Financial Engineering & Risk Management

Pricing American Options

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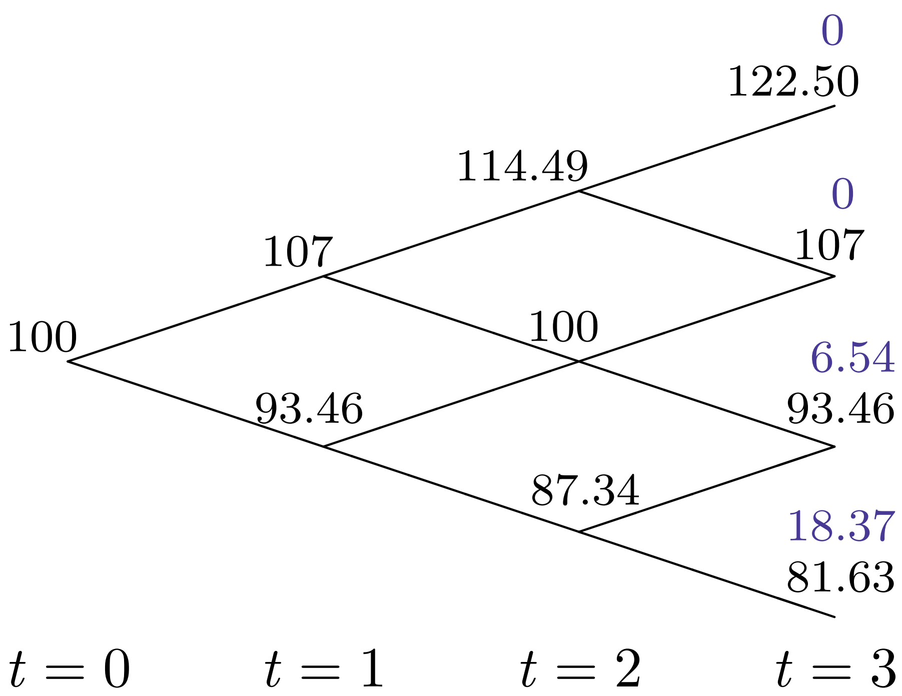
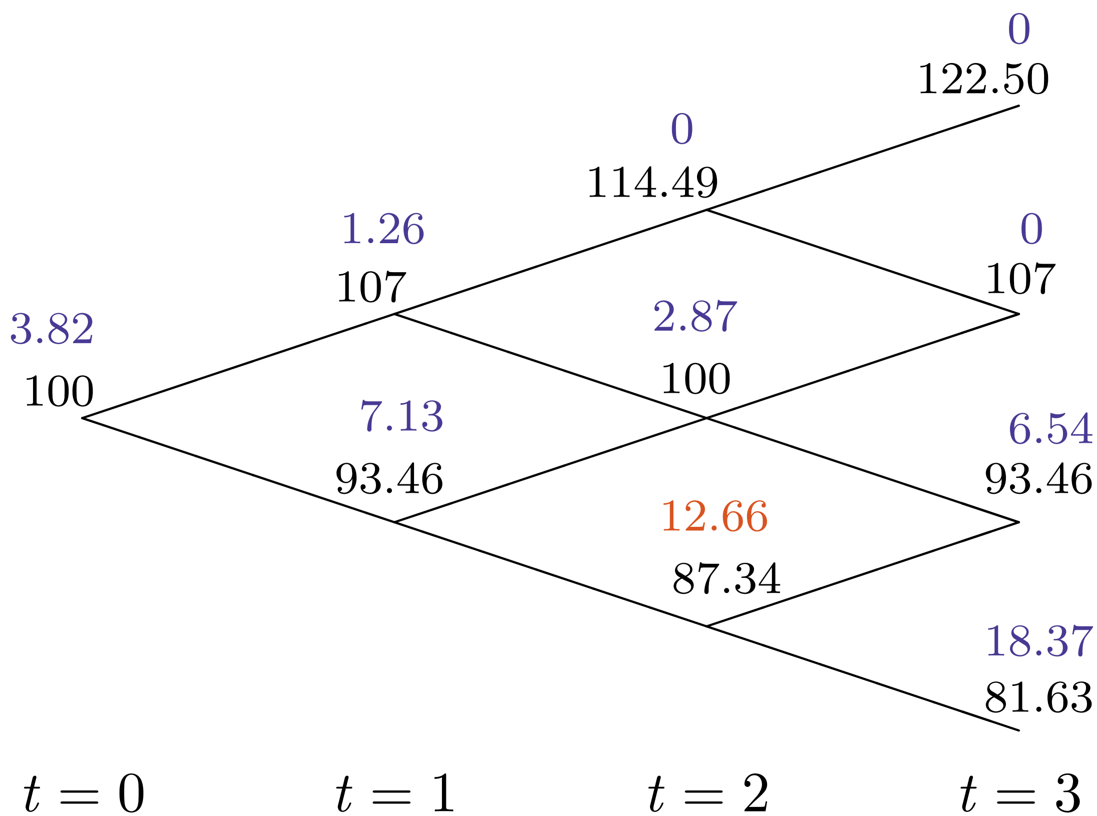
Pricing American Options

