Makassar, 6 Oktober 2020

STRUKTUR ALJABAR II

— Pertemuan V —

(Catatan)

Imanuel AS

1811141008

Struktur Aljabar II / Pertemuan ke-5 / Catatan

Makassar, 20 Oktober 2020

Nama : Imanuel AS / 181114100?

NIM : 181114100 8

Struktur Aljabar II : Catatan pertemuan ke -5

Karakteristik Ring

$\boxed{D}$

Misal R ring dan $a \in R$; $m \in \mathbb{Z}^+$

$ma = \underbrace{a+a+a+\dots+a}_{m \text{ suku}}$, $(-m)a = \underbrace{(-a)+(-a)+\dots+(-a)}_{m \text{ suku}}$

$\boxed{0_a = 0_R}$; 0_a unsur nol di $\mathbb{Z}$, 0_R unsur nol di R

$\boxed{T_1}$ Misal $a,b \in R$, R ring, untuk setiap $m,n \in \mathbb{Z}$ berlaku

(1) $(m+n)a = ma + na$

(2) $m(a+b) = ma + mb$ $\boxed{D_1}$

(3) $m(na) = (mn)a$ $\boxed{D_2}$

Bukti

(1) Ambil $a \in R$, $m,n \in \mathbb{Z}$ sebarang

Adb. $(m+n) = ma + na$

(i) Kasus $m = n = 0$

Note that

$$(m+n)a = (0+0)a$$
$$= (0)a$$
$$= \underbrace{a+a+a+\dots+a}_{0 \text{ suku}}$$
$$= 0$$

//

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(ii) Kasus $m>0, n>0$

Note that

$$(m+n)a = \underbrace{a+a+a+\dots+a}_{(m+n)\text{ suku}}$$

$$= \underbrace{[a+a+a+\dots+a]}_{m\text{ suku}} + \underbrace{[a+a+a+\dots+a]}_{n\text{ suku}}$$

$$= ma + na$$

(iii) Kasus $m<0, n<0$

misal, $m=-p$, $n=-q$, $p,q \in \mathbb{Z}^+$

Note that

$$(m+n)a = [-p+(-q)]a$$

$$= [-(p+q)]a$$

$$= \underbrace{(-a)+(-a)+(-a)+\dots+(-a)}_{(p+q)\text{ suku}}$$

$$= \underbrace{((-a)+(-a)+\dots+(-a))}_{p\text{ suku}} + \underbrace{((-a)+(-a)+\dots+(-a))}_{q\text{ suku}}$$

$$= (-p)a + (-q)a$$

$$= ma + na$$

(iv) Kasus $m<0, n>0$ atau $m>0, n<0$

Without Lost of Generality (Tanpa mengurangi perumuman)

$m<0, n>0$, misal $m=-p$, $p \in \mathbb{Z}^+$

Note that

$$ma + na = (-p)a + na$$

$$= \underbrace{[(-a)+(-a)+\dots+(-a)]}_{p\text{ suku}} + \underbrace{[a+a+\dots+a]}_{n\text{ suku}}$$

$$= \underbrace{a+a+\dots+a}_{(n-p)\text{ suku}} \quad [-a+a = 0_R]$$

$$= [n+(-p)]a$$

$$= [n+m]a$$

$$= (m+n)a$$

$\therefore \forall a \in R, m,n \in \mathbb{Z} \Rightarrow (m+n)a = ma+na$ // $\boxed{\text{Q.E.D}}$

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(2) Ambil $a, b \in \mathbb{R}$, $m \in \mathbb{Z}$ sebarang

Adb. $m(a+b) = ma + mb$

(i) Kasus $m = 0$

Note that

$$m(a+b) = 0(a+b)$$
$$= \underbrace{(a+b) + (a+b) + \dots + (a+b)}_{0 \text{ suku}}$$
$$= 0 \quad \dots\dots\dots\dots (\star)$$

$$ma + mb = 0(a) + 0(b)$$
$$= [\underbrace{a+a+\dots+a}_{0 \text{ suku}}] + [\underbrace{b+b+\dots+b}_{0 \text{ suku}}]$$
$$= 0 \quad \dots\dots\dots\dots (\star\star)$$

