Makassar, 2 Juni 2021

Analisis Kompleks

Pertemuan ke - 9

Nama : Imanuel Agung Sembe

NIM : 181114100 8

Analisis Kompleks / Pertemuan ke - 9 / catatan — Makassar, 18 Mei 2021

Imanuel Agung Sembe

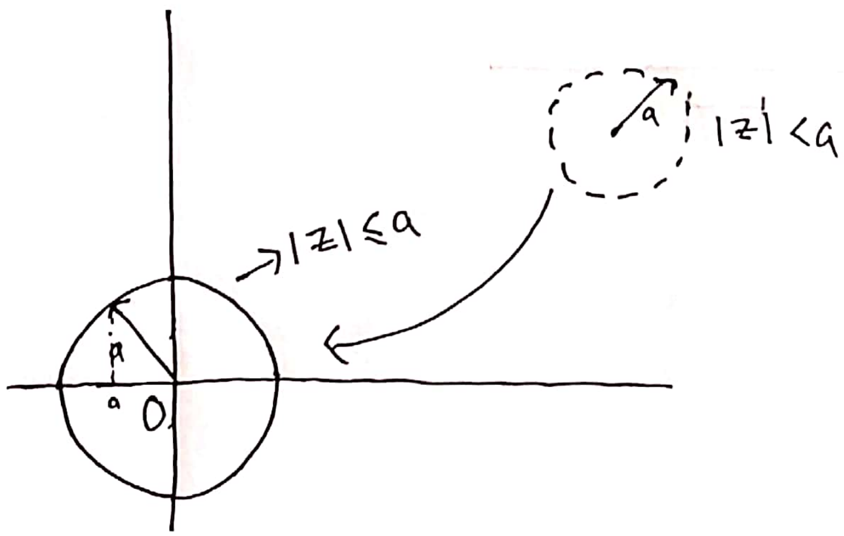
1811141008

Imanuel

Region di Bilangan Kompleks

1. Lingkaran dan cakram di bidang kompleks

- Persamaan lingkaran pusat O dengan jari-jari $a > 0$ ditulis $|z| = a$
- Cakram buka adalah $|z| < a$
- Cakram tutup adalah $|z| \leq a$



$|z| = a$

lingkaran pusat O (0,0)
Jari-jari $a > 0$

$z = x + iy$, $x, y \in \mathbb{R}$

$|z| = a \Rightarrow \sqrt{x^2 + y^2} = a$

$\Rightarrow x^2 + y^2 = a^2$

Persamaan lingkaran pusat O (0,0) dengan jari-jari a.

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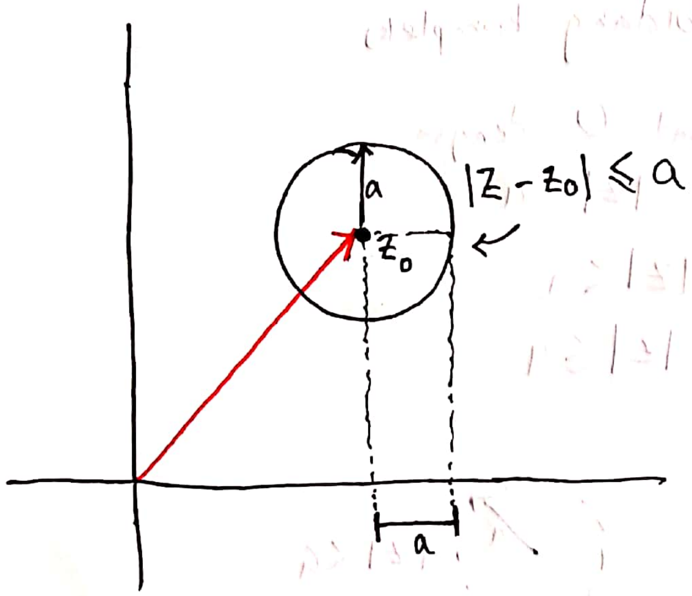
- Persamaan kompleks untuk lingkaran pusat z_0 dan jari-jari $a > 0$ adalah $|z - z_0| = a$
- Cakram buka $\Rightarrow |z - z_0| < a$
- Cakram tutup $\Rightarrow |z - z_0| \leq a$



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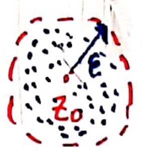
2. Persekitaran / Lingkungan

– Persekitaran – ε dari $z_0 \in \mathbb{C}$ ditulis sebagai

$$V_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$$

$|z - z_0| < \varepsilon$



E

Misal $z_0 = 1 + 2i$, $\varepsilon = 2$, $V_\varepsilon(z_0) = \ldots$?

Jawab: $V_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$

$V_\varepsilon(1+2i) = \{ z \in \mathbb{C} \mid |z - (1+2i)| < 2 \}$

Misal $z = a + bi = (a, b)$ (a: Real, b: Im)

$|z - (1+2i)| < 2$

$\Rightarrow |(a+bi) - (1+2i)| < 2$

$\Rightarrow |(a-1) + (b-2)i| < 2$

$\Rightarrow \sqrt{(a-1)^2 + (b-2)^2} < 2$

$\Rightarrow (a-1)^2 + (b-2)^2 < 4$ .... (*)

Pers. lingkaran di bid. kompleks, $r = 2$, pusat $(1,2)$ dan merupakan cakram buka.

Misal pilih nilai $a = 1$ dan $b = 1$, Maka $z = a + bi = 1 + i$

Perhatikan bahwa $z = 1 + i \in V_2(1+2i)$ karena memenuhi $(a-1)^2 + (b-2)^2 < 4$ dimana $a = 1$ dan $b = 1$, yakni

$(a-1)^2 + (b-2)^2 < 4$

$(1-1)^2 + (1-2)^2 < 4$

$0^2 + 1^2 < 4$

$1 < 4$ (Memenuhi)

Apakah ada nilai z yang lain? Tentu ada. Silahkan pilih nilai a dan b yg memenuhi persamaan (*)

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1. Misalkan $\varepsilon = 2$, $z_0 = 2+2i$, $V_\varepsilon(z_0) = \dots$?

Jawab: $V_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$

$V_2(2+2i) = \{ z \in \mathbb{C} \mid |z-(2+2i)| < 2 \}$

Misal $z = a+bi = (a,b)$

Note that,

$$|z-(2+2i)| < 2$$

$$\Rightarrow |(a+bi)-(2+2i)| < 2$$

$$\Rightarrow |(a-2)+(b-2)i| < 2$$

$$\Rightarrow \sqrt{(a-2)^2+(b-2)^2} < 2$$

$$\Rightarrow (a-2)^2+(b-2)^2 < 4 \quad \text{.........} (*)$$

Misal pilih nilai $a=1$ dan $b=2$, Maka $z = a+bi = 1+2i$

Perhatikan bahwa, $z = 1+2i \in V_2(2+2i)$ karena memenuhi $(a-2)^2+(b-2)^2 < 4$ dimana $a=1$ dan $b=2$, yakni

$$(a-2)^2+(b-2)^2 < 4$$

$$(1-2)^2+(2-2)^2 < 4$$

$$1^2 + 0^2 < 4$$

$$1 < 4 \quad \text{(Memenuhi)}$$

Apakah ada nilai z yang lain? Tentu ada. Silahkan cari nilai a dan b yang lain sedemikian sehingga memenuhi pers. $(*)$

BTW, Pers. $(*)$ merupakan pers. lingkaran ~~dalam~~ di bidang kompleks dengan jari-jari 2, dan merupakan cakram buka.

pusat $(2,2)$

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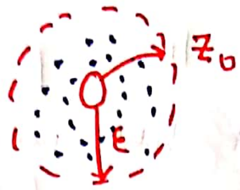
- Lingkaran tanpa pusat

$V_\varepsilon^*(z_0) = V_\varepsilon(z_0) \setminus \{z_0\}$



3.) Titik Dalam (Interior Point)

Titik $z_0 \in \mathbb{C}$ adalah titik dalam dari himpunan $S \subseteq \mathbb{C}$,
Jika terdapat $\varepsilon > 0$ sehingga $V_\varepsilon(z_0) \subseteq S$,
Himpunan semua titik dalam disebut $Int(S)$.

4.) Titik Luar (Exterior Point)

Titik $z_0 \in \mathbb{C}$ adalah titik luar dari himpunan $S \subseteq \mathbb{C}$,
Jika terdapat $\varepsilon > 0$ sehingga $V_\varepsilon(z_0) \subseteq S^c$, $S^c = \mathbb{C} - S$.
Himpunan semua titik luar S ditulis $Eks(S)$.

5.) Titik Batas

Titik $z_0 \in \mathbb{C}$ disebut titik batas dari himpunan $S \subseteq \mathbb{C}$,
Jika z_0 bukan titik dalam dan bukan titik luar.

$\Updownarrow$

$$\forall \varepsilon > 0,\ V_\varepsilon(z_0) \cap S \neq \phi \text{ dan }$$
$$V_\varepsilon^*(z_0) \cap S^c \neq \phi$$

Himpunan semua titik batas S ditulis sebagai ∂S (baca : do s)

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⑥ Titik Limit

Titik $z_0 \in \mathbb{C}$ disebut titik limit dari $S \subseteq \mathbb{C}$,

Jika $\forall \varepsilon > 0$, $V_\varepsilon^*(z_0) \cap S \neq \phi$

Himpunan semua titik limit S ditulis S'.

⑦ Titik Pencil

Titik $z_0 \in \mathbb{C}$ disebut titik pencil dari himpunan $S \subseteq C$

Jika z_0 bukan titik limit S.

Atau ingkaran dari no. 6 diatas, atau ~~dapat~~ dengan n kata lain:

Jika $\exists\, \varepsilon > 0$ sehingga $V_\varepsilon^*(z_0) \cap S \neq \phi$.

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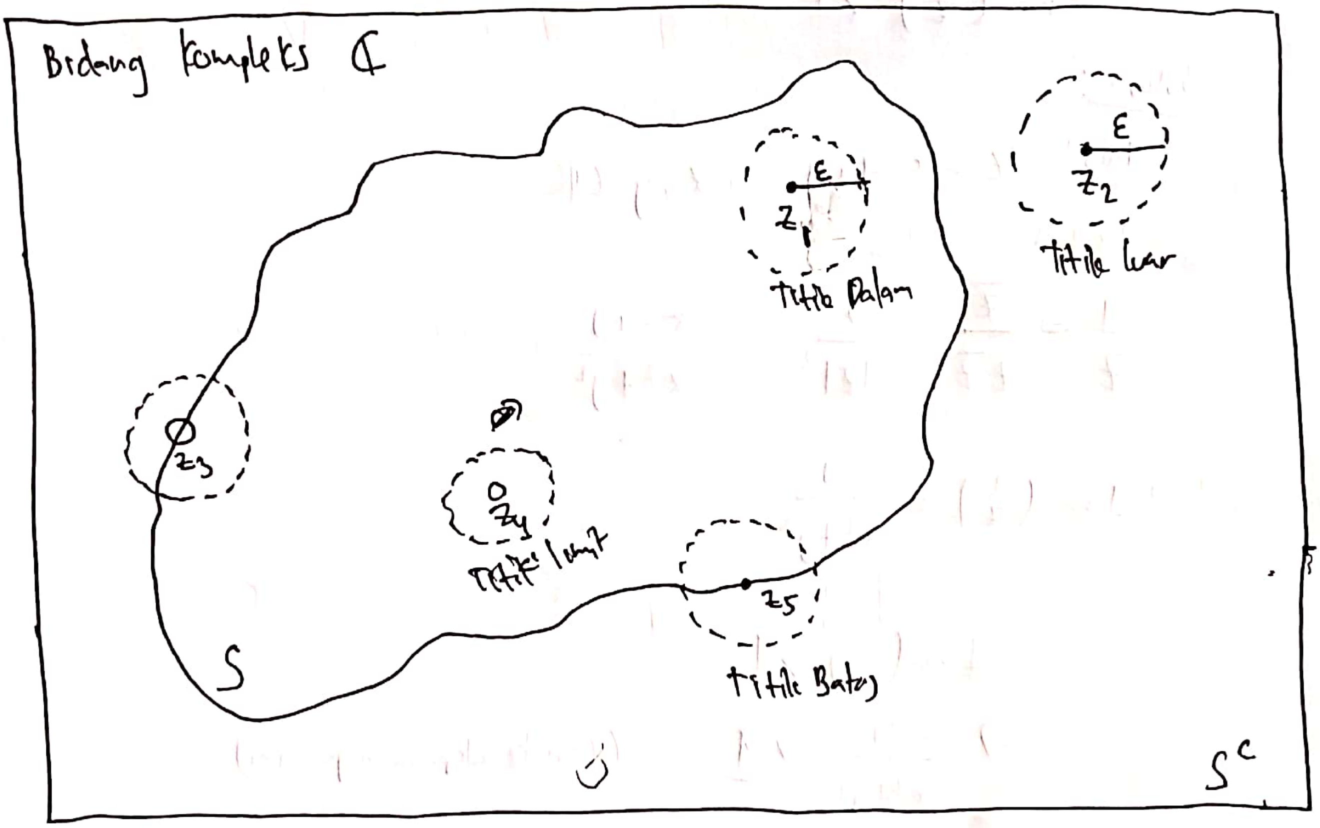

