

Fibrations and Logical Relations

*Claudio Hermida**

*University of Birmingham, Department of Computer Science

Polymorphic Lambda Calculus

$$\tau ::= X \mid \kappa \mid \tau \times \tau' \mid \tau \Rightarrow \tau' \mid \forall X . \tau$$

Logical Relations: type constructors acting on relations (over polymorphic types):

$$(R \Rightarrow Q)_{f,g:\sigma \rightarrow \tau} \equiv \forall x, y : \sigma . R(x, y) \implies Q(fx, gy)$$

$$\begin{aligned} & (\tilde{\forall} R . \Phi)_{F,G:\forall X . \tau} \equiv \\ & \forall Y, Y' : \text{Type}. \forall S \subset Y \times Y' . \Phi[S/R](FY, GY') \end{aligned}$$

Applications: Structural (co)induction principles, definability, parametricity.

What do these formulas mean?

[Ma-Reynolds] Let \mathbf{C} be a ccc with pullbacks.
Then

$$\begin{array}{ccc} \text{Sub}(\mathbf{C}) & & \text{ccc} \\ \text{cod} \downarrow & \text{preserves} & \text{ccc} \\ \mathbf{C} & & \end{array}$$

- The exponentials in $\text{Sub}(\mathbf{C})$ agree with the ‘logical exponential’ formula when $\mathbf{C} = \mathcal{Set}$.
- ad-hoc generalisation to ‘PL-categories’.

\implies seek:

– a *unified treatment* of ‘logical relations’
with a *precise correspondence* $\text{syntax} \leftrightarrow \text{categorical structure}$.

‘Corollary’: (sound and complete) categorical semantics for Plotkin-Abadi’s ‘logic for parametricity’.

Fibred category theory:

study of structures varying over a base
category

Examples:

– Category $\mathbf{C} \longrightarrow$ fibration $dom_C : [\mathbf{C}] \rightarrow \mathcal{Set}$
of families: $[\mathbf{C}]_I = \mathcal{Cat}(I, \mathbf{C})$, with reindexing
given by precomposition:

$$u : J \rightarrow I \rightsquigarrow (\{c_i\}_{i \in I} \mapsto \{c_{uj}\}_{j \in J})$$

– For $\mathbf{C} = \mathcal{Set}$, the above externalisation has
a more compact description:

$$\begin{array}{ccc} [\mathcal{Set}] & \xrightarrow{\cong} & \mathcal{Set}^{\rightarrow} \\ & \searrow \text{dom} & \swarrow \text{cod} \\ & \mathcal{Set} & \mathcal{Set} \end{array}$$

whereby we regard a family of sets $\{X_i\}_{i \in I}$ à
la Bourbaki $\sum_{i \in I} X_i \rightarrow I$, and hence reindex-
ing is realized by pullback

$$\begin{array}{ccc} \{X_{uj}\}_{j \in J} & \xrightarrow{\bar{u}} & \{X_i\}_{i \in I} \\ \pi_u \downarrow & & \downarrow \pi \\ J & \xrightarrow{u} & I \end{array}$$

– For $\mathbf{C} = \mathbf{2} = (0 \rightarrow 1)$, its externalisation yields the fibration of subobjects:

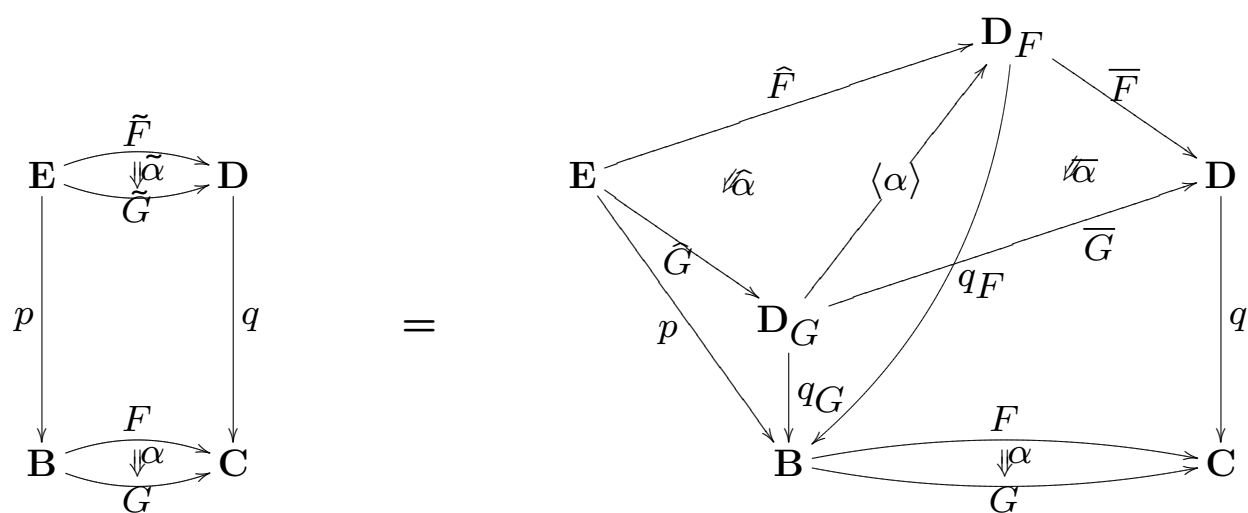
$$\begin{array}{ccc} [2] & \xrightarrow{\cong} & \text{Sub}(\mathcal{S}et) \\ & \searrow \text{dom}_{\mathbf{2}} & \swarrow \text{cod} \\ & \mathcal{S}et & \end{array}$$

which amounts to the characterisation of $\mathbf{2}$ as a *subobject classifier* (this makes sense for any topos instead of $\mathcal{S}et$).

