

## Project 2 – Deep hedging : the multi-asset case

Deadline : March 24 (5 pm). Please see the course Moodle page for detailed instructions on the project delivery.

### The problem

- **Two tradable assets** with values  $S_t^1$  and  $S_t^2$ . Example : two stocks in the same stock index (e.g. two components of the SP500).
- We want to **delta-hedge an option with payoff**  $g(S_T^1, S_T^2)$  over the time grid  $t_j = j \frac{T}{N} = j \Delta t$ ,  $j = 0, \dots, N - 1$ . We will consider an at-the-money (ATM) call on the product :

$$g(S_T^1, S_T^2) = \left( \frac{S_T^1 S_T^2}{S_0^1 S_0^2} - 1 \right)^+$$

- The **hedging portfolio** now has a delta component  $\delta_t^i$  in each of the two assets :

$$V_t = V_t^0 + \delta_t^1 S_t^1 + \delta_t^2 S_t^2 = \left( V_t - \sum_{k=1}^2 \delta_t^k S_t^k \right) + \sum_{k=1}^2 \delta_t^k S_t^k.$$

The evolution in time of this self-financing portfolio is still made of the variation of the cash part (with interest rate  $r$ ) plus the variation of the delta part :

$$V_{t_{j+1}} = V_{t_j} + r \left( V_{t_j} - \sum_{k=1}^2 \delta_{t_j}^k S_{t_j}^k \right) \Delta t + \sum_{k=1}^2 \delta_{t_j}^k (S_{t_{j+1}}^k - S_{t_j}^k), \quad j = 0, \dots, N - 1$$

- **Dynamical model** : a two-dimensional log-normal model

$$\begin{cases} \log S_{t_{j+1}}^1 = \log S_{t_j}^1 + \mu_1 \Delta t + \sigma_1 \sqrt{\Delta t} G_j^1 \\ \log S_{t_{j+1}}^2 = \log S_{t_j}^2 + \mu_2 \Delta t + \sigma_2 \sqrt{\Delta t} G_j^2, \end{cases}$$

where each asset evolves in time with its own drift parameter  $\mu_k$  and volatility parameter  $\sigma_k$ . For every  $j$ , the two increments  $(G_j^1, G_j^2)$  form a standard Gaussian vector in  $\mathbb{R}^2$  with zero mean and correlation parameter  $\rho$ , that is

$$\mathbf{G}_j := \begin{pmatrix} G_j^1 \\ G_j^2 \end{pmatrix} \sim \mathcal{N}_{\mathbb{R}^2}(0, C), \quad \text{where } C = \text{covariance matrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

The vector increments  $(\mathbf{G}_0, \mathbf{G}_1, \dots, \mathbf{G}_{N-1})$  form an independent sequence.

- Additional parameter values :  $T = 1$  year,  $N = 100$ ,  $\mu_1 = 0.025$ ,  $\mu_2 = -0.01$ ,  $\sigma_1 = 0.22$ ,  $\sigma_2 = 0.3$ ,  $\rho = 0.5$ ,  $r = 0.04$ .

### Approach

- We aim at learning the appropriate two-dimensional hedging strategy

$$(\delta_t^1, \delta_t^2)$$

over the time grid  $\{t_j\}_j$  using a deep hedging approach : we parameterize both deltas with neural networks

$$\begin{aligned}\delta_{t_j}^1 &= h_j^1(S_{t_j}^1, S_{t_j}^2) \\ \delta_{t_j}^2 &= h_j^2(S_{t_j}^1, S_{t_j}^2),\end{aligned}$$

where each NN  $h_j^k$  is a function of the values  $(S_{t_j}^1, S_{t_j}^2)$  of the two assets at current time  $t_j$ . We also parameterize the initial option price (or option premium) with a NN  $\pi_\theta(S_0^1, S_0^2)$ .

- As in the 1-dim application we have considered in session 6, the criterion to be minimized is the square of the final hedging error

$$\text{hedging error} = \text{option payoff} - (\text{option premium} + \text{contribution from hedging})$$

along the trajectories in the training set.

## Your task

1. Construct a training set of  $N_{train}$  two-dimensional trajectories

$$\left( S_{t_j}^{1,i}, S_{t_j}^{2,i} \right)_{0 \leq j \leq N}^{1 \leq i \leq N_{train}}$$

for the two assets under the log-normal dynamical model above. This will require to simulate a sequence of i.i.d. Gaussian vectors  $\begin{pmatrix} G_j^1 \\ G_j^2 \end{pmatrix}$  with correlation parameter  $\rho$ .

2. Construct and train a deep learning model tackling the deep hedging problem described above. You are free to choose the architecture of the neural networks entering into the model.
3. Assess and display the performance of the resulting model. Similarly to what we did in session 6, this can be done by providing some statistics of the final hedging error (mean, std, histogram), and displaying the profiles of the delta functions yield by the deep hedging model.