- Can also price American options in same way as European options
  - but now must also check if it's optimal to early exercise at each node.
- But recall never optimal to early exercise an American call option on non-dividend paying stock.

e.g. Price American put option: expiration at $t = 3$, $K = \$100$ and $R = 1.01$.



Pricing American Options



- Price option by working backwards in binomial the lattice.

e.g. $$12.66 = \max\left[12.66,\ \frac{1}{R}\left(q \times 6.54 + (1-q) \times 18.37\right)\right]$$

A Simple Die-Throwing Game

Consider the following game:

1. You can throw a fair 6-sided die up to a maximum of three times.
2. After any throw, you can choose to 'stop' and obtain an amount of money equal to the value you threw.

e.g. if 4 thrown on second throw and choose to 'stop', then obtain $\$4$.

Question: If you are risk-neutral, how much would you pay to play this game?

Solution: Work backwards, starting with last possible throw:

1. You have just 1 throw left so fair value is 3.5.
2. You have 2 throws left so must figure out a strategy determining what to do after 1^{st} throw. We find

$$\text{fair value} = \frac{1}{6} \times (4+5+6) + \frac{1}{2} \times 3.5 = 4.25.$$

3. Suppose you are allowed 3 throws. Then ...

Question: What if you could throw the die 1000 times?

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Replicating Strategies in the Binomial Model

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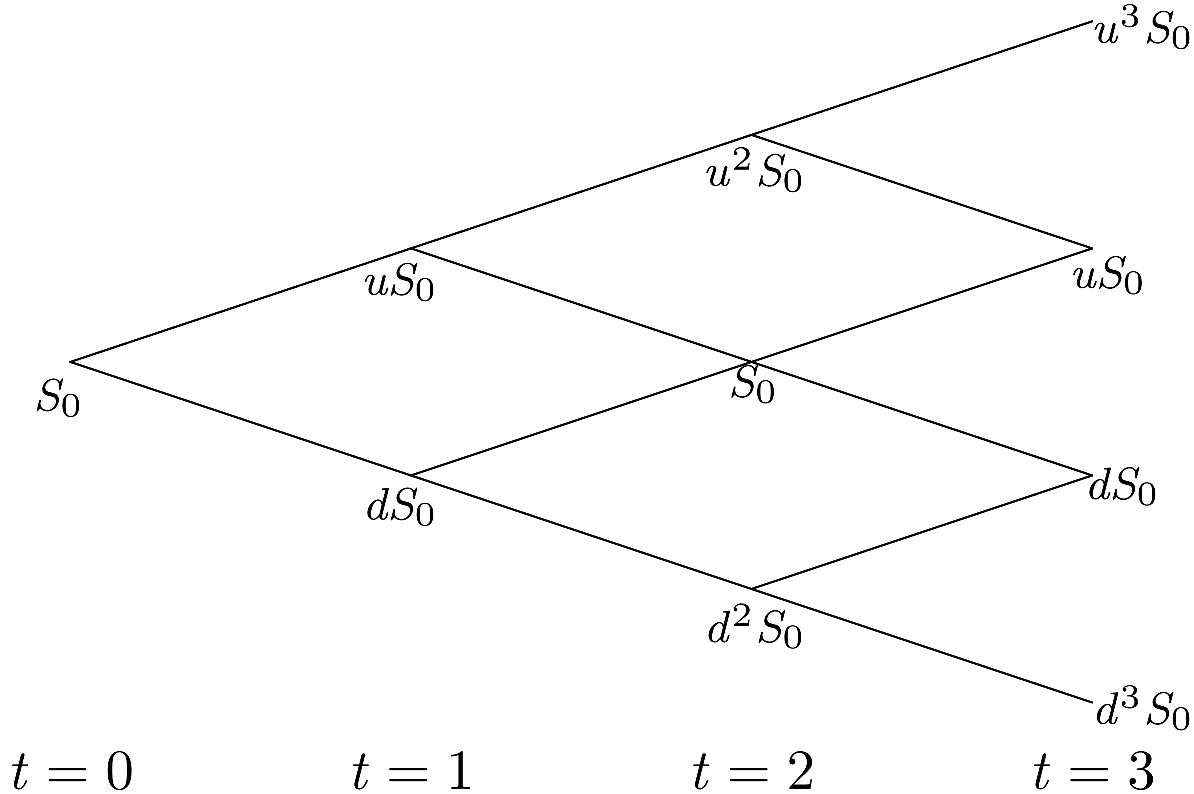
Trading Strategies in the Binomial Model

