Feynman's Path integrals

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1 Introduction

Feynman's path integral formulation offers an alternative perspective to quantum mechanics by expressing the probability amplitude of a particle's transition between two points as a sum over all possible paths connecting these states. This approach provides deep insights into quantum phenomena and establishes a connection between classical and quantum descriptions of physical systems. In quantum mechanics, two primary formulations describe the evolution of systems:

1.1 Schrödinger Approach

In the early twentieth century, Erwin Schrödinger formulated an equation that describes how the wavefunction of a quantum system evolves over time. This equation, known as the time-dependent Schrödinger equation, is expressed as:

$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle(0)$

Here, $|\psi(t)\rangle$ represents the state vector of the system at time t, $\hbar$ is the reduced Planck constant, and $\hat{H}$ denotes the Hamiltonian operator, which encapsulates the total energy (kinetic and potential) of the system.

If the Hamiltonian does not explicitly depend on time, the solutions to the Schrödinger equation can be expressed in terms of the Hamiltonian's eigenstates. These eigenstates $|n\rangle$ satisfy the time-independent Schrödinger equation:

$$\hat{H}|n\rangle = E_n|n\rangle$$

where E_n are the corresponding energy eigenvalues. The general state of the system at any time t can then be represented as a linear combination of these eigenstates:

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|n\rangle$$

In this expression, c_n are coefficients determined by the initial conditions of the system.

1.2 Path Integral Approach

In 1920, P.A.M. Dirac made an observation that later inspired Richard Feynman's development of an alternative formulation of quantum mechanics. Dirac suggested a connection between the probability amplitude of a quantum system's evolution and the classical action associated with its trajectory. However, this idea remained largely unexplored until 1948, when Feynman, formalized it into what is now known as the path integral formulation of quantum mechanics.

Feynman proposed that instead of considering a single trajectory for a particle, one should sum over all possible paths connecting the initial and final states. The total transition probability amplitude is then given by the sum of the amplitudes corresponding to each individual path. In this formulation, the transition probability amplitude between two points, denoted by $\langle\psi(x',t')|\psi(x_0,t_0)\rangle$, is expressed as an integral over all possible paths:

$$|\psi(x,t')\rangle = \int_{-\infty}^{\infty} \langle\psi(x',t')|\psi(x_0,t_0)\rangle dx' |\psi(x',t')\rangle$$

This formulation, known as the path integral approach, provides the same results as the Schrödinger picture but offers deeper insights into quantum mechanics. The fundamental object in this formulation is the **propagator**, which describes the probability transition amplitude between two spacetime points (x_0,t_0) and (x',t'):

$$U(x',t';x_0,t_0) = \langle\psi(x',t')|\psi(x_0,t_0)\rangle$$

If the Hamiltonian does not explicitly depend on time, we can redefine the initial time as $t_0 = 0$ and express the propagator as:

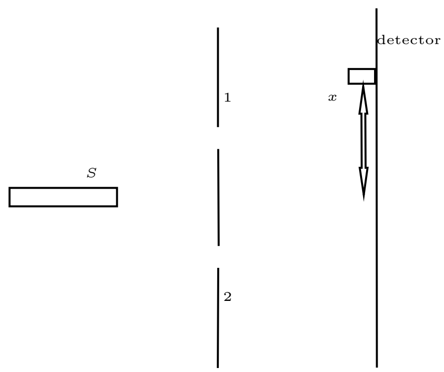
$$U(x',t;x_0) = A(t) \sum_{\text{all trajectories}} \exp\left(\frac{i}{\hbar} S[x(t)]\right)$$

Here, $A(t)$ is a normalization factor that depends only on time, and $S[x(t)]$ represents the classical action along a given trajectory. Feynman's insight was that every possible path contributes equally to the propagator but with a phase factor determined by the classical action.

Interlude: Probability amplitude

When we talk about probability in quantum mechanics, the concept remains the same: if the result of an experiment is p, then we expect to obtain a fraction, which is p, if the experiment is performed many times. What changes is how we calculate the probability.