Ket:

1. Titik Dalam : $\exists \varepsilon > 0 \ni V_\varepsilon(z_0) \subseteq S$

dimana diketahui sebelumnya : $V_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$

2. Titik luar : $\exists \varepsilon > 0 \ni V_\varepsilon(z_0) \subseteq S^c$

3. Titik Batas : $\forall \varepsilon > 0$, $V_\varepsilon(z_0) \cap S \neq \phi$ dan

$V_\varepsilon^*(z_0) \cap S^c \neq \phi$

4. Titik Limit : $\forall \varepsilon > 0$, $V_\varepsilon^*(z_0) \cap S \neq \phi$

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E Sketsa himpunan

$$\text{Im}\left(\frac{1}{z}\right) > 1$$

Solusi :

Misal $z = x + iy$, $x, y \in \mathbb{R}$

Perhatikan bahwa

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2+y^2}$$

$$\text{Im}\left(\frac{1}{z}\right) = \frac{-y}{x^2+y^2}$$

$$\text{Im}\left(\frac{1}{z}\right) > 1$$

$$\Rightarrow \frac{-y}{x^2+y^2} > 1$$ (Cari ketaksamaannya ini)

Diperoleh $\dfrac{-y}{x^2+y^2} > 1$

$$\Rightarrow -y > x^2+y^2$$

$$\Rightarrow x^2+y^2+y < 0$$

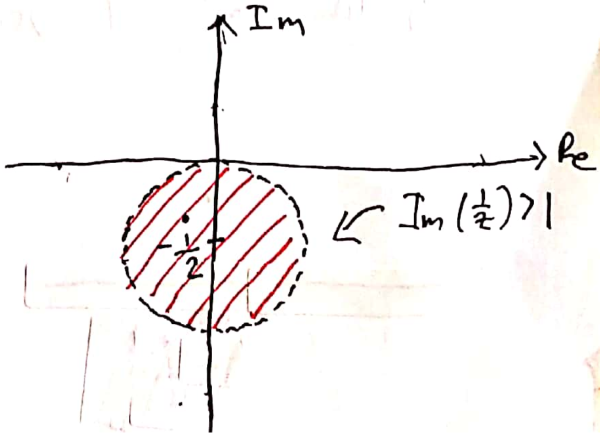
$$\Rightarrow x^2+y^2+y+\frac{1}{4} < \frac{1}{4}$$

$$\Rightarrow (x-0)^2+\left(y+\frac{1}{2}\right)^2 < \left(\frac{1}{2}\right)^2$$

Merupakan lingkaran dengan pusat $\left(0, -\frac{1}{2}\right)$ dengan jari-jari $\frac{1}{2}$.

Pusat $\left(0, -\frac{1}{2}\right) \Rightarrow z = -\frac{i}{2}$

sketsa (lihat gambar di samping)



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Sketsa himpunan berikut

① $|z-2+i| \leq 1$

Penyelesaian :

Misal $z = x+yi$; U/suatu $x, y \in \mathbb{R}$

Perhatikan bahwa

$$|z-2+i| = |(x+yi)-2+i| = |(x-2)+(y+1)i|$$
$$= \sqrt{(x-2)^2+(y+1)^2}$$

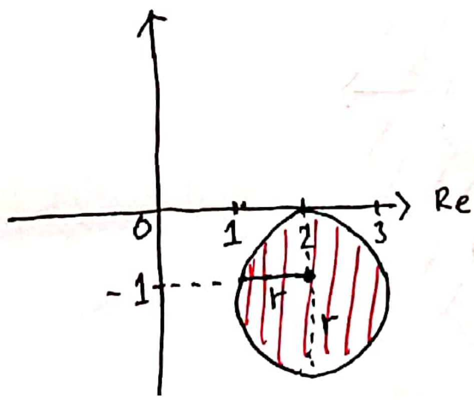
Maka diperoleh

$$|z-2+i| \leq 1$$
$$\sqrt{(x-2)^2+(y+1)^2} \leq 1$$
$$(x-2)^2+(y+1)^2 \leq (1)^2$$ [Kedua ruas dipangkatkan 2]

Ingat! Pers lingkaran dgn pusat $P(a,b)$ adalah
$$(x-a)^2+(y-b)^2 = r^2$$

→ Merupakan lingkaran dengan pusat $P(2,-1)$ dengan jari jari $r = 1$.

Gambar :



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② $|2z+3| > 4$

<u>Penyelesaian</u> :

Misal $z = x + yi$; u/suatu $x, y \in \mathbb{R}$

Perhatikan bahwa,

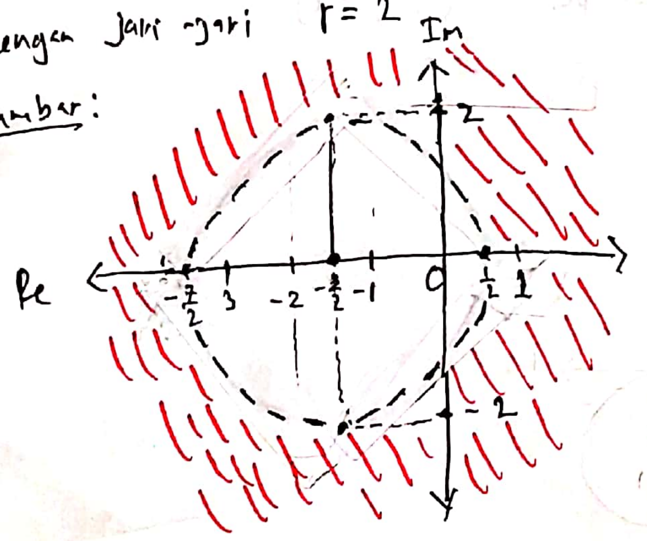
$$|2z+3| = |2(x+yi)+3| = |2x+2yi+3| = |(2x+3)+(2y)i|$$
$$= \sqrt{(2x+3)^2+(2y)^2}$$

Maka diperoleh,

$$|2z+3| > 4$$
$$\sqrt{(2x+3)^2+(2y)^2} > 4$$
$$(2x+3)^2+(2y)^2 > (4)^2 \quad \text{[Kedua ruas dipangkatkan 2]}$$
$$\frac{1}{4}\left[(2x+3)^2+(2y)^2\right] > (4)^2 \cdot \frac{1}{4}$$
$$\frac{(2x+3)(2x+3)}{4} + \frac{(2y)(2y)}{4} > 4$$
$$\frac{(2x+3)}{2}\frac{(2x+3)}{2} + \frac{2y}{2}\cdot\frac{2y}{2} > 4$$
$$(x+\tfrac{3}{2})(x+\tfrac{3}{2}) + y \cdot y > 4$$
$$(x+\tfrac{3}{2})^2 + (y-0)^2 > (2)^2$$

↳ Merupakan persamaan lingkaran dengan pusat $P(-\frac{3}{2}, 0)$ dengan jari-jari $r = 2$

<u>Gambar</u>:



Inanuel Agung Senke / 1811141008

Makassar, 4 April 2021

(3) $Im(z) > 1$

Penyelesaian :

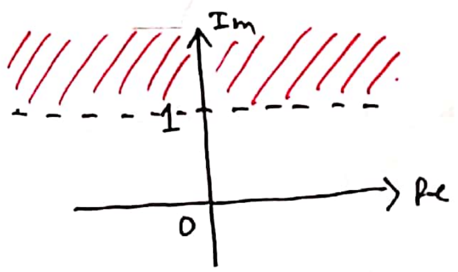
Misal $z = x + yi$; u/ suatu $x, y \in \mathbb{R}$

Maka, $Im(z) = y$

Sehingga diperoleh,

$$Im(z) > 1$$
$$y > 1$$

Diperoleh gambar sketsa sebagai berikut



(4) $Im(z) = 1$

Penyelesaian :

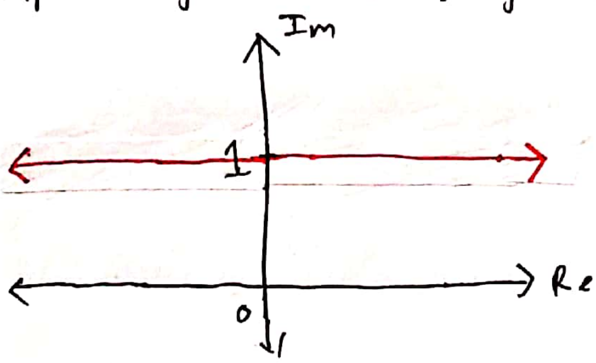
Misal $z = x + yi$; u/ suatu $x, y \in \mathbb{R}$

Maka, $Im(z) = y$

sehingga diperoleh,

$$Im(z) = 1$$
$$y = 1$$

Diperoleh gambar sketsa sebagai berikut



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(5) $|z-4| \geq |z|$

Penyelesaian :

Misal $z = x + yi$; u/suatu $x, y \in \mathbb{R}$

Perhatikan bahwa

$$|z-4| = |(x+yi)-4| = |(x-4)+(y)i| = \sqrt{(x-4)^2+(y)^2}$$

dan

$$|z| = |x+yi| = \sqrt{x^2+y^2}$$

Maka,

$$|z-4| \geq |z|$$

$$\sqrt{(x-4)^2+(y)^2} \geq \sqrt{x^2+y^2}$$

$$(x-4)^2+(y)^2 \geq x^2+y^2$$ [Kedua ruas dipangkatkan 2]

$$(x^2-8x+16)+y^2 \geq x^2+y^2$$

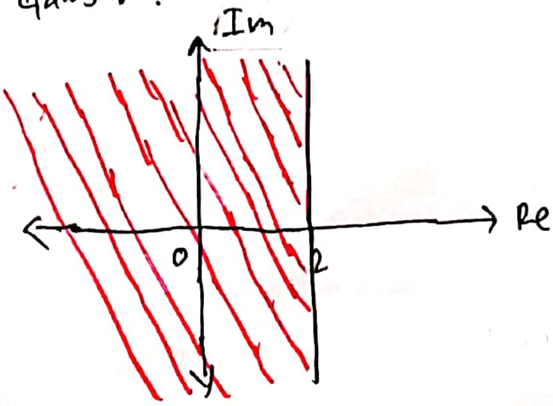
$$-8x+16 \geq 0$$ [Kedua ruas dikurangkan x^2+y^2]

$$-x+2 \geq 0$$

$$-x \geq -2$$

$$x \leq 2$$

Gambar :



Makassar, 2 Juni 2024

Analisis Kompleks

Pertemuan ke - 10

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Analisis Kompleks / Pertemuan ke-10 / Catatan

Malassar, 18 Mei 2024

Imanuel Agung Sembe

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Fungsi Kompleks

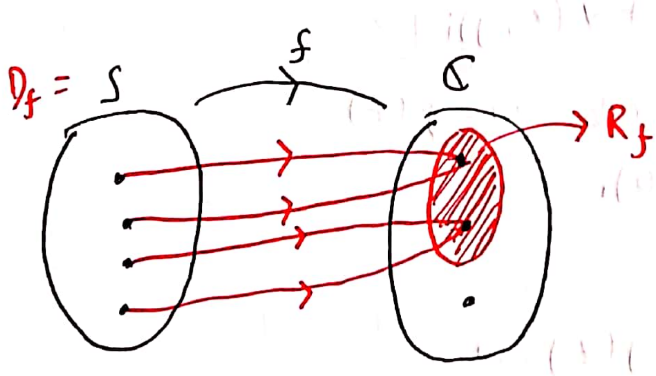
$\boxed{D}$

Fungsi $f : S \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ kompleks adalah suatu aturan yang mengaitkan setiap $z \in S$ dengan tepat satu $w \in \mathbb{C}$

