Answers to Odd-Numbered Exercises

Section 1.1

1. **a.** well-ordered **b.** well-ordered **c.** not well-ordered **d.** well-ordered **e.** not well-ordered

3. Suppose that x and y are rational numbers. Then $x = a/b$ and $y = c/d$, where a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$. Then $xy = (a/b) \cdot (c/d) = ac/bd$ and $x + y = a/b + c/d = (ad + bc)/bd$ where $bd \neq 0$. Because both $x + y$ and xy are ratios of integers, they are both rational.

5. Suppose that $\sqrt{3}$ were rational. Then there would exist positive integers a and b with $\sqrt{3} = a/b$. Consequently, the set $S = \{k\sqrt{3} \mid k \text{ and } k\sqrt{3} \text{ are positive integers}\}$ is nonempty because $a = b\sqrt{3}$. Therefore, by the well-ordering property, S has a smallest element, say, $s = t\sqrt{3}$. We have $s\sqrt{3} - s = s\sqrt{3} - t\sqrt{3} = (s - t)\sqrt{3}$. Because $s\sqrt{3} = 3t$ and s are both integers, $s\sqrt{3} - s = (s - t)\sqrt{3}$ must also be an integer. Furthermore, it is positive, because $s\sqrt{3} - s = s(\sqrt{3} - 1)$ and $\sqrt{3} > 1$. It is less than s because $s = t\sqrt{3}$, $s\sqrt{3} = 3t$, and $\sqrt{3} < 3$. This contradicts the choice of s as the smallest positive integer in S. It follows that $\sqrt{3}$ is irrational.

7. **a.** 0 **b.** -1 **c.** 3 **d.** -2 **e.** 0 **f.** -4

9. **a.** $\{8/5\} = 3/5$ **b.** $\{1/7\} = 1/7$ **c.** $\{-11/4\} = 1/4$ **d.** $\{7\} = 0$

11. 0 if x is an integer; -1 otherwise

13. We have $[x] \leq x$ and $[y] \leq y$. Adding these two inequalities gives $[x] + [y] \leq x + y$. Hence, $[x + y] \geq [[x] + [y]] = [x] + [y]$.

15. Let $x = a + r$ and $y = b + s$, where a and b are integers and r and s are real numbers such that $0 \leq r, s < 1$. Then $[xy] = [ab + as + br + sr] = ab + [as + br + sr]$, whereas $[x][y] = ab$. Thus we have $[xy] \geq [x][y]$ when x and y are both positive. If x and y are both negative, then $[xy] \leq [x][y]$. If one of x and y is positive and the other negative, then the inequality could go either direction.

17. Let $x = [x] + r$. Because $0 \leq r < 1$, $x + \frac{1}{2} = [x] + r + \frac{1}{2}$. If $r < \frac{1}{2}$, then $[x]$ is the integer nearest to x and $[x + \frac{1}{2}] = [x]$ because $[x] \leq x + \frac{1}{2} = [x] + r + \frac{1}{2} < [x] + 1$. If $r \geq \frac{1}{2}$, then $[x] + 1$ is the integer nearest to x (choosing this integer if x is midway between $[x]$ and $[x + 1]$) and $[x + \frac{1}{2}] = [x] + 1$ because $[x] + 1 \leq x + r + \frac{1}{2} < [x] + 2$.

19. Let $x = k + \epsilon$ where k is an integer and $0 \leq \epsilon < 1$. Further, let $k = a^2 + b$, where a is the largest integer such that $a^2 \leq k$. Then $a^2 \leq k = a^2 + b \leq x = a^2 + b + \epsilon < (a + 1)^2$. Then $[\sqrt{x}] = a$ and $[\sqrt{[x]}] = [\sqrt{k}] = a$ also, proving the theorem.

21. **a.** $8n - 5$ **b.** $2^n + 3$ **c.** $\left[[\sqrt{n}]/\sqrt{n}\right]$ **d.** $a_n = a_{n-1} + a_{n-2}$, for $n \geq 3$, and $a_1 = 1$, and $a_2 = 3$

23. $a_n = 2^{n-1}$; $a_n = (n^2 - n + 2)/2$; and $a_n = a_{n-1} + 2a_{n-2}$, for $n \geq 3$

25. This set is exactly the sequence $a_n = n - 100$, and hence is countable.

27. The function $f(a + b\sqrt{2}) = 2^a 3^b$ is a one-to-one map of this set into the rational numbers, which is countable.

29. Suppose $\{A_i\}$ is a countable collection of countable sets. Then each A_i can be represented by a sequence, as follows:

$$\begin{array}{lcllll} A_1 & = & a_{11} & a_{12} & a_{13} & \dots \\ A_2 & = & a_{21} & a_{22} & a_{23} & \dots \\ A_3 & = & a_{31} & a_{32} & a_{33} & \dots \\ & \vdots & & & & \end{array}$$

Consider the listing $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \dots$, in which we first list the elements with subscripts adding to 2, then the elements with subscripts adding to 3, and so on. Further, we order the elements with subscripts adding to k in order of the first subscript. Form a new sequence c_i as follows. Let $c_1 = a_1$. Given that c_n is determined, let c_{n+1} be the next element in the listing that is different from each c_i with $i = 1, 2, \dots, n$. It follows that the terms of this sequence are exactly the elements of $\bigcup_{i=1}^{\infty} A_i$, which is therefore countable.

31. **a.** $a = 4, b = 7$ **b.** $a = 7, b = 10$ **c.** $a = 7, b = 69$ **d.** $a = 1, b = 20$

33. The number α must lie in some interval of the form $r/k \le \alpha < (r+1)/k$. If we divide this interval into equal halves, then α must lie in one of the halves, so either $r/k \le \alpha < (2r+1)/2k$ or $(2r+1)/2k \le \alpha < (r+1)/k$. In the first case, we have $|\alpha - r/k| < 1/2k$, so we take $u = r$. In the second case, we have $|\alpha - (r+1)/k| < 1/2k$, so we take $u = r + 1$.

35. First, we have $|\sqrt{2} - 1/1| = 0.414\dots < 1/1^2$. Second, Exercise 30, part a, gives us $|\sqrt{2} - 7/5| < 1/50 < 1/5^2$. Third, observing that $3/7 = 0.428\dots$ leads us to try $|\sqrt{2} - 10/7| = 0.014\dots < 1/7^2 = 0.0204\dots$. Fourth, observing that $5/12 = 0.4166\dots$ leads us to try $|\sqrt{2} - 17/12| = 0.00245\dots < 1/12^2 = 0.00694\dots$.

