Consistency, Stability and Convergence of Finite Difference Schemes

Emmanuel Adebayo
adebayo@udel.edu

November 27, 2024

Contents

Consider the Heat Problem;

$$\begin{aligned} &u_t = u_{xx} \\ &u\left(0,t\right) = u\left(1,t\right) = 0 \quad 0 < t \\ &u\left(x,0\right) = \begin{cases} \sin(\pi x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{1}$$

The exact solution is $u(x,t) = \exp(-\pi^2 x)\sin(\pi x)$

1 Crank-Nicolson scheme

The Crank-Nicolson scheme is given as:

$$\frac{U_i^{n+1} - U_i^n}{\tau} = \frac{1}{2h^2}\left[U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^n - 2U_i^n + U_{i-1}^n\right] \tag{2}$$

if we take $\lambda = \frac{\tau}{2h^2}$

$$\Rightarrow U_i^{n+1} - U_i^n = \lambda\left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) \tag{3}$$

$$\Rightarrow -\lambda U_{i-1}^{n+1} + \left(1+2\lambda\right)U_i^{n+1} - \lambda U_{i+1}^{n+1} = \lambda U_{i-1}^n + \left(1-2\lambda\right)U_i^n + \lambda U_{i+1}^n \tag{4}$$

Given that

$$u(0,t) = u(1,t) = 0 \Rightarrow U_0^n = U_m^n$$

Hence we have

$$\begin{cases} \Rightarrow AU_i^{n+1} = BU_i^n \\ U_i^0 = \nu(x_i) \quad \forall i \in [0,m] \end{cases} \tag{5}$$

Where

$$A = \begin{bmatrix} 1+2\lambda & -\lambda & & & \\ -\lambda & 1+2\lambda & -\lambda & & \\ & -\lambda & 1+2\lambda & -\lambda & \\ & & & \ddots & \\ & & & -\lambda & 1+2\lambda \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1-2\lambda & \lambda & & & \\ \lambda & 1-2\lambda & \lambda & & \\ & \lambda & 1-2\lambda & \lambda & \\ & & & \ddots & \\ & & & \lambda & 1-2\lambda \end{bmatrix}$$

1.1 Consistency

To analyse the consistency we need to compute the Local Truncation error which is given by:

$$T_i^n := \frac{u_i^{n+1} - u_i^n}{\tau} - \frac{1}{2h^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n\right] \tag{6}$$

Of course, we don't know the solution, but we can use the assumption that the solution is smooth and then expand using taylor series.
Consider the following expansions:

$$u_i^{n+1} = u(x_i,t_n) + \tau u_t(x_i,t_n) + \frac{\tau^2}{2}u_{tt}(x_i,t_n) + \frac{\tau^3}{6}u_{ttt}(x_i,t_1) \tag{7}$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\tau} = u_t(x_i,t_n) + \frac{\tau}{2}u_{tt}(x_i,t_n) + \frac{\tau^2}{6}u_{ttt}(x_i,t_1) \tag{8}$$

$$\begin{aligned} u_{i+1}^n &= u(x_i+h,t_n) = u(x_i,t_n) + hu_x(x_i,t_n) + \frac{h^2}{2}u_{xx}(x_i,t_n) + \frac{h^3}{6}u_{xxx}(x_i,t_n) + \frac{h^4}{24}u_{xxxx}(x_1,t_n) \\ u_{i-1}^n &= u(x_i-h,t_n) = u(x_i,t_n) - hu_x(x_i,t_n) + \frac{h^2}{2}u_{xx}(x_i,t_n) - \frac{h^3}{6}u_{xxx}(x_i,t_n) + \frac{h^4}{24}u_{xxxx}(x_1,t_n) \end{aligned} \tag{9}$$

$$\Rightarrow \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2h^2} = \frac{1}{2}u_{xx}(x_i,t_n) + \frac{h^2}{24}u_{xxxx}(x_1,t_n) \tag{10}$$

$$\begin{aligned} u_{i+1}^{n+1} = u(x_i+h,t_n+\tau) &= u(x_i,t_n) + hu_x(x_i,t_n) + \tau u_t(x_i,t_n) + \frac{h^2}{2}u_{xx}(x_i,t_n) + h\tau u_{xt}(x_i,t_n) \\ &+ \frac{\tau^2}{2}u_{tt}(x_i,t_n) + \frac{h^3}{6}u_{xxx}(x_i,t_n) + \frac{h^2\tau}{2}u_{xxt}(x_i,t_n) + \frac{h\tau^2}{2}u_{xtt}(x_i,t_n) + \frac{\tau^3}{6}u_{ttt}(x_i,t_n) + \\ &+ \frac{h^4}{24}u_{xxxx}(x_2,t_n) + \frac{h^2\tau^2}{4}u_{xxtt}(x_i,t_2) \end{aligned} \tag{11}$$