Fibrations in a 2-category

Definition: A morphism $p : E \rightarrow B$ in a 2-category \mathcal{K} is a *fibration* if for every object X , the functor $\mathcal{K}(X, p) : \mathcal{K}(X, E) \rightarrow \mathcal{K}(X, B)$ is a fibration of categories.

$\mathcal{Fib}(\mathcal{K})$:



Structural decomposition of adjunctions between fibrations

Theorem:[?, Thm 4.3]

Consider the following data:

$$\begin{array}{ccc}
 \mathbf{E} & \xrightarrow{\tilde{F}} & \mathbf{D} \\
 p \downarrow & & \downarrow q \\
 \mathbf{B} & \xrightarrow{F} & \mathbf{A} \\
 & \underset{G}{\curvearrowright} & \\
 & \perp &
 \end{array}$$

and let $\hat{F} : p \rightarrow F^*(q)$ in \mathcal{Fib}/\mathbf{B} be the unique mediating morphism into the pullback:

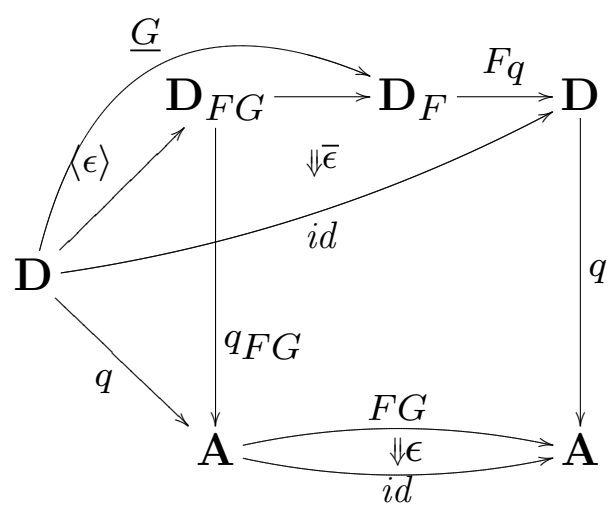
$$\begin{array}{ccccc}
 \mathbf{E} & & & \tilde{F} & \\
 & \searrow \hat{F} & & \nearrow & \\
 & \mathbf{D}_F & \xrightarrow{\tilde{F}} & \mathbf{D} & \\
 & \downarrow q_F & & \downarrow q & \\
 & \mathbf{B} & \xrightarrow{F} & \mathbf{A} & \\
 p \swarrow & & & & \\
 & & & &
 \end{array}$$

Then, the following are equivalent:

Lifting $\exists \tilde{G} : \mathbf{D} \rightarrow \mathbf{E}. \tilde{F} \dashv \tilde{G}$ (in \mathcal{Cat}) such that $(\tilde{F}, F) \dashv (\tilde{G}, G) : q \rightarrow p$ (in \mathcal{Fib}).

Factorisation $\exists \hat{G} : q_F \rightarrow p. \hat{F} \dashv \hat{G}$ (in \mathcal{Fib}/\mathbf{B}).

Lifting:



Fibrations over a base category

Consider the 2-category \mathcal{Fib}/\mathbf{B} of fibrations over a base category \mathbf{B} . Recall that

$$\mathcal{Fib}/\mathbf{B} \simeq \mathbf{Ps}[\mathbf{B}^{op}, \mathbf{Cat}]$$

Theorem[Bénabou]: Given a morphism in \mathcal{Fib}/\mathbf{B}

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{F} & \mathbf{D} \\ & \searrow p \quad \swarrow q & \\ & \mathbf{B} & \end{array}$$

the following are equivalent:

- F is a fibration in \mathcal{Fib}/\mathbf{B} .
- The associated pseudo-natural transformation $\mathcal{F}_F : \mathcal{F}_p \Rightarrow \mathcal{F}_q$ takes values in \mathcal{Fib} , i.e. every fibre functor is a fibration and the functors between fibres (for \mathcal{F}_p) preserve cartesianes.
- The functor $F : \mathbf{E} \rightarrow \mathbf{D}$ is a fibration of categories.

Example: Given an internal fibration $f : \underline{\mathbf{E}} \rightarrow \underline{\mathbf{D}}$ in a category \mathbf{B} , its externalisation

$$\begin{array}{ccc} [\underline{\mathbf{E}}] & \xrightarrow{[f]} & [\underline{\mathbf{D}}] \\ \text{\textit{dom}}_E \searrow & & \swarrow \text{\textit{dom}}_E \\ & \mathbf{B} & \end{array}$$

is a fibration in \mathcal{Fib}/\mathbf{B} . Notice that $[f] : (x : X \rightarrow E_0) \mapsto (f_0 x : X \rightarrow D_0)$ *does not preserve generic objects*.

Concretely, take

$$\mathbf{B} = \omega - \mathcal{Set}$$

$\underline{\mathbf{D}} = \underline{\mathbf{PER}}$, the category of PER's internally in $\omega - \mathcal{Set}$.