37. We may assume that b and q are positive. Note that if $q > b$, we have $|p/q - a/b| = |pb - aq|/qb \ge 1/qb > 1/q^2$. Therefore, solutions to the inequality must have $1 \le q \le b$. For a given q, there can be only finitely many p such that the distance between the rational numbers a/b and p/q is less than $1/q^2$ (indeed there is at most one.) Therefore, there are only finitely many p/q satisfying the inequality.

39. **a.** 3, 6, 9, 12, 15, 18, 21, 24, 27, 30 **b.** 1, 3, 5, 6, 8, 10, 12, 13, 15, 17 **c.** 2, 4, 7, 9, 11, 14, 16, 18, 21, 23 **d.** 3, 6, 9, 12, 15, 18, 21, 25, 28, 31

41. Assume that $1/\alpha + 1/\beta = 1$. First, show that the sequences $m\alpha$ and $n\beta$ are disjoint. Then, for an integer k, define $N(k)$ to be the number of elements of the sequences $m\alpha$ and $n\beta$ that are less than k. Then $N(k) = [k/\alpha] + [k/\beta]$. By definition of the greatest integer function, $k/\alpha - 1 < [k/\alpha] < k/\alpha$ and $k/\beta - 1 < [k/\beta] < k/\beta$. Add these inequalities to deduce that $k - 2 < N(k) < k$. Hence $N(k) = k - 1$, and the conclusion follows. To prove the converse, note that if $1/\alpha + 1/\beta \ne 1$, then the spectrum sequence can not partition the positive integers.

43. Assume that there are only finitely many Ulam numbers. Let the two largest Ulam numbers be u_{n-1} and u_n. Then the integer $u_n + u_{n-1}$ is an Ulam number larger than u_n. It is the unique sum of two Ulam numbers because $u_i + u_j < u_n + u_{n-1}$ if $j < n$ or $j = n$ and $i < n - 1$.

45. To get a contradiction, suppose that the set of real numbers is countable. Then the subset of real numbers strictly between 0 and 1 is also countable. Then there is a one-to-one correspondence $f : \mathbf{Z}^+ \to (0, 1)$. Each real number $b \in (0, 1)$ has a decimal representation of the form $b = 0.b_1b_2b_3\dots$, where b_i is the ith digit after the decimal point. For each $k = 1, 2, 3, \dots$, let $f(k) = a_k \in (0, 1)$. Then each a_k has a decimal representation of the form $a_k = a_{k1}a_{k2}a_{k3}\dots$. Form the real number $c = c_1c_2c_3\dots$ as follows: If $a_{kk} = 5$, then let $c_k = 4$. If $a_{kk} \ne 5$, then let $c_k = 5$. Then $c \ne a_k$ for every k because it differs in the kth decimal place. Therefore $f(k) \ne c$ for all k, and so f is not a one-to-one correspondence.

Section 1.2

1. **a.** 55 **b.** -15 **c.** 29/20

3. **a.** 510 **b.** 24,600 **c.** $-255/256$

5. The sum $\sum_{k=1}^{n}[\sqrt{k}]$ counts 1 for every value of k with $\sqrt{k} \geq 1$. There are n such values of k in the range $k = 1, 2, 3, \ldots, n$. It counts another 1 for every value of k with $\sqrt{k} \geq 2$. There are $n - 3$ such values in the range. The sum counts another 1 for each value of k with $\sqrt{k} \geq 3$. There are $n - 8$ such values in the range. In general, for $m = 1, 2, 3, \ldots, [\sqrt{n}]$ the sum counts a 1 for each value of k with $\sqrt{k} \geq m$, and there are $n - (m^2 - 1)$ values in the range. Therefore, $\sum_{k=1}^{n}[\sqrt{k}] = \sum_{m=1}^{[\sqrt{n}]} n - (m^2 - 1) = [\sqrt{n}](n+1) - \sum_{m=1}^{[\sqrt{n}]} m^2 = [\sqrt{n}](n+1) - ([\sqrt{n}]([\sqrt{n}]+1)(2[\sqrt{n}]+1))/6$.

7. The total number of dots in the n by $n+1$ rectangle, namely, $n(n+1)$, is $2t_n$ because the rectangle is made from two triangular arrays. Dividing both sides by 2 gives the desired formula.

9. From the closed formula for the nth triangular number, we have $t_{n+1}^2 - t_n^2 = ((n+1)(n+1+1)/2)^2 - (n(n+1)/2)^2 = (n+1)^2((n+2)^2/4 - n^2/4) = (n+1)^2(n^2+4n+4-n^2)/4 = (n+1)^2(4n+4)/4 = (n+1)^3$, as desired.

11. From Exercise 10, we have $p_n = (3n^2 - n)/2$. On the other hand, $t_{n-1} + n^2 = (n-1)n/2 + n^2 = (3n^2 - n)/2$, which is the same as above.

13. **a.** Consider a regular heptagon that we border successively by heptagons with 3, 4, 5, . . . on each side. Define the heptagonal numbers s_k to be the number of dots contained in the k nested heptagons. **b.** $(5k^2 - 3k)/2$

15. From Exercise 10, we have $p_n = (3n^2 - n)/2$. Also, $t_{3n-1}/3 = (1/3)(3n-1)(3n)/2 = (3n-1)(n)/2 = (3n^2 - n)/2 = p_n$.

17. By Exercise 16, we have $T_n = \sum_{k=1}^{n} t_k = \sum_{k=1}^{n} k(k+1)/2$. Note that $(k+1)^3 - k^3 = 3k^2 + 3k + 1 = 3(k^2 + k) + 1$ so that $k^2 + k = ((k+1)^3 - k^3)/3 - (1/3)$. Then $T_n = (1/2)\sum_{k=1}^{n} k(k+1) = (1/6)\sum_{k=1}^{n}((k+1)^3 - k^3) - (1/6)\sum_{k=1}^{n} 1$. The first sum is telescoping and the second sum is trivial, so we have $T_n = (1/6)((n+1)^3 - 1^3) - (n/6) = (n^3 + 3n^2 + 2n)/6$.

19. Each of these four quantities are products of 100 integers. The largest product is 100^{100}, because it is the product of 100 factors of 100. The second largest is 100!, which is the product of the integers 1, 2, . . . , 100, and each of these terms is less or equal to 100. The third largest is $(50!)^2$, which is the product of $1^2, 2^2, \ldots, 50^2$, and each of these factors j^2 is less than $j(50+j)$, whose product is 100!. The smallest is 2^{100}, which is the product of 100 twos.

