Fibrations and Logical Relations

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Polymorphic Lambda Calculus

$$\tau ::= X \mid \kappa \mid \tau \times \tau' \mid \tau \Rightarrow \tau' \mid \forall X\,.\tau$$

Logical Relations: type constructors acting on relations (over polymorphic types):

$$(R \tilde{\Rightarrow} Q)_{f,g:\sigma\to\tau} \equiv \forall x, y : \sigma\,.R(x,y) \Longrightarrow Q(fx, gy)$$

$$(\tilde{\forall} R.\Phi)_{F,G:\forall X\,.\tau} \equiv$$
$$\forall Y, Y' : \mathsf{Type}.\,\forall S \subset Y \times Y'\,.\Phi[S/R](FY, GY')$$

Applications: Structural (co)induction principles, definability, parametricity.

What do these formulas mean?

[Ma-Reynolds] Let $\mathbf{C}$ be a ccc with pullbacks. Then

$$\begin{array}{ll} \mathsf{Sub}(\mathbf{C}) & \text{ccc} \\ {\scriptstyle cod}\big\downarrow & \text{preserves ccc} \\ \mathbf{C} & \end{array}$$

• The exponentials in $\mathsf{Sub}(\mathbf{C})$ agree with the 'logical exponential' formula when $\mathbf{C} = \mathcal{S}et$.

• ad-hoc generalisation to 'PL-categories'.

$\Longrightarrow$ seek:

– a *unified treatment* of 'logical relations' with a *precise correspondence syntax*$\leftrightarrow$*categorical structure*.

<u>'Corollary'</u>: (sound and complete) categorical semantics for Plotkin-Abadi's 'logic for parametricity'.

Fibred category theory:
study of structures varying over a base category

<u>Examples:</u>
$-$Category $\mathbf{C} \longrightarrow$ fibration $dom_C : [\mathbf{C}] \to \mathcal{S}et$ of families: $[\mathbf{C}]_I = \mathcal{C}at(I, \mathbf{C})$, with reindexing given by precomposition:

$$u : J \to I \rightsquigarrow (\{c_i\}_{i \in I} \mapsto \{c_{uj}\}_{j \in J})$$

$-$ For $\mathbf{C} = \mathcal{S}et$, the above externalisation has a more compact description:

$$\begin{tikzcd} {[\mathcal{S}et]} \arrow[rr, "\cong"] \arrow[dr, "dom"'] & & \mathcal{S}et^{\to} \arrow[dl, "cod"] \\ & \mathcal{S}et & \end{tikzcd}$$

whereby we regard a family of sets $\{X_i\}_{i \in I}$ à la Bourbaki $\sum_{i \in I} X_i \to I$, and hence reindexing is realized by pullback

$$\begin{tikzcd} \{X_{uj}\}_{j \in J} \arrow[r, "\overline{u}"] \arrow[d, "\pi_u"'] & \{X_i\}_{i \in I} \arrow[d, "\pi"] \\ J \arrow[r, "u"'] & I \end{tikzcd}$$

– For $\mathbf{C} = \mathbf{2} = (0 \to 1)$, its externalisation yields the fibration of subobjects:

$$\begin{tikzcd}[\mathbf{2}] \arrow[rr, "\cong"] \arrow[dr, "dom_{\mathbf{2}}"'] & & \mathsf{Sub}(\mathcal{S}et) \arrow[dl, "cod"] \\ & \mathcal{S}et & \end{tikzcd}$$

which amounts to the characterisation of $\mathbf{2}$ as a *subobject classifier* (this makes sense for any topos instead of $\mathcal{S}et$).

Fibrations in a 2-category

<u>Definition</u>: A morphism $p : E \to B$ in a 2-category $\mathcal{K}$ is a *fibration* if for every object X, the functor $\mathcal{K}(X, p) : \mathcal{K}(X, E) \to \mathcal{K}(X, B)$ is a fibration of categories.

<u>$\mathcal{F}ib(\mathcal{K})$:</u>

$$
\begin{tikzcd}
\mathbf{E} \arrow[r, bend left, "\widetilde{F}"] \arrow[r, bend right, "\widetilde{G}"'] \arrow[r, phantom, "\Downarrow\widetilde{\alpha}"] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & & \mathbf{D}_F \arrow[rrd, "\overline{F}"] \arrow[dddl, bend left, "q_F"] \arrow[rd, phantom, "\Swarrow\overline{\alpha}"] & & \\
\mathbf{E} \arrow[urrr, "\widehat{F}"] \arrow[drr, "\widehat{G}"'] \arrow[ddrr, "p"'] \arrow[r, phantom, "\Swarrow\widehat{\alpha}"] & {} & & & & \mathbf{D} \arrow[dd, "q"] \\
 & & \mathbf{D}_G \arrow[uur, "\langle\alpha\rangle"] \arrow[urrr, "\overline{G}"'] \arrow[d, "q_G"] & & & \\
 & & \mathbf{B} \arrow[rrr, bend left, "F"] \arrow[rrr, bend right, "G"'] \arrow[rrr, phantom, "\Downarrow\alpha"] & & & \mathbf{C}
\end{tikzcd}
$$