$\underline{\mathbf{E}} = \underline{\mathbf{RegSub(PER)}}$, the internal category of regular subobjects of PER's.

$f = \text{\textit{cod}}$, the internal functor taking a subobject to its codomain.

Fibrations in \mathcal{Fib}

Proposition: A morphism $(\tilde{F}, F) : p \rightarrow q$

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\tilde{F}} & \mathbf{D} \\ p \downarrow & & \downarrow q \\ \mathbf{B} & \xrightarrow{F} & \mathbf{A} \end{array}$$

in \mathcal{Fib} is a fibration iff

- both $\tilde{F} : \mathbf{E} \rightarrow \mathbf{D}$ and $F : \mathbf{B} \rightarrow \mathbf{A}$ are fibrations, and
- the functor $p : \mathbf{E} \rightarrow \mathbf{B}$ takes \tilde{F} -cartesian morphisms to F -cartesian morphisms. □

Notice the pleasant symmetry of the situation above: if $(\tilde{F}, F) : p \rightarrow q$ is a fibration in \mathcal{Fib} so is $(p, q) : \tilde{F} \rightarrow F$.

Example: Given an internal fibration $f : \underline{\mathbf{E}} \rightarrow \underline{\mathbf{D}}$ in a category \mathbf{B} , we regard it as an internal category $\underline{f} \in \mathcal{Cat}(\mathbf{B}^{\rightarrow})$, and we get the following externalisation:

$$\begin{array}{ccc} [\underline{f}] & \xrightarrow{\text{dom}[\underline{f}]} & \mathbf{B}^{\rightarrow} \\ [\text{cod}] \downarrow & & \downarrow \text{cod} \\ [\underline{\mathbf{D}}] & \xrightarrow{\text{dom}_D} & \mathbf{B} \end{array}$$

which is a fibration in \mathcal{Fib} , where the induced functor $[\text{cod}]$ has action:

$$\begin{array}{ccc} F & \xrightarrow{f} & E_0 \\ u \downarrow & & \downarrow f_0 \\ X & \xrightarrow{x} & D_0 \end{array} \quad \mapsto \quad X \xrightarrow{x} D_0$$

Theorem: Consider the following data

$$\begin{array}{ccccc}
 \tilde{\mathbf{P}} & & & & \\
 \searrow \hat{p} & & p & \searrow & \\
 & \mathbf{P}_b & \xrightarrow{\bar{b}} & \mathbf{P} & \\
 \searrow \tilde{q} & \downarrow q_b & & \downarrow q & \\
 & \tilde{\mathbf{B}} & \xrightarrow{b} & \mathbf{B} &
 \end{array}$$

where both b and q are fibrations, and the square is a pullback. The following are equivalent:

- The functor $\tilde{q} : \tilde{\mathbf{P}} \rightarrow \tilde{\mathbf{B}}$ is a fibration and the morphism $(p, b) : \tilde{q} \rightarrow q$ is a fibration in \mathcal{Fib} .
- The functor $\tilde{q} : \tilde{\mathbf{P}} \rightarrow \tilde{\mathbf{B}}$ is a fibration and the morphism $\hat{p} : \tilde{\mathbf{P}} \rightarrow \mathbf{P}_b$ is a fibration in $\mathcal{Fib}/\tilde{\mathbf{B}}$.
- The functor $\hat{p} : \tilde{\mathbf{P}} \rightarrow \mathbf{P}_b$ is a fibration.
- The functor $p : \tilde{\mathbf{P}} \rightarrow \mathbf{B}$ is a fibration and the morphism $(\tilde{q}, q) : p \rightarrow b$ is a fibration in \mathcal{Fib} .
- The functor $\tilde{q} : \tilde{\mathbf{P}} \rightarrow \tilde{\mathbf{B}}$ is a fibration and the morphism $\hat{p} : \tilde{\mathbf{P}} \rightarrow \mathbf{P}_b$ is a fibration in \mathcal{Fib}/\mathbf{P} .

□

Example: Given an internal fibration $f : \underline{\mathbf{E}} \rightarrow \underline{\mathbf{D}}$ in a category \mathbf{B} , we regard it as an internal category $\underline{f} \in \mathcal{Cat}(\mathbf{B}^{\rightarrow})$, and we get the following externalisation:

$$\begin{array}{ccc} & \xrightarrow{\text{dom}[\underline{f}]} & \mathbf{B}^{\rightarrow} \\ [\underline{f}] \downarrow [\text{cod}] & & \downarrow \text{cod} \\ [\underline{\mathbf{D}}] & \xrightarrow{\text{dom}_D} & \mathbf{B} \end{array}$$

which is a fibration in \mathcal{Fib} , where the induced functor $[\text{cod}]$ has action:

$$\begin{array}{ccc} F & \xrightarrow{f} & 0 \\ u \downarrow & & \downarrow f_0 \\ X & \xrightarrow{x} & D_0 \end{array} \mapsto X \xrightarrow{x} D_0$$

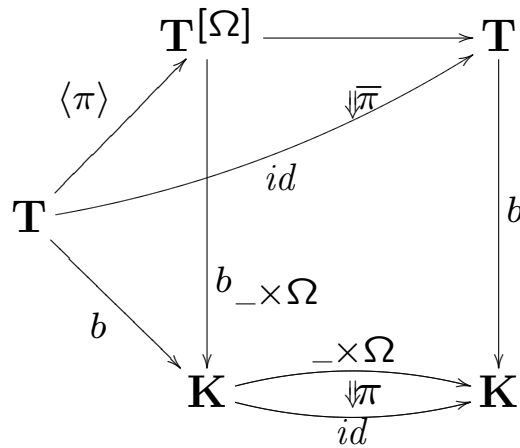
Notice that the morphism of fibrations $([\underline{f}], \text{cod})$ preserves the standard generic objects.

