Financial Engineering Notes

Swaps in Financial Markets

Swaps are contracts that convert one type of cash flow into another. The key types of swaps include:
- Plain Vanilla Swap: Converts fixed interest rate cash flows to floating rates.
- Commodity Swap: Exchanges floating prices for a fixed price of a commodity (e.g., gold, oil).
- Currency Swap: Swaps cash flows in one currency for another.

Purpose: Companies use swaps to alter the nature of their cash flows and take advantage of their strengths in different markets.
Example: Company A can borrow at 4% fixed or LIBOR + 0.3% floating. Company B can borrow at 5.2% fixed or LIBOR + 1% floating. By entering a swap, both can secure more favorable rates.

Counterparty Risk: Financial intermediaries are often involved to reduce default risk.
Pricing: Swaps are priced based on the present value of exchanged cash flows with the fixed rate set to make both parties indifferent.

Futures Contracts and Their Advantages Over Forwards

Futures contracts are standardized and traded on exchanges, solving the problems of pricing and counterparty risk seen in forwards. Key differences include:

- Futures are marked-to-market, ensuring a single transparent price for a maturity date.
- Margin Accounts: Require an initial margin (5-10% of contract value) and daily settlement.

Pros:
- High leverage
- Liquidity
- Broad asset coverage

Cons:
- Higher risk from leverage
- Linear payoffs limit hedging
- Less flexible than forwards

Pricing: Based on martingale pricing under stochastic interest rates. At maturity, futures prices equal the spot price.

Hedging with Futures Contracts

A real-world example involves a baker locking in wheat prices:
- Long Hedge: Buys 100 futures contracts (5,000 bushels each = 500,000 total).
- Cash Flow: Effective price is fixed using futures. The cash flow at maturity considers spot vs. initial futures price.
- Margin Account: Required to handle potential daily losses.
- Basis Risk: Caused by mismatched expiration dates or unavailable contracts (e.g., kidney beans).
- When perfect hedging isn't possible, variance minimization is used. The optimal number of contracts is determined using covariance and variance of asset and futures prices.

Margin Account Mechanics in Futures Trading

Using corn futures as an example:
- Initial Margin: $1,688
- Maintenance Margin: $1,250
- Profits increase margin account balance; losses reduce it.
- Margin Call: Triggered when balance falls below maintenance margin, requiring deposit back to initial margin.
- Simulated scenarios show how daily changes impact account balance and margin call frequency.

Derivative Securities: Options

1. Types of Options

- Call Option: Gives the buyer the right to purchase the underlying asset.
- Put Option: Gives the buyer the right to sell the underlying asset.

2. European vs. American Options

- European Options: Can only be exercised at expiration (time T).
- American Options: Can be exercised at any time up to expiration.

3. Payoff and Intrinsic Value

- Call Option Payoff: $\max(S_T - K, 0)$
- Put Option Payoff: $\max(K - S_T, 0)$
- Intrinsic Value: Same as the payoff if exercised immediately.

4. No-Arbitrage Pricing

The prices of European call and put options are linked through put-call parity, ensuring no arbitrage opportunities exist.

Put-Call Parity and Option Bounds

This section explores the concept of put-call parity in relation to American and European options.

1. American vs. European Options

- American Options: Can be exercised at any time before expiration.
- European Options: Can only be exercised at expiration.
- American options are generally more valuable due to the added flexibility.

2. Value Relationships

- American Call (cA) ≥ European Call (cE)
- American Put (pA) ≥ European Put (pE)

3. Put-Call Parity and Bounds

- European Call (cE) ≥ 0
- European Put (pE) ≥ 0
- These represent the minimum value bounds of European options.

4. Intrinsic Value and Early Exercise

- For American Calls on non-dividend paying stocks: cA > Intrinsic Value ⇒ Early exercise is not optimal.
- For European Puts: Early exercise may be optimal under certain conditions.

Understanding these relationships is critical for determining optimal execution strategies in options trading.

Option Pricing and the Binomial Model

This section introduces the binomial model for option pricing and lays the groundwork for more advanced strategies.

1. Overview of the Binomial Model

- Initial Stock Price: $100
- Possible future prices: $107 (up) or $93.46 (down)
- The down factor (d) is the inverse of the up factor (u)

2. Recombining Tree Structure

- The model uses a recombining tree, where the stock can reach the same price through different sequences of up and down moves

3. Risk-Free Asset

- A cash account grows at a constant risk-free interest rate (R)

4. Option Payoff

- Considers the payoff of an option defined as max(0, S - $100), where S is the stock price at the option's expiration

5. Utility and True Probabilities

- Investigates whether option pricing is influenced by utility functions of market participants or by the actual probabilities of stock movements

This module builds foundational understanding for pricing options using discrete models like the binomial model.