- Let S_t denote the stock price at time t.
- Let B_t denote the value of the cash-account at time t
  - assume without any loss of generality that $B_0 = 1$ so that $B_t = R^t$
  - so now explicitly viewing the cash account as a security.
- Let x_t denote $\#$ of shares held between times $t-1$ and t for $t = 1, \ldots, n$.
- Let y_t denote $\#$ of units of cash account held between times $t-1$ and t for $t = 1, \ldots, n$.
- Then $\boldsymbol{\theta}_t := (x_t, y_t)$ is the portfolio held:
  (i) immediately **after** trading at time $t-1$ so it is known at time $t-1$
  (ii) and immediately **before** trading at time t.
- $\boldsymbol{\theta}_t$ is also a random process and in particular, a **trading strategy**.

Trading Strategies in the Binomial Model



Self-Financing Trading Strategies

Definition. The value process, $V_t(\boldsymbol{\theta})$, associated with a trading strategy, $\boldsymbol{\theta}_t = (x_t, y_t)$, is defined by

$$V_t = \begin{cases} x_1 S_0 + y_1 B_0 & \text{for } t = 0 \\ \\ x_t S_t + y_t B_t & \text{for } t \geq 1. \end{cases} \tag{3}$$

Definition. A self-financing trading strategy is a trading strategy, $\boldsymbol{\theta}_t = (x_t, y_t)$, where changes in V_t are due entirely to trading gains or losses, rather than the addition or withdrawal of cash funds. In particular, a self-financing strategy satisfies

$$V_t \;=\; x_{t+1} S_t + y_{t+1} B_t\,, \qquad t = 1, \ldots, n-1. \tag{4}$$

The definition states that the value of a self-financing portfolio just before trading is equal to the value of the portfolio just after trading

– so no funds have been deposited or withdrawn.

Self-Financing Trading Strategies

Proposition. If a trading strategy, $\boldsymbol{\theta}_t$, is self-financing then the corresponding value process, V_t, satisfies

$$V_{t+1} - V_t \;=\; x_{t+1}\left(S_{t+1} - S_t\right) \;+\; y_{t+1}\left(B_{t+1} - B_t\right)$$

so that changes in portfolio value can only be due to capital gains or losses and not the injection or withdrawal of funds.

Proof. For $t \geq 1$ we have

$$\begin{aligned} V_{t+1} - V_t \;&=\; \left(x_{t+1}S_{t+1} + y_{t+1}B_{t+1}\right) \;-\; \left(x_{t+1}S_t + y_{t+1}B_t\right) \\ &=\; x_{t+1}(S_{t+1} - S_t) \;+\; y_{t+1}(B_{t+1} - B_t) \end{aligned}$$

and for $t = 0$ we have

$$\begin{aligned} V_1 - V_0 \;&=\; (x_1 S_1 + y_1 B_1) - (x_1 S_0 + y_1 B_0) \\ &=\; x_1(S_1 - S_0) + y_1(B_1 - B_0). \end{aligned}$$

