CSE 100: Algorithm Design and Analysis
Chapter 03: Growth of Functions

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Never allow waiting to become a habit. Live your dreams and take risks.
Life is happening now.

Unknown?

Outline

- Asymptotic notations. Comparing asymptotic quantities.
- Mathematical notations.

Asymptotic notations

Asymptotically,

- O: at most, $\leq$.
- Ω: at least, $\geq$.
- Θ: equal, $=$.
- o: $<$ (will be skipped).
- w: $>$ (will be skipped).

Notation $T(n)$

We often use $T(n)$ to denote the (maximum) running time of our algorithm for any input of size n. T is usually assumed to be a function whose domain is $\{0, 1, 2, ..., \}$.
(You can also use $A, B, C, ...$ whatever you like..)

Eg. In the sorting problem, n is the number of 'elements/numbers.'

(The input size can be described by multiple parameters. For example, if the input is a graph, then we often use $T(m, n)$ where m and n refer to the numbers of edges and vertices, respectively.)

O-notation

$O(n) = O(2n) = O(3n+5)$.

$\sqrt{n}$ $\quad 1 \quad \in O(n)$

$8n \quad n$

$2n$

Definition

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 s.t. $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$.

So, technically, $f(n) \in O(g(n))$ makes more sense than $f(n) = O(g(n))$. But the latter is more popular.

We say that g is an *asymptotic upper bound* for f.

$\exists c, n_0$

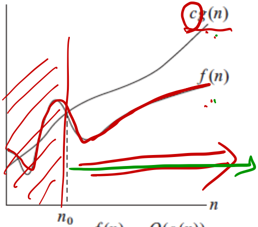
$2n^2 \leq cn^3$

for all $n \geq n_0$

$c=2 \quad n_0 = 1$

$c=1 \quad n_0 = 2$

$c = \frac{1}{2} \quad n_0 = 10$



$f(n) = O(g(n))$

for all $n \geq n_0$

for some n_0.

Eg. $2n^2 = O(n^3)$ since the condition holds

O-notation

Definition

$O(g(n)) = \{f(n) :$ there exist positive constants c and n_0 s.t. $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$.

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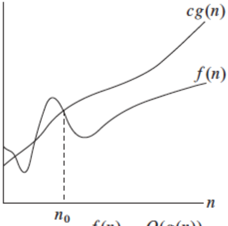

$f(n) = O(g(n))$

Eg. $2n^2 = O(n^3)$ since the condition holds for $c = 1$ and $n_0 = 2$.

O-notation
Informal definition

$O(g(n))$ is the family of functions that are asymptotically smaller than or equal to $g(n)$.

So, $f(n) = O(g(n))$ means $f(n) \leq g(n)$ 'asymptotically'.

O-notation

Examples

Examples of (functions in) $O(n^2)$:

O-notation

Examples

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$

O-notation

Examples

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$
- $\cancel{50}(\lceil n/2 \rceil)^2 + 2\cancel{01}5$

O-notation

Examples

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 $\leq_{\text{"a"}} n^2$ $= O(n^2)$

O-notation

Examples

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100
- $n^{1.99}$ $\leq$ n^2

O-notation

Examples

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100
- $n^{1.99}$
- $n \log n$ $\le_a n \cdot n = n^2$

O-notation

Examples

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100
- $n^{1.99}$
- $n \log n$
- $1000n^2 \log n$

O-notation

Examples

① Intuitively

$n^2 + \cancel{n} \le_a n^2$

② formally

$10000n^2 + 500n \le \boxed{20000} n^2$

for all $n \ge \boxed{1}$ 100

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100
- $n^{1.99}$
- $n \log n$
- $1000n^2 \log n$ (No!!!!)

vs

$\ge_a 10000 n^2$

O-notation

Examples

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100
- $n^{1.99}$
- $n \log n$
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- $n^2 / \log n$

O-notation

Examples

Examples of (functions in) $O(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100
- $n^{1.99}$
- $n \log n$
- $1000n^2 \log n$ (No!!!!)
- $n^2 / \log n$

So, you can say $n^2/\log n = O(n^2)$, but not $n^2 \log n = O(n^2)$.
Note that $f = O(g)$ does not imply $g = O(f)$.

O-notation

Intuitive understanding

$2^n = x.$

$n \ll 2^n \Rightarrow \lg x \ll x$

1. $\log n << n << 2^n$.
So, $(\log n)^{100} = O(n)$ and $n^{10000} = O(2^n)$.

2. Substitution.
We can derive $\log n = O(n)$ from $n = O(2^n)$ by setting $n = \log_2 k$.

3. Keep the most significant term and drop its coefficient along with floor/ceiling.
Eg. $100 + 50n^3 \log n + 50n^5$ $= O(n^5)$
But don't mess up with exponents! Eg. 2^{4n} vs 2^n.

