# CSE 100: Algorithm Design and Analysis
# Chapter 07: Quicksort

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My brain is open.

– *Paul Erdös*

# Quicksort

## High-level view

Uses $O(1)$ auxiliary memory.

- Introduction of quicksort: *in-place* divide-and-conquer sorting
- Introduction of randomized algorithms
- Revisiting expected (average-case) running time

# Quicksort

## High-level view

To sort $A[p \cdots r]$.

- Divide: Partition $A[p \cdots r]$ into two subarrays.
  $A[p \cdots q-1] \quad \leq \quad A[q] \quad < \quad A[q+1 \cdots r]$.
- Conquer: Sort $A[p \cdots q-1]$ and $A[q+1 \cdots r]$ by recursive calls to Quicksort.
- Combine: Free!

# Quicksort

To sort $A[p \cdots r]$:

Deterministic.

```
QUICKSORT(A, p, r)
if p < r
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
```

Example: $A[1...7] = \langle 9, 7, 3, 2, 5, 1, 4 \rangle$

# Partitioning in Quicksort

Partition$(A, p, r)$
Input: $A[p...r]$
Do: Let $x = A[r]$ (pivot value) and (after shuffling)
Return the pivot index, $p \leq q \leq r$ with

$$A[p \cdots q-1] \leq A[q] = x < A[q+1 \cdots r].$$

Example: $A[1...7] = \langle 9, 7, 3, 2, 5, 1, 4 \rangle$ Partition$(A, 1, 7)$

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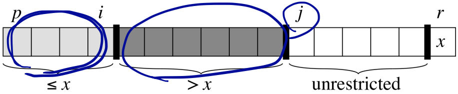
We can implement Partition without using more than $O(1)$ auxiliary memory. Thus, Quicksort is an *in-place* sorting algorithm.