Karena $(\star) = (\star\star)$ maka $m(a+b) = ma + mb$

(ii) Kasus $m > 0$

Note that

$$m(a+b) = \underbrace{(a+b) + (a+b) + \dots + (a+b)}_{m \text{ suku}}$$
$$= \underbrace{a+b + a+b + \dots + a+b}_{m+m \text{ suku}}$$ → [Karena $a, b \in R$, R Ring ⇒ Sifat tutup] → [Karena a dan b di hitung dua]

$$= \underbrace{a + a + \dots + a}_{m \text{ suku}} + \underbrace{b + b + \dots + b}_{m \text{ suku}}$$ [Sifat komutatif pada Ring]

$$= m(a) + m(b)$$ [Definisi]

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(iii) Kasus $m < 0$

misalkan $m = -p$; $p \in \mathbb{Z}^+$

Note that

$$m(a+b) = -p(a+b)$$

$$= \underbrace{[-(a+b)] + [-(a+b)] + \dots + [-(a+b)]}_{p \text{ suku}}$$

$$= \underbrace{[(-a)+(-b)] + [(-a)+(-b)] + \dots [(-a)+(-b)]}_{p \text{ suku}}$$

[karena $(-a),(-b) \in \mathbb{R}$ berlaku sifat tutup]

$$= \underbrace{(-a)+(-b)+(-a)+(-b)+\dots+(-a)+(-b)}_{(p+p) \text{ suku}}$$

[krn $(-a),(-b)$ dihitung dari suku]

[$a,b \in \mathbb{R}$, berlaku sifat komutatif]

$$= \underbrace{(-a)+(-a)+\dots+(-a)}_{p \text{ suku}} + \underbrace{(-b)+(-b)+\dots+(-b)}_{p \text{ suku}}$$

$$= -p(a) + -p(b)$$

$$= m(a) + m(b)$$

$\therefore \forall a,b \in \mathbb{R}, m \in \mathbb{Z} \Rightarrow m(a+b) = ma + mb$ //

$\boxed{\text{Q.E.D}}$

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(3) Ambil $a \in R$; $m, n \in \mathbb{Z}$ sebarang

Adb. $m(na) = (mn)a$

(i) Kasus $m = n = 0$

Note that

$$m(na) = 0(0 \cdot a)$$
$$= \underbrace{(0 \cdot a) + (0 \cdot a) + \dots + (0 \cdot a)}_{0 \text{ suku}}$$
$$= 0$$

(ii) Kasus $m > 0$, $n > 0$

Note that

$$m(na) = \underbrace{(n \cdot a) + (na) + \dots + (na)}_{m \text{ suku}}$$

[Definisi (na)]

$$= \underbrace{\Big[\underbrace{a + a + \dots + a}_{n \text{ suku}}\Big] + \Big[\underbrace{a + a + \dots + a}_{n \text{ suku}}\Big] + \dots + \Big[\underbrace{a + a + \dots + a}_{n \text{ suku}}\Big]}_{m \text{ suku}}$$

[Komutatif pada Ring]

$$= \Big[\underbrace{a + a + \dots + a + a + a + \dots + a + \dots + a + a + \dots + a}_{(mn) \text{ suku}}\Big]$$

[Jelas]

$$= (mn)a$$

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(iii) Kasus $m < 0$, $n < 0$

misal, $m = -p$, $n = -q$, $p, q \in \mathbb{Z}^+$

Note that

$$m(na) = -p((-q)\cdot a)$$

$$= \underbrace{[-((-q)\cdot a)] + [-((-q)\cdot a)] + \ldots + [-((-q)\cdot a)]}_{p \text{ suku}}$$

$[q \in \mathbb{Z}^+]$ $$= \underbrace{[(q)a] + [(q)a] + \ldots + [(q)a]}_{p \text{ suku}}$$

[Jelas] $$= \underbrace{\left[\underbrace{a+a+\ldots+a}_{q \text{ suku}}\right] + \left[\underbrace{a+a+\ldots+a}_{q \text{ suku}}\right] + \ldots + \left[\underbrace{a+a+\ldots+a}_{q \text{ suku}}\right]}_{p \text{ suku}}$$