$2^n \ll 4^n = 2^{2n}$

O-notation

Intuitive understanding

4. Sufficient Condition.
If there exists a constant $c \geq 0$, such that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c$, then $f = O(g)$.
(Note: not a necessary condition)

L'Hôpital's theorem can be helpful: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ if both $f(n)$ and $g(n)$ go to infinity as $n \to \infty$.
e.g. $n = O(e^n)$.

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0 \Rightarrow n = O(e^n).$$

?

O-notation

O is transitive

If $f \overset{\leq_a}{=} O(g)$ and $g \overset{\leq_a}{=} O(h)$, then $f = O(h)$. So, if you order functions so that each function is O of the function right to it, then you asymptotically sorted functions in non-decreasing order.

Exercise: Asymptotic comparison/sorting

$n^{0.5}$ n^1

Sort the following functions in asymptotically non-decreasing order:
$n \log n + \log^2 n$, $100\sqrt{n}\log^4 n$, $4^n + 50$, $15n$, $1000^{1000^{1000}}$

$< 100\sqrt{n}\log^4 n < \quad < \quad < 4^n + 50.$

① < ② < ③

$n \log n$ vs $n^{0.5} \log^4 n.$

$n \gg (\log n)^{10000}.$

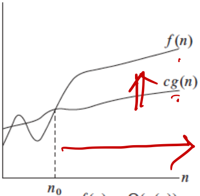
Ω-notation

Definition

$\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 s.t. $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0\}$.

So, technically, $f(n) \in \Omega(g(n))$ makes more sense than $f(n) = \Omega(g(n))$. But the latter is more popular.

We say that g is an *asymptotic lower bound* for f.



$f(n) = \Omega(g(n))$

$n^2 - 10 \geq \overset{c}{\boxed{0.1}}\, n^2$

for all $n \geq \underset{n_0}{\boxed{10}}$

Eg. $n^2 - 10n = \Omega(n^2)$ since the condition holds

Ω-notation

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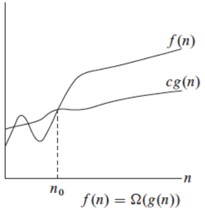

Eg. $n^2 - 10n = \Omega(n^2)$ since the condition holds for $c = 1/2$ and $n_0 = 100$.

Ω-notation

Informal definition

$\Omega(g(n))$ is the family of functions that are asymptotically greater than or equal to $g(n)$.

So, $f(n) = \Omega(g(n))$ means $f(n) \geq g(n)$ 'asymptotically'.

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $\cancel{10000}\boxed{n^2} + \cancel{500n}$

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $\cancel{5}0(\cancel{\lceil} n/\cancel{2}\cancel{\rceil})^2 + \cancel{20}15$

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 (No!!!!)

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 (No!!!!)
- $n^{1.99}$

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 (No!!!!)
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Ω-notation

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Examples of (functions in) $\Omega(n^2)$:

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Ω-notation

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Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
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- $n \log n$ (No!!!!)
- $1000n^2 \log n$ (Yes)

Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$ (No!!!!)
- $1000n^2 \log n$ (Yes)
- $n^2 / \log n$

Ω-notation
Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
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Ω-notation

Examples

Examples of (functions in) $\Omega(n^2)$:

- $10000n^2 + 500n$
- $50(\lceil n/2 \rceil)^2 + 2015$
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$ (No!!!!)
- $1000n^2 \log n$ (Yes)
- $n^2 / \log n$ (No!!!!)

So, you can say $n^2 \log n = \Omega(n^2)$, but not $n^2 / \log n = \Omega(n^2)$.
!!Note that $f = \Omega(g)$ does not imply $g = \Omega(f)$.

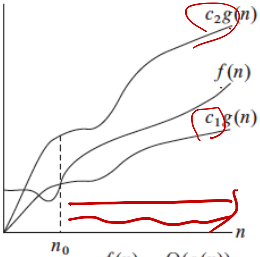
Θ-notation

Definition

$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 s.t. $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0\}$.

As before, people prefer to say $g(n) = \Theta(f(n))$.

We say that g is an *asymptotically tight bound* for f.



$f(n) = \Theta(g(n))$

Theorem

$f = \Theta(g)$ if and only if $f = O(g)$ and $f = \Omega(g)$.

Θ-notation

Informal definition

$\Theta(g(n))$ is the family of functions that are asymptotically equal to $g(n)$.

So, $f(n) = \Theta(g(n))$ means $f(n) = g(n)$ 'asymptotically'.