21. $\sum_{k=1}^{n}\left(\frac{1}{k(k+1)}\right) = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$. Let $a_j = 1/(j+1)$. Notice that this is a telescoping sum, as in Example 1.19. Therefore, we have $\sum_{k=1}^{n}\left(\frac{1}{k(k+1)}\right) = \sum_{j=1}^{n}(a_{j-1} - a_j) = a_0 - a_n = 1 - 1/(n+1) = n/(n+1)$.

23. We sum both sides of the identity $(k+1)^3 - k^3 = 3k^2 + 3k + 1$ from $k = 1$ to $k = n$. $\sum_{k=1}^{n}((k+1)^3 - k^3) = (n+1)^3 - 1$, because the sum is telescoping. $\sum_{k=1}^{n}(3k^2 + 3k + 1) = 3(\sum_{k=1}^{n} k^2) + 3(\sum_{k=1}^{n} k) + \sum_{k=1}^{n} 1 = 3(\sum_{k=1}^{n} k^2) + 3n(n+1)/2 + n$. As these two expressions are equal, solving for $\sum_{k=1}^{n} k^2$, we find that $\sum_{k=1}^{n} k^2 = (n(2n+1)(n+1))/6$.

25. **a.** $10! = (7!)(8 \cdot 9 \cdot 10) = (7!)(720) = (7!)(6!)$. **b.** $10! = (7!)(6!) = (7!)(5!) \cdot 6 = (7!)(5!)(3!)$. **c.** $16! = (14!)(15 \cdot 16) = (14!)(240) = (14!)(5!)(2!)$. **d.** $9! = (7!)(8 \cdot 9) = (7!)(6 \cdot 6 \cdot 2) = (7!)(3!)(3!)(2!)$

27. $x = y = 1$ and $z = 2$

Section 1.3

1. For $n = 1$, we have $1 < 2^1 = 2$. Now assume $n < 2^n$. Then $n + 1 < 2^n + 1 < 2^n + 2^n = 2^{n+1}$.

3. For the basis step, $\sum_{k=1}^{1} \frac{1}{k^2} = 1 \le 2 - \frac{1}{1} = 1$. For the inductive step, we assume that $\sum_{k=1}^{n} \frac{1}{k^2} \le 2 - \frac{1}{n}$. Then $\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \le 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$ by the induction hypothesis. This is less than $2 - \frac{1}{n+1} + \frac{1}{(n+1)^2} = 2 - \frac{1}{n+1}(1 - \frac{1}{n+1}) \le 2 - \frac{1}{n+1}$, as desired.

5. $\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. The basis step is trivial. For the inductive step, assume that $\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{A}^{n+1} = \mathbf{A}^n\mathbf{A} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$.

7. For the basis step, we have $\sum_{j=1}^{1} j^2 = 1 = 1(1+1)(2 \cdot 1 + 1)/6$. For the inductive step, we assume that $\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$. Then $\sum_{j=1}^{n+1} j^2 = \sum_{j=1}^{n} j^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2 = (n+1)\,(n(2n+1)/6 + n + 1) = (n+1)(2n^2 + 7n + 6)/6 = (n+1)(n+2)[2(n+1)+1]/6$.

9. For the basis step, we have $\sum_{j=1}^{1} j(j+1) = 2 = 1(2)(3)/3$. Assume it is true for n. Then $\sum_{j=1}^{n+1} j(j+1) = n(n+1)(n+2)/3 + (n+1)(n+2) = (n+1)(n+2)(n/3+1) = (n+1)(n+2)(n+3)/3$.

11. $2^{n(n+1)/2}$

13. For the basis step, we note that $12 = 4 \cdot 3$. For the inductive step, assume that postage of n cents can be formed, with $n = 4a + 5b$, where a and b are nonnegative integers. To form $n + 1$ cents postage, if $a > 0$ we can replace a 4-cent stamp with a 5-cent stamp; that is, $n + 1 = 4(a - 1) + 5(b + 1)$. If no 4-cent stamps are present, then all 5-cent stamps were used. It follows that there must be at least three 5-cent stamps and these can be replaced by four 4-cent stamps; that is, $n + 1 = 4(a + 4) + 5(b - 3)$.

15. We use mathematical induction. The inequality is true for $n = 0$ because $H_{2^0} = H_1 = 1 \ge 1 = 1 + 0/2$. Now assume that the inequality is true for n, that is, $H_{2^n} \ge 1 + n/2$. Then $H_{2^{n+1}} = \sum_{j=1}^{2^n} 1/j + \sum_{j=2^n+1}^{2^{n+1}} 1/j \ge H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} 1/2^{n+1} \ge 1 + n/2 + 2^n \cdot 1/2^{n+1} = 1 + n/2 + 1/2 = 1 + (n+1)/2$.

17. For the basis step, we have $(2 \cdot 1)! = 2 < 2^{2 \cdot 1}(1!)^2 = 4$. For the inductive step, we assume that $(2n)! < 2^{2n}(n!)^2$. Then $[2(n+1)]! = (2n)!(2n+1)(2n+2) < 2^{2n}(n!)^2(2n+1)(2n+2) < 2^{2n}(n!)^2(2n+2)^2 = 2^{2(n+1)}[(n+1)!]^2$.

19. Let A be such a set. Define B as $B = \{x - k + 1 \mid x \in A \text{ and } x \ge k\}$. Because $x \ge k$, B is a set of positive integers. Because $k \in A$ and $k \ge k$, $k - k + 1 = 1$ is in B. Because $n + 1$ is in A whenever n is, $n + 1 - k + 1$ is in B whenever $n - k + 1$ is. Thus, B satisfies the hypothesis for mathematical induction, i.e., B is the set of positive integers. Mapping B back to A in the natural manner, we find that A contains the set of integers greater than or equal to k.

21. For the basis step, we have $4^2 = 16 < 24 = 4!$. For the inductive step, we assume that $n^2 < n!$. Then $(n+1)^2 = n^2 + 2n + 1 < n! + 2n + 1 < n! + 3n < n! + n! = 2n! < (n+1)n! = (n+1)!$.

23. We use the second principle of mathematical induction. For the basis step, if the puzzle has only one piece, then it is assembled with exactly 0 moves. For the induction step, assume that all puzzles with $k \le n$ pieces require $k - 1$ moves to assemble. Suppose it takes m moves to assemble a puzzle with $n + 1$ pieces. Then the m move consists of joining two blocks of size a and b, respectively, with $a + b = n + 1$. But by the induction hypothesis, it requires exactly $a - 1$ and $b - 1$ moves to assemble each of these blocks. Thus, $m = (a - 1) + (b - 1) + 1 = a + b + 1 = n + 1$.

