Financial Engineering Notes

No-Arbitrage Principle in Financial Pricing

1. Introduction

This module introduces the concept of no-arbitrage and its crucial role in pricing financial contracts. It ensures that securities are fairly priced in efficient markets.

2. No-Arbitrage Principle

The no-arbitrage principle states that there should be no possibility of making a riskless profit. It is used to determine the price (p) of a financial contract that pays cash flows (c_k) at specific future times (t).

Mathematically, the principle implies:

If arbitrage opportunities exist, market forces will eliminate them, leading to equilibrium prices.

3. Weak No-Arbitrage

If a financial contract has non-negative cash flows at all future times (i.e., $c_k \geq 0$ for all k), then the price of the contract must be at least zero:

$$p \geq 0$$

4. Strong No-Arbitrage

If a financial contract has non-negative cash flows and at least one cash flow is strictly positive (i.e., $\exists k$ such that $c_k > 0$), then the price of the contract must be strictly positive:

$$p > 0$$

5. Market Dynamics

The no-arbitrage conditions assume the existence of liquid markets with a sufficient number of buyers and sellers. Moreover, it is assumed that price information is publicly accessible, enabling fair competition and efficient pricing.

6. Example: Bond Pricing

Consider a bond that pays $8 in one year. Using the no-arbitrage conditions, the price of this bond must be bounded appropriately. If the bond has no risk and the market is efficient, its price should reflect the present value of the future $8 cash flow, adjusted by the appropriate discount rate.

7. Conclusion

The no-arbitrage framework is essential for fair valuation in financial markets. It underpins much of modern financial theory, including pricing models and market equilibrium.

Interest Rates and Present Value

1. Introduction

This module explores the relationship between interest rates, present values, and the no-arbitrage principle. It highlights how different compounding methods affect the value of future cash flows and the valuation of financial contracts.

2. Interest Rates and Present Values

Interest rates determine the present and future values of investments and financial contracts. The no-arbitrage principle ensures that present values are consistent with fair pricing in efficient markets.

3. Types of Interest

• Simple Interest:

$$\text{Future Value} = A \times (1 + nr)$$

• Compound Interest:

$$\text{Future Value} = A \times (1 + r)^n$$

• Continuous Compounding:

$$\text{Future Value} = A \times e^{(ry)}$$

4. Present Value Calculation

The present value of a series of future cash flows is determined using the discounting formula. This is closely related to the no-arbitrage price of a financial contract:

$$PV = c_0 + c_1 / (1 + r) + c_2 / (1 + r)^2 + \ldots + c_N / (1 + r)^N$$

5. Borrowing and Lending Rates

In some markets, borrowing and lending rates differ. This affects the computation of present values, especially when considering investor constraints or frictions in financial markets. Such scenarios lead to incomplete markets where the exact pricing of financial contracts may not be uniquely determined.

6. Conclusion

Understanding interest rates and their impact on present value is fundamental in financial engineering. These concepts enable the valuation of financial contracts and form the basis of more complex pricing models.

Fixed Income Instruments

1. Introduction

Fixed income instruments are financial securities that provide predetermined cash flows over time. Despite the term 'fixed', these instruments carry certain risks that affect their valuation and performance.

2. Risks in Fixed Income Instruments

• Default Risk:

> ***Fixed income securities can default if the issuer becomes insolvent. The U.S. government is generally regarded as the only risk-free issuer.***

• Inflation Risk:

> ***Even in the absence of default, inflation can diminish the real value of future cash flows received from the investment.***

• Market Risk:

> ***The market value of fixed income securities can fluctuate due to changes in interest rates or market conditions, influencing their resale price.***

3. Types of Fixed Income Instruments

• Perpetuity:

A perpetuity offers a constant cash flow forever. Its price is calculated as:

> ***Price = A / r***

• Annuity:

> ***An annuity pays a fixed amount for a finite number of periods. It is typically valued as the difference between two perpetuities.***

• Bonds:

> ***Bonds are defined by their face value, coupon rate, maturity date, price, and credit quality rating. The yield to maturity (YTM) summarizes these attributes and serves as an indicator of the bond's risk and expected return.***

4. Conclusion

Understanding the types and associated risks of fixed income instruments is essential for evaluating their performance. These tools form the basis of conservative investment strategies and play a key role in portfolio construction and risk management.

Floating Rate Bonds and No-Arbitrage Pricing

1. Introduction

This module explores the pricing of floating rate bonds using the no-arbitrage principle and the linear pricing theorem. Floating rate bonds are distinct in that their coupon payments vary with market interest rates, introducing uncertainty until the payment date.

2. Floating Rate Bonds

Floating rate bonds are debt instruments whose coupon payments adjust based on current market interest rates. This feature links their cash flows to short-term interest rate movements, reducing interest rate risk for investors.