$$\begin{aligned}u_{i-1}^{n+1} = u\left(x_i - h, t_n + \tau\right) = u\left(x_i, t_n\right) - h u_x\left(x_i, t_n\right) + \tau u_t\left(x_i, t_n\right) + \frac{h^2}{2} u_{xx}\left(x_i, t_n\right) - h\tau u_{xt}\left(x_i, t_n\right) \\ + \frac{\tau^2}{2} u_{tt}\left(x_i, t_n\right) - \frac{h^3}{6} u_{xxx}\left(x_i, t_n\right) + \frac{h^2\tau}{2} u_{xxt}\left(x_i, t_n\right) - \frac{h\tau^2}{2} u_{xtt}\left(x_i, t_n\right) + \frac{\tau^3}{6} u_{ttt}\left(x_i, t_n\right) + \\ + \frac{h^4}{24} u_{xxxx}\left(x_2, t_n\right) + \frac{h^2\tau^2}{4} u_{xxtt}\left(x_i, t_2\right)\end{aligned} \tag{12}$$

$$-2u_i^{n+1} = -u\left(x_i, t_n + \tau\right) = -2u\left(x_i, t_n\right) - 2\tau u_t\left(x_i, t_n\right) - \tau^2 u_{tt}\left(x_i, t_n\right) - \frac{\tau^3}{3} u_{ttt}\left(x_i, t_2\right) \tag{13}$$

Adding equ $(11) - (13)$ and then dividing by $2h^2$

$$\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2h^2} = \frac{1}{2} u_{xx}\left(x_i, t_n\right) + \frac{\tau}{2} u_{xxt}\left(x_i, t_n\right) + \frac{h^2}{24} u_{xxxx}\left(x_2, t_n\right) + \frac{\tau^2}{4} u_{xxtt}\left(x_i, t_2\right) \tag{14}$$

Substituting the terms in equation $(8), (10), (14)$ into (6).

$$\begin{aligned}\Rightarrow T_i^n := u_t\left(x_i, t_n\right) + \frac{\tau}{2} u_{tt}\left(x_i, t_n\right) + \frac{\tau^2}{6} u_{ttt}\left(x_i, t_1\right) - \frac{1}{2} u_{xx}\left(x_i, t_n\right) - \frac{h^2}{24} u_{xxxx}\left(x_1, t_n\right) \\ -\frac{1}{2} u_{xx}\left(x_i, t_n\right) - \frac{\tau}{2} u_{xxt}\left(x_i, t_n\right) - \frac{h^2}{24} u_{xxxx}\left(x_2, t_n\right) - \frac{\tau^2}{4} u_{xxtt}\left(x_i, t_2\right)\end{aligned} \tag{15}$$

$$\begin{aligned}\Rightarrow T_i^n := \left(u_t\left(x_i, t_n\right) - u_{xx}\left(x_i, t_n\right)\right) + \frac{\tau}{2} u_{tt}\left(x_i, t_n\right) + \frac{\tau^2}{6} u_{ttt}\left(x_i, t_1\right) - \frac{\tau}{2} u_{xxt}\left(x_i, t_n\right) - \frac{\tau^2}{4} u_{xxtt}\left(x_i, t_2\right) \\ -\frac{h^2}{24}\left[u_{xxxx}\left(x_1, t_n\right) + u_{xxxx}\left(x_2, t_n\right)\right]\end{aligned} \tag{16}$$

The first term cancels out as a result of the heat equation.
Now, using the fact that $u_t = u_{xx} \Rightarrow u_{tt} = u_{xxt}$

$$\Rightarrow T_i^n := \frac{\tau^2}{6} u_{ttt}\left(x_i, t_1\right) - \frac{\tau^2}{4} u_{xxtt}\left(x_i, t_2\right) - \frac{h^2}{24}\left[u_{xxxx}\left(x_1, t_n\right) + u_{xxxx}\left(x_2, t_n\right)\right] \tag{17}$$

The analysis of its local truncation error shows that it is second order accurate in both space and time.

$$T_i^n = O\left(h^2 + \tau^2\right)$$

1.2 Stability

Consider the equation

$$-\lambda U_{i-1}^{n+1} + (1 + 2\lambda) U_i^{n+1} - \lambda U_{i+1}^{n+1} = \lambda U_{i-1}^n + (1 - 2\lambda) U_i^n + \lambda U_{i+1}^n \tag{18}$$

Using the Von-Nemann stability, we consider a single mode with wave number k.