25. Suppose that $f(n)$ is defined recursively by specifying the value of $f(1)$ and a rule for finding $f(n+1)$ from $f(n)$. We will prove by mathematical induction that such a function is well-defined. First, note that $f(1)$ is well-defined because this value is explicitly stated. Now assume that $f(n)$ is well-defined. Then $f(n+1)$ also is well-defined because a rule is given for determining this value from $f(n)$.

27. 65,536

29. We use the second principle of mathematical induction. The basis step consists of verifying the formula for $n = 1$ and $n = 2$. For $n = 1$, we have $f(1) = 1 = 2^1 + (-1)^1$, and for $n = 2$, we have $f(2) = 5 = 2^2 + (-1)^2$. Now assume that $f(k) = 2^k + (-1)^k$ for all positive integers k with $k < n$ where $n > 2$. By the induction hypothesis, it follows that $f(n) = f(n-1) + 2f(n-2) = (2^{n-1} + (-1)^{n-1}) + 2(2^{n-2} + (-1)^{n-2}) = (2^{n-1} + 2^{n-1}) + (-1)^{n-2}(-1+2) = 2^n + (-1)^n$.

31. We use the second principle of mathematical induction. We see that $a_0 = 1 \le 3^0 = 1$, $a_1 = 3 \le 3^i = 3$, and $a_2 = 9 \le 3^2 = 9$. These are the basis cases. Now assume that $a_k \le 3^k$ for all integers k with $0 \le k < n$. It follows that $a_n = a_{n-1} + a_{n-2} + a_{n-3} \le 3^{n-1} + 3^{n-2} + 3^{n-3} = 3^{n-3}(1+3+9) = 13 \cdot 3^{n-3} < 27 \cdot 3^{n-3} = 3^n$.

33. Let P_n be the statement for n. Then P_2 is true, because we have $((a_1 + a_2)/2)^2 - a_1a_2 = ((a_1 - a_2)/2)^2 \ge 0$. Assume P_n is true. Then by P_2, for $2n$ positive real numbers $a_1, \ldots, a_{2n}$ we have $a_1 + \cdots + a_{2n} \ge 2(\sqrt{a_1a_2} + \sqrt{a_3a_4} + \cdots + \sqrt{a_{2n-1}a_{2n}})$. Apply P_n to this last expression to get $a_1 + \cdots + a_{2n} \ge 2n(a_1a_2 \cdots a_{2n})^{1/2n}$, which establishes P_n for $n = 2^k$ for all k. Again, assume P_n is true. Let $g = (a_1a_2 \cdots a_{n-1})^{1/(n-1)}$. Applying P_n, we have $a_1 + a_2 + \cdots + a_{n-1} + g \ge n(a_1a_2 \cdots a_{n-1}g)^{1/n} = n(g^{n-1}g)^{1/n} = ng$. Therefore, $a_1 + a_2 + \cdots + a_{n-1} \ge (n-1)g$, which establishes P_{n-1}. Thus P_{2^k} is true and P_n implies P_{n-1}. This establishes P_n for all n.

35. Note that because $0 < p < q$ we have $0 < p/q < 1$. The proposition is trivially true if $p = 1$. We proceed by strong induction on p. Let p and q be given and assume the proposition is true for all rational numbers between 0 and 1 with numerators less than p. To apply the algorithm, we find the unit fraction $1/s$ such that $1/(s-1) > p/q > 1/s$. When we subtract, the remaining fraction is $p/q - 1/s = (ps - q)/qs$. On the other hand, if we multiply the first inequality by $q(s-1)$, we have $q > p(s-1)$, which leads to $p > ps - q$, which shows that the numerator of p/q is strictly greater than the numerator of the remainder $(ps - q)/qs$ after one step of the algorithm. By the induction hypothesis, this remainder is expressible as a sum of unit fractions, $1/u_1 + \cdots + 1/u_k$. Therefore, $p/q = 1/s + 1/u_1 + \cdots + 1/u_k$, which completes the induction step.

Section 1.4

1. **a.** 55 **b.** 233 **c.** 610 **d.** 2584 **e.** 6765 **f.** 75025

3. Note that $2f_{n+2} - f_n = f_{n+2} + (f_{n+2} - f_n) = f_{n+2} + f_{n+1} = f_{n+3}$. Add f_n to both sides.

5. For $n = 1$, we have $f_{2\cdot 1} = 1 = 1^2 + 2 \cdot 1 \cdot 0 = f_1^2 + 2f_0f_1$, and for $n = 2$, we have $f_{2\cdot 2} = 3 = 1^2 + 2 \cdot 1 \cdot 1 = f_2^2 + 2f_1f_2$. So the basis step holds for strong induction. Assume, then, that $f_{2n-4} = f_{n-2}^2 + 2f_{n-3}f_{n-2}$ and $f_{2n-2} = f_{n-1}^2 + 2f_{n-2}f_{n-1}$. Now compute $f_{2n} = f_{2n-1} + f_{2n-2} = 2f_{2n-2} + f_{2n-3} = 3f_{2n-2} - f_{2n-4}$. We may now substitute in our induction hypotheses to set this last expression equal to $3f_{n-1}^2 + 6f_{n-2}f_{n-1} - f_{n-2}^2 - 2f_{n-3}f_{n-2} = 3f_{n-1}^2 + 6(f_n - f_{n-1})f_{n-1} - (f_n - f_{n-1})^2 - 2(f_{n-1} - f_{n-2})(f_n - f_{n-1}) = -2f_{n-1}^2 + 6f_nf_{n-1} - f_n^2 + 2f_n(f_n - f_{n-1}) - 2f_{n-1}(f_n - f_{n-1}) = f_n^2 + 2f_{n-1}f_n$, which completes the induction step.

7. $\sum_{j=1}^{n} f_{2j-1} = f_{2n}$. The basis step is trivial. Assume that our formula is true for n, and consider $f_1 + f_3 + f_5 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2}$, which is the induction step.

9. First suppose $n = 2k$ is even. Then $f_n - f_{n-1} + \cdots + (-1)^{n+1}f_1 = (f_{2k} + f_{2k-1} + \cdots + f_1) - 2(f_{2k-1} + f_{2k-3} + \cdots + f_1) = (f_{2k+2} - 1) - 2(f_{2k})$ by the formulas in Example 1.27 and

Exercise 7. This last equals $(f_{2k+2} - f_{2k}) - f_{2k} - 1 = f_{2k+1} - f_{2k} - 1 = f_{2k-1} - 1 = f_{n-1} - 1$. Now suppose $n = 2k + 1$ is odd. Then $f_n - f_{n-1} + \cdots + (-1)^{n+1} = f_{2k+1} - (f_{2k} - f_{2k-1} + \cdots - (-1)^{n+1} f_1) = f_{2k+1} - (f_{2k-1} - 1)$ by the formula just proved for the even case. This last equals $(f_{2k+1} - f_{2k-1}) + 1 = f_{2k} + 1 = f_{n-1} + 1$. We can unite the formulas for the odd and even cases by writing the formula as $f_{n-1} - (-1)^n$.