# Partitioning in Quicksort

```
PARTITION(A, p, r)
x = A[r]
i = p − 1
for j = p to r − 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```

Loop invariant: $A[p \cdots i] \leq A[r] < A[i+1 \cdots j-1]$



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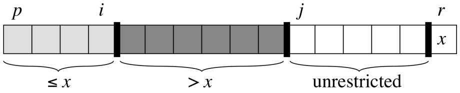


Running time of Partitioning:

# Partitioning in Quicksort

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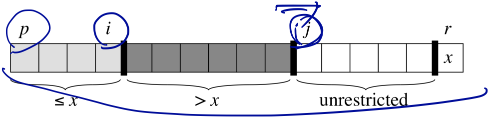
PARTITION(A, p, r)
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```

$O(1)$

Loop invariant: $A[p \cdots i] \leq A[r] < A[i+1 \cdots j-1]$




Running time of Partitioning: $\Theta(r-p)$, so if the subarray has $n$ elements, then $\Theta(n)$.

# Effect of Partitioning on Quicksort's Running Time

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# Effect of Partitioning on Quicksort's Running Time

Running time of Partitioning: $\Theta(n)$.

Running time of Quicksort: depends on the partition... If the pivot is close to a median, the partition will be effective.

- Worst-case partitioning: one subproblem with $n - 1$ elements and one with 0 elements.
- Best-case partitioning: each subproblem with less than $n/2$ elements.
- 'Balanced' partitioning: each subproblem with less than say $9/10$ elements.

# Effect of Partitioning on Quicksort's Running Time

(For simplicity, we assume that all elements in the input are distinct. See Problems 7-2 in the textbook to remove this assumption.)

$A = \langle 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \rangle$

Worst-case partitioning:

$\langle \cdots \rangle\ \boxed{1}\ \underbrace{\langle 8\ 7\ 6\ \cdots\ 2 \rangle}$

$(\underbrace{n-1}$ smaller elements) Pivot (no larger elements)

$(\cdot\cdot)\ \boxed{2}\ \langle 8\ 7\ 6\ \cdots\ 3 \rangle$

or

(no smaller elements) Pivot ($n-1$ larger elements)

# Effect of Partitioning on Quicksort's Running Time

(For simplicity, we assume that all elements in the input are distinct. See Problems 7-2 in the textbook to remove this assumption.)

Worst-case partitioning:

$$(\underline{n-1 \text{ smaller elements})}\ \text{Pivot (no larger elements)}$$

or

$$\text{(no smaller elements) Pivot } (n-1 \text{ larger elements})$$

If the worst-case partitioning keeps occurring,

$$T(n) = T(n-1) + \Theta(n)$$

$T(n) = T(n-1) + n$

$= T(n-2) + n-1 + n$

$= T(n-3) + n-2 + n-1 + n$

$= 1 + 2 + \cdots + n = \Theta(n^2)$

# Effect of Partitioning on Quicksort's Running Time

(For simplicity, we assume that all elements in the input are distinct. See Problems 7-2 in the textbook to remove this assumption.)

Worst-case partitioning:

($n-1$ smaller elements) Pivot (no larger elements)

or

(no smaller elements) Pivot ($n-1$ larger elements)

If the worst-case partitioning keeps occurring,

$$T(n) = T(n-1) + \Theta(n) \to T(n) = \Theta(n^2).$$

In the worst case, RT $= \Theta(n^2)$

# Effect of Partitioning on Quicksort's Running Time

Best-case partitioning:

($n/2$ smaller elements) Pivot ($n/2$ larger elements)

# Effect of Partitioning on Quicksort's Running Time

Best-case partitioning:

$$(n/2 \text{ smaller elements}) \text{ Pivot } (n/2 \text{ larger elements})$$

If the best-case partitioning keeps occurring,

$$T(n) = 2T(n/2) + \Theta(n)$$

# Effect of Partitioning on Quicksort's Running Time

Best-case partitioning:

$$(n/2 \text{ smaller elements}) \text{ Pivot } (n/2 \text{ larger elements})$$

If the best-case partitioning keeps occurring,

$$T(n) = 2T(n/2) + \Theta(n) \rightarrow T(n) = \Theta(n \log n).$$

In the best case, RT $= \Theta(n \log n)$

# Effect of Partitioning on Quicksort's Running Time

$\frac{1}{10}n \le \# \le \frac{9}{10}n$ $\frac{1}{10}n \le \# \le \frac{9}{10}n$

'Balanced'-case partitioning:

$$(\tfrac{1}{10}n \text{ smaller elements}) \text{ Pivot } (\tfrac{9}{10}n \text{ larger elements})$$

or

$$(\tfrac{1}{10}n + 1 \text{ smaller elements}) \text{ Pivot } (\tfrac{9}{10}n - 1 \text{ larger elements})$$

or

$$\ldots$$

$$(\tfrac{9}{10}n \text{ smaller elements}) \text{ Pivot } (\tfrac{1}{10}n \text{ larger elements})$$

# Effect of Partitioning on Quicksort's Running Time

'Balanced'-case partitioning:

$$(\tfrac{1}{10}n \text{ smaller elements}) \text{ Pivot } (\tfrac{9}{10}n \text{ larger elements})$$

or

$$(\tfrac{1}{10}n+1 \text{ smaller elements}) \text{ Pivot } (\tfrac{9}{10}n-1 \text{ larger elements})$$

or

$$\ldots$$

$$(\tfrac{9}{10}n \text{ smaller elements}) \text{ Pivot } (\tfrac{1}{10}n \text{ larger elements})$$

If the first case (worst balanced partitioning) keeps occurring,

$$T(n) = T(\tfrac{1}{10}n) + T(\tfrac{9}{10}n) + \Theta(n)$$

# Effect of Partitioning on Quicksort's Running Time

'Balanced'-case partitioning:

$$(\tfrac{1}{10}n \text{ smaller elements}) \text{ Pivot } (\tfrac{9}{10}n \text{ larger elements})$$

or

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or

$$\ldots$$

$$(\tfrac{9}{10}n \text{ smaller elements}) \text{ Pivot } (\tfrac{1}{10}n \text{ larger elements})$$

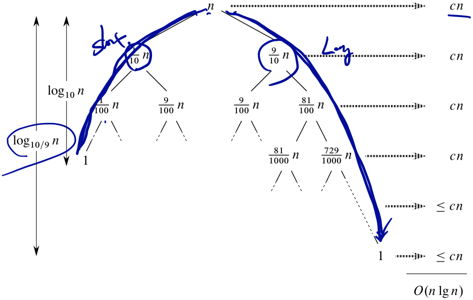
If the first case (worst balanced partitioning) keeps occurring,

$$T(n) = T(\tfrac{1}{10}n) + T(\tfrac{9}{10}n) + \Theta(n) \rightarrow T(n) = O(n \log n).$$

In the 'balanced' case, RT $= \Theta(n \log n)$

# Effect of Partitioning on Quicksort's Running Time

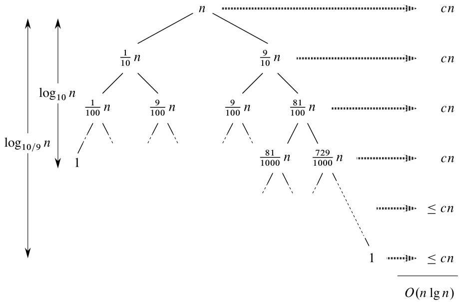
'Balanced'-case partitioning:




$$T(n) = T(\tfrac{1}{10}n) + T(\tfrac{9}{10}n) + \Theta(n)$$

# Effect of Partitioning on Quicksort's Running Time

'Balanced'-case partitioning:




$$T(n) = T(\tfrac{1}{10}n) + T(\tfrac{9}{10}n) + \Theta(n) \to T(n) = O(n \log n).$$

In the 'balanced' case, RT $= \Theta(n \log n)$

# Effect of Partitioning on Quicksort's Running Time

If the input is a random permutation of $n$ elements, we will have a 'balanced'-case with probability $\geq \frac{8}{10}$. So in expectation, the tree depth will be $\frac{10}{8} \cdot O(\log_{10/9} n) = O(\log n)$.

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The average RT $= O(n \log n)$ when the input is a random permutation of $n$ elements. But can we get similar running time for any arbitrary instances? Yes! by using randomized algorithms.

# Quicksort

## Randomized Quicksort

of Deterministic Quicksort.

- The average RT $= O(n \log n)$ when the input is a random permutation of $n$ elements (i.e. when all permutations are equally likely).
- But Deterministic quicksort's worst case running time is still $\Theta(n^2)$.
- We add randomization to quicksort so that the average running time is $\Theta(n \log n)$ for **any** input

Deterministic algorithms: you always get the same output for the same input.
Randomized algorithms: deterministic algorithms with access to "random" coins/functions

# Quicksort

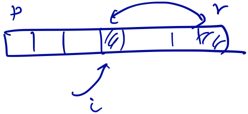
## Randomized partitioning

*Randomly* pick an element as the pivot. The random pivot will lead to a ‘balanced’ partition with probability $\geq \frac{8}{10}$.