$$U_j^n = \xi^n e^{ikx_j} \quad \text{and } \xi \in \mathbb{C} \tag{19}$$

Assume U_j^n satisfy equation (18)

$$\Rightarrow \xi^{n+1} e^{ikx_j} - \xi^n e^{ikx_j} = \lambda\left[\xi^{n+1} e^{ikx_{j+1}} - 2\xi^{n+1} e^{ikx_j} + \xi^{n+1} e^{ikx_{j-1}} + \xi^n e^{ikx_{j+1}} - 2\xi^n e^{ikx_j} + \xi^n e^{ikx_{j-1}}\right] \tag{20}$$

$$\Rightarrow \xi^n\left[\xi e^{ikx_j} - e^{ikx_j}\right] = \lambda \xi^n\left[\xi e^{ikx_{j+1}} - 2\xi e^{ikx_j} + \xi e^{ikx_{j-1}} + e^{ikx_{j+1}} - 2e^{ikx_j} + e^{ikx_{j-1}}\right] \tag{21}$$

$$\Rightarrow \xi e^{ikx_j} - e^{ikx_j} = \lambda \left[\xi e^{ikx_{j+1}} - 2\xi e^{ikx_j} + \xi e^{ikx_{j-1}} + e^{ikx_{j+1}} - 2e^{ikx_j} + e^{ikx_{j-1}}\right] \tag{22}$$

Using $e^{ikx_{j+1}} = e^{ikx_j}e^{ikh}$, $e^{ikx_{j-1}} = e^{ikx_j}e^{-ikh}$

$$\Rightarrow e^{ikx_j}\left(\xi - 1\right) = e^{ikx_j}\lambda\left[\xi\left(e^{ikh} - 2 + e^{-ikh}\right) + \left(e^{ikh} - 2 + e^{-ikh}\right)\right] \tag{23}$$

Using $e^{ikh} + e^{-ikh} = 2\cos\left(kh\right)$

$$\Rightarrow \left(\xi - 1\right) = 2\lambda\left[\xi\cos\left(kh\right) - \xi + \cos\left(kh\right) - 1\right] \tag{24}$$

Solving for the amplification factor ξ

$$\xi = \frac{1 + 2\lambda\left(\cos\left(kh\right) - 1\right)}{1 - 2\lambda\left(\cos\left(kh\right) - 1\right)} \tag{25}$$

Let $z = \cos\left(kh\right) - 1$

$$\Rightarrow \xi = \frac{1 + 2\lambda z}{1 - 2\lambda z} \tag{26}$$

We want to show that $|\xi| \leq 1$
Since $\cos\left(kh\right) \leq 1 \Rightarrow \cos\left(kh\right) - 1 \leq 0$ or $z \leq 0 \quad \forall k$. So,

$$\Rightarrow |\xi| = \left|\frac{1 + 2\lambda z}{1 - 2\lambda z}\right| \leq 1 \tag{27}$$

This results shows that **Crank–Nicolson method is unconditionally stable for all** τ and h

1.3 Implementation of the Scheme

N = [5,10,20,40];