Structural decomposition of adjunctions between fibrations

Theorem:[**?**, Thm 4.3]
Consider the following data:

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "\tilde{F}"] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, "F"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{A} \arrow[l, bend left, "G"]
\end{tikzcd}$$

and let $\widehat{F} : p \to F^*(q)$ in $\mathcal{F}ib/\mathbf{B}$ be the unique mediating morphism into the pullback:

$$\begin{tikzcd}
\mathbf{E} \arrow[dr, dashed, "\widehat{F}"] \arrow[drr, bend left, "\tilde{F}"] \arrow[ddr, bend right, "p"'] & & \\
& \mathbf{D}_F \arrow[r, "\overline{F}"] \arrow[d, "q_F"] & \mathbf{D} \arrow[d, "q"] \\
& \mathbf{B} \arrow[r, "F"'] & \mathbf{A}
\end{tikzcd}$$

Then, the following are equivalent:
Lifting $\exists \tilde{G} : \mathbf{D} \to \mathbf{E}. \tilde{F} \dashv \tilde{G}$ (in $\mathcal{C}at$) such that $(\tilde{F}, F) \dashv (\tilde{G}, G) : q \to p$ (in $\mathcal{F}ib$).
Factorisation $\exists \widehat{G} : q_F \to p. \widehat{F} \dashv \widehat{G}$ (in $\mathcal{F}ib/\mathbf{B}$).

Lifting:

$$
\begin{tikzcd}
 & \mathbf{D}_{FG} \arrow[r] \arrow[dd, "q_{FG}"] \arrow[dr, phantom, "\Downarrow\overline{\epsilon}"] & \mathbf{D}_F \arrow[r, "Fq"] & \mathbf{D} \arrow[dd, "q"] \\
\mathbf{D} \arrow[ur, "\langle\epsilon\rangle"] \arrow[urr, bend left, "\underline{G}"] \arrow[urrr, bend right=10, "id"'] \arrow[dr, "q"'] & & & \\
 & \mathbf{A} \arrow[rr, bend left=10, "FG"] \arrow[rr, bend right=10, "id"'] \arrow[rr, phantom, "\Downarrow\epsilon"] & & \mathbf{A}
\end{tikzcd}
$$

Fibrations over a base category

Consider the 2-category $\mathcal{F}ib/\mathbf{B}$ of fibrations over a base category $\mathbf{B}$. Recall that

$$\mathcal{F}ib/\mathbf{B} \simeq \mathsf{Ps}[\mathbf{B}^{op}, \mathcal{C}at]$$

<u>**Theorem**[Bénabou]:</u> Given a morphism in $\mathcal{F}ib/\mathbf{B}$

$$\begin{tikzcd} \mathbf{E} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathbf{D} \arrow[dl, "q"] \\ & \mathbf{B} & \end{tikzcd}$$

the following are equivalent:

- F is a fibration in $\mathcal{F}ib/\mathbf{B}$.

- The associated pseudo-natural transformation $\mathcal{F}_F : \mathcal{F}_p \Rightarrow \mathcal{F}_q$ takes values in $\mathcal{F}ib$, *i.e.* every fibre functor is a fibration and the functors between fibres (for $\mathcal{F}_p$) preserve cartesiannes.

- The functor $F : \mathbf{E} \to \mathbf{D}$ is a fibration of categories.

<u>Example</u>: Given an internal fibration $f : \underline{\mathbf{E}} \to \underline{\mathbf{D}}$ in a category $\mathbf{B}$, its externalisation

$$\begin{tikzcd} {[\underline{\mathbf{E}}]} \arrow[rr, "{[f]}"] \arrow[dr, "dom_E"'] & & {[\underline{\mathbf{D}}]} \arrow[dl, "dom_E"] \\ & \mathbf{B} & \end{tikzcd}$$

is a fibration in $\mathcal{F}ib/\mathbf{B}$. Notice that $[f]$: $(x : X \to E_0) \mapsto (f_0 x : X \to D_0)$ *does not preserve generic objects*.

Concretely, take

$\mathbf{B} = \omega - \mathcal{S}et$

$\underline{\mathbf{D}} = \underline{\mathsf{PER}}$, the category of PER's internally in $\omega - \mathcal{S}et$.

$\underline{\mathbf{E}} = \underline{\mathsf{RegSub(PER)}}$, the internal category of regular subobjects of PER's.

$f = cod$, the internal functor taking a subobject to its codomain.