Fibrations for polymorphic lambda-calculus

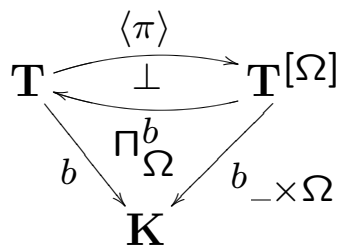
Definition: A $\lambda 2$ -fibration is a fibred-ccc with simple $(\Omega\text{-})$ products and a generic object (there is a distinguished object T over Ω such that for every object X there is a cartesian morphism $X \rightarrow T$).

Simple Products:

Consider the *diagonal morphism* $\langle \pi \rangle : b \rightarrow b^{[\Omega]}$:



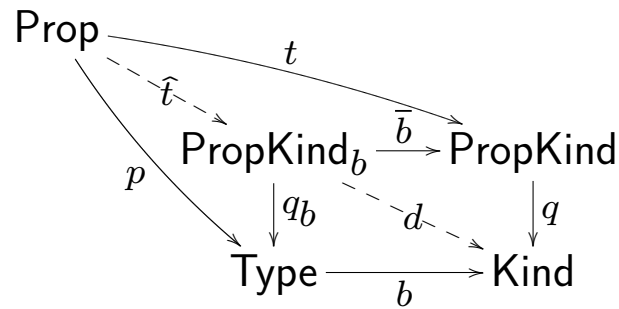
and demand a right adjoint (over \mathbf{K}):



The fibred nature of the adjunction encompasses both the existence of right adjoints to π_{Ω}^* and the Beck-Chevalley condition of pullback stability.

Classical example: The fibration obtained by externalisation of PER, the category of PERs on the natural numbers regarded as an internal category in $\omega - \mathcal{Set}$. This fibration is equivalently described as $u : \mathbf{UFam}(\mathbf{PER}) \rightarrow \omega - \mathcal{Set}$, where the fiber over an ω -set (I, E) has as objects collection of PERs $\{R_i\}_{i \in I}$. Its morphisms are I -families of PER maps *with a common realizer* (uniform realizability *cf.*[?]).

Fibrations for relational polymorphism



A **relational $\lambda 2$ -fibration** is a fibration in \mathcal{Fib} with the following:

- the base $b : \text{Type} \rightarrow \text{Kind}$ is a $\lambda 2$ -fibration,
- $q : \text{PropKind} \rightarrow \text{Kind}$ has (fibred) finite products,
- $p : \text{Prop} \rightarrow \text{Type}$, as a fibration over q , is a fibred-ccc with simple products ($\top, \wedge, \Rightarrow, \forall x : \sigma. _$),
- $t : \text{Prop} \rightarrow \text{PropKind}$, as a fibration over b , has fibred generic objects $\{P_\tau\}_{\tau \in \text{Type}}$ and fibred simple P_τ products ($\forall \rho \subset \tau. _$), and
- $\hat{t} : \text{Prop} \rightarrow \text{PropKind}_b$ has π^d -products, that is, cartesian simple products with respect to the diagonal fibration $d : \text{PropKind}_b \rightarrow \text{Kind}$ ($\forall X. _$)

Interpretation of Syntax

$$\frac{\psi \text{ Predicate_Kind}}{\kappa \text{ Kind}} \mid \frac{\phi \text{ Predicate}}{\tau \text{ Type}}$$

- κ is an object of Kind ,
- ψ is an object of PropKind_{κ} ,
- τ is an object of Type_{κ} , and
- ϕ is an object of $\text{Prop}\left(\frac{\psi \mid \tau}{\kappa}\right)$, where $\left(\frac{\psi \mid \tau}{\kappa}\right)$ represents the evident object in the pullback $\text{PropKind}_b = \text{Type} \times_{\text{Kind}} \text{PropKind}$.