[Komutatif pada ring] $$= \underbrace{[a+a+\ldots+a+a+a+\ldots+a+\ldots+a+a+\ldots+a]}_{(p\cdot q) \text{ suku}}$$

[Jelas]

[Definisi] $$= (pq)\cdot a$$

[Asumsi] $$= (-m\cdot -n)\cdot a$$

$[m,n \in \mathbb{Z}]$ $$= (mn)\cdot a$$

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(iv) Kasus $m<0, n>0$ atau $m>0, n<0$

Without lost of generality [Tanpa mengurangi perumuman]

$m<0, n>0$, misal $m=-p$, $p \in \mathbb{Z}^+$

Note that

$$m(na) = -p(na)$$

$$= \underbrace{[-(na)] + [-(na)] + \dots + [-(na)]}_{p \text{ suku}}$$

[$n \in \mathbb{Z}$] $$= \underbrace{[(-n)a] + [(-n)a] + \dots + [(-n)a]}_{p \text{ suku}}$$

[Jelas] $$= \underbrace{\left[\underbrace{(-a)+(-a)+\dots+(-a)}_{n \text{ suku}}\right] + \left[\underbrace{(-a)+(-a)+\dots+(-a)}_{n \text{ suku}}\right] + \dots + \left[\underbrace{(-a)+(-a)+\dots+(-a)}_{n \text{ suku}}\right]}_{p \text{ suku}}$$

[Sifat Komutatif Ring] $$= \underbrace{[(-a)+(-a)+\dots+(-a)+(-a)+(-a)+\dots+(-a)+\dots+(-a)+(-a)+\dots+(-a)]}_{(p\cdot n) \text{ suku}}$$

[Jelas]

[Definisi] $$= (-(pn)).a$$

[$p,n \in \mathbb{Z}$] $$= [(-p).(n)].a$$

[Asumsi $m=-p$] $$= [(m).(n)].a$$

$$= (mn).a \;//$$

$$\therefore \forall a \in R, m,n \in \mathbb{Z} \Rightarrow m(na) = (mn)a \;//$$

[Q.E.D]

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$\boxed{D}$ Misal R ring, $a \in R$

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ suku}} \quad , m \in \mathbb{Z}^+$$

$\boxed{T_2}$ Jika R ring maka untuk setiap $a \in R$, $m, n \in \mathbb{Z}^+$ maka

(1) $a^m \cdot a^n = a^{m+n}$
(2) $(a^m)^n = a^{mn}$ } [D_3]

Bukti

(1) $a^m \cdot a^n = [\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ suku}}] \cdot [\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ suku}}]$ [Definisi]

$= [\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ suku}} \cdot \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ suku}}]$ [$ab \in R$] [Jelas]

$= a^{m+n}$ // ∎ [Sifat eksponen]

(2) $(a^m)^n = \left[[\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ suku}}] \right]^n$ [Sifat eksponen]

$= \underbrace{[\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ suku}}] \cdot [\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ suku}}] \cdot [\ldots] \cdot [\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ suku}}]}_{n \text{ suku}}$ [Jelas]

$= [\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a \cdot a \cdot a \cdot a \cdot \ldots \cdot a \cdot \ldots \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{(mn) \text{ suku}}]$ [Jelas]

$= a^{mn}$ // ∎ [Sifat Eksponen]

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Makassar, 20 Feb 2020

$\boxed{D}$ Misal R ring. Karakteristik dari R

$char(R) = \min\{n \in \mathbb{N} \mid n.a = 0_R, \forall a \in R\}$

Jika, $n \in \mathbb{N}$ tidak ada, maka $char(R) = 0$

$\boxed{E}$

(1) $\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, $\mathbb{Z}_5$ ring

$\bar{0}$ unsur "nol" di $\mathbb{Z}_5$

$\bar{0} \to n.\bar{0} = \bar{0} \to n = 5$

$\bar{1} \to n.\bar{1} = \bar{0} \to n = 5$

$\bar{2} \to n.\bar{2} = \bar{0} \to n = 5$

$\bar{3} \to n.\bar{3} = \bar{0} \to n = 5$

$\bar{4} \to n.\bar{4} = \bar{0} \to n = 5$

$Char(\mathbb{Z}_{(5)}) = 5$

(2) $\mathbb{Z}$ ring, 0 unsur "nol" di $\mathbb{Z}$,

$\nexists\, n \in \mathbb{N}$ sehingga $\forall a \in \mathbb{Z}$ $na = 0$

$Char(\mathbb{Z}) = 0$

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D4 Cari Karakteristik dari Ring berikut