Consider the classic example of the electron gun and the double slit:



Here, we know that we get an interference pattern similar to light waves, even though we classically have two assumptions.

1. The electron passes through either one of the two slits when it goes from S to x.

2. The probability of arrival at x is the sum of two parts: $P = P_1 + P_2$, where P_1 is the chance of arrival through slit 1 and P_2 is the chance of arrival through slit 2.

But $P \neq P_1 + P_2$ (determined experimentally). This means that our assumptions (2) and (1) are false.

The Probability Amplitude

The probability $P(x)$ is similar to the interference pattern in the double-slit experiment. In optics, we learned that the easiest way to represent wave amplitudes is by complex numbers, and we define $P(x)$ as the absolute square of the complex number $\phi(x)$. This is the probability amplitude of arrival at x. More precisely,

$$\phi(x) = \phi_1(x) + \phi_2(x),$$

where $\phi_1(x)$ is the contribution of slit 1, and $\phi_2(x)$ is the contribution of slit 2.

Our new assumption (3) is as follows.

$$P = |\phi(x)|^2,$$

$$\phi = \phi_1 + \phi_2,$$

$$P = |\phi_1 + \phi_2|^2.$$

Now, we want to precisely quantify how probability amplitudes are calculated. We use the Dirac notation for this experiment:

$$\langle \text{particle arrives at } x \mid \text{particle leaves } S \rangle$$

Here, the two brackets $\langle\cdot|\cdot\rangle$ are a sign equivalent to saying "amplitude that". Recall that these expressions are always read from right to left: $S \to x$. Thus, we can write this in compact form as follows:

$$\langle x|S\rangle$$

Also, we know that this quantity is just a complex number.

Now, if we want to write the results of Assumption (3) in this format, we write:

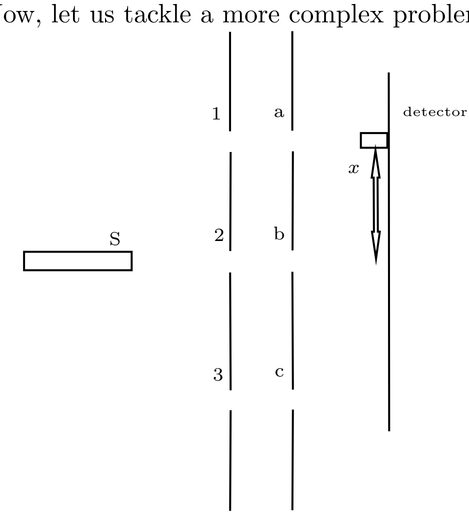
$$\langle x|s\rangle_{\text{both holes open}} = \langle x|s\rangle_{\text{through 1}} + \langle x|s\rangle_{\text{through 2}}.$$

$$\langle x|s\rangle_{\text{via 1}} = \langle x|1\rangle\langle 1|s\rangle.$$

$$\langle x|S\rangle = \langle x|1\rangle\langle 1|S\rangle + \langle x|2\rangle\langle 2|S\rangle.$$

Extending the Problem

Now, let us tackle a more complex problem.



Here,

$$\langle x|S\rangle = \langle x|a\rangle\langle a|1\rangle\langle 1|s\rangle + \langle x|b\rangle\langle b|1\rangle\langle 1|s\rangle + \langle x|c\rangle\langle c|1\rangle\langle 1|s\rangle + \ldots + \langle x|c\rangle\langle c|2\rangle\langle 3|s\rangle$$

More generally,

$$\langle x|S\rangle = \sum_{\substack{i=1,2,3\\ \alpha=a,b,c}} \langle x|\alpha\rangle\langle\alpha|i\rangle\langle i|S\rangle.$$

If we have infinite slits and infinite walls, the space between the source and the detector becomes a free space. We will also add the probability amplitude for each path between them.