Syntax	Categorical Interpretation
$\underbrace{X, Y, \dots}_{\Gamma}$	$\llbracket \Gamma \rrbracket = \Omega \times \Omega \dots \in \text{Kind}$
$\Gamma \mid \underbrace{x: \sigma, y: \tau, \dots}_{\gamma}$	$\llbracket \gamma \rrbracket = \llbracket \sigma \rrbracket \times \llbracket \tau \rrbracket \dots \in \text{Type}_{\llbracket \Gamma \rrbracket}$
$\Gamma \mid \underbrace{R \subset \sigma \times \tau, S \subset \mu \times \nu}_{\Psi}$	$\llbracket \Psi \rrbracket = P_{\llbracket \sigma \times \tau \rrbracket} \times P_{\llbracket \mu \times \nu \rrbracket} \dots \in \text{PropKind}_{\llbracket \Gamma \rrbracket}$
$\forall X. \phi$ with $\frac{\Psi \mid \phi}{\Gamma, X \mid \sigma}$	d -cart. simple prod. for $\hat{t}: \text{Prop} \rightarrow \text{PropKind}_b$ for the projection $\pi: \llbracket \Gamma \rrbracket \times \Omega \rightarrow \llbracket \Gamma \rrbracket \in \text{Kind}$
$\forall x: \sigma. \phi$ with $\frac{\Psi \mid \phi}{\Gamma \mid \gamma, \sigma}$	simple prod. for $p_{\llbracket \Psi \rrbracket}: \text{Prop}_{\llbracket \Psi \rrbracket} \rightarrow \text{Type}_{\llbracket \Gamma \rrbracket}$ for the projection $\pi: \llbracket \gamma \rrbracket \times \llbracket \sigma \rrbracket \rightarrow \llbracket \gamma \rrbracket$
$\forall R \subset \sigma \times \tau. \phi$ with $\frac{\Psi, R \subset \sigma \times \tau \mid \phi}{\Gamma \mid \sigma}$	simple prod. for $t_{\llbracket \sigma \rrbracket}: \text{Prop}_{\llbracket \sigma \rrbracket} \rightarrow \text{PropKind}_{\llbracket \Gamma \rrbracket}$ for the projection $\pi: \llbracket \Psi \rrbracket \times P_{\llbracket \sigma \times \tau \rrbracket} \rightarrow \llbracket \Psi \rrbracket$

Fibred generic objects

Proposition: Given a fibration in \mathcal{Fib}

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{t} & \mathbf{P} \\ p \downarrow & & \downarrow q \\ \mathbf{T} & \xrightarrow{b} & \mathbf{K} \end{array}$$

where the base fibration $b : \mathbf{T} \rightarrow \mathbf{K}$ has a generic object \mathbf{T} over Ω . Then:

- (i) (p, q) has *fibred generic objects*, i.e. every fibre fibration $p_\sigma : \mathbf{L}_\sigma \rightarrow \mathbf{P}_{b\sigma}$ has a generic object \mathbf{L}_σ (over $G_{b\sigma}$), and the pseudo-morphisms $(u^*p, u^*q) : p_\sigma \rightarrow p_\tau$ in \mathcal{Fib} (induced by morphisms $u : \tau \rightarrow \sigma$ in \mathbf{T}) preserve them.
- (ii) The fibration $p : \mathbf{L} \rightarrow \mathbf{P}$ has a generic object, which is preserved by (p, q) . □

The object $\mathbf{L}_{\mathbf{T}}$ over G_Ω is generic for t .

Example: Since the total and base fibrations of the externalisation

$$\begin{array}{ccc} \underline{[f]} & \xrightarrow{\text{dom}[\underline{f}]} & \mathbf{B} \\ \text{[cod]} \downarrow & & \downarrow \text{cod} \\ \underline{[D]} & \xrightarrow{\text{dom}_D} & \mathbf{B} \end{array}$$

have generic objects, the fibration $([\text{cod}], \text{cod})$ has fibred generic objects: given $x : X \rightarrow D_0$, the fibre fibration $(\text{dom}[\underline{f}])_x : \underline{[f]}_x \rightarrow \mathbf{B}/X$ has a generic object given by the pullback square

$$\begin{array}{ccc} (E_0)_x & \xrightarrow{\overline{x}} & E_0 \\ (f_0)_x \downarrow & & \downarrow f_0 \\ X & \xrightarrow{x} & D_0 \end{array}$$

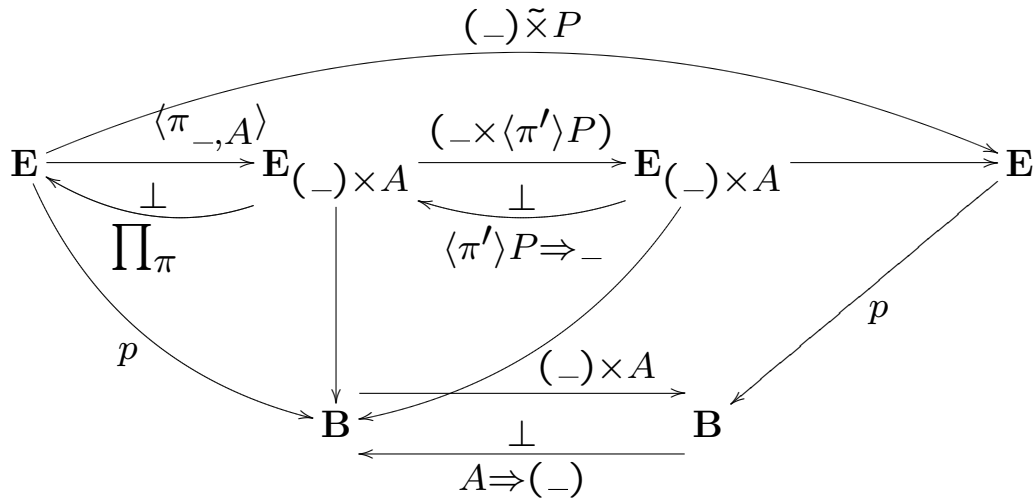
over $(f_0)_x : (E_0)_x \rightarrow X \in \mathbf{B}/X$.

Lifting cartesian-closure

Proposition:[?]

Given $p : \mathbf{E} \rightarrow \mathbf{B}$ which is a fibred-ccc with simple products, if \mathbf{B} is a ccc then \mathbf{E} is a ccc and p strictly preserves the cartesian closed structure.

