Struktur Aljabar II / Pertemuan ke-2 / Catatan

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Struktur Aljabar II : Catatan Pertemuan ke-2

$G \neq \emptyset$, $(G,*)$ grup
G Himpunan

- $(G,*)$ Tertutup
  $\forall a,b \in G \Rightarrow a,b \in G$
- $(G,*)$ Assosiatif
  $\forall a,b,c \in G \Rightarrow a*(b*c) = (a*b)*c$
- Ada unsur identitas
  $\exists e \in G, \forall a \in G \Rightarrow a*e = e*a = a$
- setiap unsur di G harus punya invers
  $\forall a \in G, \exists a' \in G \ni a*a' = a'*a = e$

$\boxed{T}$ Subhimpunan H dari grup G, $H \neq \emptyset$ disebut subgrup G jika dan hanya jika

$$\boxed{\forall a,b \in H \Rightarrow ab^{-1} \in H}$$

$\boxed{T}$ Subgrup H dari G disebut subgrup normal jika dan hanya jika

$$\boxed{\forall g \in G \text{ berlaku } gHg^{-1} \subseteq H}$$

$\mathbb{Z} \leftarrow$ Himpunan bilangan bulat

$(\mathbb{Z},+) \rightarrow$ Grup $\leftarrow (\mathbb{R}\setminus\{0\}, \times)$, $(\mathbb{Q}\setminus\{0\}, \times)$, $(\mathbb{R},+)$, $(\mathbb{Q},+)$, $(M_2(\mathbb{R}),+)$

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Misal G himpunan, $G \neq \phi$, $(G, *)$

(1) G Tertutup
$\forall a, b \in G \Rightarrow a * b \in G$ } G grupoid

(2) G Grupoid + Assosiatif } G semigrup
$\forall a, b, c \in G \Rightarrow a * (b * c) = (a * b) * c$

(3) G Semigrup + Punya unsur identitas } G Monoid
$\exists e \in G, \forall a \in G \Rightarrow a * e = e * a = a$

(4) G Monoid + Setiap unsur punya invers } G grup
$\forall a \in G, \exists a' \in G \ni a * a' = a' * a = e$

$\boxed{\text{D}}$ Grup $(G, *)$ disebut grup komutatif / Abelian
Jika $\forall a, b \in G \Rightarrow a * b = b * a$

Fenomena I $\rightarrow$ $(\mathbb{R}, \times)$ $\rightarrow$ Bukan Grup

$(\mathbb{R}, \times)$ $\rightarrow$ Monoid

Komutatif

$(\mathbb{R}, +)$ $\rightarrow$ Grup komutatif

$(\mathbb{R}, \times)$ $\rightarrow$ Monoid (Semigrup)!

Distributif

$\forall a, b, c \in \mathbb{R}$
$a \times (b + c) = (a \times b) + (a \times c)$
$(a + b) \times c = (a \times c) + (b \times c)$

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Fenomena II

Distributif

$$\forall a,b,c \in \mathbb{Z}$$
$$a \times (b+c) = (a \times b) + (a \times c)$$
$$(a+b) \times c = (a \times c) + (b \times c)$$

$(\mathbb{Z},+)$ → Grup Komutatif

$(\mathbb{Z},\times)$ → Semigrup

Fenomena III

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a,b,c,d \in \mathbb{R} \right\}$$

$(M_2(\mathbb{R}),+)$ $(M_2(\mathbb{R}),\times)$

Distributif

↓

Dan Banyak Fenomena Lainnya !

↓

$\boxed{\text{Ring (Gelanggang)}}$

(Generalisasi dari Sistem aljabar yang diamati)

"Melibatkan dua operasi"

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[1]

Ring / Gelanggang

Misalkan R suatu himpunan yang tidak kosong dan $+$ dan $\cdot$ adalah dua operasi di R

$$+ : R \times R \longrightarrow R \qquad \cdot : R \times R \longrightarrow R$$
$$(a,b) \longmapsto a+b \qquad (a,b) \longmapsto ab$$

yang berturut-turut merupakan operasi "tambah" dan "kali". Sistem matematika $(R,+,\cdot)$ disebut <u>Ring / Gelanggang</u>.

(1) $(R,+)$ merupakan grup komutatif / grup abel.