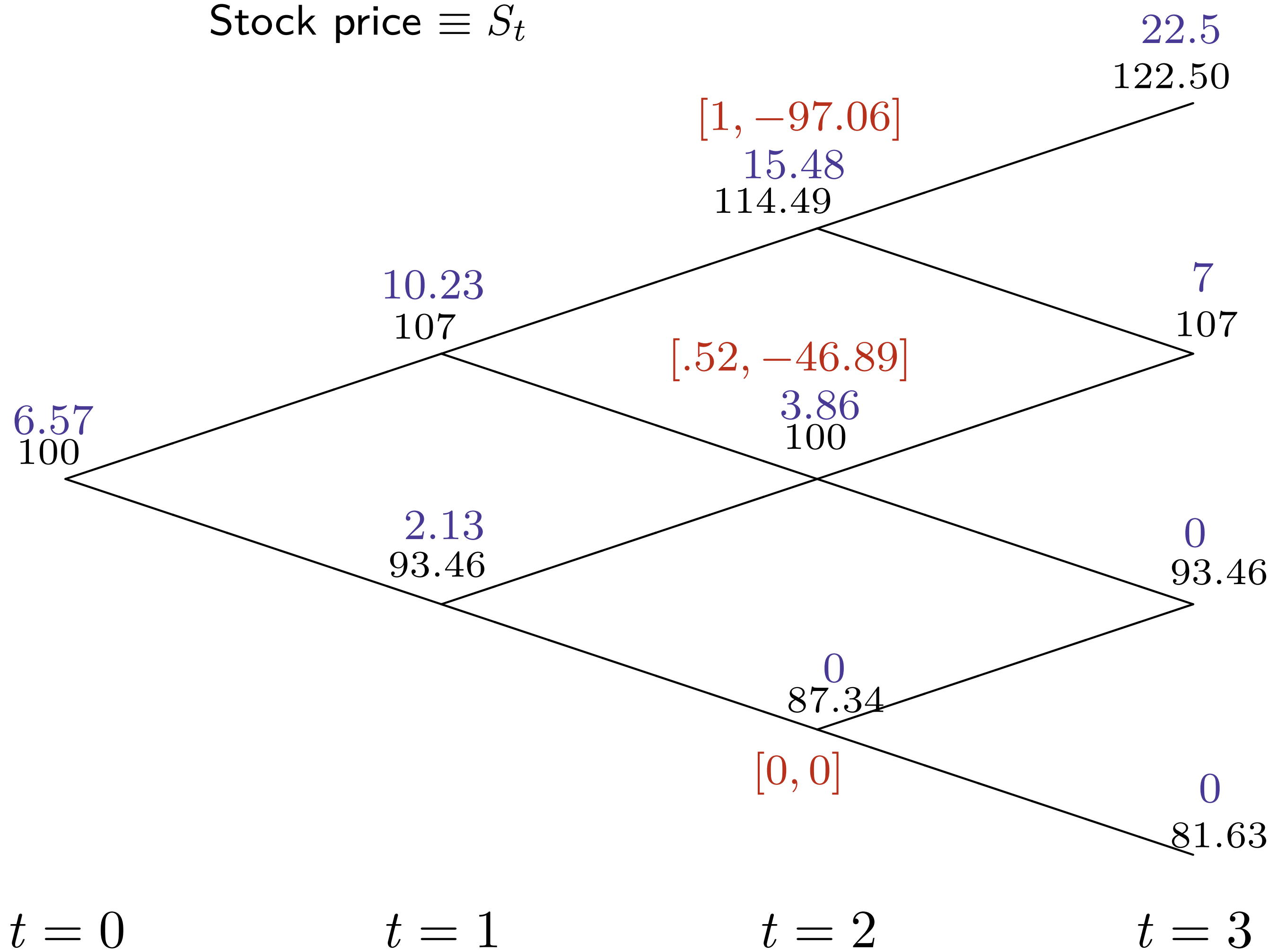
$\square$

Risk-Neutral Price $\equiv$ Price of Replicating Strategy

- We have seen how to price derivative securities in the binomial model.
- The key to this was the use of the 1-period risk neutral probabilities.
- But we first priced options in 1-period models using a replicating portfolio
  - and we did this without needing to define risk-neutral probabilities.
- In the multi-period model we can do the same, i.e., can construct a self-financing trading strategy that replicates the payoff of the option
  - this is called dynamic replication.
- The initial cost of this replicating strategy must equal the value of the option
  - otherwise there's an arbitrage opportunity.
- The dynamic replication price is of course equal to the price obtained from using the risk-neutral probabilities and working backwards in the lattice.
- And at any node, the value of the option is equal to the value of the replicating portfolio at that node.