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $\cancel{10000}n^2 + \cancel{500n}$

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes) $\in O(n^2), \Omega(n^2)$

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!) $\because \in O(n^2)$, $\notin \Omega(n^2)$

Θ-notation
Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!) $\because$ $\notin$ $\Omega(n^2)$

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$ (No!!!!)

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$ (No!!!!)
- $1000n^2 \log n$ $\because \notin O(n^2)$? $\in \Omega(n^2)$

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$ (No!!!!)
- $1000n^2 \log n$ (No!!!)

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$ (No!!!!)
- $1000n^2 \log n$ (No!!!)
- $n^2 / \log n$ $\because \notin \Omega(n^2)$

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$ (No!!!!)
- $1000n^2 \log n$ (No!!!)
- $n^2 / \log n$ (No!!!!)

Θ-notation

Examples

Examples of (functions in) $\Theta(n^2)$:

- $10000n^2 + 500n$ (Yes)
- $50(\lceil n/2 \rceil)^2 + 2015$ (Yes)
- 100 (No!!!!)
- $n^{1.99}$ (No!!!!)
- $n \log n$ (No!!!!)
- $1000n^2 \log n$ (No!!!)
- $n^2 / \log n$ (No!!!!)

Relational Properties

Transitivity

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- $f = \Theta(g)$ and $g = \Theta(h)$ imply

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Relational Properties

Transitivity

- $f = \Theta(g)$ and $g = \Theta(h)$ imply $f = \Theta(h)$.
- $f = O(g)$ and $g = O(h)$

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- $f = \Theta(g)$ and $g = \Theta(h)$ imply $f = \Theta(h)$.
- $f = O(g)$ and $g = O(h)$ imply $f = O(h)$.

Relational Properties

Transitivity

- $f = \Theta(g)$ and $g = \Theta(h)$ imply $f = \Theta(h)$.
- $f = O(g)$ and $g = O(h)$ imply $f = O(h)$.
- $f = \Omega(g)$ and $g = \Omega(h)$

Relational Properties

Transitivity

- $f = \Theta(g)$ and $g = \Theta(h)$ imply $f = \Theta(h)$.
- $f = O(g)$ and $g = O(h)$ imply $f = O(h)$.
- $f = \Omega(g)$ and $g = \Omega(h)$ imply $f = \Omega(h)$.

Symmetry

- $f = \Theta(g)$ if and only if $g = \Theta(f)$.

Transpose Symmetry

- $f = O(g)$ if and only if $g = \Omega(f)$.

O and Θ in equations and inequalities

In equation, we may use $O()$ or $\Theta()$ to refer to a function that we do not care to name. For example, say we want to hide messy details of $2n^3 + n^2 + 10n + 5$. Then, we can simply say
$2n^3 + \cancel{n^2 + 10n + 5} = 2n^3 + O(n^2)$ or
$2n^3 + n^2 + 10n + 5 = 2n^3 + \Theta(n^2)$.

O and Θ in equations and inequalities

$2n^3 + n^2 + \Theta(n)$

In equation, we may use $O()$ or $\Theta()$ to refer to a function that we do not care to name. For example, say we want to hide messy details of $2n^3 + n^2 + 10n + 5$. Then, we can simply say
$2n^3 + n^2 + 10n + 5 = 2n^3 + O(n^2)$ or
$2n^3 + n^2 + 10n + 5 = 2n^3 + \Theta(n^2)$.

If we want to simplify it further, we can say $2n^3 + \Theta(n^2) = \Theta(n^3)$, meaning that for any function in $\Theta(n^2)$, the LHS is in $\Theta(n^3)$.

Asymptotic notation in summation or recursion

Examples:

Asymptotic notation in summation or recursion

Examples:

- $T(n) = 2T(n/2) + O(n)$ $\leq cn$

Asymptotic notation in summation or recursion

Examples:

- $T(n) = 2T(n/2) + O(n)$
- $\sum_{i=1}^{n} O(i)$. $\leq ci$.

Mathematical notation

- f is monotonically increasing (or equivalently non-decreasing) if $m \leq n$ implies $f(m) \leq f(n)$.
- f is monotonically decreasing (or equivalently non-increasing) if $m \leq n$ implies $f(m) \geq f(n)$.
- f is strictly increasing if $m < n$ implies $f(m) < f(n)$.
- f is strictly decreasing if $m < n$ implies $f(m) > f(n)$.

Mathematical notation in this book

- $\lg n = \log_2 n$ (binary log).
- $\ln n = \log_e n$ (natural log).
- $\lg^k n = (\lg n)^k$.
- $\lg \lg n = \lg(\lg n)$.