Space Lebesgue (Ruang Lebesgue)

Ruang Lebesgue dinotasikan dengan $L^p(\mathbb{R}) = \left\{ f \mid \int_{\mathbb{R}} |f|^p < \infty; 1 \le p \le \infty \right\}$. Jika $f \in L^P(\mathbb{R})$, norm dinotasikan dengan $\|f\|_p = (\int |f(x)| dx)^{\frac{1}{p}}$.

Contoh :

Misalkan fungsi $f(x)$ adalah

$$f(x) = \begin{cases} 1 \,; jika -1 \le x \le 1 \\ 0 \,; selainnya \end{cases}$$

Jawab :

$\int_{-\infty}^{\infty} |f(x)| \, dx$

$= \int_{-\infty}^{-1} 0 \, dx + \int_{-1}^{1} 1 \, dx + \int_{1}^{\infty} 0 \, dx$

$= 0 + x|_{-1}^{1} + 0$

$= 1 - (-1) = 2$

$\therefore f(x) \in L^1(\mathbb{R})$

Contoh :

$f(x) = 1, \forall x \in [-2,2]$

Jawab :

$\int_{-\infty}^{\infty} |f(x)| dx$

$= \int_{-2}^{2} 1 \, dx$

$= x|_{-2}^{2}$

$= 2 - (-2) = 4$

$\therefore f(x) \in L^1[-2,2]$

Contoh :

$f(x) = e^{-|x|}, x \in \mathbb{R}$

Buktikan $f \in L^1(\mathbb{R})$

Jawab :

Karena $|x| = \begin{cases} x\,; jika\ x \geq 0 \\ -x; jika\ x < 0 \end{cases}$

$\int_{-\infty}^{\infty} |f(x)|\, dx$

$= \int_{-\infty}^{0} |f(x)| dx + \int_{0}^{\infty} |f(x)|\, dx$

$= \int_{-\infty}^{0} \left|e^{-(-x)}\right| dx + \int_{0}^{\infty} \left|e^{(-x)}\right|\, dx$

$= \lim_{n\to\infty} e^{x}\ |_{-\infty}^{0} + \lim_{n\to\infty} -e^{-x}\ |_{0}^{\infty}$

$= (e^{0} - e^{-\infty}) + (-e^{-\infty} + e^{0})$

$= (1 - 0) + (0 + 1)$

$= 2$

$\therefore f(x) \in L^{1}(\mathbb{R})$

Contoh :

$f(x) = e^{-x} dx$

Jawab :

$\int_{-\infty}^{\infty} e^{-x} dx$

$= \lim_{n\to\infty} \int_{-n}^{n} e^{-x} dx$

$= \lim_{n\to\infty}\left(-e^{-n} - (-e^{n})\right)$

$= \lim_{n\to\infty}(-e^{-n} + e^{n})$

$= 0 + \infty = \infty$

$\therefore f(x) \notin L^{1}(\mathbb{R})$

Contoh :

$f(x) = e^{-x} dx\,,\ x \in \mathbb{R}$

Buktikan apakah $f(x) \in L^{2}(\mathbb{R})$

Jawab :

$\lim_{n\to\infty} \left(\int_{-\infty}^{\infty} \left|e^{-|x|}\right|^{p}\right)^{\frac{1}{p}}$

$$= \lim_{n\to\infty}\left(\left(\int_{-\infty}^{0}|e^{2x}|\right)^{\frac{1}{2}} + \left(\int_{0}^{\infty}|e^{-2x}|\right)^{\frac{1}{2}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}e^{2x}\Big|\begin{matrix}0\\-n\end{matrix}\right)^{\frac{1}{2}} + \lim_{m\to\infty}\left(\frac{1}{2}e^{-2x}\Big|\begin{matrix}m\\0\end{matrix}\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}\lim_{n\to\infty}(e^{0} - e^{2n})\right)^{\frac{1}{2}} + \left(-\frac{1}{2}\lim_{m\to\infty}(e^{2m} - e^{0})\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1-0) + \frac{1}{2}(0-1) = \left(\frac{1}{2} + \frac{1}{2}\right)^{\frac{1}{2}} = 1$$

$\therefore f(x) \in L^2(\mathbb{R})$

Ruang L^p (L^p Spaces)

Definisi

Misalkan $(X, \mathcal{A}, \mu)$ adalah ruang ukuran berhingga-σ. Norm dari L^p didefiniskan

$$\|f\|_p = \left(\int |f(x)|^p \, d\mu\right)^{\frac{1}{p}}; 1 < p < \infty$$

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, d\mu\right)^{\frac{1}{p}} = \|f\|_p^p = \int_{-\infty}^{\infty} |f(x)|^p \, d\mu$$

Contoh :

$$\|f\|_{3/2} = \left(\int_{-\infty}^{\infty} |f(x)|^{3/2} \, d\mu\right)^{\frac{1}{3/2}} = \|f\|_{3/2}^{3/2} = \int_{-\infty}^{\infty} |f(x)|^{3/2} \, d\mu$$

Untuk $p = \infty$, norma L^p didefinisikan

$$\|f\|_\infty = \mathrm{Inf}\{M: \mu(\{x: |f(x) \geq M|\}) = 0\}$$

M adalah bilangan real positif terkecil sehingga $f(x) \leq M \; a.e$

Proposisi 15.1 (Hölder's Inequality)