$D_f = S$

$R_f = \{ w \in \mathbb{C} \mid w = f(z) \text{ u/suatu } z \in D_f \}$

$\quad = f(S)$



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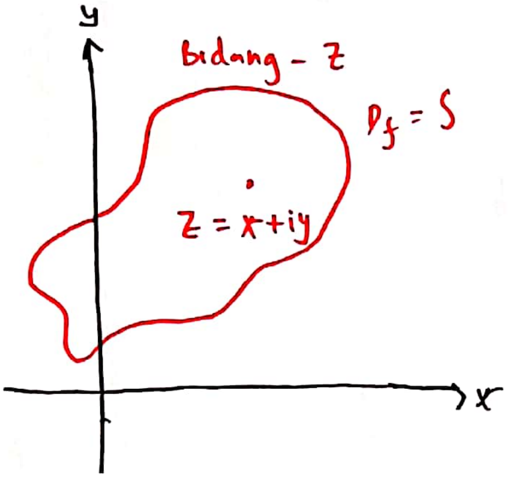
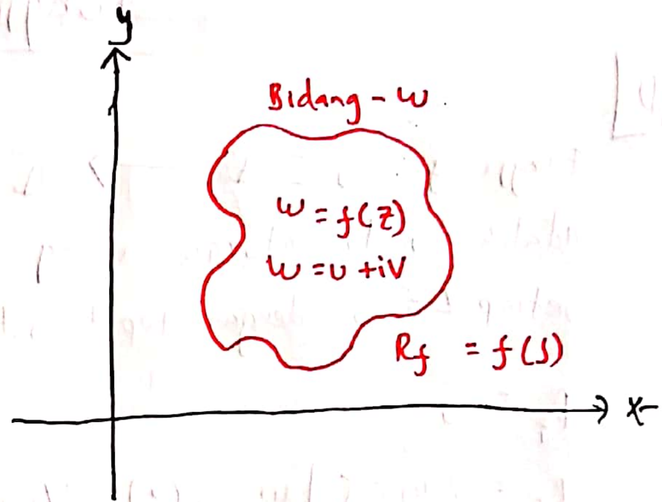
$z = x + iy = (x, y)$

$\nearrow$ Re(z) $\nwarrow$ Im(z)





(1) Jika $z = x + iy$ maka $w = f(z)$

Berbentuk $f(z) = u(x, y) + v(x, y)i$

(2) Jika $z = r \operatorname{cis} \theta = re^{i\theta}$ maka $w = f(z)$

$w = f(z) = u(r, \theta) + v(r, \theta)i$

$\boxed{N}$

Jika $v(x, y) = 0$ maka $f(z) = w$

Fungsi kompleks bernilai real.

$\boxed{E}$ $f(z) = z^2$

(1) Misal $z = x + yi$, maka

$$
\begin{aligned}
f(z) &= f(x + yi) \\
&= (x + yi)^2 \\
&= \underbrace{(x^2 - y^2)}_{u(x,y)} + \underbrace{(2xy)}_{v(x,y)} i
\end{aligned}
$$

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$\boxed{E}$ $f(z) = |z|^2$

Misal $z = x + iy$

$f(z) = f(x+iy) = |x+iy|^2$

$= (x^2+y^2) + 0i$

① Fungsi Polinom (Suku banyak) di Bil. kompleks

$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$

$a_n \neq 0$, $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$

$f(z) = z^2 + 1$ ada pembuat nol

$z^2 + 1 = 0$

$z = \pm\sqrt{-1} \begin{cases} i \\ -i \end{cases}$

secara umum

$f(z) = z^n + 1$

$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$

$f(z) = 0$

Banyaknya solusi

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② Fungsi rasional di Bil. kompleks

$f(z) = \frac{P(z)}{Q(z)}$, P, Q suku banyak

$Q(z) \neq 0$

$\boxed{E}$ $f(z) = \frac{z^2+1}{2z}$

$f(x) = x^{\frac{1}{2}}$

$z \in \mathbb{C} \setminus \{0\}$, $(z = re^{i\theta})$

$f(z) = z^{\frac{1}{2}}$ mempunyai 2 nilai

akar 1 : $z^{\frac{1}{2}} = \sqrt{r} \cdot e^{\frac{i\theta}{2}}$ $r = |z|$

akar 2 : $z^{\frac{1}{2}} = -\sqrt{r} \cdot e^{\frac{i\theta}{2}}$ $-\pi < \theta < \pi$

$f(z) = \sqrt{r} \cdot e^{\frac{i\theta}{2}}$ adalah fungsi

$\boxed{D}$ Misal $A \subseteq \mathbb{C}$, $B \subseteq \mathbb{C}$, f, g fungsi yang didefinisikan

$w = f(z)$, $z \in A$

$t = g(z)$, $z \in B$

Operasi f dan g pada $D = A \cap B$ didefinisikan

(1) $(f+g)(z) = f(z) + g(z)$

(2) $(f-g)(z) = f(z) - g(z)$

(3) $(fg)(z) = f(z) \cdot g(z)$

(4) $(cf)(z) = c \cdot f(z)$, $c \in \mathbb{R}$

(5) $\left(\frac{f}{g}\right)(z) = \frac{f(z)}{g(z)}$, $g(z) \neq 0$

(6) $(f^n)(z) = (f(z))^n$, $n \in \mathbb{N}$

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$\boxed{D}$ Misal $f: D_f \to R_f$ $g: D_g \to R_g$

adalah fungsi kompleks.

Jika $R_f \cap D_g \neq \phi$ maka terdapat suatu

fungsi kompleks $h: E \to R_g$ dengan $E \subseteq D_f$

disebut fungsi komposisi f dan g ditulis $g \circ f$ yaitu

$$h(z) = (g \circ f)(z) = g(f(z))$$

D_f dari $g \circ f$ adalah prapeta $R_f \cap D_g$ terhadap fungsi f

$$D_{g \circ f} = f^{-1}(R_f \cap D_g) = \{ z \in D_f \mid f(z) \in R_f \cap D_g \}$$

R_f dari $g \circ f$ adalah peta $R_f \cap D_g$ terhadap fungsi g

$$R_{g \circ f} = g(R_f \cap D_g) = \{ g(z) \mid z \in R_f \cap D_g \}$$

$\boxed{E}$ Misal $f(z) = 3z + 1$ dan $g(z) = z^2 + z + 1 - i$

(1) Tentukan $(f+g)(z)$

(2) Selidiki apakah fungsi $g \circ f$ terdefinisi?
Jika ya, tentukan $(g \circ f)(z)$

Solusi:

(1) $D_f = \mathbb{C}$, $D_g = \mathbb{C}$, $D_f \cap D_g \neq \phi$

$f + g$ terdefinisi pada $D_f \cap D_g \neq \phi$

sehingga

$$(f+g)(z) = f(z) + g(z)$$
$$= (3z+1) + (z^2 + z + 1 - i)$$
$$= z^2 + 4z + 1$$ //

$\to$ next

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b) $R_f = \mathbb{C}$; $D_g = \mathbb{C}$, karena

$R_f \cap D_g = \mathbb{C} \neq \phi$, akibatnya

$g \circ f$ terdefinisi dengan

$$(g \circ f)(z) = g(f(z))$$

$$= g(3z+i)$$

$$= (3z+i)^2 + (3z+i) + 1 - i$$

$$= 9z^2 + 6zi - 1 + (3z+i) + 1 - i$$

$$= 9z^2 + (6i+3)z$$

$$= 9z^2 + (3+6i)z$$

Kalau diperhatikan, fungsi kompleks mirip dengan fungsi dua variabel rel. Kenapa? karena z itu bisa dipandang sebagai $z = x + iy$.

Perhatikan bahwa,

$$z = x + iy \quad , x, y \in \mathbb{R}$$

$$z = r \operatorname{cis} \theta = re^{i\theta} \quad , r = |z| = \sqrt{x^2+y^2}$$

$$\theta = \operatorname{Arg}(z)$$

Kalau dimisalkan $f(z) = w$, $w = u + iv$

Maka diperoleh $f(x+iy) = u + iv$ ← $u = u(x,y)$, $v = v(x,y)$

bisa juga $f(r \operatorname{cis} \theta) = u + iv$ ← $u = u(r,\theta)$, $v = v(r,\theta)$

$$w = f(z) = u(x,y) + iv(x,y)$$

$$w = f(z) = u(r,\theta) + iv(r,\theta)$$

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$\boxed{E}$ [Coba sendiri]

Misal $f(z) = z^2 + z - 3$,

Nyatakan f dalam bentuk

$f(z) = u(x,y) + iv(x,y)$ dan

$f(z) = u(r,\theta) + iv(r,\theta)$

Penyelesaian:

• Misal $z = x + yi$; u/ suatu $x,y \in \mathbb{R}$ maka :

$$f(z) = (x+yi)^2 + (x+yi) - 3$$
$$= (x^2 + 2xyi - y^2) + (x+yi) - 3$$
$$= \underbrace{(x^2 - y^2 + x - 3)}_{u(x,y)} + \underbrace{(2xy + y)}_{v(x,y)} i$$

• Misal $z = r.\text{cis}\,\theta = r.e^{i\theta}$, maka

$$f(z) = f(r.\text{cis}\,\theta)$$
$$= (r.\text{cis}\,\theta)^2 + (r.\text{cis}\,\theta) - 3$$
$$= (r^2.\text{cis}\,2\theta) + (r.\text{cis}\,\theta) - 3$$
$$= [r^2(\cos 2\theta + i\cdot\sin 2\theta)] + [r(\cos\theta + i\cdot\sin\theta)] - 3$$
$$= [r^2.\cos 2\theta + (r^2.\sin 2\theta)i] + [r.\cos\theta + (r.\sin\theta)i] - 3$$
$$= \underbrace{(r^2.\cos 2\theta + r.\cos\theta - 3)}_{u(r,\theta)} + \underbrace{(r^2.\sin 2\theta + r.\sin\theta)}_{v(r,\theta)} i$$

Analisis Kompleks / Pertemuan ke-10 / PR Imanuel Agung Sembe Makassar, 26 April 2021
1811141008

1. Tentukan

(a) $f(2i)$ jika $f(z) = z^2 - 2z - 1$

Penyelesaian:

Misal $z = a + bi$; u/suatu $a, b \in \mathbb{R}$

Maka $f(z) = f(a + bi)$

$= (a + bi)^2 - 2(a + bi) - 1$

Sehingga untuk $a = 0$ dan $b = 2$, diperoleh

$$
\begin{aligned}
f(2i) &= f(0 + 2i) \\
&= (0 + 2i)^2 - 2(0 + 2i) - 1 \\
&= (2i)^2 - 2(2i) - 1 \\
&= 4(i)^2 - 4i - 1 \\
&= -4 - 4i - 1 \\
&= -4i - 5 \\
&= -5 - 4i \quad //
\end{aligned}
$$

(b.) $f(2-i)$ jika $f(z) = 3z^2 - i\bar{z}$

Penyelesaian:

$$
\begin{aligned}
f(2-i) &= 3(2-i)^2 - i\,\overline{(2-i)} \\
&= 3(2-i)^2 - i(2+i) \\
&= 3(-4i + 5) - (2i + i^2) \\
&= -12i + 15 - (2i + (-1)) \\
&= -12i + 15 - 2i + 1 \\
&= -14i + 16 \\
&= 16 - 14i \quad //
\end{aligned}
$$

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(c) $f(-i)$ jika $f(z) = \frac{z+1}{z-1}$

Penyelesaian :

$$f(-i) = \frac{(-i)+1}{(-i)-1} = \frac{(-i)+1}{(-i)-1} \cdot \frac{(-i)+1}{(-i)+1} = \frac{(i)^2+(-i)+(-i)+1}{(i)^2 -i+i-1} = -i$$

//

(d) $f(-4-4i)$ jika $f(z) = |z|^2 - [Re(z)]^2$

Penyelesaian :

$$f(-4-4i) = |-4-4i|^2 - [-4]^2$$

$$= \left(\sqrt{(-4)^2+(-4)^2}\right)^2 - [16]$$

$$= 16+16 - 16$$

$$= 16$$

//

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(2.) Nyatakan f kedalam bentuk

$u(x,y) + iv(x,y)$ dan $u(r,\theta) + iv(r,\theta)$

(a.) $f(z) = z^2 + 3z^3$

Penyelesaian :

•) Misal $z = x + yi$ maka

$$f(z) = (x+yi)^2 + 3(x+yi)^3$$
$$= (x^2 + 2xyi - y^2) + 3(x^3 - 3xy^2 + (3x^2y - y^3)i)$$
$$= x^2 + 2xyi - y^2 + 3x^3 - 9xy^2 + (9x^2y - 3y^3)i$$
$$= \underbrace{(x^2 - y^2 + 3x^3 - 9xy^2)}_{u(x,y)} + \underbrace{(2xy + 9x^2y - 3y^3)}_{v(x,y)}i$$

