CSE 100: Algorithm Design and Analysis
Chapter 09: Median and Order Statistics

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Last Update: 03-03-2023

It is the time you have wasted for your rose that makes your rose so important.

Antoine de Saint-Exupéry, The Little Prince

Medians and Order Statistics

- ith order statistic is the ith smallest element of a set of n elements.
- The minimum is the 1st order statistic.
- The maximum is the nth order statistic.
- A median is the 'halfway point' of the set.
  - If n is odd, the median is unique, and is $(n+1)/2$ order statistic.
  - If n is even, then there are two medians.
    - The lower median: $i = n/2$.
    - The upper median: $i = n/2 + 1$.

The Selection Problem

Input: A set A of n (distinct) numbers and an integer i, with $1 \leq i \leq n$.
Output: The ith smallest element of A.

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- If we use sorting, we can solve this problem in $O(n \log n)$ time.
- We will learn $O(n)$ time algorithms.
  - A randomized algorithm with $O(n)$ expected running time
  - A deterministic algorithm with $O(n)$ running time.

The Selection Problem

Warm-up: Minimum

```
MINIMUM(A)
min = A[1]
for i = 2 to A.length
    if min > A[i]
        min = A[i]
return min
```

Time:

The Selection Problem

Warm-up: Minimum

```
MINIMUM(A)
min = A[1]
for i = 2 to A.length
    if min > A[i]
        min = A[i]
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```

Time: $\Theta(n)$.

The Selection Problem

Warm-up: Maximum

Time: $\Theta(n)$.

Selection in linear time in expectation

Based on (randomized) partitioning used in Quicksort.

R-Select(A, p, r, i): Return ith smallest element in $A[p...r]$.

Ex. $A[1...9] = \langle 1, 8, 4, 9, 7, 2, 3, 6, 5 \rangle$.

R-Quicksort($A, 1, 9$)

R-Select($A, 1, 9, 2$)

Selection in linear time in expectation

Based on (randomized) partitioning used in Quicksort

```
RANDOMIZED-PARTITION(A, p, r)
    i = RANDOM(p, r)
    exchange A[r] with A[i]
    return PARTITION(A, p, r)
```

Selection in linear time in expectation