1.4 Results from Crank-Nicolson Scheme

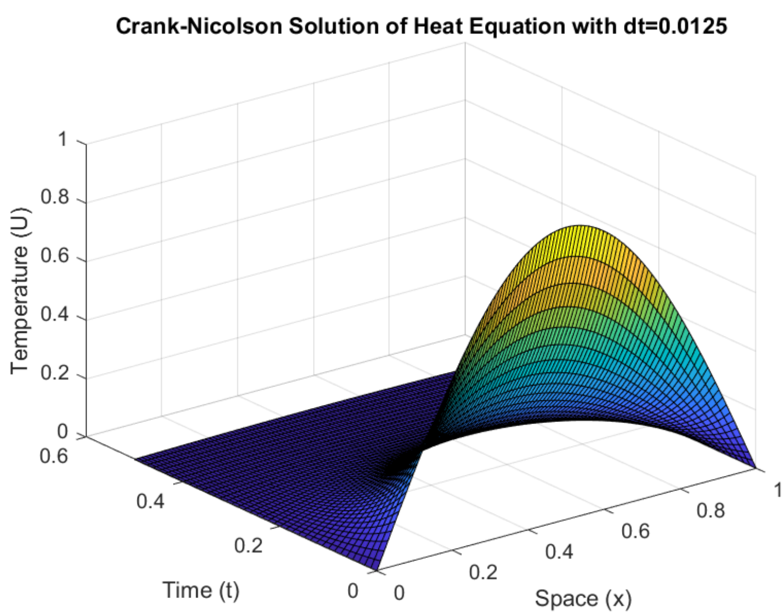

Figure 1: Crank-Nicolson solution with dt=0.0125

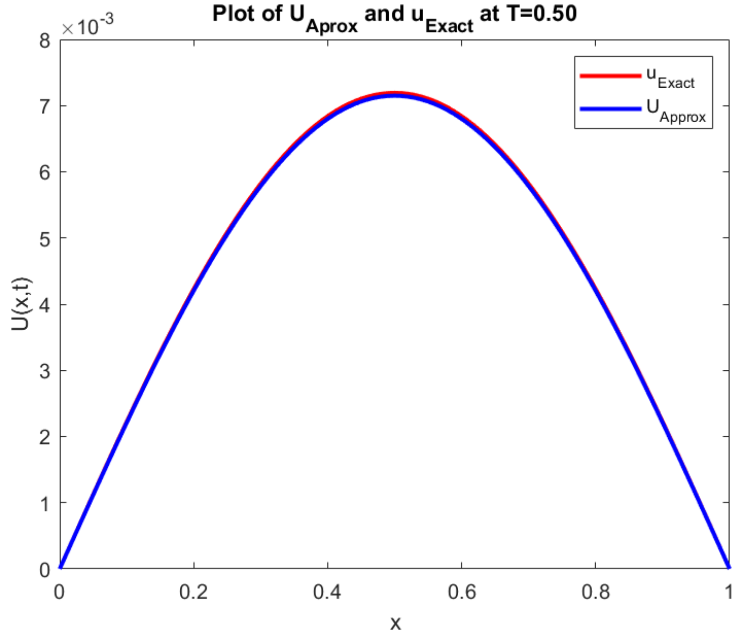

Figure 2: Plot of Exact and Approximated solution at T = 0.5

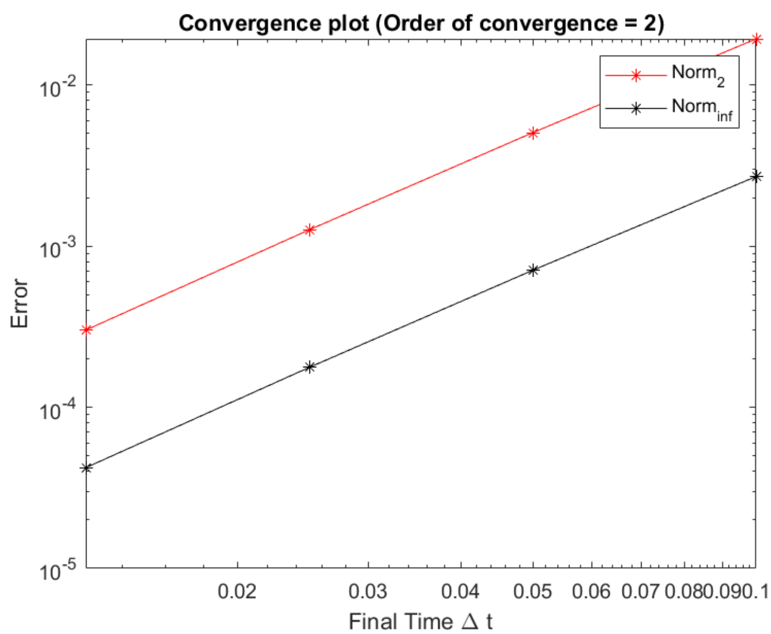

Figure 3: Order of Convergence is 2

2 Forward Euler Scheme

The Forward Euler Scheme is given as:

$$\frac{U_i^{n+1} - U_i^n}{\tau} = \frac{1}{h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right) \tag{28}$$

If we take $\lambda = \frac{\tau}{h^2}$

$$\Rightarrow U_i^{n+1} = \lambda U_{i-1}^n + (1 - 2\lambda)\, U_i^n + \lambda U_{i+1}^n \tag{29}$$

Given that

$$u\,(0,t) = u\,(1,t) = 0 \Rightarrow U_0^n = U_m^n$$

Hence we have

$$\begin{cases} \Rightarrow U_i^{n+1} = AU_i^n \quad \forall n = 0, 1, \ldots \\ U_i^0 = \nu(x_i) \quad \forall i \in [1, m-1] \end{cases} \tag{30}$$

Where

$$A = \begin{bmatrix} 1-2\lambda & \lambda & & & \\ \lambda & 1-2\lambda & \lambda & & \\ & \lambda & 1-2\lambda & \lambda & \\ & & & \ddots & \\ & & & \lambda & 1-2\lambda \end{bmatrix}$$

2.1 Consistency

To analyse the consistency we need to compute the Local Truncation error which is given by:

$$T_i^n := \frac{u_i^{n+1} - u_i^n}{\tau} - \frac{1}{h^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) \tag{31}$$