Proof:



\therefore Logical predicate formula for exponentials:
 $P \subset A, Q \subset B$

$$[P \Rightarrow Q](f : A \Rightarrow B) \equiv \underbrace{\forall a \in A.}_{\Pi_\pi} \underbrace{P(a)}_{\langle \pi' \rangle P} \Rightarrow \underbrace{Q(f(a))}_{ev^* Q}$$

□

Products in a composite fibration

Proposition: Consider the composite of fibrations

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{t} & \mathbf{D} \\ & \searrow q & \downarrow b \\ & & \mathbf{B} \end{array}$$

with an exact* 2-cell $\pi : d \Rightarrow c : \Delta_\pi \rightarrow \mathbf{B}$, and its b -cartesian lifting, $\bar{\pi} : \bar{d} \langle \pi \rangle_b \Rightarrow \bar{c} : \mathbf{D}_c \rightarrow \mathbf{D}$. Assuming b has π -products, then

t admits $\bar{\pi}$ -products (over \mathbf{B}) iff q admits π -products and t preserves them.

Proof: By lifting and factorisation of adjunc-

*cartesian natural transformation, e.g. projections

tions in \mathcal{Fib} :

$$\begin{array}{ccccc}
 & & \langle \pi \rangle_q & & \\
 & \curvearrowright & & \curvearrowright & \\
 \mathbf{E}_c & \xrightarrow{\langle \bar{\pi} \rangle_t} & \mathbf{E}_{\bar{d}} & \xrightarrow{\langle \pi \rangle_b} & \mathbf{E}_d \\
 & \downarrow \perp & & & \downarrow t_{\bar{d}} \\
 & \Pi_{\bar{\pi}}^t & & & \\
 & \curvearrowleft & & \curvearrowleft & \\
 & & \mathbf{D}_c & \xrightarrow{\langle \pi \rangle_b} & \mathbf{D}_d \\
 & & & \downarrow \perp & \\
 & & & \Pi_{\pi}^b &
 \end{array}$$

□

Proposition: Consider a fibration $b : \mathbf{D} \rightarrow \mathbf{B}$ and (exact) 2-cells $\pi : d' \Rightarrow c'$ (into \mathbf{D}) and $\kappa : d \Rightarrow c$ (into \mathbf{B}) with $b\pi = \kappa$, and let $\hat{\pi}$ be the vertical factor of π through the cartesian lifting of κ . Let $t : \mathbf{E} \rightarrow \mathbf{D}$ be a fibration and assume that:

1. Both the composite fibration $q = bt : \mathbf{E} \rightarrow \mathbf{B}$ and b admit κ -products, and t preserves them.
2. The change-of-base fibration $t_{\bar{d}} : \mathbf{E}_{d_b} \rightarrow \mathbf{D}_d$ admits $\hat{\pi}$ -products (over \mathbf{B}).

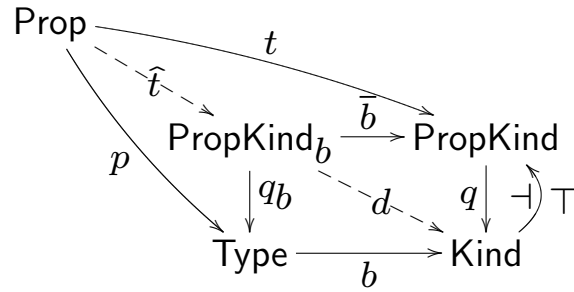
Then, the fibration t admits π -products (over B).

Proof: (??) is equivalent to t having $\overline{\kappa}^b$ -products, so that π -products are obtained by composition of the vertical ones, $\hat{\pi}$, and the cartesian ones $\overline{\kappa}^b$ □

Theorem: For a relational $\lambda 2$ -fibration $(p, q) : t \rightarrow b$:

- (i) $t : \text{Prop} \rightarrow \text{PropKind}$ is a $\lambda 2$ -fibration
- (ii) $(p, q) : t \rightarrow b$ is a morphism of $\lambda 2$ -fibrations.

Proof:



$b \text{ fibred-ccc}$	$\xRightarrow{\text{ch-o-base}}$	$\bar{b} \text{ fibred-ccc}$
$b \text{ simp. prod., } q \text{ pres. } \times$	$\xRightarrow{\text{ch-o-base}}$	$\bar{b} \text{ simp. prod.}$
$q \dashv \top, b \text{ has g. o.}$	$\xRightarrow{\text{ch-o-base}}$	$\bar{b} \text{ has g. o.}$
$\text{and } (\epsilon :)_q \top = id$		$(q_b, q) \text{ pres. g. o.}$

$\therefore (q_b, q) : \bar{b} \rightarrow b$ is a morphism of $\lambda 2$ -fibrations.

So, we are reduced to proving that t is a $\lambda 2$ -fibration and $\hat{t} : t \rightarrow b_q$ is a morphism of such (over the same base).

- a generic object for t is induced by the fibred generic objects of its fibres.

- t is a fibred-ccc since the fibre fibrations $p_\Psi : \text{Prop}_\Psi \rightarrow \text{Type}_{q\Psi}$ admit the ‘lifting’ of cartesian closed structure.