•) Misal $z = r \text{ cis } \theta = re^{i\theta}$, maka

$$f(z) = f(r \text{ cis } \theta) =$$
$$= (r \text{ cis } \theta)^2 + 3(r \text{ cis } \theta)^3$$
$$= (r^2 \cdot \text{cis } 2\theta) + 3(r^3 \text{ cis } 3\theta)$$
$$= [r^2(\cos 2\theta + i \cdot \sin 2\theta)] + [3 \cdot (r^3(\cos 3\theta + i \cdot \sin 3\theta))]$$
$$= [r^2 \cdot \cos 2\theta + (r^2 \cdot \sin 2\theta)i] + [3r^3 \cdot \cos 3\theta + (3r^3 \cdot \sin 3\theta)i]$$
$$= \underbrace{(r^2 \cdot \cos 2\theta + 3r^3 \cdot \cos 3\theta)}_{u(r,\theta)} + \underbrace{(r^2 \cdot \sin 2\theta + 3r^3 \cdot \sin 3\theta)i}_{v(r,\theta)}$$

Dipindai dengan CamScanner

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(b) $f(z) = i\bar{z} + \mathrm{Im}\left(\frac{i}{z}\right)$

Penyelesaian :

→ Misal $z = x + yi$, maka

$$f(z) = i\,\overline{(x+yi)} + \mathrm{Im}\left(\frac{i}{x+yi}\right)$$

$$= i(x-yi) + \mathrm{Im}\left(\frac{i}{x+yi}\cdot\frac{x-yi}{x-yi}\right)$$

$$= i(x-yi) + \mathrm{Im}\left(\frac{i(x-yi)}{x^2+y^2}\right)$$

$$= (x-yi)i + \mathrm{Im}\left(\frac{xi - y(i)^2}{x^2+y^2}\right)$$

$$= ix - y(i)^2 + \mathrm{Im}\left(\frac{y+xi}{x^2+y^2}\right)$$

$$= y + xi + \mathrm{Im}\left(\frac{y}{x^2+y^2} + \left(\frac{x}{x^2+y^2}\right)\cdot i\right)$$

$$= y + xi + \frac{x}{x^2+y^2}$$

$$= \underbrace{\left(y + \frac{x}{x^2+y^2}\right)}_{u(x,y)} + \underbrace{x}_{v(x,y)}\,i$$

→ Misal $z = r\,\mathrm{cis}\,\theta$, maka

$$f(z) = i\,\overline{(r\cdot\mathrm{cis}\,\theta)} + \mathrm{Im}\left(\frac{i}{r\cdot\mathrm{cis}\,\theta}\right)$$

$$= i\left[\overline{r\cdot\cos\theta + (r\cdot\sin\theta)i}\right] + \mathrm{Im}\left(\frac{i}{r\cdot\cos\theta + (r\cdot\sin\theta)i}\cdot\frac{r\cdot\cos\theta - (r\cdot\sin\theta)i}{r\cdot\cos\theta - (r\cdot\sin\theta)i}\right)$$

$\boxed{\overline{r\cdot\cos\theta + (r\cdot\sin\theta)i} = r\cdot\cos\theta - (r\cdot\sin\theta)i \;//}$

$$= (r\cdot\cos\theta)i - (r\cdot\sin\theta)(i)^2 + \mathrm{Im}\left(\frac{(r\cdot\cos\theta)i - (r\cdot\sin\theta)\cdot(i)^2}{(r\cdot\cos\theta)^2 - (r\cdot\sin\theta)^2\cdot(i)^2}\right)$$

$$= (r\cdot\cos\theta)i - (r\cdot\sin\theta)(-1) + \mathrm{Im}\left(\frac{(r\cdot\cos\theta)i - (r\cdot\sin\theta)(-1)}{(r\cdot\cos\theta)^2 - (r\cdot\sin\theta)^2(-1)}\right)$$

$$= r\cdot\sin\theta + (r\cdot\cos\theta)i + \mathrm{Im}\left(\frac{r\cdot\sin\theta + (r\cdot\cos\theta)i}{(r\cdot\cos\theta)^2 + (r\cdot\sin\theta)^2}\right)$$

$$= r\cdot\sin\theta + (r\cdot\cos\theta)i + \frac{r\cdot\cos\theta}{(r\cdot\cos\theta)^2 + (r\cdot\sin\theta)^2}$$

$$= \underbrace{\left(r\cdot\sin\theta + \frac{r\cdot\cos\theta}{(r\cdot\cos\theta)^2 + (r\cdot\sin\theta)^2}\right)}_{u(r,\theta)} + \underbrace{(r\cdot\cos\theta)}_{v(r,\theta)}i$$

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(C.) $f(z) = 2\pi i$

Penyelesaian:

- Misal $z = x + yi$ u/ suatu $x = 0$ dan $y = 2\pi$.

$z = 0 + 2\pi i$

Maka

$$f(z) = 2\pi i$$
$$= \underbrace{0}_{U(x,y)} + \underbrace{(2\pi)}_{V(x,y)} i$$

- Misal $z = r \cdot cis\ \theta$, maka

$$f(z) = 2\pi i$$
$$= \underbrace{0}_{U(r,\theta)} + \underbrace{(2\pi)}_{V(r,\theta)} i$$

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(d) $f(z) = \dfrac{z-i}{z+i}$

<u>Penyelesaian</u>.

∘) Misal $z = x+yi$, maka

$$f(z) = \frac{(x+yi)-i}{(x+yi)+i}$$

$$= \frac{(x+yi)-i}{(x+yi)+i} \cdot \frac{(x+yi)-i}{(x+yi)-i}$$

$$= \frac{(x+yi)^2 - 2(x+yi) + 1}{(x+yi)^2 + 1}$$

$$= \frac{(x^2+2xyi-y^2) - 2(x+yi) + 1}{(x^2+2xyi-y^2)+1}$$

$$= \frac{(x^2+2xyi-y^2) - 2(x+yi) + 1}{(x^2-y^2+1)+2xyi} \cdot \frac{(x^2-y^2+1) - 2xyi}{(x^2-y^2+1)-2xyi}$$

$$= \underbrace{\frac{x^5 - 2x^3y^2 + 2x^3 + xy^4 - 2xy^2 - 2x^4 - 2x^2y^2 - 2x^2 + x}{(x^2-y^2+1)^2 + 4x^2y^2}}_{u(x,y)} + \underbrace{\left(\frac{2x^3y + 2xy^3 - 2xy}{(x^2-y^2+1)^2 + 4x^2y^2}\right)}_{v(x,y)} i$$

Imanuel AS / 1811141008

Makassar, 26 April 2021

•) Misal $z = r \cdot \text{cis}\,\theta$, maka

$$f(z) = \frac{(r \cdot \text{cis}\,\theta) - i}{(r \cdot \text{cis}\,\theta) + i}$$

$$= \frac{(r \cdot \cos\theta + (r \cdot \sin\theta)\,i) - i}{(r \cdot \cos\theta + (r \cdot \sin\theta)\,i) + i}$$

$$= \frac{(r \cdot \cos\theta) + (r \cdot \sin\theta - 1)\,i}{(r \cdot \cos\theta) + (r \cdot \sin\theta + 1)\,i} \cdot \frac{(r \cdot \cos\theta) - (r \cdot \sin\theta + 1)\,i}{(r \cdot \cos\theta) - (r \cdot \sin\theta + 1)\,i}$$

$$= \frac{(r \cdot \cos\theta)^2 - (r \cdot \cos\theta)(r \cdot \sin\theta + 1)i + (r \cdot \sin\theta - 1)i \cdot (r \cdot \cos\theta) + r^2 \sin^2\theta - 1}{(r \cdot \cos\theta)^2 - (r \cdot \sin\theta + 1)^2 \cdot (i)^2}$$

$$= \underbrace{\frac{\left[(r \cdot \cos\theta)^2 + (r \cdot \sin\theta)^2 - 1\right]}{(r \cdot \cos\theta)^2 + (r \cdot \sin\theta + 1)^2}}_{U(r,\theta)} + \underbrace{\left(\frac{\left[(r \cdot \sin\theta - 1)(r \cdot \cos\theta) - (r \cdot \sin\theta + 1)(r \cdot \cos\theta)\right]}{(r \cdot \cos\theta)^2 + (r \cdot \sin\theta + 1)^2}\right)}_{V(r,\theta)} \cdot i$$

Makassar, 2 Juni 2024

Analisis Kompleks

Pertemuan ke - 11

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Analisis Kompleks / Pertemuan ke - 11 / catatan

Makassar, 28 Mei 2021

Imanuel Agung Sembe

1811141008

Fungsi Kompleks sebagai Pemetaan

$w = f(z)$

↳ dipandang sebagai pemetaan titik (x, y) di bidang z ke titik (u, v) di bidang w.

E

Akan ditentukan peta dari daerah

$D = \{(x,y) \mid 0 < x < 1, -x < y < x\}$

oleh $f(z) = (x^2 - y^2) + 2xyi$

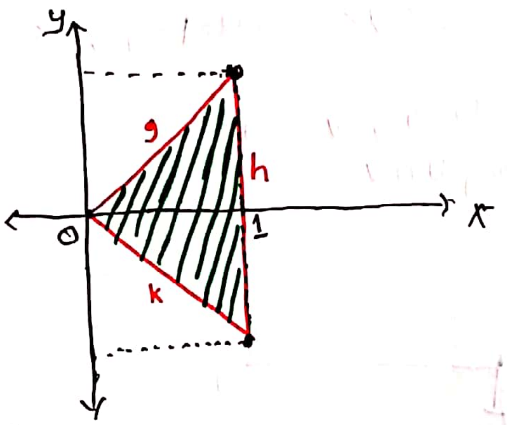
$z = x + iy = (x, y)$

Solusi :

$D = \{(x,y) \mid 0 \leq x \leq 1, -x < y < x\}$

$$f(z) = \underbrace{(x^2 - y^2)}_{u(x,y)} + \underbrace{2xy}_{v(x,y)} i$$

$u = x^2 - y^2 \qquad v = 2xy$



① Peta garis $g : y = x, \; 0 < x \leq 1$ oleh f adalah
$u = x^2 - y^2 = x^2 - x^2 = 0$
$v = 2xy = \boxed{2x^2} \qquad \boxed{0 < v \leq 2}$

dari sini (interval x)

Jadi peta g adalah

$g^* : u = 0 \qquad 0 < v \leq 2$

(2) Peta garis h : $x = 1$ $\boxed{-1 \le y \le 1}$

oleh f adalah   dgn batas ini

$U = x^2 - y^2 = \boxed{1 - y^2}$ $V = 2xy = 2y$

diperoleh   dari sini

$V^2 = (2y)^2 = 4y^2 = 4(1-U)$ $\boxed{0 \le U \le 1}$

dan $\boxed{-2 \le V \le 2}$ dari $V = 2y$ $-1 \le y \le 1$

Jadi, peta h adalah

$h^* : V^2 = 4(1-U)$, $-2 \le V \le 2$

(3) Peta garis k : $y = -x$, $\boxed{0 \le x \le 1}$ ini

oleh f adah

$U = x^2 - y^2 = x^2 - (-x)^2 = 0$

$V = 2xy = \boxed{-2x^2}$, $\boxed{-2 \le V \le 0}$

dari sini

dgn interval

Jadi peta k adalah

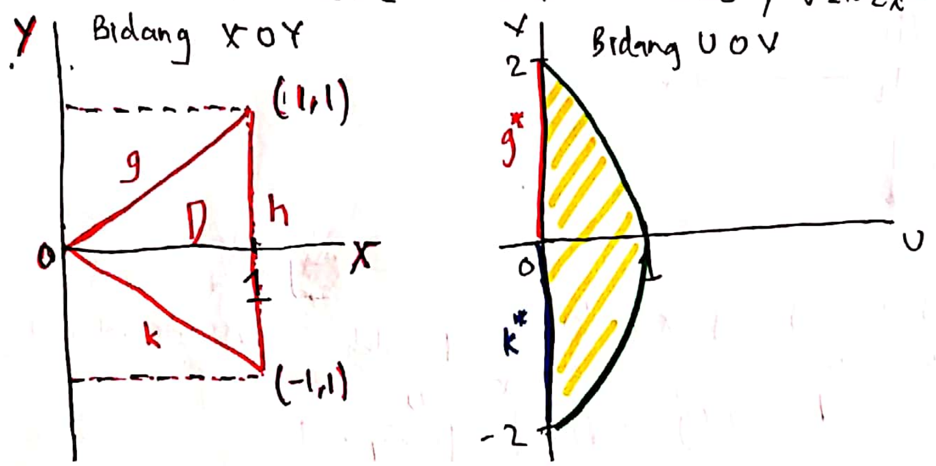
$k^* : U = 0$, $V = -2x^2$, $-2 \le V \le 0$.