The St. Petersburg Paradox and Utility Theory

This section discusses the St. Petersburg Paradox and how it led to the development of utility theory in decision-making under uncertainty.

1. The Paradox Setup

- A fair coin is tossed until the first head appears.
- The payoff doubles with each toss: $2, $4, $8, ..., $2^n.
- Expected payoff is calculated as an infinite sum: $\sum(1/2)^n * 2^n = \infty$.
- Despite the infinite expected value, people are unwilling to pay large amounts to play.

2. Introduction of Utility Function

- Daniel Bernoulli proposed that people evaluate outcomes using a utility function instead of just monetary value.
- Utility functions are increasing and concave, reflecting diminishing marginal utility of wealth.

3. Logarithmic Utility Function

- Bernoulli suggested using a log utility function: $U(W) = \log(W)$.
- This results in a finite expected utility for the St. Petersburg game, resolving the paradox.

4. Implications for Option Pricing

- The paradox highlights that pricing based on expected value may not reflect actual behavior.
- Raises the question of whose utility function to use in financial modeling.
- Future sections will explore simpler and more universal pricing methods.

Option Pricing: 1-Period Binomial Model

This section covers option pricing using the 1-Period Binomial Model and introduces foundational arbitrage concepts.

1. Model Assumptions

- Initial stock price (S_0) = $100
- At time t = 1, stock price can go up to $107 or down to $93.46
- Up factor (u) = 1.07, Down factor (d) = 93.46 / 100
- Risk-free rate (r) applies to borrowing or lending via a cash account
- Short selling is allowed

2. Call Option Pricing Examples

- Call Option with strike price $107:
  - Payoff is 0 in both up and down states, so the option value is $0
- Call Option with strike price $92:
  - Since the stock is always above $92, payoff = ($S_0$ - 92) / r

3. Arbitrage Concepts

- Type A Arbitrage: Zero initial investment with guaranteed profit in at least one state and no loss in others
- Type B Arbitrage: Positive initial inflow with guaranteed non-negative payoff in the future
- The model introduces conditions for no arbitrage to ensure fair pricing

Option Pricing with Replicating Portfolio: 1-Period Binomial Model

This section expands on option pricing using the 1-Period Binomial Model by introducing the concept of a replicating portfolio.

1. Model Setup

- Initial stock price (S_0) = $100
- Up state price = $107, Down state price = $93.46
- Gross risk-free rate (R) = 1.01 (i.e., 1% per period)

2. Call Option Payoff

- Strike price (K) = $102
- Payoff in up state = max(107 - 102, 0) = $5
- Payoff in down state = max(93.46 - 102, 0) = $0

3. Replicating Portfolio

- Let x be the number of shares and y be the amount in the risk-free asset
- Solve for x and y to match the option's payoffs in both states:
  - 107x + 1.01y = 5
  - 93.46x + 1.01y = 0
- Solution: x = 0.3693, y = -34.1708
- This implies borrowing $34.1708 and buying 0.3693 shares

4. Option Value

- Cost of replicating portfolio = 100 * 0.3693 + (-34.1708) = $2.76
- Therefore, the arbitrage-free price of the call option is $2.76

Pricing Derivative Securities Using the 1-Period Binomial Model

Stock Price Movements: The stock can either move up to ($u \times S_0$) or down to ($d \times S_0$) after one period.

Payoff Structure: The derivative security has payoffs (C_u) if the stock price goes up and (C_d) if it goes down.

Replicating Portfolio: To find the fair value of the derivative, a replicating portfolio is constructed by purchasing (x) units of the stock and investing (y) dollars in a cash account.

Equations: Two equations are set up based on the payoffs in both scenarios (up and down), allowing for the calculation of (x) and (y).

Risk-Neutral Pricing: The fair value of the derivative can be expressed using risk-neutral probabilities, leading to the formula: $C_0 = (1/R) \times [qC_u + (1 - q)C_d]$.

Independence from True Probabilities: Interestingly, the option price does not depend on the actual probabilities of stock movements, (p) and (1 - p), but rather on the risk-free rate (R), and the up and down factors (u and d).

This concept of risk-neutral pricing is crucial in finance and will be explored further in multi-period models and more complex scenarios later in the course.