- Simple products for t are induced by $\overline{\pi}^{b_q}$ -products for \hat{t} via our previous lifting result. These latter are obtained by composition of $\overline{\pi}^d$ -products and simple products for the fibres of t : for $\tilde{\pi} = \overline{\pi}^q \hat{\pi} : \Psi \rightarrow \Phi$ in PropKind and $\sigma \in (\text{PropKind}_b)_\Phi$

$$\begin{array}{ccccc}
 & & \langle \overline{\pi}^{b_q} \rangle_{\hat{t}} & & \\
 & \nearrow & & \searrow & \\
 P_\sigma & \xrightarrow{\langle \overline{\pi}^d \rangle} & P_{\langle \overline{\pi}^d \rangle \sigma} & \xrightarrow{\langle \hat{\pi}^{b_q} \rangle} & P_{\langle \hat{\pi} \rangle_{b_q} \sigma} \\
 & \nwarrow \perp & \nwarrow \perp & & \\
 & \Pi_{\overline{\pi}^d}^{\hat{t}} & & \Pi_{\hat{\pi}}^t &
 \end{array}$$

with $\overline{\pi}^{b_q} = \overline{\pi}^d$ by the nature of liftings in a composite fibration and $\langle \hat{\pi}^{b_q} \rangle_{\hat{t}} = \langle \hat{\pi} \rangle_t$ by definition of \hat{t} as a factor into a pullback. \square

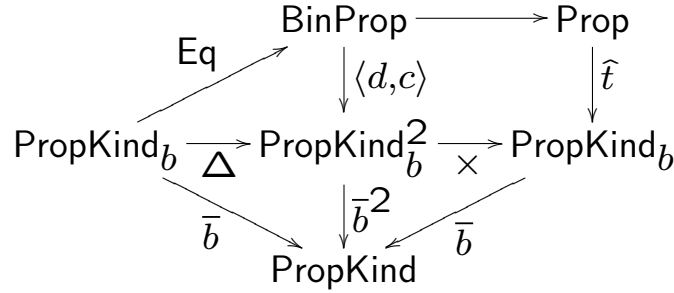
\therefore Logical predicate formula for simple products:

Given $\frac{\Psi, P \subset X \mid \phi}{\kappa, X \mid \tau}$ we get $\frac{\Psi \mid \tilde{\Pi}(\lambda P \subset X. \phi)}{\kappa \mid \prod X. \tau}$
with

$$[\tilde{\Pi}(\lambda P \subset X. \phi)](f : \prod X. \tau) \equiv \underbrace{\forall X.}_{\bar{\pi}^d - prod} \underbrace{\forall P \subset X.}_{\hat{\pi} - prod} \underbrace{\phi(f[X])}_{\langle \epsilon_\sigma \rangle_p \phi}$$

where the counit $\epsilon_\sigma : \pi^* \Pi(\sigma) \rightarrow \sigma$ yields the *generic instance* of a polymorphic function.

Structural Equality

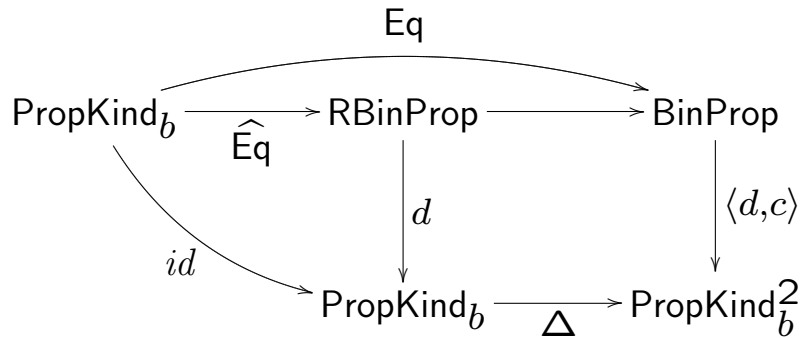


\hat{t} has fibred equality (over PropKind), so that for every $\psi \in \text{PropKind}$

$$\text{Eq}_\psi : \text{PropKind}_b \psi \rightarrow \text{BinProp}_\psi$$

$$\left(\frac{\psi \mid \tau}{\kappa} \right) \mapsto \Sigma_{\delta_\tau} (\top_\tau)$$

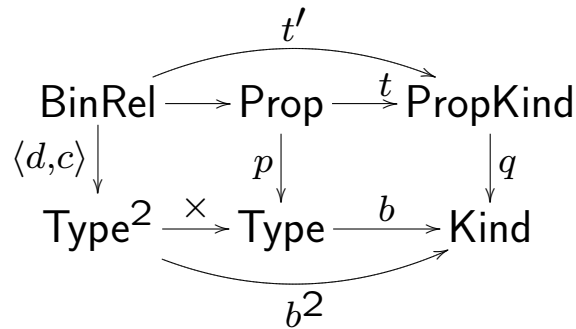
endows the fibration $\hat{t}_\psi : \text{Prop}_\psi \rightarrow (\text{PropKind}_b)_\psi$ with a Lawvere equality.