Expanding the solution using Taylor series,
Consider the following expansions:

$$u_i^{n+1} = u(x_i, t_n) + \tau u_t(x_i, t_n) + \frac{\tau^2}{2} u_{tt}(x_i, t_n) + \frac{\tau^3}{6} u_{ttt}(x_i, t_1) \tag{32}$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\tau} = u_t(x_i, t_n) + \frac{\tau}{2} u_{tt}(x_i, t_1) \tag{33}$$

$$\begin{aligned} u_{i+1}^n = u(x_i + h, t_n) = u(x_i, t_n) + hu_x(x_i, t_n) + \frac{h^2}{2} u_{xx}(x_i, t_n) + \frac{h^3}{6} u_{xxx}(x_i, t_n) + \frac{h^4}{24} u_{xxxx}(x_1, t_n) \\ u_{i-1}^n = u(x_i - h, t_n) = u(x_i, t_n) - hu_x(x_i, t_n) + \frac{h^2}{2} u_{xx}(x_i, t_n) - \frac{h^3}{6} u_{xxx}(x_i, t_n) + \frac{h^4}{24} u_{xxxx}(x_1, t_n) \end{aligned} \tag{34}$$

$$\Rightarrow \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = u_{xx}(x_i, t_n) + \frac{h^2}{12} u_{xxxx}(x_1, t_n) \tag{35}$$

$$\Rightarrow T_i^n = u_t(x_i, t_n) + \frac{\tau}{2} u_{tt}(x_i, t_1) - u_{xx}(x_i, t_n) - \frac{h^2}{12} u_{xxxx}(x_1, t_n) \tag{36}$$

Since $u_t = u_{xx}$

$$\Rightarrow T_i^n = \frac{\tau}{2} u_{tt}(x_i, t_1) - \frac{h^2}{12} u_{xxxx}(x_1, t_n) \tag{37}$$

The analysis of its local truncation error shows that it is first and second order accurate in time and space respectively

$$T_i^n = O\left(h^2 + \tau^2\right)$$

2.2 Stability

Consider the equation

$$U_i^{n+1} = \lambda U_{i-1}^n + (1 - 2\lambda) U_i^n + \lambda U_{i+1}^n \tag{38}$$

Using the Von-Nemann stability, we consider a single mode with wave number k.

$$U_j^n = \xi^n e^{ikx_j} \quad \text{and } \xi \in \mathbb{C} \tag{39}$$

Assume U_j^n satisfy equation (38)

$$\Rightarrow \xi^{n+1}e^{ikx_j} - \xi^n e^{ikx_j} = \lambda \left[\xi^n e^{ikx_{j+1}} - 2\xi^n e^{ikx_j} + \xi^n e^{ikx_{j-1}}\right] \tag{40}$$

$$\Rightarrow \xi^n \left[\xi e^{ikx_j} - e^{ikx_j}\right] = \lambda \xi^n \left[e^{ikx_{j+1}} - 2e^{ikx_j} + e^{ikx_{j-1}}\right] \tag{41}$$

$$\Rightarrow \xi e^{ikx_j} - e^{ikx_j} = \lambda \left[e^{ikx_{j+1}} - 2e^{ikx_j} + e^{ikx_{j-1}}\right] \tag{42}$$

Using $e^{ikx_{j+1}} = e^{ikx_j}e^{ikh}$, $e^{ikx_{j-1}} = e^{ikx_j}e^{-ikh}$

$$\Rightarrow e^{ikx_j}\left(\xi - 1\right) = e^{ikx_j}\lambda \left[\left(e^{ikh} - 2 + e^{-ikh}\right)\right] \tag{43}$$

Using $e^{ikh} + e^{-ikh} = 2\cos\left(kh\right)$

$$\Rightarrow \left(\xi - 1\right) = 2\lambda \left[\cos\left(kh\right) - 1\right] \tag{44}$$

Solving for the amplification factor ξ

$$\xi = 1 + 2\lambda \left[\cos\left(kh\right) - 1\right] \tag{45}$$

Since $-1 \leq \cos\left(kh\right) \leq 1$

$$\begin{aligned}
&\Rightarrow -2 \leq \cos\left(kh\right) - 1 \leq 0 \\
&\Rightarrow -4 \leq 2\left(\cos\left(kh\right) - 1\right) \leq 0 \\
&\Rightarrow -4\lambda \leq 2\lambda\left(\cos\left(kh\right) - 1\right) \leq 0 \\
&\Rightarrow 1 - 4\lambda \leq 1 + 2\lambda\left(\cos\left(kh\right) - 1\right) \leq 1 \\
&\Rightarrow 1 - 4\lambda \leq \xi \leq 1
\end{aligned} \tag{46}$$

for $|\xi| \leq 1$, we would require

$$4\lambda \leq 2 \Rightarrow \frac{\tau}{h^2} \leq \frac{1}{2} \Rightarrow \tau \leq \frac{h^2}{2} \tag{47}$$

This results shows that **Forward Euler Scheme is conditionally stable** based on the restriction in (47)