11. From Exercise 5, we have $f_{2n} = f_n^2 + 2f_{n-1}f_n = f_n(f_n + f_{n-1} + f_{n-1}) = (f_{n+1} - f_{n-1})(f_{n+1} + f_{n-1}) = f_{n+1}^2 - f_{n-1}^2$.

13. We use mathematical induction. For the basis step, $\sum_{j=1}^{1} f_j^2 = f_1^2 = f_1 f_2$. To make the inductive step, we assume that $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$. Then $\sum_{j=1}^{n+1} f_j^2 = \sum_{j=1}^{n} f_j^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1} f_{n+2}$.

15. From Exercise 13, we have $f_{n+1}f_n - f_{n-1}f_{n-2} = (f_1^2 + \cdots + f_n^2) - (f_1^2 + \cdots f_{n-2}^2) = f_n^2 + f_{n-1}^2$. The identity in Exercise 10 shows that this is equal to f_{2n-1} when n is a positive integer, and in particular when n is greater than 2.

17. For fixed m, we proceed by induction on n. The basis step is $f_{m+1} = f_m f_2 + f_{m-1} f_1 = f_m \cdot 1 + f_{m-1} \cdot 1$, which is true. Assume the identity holds for $1, 2, \ldots, k$. Then $f_{m+k} = f_m f_{k+1} + f_{m-1} f_k$ and $f_{m+k-1} = f_m f_k + f_{m-1} f_{k-1}$. Adding these equations gives us $f_{m+k} + f_{m+k-1} = f_m(f_{k+1} + f_k) + f_{m-1}(f_k + f_{k-1})$. Applying the recursive definition yields $f_{m+k+1} = f_m f_{k+2} + f_{m-1} f_{k+1}$.

19. $\sum_{i=1}^{n} L_i = L_{n+2} - 3$. We use mathematical induction. The basis step is $L_1 = 1 = L_3 - 3$. Assume that the formula holds for n and compute $\sum_{i=1}^{n+1} L_i = \sum_{i=1}^{n} L_i + L_{n+1} = L_{n+2} - 3 + L_{n+1} = (L_{n+2} + L_{n+1}) - 3 = L_{n+3} - 3$.

21. $\sum_{i=1}^{n} L_{2i} = L_{2n+1} - 1$. We use mathematical induction. The basis step is $L_2 = 3 = L_3 - 1$. Assume that the formula holds for n and compute $\sum_{i=1}^{n+1} L_{2i} = \sum_{i=1}^{n} L_{2i} + L_{2n+2} = L_{2n+1} - 1 + L_{2n+2} = L_{2n+3} - 1$.

23. We proceed by induction. The basis step is $L_1^2 = 1 = L_1 L_2 - 2$. Assume the formula holds for n and consider $\sum_{i=1}^{n+1} L_i^2 = \sum_{i=1}^{n} L_i^2 + L_{n+1}^2 = L_n L_{n+1} - 2 + L_{n+1}^2 = L_{n+1}(L_n + L_{n+1}) - 2 = L_{n+1} L_{n+2} - 2$.

25. For the basis step, we check that $L_1 f_1 = 1 \cdot 1 = 1 = f_2$ and $L_2 f_2 = 3 \cdot 1 = 3 = f_4$. Assume the identity is true for all positive integers up to n. Then we have $f_{n+1} L_{n+1} = (f_{n+2} - f_n)(f_{n+2} - f_n)$ from Exercise 16. This equals $f_{n+2}^2 - f_n^2 = (f_{n+1} + f_n)^2 - (f_{n-1} + f_{n-2})^2 = f_{n+1}^2 + 2f_{n+1}f_n + f_n^2 - f_{n-1}^2 - 2f_{n-1}f_{n-2} - f_{n-2}^2 = (f_{n+1}^2 - f_{n-1}^2) + (f_n^2 - f_{n-2}^2) + 2(f_{n+1}f_n - f_{n-1}f_{n-2}) = (f_{n+1} - f_{n-1})(f_{n+1} + f_{n-1}) + (f_n - f_{n-2})(f_n + f_{n-2}) + 2(f_{2n-1})$, where the last parenthetical expression is obtained from Exercise 8. This equals $f_n L_n + f_{n-1} L_{n-1} + 2f_{2n-1}$. Applying the induction hypothesis yields $f_{2n} + f_{2n-2} + 2f_{2n-1} = (f_{2n} + f_{2n-1}) + (f_{2n-1} + f_{2n-2}) = f_{2n+1} + f_{2n} = f_{2n+2}$, which completes the induction.

27. We prove this by induction on n. Fix m a positive integer. If $n = 2$, then for the basis step we need to show that $L_{m+2} = f_{m+1} L_2 + f_m L_1 = 3f_{m+1} + f_m$, for which we will use induction on m. For $m = 1$ we have $L_3 = 4 = 3 \cdot f_2 + f_1$, and for $m = 2$ we have $L_4 = 7 = 3 \cdot f_3 + f_2$, so the basis step for m holds. Now assume that the basis step for n holds for all values of m less than and equal to m. Then $L_{m+3} = L_{m+2} + L_{m+1} = 3f_{m+1} + f_m + 3f_m + f_{m-1} = 3f_{m+2} + f_{m+1}$, which completes the induction step on m and proves the basis step for n. To prove the induction step on n, we compute $L_{m+n+1} = L_{m+n} + L_{m+n-1} = (f_{m+1} L_n + f_m L_{n-1}) + (f_{m+1} L_{n-1} + f_m L_{n-2}) = f_{m+1}(L_n + L_{n-1}) + f_m(L_{n-1} + L_{n-2}) = f_{m+1} L_{n+1} + f_m L_n$, which completes the induction on n and proves the identity.

29. $50 = 34 + 13 + 3 = f_9 + f_7 + f_4$, $85 = 55 + 21 + 8 + 1 = f_{10} + f_8 + f_6 + f_2$, $110 = 89 + 21 = f_{11} + f_8$ and $200 = 144 + 55 + 1 = f_{12} + f_{10} + f_2$.

