r\cdot\sin\theta)^2}\right)}_{U(r,\theta)} + \underbrace{(r\cdot\cos\theta)}_{V(r,\theta)}i$$

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(c.) $f(z) = 2\pi i$

<u>Penyelesaian</u>:

- Misal $z = x + yi$ u/ suatu $x = 0$ dan $y = 2\pi$.

$z = 0 + 2\pi i$

Maka

$$f(z) = 2\pi i = \underbrace{0}_{u(x,y)} + \underbrace{(2\pi)}_{v(x,y)} i$$

- Misal $z = r \cdot cis\, \theta$, maka

$$f(z) = 2\pi i = \underbrace{0}_{u(r,\theta)} + \underbrace{(2\pi)}_{v(r,\theta)} i$$

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(d) $f(z) = \dfrac{z-i}{z+i}$

<u>Penyelesaian</u>.

•) Misal $z = x+yi$, maka

$$f(z) = \frac{(x+yi)-i}{(x+yi)+i}$$

$$= \frac{(x+yi)-i}{(x+yi)+i} \cdot \frac{(x+yi)-i}{(x+yi)-i}$$

$$= \frac{(x+yi)^2 - 2(x+yi) + 1}{(x+yi)^2 + 1}$$

$$= \frac{(x^2+2xyi-y^2) - 2(x+yi) + 1}{(x^2+2xyi-y^2)+1}$$

$$= \frac{(x^2+2xyi-y^2) - 2(x+yi) + 1}{(x^2-y^2+1)+2xyi} \cdot \frac{(x^2-y^2+1) - 2xyi}{(x^2-y^2+1)-2xyi}$$

$$= \underbrace{\frac{x^5 - 2x^3y^2 + 2x^3 + xy^4 - 2xy^2 - 2x^4 - 2x^2y^2 - 2x^2 + x}{(x^2-y^2+1)^2 + 4x^2y^2}}_{u(x,y)} + \underbrace{\left(\frac{2x^3y + 2xy^3 - 2xy}{(x^2-y^2+1)^2 + 4x^2y^2}\right)}_{v(x,y)} i$$

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∘) Misal $z = r \cdot \text{cis}\,\theta$, maka

$$f(z) = \frac{(r \cdot \text{cis}\,\theta) - i}{(r \cdot \text{cis}\,\theta) + i}$$

$$= \frac{(r \cdot \cos\theta + (r \cdot \sin\theta)\,i) - i}{(r \cdot \cos\theta + (r \cdot \sin\theta)\,i) + i}$$

$$= \frac{(r \cdot \cos\theta) + (r \cdot \sin\theta - 1)\,i}{(r \cdot \cos\theta) + (r \cdot \sin\theta + 1)\,i} \cdot \frac{(r \cdot \cos\theta) - (r \cdot \sin\theta + 1)\,i}{(r \cdot \cos\theta) - (r \cdot \sin\theta + 1)\,i}$$

$$= \frac{(r \cdot \cos\theta)^2 - (r \cdot \cos\theta)(r \cdot \sin\theta + 1)i + (r \cdot \sin\theta - 1)i \cdot (r \cdot \cos\theta) + r^2 \sin^2\theta - 1}{(r \cdot \cos\theta)^2 - (r \cdot \sin\theta + 1)^2 \cdot (i)^2}$$

$$= \underbrace{\frac{\left[(r \cdot \cos\theta)^2 + (r \cdot \sin\theta)^2 - 1\right]}{(r \cdot \cos\theta)^2 + (r \cdot \sin\theta + 1)^2}}_{U(r,\theta)} + \underbrace{\left(\frac{\left[(r \cdot \sin\theta - 1)(r \cdot \cos\theta) - (r \cdot \sin\theta + 1)(r \cdot \cos\theta)\right]}{(r \cdot \cos\theta)^2 + (r \cdot \sin\theta + 1)^2}\right)}_{V(r,\theta)} \cdot i$$