3. Linear Pricing Theorem

The linear pricing theorem states that the price of a combined cash flow equals the sum of the prices of its individual components. If CA and CB are two cash flows with respective prices PA and PB, then the price of the combined cash flow C is:

$$P = PA + PB$$

4. No-Arbitrage Principle

The no-arbitrage principle asserts that in an efficient and liquid market, there should be no opportunity for risk-free profit. If the price of a cash flow is lower than the sum of its component prices, an arbitrage opportunity exists, prompting immediate market correction.

5. Pricing Floating Rate Bonds

To price floating rate bonds, complex cash flows are decomposed into simpler components. Each component is priced independently using market data and no-arbitrage arguments, and the total price is obtained by summing these component prices.

6. Final Result

Under no-arbitrage conditions, the price of a floating rate bond equals its face value. This result highlights the power of no-arbitrage pricing even in stochastic interest rate environments.

7. Conclusion

Floating rate bonds illustrate the application of fundamental pricing principles in uncertain environments. The linear pricing theorem and no-arbitrage condition provide a robust framework for valuing these instruments.

Term Structure of Interest Rates

1. Introduction

This module covers the term structure of interest rates, which describes how interest rates vary with different loan durations. Understanding the term structure is essential for pricing bonds and other fixed income instruments accurately.

2. Interest Rate Dependency

Interest rates generally increase with the length of the loan. This is because investors require higher compensation for tying up their funds over longer periods due to increased risks and decreased liquidity.

3. Influencing Factors

• Investor Preferences:

> ***Investors prefer liquid assets. To persuade them to commit to longer durations, issuers must offer higher interest rates.***

• Expectations and Market Segmentation:

> ***Interest rates are also influenced by expectations about future rates and the characteristics of different investor groups in various market segments.***

4. Spot Rates

A spot rate (S_t) is the interest rate for a loan that matures in exactly t years. These rates are crucial for valuing future cash flows precisely.

5. Present Value Calculation

The present value (PV) of receiving an amount A in t years using the spot rate S_t is given by:

$$PV = A / (1 + S_t)^t$$

6. Forward Rates

Forward rates represent the interest rate agreed today for lending between future dates, from year u to year v. They are derived from existing spot rates and are used to avoid arbitrage.

7. No Arbitrage Condition

The no-arbitrage condition ensures that equivalent financial positions have the same value regardless of the method of construction. This condition maintains consistency and fairness in pricing across various time horizons.

8. Conclusion

The term structure of interest rates provides a foundational concept for bond pricing and interest rate modeling. Spot and forward rates, under no-arbitrage assumptions, allow for accurate valuation and strategic financial planning.

Forward Contracts and No-Arbitrage Pricing

1. Introduction

This module focuses on forward contracts and how the no-arbitrage principle is used to price them. Forward contracts are fundamental derivatives in financial markets, used for hedging and speculation.

2. Definition and Examples

A forward contract is an agreement made at time $t = 0$ to buy or sell a specified amount of an asset at a fixed price (the forward price, F) at a future date (T). Examples include contracts involving stocks, gold, currencies, and Treasury Bills.

3. Payoff at Maturity

At maturity (time T), the value of the forward contract to the buyer is the difference between the market price of the asset (S_T) and the forward price (F):

$$Payoff = S_T - F$$

4. Initial Pricing

The forward price is set so that the initial value of the contract is zero. This ensures that both buyers and sellers are indifferent to entering the contract at inception.

5. No-Arbitrage Pricing

The no-arbitrage principle is used to derive the forward price. It states that there should be no risk-free profit opportunity. In an arbitrage-free market, any discrepancies in pricing would be quickly corrected by market forces.

6. Short Selling and Cost of Carry

Short selling involves borrowing an asset and selling it immediately, with the obligation to repurchase it later. This strategy helps explain forward pricing under the no-arbitrage condition. The cost of carry, including financing and storage costs, is reflected in the forward price, typically making it higher than the spot price.

7. Conclusion

Forward contracts exemplify how financial engineering utilizes no-arbitrage principles to ensure fair pricing. Understanding their valuation and the forces behind the forward price is essential for managing financial risk.

Forward Contracts on Non-Dividend Paying Stocks

1. Scenario Setup

This module discusses pricing a forward contract on a non-dividend paying stock with a six-month maturity.

2. Market Parameters

• Current Stock Price (S_0): $50

• Annual Interest Rate: 4% per annum, semi-annual compounding

3. Discount Rate Calculation

For semi-annual compounding, the discount factor for 6 months is calculated as:

$$\textit{Discount Rate} = 1 / (1 + 0.04 / 2) = 0.9804$$

4. Forward Price Calculation

The forward price is obtained using the formula:

$$F = S_0 / \textit{Discount Rate} = 50 / 0.9804 \approx 51$$

5. Forward Contract Valuation at Time t > 0

To value a forward contract at a future time ($t > 0$), one can use the difference between two forward contracts. This construction considers cash flows from short and long positions and leads to a deterministic valuation, based on the spot-forward parity relationship.

6. Conclusion

This analysis shows how forward pricing incorporates interest rates and discounting, ensuring consistency through the no-arbitrage framework. Understanding how to value forwards at future times helps manage derivative positions over time.