Fibrations in $\mathcal{F}ib$

<u>Proposition</u>: A morphism $(\tilde{F}, F) : p \to q$

$$\begin{tikzcd} \mathbf{E} \arrow[r, "\tilde{F}"] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\ \mathbf{B} \arrow[r, "F"'] & \mathbf{A} \end{tikzcd}$$

in $\mathcal{F}ib$ is a fibration iff

- both $\tilde{F} : \mathbf{E} \to \mathbf{D}$ and $F : \mathbf{B} \to \mathbf{A}$ are fibrations, and

- the functor $p : \mathbf{E} \to \mathbf{B}$ takes $\tilde{F}$-cartesian morphisms to F-cartesian morphisms. $\square$

Notice the pleasant symmetry of the situation above: if $(\tilde{F}, F) : p \to q$ is a fibration in $\mathcal{F}ib$ so is $(p, q) : \tilde{F} \to F$.

Example: Given an internal fibration $f : \underline{\mathbf{E}} \to \underline{\mathbf{D}}$ in a category $\mathbf{B}$, we regard it as an internal category $\underline{f} \in \mathcal{C}at(\mathbf{B}^{\to})$, and we get the following externalisation:

$$\begin{tikzcd}
[\underline{f}] \arrow[r, "dom_{[\underline{f}]}"] \arrow[d, "{[cod]}"'] & \mathbf{B}^{\to} \arrow[d, "cod"] \\
[\underline{\mathbf{D}}] \arrow[r, "dom_D"'] & \mathbf{B}
\end{tikzcd}$$

which is a fibration in $\mathcal{F}ib$, where the induced functor $[cod]$ has action:

$$\begin{tikzcd}
F \arrow[r, "f"] \arrow[d, "u"'] & E_0 \arrow[d, "f_0"] \\
X \arrow[r, "x"'] & D_0
\end{tikzcd} \quad \mapsto \quad \begin{tikzcd}
X \arrow[r, "x"'] & D_0
\end{tikzcd}$$

<u>**Theorem**</u>: Consider the following data

$$\begin{tikzcd}
\tilde{\mathbf{P}} \arrow[dr, "\widehat{p}"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "\tilde{q}"'] & & \\
& \mathbf{P}_b \arrow[r, "\overline{b}"] \arrow[d, "q_b"] & \mathbf{P} \arrow[d, "q"] \\
& \tilde{\mathbf{B}} \arrow[r, "b"'] & \mathbf{B}
\end{tikzcd}$$

where both b and q are fibrations, and the square is a pullback. The following are equivalent:

• The functor $\tilde{q} : \tilde{\mathbf{P}} \to \tilde{\mathbf{B}}$ is a fibration and the morphism $(p, b) : \tilde{q} \to q$ is a fibration in $\mathcal{F}ib$.

• The functor $\tilde{q} : \tilde{\mathbf{P}} \to \tilde{\mathbf{B}}$ is a fibration and the morphism $\widehat{p} : \tilde{\mathbf{P}} \to \mathbf{P}_b$ is a fibration in $\mathcal{F}ib/\tilde{\mathbf{B}}$.

• The functor $\widehat{p} : \tilde{\mathbf{P}} \to \mathbf{P}_b$ is a fibration.

• The functor $p : \tilde{\mathbf{P}} \to \mathbf{B}$ is a fibration and the morphism $(\tilde{q}, q) : p \to b$ is a fibration in $\mathcal{F}ib$.

• The functor $\tilde{q} : \tilde{\mathbf{P}} \to \tilde{\mathbf{B}}$ is a fibration and the morphism $\widehat{p} : \tilde{\mathbf{P}} \to \mathbf{P}_b$ is a fibration in $\mathcal{F}ib/\mathbf{P}$.

$\square$

Example: Given an internal fibration $f : \underline{\mathbf{E}} \to \underline{\mathbf{D}}$ in a category $\mathbf{B}$, we regard it as an internal category $\underline{f} \in \mathcal{C}at(\mathbf{B}^{\to})$, and we get the following externalisation:

$$\begin{tikzcd}
[\underline{f}] \arrow[r, "dom_{[\underline{f}]}"] \arrow[d, "{[cod]}"'] & \mathbf{B}^{\to} \arrow[d, "cod"] \\
[\underline{\mathbf{D}}] \arrow[r, "dom_D"'] & \mathbf{B}
\end{tikzcd}$$

which is a fibration in $\mathcal{F}ib$, where the induced functor $[cod]$ has action:

$$\begin{tikzcd}
F \arrow[r, "f"] \arrow[d, "u"'] & {}_0 \arrow[d, "f_0"] & \mapsto & \\
X \arrow[r, "x"'] & D_0 & & X \arrow[r, "x"'] & D_0
\end{tikzcd}$$

Notice that the morphism of fibrations $([\underline{f}], cod)$ preserves the standard generic objects.

Fibrations for polymorphic lambda-calculus

<u>Definition</u>: A $\lambda 2$-fibration is a fibred-ccc with simple (Ω-)products and a generic object (there is a distinguished object T over Ω such that for every object X there is a cartesian morphism $X \to T$).