From the above examples, we understand the need to add probability amplitudes to calculate the probability.

also, we can get an intuitive understanding of why Feynman defined the propagator as:

$$U(x', t'; x_0, t_0) = \langle \psi(x', t') | \psi(x_0, t_0) \rangle.$$

It signifies the propagation of the state:

$$\psi(x_0, t_0) \to \psi(x', t').$$

And we get it by adding probability amplitudes for all the paths between (x_0, t_0) and (x', t').

2 Propagator

In a classical system, suppose a particle is found at point x_a at time t_a and at point x_b at time t_b. The path followed by the particle to travel between the given points could be found by using the *principle of least action*. By this principle, each of the possible path has a certain quantity called *action*, denoted by S. The path followed by the particle or the classical path, denoted by $\tilde{x}(t)$ is the path for which the action is stationary. Mathematically, action is defined as

$$S = \int_{t_a}^{t_b} L(x, \dot{x}, t) dt$$

where L is the Lagrangian of the system given by

$$L(x, \dot{x}, t) = T - V$$

where T and V are total kinetic and potential energies of the object respectively.

The concept of action carries over to quantum mechanics as well. Suppose a particle in free space is observed initially at point x_a at time t_a and later at point x_b at time t_b. Currently we do not know the exact path the particle traveled from point x_a to point x_b, as there are infinite possible paths or trajectories the particle could take in free space. We assume that each of these trajectories contributes to the total probability amplitude for the particle to travel from point x_a to point x_b. The sum of all these trajectories is called a **kernel** or a **propagator** and is denoted as $K(b, a)$ or K_{ba}. The probability of the particle to go from point x_a to x_b or $P(b, a)$ is therefore proportional to the absolute square of the propagator or

$$P(b, a) \alpha |K_{ba}|^2$$

The particle wave can be represented by $e^{i\phi}$, where ϕ is the wave's phase. Then the propagator K_{ba} can be represented by

$$K_{ba} = C \sum_{paths} e^{i\phi}$$

Where C is normalization constant. Now each path $x(t)$ can be broken down into many small paths of length Δx and the time interval to pass it being Δt, therefore overall the propagator can be written as

$$K_{ba} = C \sum_{paths} e^{i \sum \Delta\phi} \tag{1}$$

where,

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x - 2\pi f \Delta t$$

where λ and f are wavelength and frequency of the wave respectively. For a particle of mass m, moving with velocity v using,

$$\lambda = \frac{h}{mv}(debroglie's relation)$$

$$E = hf(Plank's relation)$$

$\Delta\phi$ can be rewritten as,

$$\sum \Delta\phi = \sum (\frac{2\pi}{\lambda}\Delta x - 2\pi f \Delta t) = \frac{\sum(mv\Delta x - E\Delta t)}{\hbar} = \frac{\sum(mv\frac{\Delta x}{\Delta t} - E)\Delta t}{\hbar}$$

Now for infinitesimal Δt, the summation could be replaced with integration. Also as E is the total energy of the particle, it could be written as $E = \frac{1}{2}mv^2 + V$. Hence, now $\Delta\phi$ becomes,

$$\phi = \int d\phi = \frac{\int(mv\frac{dx}{dt} - \frac{1}{2}mv^2 - V)dt}{\hbar} = \frac{\int(\frac{1}{2}mv^2 - V)dt}{\hbar} = \frac{S}{\hbar} \tag{2}$$

Hence, summing over all the paths, we can write,

$$K_{ba} = C \sum_{paths} e^{iS/\hbar} \tag{3}$$

where C is a normalization constant.

3 The Classical Limit

We can imagine $e^{i\phi}$ as a phase vector of length one, centered at the origin, making an angle ϕ with the x-axis.