(1) $\mathbb{R}$
(2) $M_2(\mathbb{R})$
(3) $M_2(\mathbb{Z})$
(4) $\mathbb{Z}[i]$
(5) $\mathbb{Z}_3[i]$
(6) $\mathbb{Z}_7$
(7) $\mathbb{Q}$
(8) $\mathbb{Q} \times \mathbb{Q}$
(9) $\mathbb{Z} \times \mathbb{Z}$
(10) $\mathbb{R} \times \mathbb{R}$

Note: $\mathbb{Z}[i] = \{a+bi \mid a,b \in \mathbb{Z}\}$ ← Himpunan bilangan bulat Gauss

$\mathbb{Z}_3[i] = \{a+bi \mid a,b \in \mathbb{Z}_3\}$

$= \{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}$

$\boxed{i^2 = -1}$ $\qquad \mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

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(4) $\mathbb{Z}[i]$ ring

$\mathbb{Z}[i] = \{ a+bi \mid a,b \in \mathbb{Z},\ i = \sqrt{-1} \}$

$0_{\mathbb{Z}[i]} = (0+0i) \in \mathbb{Z}[i]$

Ambil sebarang $a \in \mathbb{Z}[i]$

Tulis $a = (x_1 + y_1 i)$

Note that, $\forall a \in \mathbb{Z}[i] \Rightarrow$

$\nexists n \in \mathbb{N} \ni n.a = (0+0i)$

$n.(x_1+y_1 i) = (0+0i)$

$(nx_1 + ny_1 i) = (0+0i)$

Misal $a = (5+7i) \in \mathbb{Z}[i]$

Note that,

$\nexists n \in \mathbb{N} \ni n.a = (0+0i)$

$n(5+7i) = (0+0i)$

$(5.n + 7n.i) = (0+0i)$

$\therefore$ Char $(\mathbb{Z}[i]) = 0$

(5) $\mathbb{Z}_3[i]$ ring

$\mathbb{Z}_3[i] = \{a+bi \mid a,b \in \mathbb{Z}_3\}$ $i = \sqrt{-1}$

$= \{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}$

$0_{\mathbb{Z}_3[i]} = (0+0i) \in \mathbb{Z}_3[i]$; $(0+0i) = (0+0) = 0$

Ambil sebarang $a \in \mathbb{Z}_3[i]$

Tulis $a = (x_1 + y_1 i)$

Note that,

$\nexists n \in \mathbb{N} \ni \forall a \in \mathbb{Z}_3[i] \Rightarrow n.a = (0+0i)$

$n(x_1+y_1 i) = (0+0i)$

$(nx_1 + ny_1 i) = (0+0i)$

Misal $a = (2+2i)$

Note that,

$\nexists n \in \mathbb{N} \ni n.a = (0+0i)$

$n.(2+2i) = (0+0i)$

$(2n + 2n.i) = (0+0i)$

$\therefore$ Char $(\mathbb{Z}_3[i]) = 0$

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Makassar, ...