Simple Products:

Consider the *diagonal morphism* $\langle \pi \rangle : b \to b^{[\Omega]}$:

$$
\begin{tikzcd}
& \mathbf{T}^{[\Omega]} \arrow[r] \arrow[dd, "b_{\_\times\Omega}"] & \mathbf{T} \arrow[dd, "b"] \\
\mathbf{T} \arrow[ur, "\langle\pi\rangle"] \arrow[urr, bend right, "id"'] \arrow[urr, phantom, "\Downarrow\overline{\pi}"] \arrow[dr, "b"'] & & \\
& \mathbf{K} \arrow[r, bend left, "\_\times\Omega"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow\pi"] & \mathbf{K}
\end{tikzcd}
$$

and demand a right adjoint (over $\mathbf{K}$):

$$
\begin{tikzcd}
\mathbf{T} \arrow[rr, bend left, "\langle\pi\rangle"] \arrow[rr, phantom, "\perp"] \arrow[dr, "b"'] & & \mathbf{T}^{[\Omega]} \arrow[ll, bend left, "\sqcap^{b}_{\Omega}"] \arrow[dl, "b_{\_\times\Omega}"] \\
& \mathbf{K} &
\end{tikzcd}
$$

The fibred nature of the adjunction encompasses both the existence of right adjoints to π_Ω^* and the Beck-Chevalley condition of pullback stability.

Classical example: The fibration obtained by externalisation of PER, the category of PERs on the natural numbers regarded as an internal category in $\omega - \mathcal{S}et$. This fibration is equivalently described as $u : \mathsf{UFam}(\mathrm{PER}) \to \omega - \mathcal{S}et$, where the fiber over an ω-set (I, E) has as objects collection of PERs $\{R_i\}_{i \in I}$. Its morphisms are I-families of PER maps *with a common realizer* (uniform realizability *cf.*[**?**]).

Fibrations for relational polymorphism

$$\begin{tikzcd}
\mathsf{Prop} \arrow[drr, bend left, "t"] \arrow[dr, dashed, "\widehat{t}"] \arrow[ddr, bend right, "p"'] & & \\
& \mathsf{PropKind}_b \arrow[r, "\overline{b}"] \arrow[d, "q_b"] \arrow[dr, dashed, "d"] & \mathsf{PropKind} \arrow[d, "q"] \\
& \mathsf{Type} \arrow[r, "b"'] & \mathsf{Kind}
\end{tikzcd}$$

A **relational $\lambda 2$-fibration** is a fibration in $\mathcal{F}ib$ with the following:

– the base b : Type $\to$ Kind is a $\lambda 2$-fibration,

– q : PropKind $\to$ Kind has (fibred) finite products,

– p : Prop $\to$ Type, *as a fibration over* q, is a fibred-ccc with simple products ($\top, \wedge, \Rightarrow, \forall x : \sigma. \_$),

– t : Prop $\to$ PropKind, *as a fibration over* b, has fibred generic objects $\{P_\tau\}_{\tau \in \mathsf{Type}}$ and fibred simple P_τ products ($\forall \rho \subset \tau. \_$), and

– $\widehat{t}$: Prop $\to$ PropKind$_b$ has $\overline{\pi}^d$-products, that is, cartesian simple products with respect to the diagonal fibration d : PropKind$_b \to$ Kind ($\forall X. \_$)

Interpretation of Syntax

$$\frac{\Psi \ \text{Predicate\_Kind} \;\Big|\; \phi \ \text{Predicate}}{\kappa \ \text{Kind} \;\Big|\; \tau \ \text{Type}}$$

- κ is an object of Kind,
- Ψ is an object of PropKind_κ,
- τ is an object of Type_κ, and
- ϕ is an object of $\text{Prop}_{\left(\frac{\Psi \mid \tau}{\kappa}\right)}$, where $\left(\frac{\Psi \mid \tau}{\kappa}\right)$ represents the evident object in the pullback $\text{PropKind}_b = \text{Type} \times_{\text{Kind}} \text{PropKind}$.

