PERTEMUAN 3

Ruang Vektor atas Lapangan

$\boxed{1}$

Misalkan V suatu himpunan, $V \neq \emptyset$ dengan F suatu lapangan, V dilengkapi dengan operasi penjumlahan

$$+ : V \times V \longrightarrow V$$
$$(\vec{a}, \vec{b}) \longmapsto \vec{a} + \vec{b}$$

dan operasi perkalian skalar

$$\cdot : F \times V \longrightarrow V$$
$$(\alpha, \vec{a}) \longmapsto \alpha\vec{a}$$

disebut **Ruang Vektor atas lapangan F** jika memenuhi :

(1) $\forall\ \vec{a}, \vec{b} \in V \Rightarrow \vec{a} + \vec{b} \in V$

(2) $\forall\ \vec{a}, \vec{b}, \vec{c} \in V \Rightarrow \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

(3) Terdapat $\vec{0} \in V$ sehingga untuk setiap $\vec{a} \in V$ memenuhi

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

(4) $\forall\ \vec{a} \in V\ \exists\ -\vec{a} \in V$ sehingga $\vec{a} + (-\vec{a}) = -\vec{a} + \vec{a} = \vec{0}$

(5) $\forall\ \vec{a}, \vec{b} \in V \Rightarrow \vec{a} + \vec{b} = \vec{b} + \vec{a}$

(6) $\forall\ \vec{a} \in V, \alpha \in F \Rightarrow \alpha\vec{a} \in V$

(7) $\forall\ \vec{a}, \vec{b} \in V, \alpha \in F \Rightarrow \alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b}$

(8) $\forall\ \vec{a} \in V,\ \alpha, \beta \in F \Rightarrow (\alpha + \beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$

(9) $\forall\ \vec{a} \in V,\ \alpha, \beta \in F \Rightarrow (\alpha\beta)\vec{a} = \alpha(\beta\vec{a})$

(10) $\exists\ \vec{1} \in F, \forall\ \vec{a} \in V \Rightarrow 1\vec{a} = \vec{a}$.

$\boxed{E}$

① Himpunan $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$ dengan operasi penjumlahan dan perkalian skalar

$$\vec{x}+\vec{y} = (a_1,b_1)+(a_2,b_2) = (a_1+a_2, b_1+b_2)$$

$$\alpha\vec{x} = \alpha(a_1,b_1) = (\alpha a_1, \alpha b_1)$$

Buktikan $\mathbb{R}^2$ ruang vektor atas $\mathbb{R}$

Bukti

Ambil $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$, $\alpha, \beta \in \mathbb{R}$ sebarang

tulis

$\vec{x} = (a_1,b_1)$ untuk suatu $a_1,b_1 \in \mathbb{R}$

$\vec{y} = (a_2,b_2)$ untuk suatu $a_2,b_2 \in \mathbb{R}$

$\bar{z} = (a_3,b_3)$ untuk suatu $a_3,b_3 \in \mathbb{R}$

(1) adb $\mathbb{R}^2 \neq \phi$

misal $(1,2) \in \mathbb{R}^2$ jadi $\mathbb{R}^2 \neq \phi$

(2) adb $\vec{x}+\vec{y} \in \mathbb{R}^2$

$$\vec{x}+\vec{y} = (a_1,b_1)+(a_2,b_2) = (a_1+a_2, b_1+b_2) \in \mathbb{R}^2$$

(3) ads $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

Perhatikan bahwa

$$
\begin{aligned}
(\vec{x}+\vec{y})+\vec{z} &= [(a_1,b_1)+(a_2,b_2)]+(a_3,b_3) \\
&= (a_1+a_2, b_1+b_2)+(a_3,b_3) \\
&= (a_1+a_2+a_3, b_1+b_2+b_3) \\
&= (a_1+(a_2+a_3), b_1+(b_2+b_3)) \\
&= (a_1,b_1)+(a_2+a_3, b_2+b_3) \\
&= (a_1,b_1)+[(a_2,b_2)+(a_3,b_3)] \\
&= \vec{x}+(\vec{y}+\vec{z})
\end{aligned}
$$

(4) Pilih $\vec{0}=(0,0) \in \mathbb{R}^2$ sehingga untuk setiap $\vec{x}=(a_1,b_1) \in \mathbb{R}^2$