(6) $\mathbb{Z}_7$ ring

$0_{\mathbb{Z}_7} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$

$\bar{0} \to n.\bar{0} = \bar{0} \to n = 7, 14, 21, 28, \ldots$

$\bar{1} \to n.\bar{1} = \bar{0} \to n = 7, 14, 21, 28, \ldots$

$\bar{2} \to n.\bar{2} = \bar{0} \to n = 7, 14, 21, 28, \ldots$

$\bar{3} \to n.\bar{3} = \bar{0} \to n = 7, 14, 21, 28, \ldots$

$\bar{4} \to n.\bar{4} = \bar{0} \to n = 7, 14, 21, 28, \ldots$

$\bar{5} \to n.\bar{5} = \bar{0} \to n = 7, 14, 21, 28, \ldots$

$\bar{6} \to n.6 = \bar{0} \to n = 7, 14, 21, 28, \ldots$

$\bar{7} \to n.7 = \bar{0} \to n = 7, 14, 21, 28, \ldots$

n minimum $= 7$, $n \in \mathbb{N}$

$\therefore$ Char $(\mathbb{Z}_7) = 7$

(7) $\mathbb{Q}$ ring

$0_{\mathbb{Q}} = 0$

$\nexists n \in \mathbb{N} \ni \forall a \in \mathbb{Q} \Rightarrow na = 0$

$\therefore$ Char $(\mathbb{Q}) = 0$

(8) $\mathbb{Q} \times \mathbb{Q}$ ring

$\mathbb{Q} \times \mathbb{Q} = \{(a,b) \mid a \in \mathbb{Q}, b \in \mathbb{Q}\}$ [Pasangan berurutan]

$+ : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$

$(a,b) \longmapsto a+b$

$\times : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$

$(a,b) \longmapsto a \times b$

$0_{\mathbb{Q}\times\mathbb{Q}} = (0,0)$ [Identitas Penjumlahan R^2]

Note that, $na = (0,0)$

$n(x_1, y_1) = (0,0)$

[Scalar multiple R^2 definition] $(nx_1, ny_1) = (0,0) \Rightarrow \nexists n \in \mathbb{N} \ni (nx_1, ny_1) = (0,0)$

$\therefore$ Char $(\mathbb{Q} \times \mathbb{Q}) = 0$

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(9) $\mathbb{Z} \times \mathbb{Z}$ ring

$\mathbb{Z} \times \mathbb{Z} = \{(a,b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}\}$ [Pasangan berurutan]

$+ : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$

$(a,b) \longmapsto (a+b)$

$\times : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$

$(a,b) \longmapsto (a \times b)$

$0_{\mathbb{Z} \times \mathbb{Z}} = (0,0)$ [Identity Penjumlahan $\mathbb{R}^2$]

Note that,

$n \cdot a = (0,0)$

$n \cdot (x_1, y_1) = (0,0)$ [$a = (x_1, y_1) \mid x_1 \in \mathbb{Z}, y_1 \in \mathbb{Z}$]

$(nx_1, ny_1) = (0,0)$ [Scalar Multiple $\mathbb{R}^2$ Definition]

$\Rightarrow \nexists\, n \in \mathbb{N} \ni (nx_1, ny_1) = (0,0)$

$\therefore$ Char $(\mathbb{Z} \times \mathbb{Z}) = 0$

(10.) $\mathbb{R} \times \mathbb{R}$ ring

$\mathbb{R} \times \mathbb{R} = \{(a,b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$ [Pasangan berurut]

$+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$(a,b) \longmapsto (a+b)$

$\times : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$(a,b) \longmapsto (a \times b)$

$0_{\mathbb{R} \times \mathbb{R}} = (0,0)$ [Identitas Penjumlahan $\mathbb{R}^2$]

Note that,

$n \cdot a = (0,0)$

$n(x_1, y_1) = (0,0)$ [$a = (x_1, y_1) \mid x \in \mathbb{R}, y \in \mathbb{R}$]

$(nx_1, ny_1) = (0,0)$ [Scalar Multiple $\mathbb{R}^2$ Definition]

$\Rightarrow \nexists\, n \in \mathbb{N} \ni (nx_1, ny_1) = (0,0)$

$\therefore$ Char $\mathbb{R} \times \mathbb{R} = 0$

Imanuel AS / 1811141008 Gatta Mau

Makassar, [illegible] Oktober

Definisi: Order Elemen Suatu Grup

Misal $(G, \cdot)$ adalah sebarang grup. Misal a adalah sebarang elemen dari G. Untuk suatu bilangan terkecil m yang memenuhi $a^m = e$ (e adalah elemen identitas di G) maka m dikatakan sebagai order dari a, dan dituliskan sebagai $|a| = m$.

Dalam kasus ini, jika tidak ada m yang memenuhi $a^m = e$, kita katakan bahwa a berorder infinite atau nol.