(a) $\forall a,b,c \in R \Rightarrow a+(b+c) = (a+b)+c$

(b) $\exists e \in R, \forall a \in R \Rightarrow a+e = e+a \quad [e = 0_R]$

(c) $\forall a \in R, \exists a' \in R \Rightarrow a+a' = a'+a = 0_R$

(d) $\forall a,b \in R \Rightarrow a+b = b+a$

(2) $(R,\cdot)$ merupakan semigrup

$\forall a,b,c \in R \Rightarrow a(bc) = (ab)c$

(3) $(R,+,\cdot)$ bersifat distributif

$\forall a,b,c \in R \Rightarrow a(b+c) = ab+ac$

$(a+b)c = ac+bc$

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[N]

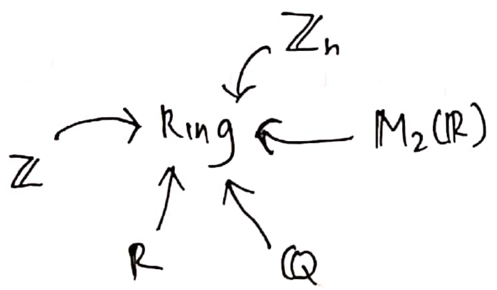
(1) Notasi $(R, +, \times)$ Ring $\xrightarrow{\text{Tulis}}$ R ring
(kecuali disebutkan khusus)

(2) $(R, +, \times)$ ⟵ tidak selamanya operasinya harus "tambah" dan "kali"

└ Pandang $+$ ⟵ "Operasi pertama"
$\times$ ⟵ "Operasi kedua"

[E]

(1)

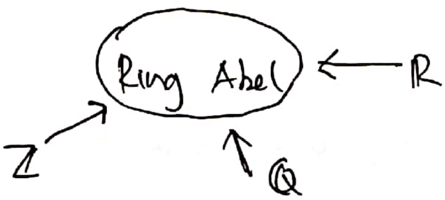


[D] Ring / Gelanggang R disebut Ring abel jika

$$\boxed{\forall a,b \in R \Rightarrow ab = ba}$$

[E]

(1)



(2) $M_2(\mathbb{R})$ Ring tidak abel.

Misal $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \in M_2(\mathbb{R})$, $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in M_2(\mathbb{R})$

$$\left.\begin{array}{l} AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 11 & 4 \end{pmatrix} \\ BA = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 5 & 8 \end{pmatrix} \end{array}\right\} AB \neq BA$$

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$\boxed{R}$

$(R, +, \times)$

$R \neq \phi$

Ring

(1) $\forall\, a, b \in R \Rightarrow a+b \in R$

(2) $\forall\, a, b, c \in R \Rightarrow a+(b+c) = (a+b)+c$

(3) $\exists\, 0_R \in R,\ \forall\, a \in R \Rightarrow a + 0_R = 0_R + a = a$

(4) $\forall\, a \in R,\ \exists\, (-a) \in R \ni a+(-a) = (-a)+a = 0_R$

(5) $\forall\, a, b \in R \Rightarrow a+b = b+a$

} $(R, +)$ Grup Abel

(6) $\forall\, a, b \in R \Rightarrow ab \in R$

(7) $\forall\, a, b, c \in R \Rightarrow a(bc) = (ab)c$

} $(R, \times)$ Semigrup

(8) $\forall\, a, b, c \in R$ berlaku

(a) $a(b+c) = ab + ac$

(b) $(a+b)c = ac + bc$

} $(R, +, \times)$ Distributif

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Topik Diskusi

Diberikan himpunan

$$Gl_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}$$

Periksa apakah $Gl_2(\mathbb{R})$ gelanggang

Penyelesaian :

~~Akan dibuktikan~~ : $Gl_2(\mathbb{R})$, bukan Gelanggang / Ring

Akan ditunjukkan : $Gl_2(\mathbb{R})$ tidak memenuhi aksioma ke-3 ($\exists$ identitas)

Adb : $Gl_2(\mathbb{R}) \neq \phi$

Misal $A = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix} \in Gl_2(\mathbb{R})$; $(5.2) - (3.4) = 10 - 12 = -2 \neq 0$

Ambil sebarang $A, B, C \in Gl_2(\mathbb{R})$

Tulis $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ untuk suatu $a_1, b_1, c_1, d_1 \in \mathbb{R}$; $(a_1 . d_1) - (b_1 . c_1) \neq 0$

$B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ untuk suatu $a_2, b_2, c_2, d_2 \in \mathbb{R}$; $(a_2 . d_2) - (b_2 . c_2) \neq 0$

$C = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$ untuk suatu $a_3, b_3, c_3, d_3 \in \mathbb{R}$; $(a_3 . d_3) - (b_3 . c_3) \neq 0$

Note that,

(1) Adb. $\forall A, B \in Gl_2(\mathbb{R}) \Rightarrow A+B \in Gl_2(\mathbb{R})$