31. We proceed by mathematical induction. The basis steps ($n = 2$ and 3) are easily seen to hold. For the inductive step, we assume that $f_n \le \alpha^{n-1}$ and $f_{n-1} \le \alpha_{n-2}$. Now $f_{n+1} = f_n + f_{n-1} \le \alpha^{n-1} + \alpha^{n-2} = \alpha^n$, because α satisfies $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$.

33. We use Theorem 1.3. Note that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$, because they are roots of $x^2 - x - 1 = 0$. Then we have $f_{2n} = (\alpha^{2n} - \beta^{2n})/\sqrt{5} = (1/\sqrt{5})((\alpha+1)^n - (\beta+1)^n) = (1/\sqrt{5})\left(\sum_{j=0}^{n}\binom{n}{j}\alpha^j - \sum_{j=0}^{n}\binom{n}{j}\beta^j\right) = (1/\sqrt{5})\sum_{j=0}^{n}\binom{n}{j}(\alpha^j - \beta^j) = \sum_{j=1}^{n}\binom{n}{j}f_j$ because the first term is zero in the second-to-last sum.

35. On one hand, $\det(\mathbf{F}^n) = \det(\mathbf{F})^n = (-1)^n$. On the other hand,

$$\det\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = f_{n+1}f_{n-1} - f_n^2.$$

37. $f_0 = 0$, $f_{-1} = 1$, $f_{-2} = -1$, $f_{-3} = 2$, $f_{-4} = -3$, $f_{-5} = 5$, $f_{-6} = -8$, $f_{-7} = 13$, $f_{-8} = -21$, $f_{-9} = 34$, $f_{-10} = -55$

39. The square has area 64 square units, while the rectangle has area 65 square units. This corresponds to the identity in Exercise 14, which tells us that $f_7f_5 - f_6^2 = 1$. Notice that the slope of the hypotenuse of the triangular piece is 3/8, while the slope of the top of the trapezoidal piece is 2/5. We have $2/5 - 3/8 = 1/40$. Thus, the "diagonal" of the rectangle is really a very skinny parallelogram of area 1, hidden visually by the fact that the two slopes are nearly equal.

41. We solve the equation $r^2 - r - 1 = 0$ to discover the roots $r_1 = (1+\sqrt{5})/2$ and $r_2 = (1-\sqrt{5})/2$. Then, according to the theory in the paragraph above, $f_n = C_1r_1^n + C_2r_2^n$. For $n = 0$, we have $0 = C_1r_1^0 + C_2r_2^0 = C_1 + C_2$. For $n = 1$, we have $1 = C_1r_1 + C_2r_2 = C_1(1+\sqrt{5})/2 + C_2(1-\sqrt{5})/2$. Solving these two equations simultaneously yields $C_1 = 1/\sqrt{5}$ and $C_2 = -1/\sqrt{5}$. So the explicit formula is $f_n = (1/\sqrt{5})r_1^n - (1/\sqrt{5})r_2^n = (r_1^n - r_2^n)/\sqrt{5}$.

43. We seek to solve the recurrence relation $L_n = L_{n-1} + L_{n-1}$ subject to the initial conditions $L_1 = 1$ and $L_2 = 3$. We solve the equation $r^2 - r - 1 = 0$ to discover the roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then, according to the theory in the paragraph above Exercise 41, $L_n = C_1\alpha^n + C_2\beta^n$. For $n = 1$, we have $L_1 = 1 = C_1\alpha + C_2\beta$. For $n = 2$, we have $3 = C_1\alpha^2 + C_2\beta^2$. Solving these two equations simultaneously yields $C_1 = 1$ and $C_2 = 1$. So the explicit formula is $L_n = \alpha^n + \beta^n$.

45. First check that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. We proceed by induction. The basis steps are $(1/\sqrt{5})(\alpha - \beta) = (1/\sqrt{5})(\sqrt{5}) = 1 = f_1$ and $(1/\sqrt{5})(\alpha^2 - \beta^2) = (1/\sqrt{5})((1+\alpha) - (1+\beta)) = (1/\sqrt{5})(\alpha - \beta) = 1 = f_2$. Assume the identity is true for all positive integers up to n. Then $f_{n+1} = f_n + f_{n-1} = (1/\sqrt{5})(\alpha^n - \beta^n) + (1/\sqrt{5})(\alpha^{n-1} - \beta^{n-1}) = (1/\sqrt{5})(\alpha^{n-1}(\alpha+1) - \beta^{n-1}(\beta + 1)) = (1/\sqrt{5})(\alpha^{n-1}(\alpha^2) - \beta^{n-1}(\beta^2)) = (1/\sqrt{5})(\alpha^{n+1} - \beta^{n+1})$, which completes the induction.

Section 1.5

1. $3 \mid 99$ because $99 = 3 \cdot 33$, $5 \mid 145$ because $145 = 5 \cdot 29$, $7 \mid 343$ because $343 = 7 \cdot 49$, and $888 \mid 0$ because $0 = 888 \cdot 0$

3. **a.** yes **b.** yes **c.** no **d.** no **e.** no **f.** no

5. **a.** $q = 5$, $r = 15$ **b.** $q = 17$, $r = 0$ **c.** $q = -3$, $r = 7$ **d.** $q = -6$, $r = 2$

7. **a.** 1 and 13 **b.** 1, 3, 7, and 21 **c.** 1, 2, 3, 4, 6, 9, 12, 18, and 36 **d.** 1, 2, 4, 11, 22, and 44

9. **a.** $(11, 22) = 11$ **b.** $(36, 42) = 6$ **c.** $(21, 22) = 1$ **d.** $(16, 64) = 16$

11. Each of 1, 2, 3, . . . , 10 is relatively prime to 11.

13. (10, 11), (10, 13), (10, 17), (10, 19), (11, 12), (11, 13), . . . , (11, 20), (12, 13), (12, 17), (12, 19), (13, 14), (13, 15), . . . , (13, 20), (14, 15), (14, 17), (14, 19), (15, 16), (15, 17), (15, 19), (16, 17), (16, 19), (17, 18), (17, 19), (17, 20), (18, 19) and (19, 20)

15. By hypothesis, $b = ra$ and $d = sc$, for some r and s. Thus, $bd = rs(ac)$ and $ac \mid bd$.

17. If $a \mid b$, then $b = na$ and $bc = n(ca)$, i.e., $ac \mid bc$. Now suppose $ac \mid bc$. Thus, $bc = nac$ and, as $c \neq 0$, $b = na$, i.e., $a \mid b$.

19. By definition, $a \mid b$ if and only if $b = na$ for some integer n. Then raising both sides of this equation to the kth power yields $b^k = n^k a^k$ whence $a^k \mid b^k$.