The Replicating Strategy For Our European Option

Key: Replicating strategy $\equiv [x_t, y_t]$
Option price $\equiv C_t$
Stock price $\equiv S_t$



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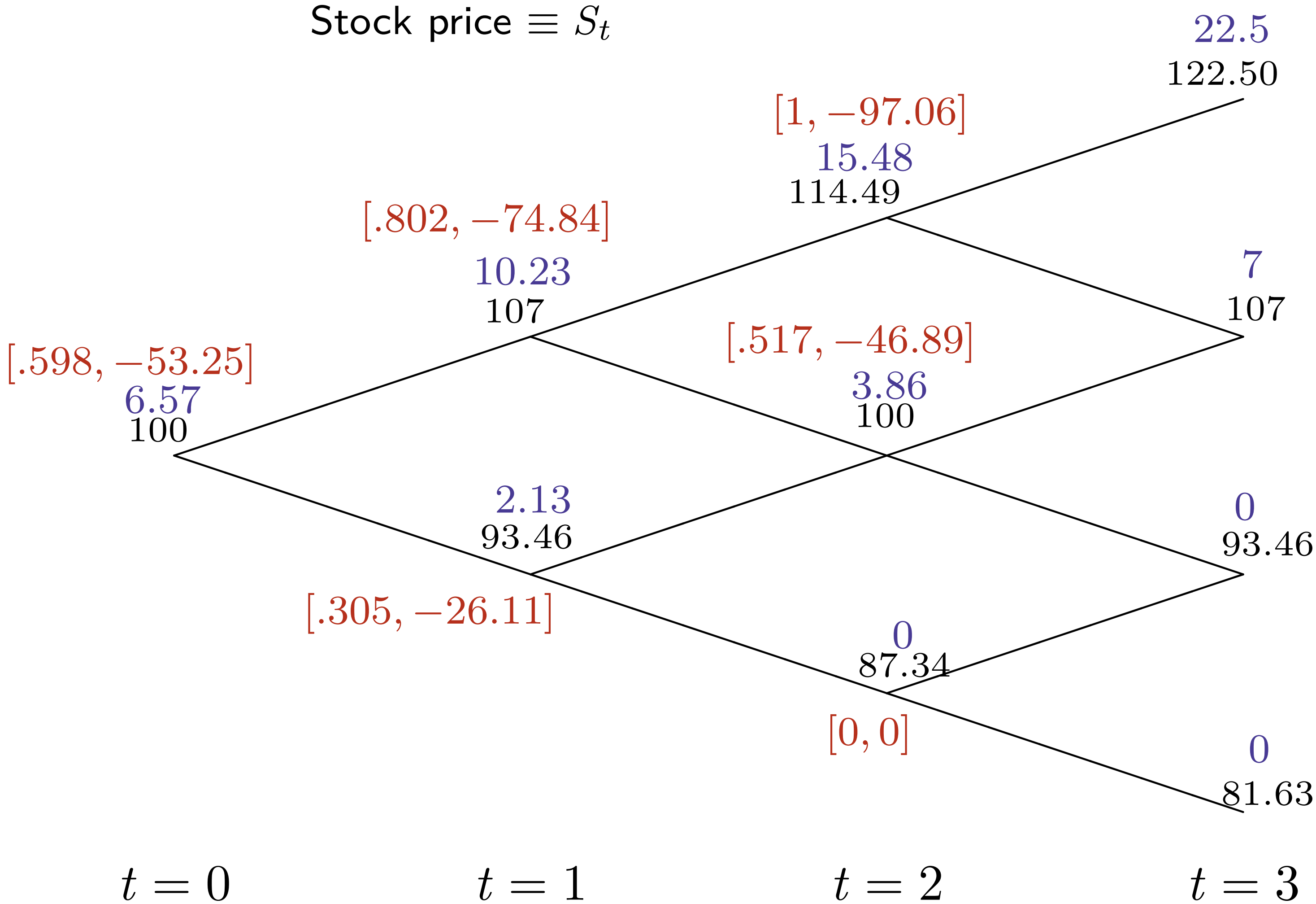

e.g. $.802 \times 107 + (-74.84) \times 1.01 = 10.23$ at upper node at time $t = 1$