Session 4: Partial Differential Equations

$y = f(x)$

$x(t), v(t), a(t), F(t)$

$F(x)$

$u(x, y, z, t)$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial t^2}$

$\int \frac{\partial u}{\partial x} dx$

$u + g(y, z, t)$

Laplace: $\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$ (z, t) or (x, y)

Heat: $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

Wave: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ $\quad u(x,t) = ?$ Temperature

$u(x,t) = X(x) T(t)$

$X(x) T'(t) = \alpha X''(x) T(t)$

$\frac{1}{\alpha} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -K^2$

$T'(t) = -\alpha K^2 T(t)$

$X''(x) = -K^2 X$

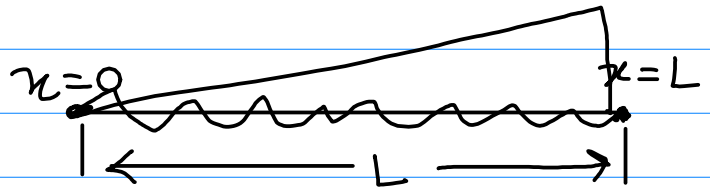
$$T(t) = Ae^{-\alpha K^2 t}$$

$$X(x) = C_1 \sin Kx + C_2 \cos Kx$$

$$u(x,t) = e^{-\alpha K^2 t}(a \sin Kx + b \cos Kx)$$

$$u(x,0) = f(x)$$

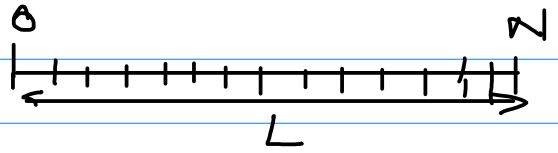
$$u(0,t) = 0 \qquad u(L,t) = 0$$



$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin K_n x \cdot e^{-\alpha K_n^2 t} \qquad K_n = \frac{n\pi}{L}$$

$$A_n = \int_0^L f(x) \sin K_n x \, dx$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

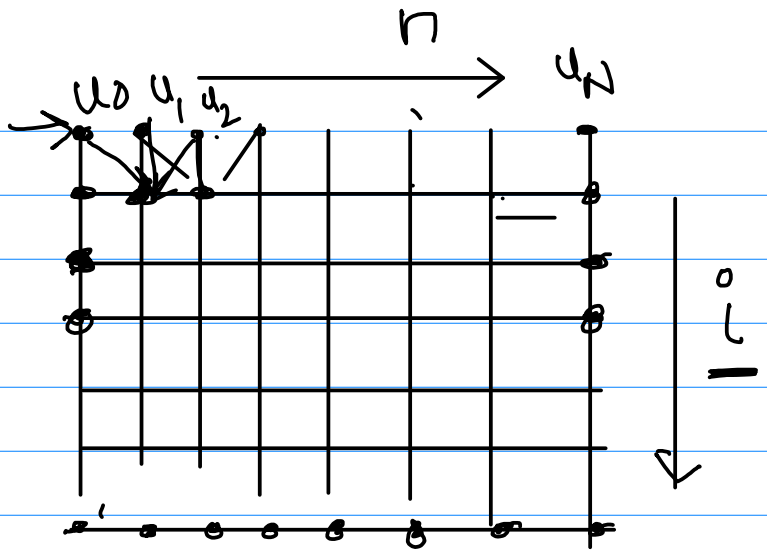


$$\frac{u_n^{i+1} - u_n^i}{\Delta t} = \alpha \left(\frac{u_{n+1}^i - 2u_n^i + u_{n-1}^i}{\Delta x^2} \right)$$

$$u_n^{i+1} = u_n^i - \underbrace{\frac{\alpha \Delta t}{\Delta x^2}}_{k} \left(\frac{u_{n+1}^i - 2u_n^i + u_{n-1}^i}{\Delta x^2} \right)$$

$$u_n^0 = f(x_n)$$

$$u_0^i = 0 \qquad u_N^i = 0$$


n
u_0 u_1 u_2
u_N
0
L