2.3 Implementation of FE Scheme

In order to satisfy the stability requirement, we required,

$$N \geq \frac{2 \cdot Tf}{\left(\frac{b-a}{M+1}\right)^2} \tag{48}$$

If we take $M = 11 \Rightarrow N \geq 144$

2.4 Result from Forward Euler Scheme

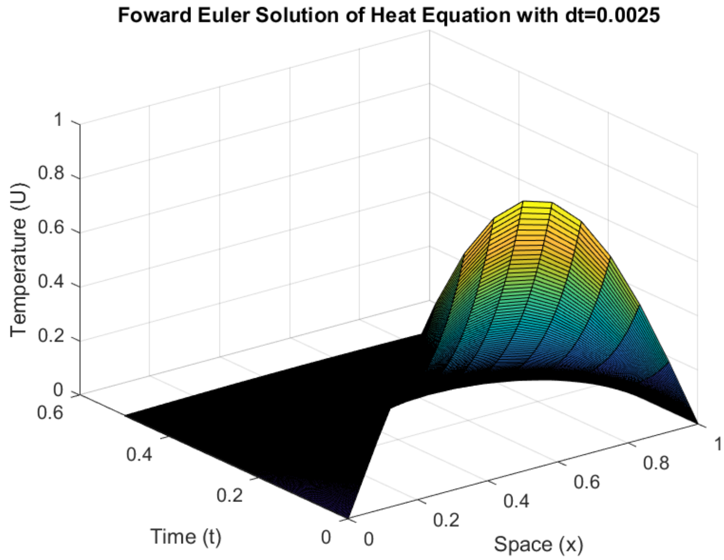

Figure 4: Forward Euler solution with dt=0.0025

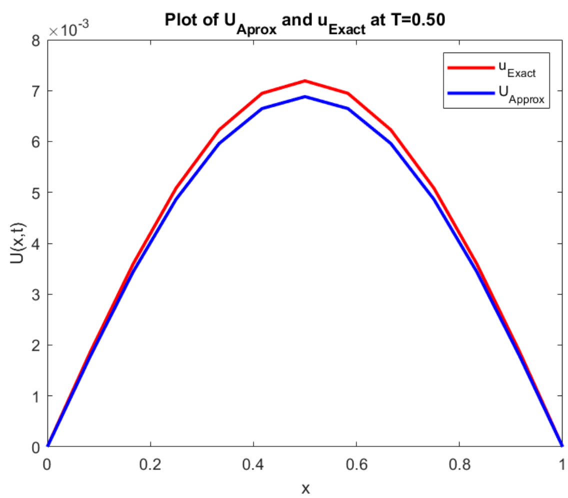

Figure 5: Plot of Exact and Approximated solution at T = 0.5

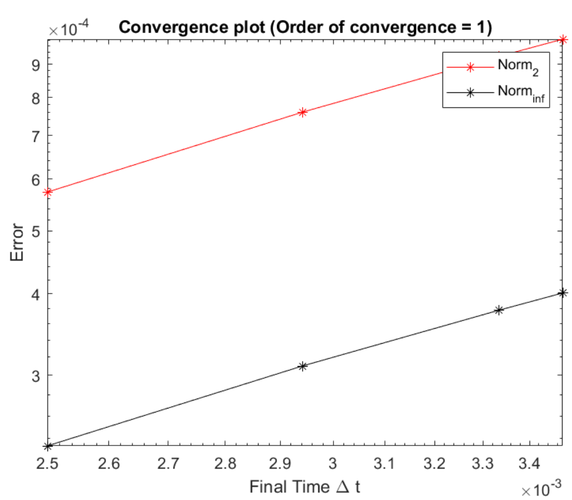

Figure 6: Order of Convergence is 1

3 Stabilized Centered Time

Consider the scheme

$$\frac{U_i^{n+1} - U_i^{n-1}}{2\tau} = \frac{1}{h^2}\left(U_{i+1}^n - U_i^{n+1} - U_i^{n-1} + U_{i-1}^n\right) \tag{49}$$

Where $\lambda = \frac{2\tau}{h^2}$

$$\Rightarrow U_i^{n+1} - U_i^{n-1} = \lambda\left(U_{i+1}^n - U_i^{n+1} - U_i^{n-1} + U_{i-1}^n\right) \tag{50}$$

$$\Rightarrow (1+\lambda)\, U_i^{n+1} = \lambda\left(U_{i+1}^n + U_{i-1}^n\right) + (1-\lambda)\, U_i^{n-1} \tag{51}$$

$$\Rightarrow U_i^{n+1} = \frac{\lambda}{1+\lambda}\left(U_{i+1}^n + U_{i-1}^n\right) + \frac{1-\lambda}{1+\lambda} U_i^{n-1} \tag{52}$$

Since $U_0^n = U_M^n = 0$

$$\Rightarrow U_i^{n+1} = AU_i^n + \beta U_i^{n-1} \quad \forall 1 \leq i \leq M-1 \tag{53}$$

Where $A \in \mathbb{R}^{(M-1)\times(M-1)}$

$$A = \begin{bmatrix} 0 & \alpha & 0 & \cdots & 0 \\ \alpha & 0 & \alpha & \ddots & \vdots \\ 0 & \alpha & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \alpha \\ 0 & 0 & 0 & \alpha & 0 \end{bmatrix} \tag{54}$$