Jika $1 < p < \infty$ dan $1 < q < \infty$, dan $\frac{1}{p} + \frac{1}{q} = 1$, maka

$$\int |fg| d\mu \leq \|f\|_p \|g\|_q = \left(\int |f(x)|^p \, d\mu\right)^{\frac{1}{p}} \left(\int |g(x)|^q \, d\mu\right)^{\frac{1}{q}}$$

Misal :

$$p = 2, \qquad q = 2$$

$$\|fg\|_1 = \|f\|_2 \|g\|_2$$

$$p = 3, \qquad q = 3/2$$

$$\|fg\|_1 = \|f\|_3 \|g\|_{3/2}$$

Kasus 1 : $p = \infty, q = 1$

$$\|f\|_\infty = \sup_{\mathrm{x}} |f(x)|$$

$= |f| \leq M$, untuk suatu $M > 0$ atau

$$\int |fg|\, d\mu = \int |f|\, |g| d\mu \leq M|g| d\mu$$

$$= M \int |g|\, d\mu = M|g|_1 = \|f\|_\infty \|g\|_1$$

Kasus 2 : $\|f\|_p = 0$ atau $\|g\|_q = 0$

Jika $\|f\|_p = 0$, maka $f = 0$

$$\|fg\| = \int |fg| d\mu \leq \|f\|_p \|g\|_q$$

Karena $f = 0$, maka

$$\|0g\| = 0$$

Kasus 3 : $F(x) = \frac{|f(x)|}{\|f\|_p}$ dan $G(x) = \frac{|g(x)|}{\|g\|_q}$

$$\|f\|_p = \left(\int |F(x)|^p\, d\mu\right)^{\frac{1}{p}} = \left(\int \left|\frac{|f(x)|}{\|f\|_p}\right|^p d\mu\right)^{\frac{1}{p}} = \left(\int \frac{|f(x)|^p}{\|f\|_p^p}\right)^{\frac{1}{p}}$$

$$= \frac{1}{\|f\|_p} \left(\int |f(x)|^p\, d\mu\right)^{\frac{1}{p}} = \frac{1}{\|f\|_p} \|f\|_p = 1$$

$$\|g\|_q = \left(\int |G(x)|^q\, d\mu\right)^{\frac{1}{q}} = \left(\int \left|\frac{|g(x)|}{\|g\|_q}\right|^q d\mu\right)^{\frac{1}{q}} = \left(\int \frac{|g(x)|^q}{\|f\|_q^q}\right)^{\frac{1}{q}}$$

$$= \frac{1}{\|g\|_q} \left(\int |g(x)|^q\, d\mu\right)^{\frac{1}{q}} = \frac{1}{\|g\|_q} \|g\|_q = 1$$

Akan ditunjukkan $\int FG\, d\mu \leq 1$

Untuk setiap λ dengan $a, b \in \mathbb{R}$, dimana $0 \leq \lambda \leq 1$. Berlaku

$$e^{\lambda a + (1-b)\lambda} \leq \lambda e^a + (1-\lambda) e^b$$

Untuk setiap $a \leq b$, jika $F(x), G(x) \neq 0$, ambil

$$a = p \log F(x)$$

$$b = q \log G(x)$$

$$\lambda = \frac{1}{p} \text{ dan } 1 - \lambda = \frac{1}{q}$$

Jadi

$$e^{\frac{1}{p} \cdot p \log F(x) + \frac{1}{q} \cdot q \log G(x)} \leq \frac{1}{p} e^{p \log F(x)} + \frac{1}{q} e^{q \log G(x)}$$

Atau

$$e^{\log F(x) + \log G(x)} \leq \frac{1}{p} e^{\log F(x)^p} + \frac{1}{q} e^{\log G(x)^q}$$

Ini berarti

$$e^{\log F(x).G(x)} \leq \frac{1}{p} e^{\log F(x)^p} + \frac{1}{q} e^{\log G(x)^q}$$

Jadi diperoleh

$$F(x) \cdot G(x) \leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q$$

$$\int FG \, d\mu \leq \int \frac{1}{p} F(x)^p d\mu + \int \frac{1}{q} G(x)^q d\mu$$

$$= \int \frac{1}{p} \cdot \left(\frac{|f(x)|}{||f||_p} \right)^p d\mu + \int \frac{1}{q} \cdot \left(\frac{|g(x)|}{||g||_q} \right)^q d\mu$$

$$= \frac{1}{p} \int \frac{|f(x)|^p}{||f||_p^p} d\mu + \frac{1}{q} \int \frac{|g(x)|^q}{||g||_q^q} d\mu$$

$$= \frac{1}{p} \cdot \frac{1}{||f||_p^p} \int |f(x)|^p d\mu + \frac{1}{q} \cdot \frac{1}{||g||_q^q} \int |g(x)|^q d\mu$$

$$= \frac{1}{p} \cdot \frac{1}{||f||_p^p} \cdot ||f||_p^p + \frac{1}{q} \cdot \frac{1}{||g||_q^q} ||g||_q^q$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

Karena $\int FG \, d\mu \leq 1$

$$\int \frac{|f(x)|^p}{||f||_p^p} \cdot \frac{|g(x)|^q}{||g||_q^q} d\mu \leq 1$$

$$\frac{1}{||f||_p} \cdot \frac{1}{||g||_q} \int |fg| \, d\mu \leq 1$$

$$\int |fg| \, d\mu \leq ||f||_p ||g||_q$$