∴ Diperoleh:

$g : y = x$, $0 \le x \le 1 \rightarrow g^* : U = 0$ $V = 2x^2$
$0 \le V \le 2$

$h : x = 1$, $-1 \le y \le 1 \rightarrow h^* : V^2 = 4(1-U)$
$-2 \le V \le 2$, $U = 1 - y^2$, $0 \le U \le 1$

$k : y = -x$, $0 \le x \le 1 \rightarrow k^* : U = 0$, $V = -2x^2$



∴ Peta D oleh f adalah $D^* = \{ (U,V) \mid V^2 = 4(1-U) ; 0 \le U \le 1 , -2 \le V \le 2 \}$

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Pemetaan Linear Kompleks

Pemetaan Linear $w = f(z) = az$ memenuhi

$$f(z_1 + z_2) = f(z_1) + f(z_2) \quad \forall z_1, z_2 \in \mathbb{C}$$

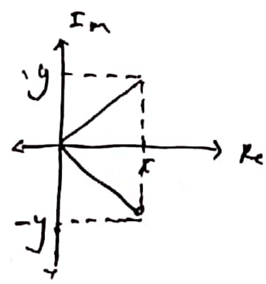
$$f(\alpha z) = \alpha \cdot f(z) \quad \forall k \in \mathbb{C},\ z \in \mathbb{C}$$

Pemetaan ini dapat ditulis dalam bentuk

$$\omega = u + vi = \begin{pmatrix} u \\ v \end{pmatrix} = f(x+yi) = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\boxed{E}$

1. $f(z) = \omega = \bar{z}$ ini memetakan $z = x + yi$ ke $\bar{z} = x - yi$ dan ini merupakan pencerminan dengan sumbu x.



Tetapi yang menarik bahwa

$$w = u + vi = \begin{pmatrix} u \\ v \end{pmatrix} = f(x+yi) = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x - yi$$

2. $f(z) = -i\bar{z}$ memetakan $z = x + yi \longrightarrow f(z) = i(x - yi) = -y - xi$
(Cari Matriksnya)

Solusi:

$$w = u + vi = \begin{pmatrix} u \\ v \end{pmatrix} = f(x+yi) = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -y - xi$$

//

Makassar, 2 Juni 2024

Analisis Kompleks

Pertemuan ke - 12

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Analisis Kompleks / Pertemuan ke-12 / Catatan

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Persekitaran / lingkungan (Repeat Please)

* Persekitaran - ε dari $z_0 \in \mathbb{C}$ ditulis

$$V_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$$

$|z - z_0| < \varepsilon$

* Lingkungan tanpa pusat

$$V_\varepsilon^*(z_0) = V_\varepsilon(z_0) \setminus \{ z_0 \}$$

$\boxed{E}$

Misal $z_0 = 1 + 2i$, $\varepsilon = 2$, $V_\varepsilon(z_0) = \dots ?$

$$V_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$$

$$V_2(1+2i) = \{ z \in \mathbb{C} \mid |z - (1+2i)| < 2 \}$$

Misal $z = a + bi = (a, b)$
$\quad\quad\quad\quad\quad\quad\quad\quad$ Re $\;$ Im

$$|z - (1+2i)| < 2$$

$$\Rightarrow |(a+bi) - (1+2i)| < 2$$

$$\Rightarrow |(a-1) + (b-2)i| < 2$$

$$\Rightarrow \sqrt{(a-1)^2 + (b-2)^2} < 2$$

$$\Rightarrow (a-1)^2 + (b-2)^2 < 4$$

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Makassar, 20 Mei 2021

Limit Fungsi Kompleks

Definisi

Fungsi $f : V_r^*(z_0) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$, notasi

$\lim_{z \to z_0} f(z) = w_0$ didefinisikan sebagai

$\forall \varepsilon > 0 \ \exists \delta > 0$ sehingga

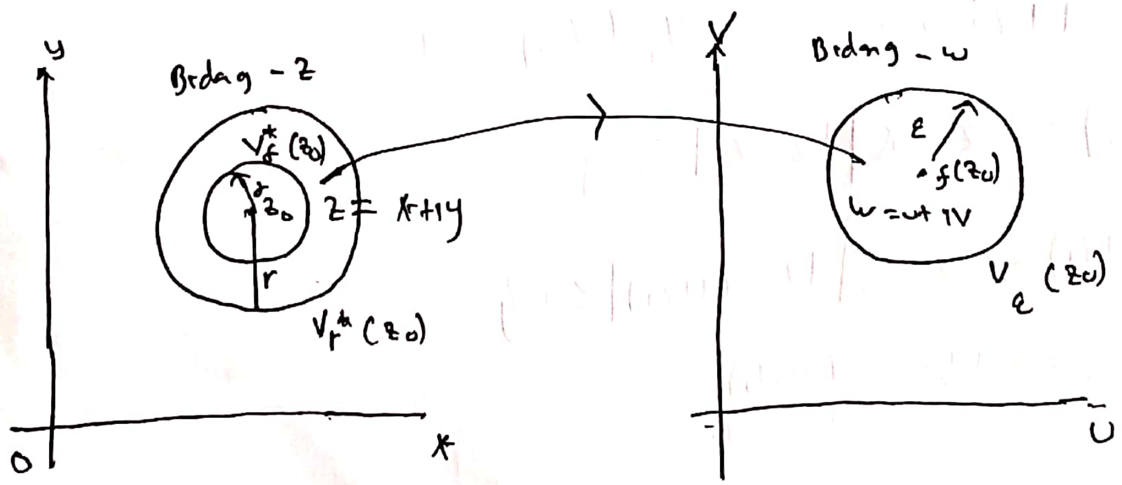
Jika $0 < |z - z_0| < \delta$ maka

$|f(z) - w_0| < \varepsilon$

disini:

z_0 = Pusat $V_r^*(z_0)$

$f(z_0) \in V_\varepsilon(w_0) \Leftrightarrow z \in V_r^*(z_0) \cap V_\delta^*(z_0)$



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Makassar, 20 Mei 2021

Contoh

Buktikan $\lim_{z \to z_0} iz^2 = iz_0^2$

Bukti:

Analisis Pendahuluan

Ambil $\varepsilon > 0$ sebarang akan dicari

$\delta > 0$ sehingga jika $0 < |z - z_0| < \delta$

maka $|iz^2 - iz_0^2| < \varepsilon$

Perhatikan bahwa

$$|(z^2 - iz_0^2)| = |i|\,|z^2 - z_0^2|$$
$$= |(z - z_0)(z + z_0)|$$
$$= |z - z_0|\,|z + z_0| \quad \ldots\ldots (*)$$

Misal $|z - z_0| < \delta \leq 1$ akibatnya

$$|z + z_0| = |z - z_0 + 2z_0|$$
$$\leq |z - z_0| + |2z_0|$$
$$= |z - z_0| + 2|z_0|$$
$$< 1 + 2|z_0| \quad \text{, sehingga}$$

berdasarkan (*) diperoleh

$$|iz^2 - iz_0^2| = |z - z_0|\,|z + z_0|$$
$$< |z - z_0|\,(1 + 2|z_0|) < \varepsilon$$

Lp Jika $\delta = \dfrac{\varepsilon}{1 + 2|z_0|}$,

Pilih $\delta \leq \min\left\{1, \dfrac{\varepsilon}{1 + 2|z_0|}\right\}$.

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Makassar, 28 Mei 2024.

Bukti Formal

Diberikan $\varepsilon > 0$ sebarang, Pilih

$\delta = \min\left\{1, \frac{\varepsilon}{1+2|z_0|}\right\}$ sehingga

untuk $|z - z_0| < \delta$ maka

$$|iz^2 - iz_0^2| = |i|\,|z^2 - z_0^2|$$

$$= |z - z_0|\,|z + z_0|$$

$$< \delta\,(1 + 2|z_0|)$$

$$< \frac{\varepsilon}{(1+2|z_0|)} \cdot (1 + 2|z_0|) = \varepsilon$$

$\therefore \lim_{z \to z_0} iz^2 = iz_0^2$ //

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(2.) $\lim_{z \to z_0} (2z^2 - 2) = 2z_0^2 - 2$

Penyelesaian :

Bukti :

-> Analisis Pendahuluan :

Ambil $\epsilon > 0$ sebarang
Akan dicari $\delta > 0 \ni 0 < |z - z_0| < \delta \Rightarrow |2z^2 - 2 - (2z_0^2 - 2)| < \epsilon$
Perhatikan bahwa

$$\begin{aligned} |2z^2 - 2 - (2z_0^2 - 2)| &= |2z^2 - 2 - 2z_0^2 + 2| \\ &= |2z^2 - 2z_0^2| \\ &= |2||z^2 - z_0^2| \\ &= 2 \cdot |z^2 - z_0^2| \\ &= 2\, |(z - z_0)(z + z_0)| \\ &= 2\, |z - z_0|\, |z + z_0| \quad \ldots (*) \end{aligned}$$

Misal, $|z - z_0| < \delta \leq 1$, akibatnya

$$\begin{aligned} |z + z_0| &= |z + z_0 - z_0 + z_0| \\ &= |z - z_0 + 2z_0| \\ &\leq |z - z_0| + |2z_0| \quad \text{[Ketaksamaan } \Delta\text{]} \\ &= |z - z_0| + 2|z_0| \\ &< 1 + 2|z_0| \end{aligned}$$

sehingga berdasarkan $(*)$ diperoleh

$$\begin{aligned} |2z^2 - 2 - (2z_0^2 - 2)| &= 2 \cdot |z - z_0| \cdot |z + z_0| \\ &< 2 \cdot |z - z_0| \cdot (1 + 2|z_0|) \leq \epsilon \end{aligned}$$

~~$\delta \epsilon$~~ Bila $\delta = \dfrac{\epsilon}{2(1 + 2|z_0|)}$

Pilih $\delta \leq \min\left\{1, \dfrac{\epsilon}{2(1 + 2|z_0|)}\right\}$

-> Bukti Formal

Diberikan $\epsilon > 0$ sebarang, pilih $\delta = \min\left\{1, \dfrac{\epsilon}{2(1 + 2|z_0|)}\right\}$, sehingga untuk $|z - z_0| < \delta$ berlaku

$$\begin{aligned} |2z^2 - 2 - (2z_0^2 - 2)| &= 2\,|z - z_0|\,|z + z_0| \\ &< 2\, \delta \cdot (1 + 2|z_0|) \\ &< 2 \cdot \frac{\epsilon}{2(1 + 2|z_0|)} \cdot (1 + 2|z_0|) = \epsilon \end{aligned}$$

$\therefore \lim_{z \to z_0} 2z^2 - 2 = 2z_0^2 - 2$ ▨

Makassar, 2 Juni 2024

Analisis Kompleks

Pertemuan ke - 13

Nama : Imanuel Agung Sembe

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Analisis Kompleks / Pertemuan ke - 13 Makassar, 28 Mei 2021

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Teorema Limit

(1) Misalkan $f(z) = u(x,y) + iv(x,y)$

$z = x + iy$ dan $z_0 = x_0 + iy_0$

$w_0 = u_0 + iv_0$ maka

(*) $\lim_{z \to z_0} f(z) = w_0$ jika dan hanya jika

(**) $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ dan $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Proof :

$\Leftarrow$ Ambil $\varepsilon > 0$ sebarang karena

$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ akibatnya terdapat $\delta_1 > 0$ sehingga untuk

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1$ berlaku $|u - u_0| < \frac{\varepsilon}{2}$ ....(1)

di lain pihak

$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ akibatnya terdapat $\delta_2 > 0$ sehingga untuk

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2$ berlaku $|v - v_0| < \frac{\varepsilon}{2}$) .... (2)

Pilih $\delta = \min\{\delta_1, \delta_2\}$ sehingga

untuk $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ berlaku

$$\begin{aligned} |f(z) - w_0| &= |(u+iv) - (u_0 + iv_0)| \\ &= |(u-u_0) + i(v-v_0)| \\ &\leq |u-u_0| + |v-v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

$\therefore \lim_{z \to z_0} f(z) = w_0$

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$\Rightarrow$ Diketahui (*) berlaku.