$a_1, b_1 \in \mathbb{R}$ Berlaku

$$\vec{x}+\vec{0} = (a_1,b_1)+(0,0) = (a_1+0, b_1+0) = (a_1,b_1) = \vec{x}$$

dilain pihak

$$\vec{0}+\vec{x} = (0,0)+(a_1,b_1) = (0+a_1, 0+b_1) = (a_1,b_1) = \vec{x}$$

sehingga

$$\vec{x}+\vec{0} = \vec{0}+\vec{x} = \vec{x}.$$

(5) Ambil $\vec{x} = (a_1, b_1) \in \mathbb{R}^2$, $a_1, b_1 \in \mathbb{R}$, pilih

$-\vec{x} = (-a_1, -b_1) \in \mathbb{R}^2$ sehingga

$$\vec{x} + (-\vec{x}) = (a_1, b_1) + (-a_1, -b_1)$$
$$= (a_1 - a_1, b_1 - b_1) = (0,0) = \vec{0}$$

dilain pihak

$$(-\vec{x}) + \vec{x} = (-a_1, -b_1) + (a_1, b_1)$$
$$= (-a_1 + a_1, -b_1 + b_1) = (0,0) = \vec{0}$$

Jadi

$\forall x \in \mathbb{R}^2 \ \exists \ \vec{0} \in \mathbb{R}^2$ sehingga $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$

(6) adb $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

$$\vec{x} + \vec{y} = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) = (a_2 + a_1, b_2 + b_1)$$
$$= (a_2, b_2) + (a_1, b_1)$$
$$= \vec{y} + \vec{x}$$

(7) adb $\alpha\vec{x} \in \mathbb{R}^2$

$$\alpha\vec{x} = \alpha(a_1, b_1) = (\alpha a_1, \alpha b_1) \in \mathbb{R}^2$$

(8) adb $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$

$$\alpha(\vec{x} + \vec{y}) = \alpha((a_1, b_1) + (a_2, b_2))$$
$$= \alpha(a_1 + a_2, b_1 + b_2)$$
$$= (\alpha(a_1 + a_2), \alpha(b_1 + b_2))$$
$$= (\alpha a_1 + \alpha a_2, \alpha b_1 + \alpha b_2)$$
$$= (\alpha a_1, \alpha b_1) + (\alpha a_2, \alpha b_2)$$
$$= \alpha(a_1, b_1) + \alpha(a_2, b_2)$$
$$= \alpha\vec{x} + \alpha\vec{y}$$

(9) adb $(\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$

$$(\alpha+\beta)\vec{x} = (\alpha+\beta)(a_1,b_1)$$

$$= ((\alpha+\beta)a_1, (\alpha+\beta)b_1)$$

$$= (\alpha a_1+\beta a_1, \alpha b_1+\beta b_1)$$

$$= (\alpha a_1, \alpha b_1) + (\beta a_1, \beta b_1)$$

$$= \alpha(a_1, b_1) + \beta(a_1, b_1) = \alpha\vec{x}+\beta\vec{y}$$

(10) adb $(\alpha\beta)\vec{x} = \alpha(\beta\vec{x})$

$$(\alpha\beta)\vec{x} = (\alpha\beta)(a_1, b_1)$$

$$= ((\alpha\beta)a_1, (\alpha\beta)b_1)$$

$$= (\alpha(\beta a_1), \alpha(\beta b_1))$$

$$= \alpha(\beta a_1, \beta b_1) = \alpha[\beta(a_1,b_1)] = \alpha[\beta\vec{x}]$$

(11) Terdapat $1 \in \mathbb{R}$ sehingga untuk setiap $\bar{x} \in \mathbb{R}^2$

berlaku $1\bar{x} = 1(a_1,b_1) = (a_1,b_1) = \bar{x}$

$\therefore$ $\mathbb{R}^2$ ruang vektor atas $\mathbb{R}$

(2) Periksa yang manakah diantara himpunan berikut yang merupakan ruang vektor.

(a) $\mathbb{Z}$ ruang vektor atas $\mathbb{R}$?

Jawab

Bukan, karena $\alpha = \sqrt{2}$, $a = 2 \in \mathbb{Z}$

tetapi $\alpha a = 2\sqrt{2} \notin \mathbb{Z}$

$\therefore$ $\mathbb{Z}$ bukan ruang vektor atas $\mathbb{R}$

(b) $\mathbb{R}$ ruang vektor atas $\mathbb{Z}$?

Jawab

Bukan, karena $\mathbb{Z}$ bukan lapangan.

(c) $\mathbb{M}_2(\mathbb{R})$ ruang vektor atas $\mathbb{R}$?

Jawab

Ya, coba anda buktikan detail.

Tugas

Buktikan bahwa

$$\mathbb{M}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \right\}$$

merupakan ruang vektor atas lapangan F