21. Let a and b be odd, and c even. Then $ab = (2x + 1)(2y + 1) = 4xy + 2x + 2y + 1 = 2(2xy + x + y) + 1$, so ab is odd. On the other hand, for any integer n, we have $cn = (2z)n = 2(zn)$, which is even.

23. By the division algorithm, $a = bq + r$, with $0 \le r < b$. Thus $-a = -bq - r = -(q + 1)b + b - r$. If $0 \le b - r < b$, then we are done. Otherwise, $b - r = b$, or $r = 0$ and $-a = -qb + 0$.

25. **a.** The division algorithm covers the case when b is positive. If b is negative, then we may apply the division algorithm to a and $|b|$ to get a quotient q and remainder r such that $a = q|b| + r$ and $0 \le r < |b|$. But because b is negative, we have $a = q(-b) + r = (-q)b + r$, as desired. **b.** 3

27. By the division algorithm, let $m = qn + r$, with $0 \le r < n - 1$ and $q = [m/n]$. Then $\big[(m + 1)/n\big] = \big[(qn + r + 1)/n\big] = \big[q + (r + 1)/n\big] = q + \big[(r + 1)/n\big]$, as in Example 1.31. If $r = 0, 1, 2, \ldots, n - 2$, then $m \neq kn - 1$ for any integer k and $1/n \le (r + 1)/n < 1$ and so $[(r + 1)/n] = 0$. In this case, we have $[(m + 1)/n] = q + 0 = [m/n]$. On the other hand, if $r = n - 1$, then $m = qn + n - 1 = n(q + 1) - 1 = nk - 1$, and $[(r + 1)/n] = 1$. In this case, we have $[(m + 1)/n] = q + 1 = [m/n] + 1$.

29. The positive integers divisible by the positive integer d are those integers of the form kd where k is a positive integer. The number of these that are less than x is the number of positive integers k with $kd \le x$, or equivalently with $k \le x/d$. There are $[x/d]$ such integers.

31. 128; 18

33. 457

35. It costs $44 - [1 - w]17$ cents to mail a letter weighing x ounces. It can not cost \$1.81; a 13-ounce letter costs \$2.65.

37. Multiplying two integers of this form gives us $(4n + 1)(4m + 1) = 16mn + 4m + 4n + 1 = 4(4mn + m + n) + 1$. Similarly, $(4n + 3)(4m + 3) = 16mn + 12m + 12n + 9 = 4(4mn + 3m + 3n + 2) + 1$.

39. Every odd integer may be written in the form $4k + 1$ or $4k + 3$. Observe that $(4k + 1)^4 = 16^2k^4 + 4(4k)^3 + 6(4k)^2 + 4(4k) + 1 = 16(16k^4 + 16k^3 + 6k^2 + k) + 1$. Proceeding further, $(4k + 3)^4 = (4k)^4 + 12(4k)^3 + 54(4k)^2 + 108(4k) + 3^4 = 16(16k^4 + 48k^3 + 54k^2 + 27k + 5) + 1$.

41. Of any consecutive three integers, one is a multiple of three. Also, at least one is even. Therefore, the product is a multiple of $2 \cdot 3 = 6$.

43. For the basis step, note that $0^3 + 1^3 + 2^3 = 9$ is a multiple of 9. Suppose that $n^3 + (n + 1)^3 + (n + 2)^3 = 9k$ for some integer k. Then $(n + 1)^3 + (n + 2)^3 + (n + 3)^3 = n^3 + (n + 1)^3 + (n + 2)^3 + (n + 3)^3 - n^3 = 9k + n^3 + 9n^2 + 27n + 27 - n^3 = 9k + 9n^2 + 27n + 27 = 9(k + n^2 + 3n + 3)$, which is a multiple of 9.

45. We proceed by mathematical induction. The basis step is clear. Assume that only f_{4n}'s are divisible by 3 for f_i, $i \le 4k$. Then, as $f_{4k+1} = f_{4k} + f_{4k-1}$, $3 \mid f_{4k}$ and $3 \mid f_{4k+1}$ gives us the contradiction $3 \mid f_{4k-1}$. Thus, $3 \nmid f_{4k+1}$. Continuing on, if $3 \mid f_{4k}$ and $3 \mid f_{4k+2}$, then $3 \mid f_{4k+1}$, which contradicts the statement just proved. If $3 \mid f_{4k}$ and $3 \mid f_{4k+3}$, then, because $f_{4k+3} = 2f_{4k+1} + f_{4k}$, we again have a contradiction. But, as $f_{4k+4} = 3f_{4k+1} + 2f_{4k}$, and $3 \mid f_{4k}$ and $3 \mid 3 \cdot f_{4k+1}$, we see that $3 \mid f_{4k+4}$.

47. First note that for $n > 5$, $5f_{n-4} + 3f_{n-5} = 2f_{n-4} + 3(f_{n-4} + f_{n-5}) = 2f_{n-4} + 3f_{n-3} = 2(f_{n-4} + f_{n-3}) + f_{n-3} = 2f_{n-2} + f_{n-3} = f_{n-2} + f_{n-2} + f_{n-3} = f_{n-2} + f_{n-1} = f_n$, which proves the first identity. Now note that $f_5 = 5$ is divisible by 5. Suppose that f_{5n} is divisible by 5. From the identity above, $f_{5n+5} = 5f_{5n+5-4} + 3f_{5n+5-5} = 5f_{5n+1} + 3f_{5n}$, which is divisible by 5 because $5f_{5n+1}$ is a multiple of 5 and, by the induction hypothesis, so is f_{5n}. This completes the induction.

49. 39, 59, 89, 134, 67, 101, 152, 76, 38, 19, 29, 44, 22, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1

51. We prove this using the second principle of mathematical induction. Because $T(2) = 1$, the Collatz conjecture is true for $n = 2$. Now assume that the conjecture holds for all integers less that n. By assumption, there is an integer k such that k iterations of the transformation T, starting at n, produces an integer m less than n. By the inductive hypothesis, there is an integer l such that iterating T l times starting at m produces the integer 1. Hence, iterating T $k + l$ times starting with n leads to 1.

53. We first show that $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an even integer. By the binomial theorem, it follows that $(2+\sqrt{3})^n + (2-\sqrt{3})^n = \sum_{j=0}^{n}\binom{n}{j}2^j\sqrt{3}^{n-j} + \sum_{j=0}^{n}\binom{n}{j}2^j(-1)^{n-j}\sqrt{3}^{n-j} = 2(2^n + \binom{n}{2}3\cdot 2^{n-2} + \binom{n}{4}3^2\cdot 2^{n-4} + \cdots) = 2l$ where l is an integer. Next, note that $(2-\sqrt{3})^n < 1$. Because $(2+\sqrt{3})^n$ is not an integer, we see that $[(2+\sqrt{3})^n] = (2+\sqrt{3})^n + (2-\sqrt{3})^n - 1$. It follows that $[(2+\sqrt{3})^n]$ is odd.

