Singular Value Decomposition

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The Singular Value Decomposition (SVD) of a matrix is a factorization of that matrix into three matrices. It says that any matrix A can be factorized as:

$$A = U \sum V^T$$

where U and V are orthogonal invertible matrices with orthonormal eigenvectors chosen from AA^T and A^TA respectively. $\sum$ is a diagonal matrix with r elements equal to the root of the positive eigenvalues of AA^T or A^TA (both matrices have the same positive eigenvalues anyway). Theses are called singular values. They are written in decreasing order in $\sum$.

$$\underset{m \times n}{\overset{\mathrm{A}}{\begin{pmatrix} x_{11} & x_{12} & & x_{1n} \\ & & \ddots & \\ x_{m1} & & & x_{mn} \end{pmatrix}}} = \underset{m \times m}{\overset{\mathrm{U}}{\begin{pmatrix} u_{11} & & u_{m1} \\ & \ddots & \\ u_{1m} & & u_{mm} \end{pmatrix}}} \underset{m \times n}{\overset{\mathrm{S}}{\begin{pmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ & & & \ddots \\ 0 & & & 0 \end{pmatrix}}} \underset{n \times n}{\overset{\mathrm{V^T}}{\begin{pmatrix} v_{11} & & v_{1n} \\ & \ddots & \\ v_{n1} & & v_{nn} \end{pmatrix}}}$$

Fig. 0: Here S represents Σ

$$A = U \sum V^T = \sigma_1 u_1 v_1^T + \ ... \ + \sigma_r u_r v_r^T$$

The major takeaway from SVD is that the vectors v_i, u_i and the singular value σ_i have decreasing orders of importance as i increases. Initial values give the most variance (the most importance) and hence this is applied in techniques such as PCA.