Syntax	Categorical Interpretation
$\underbrace{X, Y, \ldots}_{\Gamma}$	$[\![\Gamma]\!] = \Omega \times \Omega \ldots \in$ Kind
$\Gamma \mid \underbrace{x\colon \sigma, y\colon \tau, \ldots}_{\gamma}$	$[\![\gamma]\!] = [\![\sigma]\!] \times [\![\tau]\!] \ldots \in$ $\mathsf{Type}_{[\![\Gamma]\!]}$
$\Gamma \mid \underbrace{R{\subset}\sigma{\times}\tau, S{\subset}\mu{\times}\nu}_{\Psi}$	$[\![\Psi]\!] = \mathsf{P}_{[\![\sigma\times\tau]\!]} \times \mathsf{P}_{[\![\mu\times\nu]\!]} \ldots \in$ $\mathsf{PropKind}_{[\![\Gamma]\!]}$
$\forall X.\phi$ with $\begin{array}{c\|c} \Psi & \phi \\ \hline \Gamma, X & \sigma \end{array}$	d-cart. simple prod. for $\hat{t}:$ Prop $\rightarrow$ $\mathsf{PropKind}_b$ for the projection $\pi : [\![\Gamma]\!] \times \Omega \rightarrow [\![\Gamma]\!] \in$ Kind
$\forall x\colon \sigma.\, \phi$ with $\begin{array}{c\|c} \Psi & \phi \\ \hline \Gamma & \gamma, \sigma \end{array}$	simple prod. for $p_{[\![\Psi]\!]}$: $\mathsf{Prop}_{[\![\Psi]\!]} \rightarrow$ $\mathsf{Type}_{[\![\Gamma]\!]}$ for the projection $\pi : [\![\gamma]\!] \times [\![\sigma]\!] \rightarrow [\![\gamma]\!]$
$\forall R \subset \sigma \times \tau.\, \phi$ with $\begin{array}{c\|c} \Psi, R \subset \sigma \times \tau & \phi \\ \hline \Gamma & \sigma \end{array}$	simple prod. for $t_{[\![\sigma]\!]}$: $\mathsf{Prop}_{[\![\sigma]\!]} \rightarrow$ $\mathsf{PropKind}_{[\![\Gamma]\!]}$ for the projection $\pi : [\![\Psi]\!] \times \mathsf{P}_{[\![\sigma\,\times\,\tau]\!]} \rightarrow [\![\Psi]\!]$

Fibred generic objects

Proposition: Given a fibration in $\mathcal{F}ib$

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{t} & \mathbf{P} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbf{T} & \xrightarrow[b]{} & \mathbf{K} \end{array}$$

where the base fibration $b : \mathbf{T} \to \mathbf{K}$ has a generic object T over Ω. Tfae:

(i) (p, q) has *fibred generic objects*, *i.e.* every fibre fibration $p_\sigma : \mathbf{L}_\sigma \to \mathbf{P}_{b\sigma}$ has a generic object L_σ (over $G_{b\sigma}$), and the pseudo-morphisms $(u^{*p}, u^{*q}) : p_\sigma \to p_\tau$ in $\mathcal{F}ib$ (induced by morphisms $u : \tau \to \sigma$ in $\mathbf{T}$) preserve them.

(ii) The fibration $p : \mathbf{L} \to \mathbf{P}$ has a generic object, which is preserved by (p, q). □

The object L_T over G_Ω is generic for t.

<u>Example</u>: Since the total and base fibrations of the externalisation

$$\begin{tikzcd} [\underline{f}] \arrow[r, "dom_{[\underline{f}]}"] \arrow[d, "{[cod]}"'] & \mathbf{B}^{\to} \arrow[d, "cod"] \\ [\underline{\mathbf{D}}] \arrow[r, "dom_D"'] & \mathbf{B} \end{tikzcd}$$

have generic objects, the fibration $([cod], cod)$ has fibred generic objects: given $x : X \to D_0$, the fibre fibration $(dom_{[\underline{f}]})_x : [\underline{f}]_x \to \mathbf{B}/X$ has a generic object given by the pullback square

$$\begin{tikzcd} (E_0)_x \arrow[r, "\overline{x}"] \arrow[d, "(f_0)_x"'] & E_0 \arrow[d, "f_0"] \\ X \arrow[r, "x"'] & D_0 \end{tikzcd}$$

over $(f_0)_x : (E_0)_x \to X \in \mathbf{B}/X$.

Lifting cartesian-closure

Proposition:[?]

Given $p : \mathbf{E} \to \mathbf{B}$ which is a fibred-ccc with simple products, if $\mathbf{B}$ is a ccc then $\mathbf{E}$ is a ccc and p strictly preserves the cartesian closed structure.

Proof:

$$
\begin{tikzcd}
\mathbf{E} \arrow[r, shift left, "\langle \pi_{\_,A} \rangle"] \arrow[r, phantom, "\perp"] \arrow[dr, bend right, "p"'] \arrow[rrr, bend left, "(\_)\tilde{\times}P"] & \mathbf{E}_{(\_)\times A} \arrow[l, shift left, bend left, "\prod_\pi"] \arrow[r, shift left, "(\_\times\langle\pi'\rangle P)"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathbf{E}_{(\_)\times A} \arrow[l, shift left, bend left, "\langle\pi'\rangle P\Rightarrow\_"] \arrow[r] \arrow[dl, bend left] & \mathbf{E} \arrow[dl, "p"] \\
& \mathbf{B} \arrow[r, shift left, "(\_)\times A"] \arrow[r, phantom, "\perp"] & \mathbf{B} \arrow[l, shift left, "A\Rightarrow(\_)"] &
\end{tikzcd}
$$

$\therefore$ *Logical predicate formula for exponentials*:
$P \subset A, Q \subset B$

$$[P \tilde{\Rightarrow} Q](f : A \Rightarrow B) \equiv \underbrace{\forall a \in A.}_{\prod_\pi} \underbrace{P(a)}_{\langle\pi'\rangle P} \Longrightarrow \underbrace{Q(f(a))}_{ev^* Q}$$