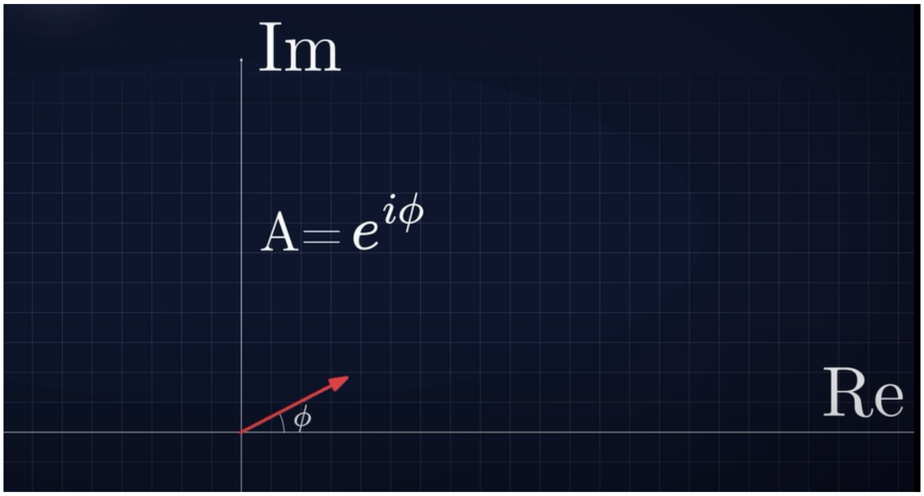


Figure 1:

Now as ϕ equals $S/\hbar$, for classical particles, ϕ will be an extremely large angle as S is much larger as compared to $\hbar$ which is of the order $10^{-34} Js$. Therefore if we choose any one of the paths and consider another path very close to it on a classical scale, ϕ will change tremendously, and essentially the phase vector will point in a random direction. Since we are considering every possible path, many phase vectors will end up canceling each other.

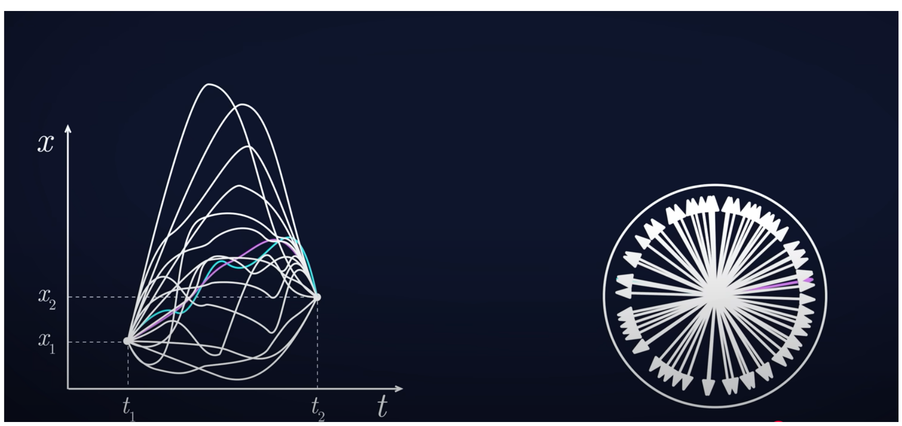


Figure 2:

Now the only exception here is the path with the least action, since if we change this path by a slight amount, the action(S) remains constant up to the first order, i.e.

$$S[x] = S[x + \delta x] + O(\delta x^2)$$

and this will give you the classical path.

4 Path Integrals

First of all remember equation (3), there we are actually doing a discrete sum over paths, but in reality It should not be a discrete sum and rather a path integral

$$K_{ba} = \int_{x_i}^{x_f} e^{iS[x]/\hbar} Dx$$

Now how do we evaluate a path integral, well we will follow a similar strategy as is done while deriving an ordinary integral, consider a general path integral

$$K_{ba} = \int_{x_i}^{x_f} \Phi[x] Dx$$

here $\Phi[x]$ is a functional i.e. unlike a function whose input is a number and output is also a number, a functional's input is a curve and the output is a number.