```
RANDOMIZED-PARTITION(A, p, r)
    i = RANDOM(p, r)
    exchange A[r] with A[i]
    return PARTITION(A, p, r)
```

i

A[p r]



# Quicksort

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RANDOMIZED-QUICKSORT(A, p, r)
    if p < r
        q = RANDOMIZED-PARTITION(A, p, r)
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```
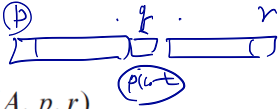

pivot's index after partitioning



# Quicksort

The average/expected/average-case RT of the randomized quicksort is $\Theta(n \log n)$ for *any* input. (The randomness is internal to the algorithm and is independent of the input).

vs.

The deterministic quicksort has an average running time $\Theta(n \log n)$ over all inputs when all permutations. are equally likely. The deterministic quicksort has a (worst-case) running time $\Theta(n^2)$.

# Quicksort

- Deterministic Quicksort. Let $T(I)$ denote the algorithm's RT on input $I$.
  - $\max_{I \text{ of size } n} T(I) = \Theta(n^2)$.
  - $\mathbb{E}_{I \text{ is a random permutation of } n \text{ elements}} T(I) = \Theta(n \log n)$.
- Randomized Quicksort. Let $T(I, r)$ denote the algorithm's RT with random coins $r$:

  function

  - For any input $I$ of size $n$,
    $\mathbb{E}_{\text{internal random coins } r} T(I, r) = \Theta(n \log n)$.

# Overview of Randomized Quicksort Analysis

- Quicksort is a comparison based algorithm: the final ordering is only determined by comparisons between the input elements.
- For simplicity, we can assume wlog that $1, 2, 3, ..., n$ are the elements to be sorted and we aim to count the number of comparisons made throughout the execution to asymptotically bound RT.
- Let $X_{ij}$ is the indicator var such that $X_{ij} = 1$ if $i$ is compared to $j$, otherwise 0. $i<j$
- We want to bound $\mathbb{E} \sum_{1 \leq i < j \leq n} X_{ij} = \sum_{1 \leq i < j \leq n} \mathbb{E} X_{ij}$ by linearity of expectation.
- Show that $\mathbb{E} X_{ij} = \frac{2}{j-i+1}$.
- Show that $\mathbb{E} \sum_{1 \leq i < j \leq n} \frac{2}{j-i+1} = O(n \log n)$.

Therefore, Randomized Quicksort makes $O(n \log n)$ comparisons in expectation, meaning that its expected running time is $O(n \log n)$.

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