```
RANDOMIZED-SELECT(A, p, r, i)
if p == r
    return A[p]
q = RANDOMIZED-PARTITION(A, p, r)
k = q - p + 1
if i == k          // the pivot value is the answer
    return A[q]
elseif i < k
    return RANDOMIZED-SELECT(A, p, q - 1, i)
else return RANDOMIZED-SELECT(A, q + 1, r, i - k)
```

Selection in linear time in expectation

$$T(n) = T(\max\{\text{left subarray size}, \text{right subarray size}\}) + \Theta(n)$$

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- Best Partitioning: $T(n) = T(n/2) + \Theta(n) \rightarrow T(n) = \Theta(n)$.
- Worst Partitioning: $T(n) = T(n-1) + \Theta(n)$

Selection in linear time in expectation

$T(n) = T(\max\{\text{left subarray size}, \text{right subarray size}\}) + \Theta(n)$

- Best Partitioning: $T(n) = T(n/2) + \Theta(n) \rightarrow T(n) = \Theta(n)$.
- Worst Partitioning: $T(n) = T(n-1) + \Theta(n)$
  $\rightarrow T(n) = \Theta(n^2)$.
- 'Balanced' Partitioning: $T(n) = T(\frac{3}{4}n) + \Theta(n)$.

Let's say that the partitioning is 'balanced' if

$$\frac{1}{4}n \leq \text{ pivot's order } \leq \frac{3}{4}n$$

Selection in linear time in expectation

- 'Balanced' Partitioning: $T(n) = T(\frac{3}{4}n) + \Theta(n)$
  $\to T(n) = \Theta(n)$.

Let's say that the partitioning is 'balanced' if

$$\frac{1}{4}n \leq \text{ pivot's order } \leq \frac{3}{4}n$$

We can show that RT of Randomized Selection is $\Theta(n)$ in expectation (for any input).

* We can even show that RT $= \Theta(n)$ with probability that tends to 1 as n grows. (Beyond the scope of this course)

Selection in linear time in expectation

Formal proof

- Let $E(n) := \max_{I:\text{input of size} n} E[RT(I)]$.
- $E(n)$ is non-decreasing in n.
- The partitioning is 'balanced' with probability $1/2$.
- If 'balanced', the bigger subproblem size $\leq \frac{3}{4}n$; else $\leq n-1$.
- $E(n) \leq \frac{1}{2}E(\frac{3}{4}n) + \frac{1}{2}E(n-1) + O(n) \leq \frac{1}{2}E(\frac{3}{4}n) + \frac{1}{2}E(n) + O(n)$.
- $E(n) \leq E(\frac{3}{4}n) + O(n) \rightarrow E(n) = O(n)$.

* Let's say that the partitioning is 'balanced' if $\frac{1}{4}n \leq$ pivot's order $\leq \frac{3}{4}n$

Selection in linear time

(Deterministic) Select

- If we can find a good pivot leading to a balanced partition in linear time, we have a linear time deterministic algorithm.
- But finding a good pivot is a kind of select problem.

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(Deterministic) Select

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- But finding a good pivot is a kind of select problem. So, recursion will be used to find such a pivot.

Key idea: Find a good pivot from a smaller problem.

Selection in linear time

(Deterministic) Select

1. Divide the n elements into groups of size 5. So $n/5$ groups.
2. Find the median of each group.
3. Find the median x of the $n/5$ medians by a recursive call to Select.
4. Call Partition with x as the pivot.
5. Make a recursive call to Select either on the smaller elements or larger elements (depending on the situation); if the pivot is the answer we are done.

Key question: Is x a good pivot?

Selection in linear time

(Deterministic) Select

Claim: $\frac{3}{10}n \leq x$'s order $\leq \frac{7}{10}n$.
NOTE: for simplicity, small additive constants are ignored.

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Selection in linear time

(Deterministic) Select: RT analysis

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Selection in linear time

(Deterministic) Select: RT analysis

Claim: $\frac{3}{10}n \le x$'s order $\le \frac{7}{10}n$.

1. Divide the n elements into groups of size 5. So $n/5$ groups. $\Theta(n)$.
2. Find the median of each group.

Selection in linear time

(Deterministic) Select: RT analysis

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1. Divide the n elements into groups of size 5. So $n/5$ groups. $\Theta(n)$.
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Selection in linear time

(Deterministic) Select: RT analysis

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2. Find the median of each group. $\Theta(n)$.
3. Find the median x of the $n/5$ medians by a recursive call to Select. $T(n/5)$.
4. Call Partition with x as the pivot.

Claim: $\frac{3}{10}n \leq x$'s order $\leq \frac{7}{10}n$.

1. Divide the n elements into groups of size 5. So $n/5$ groups. $\Theta(n)$.
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(Deterministic) Select: RT analysis

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1. Divide the n elements into groups of size 5. So $n/5$ groups. $\Theta(n)$.
2. Find the median of each group. $\Theta(n)$.
3. Find the median x of the $n/5$ medians by a recursive call to Select. $T(n/5)$.
4. Call Partition with x as the pivot. $\Theta(n)$.
5. Make a recursive call to Select either on the smaller elements or larger elements (depending on the situation); if the pivot is the answer we are done. $T(\frac{7}{10}n)$.

Selection in linear time

(Deterministic) Select: RT analysis

$$T(n) = T(n/5) + T(7n/10) + \Theta(n)$$

Selection in linear time

(Deterministic) Select: RT analysis

$$T(n) = T(n/5) + T(7n/10) + \Theta(n)$$
$$\Rightarrow T(n) = \Theta(n).$$