Consider a random path x(t) and lets break the time interval t_i to t_f into N equal parts as and assign numbers $x_1, x_2, ...$ to each of the times $t_1, t_2, ...$ as follows

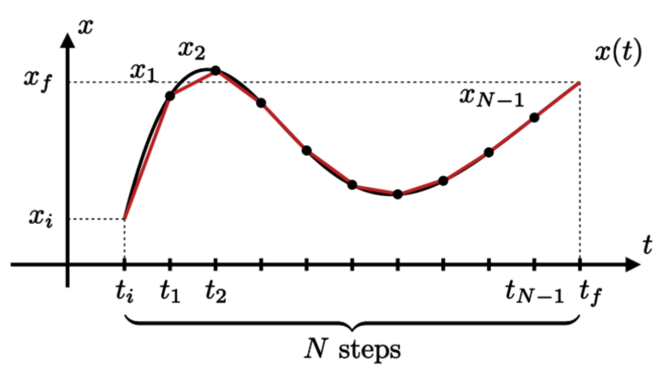


Figure 3:

Now we have broken up our path into an approximate path. Now, if we want to sum over a functional $\Phi[x]$ over this path, we just need to sum over all the possible values of $x_1, x_2, ...$ which will be from $(-\infty, +\infty)$ and if we take $N \to \infty$ we get the following integral

$$\int_{x_i}^{x_f} \Phi[x]Dx = A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\Phi[x]dx_1dx_2dx_3...dx_{N-1} \quad (4)$$

A is some constant.

5 Free Particle

Now for a free particle, potential energy V = 0 Therefore

$$S = \int_{t_i}^{t_f}(T-V)dt = \int_{t_i}^{t_f} Tdt = \int_{t_i}^{t_f}\frac{1}{2}m(\frac{dx}{dt})^2dt$$

Now, as we did earlier that we chose a random path x(t) and then broke it into N equal time intervals and approximated our curve x(t) to some zig-zag curve

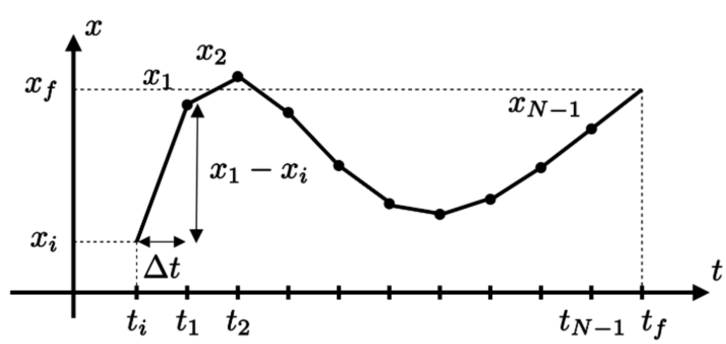


Figure 4:

Consider the first line-segment, since it's a straight line therefore the speed is constant

$$\frac{dx}{dt} = \frac{\Delta x}{\Delta t} = \frac{x_1 - x_i}{\Delta t}$$

Since the speed is constant therefore the Kinetic energy is also constant

$$K_o = \frac{1}{2}m(\frac{dx}{dt})^2 = \frac{1}{2}m(\frac{x_1-x_i}{\Delta t})^2$$

Therefore the action of this line-segment is

$$S_o = \frac{1}{2}m(\frac{x_1-x_i}{\Delta t})^2\Delta t = \frac{m}{2\Delta t}(x_1-x_i)^2$$

Similarly

$$S_1 = \frac{m}{2\Delta t}(x_2 - x_1)^2, S_2 = \frac{m}{2\Delta t}(x_3 - x_2)^2, ...$$

adding all the actions to get the action of the whole curve

$$S = \frac{m}{2\Delta t}[(x_1 - x_i)^2 + (x_2 - x_1)^2 + ... + (x_f - x_{N-1})^2]$$

putting this action into the formula for propagator

$$\int_{x_i}^{x_f} e^{iS[x]/\hbar} Dx = lim_{N\to\infty} A \int_{-\infty}^{\infty} e^{iS[x]/\hbar} dx_1 dx_2 dx_3 ... dx_{N-1}$$

$$= lim_{N\to\infty} A \int_{-\infty}^{\infty} e^{\frac{im}{2\Delta t\hbar}[(x_1-x_i)^2+(x_2-x_1)^2+...+(x_f-x_{N-1})^2]} dx_1 dx_2 dx_3 ... dx_{N-1} \tag{5}$$