Note that,

$$A + B = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} \in Gl_2(\mathbb{R})$$

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(2) Adb. $\forall A, B, C \in Gl_2(\mathbb{R}) \Rightarrow A+(B+C) = (A+B)+C$

Note that

- $A+(B+C) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \left[\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right]$

$= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2+a_3 & b_2+b_3 \\ c_2+c_3 & d_2+d_3 \end{pmatrix}$

$= \begin{pmatrix} a_1+a_2+a_3 & b_1+b_2+b_3 \\ c_1+c_2+c_3 & d_1+d_2+d_3 \end{pmatrix}$ ...... (*)

- $(A+B)+C = \left[\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right] + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$

$= \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$

$= \begin{pmatrix} a_1+a_2+a_3 & b_1+b_2+b_3 \\ c_1+c_2+c_3 & d_1+d_2+d_3 \end{pmatrix}$ ...... (**)

karena (*) = (**) maka $A+(B+C) = (A+B)+C$

(3) Adb. $\cancel{\exists}\ I \in Gl_2(\mathbb{R}), \forall A \in Gl_2(\mathbb{R}) \Rightarrow A+I = I+A = A$

Pilih $I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin Gl_2(\mathbb{R})$ karena $(0.0)-(0.0) = 0$

Di lain pihak, tidak ada $I \in Gl_2(\mathbb{R})$ lain yang dapat memenuhi sedemikian sehingga $A+I = I+A = A$

$\therefore$ Karena $Gl_2(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{R}, ad-bc \neq 0\right\}$ tidak memiliki identitas terhadap operasi penjumlahannya, maka $Gl_2(\mathbb{R})$ BUKAN GELANGGANG / RING.

$\boxed{///}$ (terbukti)

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Struktur Aljabar II

$2\mathbb{Z} = \{$ Bilangan bulat kelipatan dua $\}$
$= \{ 2z \ : z \in 2\mathbb{Z} \}$

Apakah $(2\mathbb{Z}, +, \times)$ Ring ?

Penyelesaian :

Akan ditunjukkan : 1.) $2\mathbb{Z} \neq \phi$
2) $(2\mathbb{Z}, +)$ Grup Abel
3.) $(2\mathbb{Z}, \times)$ Semigrup
4.) $(2\mathbb{Z}, +, \times)$ Distributif

Note that,

1.) $2\mathbb{Z} \neq \phi$ sebab $\exists\, 2 \in 2\mathbb{Z}$ .... $\blacksquare$

2) Akan ditunjukkan : $(2\mathbb{Z}, +)$ memenuhi sifat tutup, assosiatif, $\exists$identitas . $\exists$invers, dan komutatif.

- Tutup , $\forall a, b \in 2\mathbb{Z} \Rightarrow (a+b) \in 2\mathbb{Z}$

Ambil sebarang $a, b \in 2\mathbb{Z}$

Tulis $a = 2z_1 \quad , z_1 \in \mathbb{Z}$
$b = 2z_2 \quad , z_2 \in \mathbb{Z}$

Note that,

$(a+b) = (2z_1 + 2z_2)$
$= 2(z_1 + z_2) \in 2\mathbb{Z}$ .... $\blacksquare$

- Assosiatif, $\forall a, b, c \in 2\mathbb{Z} \Rightarrow (a+b)+c = a+(b+c)$

Ambil sebarang $a, b, c \in 2\mathbb{Z}$

Tulis $a = 2.z_1 \quad , z_1 \in \mathbb{Z}$
$b = 2z_2 \quad , z_2 \in \mathbb{Z}$
$c = 2.z_3 \quad , z_3 \in \mathbb{Z}$

Note that,

$(a+b)+c = (z_1 + z_2) + z_3$
$= z_1 + (z_2 + z_3)$
$= a + (b+c)$ .... $\blacksquare$

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•) $\exists$ identitas, yaitu $\forall a \in 2\mathbb{Z}$, $\exists b \in 2\mathbb{Z} \ni a+b = b+a = a$

. Ambil sebarang $a \in 2\mathbb{Z}$

Tulis, $a = 2z_1$, $z_1 \in \mathbb{Z}$

$\Rightarrow \exists\ b = 0 \in 2\mathbb{Z} \ni a+b = 2z_1 + 0 = 2z_1 = a$ dan

$b+a = 0 + 2z_1 = 2z_1 = a$ . ... $\blacksquare$

•) $\exists$ invers, yaitu $\forall a \in 2\mathbb{Z}$, $\exists b \in 2\mathbb{Z} \ni a+b = b+a = 0$