Kita tahu bahwa u/ setiap bilangan positif ε sebarang, selalu ada bilangan positif δ sedemikian sehingga

(***) $\quad |(u+iv)-(u_0+iv_0)| < \varepsilon$

dimana

(****) $\quad 0 < |(x+iy)-(x_0+iy_0)| < \delta$

Tapi

$$|u-u_0| \leq |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|,$$
$$|v-v_0| \leq |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|$$

dan

$$|(x+iy)-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

oleh karena itu, dari (***) dan (****) diperoleh

$$|u-u_0| < \varepsilon \text{ dan } |v-v_0| < \varepsilon$$

dimana

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

Ini menunjukkan $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ dan $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

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(2.) Misalkan $\lim_{z \to z_0} f(z) = w_0$ dan $\lim_{z \to z_0} t(z) = V_0$ maka

(a) $\lim_{z \to z_0} \left(f(z) + t(z) \right) = w_0 + V_0$

Bukti :

Analisis Pendahuluan

Jika $\varepsilon > 0$, maka $\frac{\varepsilon}{2} > 0$ pasti.

Karena diketahui $\lim_{z \to z_0} f(z) = w_0$, maka terdapat $\delta_1 > 0$ sedemikian sehingga

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - w_0| < \frac{\varepsilon}{2}$$

Karena diketahui $\lim_{z \to z_0} t(z) = V_0$, maka terdapat $\delta_2 > 0$ sedemikian sehingga

$$0 < |z - z_0| < \delta_2 \Rightarrow |t(z) - V_0| < \frac{\varepsilon}{2}$$

Pilih $\delta = \min \{ \delta_1, \delta_2 \}$

Bukti Formal

Diberikan $\varepsilon > 0$ sebarang,

Pilih $\delta = \min \{ \delta_1, \delta_2 \}$

Sehingga untuk $|z - z_0| < \delta$ maka

$$\left| \left(f(z) + t(z) \right) - \left(w_0 + V_0 \right) \right| = \left| \left(f(z) - w_0 \right) + \left(t(z) - V_0 \right) \right|$$

$$\leq |f(z) - w_0| + |t(z) - V_0| \quad [\text{Ketaksamaan } \triangle]$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \lim_{z \to z_0} \left(f(z) + t(z) \right) = w_0 + V_0$. //

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(b) $\lim_{z \to z_0} \left(f(z) - t(z)\right) = w_0 - v_0$

Bukti :

Karena bagian (a) sudah terbukti, maka hal berikut berlaku :

$$\lim_{z \to z_0} \left(f(z) - t(z)\right) = \lim_{z \to z_0} \left(f(z) + (-1) \cdot t(z)\right)$$

$$= \lim_{z \to z_0} f(z) + \lim_{z \to z_0} (-1) \cdot t(z)$$

$$= \lim_{z \to z_0} f(z) + (-1) \cdot \lim_{z \to z_0} t(z)$$

$$= \lim_{z \to z_0} f(z) - \lim_{z \to z_0} t(z)$$

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(c) $\lim_{z \to z_0} (f(z) \cdot t(z)) = (W_0) \cdot (V_0)$

Bukti :

Dengan menggunakan teorema (a) dan (b), diperoleh :

$$\lim_{z \to z_0} [f(z) - W_0] = \lim_{z \to z_0} f(z) - \lim_{z \to z_0} W_0 = W_0 - W_0 = 0$$

$$\lim_{z \to z_0} [t(z) - V_0] = \lim_{z \to z_0} t(z) - \lim_{z \to z_0} V_0 = V_0 - V_0 = 0$$

Diambil $\varepsilon > 0$ sebarang, maka terdapat $\delta_1 ; \delta_2 > 0$, sehingga :

Jika $0 < |z - z_0| < \delta_1$ maka $|(f(z) - W_0) - 0| < \sqrt{\varepsilon}$ dan

Jika $0 < |z - z_0| < \delta_2$ maka $|(t(z) - V_0) - 0| < \sqrt{\varepsilon}$

Pilih $\delta = \min\{\delta_1, \delta_2\}$

Untuk $0 < |z - z_0| < \delta$ diperoleh

$$|[f(z) - W_0][t(z) - V_0] - 0| = |f(z) - W_0| \cdot |t(z) - V_0|$$
$$< \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$$

$\therefore \lim_{z \to z_0} [f(z) - W_0][t(z) - V_0] = 0$

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Selanjutnya perhatikan bahwa,

$$[f(z) - w_0][t(z) - v_0] = f(z) \cdot t(z) - v_0 \cdot f(z) - w_0 \cdot t(z) + w_0 \cdot v_0$$

Atau,

$$f(z) \cdot t(z) = [f(z) - w_0][t(z) - v_0] + v_0 \cdot f(z) + w_0 \cdot t(z) - w_0 \cdot v_0$$

Kedua ruas dilimitkan, menjadi :

$$\lim_{z \to z_0} [f(z) \cdot t(z)] = \lim_{z \to z_0} \left([f(z) - w_0][t(z) - v_0] + v_0 \cdot f(z) + w_0 \cdot t(z) - w_0 \cdot v_0\right)$$

$$= \lim_{z \to z_0} [f(z) - w_0][t(z) - v_0] + \lim_{z \to z_0} v_0 \cdot f(z) + \lim_{z \to z_0} w_0 \cdot t(z) - \lim_{z \to z_0} w_0 \cdot v_0$$

$$= 0 + v_0 \cdot \lim_{z \to z_0} f(z) + w_0 \cdot \lim_{z \to z_0} t(z) - \lim_{z \to z_0} w_0 \cdot v_0$$

$$= v_0 \cdot w_0 + w_0 \cdot v_0 - w_0 \cdot v_0$$

$$= v_0 \cdot w_0$$

$$\lim_{z \to z_0} [f(z) \cdot t(z)] = (w_0) \cdot (v_0)$$

//

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(d) $\lim_{z \to z_0} \frac{f(z)}{t(z)} = \frac{w_0}{v_0}$, $V_0 \neq 0$

Bukti :

Pertama akan ditunjukkan :

$$\lim_{z \to z_0} \frac{1}{t(z)} = \frac{1}{v_0}$$

Diambil $\varepsilon > 0$ sebarang.

Karena $\lim_{z \to z_0} t(z) = V_0$ maka terdapat $\delta_1 > 0$ sedemikian sehingga

$$|t(z) - V_0| < \frac{|V_0|}{2} \quad \text{dimana } 0 < |z - z_0| < \delta_1$$

Untuk $0 < |z - z_0| < \delta_1$ diperoleh :

$$\begin{aligned} v_0 &= |V_0 - t(z) + t(z)| \\ &\leq |V_0 - t(z)| + |t(z)| \quad [\text{Ketaksamaan } \Delta] \\ &= |t(z) - V_0| + |t(z)| \\ &< \frac{|V_0|}{2} + |t(z)| \end{aligned}$$

Hal ini menunjukkan

$$|V_0| < \frac{|V_0|}{2} + |t(z)| \Rightarrow \frac{|V_0|}{2} < |t(z)| \Rightarrow \frac{1}{t(z)} < \frac{2}{|V_0|}$$

→ next

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Selanjutnya, terdapat juga $\delta_2 > 0$ sedemikian sehingga

$$|t(z) - V_0| < \frac{|V_0|^2}{2} \cdot \varepsilon \quad \text{dimana} \quad 0 < |z - z_0| < \delta_2$$

Pilih $\delta = \min \{\delta_1, \delta_2\}$. Jika $0 < |z - z_0| < \delta$, diperoleh:

$$\left| \frac{1}{t(z)} - \frac{1}{V_0} \right| = \left| \frac{V_0 - t(z)}{V_0 \cdot t(z)} \right|$$

$$= \frac{1}{|V_0 \cdot t(z)|} \cdot |V_0 - t(z)|$$

$$= \frac{1}{|V_0|} \cdot \frac{1}{|t(z)|} \cdot |t(z) - V_0|$$

$$< \frac{1}{|V_0|} \cdot \frac{2}{|V_0|} \cdot |t(z) - V_0|$$

$$< \frac{2}{|V_0|^2} \cdot \frac{|V_0|^2}{2} \cdot \varepsilon = \varepsilon$$

$$\therefore \lim_{z \to z_0} \frac{1}{t(z)} = \frac{1}{V_0}$$

Selanjutnya,

$$\lim_{z \to z_0} \left[\frac{f(z)}{t(z)} \right] = \lim_{z \to z_0} \left[f(z) \cdot \frac{1}{t(z)} \right]$$

$$= \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} \frac{1}{t(z)} \quad [\text{Bagian } (c)]$$

$$= W_0 \cdot \frac{1}{V_0}$$

$$\lim_{z \to z_0} \left[\frac{f(z)}{t(z)} \right] = \frac{W_0}{V_0} \quad // \quad \blacksquare$$

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③ Jika $\lim_{z \to z_0} f(z) = w_0$ maka $\lim_{z \to z_0} |f(z)| = |w_0|$

(Kebalikan teorema ini tidak berlaku)

④ $\lim_{z \to z_0} f(z) = 0 \iff \lim_{z \to z_0} |f(z)| = 0$

$\boxed{E}$

Hitung $\lim_{z \to 0} \frac{(\bar{z})^3}{|z|^2}$

Solusi :

Perhatikan bahwa

$$\lim_{z \to 0} \left| \frac{(\bar{z})^3}{|z|^2} \right| = \lim_{z \to 0} \frac{|\bar{z}|^3}{|z|^2} = \lim_{z \to 0} |z|$$

$$= \left| \lim_{z \to 0} z \right|$$

$$= 0$$

$\therefore \lim_{z \to 0} \frac{(\bar{z})^3}{|z|^2} = 0$ //

Makassar, 2 Juni 2024

Analisis Kompleks

Pertemuan ke - 14

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Analisis Kompleks / Pertemuan ke - 14 / Catatan

Makassar, 1 Juni 2021

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Limit Tak Hingga

$\lim\limits_{z \to z_0} = \infty \iff \forall \varepsilon > 0 \ \exists \delta > 0$ sehingga

$$\underbrace{0 < |z - z_0| < \delta}_{z \in V_\delta^*(z_0)} \text{ maka } \underbrace{|f(z)|}_{f(z) \in V_\varepsilon(\infty)} > \frac{1}{\varepsilon}$$

dimana $V_\delta(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \delta \}$

dan $V_\delta^*(z_0) = V_\delta(z_0) \setminus \{ z_0 \}$

dimana $V_\varepsilon(\infty) = \{ z \in \mathbb{C} \mid |z| > \frac{1}{\varepsilon} \}$

↑ lingkungan / persekitaran titik tak hingga

$$\left.\begin{aligned} \lim_{z \to z_0} f(z) = \infty &\iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \ni z \in V_\delta^*(z_0) \Rightarrow f(z) \in V_\varepsilon(\infty) \\ &\iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \ni 0 < |z - z_0| < \delta \Rightarrow |f(z)| > \frac{1}{\varepsilon} \\ &\iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \ni 0 < |z - z_0| < \delta \Rightarrow \left| \frac{1}{f(z)} \right| < \varepsilon \end{aligned}\right\} \text{ekivalen}$$

Fakta menarik:

$$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0 .$$

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E Tunjukkan bahwa

$$\lim_{z \to -1} \frac{iz+3}{z+1} = \infty$$

Bukti :

Misal $f(z) = \frac{iz+3}{z+1}$

Perhatikan bahwa,

$$\lim_{z \to -1} \left(\frac{1}{\frac{iz+3}{z+1}} \right) = \lim_{z \to -1} \left(\frac{z+1}{iz+3} \right)$$

$$= \frac{0}{-i+3}$$

$$= 0$$

$\therefore$ Karena $\lim_{z \to -1} \left(\frac{1}{\frac{iz+3}{z+1}} \right) = 0$ akibatnya $\lim_{z \to -1} \left(\frac{iz+3}{z+1} \right) = \infty$.

