The Theoretical Minimum: Classical Mechanics, lecture 1

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These notes are from a series of lectures by Leonard Susskind, entitled The Theoretical Minimum: Classical Mechanics. There is also a book based on these lectures, but my notes are a lot more concise and are the result of furious scribbles taken during lectures.

Discrete systems I

For any physical system, we have two main questions:

1. What laws are there for a particular type of system?
2. What are the rules for the allowable laws?

Consider a **coin**. It has two states/configurations/values: heads or tails. If we put a coin on a table, its initial condition is either heads or tails, and our law of motion is that it **stays the same**. In other words, at any point in time, we have that

$$H \to H, \quad T \to T.$$

Figure 1: Staying the same.

Discrete systems II

But we could also construct a system where the law of motion is that at any point of time, the coin **switches**; that is,

$$H \to T, \quad T \to H.$$

Figure 2: Switching.

We can define a function $\sigma(t)$ that tells us about the side of the coin at the next point in time $t + 1$ given some time t (for our cases, we can assume that $t \in \mathbb{Z}$). We have

$$\sigma(t) = \begin{cases} 1 & \text{heads} \\ 0 & \text{tails} \end{cases}$$

Discrete systems III

We can define our first law of motion, the one where we had the coin staying still, with the equation $\sigma(t+1) = \sigma(t)$. Similarly, we can express our second law of motion, the one where the coin switches at every time interval, as $\sigma(t+1) = -\sigma(t)$. Notice that both systems are completely deterministic (predictive). If we intervene, then we are breaking the closed system. In closed systems, laws are entirely deterministic.

Now consider a **die**. It has a set of configurations $\{1, 2, 3, 4, 5, 6\}$. We could express any laws of motion we come up with as a graph, such as
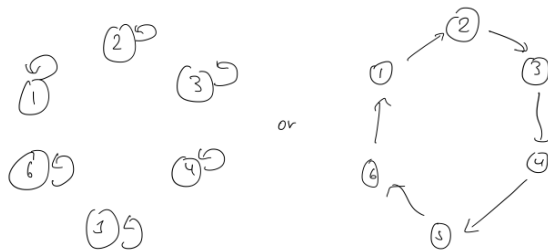
Discrete systems IV

Figure 3: Dice laws of motion.

Once again, these laws are completely deterministic. We could also consider
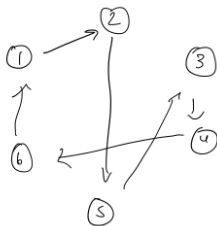
Discrete systems V

Figure 4: Dice laws of motion.

This is logically equivalent to the above; just relabel the numbers.
But one could devise something like this:
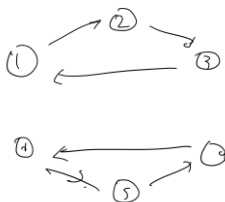
Discrete systems VI

Figure 5: Dice laws of motion.

This is **not** logically equivalent, because if you are in one cycle you won't get to the other one. This is called a **conserved quantity**, and its law is a conservation law. We can put states $1, 2, 3$ into some group A, and states $4, 5, 6$ into some group B. If we are in group A, we stay in group A. Similarly, if we are in group B, we stay in group B.
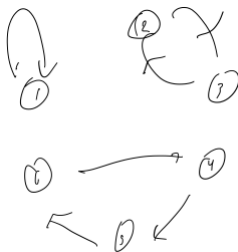
We can also consider

Discrete systems VII

Figure 6: Dice laws of motion.

If we let group A $= \{1\}$, group B $= \{2, 3\}$ and group C $= \{4, 5, 6\}$, we can also see that these are conserved quantities.

We can now consider some infinite line, where each point is $\in \mathbb{Z}$.
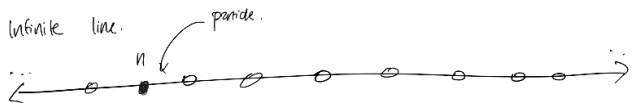
Discrete systems VIII

Figure 7: An infinite number of states.

A particle has an infinite number of states $\in \mathbb{Z}$. We could invent laws such as:

- ▶ wherever we are, move one unit;
- ▶ wherever we are, move two units (in fact, in this instance, we have a conserved quantity: oddness and evenness!);
- ▶ etc.

All of the above rules are **allowable laws**. But not all laws are legal. Which are not allowable? Consider a 3-sided coin which can land on its edge, for argument's sake.
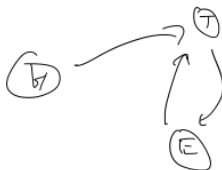
Discrete systems IX

Figure 8: "E" means edge.

We have a situation of

$$H \to T \to E \to T \to E \cdots, \quad T \to E \to T \to E \cdots, E \to T \to E \to T$$

so we can predict into the future... but we **cannot** predict into the past. This means that this is **non-reversible**. How do we check reversibility? Just try reversing each arrow.
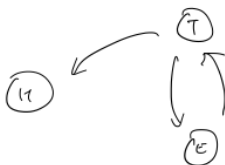
Discrete systems X

Figure 9: Reversing. . . nope.

Now we have two arrows coming out of one state? Where do we go? We don't know, so this is not deterministic. This sort of law is not allowed in *classical* physics.

If we know the initial conditions perfectly, you could predict the future and the past. Otherwise, we would like to quantify how much we can predict. For example, weather forecasters can reasonably predict the weather for 3 days from now, but since the atmosphere is a chaotic system, we might no longer be able to after a few days. The laws of classical mechanics are deterministic into the future and the past.

How does one determine predictibility concretely? It's simple: **one input, one output arrow**.