$\square$

Products in a composite fibration

Proposition: Consider the composite of fibrations

$$\begin{tikzcd} \mathbf{E} \arrow[r, "t"] \arrow[dr, "q"'] & \mathbf{D} \arrow[d, "b"] \\ & \mathbf{B} \end{tikzcd}$$

with an exact* 2-cell $\pi : d \Rightarrow c : \mathbf{\Delta}_\pi \to \mathbf{B}$, and its b-cartesian lifting, $\overline{\pi} : \overline{d}\langle\pi\rangle_b \Rightarrow \overline{c} : \mathbf{D}_c \to \mathbf{D}$. Assuming b has π-products, then

t admits $\overline{\pi}$-products (over $\mathbf{B}$) iff q admits π-products and t preserves them.

Proof: By lifting and factorisation of adjunc-

*cartesian natural transformation, *e.g.* projections

tions in $\mathcal{F}ib$:

$$\begin{tikzcd}
\mathbf{E}_c \arrow[r, shift left, "\langle\overline{\pi}\rangle_t"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left, "\langle\pi\rangle_q"] \arrow[dr, bend right, "t_{\overline{c}}"'] & \mathbf{E}_{\overline{d}\langle\pi\rangle_b} \arrow[l, shift left, bend left, "\prod^t_{\overline{\pi}}"] \arrow[r] \arrow[d] & \mathbf{E}_d \arrow[d, "t_{\overline{d}}"] \\
& \mathbf{D}_c \arrow[r, shift left, "\langle\pi\rangle_b"] \arrow[r, phantom, "\perp"] & \mathbf{D}_d \arrow[l, shift left, bend left, "\prod^b_{\pi}"]
\end{tikzcd}$$

$\square$

<u>**Proposition**</u>: Consider a fibration $b : \mathbf{D} \to \mathbf{B}$ and (exact) 2-cells $\pi : d' \Rightarrow c'$ (into $\mathbf{D}$) and $\kappa : d \Rightarrow c$ (into $\mathbf{B}$) with $b\pi = \kappa$, and let $\widehat{\pi}$ be the vertical factor of π through the cartesian lifting of κ. Let $t : \mathbf{E} \to \mathbf{D}$ be a fibration and assume that:

1. Both the composite fibration $q = bt : \mathbf{E} \to \mathbf{B}$ and b admit κ-products, and t preserves them.

2. The change-of-base fibration $t_{\overline{d}} : \mathbf{E}_{d_b} \to \mathbf{D}_d$ admits $\widehat{\pi}$-products (over $\mathbf{B}$).

Then, the fibration t admits π-products (over $\mathbf{B}$).

<u>*Proof*:</u> (**??**) is equivalent to t having $\overline{\kappa}^b$-products, so that π-products are obtained by composition of the vertical ones, $\widehat{\pi}$, and the cartesian ones $\overline{\kappa}^b$ $\square$

<u>Theorem</u>: For a relational $\lambda 2$-fibration $(p,q) : t \to b$:

$$\boxed{\begin{array}{l}\text{(i) } t : \text{ Prop} \to \text{ PropKind is a } \lambda 2\text{-fibration} \\ \text{(ii) } (p,q) : t \to b \text{ is a morphism of } \lambda 2\text{-fibrations.}\end{array}}$$

<u>*Proof*:</u>

$$\begin{tikzcd}
\text{Prop} \arrow[dr, dashed, "\widehat{t}"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "p"'] & & \\
& \text{PropKind}_b \arrow[r, "\overline{b}"] \arrow[d, "q_b"] \arrow[dr, dashed, "d"] & \text{PropKind} \arrow[d, "q"'] \arrow[d, phantom, shift left=3, "\dashv"] \\
& \text{Type} \arrow[r, "b"'] & \text{Kind} \arrow[u, bend right, "\top"']
\end{tikzcd}$$

$$\begin{array}{rcl}
b\text{fibred-ccc} & \overset{ch-o-base}{\Longrightarrow} & \overline{b} \text{ fibred-ccc} \\
b \text{ simp. prod.}, q \text{ pres. } \times & \overset{ch-o-base}{\Longrightarrow} & \overline{b} \text{ simp. prod.} \\
q \dashv \top, b \text{ has g. o.} & \overset{ch-o-base}{\Longrightarrow} & \overline{b} \text{ has g. o.} \\
\text{and } (\epsilon :) q\top = id & & (q_b, q) \text{ pres. g. o.}
\end{array}$$

$\therefore (q_b, q) : \overline{b} \to b$ is a morphism of $\lambda 2$-fibrations.

So, we are reduced to proving that t is a $\lambda 2$-fibration and $\widehat{t} : t \to b_q$ is a morphism of such (over the same base).

• a generic object for t is induced by the fibred generic objects of its fibres.