55. We prove existence of q and r by induction on a. First assume that $a \geq 0$. Assume existence in the division algorithm holds for all nonnegative integers less than a. If $a < b$, then let $q = 0$ and $r = a$, so that $a = qb + r$ and $0 \leq r = a < b$. If $a \geq b$, then $a - b$ is nonnegative and by the induction hypothesis, there exist q' and r' such that $a - b = q'b + r'$, with $0 \leq r' < b$. Then $a = (q'+1)b + r'$, so we let $q = q' + 1$ and $r = r'$. This establishes the induction step, so existence is proved for $a \geq 0$. Now suppose $a < 0$. Then $-a > 0$, so, from our work above, there exist q' and r' such that $-a = q'b + r'$ and $0 \leq r' < b$. Then $a = -q'b - r'$. If $r' = 0$, we're done. If not, then $0 \leq b - r' < b$ and $a = (-q' - 1)b + b - r'$, so letting $q = -q' - 1$ and $r = b - r'$ satisfies the theorem. Uniqueness is proved just as in the text.

Section 2.1

1. $(5554)_7$; $(2112)_{10}$

3. $(175)_{10}$; $(1111100111)_2$

5. $(8F5)_{16}$; $(74E)_{16}$

7. This is because we are using the blocks of three digits as one "digit," which has 1000 possible values.

9. -39; 26

11. If m is any integer weight less than 2^k, then by Theorem 1.10, m has a base two expansion $m = a_{k-1}2^{k-1} + a_{k-2}2^{k-2} + \cdots + a_1 2^1 + a_0 2^0$, where each a_i is 0 or 1. The 2^i weight is used if and only if $a_i = 1$.

13. Let w be the weight to be measured. By Exercise 10, w has a unique balanced ternary expansion. Place the object in pan 1. If $e_i = 1$, then place a weight of 3^i into pan 2. If $e_i = -1$, then place a weight of 3^i in pan 1. If $e_i = 0$, then do not use the weight of 3^i. Now the pans will be balanced.

15. To convert a number from base r to base r^n, take the number in blocks of size n. To go the other way, convert each digit of a base r^n number to base r, and concatenate the results.

17. $(a_k a_{k-1} \ldots a_1 a_0 00 \ldots 00)_b$, where we have placed m zeroes at the end of the base b expansion of n

19. **a.** -6 **b.** 13 **c.** -14 **d.** 0

21. If m is positive, then $a_{n-1} = 0$ and $a_{n-2}a_{n-3} \ldots a_0$ is the binary expansion of m. Hence, $m = \sum_{i=0}^{n-2} a_i 2^i$ as desired. If m is negative, then the one's complement expansion for m has its leading bit equal to 1. If we view the bit string $a_{n-2}a_{n-3} \ldots a_0$ as a a binary number, then it represents $(2^{n-1} - 1) - (-m)$, because finding the one's complement is equivalent to subtracting the binary number from $111 \cdots 1$. That is, $(2^{n-1} - 1) - (-m) = \sum_{i=0}^{n-2} a_i 2^i$. Solving for m gives us the desired identity.

23. **a.** -7 **b.** 13 **c.** -15 **d.** -1

25. Complement each of the digits in the two's complement representation for m and then add 1.

27. $4n$

29. We first show that every positive integer has a Cantor expansion. To find a Cantor expansion of the positive integer n, let m be the unique positive integer such that $m! \le n < (m+1)!$. By the division algorithm there is an integer a_m such that $n = m! \cdot a_m + r_m$ where $0 \le a_m \le m$ and $0 \le r_m < m!$. We iterate, finding that $r_m = (m-1)! \cdot a_{m-1} + r_{m-1}$ where $0 \le a_{m-1} \le m-1$ and $0 \le r_{m-1} < (m-1)!$. We iterate $m-2$ more times, where we have $r_i = (i-1)! \cdot a_{i-1} + r_{i-1}$ where $0 \le a_{i-1} \le i-1$ and $0 \le r_{i-1} < (i-1)!$ for $i = m+1, m, m-1, \ldots, 2$ with $r_{m+1} = n$. At the last stage, we have $r_2 = 1! \cdot a_1 + 0$ where $r_2 = 0$ or 1 and $r_2 = a_1$. Uniqueness is proven as in the base-b expansion.

31. Call a position *good* if the number of ones in each column is even, and *bad* otherwise. Because a player can only affect one row, he or she must affect some column sums. Thus, any move from a good position produces a bad position. To find a move from a bad position to a good one, construct a binary number by putting a 1 in the place of each column with odd sum, and a 0 in the place of each column with even sum. Subtracting this number of matches from the largest pile will produce a good position.

33. **a.** First show that the result of the operation must yield a multiple of 9. Then it suffices to check only multiples of 9 with decreasing digits. There are only 79 of these. If we perform the operation on each of these 79 numbers and reorder the digits, we will have one of the following 23 numbers: 7551, 9954, 5553, 9990, 9981, 8820, 9810, 9620, 8532, 8550, 9720, 9972, 7731, 6543, 8730, 8640, 8721, 7443, 9963, 7632, 6552, 6642, or 6174. It will suffice to check only 9810, 7551, 9990, 8550, 9720, 8640, and 7632, because the other numbers will appear in the sequences which these 8 numbers generate. **b.** 8

35. Consider $a_0 = (3043)_6$. We find that T_6 repeats with period 6. Therefore, it never goes to a Kaprekar's constant for the base 6.

37. Suppose $n = a_i + a_j = a_k + a_l$ with $i \le j$ and $k \le l$. First, suppose $i \ne j$. Then $n = a_i + a_j = 2^i + 2^j$ is the binary expansion of n. By Theorem 2.1, this expansion is unique. If $k = l$, then $a_k + a_l = 2^{k+1}$, which would be a different binary expansion of n, so $k \ne l$. Then we must have $i = k$ and $j = l$ by Theorem 2.1, so the sum is unique. Next, suppose $i = j$. Then $n = 2^{i+1}$ and so $a_k + a_l = 2^k + 2^l = 2^{i+1}$. This forces $k = l = i$, and again the sum is unique. Therefore, $\{a_i\}$ is a Sidon sequence.

Section 2.2

1. $(10010110110)_2$

3. $(1011101100)_2$

5. $(10110001101)_2$

7. $q = (11111)_2$, $r = (1100)_2$