Latihan :

Tunjukkan bahwa $\lim_{z \to -2} \left(\frac{z+5}{z+2} \right) = \infty$

Penyelesaian :

Misal $f(z) = \frac{z+5}{z+2}$

Perhatikan bahwa,

$$\lim_{z \to -2} \left(\frac{1}{\frac{z+5}{z+2}} \right) = \lim_{z \to -2} \left(\frac{z+2}{z+5} \right)$$

$$= \frac{0}{3}$$

$$= 0$$

$\therefore$ Karena $\lim_{z \to -2} \left(\frac{1}{\frac{z+5}{z+2}} \right) = 0$ akibatnya $\lim_{z \to -2} \left(\frac{z+5}{z+2} \right) = \infty$

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Limit di Tak Hingga

$$\boxed{\lim_{z\to\infty} f(z) = w_0} \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 \ni \underbrace{|z| > \frac{1}{\delta}}_{z \in V_\delta(\infty)} \Rightarrow \underbrace{|f(z) - w_0| < \varepsilon}_{f(z) \in V_\varepsilon(w_0)}$$

z diganti $\frac{1}{z} \Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$ sehingga $0 < |z| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$

$$\Updownarrow$$

$$\boxed{\lim_{z\to z_0} f\left(\frac{1}{z_0}\right) = w_0}$$

Berarti $\lim_{z\to\infty} f(z) = w_0 \Leftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$

$\boxed{E}$ Tunjukkan bahwa $\lim_{z\to\infty} \frac{2z+i}{z+1} = 2$

Bukti

Misalkan $f(z) = \frac{2z+i}{z+i}$ ~~jadi~~

Jadi $f\left(\frac{1}{z}\right) = \frac{2\left(\frac{1}{z}\right)+i}{\left(\frac{1}{z}\right)+i}$

Sehingga $\lim_{z\to 0} f\left(\frac{1}{z}\right) = \lim_{z\to 0} \left(\frac{2\left(\frac{1}{z}\right)+i}{\frac{1}{z}+i}\right)$

$$= \lim_{z\to 0} \frac{2+iz}{1+iz}$$

$$= 2$$

Akibatnya, $\lim_{z\to\infty} \frac{2z+i}{z+i} = 2$. //

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Latihan.

(1.) Tunjukkan Bahwa $\lim\limits_{z \to \infty} \frac{2z^2+i}{z^2+2i} = 2$

Jawab:

Bukti:

Misal $f(z) = \frac{2z^2+i}{z^2+2i}$

Maka, $\lim\limits_{z\to 0} f\left(\frac{1}{z}\right) = \lim\limits_{z\to 0} \frac{2\left(\frac{1}{z}\right)^2+i}{\left(\frac{1}{z}\right)^2+2i}$

$= \lim\limits_{z\to 0} \frac{2\left(\frac{1}{z^2}\right)+i}{\frac{1}{z^2}+2i}$

$= \lim\limits_{z\to 0} \frac{2+iz^2}{1+2z^2 i}$

$= 2$ //

Karena $\lim\limits_{z\to 0} f\left(\frac{1}{z}\right) = 2$ dimana $f(z) = \frac{2z^2+i}{z^2+2i}$, ~~maka~~ akibatnya $\lim\limits_{z\to\infty} \frac{2z^2+i}{z^2+2i} = 2$. //

(2.) Tunjukkan bahwa $\lim\limits_{z\to\infty} \frac{3z^3-2z^2+z}{3z^3+z} = 1$

Jawab:

Bukti:

Misal $f(z) = \frac{3z^3-2z^2+z}{3z^3+z}$

Maka, $\lim\limits_{z\to 0} f\left(\frac{1}{z}\right) = \lim\limits_{z\to 0} \frac{3\left(\frac{1}{z}\right)^3 - 2\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)}{3\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)}$

$= \lim\limits_{z\to 0} \frac{3\cdot\frac{1}{z^3} - 2\frac{1}{z^2} + \frac{1}{z}}{3\frac{1}{z^3} + \frac{1}{z}}$

$= \lim\limits_{z\to 0} \frac{\frac{3-2z+z^2}{z^3}}{\frac{3+z^2}{z^3}}$

$= \lim\limits_{z\to 0} \frac{3-2z+z^2}{3+z^2}$

$= \frac{3}{3}$

$= 1$ //

Karena $\lim\limits_{z\to 0} f\left(\frac{1}{z}\right) = 1$ dimana $f(z) = \frac{3z^3-2z^2+z}{3z^3+z}$, maka berakibat $\lim\limits_{z\to\infty} \frac{3z^3-2z^2+z}{3z^3+z} = 1$. //

Makassar, 4 Juli 2021

Analisis Kompleks

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Analisis Kompleks / Pertemuan ke - 15 / Catatan

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Kekontinuan Fungsi Kompleks

Definisi:

Misalkan z_0 titik limit dari D_f

$f : D_f \to \mathbb{C}$ kontinu di $z_0 \in D_f$

Jika $\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$

$\downarrow$

(1) $\lim_{z \to z_0} f(z)$ ada

(2) $f(z_0)$ ada

(3) $\lim_{z \to z_0} f(z) = f(z_0)$

$\lim_{z \to z_0} f(z) = f(z_0) \iff \forall \varepsilon > 0 \ \exists \delta > 0$ sehingga

jika $|z - z_0| < \delta$ maka

$|f(z) - f(z_0)| < \varepsilon$

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Sifat-sifatnya:

(1) Jika f dan g kontinu di $z_0 \in D_f \cap D_g$ maka $f+g$, $f-g$, αf (α konstanta kompleks), $f \cdot g$, $\frac{f}{g}$ $(g(z_0) \neq 0)$ kontinu di z_0.

(2) Jika f kontinu di z_0 dan g kontinu $f(z_0)$ maka $g \circ f$ kontinu di z_0.

(3) Jika $z = x + yi$, $f(z) = u(x,y) + v(x,y)i$ dan $z_0 = x_0 + y_0 i$ maka f kontinu di z_0 jika dan hanya jika $u = u(x,y)$ dan $v = v(x,y)$ keduanya kontinu di (x_0, y_0).

(4) Jika $z = r \cdot cis(\theta)$, $f(z) = u(r,\theta) + v(r,\theta)i$ dan $z_0 = r_0 \cdot cis(\theta_0)$ maka f kontinu di z_0 jika dan hanya jika $u = u(r,\theta)$ dan $v = v(r,\theta)$ kontinu di (r_0, θ_0).

(5.) Jika f kontinu di z_0 dan $f(z_0) \neq 0$ maka $\exists r > 0$ sehingga $|f(z)| > 0$ pada $V_r(z_0)$.

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f kontinu di z_0 jika $\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$

$\Updownarrow$

(1.) $\lim_{z \to z_0} f(z)$ = ada

(2) $f(z_0)$ ada

(3) $\lim_{z \to z_0} f(z) = f(z_0)$

$\boxed{E}$ (1) Buktikan bahwa $f(z) = z^3$ kontinu di $z_0 = i$

Solusi :

Adb. $\lim_{z \to z_0} f(z) = f(z_0)$

Perhatikan bahwa

(1) $\lim_{z \to z_0} f(z) = \lim_{z \to i} z^3 = (i)^3 = -i$

(2) $f(z_0) = f(i) = (i)^3 = -i$

Dari (1) dan (2) diperoleh

$\lim_{z \to z_0} f(z) = f(z_0)$.

Jadi, $f(z) = z^3$ kontinu di $z_0 = i$.

(Kalau tidak kontinu, berarti dia tidak memenuhi salah satu dari ketiga syarat di atas).

(2) Misal

$$f(z) = \begin{cases} z^2 & ; z \neq i \\ 0 & ; z = i \end{cases}$$

tidak kontinu di $z = i$, sebab

$\lim_{z \to i} f(z) = \lim_{z \to i} z^2 = (i)^2 = -1$.

Di lain pihak $f(i) = 0$

Jadi, $\lim_{z \to i} f(z) \neq f(i)$

$\therefore f(z)$ tidak kontinu.

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Makassar, 4 Juni 2021

Turunan Fungsi Kompleks

Untuk Fungsi $w = f(z)$ terdefinisi pada $V_\varepsilon(z_0)$.
Turunan f di z_0 ditulis $f'(z_0)$ dituliskan sebagai

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Asalkan Limit ini ada.

Fungsi $w = f(z)$ disebut terdiferensialkan di z_0 jika $f'(z_0)$ ada.

Substitusi $\Delta z = z - z_0$ diperoleh

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Apa hubungan antara kekontinuan dan turunan pada fungsi kompleks?
Jika dia punya turunan berarti dia kontinu.

Teorema :

Jika $w = f(z)$ punya turunan di z_0 maka f kontinu di z_0.

Bukti: $\lim_{z \to z_0} f(z) = f(z_0)$

Note that, $f(z) = \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) + f(z_0)$

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) \right) + f(z_0)$$

$$= \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right) \cdot \lim_{z \to z_0} (z - z_0) + f(z_0)$$

$$= f'(z_0) \cdot 0 + f(z_0)$$

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$\therefore f$ kontinu di z_0. //

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Makassar, 14 Juni 2021

Turunan fungsi $w = f(z)$ pada daerah D ditulis $f'(z)$ atau $\frac{dw}{dz}$ didefinisikan $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$, asalkan limitnya ada.

$\boxed{E}$ Tentukan turunan dari $f(z) = z^2$ pada $\mathbb{C}$

Solusi :

$$
\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\cancel{z^2} + 2z.\Delta z + \Delta z^2 - \cancel{z^2}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\Delta z\,(2z + \Delta z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} (2z + \Delta z) \\
&= 2z \,//
\end{aligned}
$$

$\boxed{L}$ Tentukan $f'(z)$ dari

(1) $f(z) = 3z^2 - 2z + 4$

(2) $f(z) = (1 - 4z^2)^3$

(3) $f(z) = \frac{z-1}{2z+1}$, $z \neq -\frac{1}{2}$

(4) $f(z) = \frac{(1+z^2)^4}{z^2}$, $z \neq 0$

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Penyelsaian :

(1) $f(z) = 3z^2 - 2z + 4$.

Penyelsaian :

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[3(z+\Delta z)^2 - 2(z+\Delta z) + 4] - [3z^2 - 2z + 4]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[3(z^2 + 2z \cdot \Delta z + \Delta^2 \cdot z^2) - 2(z+\Delta z) + 4] - [3z^2 - 2z + 4]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{3z^2} + 6z \cdot \Delta z + 3 \cdot \Delta^2 \cdot z^2 - \cancel{2z} - 2 \cdot \Delta z \cancel{+4} - \cancel{3z^2} + \cancel{2z} \cancel{-4}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{6z \cdot \Delta z + 3 \cdot \Delta^2 \cdot z^2 - 2 \cdot \Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(6z + \Delta z - 2)}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} (6z + \Delta z - 2)$$

$$= 6z - 2 \text{ //}$$

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(2) $f(z) = (1-4z^2)^3$

Penyelesaian :

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(1-4(z+\Delta z)^2)^3 - (1-4z^2)^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{1} - 12(z+\Delta z)^2 + 48(z+\Delta z)^4 - 64(z+\Delta z)^6 - \cancel{1} + 12z^2 - 48z^4 + 64z^6}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{-12z^2} - 24\Delta z . z - 12(\Delta z)^2 + \cancel{48z^4} + 192z^3\Delta z + 288z^2(\Delta z)^2 + 192z(\Delta z)^3}{\Delta z}$$

$$\frac{+48(\Delta z)^4 - \cancel{64z^6} - 384z^5 . \Delta z - 960z^4(\Delta z)^2 - 1280z^3(\Delta z)^3 - 960z^2(\Delta z)^4}{}$$

$$\frac{-384z(\Delta z)^5 - 64(\Delta z)^6 + \cancel{12z^2} - \cancel{48z^4} + \cancel{64z^6}}{}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(-24z - 12\Delta z + 192z^3 + 288z^2\Delta z + 192z(\Delta z)^2 + 48(\Delta z)^3 - 384z^5}{\cancel{\Delta z}}$$

$$\frac{-960z^4\Delta z - 1280z^3(\Delta z)^2 - 960z^2(\Delta z)^3 - 384z(\Delta z)^4 - 64(\Delta z)^5)}{}$$

$$= \lim_{\Delta z \to 0} -24z - 12\Delta z + 192z^3 + 288\Delta z + 192z(\Delta z)^2 + 48(\Delta z^3) - 384z^5$$

$$- 960z^4\Delta z - 1280z^3(\Delta z)^2 - 960z^2(\Delta z)^3 - 384z(\Delta z)^4 - 64(\Delta z)^5$$

$$= -24z + 192z^3 - 384z^5 \quad //$$

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(3) $f(z) = \dfrac{z-1}{2z+1} \quad ; z \neq -\frac{1}{2}$