• t is a fibred-ccc since the fibre fibrations $p_\Psi :$ $\mathsf{Prop}_\Psi \to$ $\mathsf{Type}_{q\Psi}$ admit the 'lifting' of cartesian closed structure.

• Simple products for t are induced by $\overline{\pi}^{b_q}$-products for $\widehat{t}$ via our previous lifting result. These latter are obtained by composition of $\overline{\pi}^{d}$-products and simple products for the fibres of t: for $\tilde{\pi} = \overline{\pi}^{q}\widehat{\pi} : \Psi \to \Phi$ in $\mathsf{PropKind}$ and $\sigma \in ($ $\mathsf{PropKind}_b)_\Phi$

$$\begin{tikzcd}
\mathsf{P}\sigma \arrow[r, "\langle\overline{\pi}^{d}\rangle"] \arrow[rr, bend left, "\langle\overline{\tilde{\pi}}^{b_q}\rangle_{\widehat{t}}"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{P}_{\langle\overline{\pi}^{d}\rangle\sigma} \arrow[l, bend left, "\prod^{\widehat{t}}_{\overline{\pi}^{d}}"] \arrow[r, "\langle\overline{\widehat{\pi}}^{b_q}\rangle"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{P}_{\langle\tilde{\pi}\rangle_{b_q}\sigma} \arrow[l, bend left, "\prod^{t}_{\widehat{\pi}}"]
\end{tikzcd}$$

with $\overline{\overline{\pi}^{q}}^{b_q} = \overline{\pi}^{d}$ by the nature of liftings in a composite fibration and $\langle\overline{\widehat{\pi}}^{b_q}\rangle_{\widehat{t}} = \langle\widehat{\pi}\rangle_t$ by definition of $\widehat{t}$ as a factor into a pullback. $\square$

$\therefore$ *Logical predicate formula for simple products*:

Given $\dfrac{\Psi, P\subset X}{\kappa, X} \left| \dfrac{\phi}{\tau} \right.$ we get $\dfrac{\Psi}{\kappa} \left| \dfrac{\tilde{\prod}(\lambda P\subset X.\,\phi)}{\prod X.\,\tau} \right.$ with

$$[\tilde{\prod}(\lambda P \subset X.\,\phi)](f : \prod X.\,\tau) \equiv \underbrace{\forall X.}_{\overline{\pi}^d-prod} \underbrace{\forall P\subset X.}_{\widehat{\pi}-prod} \underbrace{\phi(f[X])}_{\langle\epsilon_\sigma\rangle_p\,\phi}$$

where the counit $\epsilon_\sigma : \pi^* \prod(\sigma) \to \sigma$ yields the *generic instance* of a polymorphic function.

Structural Equality

$$\begin{tikzcd}
 & \mathsf{BinProp} \arrow[r] \arrow[d, "\langle d{,}c\rangle"] & \mathsf{Prop} \arrow[d, "\hat{t}"] \\
\mathsf{PropKind}_b \arrow[ur, "\mathsf{Eq}"] \arrow[r, "\Delta"'] \arrow[dr, "\bar{b}"'] & \mathsf{PropKind}_b^2 \arrow[r, "\times"'] \arrow[d, "\bar{b}^2"] & \mathsf{PropKind}_b \arrow[dl, "\bar{b}"] \\
 & \mathsf{PropKind} &
\end{tikzcd}$$

$\hat{t}$ has fibred equality (over PropKind), so that for every $\Psi \in \mathsf{PropKind}$

$$\begin{array}{rcl}
\mathsf{Eq}_\Psi \;:\; \mathsf{PropKind}_{b\Psi} & \to & \mathsf{BinProp}_\Psi \\
\left(\dfrac{\Psi \mid \tau}{\kappa}\right) & \mapsto & \Sigma_{\delta_\tau}(\top_\tau)
\end{array}$$

endows the fibration $\hat{t}_\Psi : \mathsf{Prop}_\Psi \to (\mathsf{PropKind}_b)_\Psi$ with a Lawvere equality.

$$\begin{tikzcd}
\mathsf{PropKind}_b \arrow[r, "\widehat{\mathsf{Eq}}"'] \arrow[rr, bend left, "\mathsf{Eq}"] \arrow[dr, bend right, "id"'] & \mathsf{RBinProp} \arrow[r] \arrow[d, "d"] & \mathsf{BinProp} \arrow[d, "\langle d{,}c\rangle"] \\
 & \mathsf{PropKind}_b \arrow[r, "\Delta"'] & \mathsf{PropKind}_b^2
\end{tikzcd}$$

- $\widehat{\mathsf{Eq}}_\Psi : \mathsf{PropKind}_{b\Psi} \to \mathsf{RBinProp}_\Psi$ preserves ccc ($\equiv$ *pointwise equality of functions*)

- $\widehat{\mathsf{Eq}} : \bar{b} \to \bar{b}d$ preserves simple products ($\equiv$ *pointwise equality of polymorphic functions*)