Let's finally get into some real world physics: **point particles in space**. What are the configuraions of a point particle?

Super basic math review I

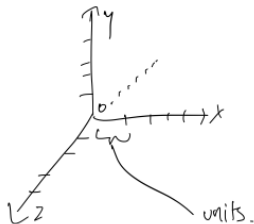Let's look at the standard Cartesian coordinate system in 3 dimensions.

Figure 10: Cartesian coordinate system.

Super basic math review II

A point is labelled with coordinates (x, y, z). The standard direction of the z is determined by the right-hand-rule.

A **vector** has both length and direction. Example: position of point relative to origin (but origin-independent). Typically, vectors are denoted $\vec{v}$. The length is denoted $|\vec{v}|$, and the direction is harder to write down. We can define the vector as well by connecting each coordinate to the basis vectors:
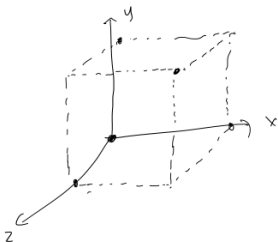

Figure 11: Coordinates.

Super basic math review III

For our cases, we denote the components v_x, v_y, v_z.

We now look at some basic facts about the algebra of vectors. If $c \in \mathbb{R}$, then $c\vec{A}$ is the vector $\vec{A}$, but c times as long. If we take $-\vec{A}$, it is $\vec{A}$ but in the opposite (negative) direction. We can **add** vectors geometrically:
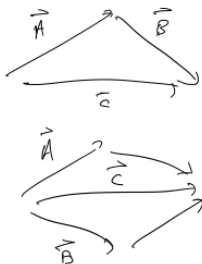

Figure 12: Adding vectors.

Super basic math review IV

or algebraically by summing the components,

$$C_x = A_x + B_x \quad \implies \quad C_i = A_i + B_i, \quad \forall i.$$

Can we multiply vectors? Yes, in two main different ways. Can we divide vectors? Uhh... not really well defined. We'll look at the dot product method of multiplying vectors.
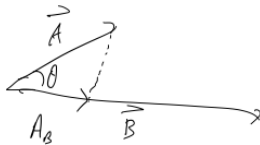

Figure 13: Dot product visually

Super basic math review V

The **dot product** is defined as $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$, where θ is the angle between the vectors. The dot product is commutative; that is, $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. We also have that $\vec{A} \cdot \vec{A} = |\vec{A}|^2$. We can also use the dot product as a nice way to compute the angle between two vectors.
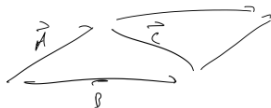
Now let us prove the law of cosines.

Figure 14: Law of cosines

Super basic math review VI

$$\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B})$$
$$= \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2(\vec{A} \cdot \vec{B})$$
$$= |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos \theta_{\vec{A}\vec{B}}$$

Particle motion I

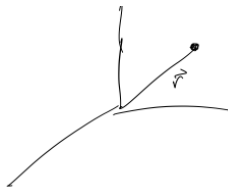Consider a particle with vector denoted by $\vec{r}$ and coordinates x, y, z.

Figure 15: Particle.

We are generally interested in the motion of particles. The three equations

$$\vec{r}_x(t) = x(t), \quad \vec{r}_y(t) = y(t), \quad \vec{r}_z(t) = z(t)$$

summarise the motion of the particle. We also know that the velocity is the derivative of the particle, so we have

Particle motion II

$$v_x = \frac{\mathrm{d}}{\mathrm{d}t} r_x = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right).$$

We can use Newton's notation

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \dot{f}(t), \quad v_i = \frac{\mathrm{d}x_i}{\mathrm{d}t} = \dot{x}_i.$$

We also have that acceleration is the rate of change of velocity.
Acceleration also has direction, so we have

$$a_i = \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = \ddot{x}_i = \dot{v}_i.$$

We can also write

Particle motion III

$$\vec{v} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r} = \dot{\vec{r}}, \quad \vec{a} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} = \ddot{\vec{r}}.$$

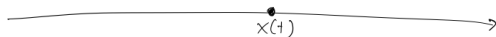Now consider simple motion along a line. We have

Figure 16: Motion along a line.

Suppose that $x(t) = a + bt + ct^2$. We then have that $v(t) = \dot{x} = b + 2ct$, and $a = \ddot{x} = 2c$.

We can also look at circular motion on the unit circle.
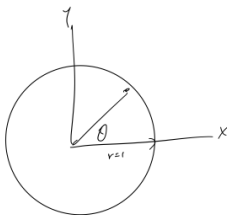
Particle motion IV

Figure 17: Circular motion on the unit circle.

Angle increases with respect to time with rule $\theta = \omega t$. How long does it take to go from 0 to 2π? We can obtain this by solving $2\pi = \omega t$, which implies

$$t = \frac{2\pi}{\omega} = \text{period}.$$

ω is called the angular frequency. Parametrising the unit circle, we have

Particle motion V

$$x = \cos \omega t, \quad y = \sin \omega t$$

and therefore

$$v_x = -\omega \sin(\omega t), \quad v_y = \omega \cos(\omega t).$$

What is the angle between position and velocity? Intuitively, perpendicular, but we can compute this explicitly using the dot product.

$$x \cdot v_x + y \cdot v_y = -\omega \sin \omega t \cos \omega t + \omega \sin \omega t \cos \omega t = 0$$

therefore the two vectors are orthogonal. For acceleration and position, we can perform a similar calculation:

Particle motion VI

$$a_x = -\omega^2 \cos(\omega t), \quad a_y = -\omega^2 \sin(\omega t).$$

Note that acceleration is parallel to the position vector, but is inwards (as is the case with centrifugal acceleration).