Contoh: (1) Diberikan grup modulo 6 dengan operasi jumlah atau $(M_6, +)$

Note that,

$M_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

Elemen identitas adalah $\bar{0}$

$\bar{1}^6 = \bar{1}+\bar{1}+\bar{1}+\bar{1}+\bar{1}+\bar{1} = \bar{6} = \bar{0}$ maka $|\bar{1}| = 6$

$\bar{2}^3 = \bar{2}+\bar{2}+\bar{2} = \bar{6} = \bar{0}$ maka $|\bar{2}| = 3$

$\bar{3}^2 = \bar{3}+\bar{3} = \bar{6} = \bar{0}$ maka $|\bar{3}| = 2$

$\bar{4}^3 = \bar{4}+\bar{4}+\bar{4} = \bar{12} = \bar{0}$ maka $|\bar{4}| = 3$

$\bar{5}^6 = \bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5} = \bar{30} = \bar{0}$ maka $|\bar{5}| = 6$

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$\boxed{T_3}$ Misal R ring dengan unsur kesatuan.

(1) Jika $(R,+)$ memiliki order 0 maka char(R) = 0

(2) Jika $(R,+)$ memiliki order n maka char(R) = n

Bukti :

Ambil R ring dengan unsur kesatuan, misal $1_R \in R$ unsur kesatuannya,

(1) Kasus order $(R,+)$ adalah 0

$\nexists\, n \in \mathbb{N}$ sehingga $n.1_R = 0_R$, $na \neq 0_R$ $\forall a \in R$.

$\nexists\, n \in \mathbb{N}$ sehingga $na = 0_R$, $\forall a \in R$,

Jadi char(R) = 0

(2) Kasus order $(R,+)$ adalah n

Terdapat $n \in \mathbb{N}$ terkecil sehingga $n.1_R = 0_R$

Ambil $a \in R$ sebarang

Akibatnya

$$
\begin{aligned}
n.a &= \underbrace{a + a + \dots + a}_{n \text{ suku}} \\
&= \underbrace{1_R \cdot a + 1_R \cdot a + \dots + 1_R \cdot a}_{n \text{ suku}} \\
&= (\underbrace{1_R + 1_R + \dots + 1_R}_{n \text{ suku}})a \\
&= (n.1_R)a \\
&= 0_R \cdot a \\
&= 0_R
\end{aligned}
$$

Jadi $n \in \mathbb{N}$ terkecil sehingga $na = 0$, $\forall a \in R$

Char(R) = n

$\boxed{\text{Q.E.D}}$

Makassar, 4 Oktober '20

Imanuel AS / 181114008

$\boxed{T_4}$

Jika D daerah integral dan $a, b \in D$, $a \neq 0_R$, $b \neq 0_R$ maka $ord(a) = ord(b)$ terhadap grup $(D, +)$ $\boxed{D_5}$

Bukti

Misalkan $a, b \in D$ sebarang, Pandang grup $(D, +)$.
Selanjutnya, tinjau kasus berikut :

(*) Kasus 1

Misalkan order dari a terhadap $(D, +)$ terhingga; misalnya $ord(a) = n$. Ini berarti n merupakan bilangan bulat positif terkecil sehingga $n.a = 0$. Karena $n.a = 0$,

$$\Rightarrow (na)b = 0b = 0$$

$$\Rightarrow (\underbrace{a + a + \dots + a}_{n \text{ suku}})b = 0$$

$$\Rightarrow (\underbrace{ab + ab + \dots + ab}_{n \text{ suku}}) = 0 \qquad [\text{Hukum Distributif}]$$

$$\Rightarrow a(\underbrace{b + b + \dots + b}_{n \text{ suku}}) = 0 \qquad [\text{Hukum Distributif}]$$

$$\Rightarrow a(nb) = 0$$

Karena D merupakan ring tanpa pembagi nol, dan $a \neq 0$, $a(nb) = 0$, maka $nb = 0$, ini berarti
$ord(b) \leq n. ord(a)$ ....(i)

Selanjutnya, kita misalkan $ord(b) = m$, maka $mb = 0$. Dengan prosedur yang sama di atas dapat ditunjukkan $(ma)b = 0$
Karena D ring tanpa pembagi nol, dan $b \neq 0$, $(ma)b = 0$, maka $ma = 0$, hal ini berarti
$ord(a) \leq m = ord(b)$ ....(ii)

Dan (i) dan (ii) diperoleh $ord(a) = ord(b)$.