Parametricity schema

Set-up the fibration of **binary relations**:

$$\begin{tikzcd}
\mathsf{BinRel} \arrow[r] \arrow[d, "\langle d,c\rangle"'] \arrow[rr, bend left, "t'"] & \mathsf{Prop} \arrow[r, "t"] \arrow[d, "p"'] & \mathsf{PropKind} \arrow[d, "q"] \\
\mathsf{Type}^2 \arrow[r, "\times"] \arrow[rr, bend right, "b^2"'] & \mathsf{Type} \arrow[r, "b"] & \mathsf{Kind}
\end{tikzcd}$$

Postulate the existence of the following 're-flexive graph' in $\lambda 2{-}\mathcal{F}ib/\mathsf{PropKind}$:

$$\begin{tikzcd}
 & \mathsf{BinRel} \arrow[d, "\langle d,c\rangle"] \arrow[dr, "t'"] & \\
\mathsf{Type} \arrow[ur, "\mathsf{Eq}_{(X\mapsto R)}"] \arrow[r, "\Delta"'] \arrow[dr, "b"'] & \mathsf{Type}^2 & \mathsf{PropKind} \\
 & \mathsf{Kind} \arrow[ur, "(\Omega\mapsto \mathsf{P}_\Omega)"'] &
\end{tikzcd}$$

Relational Parametricity principle:

There exists a morphism of $\lambda 2$-fibrations $(\mathsf{Eq}_{(X\mapsto R)}, (\Omega \mapsto \mathsf{P}_\Omega) : b \to t$ such that $\langle d, c\rangle \mathsf{Eq}_{(X\mapsto R)} = \Delta$ and *which extends equality on closed types*: $[\mathsf{Eq}_{(X\mapsto R)}]_{\mathbf{1}} = \mathsf{Eq}_{\mathbf{1}} : \mathsf{Type}_{\mathbf{1}} \to \mathsf{BinRel}_{\mathbf{1}}$

Lifting locally cartesian closure

Proposition: Consider a fibration $b : \mathbf{D} \to \mathbf{B}$ and (exact) 2-cells $\pi : d' \Rightarrow c'$ (into $\mathbf{D}$) and $\kappa : d \Rightarrow c$ (into $\mathbf{B}$) with $b\pi = \kappa$, and let $\widehat{\pi}$ be the vertical factor of π through the cartesian lifting of κ. Let $t : \mathbf{E} \to \mathbf{D}$ be a fibration and assume that:

1. t has $\overline{\kappa}^b$-products

2. The change-of-base fibration $t_{\overline{d}} : \mathbf{E}_{d_b} \to \mathbf{D}_d$ admits $\widehat{\pi}$-products (over $\mathbf{B}$).

Then, the fibration t admits π-products (over $\mathbf{B}$).

$\square$

Theorem:

1. If $\mathbf{B}$ is lccc, so is $\mathbf{B}^{\rightarrow}$ and $cod : \mathbf{B}^{\rightarrow} \rightarrow \mathbf{B}$ preserves the structure.

2. Let $\mathbf{B}$ be an lccc and let RG be the generic **reflexive graph**.

   (a) the fibration $cod : \mathbf{B}^{RG} \rightarrow \mathbf{B}^2$ has products

   (b) $\mathbf{B}^{RG}$ is lccc and $cod : \mathbf{B}^{RG} \rightarrow \mathbf{B}^2$ preserves the structure.

Proof

(**??**) We have the following fibration in $\mathcal{F}ib$:

$$\begin{tikzcd} (\mathbf{B}^{\rightarrow})^{\rightarrow} \arrow[r, "cod"] \arrow[d, "cod^{\rightarrow}"'] & \mathbf{B}^{\rightarrow} \arrow[d, "cod"] \\ \mathbf{B}^{\rightarrow} \arrow[r, "cod"'] & \mathbf{B} \end{tikzcd}$$

By the previous proposition, we get products for $cod^{\rightarrow}$ by composition of products over vertical and cartesian morphisms:

vertical products Given $f : X \to Y$, the *vertical fibre fibration* $t_f : \mathbf{B}^{\rightarrow}/f \to \mathbf{B}/X$ has the following action: given a morphism $h : x \to y$ in $\mathbf{B}/X$

$$\begin{tikzcd} (\mathbf{B}^{\rightarrow}/f)_y \arrow[r, "h^*"] \arrow[d, "\cong"'] & (\mathbf{B}^{\rightarrow}/f)_x \arrow[d, "\cong"] \\ \mathbf{B}/(dom(y \times f)) \arrow[r, "(h\times f)^*"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{B}/(dom(x \times f)) \arrow[l, bend left, "\prod_{h\times f}"] \end{tikzcd}$$

cartesian products Cartesian morphisms for *cod* are pullback squares and thus we apply products in $\mathbf{B}$, which satisfy Beck-Chevalley.

(**??**) Product functors for $\mathbf{B}$ take reflexive graphs to reflexive graphs. $\square$

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