3) (a) Series: $2^{2^0}, 2^{2^1}, 2^{2^2}, 2^{2^3}$ ... $T(n) = \Theta(2^{S(n)})$

loop triggers: 0, 1, 2, 3 ...

$S(n)$ is no. of loops

if $n <= 2$ $S(n) = 0$

Every time n is squared, loop triggers once more.

$$n = 2^{2^{S(n)}}$$

$$\log n = \log 2^{2^{(S(n))}}$$

$$\log(n) = 2^{(S(n))} \cancel{\log 2}$$

$$\log(\log(n)) = (S(n)) \log 2$$

$$S(n) = \frac{\log(\log(n))}{\log 2}$$

$$T(n) = \sum_{i=0}^{\log(\log(n))} \Theta(2) \qquad = \log(\log(n))$$

$$= \Theta(\log(\log(n)))$$

(b) $T(n) = \sum_{i=0}^{n-1} \Theta(1) + \sum_{i} \sum_{k=0}^{i^3} \Theta(1)$

$n = 4$: 0 loops, 4^3 times loop

if triggers: i	2	4
Dummy: j	1	2
loops:	2^3	4^3

$n = 16$

4	8	12	16
1	2	3	4

$(4^3 + 8^3 + 12^3 + 16^3)$ loops

($\sqrt{n}$ times)

$n = 9$

i	3	6	9
j	1	2	3
Loops	3^3	6^3	9^3

$\sqrt{n}$ times:

$3^3 = (\sqrt{9})^3$

$6^3 = (\sqrt{9} \times 2)^3$

$9^3 = (\sqrt{9} \times 3)^3$

$= (\sqrt{n} \times j)^3$

$$T(n) = \sum_{j=1}^{\sqrt{n}} \Theta((j\sqrt{n})^3)$$

$(1 \times \sqrt{n})^3 + (2 \times \sqrt{n})^3 + (3 \times \sqrt{n})^3 \ldots \times (j\sqrt{n})^3$

$= (\sqrt{n})^3 + 8\sqrt{n}^3 + 27\sqrt{n}^3 \ldots \times (j\sqrt{n})^3$

$(\sqrt{n})^3 (1 + 8 + 27 \ldots)$

$(\sqrt{n})^3 \left(\sum_{j=1}^{\sqrt{n}} x^3 \right)$

$\Theta\left((\sqrt{n})^3 \left(\frac{(\sqrt{n})^2 (\sqrt{n}+1)^2}{4} \right) \right)$

$\Theta(n) + \Theta\left(\frac{n^2 \sqrt{n} (\sqrt{n}+1)^2}{4} \right) = T(n)$

$\Theta\left(n + \frac{n^2 \sqrt{n} (\sqrt{n}+1)^2}{4} \right)$

$= \Theta\left(\frac{n^2 \sqrt{n} (\sqrt{n}+1)^2}{4} \right)$

(c) $T(n) = \sum_{i=1}^{n} \sum_{k=1}^{n} \left(\Theta(1) + \sum_{m=1}^{n} O(1)\right)$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \Theta(1) + n \sum_{m=1}^{n} O(1)$$

k: 1 2 3 4 5 ... k

m values: 1, 2, 4, 8, 16, 32 ... 2^k

$2^k = m$

$m < n$

$2^k < n$

$z \log 2 < \log n$

$z < \log n$

At most: 'if' is true all n times

At least: 'if' doesn't occur at all

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \Theta(1) + n \sum_{m=1}^{n} \log n$$

$$\boxed{\Theta(n^2) + O(n \log n)}$$

(d)

n	i	size	times executed
15	0	10	0
	10	15	10

$10 \times x^? = 10$

n	i	size	times executed
30	0	10	
	10	15	10
	15	22	15
	22	33	22

$$10 + 15 + 22$$

$$10 \times 1 + 10 \times 3/2 + 10 \times (3/2)^2$$

$$\sum_{k=0} \left(10 \times \left(\frac{3}{2}\right)^k\right)$$

$$10 \times (3/2)^k < n$$

otherwise loop doesn't run

$$(3/2)^k < \frac{n}{10}$$

$$k < \log_{3/2}\left(\frac{n}{10}\right)$$

$$\sum_{k=0}^{\log_{3/2}(n/10)} \left(\Theta\left(10 \times \left(\frac{3}{2}\right)^k\right)\right)$$

$$\Theta\left(3/2\right)^{\log_{3/2}(n/10)}$$

$$= \Theta(n/10) = \Theta(n)$$