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(*) Kasus 2

Untuk kasus ord (a) nol, akan ditunjukkan ord (b) = 0.

Andaikan ord (b) terhingga, sebut ord (b) = m,

maka $mb = 0$.

Oleh karena itu, $a(mb) = a0 = 0$

$\Rightarrow a(\underbrace{b+b+\dots+b}_{m \text{ suku}}) = 0$

$\Rightarrow (\underbrace{ab+ab+\dots+ab}_{m \text{ suku}}) = 0$

$\Rightarrow (\underbrace{a+a+\dots+a}_{m \text{ suku}})b = 0$

$\Rightarrow (ma)b = 0$

Karena D daerah integral, $(ma)b = 0$ dan $b \neq 0$.

Maka $ma = 0$, akibatnya ord (b) $\leq m$.

Hal di atas berarti order dari a terhingga, kontradiksi dengan ord (a) = 0 (atau tak terhingga).

Jadi harus ord (b) = 0.

$\therefore$ Dari Kasus 1 dan Kasus 2 disimpulkan bahwa ternyata ord (a) = ord (b)

Imanuel AS/1811141008 Mat

Makassar, 4

Lemma 1

Jika $m, n \in \mathbb{Z}$ dan $a, b \in R$, R ring, maka

$$(m \cdot a)(n \cdot b) = (mn)ab \qquad \boxed{D_6}$$

Bukti

Ambil $m, n \in \mathbb{Z}$ sebarang

Ambil $a, b \in R$ sebarang

Note that

$$(ma)(nb) = [\underbrace{a+a+\dots+a}_{m \text{ suku}}] \cdot [\underbrace{b+b+\dots+b}_{n \text{ suku}}] \qquad [\text{Definisi}]$$

$$= (a+a+\dots+a)b + (a+a+\dots+a)b + \dots + (a+a+\dots+a)b$$

$$= \underbrace{(ab+ab+\dots+ab) + (ab+ab+\dots+ab) + \dots + (ab+ab+\dots+ab)}_{(mn) \text{ suku}}$$

$$= (mn)(ab) \qquad [\text{Definisi}]$$

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Imanuel AS / 1811141008

$\boxed{T_5}$ Karakteristik daerah integral adalah 0 atau prima

Proof

Adb. karakteristik daerah integral adalah 0 atau prima.

Ambil D daerah integral sebarang, karena D merupakan ring dengan unsur kesatuan.

(1) Tinjau $(D,+)$ memiliki order 0, akibatnya berdasarkan $\boxed{T_3}$,

$$\text{char}(D) = 0$$

(2) Tinjau $(D,+)$ memiliki order n, akibatnya berdasarkan $\boxed{T_3}$,

$$\text{char}(D) = n \quad , n \in \mathbb{N}$$

Akan dibuktikan: n prima

Andaikan n komposit,

Akibatnya

$n = n_1 . n_2 \quad , n_1 \neq 1 , n_2 \neq 1 , n_1 < n , n_2 < n$

Karena $\text{char}(D) = n$, maka $n \in \mathbb{N}$ terkecil sehingga $\forall a \in D$, $a \neq 0_R$ diperoleh $na = 0_R$

akibatnya diperoleh

$$na = 0_R$$

$\Rightarrow (n_1.n_2)a = 0_R$

$\Rightarrow (n_1.n_2)ab = 0_R \cdot b \quad [b \in D, b \neq 0_R]$

$\Rightarrow (n_1.n_2)ab = 0_R$

$\Rightarrow (n_1.a)(n_2 b) = 0_R \quad$ [Lemma 1]

Karena D daerah integral diperoleh $n_1.a = 0_R$ atau $n_2.b = 0_R$. Padahal dari $\boxed{T_4}$ haruslah $\text{ord}(a) = \text{ord}(b)$ yang berakibat $\text{ord}(a) = \text{ord}(b) = n$, dilain pihak $n_1 < n$, $n_2 < n$.

Jadi $n_1.a \neq 0_R$ dan $n_1 b \neq 0_R$ (Kontradiksi)

Jadi, haruslah n prima.

$\therefore$ Dari (1) dan (2) maka $\boxed{T_5}$ terbukti.