Where $\alpha = \frac{\lambda}{1+\lambda}$ and $\beta = \frac{1-\lambda}{1+\lambda}$

3.1 Consistency

To analyse the consistency we need to compute the Local Truncation error which is given by:

$$T_i^n := \frac{u_i^{n+1} - u_i^{n-1}}{2\tau} - \frac{1}{h^2}\left(u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n\right) \tag{55}$$

Expanding the terms using Taylor series, we have

$$u_i^{n+1} = u\left(x_i, t_n\right) + \tau u_t\left(x_i, t_n\right) + \frac{\tau^2}{2} u_{tt}\left(x_i, t_n\right) + \frac{\tau^3}{6} u_{ttt}\left(x_i, t_n\right) + \frac{\tau^4}{24} u_{tttt}\left(x_i, t_1\right) \tag{56}$$

$$u_i^{n-1} = u\left(x_i, t_n\right) - \tau u_t\left(x_i, t_n\right) + \frac{\tau^2}{2} u_{tt}\left(x_i, t_n\right) - \frac{\tau^3}{6} u_{ttt}\left(x_i, t_n\right) + \frac{\tau^4}{24} u_{tttt}\left(x_i, t_2\right) \tag{57}$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^{n-1}}{2\tau} = u_t\left(x_i, t_n\right) + \frac{\tau^2}{6} u_{ttt}\left(x_i, t_n\right) + \frac{\tau^3}{24}\left[u_{tttt}\left(x_i, t_3\right)\right] \tag{58}$$

$$u_{i+1}^n = u\left(x_i, t_n\right) + h u_x\left(x_i, t_n\right) + \frac{h^2}{2} u_{xx}\left(x_i, t_n\right) + \frac{h^3}{6} u_{xxx}\left(x_i, t_n\right) + \frac{h^4}{24} u_{xxxx}\left(x_1, t_n\right) \tag{59}$$

$$u_{i-1}^n = u\left(x_i, t_n\right) - h u_x\left(x_i, t_n\right) + \frac{h^2}{2} u_{xx}\left(x_i, t_n\right) - \frac{h^3}{6} u_{xxx}\left(x_i, t_n\right) + \frac{h^4}{24} u_{xxxx}\left(x_2, t_n\right) \tag{60}$$

$$\Rightarrow u_{i+1}^n + u_{i-1}^n = 2u\left(x_i, t_n\right) + h^2 u_{xx}\left(x_i, t_n\right) + \frac{1}{12} h^4 u_{xxxx}\left(x_3, t_n\right) \tag{61}$$

$$u_i^{n+1} + u_i^{n-1} = 2u(x_i, t_n) + \tau^2 u_{tt}(x_i, t_n) + \frac{\tau^4}{24} u_{tttt}(x_i, t_4) \tag{62}$$

Hence, we have

$$\begin{aligned} T_i^n = u_t(x_i, t_n) + \frac{\tau^2}{6} u_{ttt}(x_i, t_n) + \frac{\tau^3}{24}\left[u_{tttt}(x_i, t_3)\right] - \frac{2}{h^2} u(x_i, t_n) - u_{xx}(x_i, t_n) \\ -\frac{1}{12} h^2 u_{xxxx}(x_3, t_n) + \frac{2}{h^2} u(x_i, t_n) + \tau^2 u_{tt}(x_i, t_n) + \frac{\tau^4}{24h^2} u_{tttt}(x_i, t_4) \end{aligned} \tag{63}$$

Which then simplifies to,

$$T_i^n = -\frac{1}{12} h^2 u_{xxxx}(x_3, t_n) + \frac{\tau^2}{6} u_{ttt}(x_i, t_n) + \frac{\tau^3}{24}\left[u_{tttt}(x_i, t_3)\right] + \frac{\tau^2}{h^2} u_{tt}(x_i, t_n) + \frac{\tau^4}{24h^2} u_{tttt}(x_i, t_4) \tag{64}$$

If we assume that $\frac{\tau}{h} = C$ as $\tau, h \to 0$

Then

$$T_i^n \approx O\left(h^2 + \tau^2\right) \tag{65}$$

3.2 Stability

Using the Von-Nemann stability, we consider a single mode with wave number k.

