Session 2: Linear Ordinary Differential Equations

$$\frac{dx}{dt} = f(x,t)$$

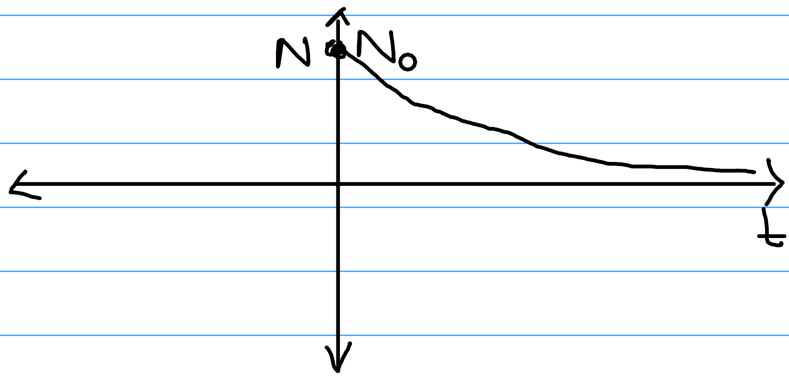
$$\text{"}x(t)\text{"}$$

$$\frac{dx}{dt} = f(t) \qquad x(t) = \int f(t)\,dt + c$$

$$\frac{dx}{dt} = f(x) \qquad \int \frac{dx}{f(x)} = t + c$$

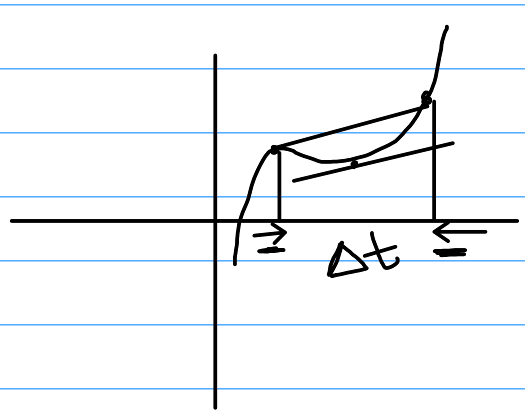
$$\frac{dN}{dt} = -\lambda N$$

$$N(t) = N_0 e^{-\lambda t}$$



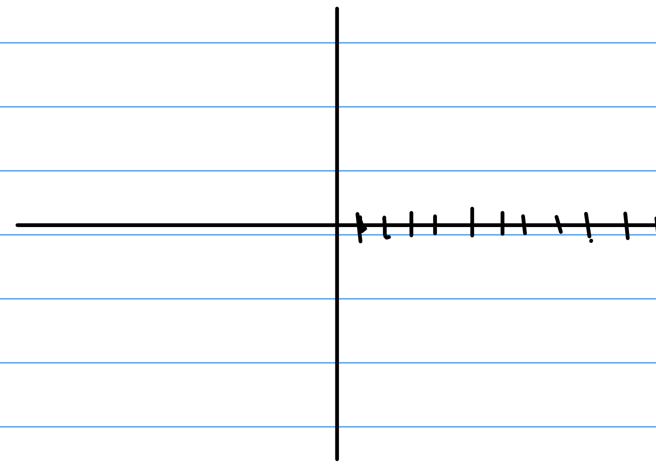
$$\frac{dx}{dt} = f(x)$$

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$



$$\frac{x(t+\Delta t) - x(t)}{\Delta t} \simeq \frac{dx}{dt}$$

$\Delta t << 1$



$$\frac{x_{n+1} - x_n}{\Delta t} \simeq \frac{dx}{dt}$$

$x_{n+1} \sim x(t_n + \Delta t)$

$x_n \sim x(t_n)$

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n) \text{ or } f(x_{n+1})$$

$\Delta t = h$ (Euler methods)

$$\boxed{x_{n+1} = x_n + h f(x_n)}$$ (backward)

$f(x) = ax$

$$x_{n+1} = (1+ah)\,x_n = (1+ah)\left[(1+ah)\,x_{n-1}\right]$$

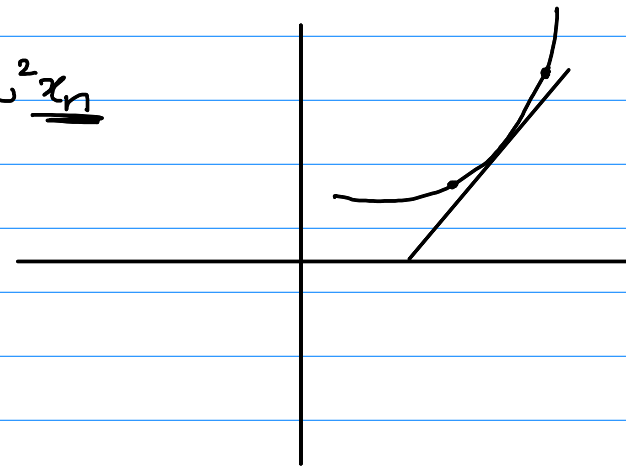
$$x_{n+1} = (1+ah)^n x_0$$

SHM

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\frac{x_{n+2} - 2x_{n+1} + x_n}{h^2} = -\omega^2 x_n$$

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} = -\omega^2 x_n$$



$$\boxed{x_0 \quad x_1}$$

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 x$$

$$\frac{dx}{dt} = v \quad \text{— (1)}$$

$$\boxed{v \overset{\sim}{=} \dot{x}}$$

$$\frac{dv}{dt} = -\omega^2 x \quad \text{— (2)}$$

Naive:

$$\boxed{\begin{aligned} x_{n+1} &= x_n + h v_n \\ v_{n+1} &= v_n - h\omega^2 x_n \end{aligned}}$$

$x_0 \rightarrow$

$v_0 \rightarrow$

Improved:

$$x_{n+1} = x_n + h v_n$$
$$v_{n+1} = v_n - h\omega^2 x_{n+1}$$

$\boxed{\text{Euler-Symplectic Method}}$

(Hamiltonian mechanics)

Vectorization

$a = [a_1, a_2, a_3 \cdots a_n]$
$b = [b_1, b_2, b_3 \cdots b_n]$

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for i in range(n):
    c.append(ai+bi)
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$O(n)$

$a + b \longrightarrow O(1)$

$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$x_{n+1} = x_n + h v_n$$
$$v_{n+1} = v_n - h\omega^2 x_n$$

$$\begin{bmatrix} x_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1x_n + h v_n \\ 1v_n - h\omega^2 x_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & h \\ -h\omega^2 & 1 \end{bmatrix}}\begin{bmatrix} x_n \\ v_n \end{bmatrix}$$

$$\frac{dx}{dt} = v \qquad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$\frac{dv}{dt} = -\omega^2 x \qquad \boxed{\frac{d\vec{x}}{dt} = \underline{A}\vec{x}} \qquad \vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\vec{x}_{n+1} = \vec{x}_n + Ah\vec{x}_n = (I + hA)\vec{x}_n$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & h \\ -h\omega^2 & 1 \end{bmatrix} = I + hA$$

$$|I + hA| = \underbrace{(1 + h^2\omega^2)}$$

$$x_{n+1} = x_n + hv_n \quad \text{--- (1)}$$

$$v_{n+1} = v_n - h\omega^2 x_{n+1} = v_n - h\omega^2(x_n + hv_n)$$

$$v_{n+1} = (1 - h^2\omega^2)v_n - h\omega^2 x_n \quad \text{--- (2)}$$

$$A = \begin{bmatrix} 1 & h \\ -h\omega^2 & 1 - h^2\omega^2 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$$

$$|A| = 1$$