We can see that the integral above is a gaussian integral for each variable $x_1, x_2, x_3, ...$, we know that

$$\int_{-\infty}^{\infty} e^{ia(x^2+bx)} dx = \sqrt{\frac{\pi i}{a}} e^{\frac{-iab^2}{4}} \tag{6}$$

Therefore for example for the variable x_1

$$\int_{-\infty}^{\infty} e^{\frac{im}{2\Delta t\hbar}[(x_1-x_i)^2+(x_2-x_1)^2]} dx = e^{\frac{im}{2\Delta t\hbar}(x_i^2+x_2^2)} \int_{-\infty}^{\infty} e^{\frac{im}{\Delta t\hbar}[x_1^2 - x_1(x_i+x_2)]} dx_1 = \sqrt{\frac{\pi i\hbar\Delta t}{m}} e^{\frac{im}{4\Delta t\hbar}(x_2-x_i)^2}$$

similarly for x_2 we get

$$\sqrt{\frac{\pi i\hbar\Delta t}{m}} \int_{-\infty}^{\infty} dx_2\, e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x_3-x_2)^2} e^{\frac{i}{\hbar}\frac{m}{4\Delta t}(x_2-x_i)^2} = \sqrt{\frac{\pi i\hbar\Delta t}{m}} \sqrt{\frac{4\pi i\hbar\Delta t}{3m}} e^{\frac{i}{\hbar}\frac{m}{6\Delta t}(x_3-x_i)^2}.$$

for x_3

$$\sqrt{\frac{\pi i\hbar\Delta t}{m}} \sqrt{\frac{4\pi i\hbar\Delta t}{3m}} \int_{-\infty}^{\infty} dx_3\, e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x_4-x_3)^2} e^{\frac{i}{\hbar}\frac{m}{6\Delta t}(x_3-x_i)^2} = \sqrt{\frac{\pi i\hbar\Delta t}{m}} \sqrt{\frac{4\pi i\hbar\Delta t}{3m}} \sqrt{\frac{6\pi i\hbar\Delta t}{4m}} e^{\frac{i}{\hbar}\frac{m}{8\Delta t}(x_4-x_i)^2}.$$

similarly continuing on with pattern we get

$$N = 2: \quad \sqrt{\frac{\pi i\hbar\Delta t}{m}}\, e^{\frac{i}{\hbar}\frac{m}{4\Delta t}(x_2-x_i)^2}$$

$$N = 3: \quad \sqrt{\frac{4}{3}\left(\frac{\pi i\hbar\Delta t}{m}\right)^2}\, e^{\frac{i}{\hbar}\frac{m}{6\Delta t}(x_3-x_i)^2}$$

$$N = 4: \quad \sqrt{\frac{8}{4}\left(\frac{\pi i\hbar\Delta t}{m}\right)^3}\, e^{\frac{i}{\hbar}\frac{m}{8\Delta t}(x_4-x_i)^2}$$

$$N = 5: \quad \sqrt{\frac{16}{5}\left(\frac{\pi i\hbar\Delta t}{m}\right)^4}\, e^{\frac{i}{\hbar}\frac{m}{10\Delta t}(x_5-x_i)^2}.$$