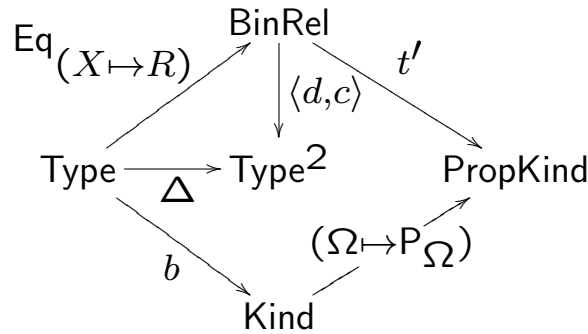
- $\hat{\text{Eq}}_\psi : \text{PropKind}_b \psi \rightarrow \text{RBinProp}_\psi$ preserves ccc (\equiv pointwise equality of functions)
- $\hat{\text{Eq}} : \bar{b} \rightarrow \bar{b}d$ preserves simple products (\equiv pointwise equality of polymorphic functions)

Parametricity schema

Set-up the fibration of **binary relations**:



Postulate the existence of the following ‘reflexive graph’ in $\lambda 2\text{-Fib}/\text{PropKind}$:



Relational Parametricity principle:

There exists a morphism of $\lambda 2$ -fibrations $(\text{Eq}_{(X \mapsto R)}, (\Omega \mapsto P_\Omega)) : b \rightarrow t$ such that $\langle d, c \rangle \text{Eq}_{(X \mapsto R)} = \Delta$ and which extends equality on closed types: $[\text{Eq}_{(X \mapsto R)}]_1 = \text{Eq}_1 : \text{Type}_1 \rightarrow \text{BinRel}_1$

Lifting locally cartesian closure

Proposition: Consider a fibration $b : \mathbf{D} \rightarrow \mathbf{B}$ and (exact) 2-cells $\pi : d' \Rightarrow c'$ (into \mathbf{D}) and $\kappa : d \Rightarrow c$ (into \mathbf{B}) with $b\pi = \kappa$, and let $\hat{\pi}$ be the vertical factor of π through the cartesian lifting of κ . Let $t : \mathbf{E} \rightarrow \mathbf{D}$ be a fibration and assume that:

1. t has $\overline{\kappa}^b$ -products
2. The change-of-base fibration $t_{\overline{d}} : \mathbf{E}_{d_b} \rightarrow \mathbf{D}_d$ admits $\hat{\pi}$ -products (over \mathbf{B}).

Then, the fibration t admits π -products (over \mathbf{B}).

□

Theorem:

1. If \mathbf{B} is lccc, so is \mathbf{B}^{\rightarrow} and $cod : \mathbf{B}^{\rightarrow} \rightarrow \mathbf{B}$ preserves the structure.
2. Let \mathbf{B} be an lccc and let RG be the generic **reflexive graph**.
 - (a) the fibration $cod : \mathbf{B}^{RG} \rightarrow \mathbf{B}^2$ has products
 - (b) \mathbf{B}^{RG} is lccc and $cod : \mathbf{B}^{RG} \rightarrow \mathbf{B}^2$ preserves the structure.

Proof

(??) We have the following fibration in \mathcal{Fib} :

$$\begin{array}{ccc} (\mathbf{B}^{\rightarrow})^{\rightarrow} & \xrightarrow{cod} & \mathbf{B}^{\rightarrow} \\ \downarrow cod^{\rightarrow} & & \downarrow cod \\ \mathbf{B}^{\rightarrow} & \xrightarrow{cod} & \mathbf{B} \end{array}$$

By the previous proposition, we get products for cod^{\rightarrow} by composition of products over vertical and cartesian morphisms:

vertical products Given $f : X \rightarrow Y$, the *vertical fibre fibration* $t_f : \mathbf{B}^{\rightarrow}/f \rightarrow \mathbf{B}/X$ has the following action: given a morphism $h : x \rightarrow y$ in \mathbf{B}/X

$$\begin{array}{ccc} (\mathbf{B}^{\rightarrow}/f)_y & \xrightarrow{h^*} & (\mathbf{B}^{\rightarrow}/f)_x \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{B}/(\text{dom}(y \times f)) & \xrightarrow{(h \times f)^*} & \mathbf{B}/(\text{dom}(x \times f)) \\ & \nwarrow \perp & \\ & \prod_{h \times f} & \end{array}$$

cartesian products Cartesian morphisms for cod are pullback squares and thus we apply products in \mathbf{B} , which satisfy Beck-Chevalley.

(??) Product functors for \mathbf{B} take reflexive graphs to reflexive graphs. \square

C. Hermida. Some properties of **fib** as a fibred 2-category. *Journal of Pure and Applied Algebra*, 134(1):83–109, 1999. Presented at ECCT'94, Tours, France.

Claudio Hermida. Fibrations for abstract multicategories. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 281–293. Amer. Math. Soc., Providence, RI, 2004.

B. Jacobs. *Categorical logic and type theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics*. North Holland, 1999.

Q. Ma and J. C. Reynolds. Types, abstraction and parametric polymorphism 2. In *Math. Found. of Prog. Lang. Sem.*, Lecture Notes in Computer Science. Springer Verlag, 1991.

Peter W. O'Hearn and John C. Reynolds. From Algol to polymorphic linear lambda-calculus. *J. ACM*, 47(1):167–223, 2000.

G. Plotkin and M. Abadi. A logic for parametric polymorphism. In M. Bezen and J. F. Groote, editors, *Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*, pages 361–375, Utrecht, The Netherlands, March 1993. Springer-Verlag, Berlin.

E. Robinson and G Rosolini. Reflexive graphs and parametric polymorphism. In *Proceedings, Ninth Annual IEEE Symposium on Logic in Computer Science*, pages 364–371. IEEE Computer Society Press, 1994.