Ambil sebarang $a \in 2\mathbb{Z}$

Tulis, $a = 2$, $\in \mathbb{Z}$

$\Rightarrow \exists\ b = -a \ni a+b = a+(-a) = 2z_1 + (-2z_1)$

$= 2z_1 - 2z_1$

$= 0$

dan

$b+a = (-a)+a = (-2z_1) + z_1$

$= 0$ .... $\blacksquare$

•) Komutatif, $\forall a,b \in 2\mathbb{Z} \Rightarrow (a+b) = (b+a)$

Ambil sebarang $a,b \in 2\mathbb{Z}$

Tulis, $a = 2z_1$, $z_1 \in \mathbb{Z}$

$b = 2z_2$, $z_2 \in \mathbb{Z}$

Note that,

$a+b = 2z_1 + 2z_2$

$= 2z_2 + 2z_1$

$= b + a$ .... $\blacksquare$

$\therefore (2\mathbb{Z},+)$ adalah grup abelian.

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3) Akan ditunjukkan : $(2\mathbb{Z}, \times)$ memenuhi sifat tutup dan assosiatif.

•) Tutup, $\forall a, b \in 2\mathbb{Z} \Rightarrow a \times b \in 2\mathbb{Z}$

Ambil sebarang $a, b \in 2\mathbb{Z}$

Tulis $a = 2z_1$, $z_1 \in \mathbb{Z}$

$b = 2z_2$, $z_2 \in \mathbb{Z}$

Note that,

$$
\begin{aligned}
(a \times b) &= (2z_1 \times 2z_2) \\
&= (2z_2 \times 2z_1) \\
&= (b \times a) \quad \ldots\ldots\ldots\ldots\ldots\ldots \boxtimes
\end{aligned}
$$

•) Assosiatif, $\forall a, b, c \in 2\mathbb{Z} \Rightarrow a \times (b \times c) = (a \times b) \times c$

Ambil sebarang $a, b, c \in 2\mathbb{Z}$

Tulis, $a = 2z_1$, $z_1 \in \mathbb{Z}$

$b = 2z_2$, $z_2 \in \mathbb{Z}$

$c = 2z_3$, $z_3 \in \mathbb{Z}$

Note that,

$$
\begin{aligned}
a \times (b \times c) &= z_1 \times (z_2 \times z_3) \\
&= (z_1 \times z_2) \times z_3 \\
&= (a \times b) \times c \quad \ldots\ldots\ldots\ldots\ldots\ldots \boxtimes
\end{aligned}
$$

$\therefore$ $(2\mathbb{Z}, \times)$ adalah semigrup.

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4.) Akan ditunjukkan: $\forall\, a,b,c \in 2\mathbb{Z}$ berlaku

- $a \times (b+c) = ab + ac$
- $(a+b) \times c = ac + bc$

Penyelesaian:

- Ambil sebarang $a,b,c \in 2\mathbb{Z}$

Tulis, $a = 2z_1$, $z_1 \in \mathbb{Z}$

$b = 2z_2$, $z_2 \in \mathbb{Z}$

$c = 2z_3$, $z_3 \in \mathbb{Z}$

Note that,

$$
\begin{aligned}
a \times (b+c) &= 2z_1 \times (2z_2 + 2z_3) \\
&= (2z_1 \cdot 2z_2) + (2z_1 \cdot 2z_3) \\
&= ab + ac \quad \ldots\ldots\ldots\ldots\ldots \blacksquare
\end{aligned}
$$

- Ambil sebarang $a,b,c \in 2\mathbb{Z}$

Tulis, $a = 2z_1$, $z_1 \in \mathbb{Z}$

$b = 2z_2$, $z_2 \in \mathbb{Z}$

$c = 2z_3$, $z_3 \in \mathbb{Z}$

Note that,

$$
\begin{aligned}
(a+b) \times c &= (2z_1 + 2z_2) \times 2z_3 \\
&= (2z_1 \cdot 2z_3) + (2z_2 2z_3) \\
&= ac + bc \quad \ldots\ldots\ldots\ldots\ldots \blacksquare
\end{aligned}
$$

$\therefore (2\mathbb{Z}, +, \times)$ adalah distributif.

$\therefore$ Karena $2\mathbb{Z} \neq \phi$, $(2\mathbb{Z}, +)$ Grup abelian, $(2\mathbb{Z}, \times)$ semi grup, dan $(2\mathbb{Z}, +, \times)$ distributif, maka $(2\mathbb{Z}, +, \times)$ adalah Ring.

(Terbukti) $\blacksquare$

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