Extrapolating to general N, we get

$$\int_{-\infty}^{\infty} dx_1 \cdots dx_{N-1}\, e^{\frac{i}{\hbar}\frac{m}{2\Delta t}\left\{(x_1-x_i)^2+(x_2-x_1)^2+\cdots+(x_f-x_{N-1})^2\right\}} = \sqrt{\frac{1}{N}\left(\frac{2\pi i\hbar\Delta t}{m}\right)^{N-1}}\, e^{\frac{i}{\hbar}\frac{m}{2N\Delta t}(x_f-x_i)^2}.$$

multiplying by a factor A and after rearranging we get

$$\int_{x_i}^{x_f} \mathcal{D}x\, e^{\frac{i}{\hbar}S} = \lim_{N\to\infty} A\left(\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\right)^N \sqrt{\frac{m}{2\pi i\hbar(t_f-t_i)}}\, e^{\frac{i}{\hbar}\frac{1}{2}m\frac{(x_f-x_i)^2}{t_f-t_i}}.$$

where we have used the fact that $N\Delta t = t_f - t_i$. Here we can see that only $(\sqrt{\frac{2\pi i\hbar\Delta t}{m}})^N$ has N dependence, Therefore in order for the limit to make sense we can let $A = (\sqrt{\frac{m}{2\pi i\hbar\Delta t}})^N$ hence all the N's have disappeared and we get

$$K_{fi} = \sqrt{\frac{m}{2\pi i\hbar(t_f-t_i)}}\, e^{\frac{im}{2\hbar}\frac{(x_f-x_i)^2}{t_f-t_i}} \tag{7}$$

now, Probability Density = $|K_fi|^2$ and

$$|K_fi|^2 = \frac{m}{2\pi i\hbar(t_f-t_i)^2} \tag{8}$$

here, there is no e term because the magnitude of the e term will be 1. Now if you carefully see that the probability density does not depend on x_f that is the probability of finding the particle is equal at every point. This at first feels absurd but can be explained by the "Heisenberg's Uncertainty Principle", we have fixed x_i that is the position is fixed therefore $\Delta x = 0$ therefore $\Delta p \to \infty$ i.e. the particle can have any velocity at point x_i therefore it can reach any point x_f with an equal probability.

6 The Schrödinger Equation

Assuming that $(x_i, t_i) = (0, 0)$ and let (x_f, t_f) be a general (x, t) we get our wavefunction or the propagator as

$$\Psi(x,t) = \sqrt{\frac{m}{2\pi i\hbar t}}\, e^{\frac{i}{\hbar}\frac{1}{2}m\frac{x^2}{t}}. \tag{9}$$

Now we will check if this wavefunction satisfies the schrodinger's equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}.$$

finding $\frac{\partial\Psi}{\partial t}$

$$\frac{\partial\Psi}{\partial t} = -\frac{1}{2t}\left(1+\frac{im}{\hbar t}x^2\right)\Psi$$

finding $\frac{\partial^2\Psi}{\partial x^2}$

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{im}{\hbar t}\left(1 + \frac{im}{\hbar t}x^2\right)\Psi$$

Now we multiply $\frac{\partial\Psi}{\partial t}$ by $i\hbar$,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{i\hbar}{2t}\left(1 + \frac{im}{\hbar t}x^2\right)\Psi,$$

and finally we multiply $\frac{\partial^2\Psi}{\partial x^2}$ by $-\frac{\hbar^2}{2m}$,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = -\frac{i\hbar}{2t}\left(1 + \frac{im}{\hbar t}x^2\right)\Psi,$$

we get the exact same thing, hence LHS equals RHS
Hence, we find that this wavefunction satisfies the Schrödinger equation.

References

R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals (McGraw- Hill, New York, 1965)

R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics, Volume 3*, Addison-Wesley, 1965.

https://www.youtube.com/watch?v=qJZ1Ez28C-At=1418s

https://www.youtube.com/watch?v=Sp5SvdDh2u8list=PL-IZN8QRUw-ykZ0yIZ-I5jeByyYHmcCcFindex=4

https://www.youtube.com/watch?v=UWuhMJZaiZMlist=PL-IZN8QRUw-ykZ0yIZ-I5jeByyYHmcCcF