$$U_j^n = \xi^n e^{ikx_j} \qquad \text{and } \xi \in \mathbb{C} \tag{66}$$

Assume U_j^n satisfy equation (50)

$$\Rightarrow (1+\lambda)\,\xi^{n+1} e^{ikx_j} = \lambda\left(\xi^n e^{ikx_{j+1}} + \xi^n e^{ikx_{j-1}}\right) + (1-\lambda)\,\xi^{n-1} e^{ikx_j} \tag{67}$$

$$\Rightarrow (1+\lambda)\,\xi^2 e^{ikx_j} = \lambda\xi\left(e^{ikx_{j+1}} + e^{ikx_{j-1}}\right) + (1-\lambda)\,e^{ikx_j} \tag{68}$$

$$\Rightarrow (1+\lambda)\,\xi^2 e^{ikx_j} = \lambda\xi e^{ikx_j}\left(e^{ikh} + e^{-ikh}\right) + (1-\lambda)\,e^{ikx_j} \tag{69}$$

$$\Rightarrow (1+\lambda)\,\xi^2 = \lambda\xi\left(e^{ikh} + e^{-ikh}\right) + (1-\lambda) \tag{70}$$

$$\Rightarrow (1+\lambda)\,\xi^2 = \lambda\xi\,(2\cos(kh)) + (1-\lambda) \tag{71}$$

$$\Rightarrow \xi^2 = \frac{\lambda}{1+\lambda}\,(2\cos(kh))\,\xi + \frac{1-\lambda}{1+\lambda} \tag{72}$$

$$\Rightarrow \xi^2 - \alpha\,(2\cos(kh))\,\xi - \beta = 0 \tag{73}$$

$$\Rightarrow \xi = \frac{1}{2}\left(2\alpha\cos(kh) \pm \sqrt{4\alpha^2\cos^2(kh) + 4\beta}\right) \tag{74}$$

$$\Rightarrow \xi = \alpha\cos(kh) \pm \sqrt{\alpha^2\cos^2(kh) + \beta} \tag{75}$$

Let's Analyze each of the roots

$$\xi_1 = \alpha\cos(kh) + \sqrt{\alpha^2\cos^2(kh) + \beta} \tag{76}$$

for stability,

$$|\alpha| + \left|\sqrt{\alpha^2 + \beta}\right| \leq 1$$

3.3 Results

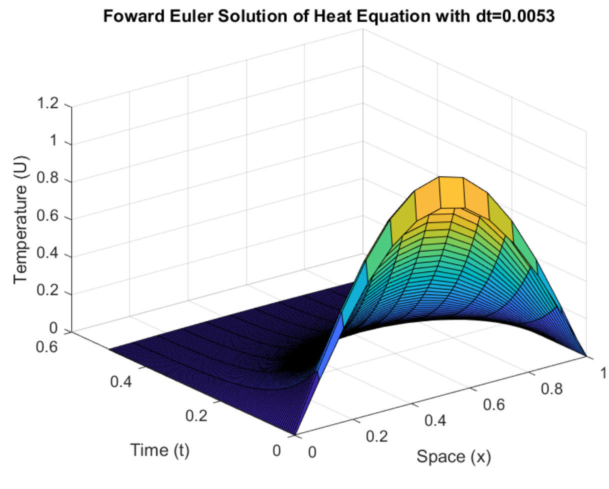

Figure 7: Crank-Nicolson solution with dt=0.0025

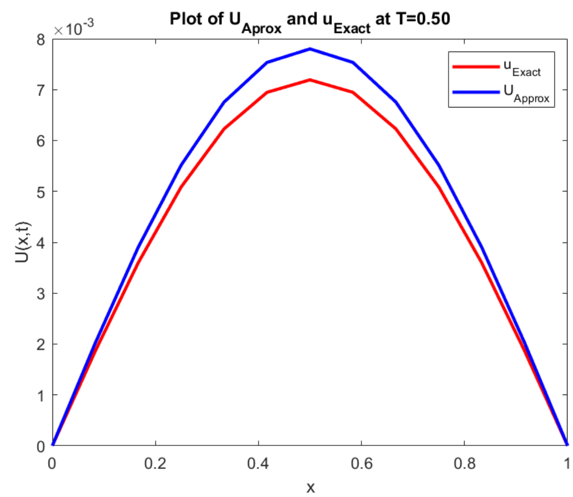

Figure 8: Plot of Exact and Approximated solution at T = 0.5

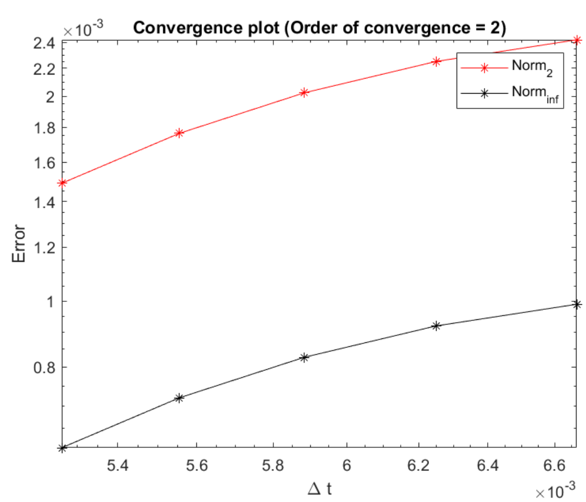

Figure 9: Order of Convergence is 2