Penyelesaian :

$$f'(z) = \lim_{x \to z} \frac{f(x) - f(z)}{x - z}$$ [Lihat buku kalkulus I hal 101]

$$= \lim_{x \to z} \frac{\frac{x-1}{2x+1} - \frac{z-1}{2z+1}}{x - z}$$

$$= \lim_{x \to z} \frac{(x-1)(2z+1) - (z-1)(2x+1)}{(2x+1)(2z+1)} \cdot \frac{1}{x-z}$$

$$= \lim_{x \to z} \frac{(2xz + x - 2z - 1) - (2xz + z - 2x - 1)}{(2x+1)(2z+1)} \cdot \frac{1}{x-z}$$

$$= \lim_{x \to z} \frac{x - 2z - z + 2x}{(2x+1)(2z+1)} \cdot \frac{1}{x-z}$$

$$= \lim_{x \to z} \frac{3(x-z)}{(2x+1)(2z+1)} \cdot \frac{1}{(x-z)}$$

$$= \lim_{x \to z} \frac{3}{(2x+1)(2z+1)}$$

$$= \frac{3}{(2z+1)(2z+1)}$$

$$= \frac{3}{(2z+1)^2} \quad //$$

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Cara bapak (Normal)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\left(\frac{(z+\Delta z)-1}{2(z+\Delta z)+1} - \frac{z-1}{2z+1} \right) \cdot \frac{1}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[\left(\frac{(z+\Delta z-1)(2z+1) - (z-1)(2z+2\Delta z+1)}{(2z+2\Delta z+1)(2z+1)} \right) \cdot \frac{1}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[\left(\frac{(\cancel{2z^2} + \cancel{2z \cdot \Delta z} - \cancel{2z} + \cancel{z} + \Delta z - \cancel{1}) - (\cancel{2z^2} + \cancel{2z \cdot \Delta z} + \cancel{z} - \cancel{2z} - 2\Delta z - \cancel{1})}{(2z+2\Delta z+1)(2z+1)} \right) \frac{1}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{\Delta z + 2\Delta z}{(2z+2\Delta z+1)(2z+1)} \cdot \frac{1}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{3\cancel{\Delta z}}{(2z+2\Delta z+1)(2z+1)} \cdot \frac{1}{\cancel{\Delta z}} \right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{3}{(2z+2\Delta z+1)(2z+1)} \right]$$

$$= \frac{3}{(2z+1)(2z+1)}$$

$$= \frac{3}{(2z+1)^2} \quad //$$

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(4) $f(z) = \dfrac{(1+z^2)^4}{z^2}$

Penyelesaian :

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\frac{(1+(z+\Delta z)^2)^4}{(z+\Delta z)^2} - \frac{(1+z^2)^4}{z^2} \right] \cdot \frac{1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\frac{(1+4(z+\Delta z)^2+6(z+\Delta z)^4+4(z+\Delta z)^6+(z+\Delta z)^8)\,z^2 - (1+4z^2+6z^4+4z^6+z^8)}{(z+\Delta z)\,z^2} \right] \cdot \frac{1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\frac{(1+4z^2+8z\cdot\Delta z+4(\Delta z)^2+6z^4+24z^3\Delta z+36z^2(\Delta z)^2+24z(\Delta z)^3 + 6(\Delta z)^4+4z^6+24z^5\Delta z+60z^4(\Delta z)^2+80z^3(\Delta z)^3+60z^2(\Delta z)^4 + 24z(\Delta z)^5+4(\Delta z)^6+z^8+8z^7\cdot\Delta z+28z^6(\Delta z)^2+56z^5(\Delta z)^3 + 70z^4(\Delta z)^4+56z^3(\Delta z)^5+28z^2(\Delta z)^6+8z(\Delta z)^7+(\Delta z)^8)\,z^2 - (1+4z^2+6z^4+4z^6+z^8)(z^2+2z\cdot\Delta z+(\Delta z)^2)}{(z+\Delta z)^2\cdot z^2} \right] \cdot \frac{1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\frac{\cancel{z^2}+\cancel{4z^4}+8z^3\Delta z+4z^2(\Delta z)^2+\cancel{6z^6}+24z^5\cdot\Delta z+36z^4(\Delta z)^2+24z^3(\Delta z)^3 + 6z^2(\Delta z)^4+\cancel{4z^8}+24z^7\cdot\Delta z+60z^6(\Delta z)^2+80z^5(\Delta z)^3+60z^4(\Delta z)^4 + 24z^3(\Delta z)^5+4z^2(\Delta z)^6+\cancel{z^{10}}+8z^9\Delta z+28z^8(\Delta z)^2+56z^7(\Delta z)^3 + 70z^6(\Delta z)^4+56z^5(\Delta z)^5+28z^4(\Delta z)^6+8z^3(\Delta z)^7+z^2(\Delta z)^8 - \cancel{z^2}-2z\Delta z-(\Delta z)^2-\cancel{4z^4}-8z^3\Delta z-4z^2(\Delta z)-\cancel{6z^6}-12z^5\Delta z-6z^4(\Delta z)^2 - \cancel{4z^8}-8z^7\cdot\Delta z-4z^6(\Delta z)^2-\cancel{z^{10}}-2z^9\Delta z-z^8(\Delta z)^2}{(z+\Delta z)^2\cdot z^2} \right] \cdot \frac{1}{\cancel{\Delta z}}$$

$$= \frac{12z^5+16z^7+6z^9-2z}{z^2\cdot z^2}$$

$$= \frac{12z^5+16z^7+6z^9-2z}{z^4}$$

//

Dipindai dengan CamScanner

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E

Seperti Teorema sebelumnya (hal 4) bahwa teorema tersebut tidak berlaku sebaliknya. Berikut contohnya:

$f(z) = |z|^2$ kontinu di pada $\mathbb{C}$ tetapi dia hanya punya turunan di $z=0$.

Buktinya :

① Adb. $f(z) = |z|^2$ kontinu pada $\mathbb{C}$

Solusi :

Adb. $\lim_{z\to 0} f(z) = 0$

Perhatikan bahwa:

(a) $\lim_{z\to 0} f(z) = \lim_{z\to 0} |z|^2 = 0$

(b.) $f(0) = 0$

$\lim_{z\to 0} f(z) = f(0)$

Artinya f kontinu di $z=0$

Adb : $f(z) = |z|^2$ kontinu pada C

Karena $f(z) = |z|^2 = x^2 + y^2$, $z = x+yi$

$u(x,y) = x^2+y^2$ (Real) $v(x,y) = 0$ (Im)

u dan v kontinu diseluruh bidang datar (Fungsi polinom pasti kontinu).

② Adb. $f(z) = |z|^2$ hanya punya turunan di $z=0$

Perhatikan bahwa, misalkan $z_0 \in D_f = \mathbb{C}$ diperoleh

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

$$= \lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0} \quad \dots (*)$$

→ Tinjau $z_0 = 0$, diperoleh

$$f'(0) = \lim_{z\to 0} \frac{|z|^2}{z} = \lim_{z\to 0} \frac{z\bar{z}}{z}$$

$$= \lim_{z\to 0} \bar{z}$$

$$= 0 \;//$$

→ Tinjau $z_0 \neq 0$

Misal $z_0 = x_0 + y_0 i$ akibatnya diperoleh

$$f'(z_0) = \lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0}$$

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{x^2+y^2-(x_0^2+y_0^2)}{(x-x_0)+i(y-y_0)}$$

Misal $y = y_0$ diperoleh

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(2) Adb. $f(z) = |z|^2$ hanya punya turunan di $z=0$.

Perhatikan bahwa, misalkan

$z_0 \in D_f = \mathbb{C}$ diperoleh

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

$$= \lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0} \quad \ldots. (*)$$

∘) Tinjau $z_0 = 0$, diperoleh

$$f'(0) = \lim_{z\to 0} \frac{|z|^2}{z}$$

$$= \lim_{z\to 0} \frac{\cancel{z}\cdot\bar{z}}{\cancel{z}}$$

$$= \lim_{z\to 0} \bar{z}$$

$$= 0_{//}$$

∘) Tinjau $z_0 \neq 0$

Misal $z_0 = x_0 + y_0 i$ akibatnya diperoleh

$$f'(z_0) = \lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0}$$

$$f'(z_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{x^2+y^2-(x_0^2+y_0^2)}{(x-x_0)+i(y-y_0)}$$

Misal $y = y_0$ diperoleh

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{x^2+y^2-x_0^2-y_0^2}{(x-x_0)+i(y-y_0)}$$

$$= \lim_{x\to x_0} \frac{x^2+\cancel{y_0^2}-x_0^2-\cancel{y_0^2}}{x-x_0+i(\cancel{y_0-y_0})}$$

$$= \lim_{x\to x_0} \frac{(x-x_0)(x+x_0)}{(x-x_0)}$$

$$= \lim_{x\to x_0} (x+x_0)$$

$$= 2x_0 \quad \ldots\ldots\ldots (**)$$

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Di lain pihak,

Misal $x = x_0$ diperoleh

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{x^2+y^2-(x_0^2+y_0^2)}{(x-x_0)+i(y-y_0)}$$

$$= \lim_{y\to y_0} \frac{\cancel{x_0^2}+y^2-\cancel{x_0^2}-y_0^2}{\cancel{(x_0-x_0)}+i(y-y_0)}$$

$$= \lim_{y\to y_0} \frac{y^2-y_0^2}{i(y-y_0)}$$

$$= \lim_{y\to y_0} \frac{(y+y_0)\cancel{(y-y_0)}}{i\cancel{(y-y_0)}}$$

$$= \lim_{y\to y_0} \frac{y+y_0}{i} \quad \to \frac{y+y_0}{i}\cdot\frac{i}{i} = -i(y+y_0)$$

$$= \lim_{y\to y_0} -i(y+y_0)$$

$$= -2iy_0 \quad \ldots\ldots\ldots\ldots\ (***)$$

Perhatikan $(***) \neq (**)$

Jadi $f'(z_0)$ tidak ada untuk $z_0 \neq 0$

$\therefore$ f kontinu di seluruh $\mathbb{C}$ tetapi

f hanya punya turunan di $z = 0$.

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Aturan Turunan

Jika f dan g punya turunan di $z \in \mathbb{C}$ maka

$f+g$, $f-g$, αf (α konstanta kompleks), fg, $\frac{f}{g}$ $(g(z) \neq 0)$

punya turunan di $z \in \mathbb{C}$ dengan

(1) $(f+g)'(z) = f'(z) + g'(z)$

(2) $(f-g)'(z) = f'(z) - g'(z)$

(3) $(\alpha f)'(z) = \alpha \cdot f'(z)$

(4) $(f \cdot g)'(z) = f(z) \cdot g'(z) + f'(z) \cdot g(z)$

(5) $\left(\frac{f}{g}\right)'(z) = \frac{g(z) \cdot f'(z) - f(z) \cdot g'(z)}{(g(z))^2}$

Aturan Rantai

Jika fungsi f punya turunan di $z \in \mathbb{C}$ dan g punya turunan di $f(z)$ maka $g \circ f$ punya turunan di z dengan

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$$

Notasi Leibniz

Jika $s = g(w)$, $w = f(z)$ $s = g(f(z))$

Maka $\frac{ds}{dz} = \frac{ds}{dw} \cdot \frac{dw}{dz}$.

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Teorema

Misal f punya turunan pada $\mathbb{C}$

(1) Jika $f(z)=k \ \forall z \in \mathbb{C}$ maka $f'(z)=0$

(2) Jika $f(z)=z \ \forall z \in \mathbb{C}$ maka $f'(z)=1$

(3) Jika $f(z)=z^n \ \forall z \in \mathbb{C}, n \in \mathbb{Z}$
maka $f'(z)=n \cdot z^{n-1}$

$\boxed{E}$

$$f(z)=z^{2021} \rightarrow f'(z)=(2021)\, z^{2020}$$

$\boxed{E}$ $f(z)=z^2+3z \rightarrow f'(z)=2z+3$

Bisa pakai cara: $f'(z)=\lim\limits_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$

atau $f'(z)=\lim\limits_{\Delta z\to 0} \dfrac{f(z+\Delta z)-f(z)}{\Delta z)}$

Contoh pakai cara yang dibawah:

$$f'(z)=\lim_{\Delta z\to 0} \frac{[(z+\Delta z)^2+3(z+\Delta z)]-(z^2+3z)}{\Delta z}$$

$$=\lim_{\Delta z\to 0} \frac{\cancel{z^2}+2z\Delta z+\Delta z^2+\cancel{3z}+3\Delta z-\cancel{z^2}-\cancel{3z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0} \frac{\cancel{\Delta z}(2z+\Delta z+3)}{\cancel{\Delta z}}$$

$$=\lim_{\Delta z\to 0} 2z+\Delta z+3$$

$$=2z+3 \,//$$