

Financial Engineering & Risk Management

Introduction to Brownian Motion

M. Haugh G. Iyengar

Department of Industrial Engineering and Operations Research
Columbia University

Brownian Motion

Definition. We say that a random process, $\{X_t : t \geq 0\}$, is a **Brownian motion** with parameters (μ, σ) if

1. For $0 < t_1 < t_2 < \dots < t_{n-1} < t_n$

$$(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}), \dots, (X_{t_n} - X_{t_{n-1}})$$

are mutually independent.

2. For $s > 0$, $X_{t+s} - X_t \sim N(\mu s, \sigma^2 s)$ and
3. X_t is a continuous function of t .

We say that X_t is a $B(\mu, \sigma)$ Brownian motion with **drift** μ and **volatility** σ

Property #1 is often called the **independent increments** property.

Remark. **Bachelier** (1900) and **Einstein** (1905) were the first to explore Brownian motion from a mathematical viewpoint whereas **Wiener** (1920's) was the first to show that it actually exists as a well-defined mathematical entity.

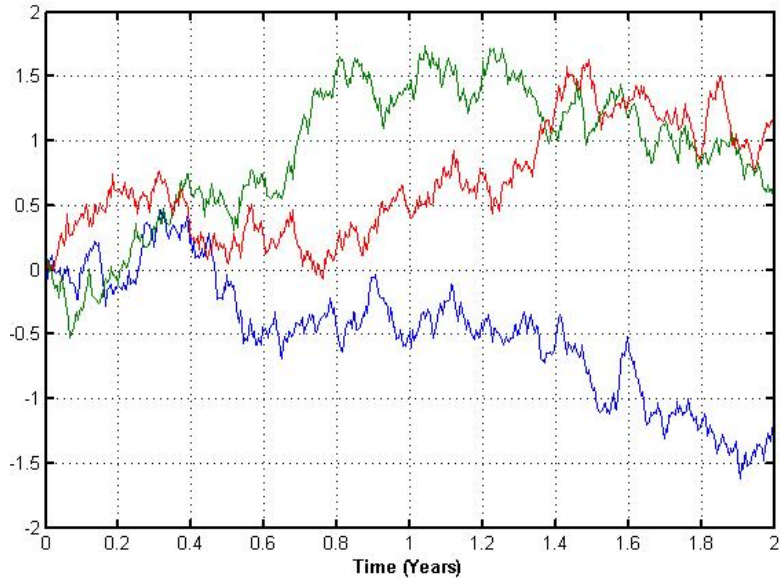
Standard Brownian Motion

- When $\mu = 0$ and $\sigma = 1$ we have a **standard** Brownian motion (SBM).
- We will use W_t to denote a SBM and we always assume that **$W_0 = 0$** .
- Note that if $X_t \sim B(\mu, \sigma)$ and $X_0 = x$ then we can write

$$X_t = x + \mu t + \sigma W_t \tag{8}$$

where W_t is an SBM. Therefore see that $X_t \sim N(x + \mu t, \sigma^2 t)$.

Sample Paths of Brownian Motion



Information Filtrations

- For any random process we will use \mathcal{F}_t to denote the **information** available at time t
 - the set $\{\mathcal{F}_t\}_{t \geq 0}$ is then the **information filtration**
 - so $E[\cdot | \mathcal{F}_t]$ denotes an expectation **conditional** on time t information available.
- Will usually write $E[\cdot | \mathcal{F}_t]$ as $E_t[\cdot]$.

Important Fact: The independent increments property of Brownian motion implies that any function of $W_{t+s} - W_t$ is **independent** of \mathcal{F}_t and that

$$(W_{t+s} - W_t) \sim N(0, s).$$

A Brownian Motion Calculation

Question: What is $E_0[W_{t+s} W_s]$?

Answer: We can use a version of the conditional expectation identity to obtain

$$\begin{aligned} E_0[W_{t+s} W_s] &= E_0[(W_{t+s} - W_s) W_s] \\ &= E_0[(W_{t+s} - W_s) W_s] + E_0[W_s^2]. \end{aligned} \quad (9)$$

Now we know (why?) $E_0[W_s^2] = s$.

To calculate first term on r.h.s. of (9) a version of the **conditional expectation identity** implies

$$\begin{aligned} E_0[(W_{t+s} - W_s) W_s] &= E_0[E_s[(W_{t+s} - W_s) W_s]] \\ &= E_0[W_s E_s[(W_{t+s} - W_s)]] \\ &= E_0[W_s 0] \\ &= 0. \end{aligned}$$

Therefore obtain $E_0[W_{t+s} W_s] = s$.

Financial Engineering & Risk Management

Geometric Brownian Motion

M. Haugh G. Iyengar

Department of Industrial Engineering and Operations Research
Columbia University

Geometric Brownian Motion

Definition. We say that a random process, X_t , is a **geometric Brownian motion** (GBM) if for all $t \geq 0$

$$X_t = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

where W_t is a **standard Brownian motion**.

We call μ the **drift**, σ the **volatility** and write $X_t \sim \text{GBM}(\mu, \sigma)$.

Note that

$$\begin{aligned} X_{t+s} &= X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)(t+s) + \sigma W_{t+s}} \\ &= X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)s + \sigma(W_{t+s} - W_t)} \\ &= X_t e^{\left(\mu - \frac{\sigma^2}{2}\right)s + \sigma(W_{t+s} - W_t)} \end{aligned} \tag{10}$$

– a representation that is very useful for **simulating** security prices.

Geometric Brownian Motion

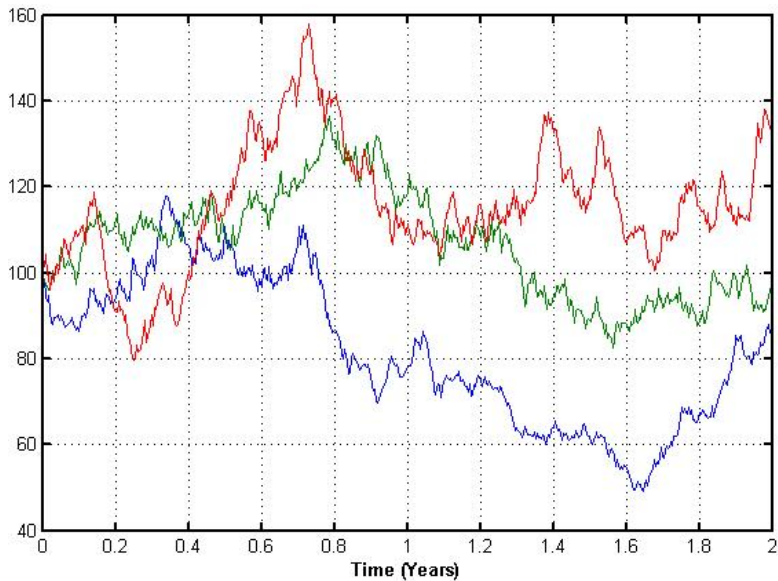
Question: Suppose $X_t \sim \text{GBM}(\mu, \sigma)$. What is $E_t[X_{t+s}]$?

Answer: From (10) we have

$$\begin{aligned} E_t[X_{t+s}] &= E_t \left[X_t e^{\left(\mu - \frac{\sigma^2}{2}\right)s} + \sigma(W_{t+s} - W_t) \right] \\ &= X_t e^{\left(\mu - \frac{\sigma^2}{2}\right)s} E_t \left[e^{\sigma(W_{t+s} - W_t)} \right] \\ &= X_t e^{\left(\mu - \frac{\sigma^2}{2}\right)s} e^{\frac{\sigma^2}{2}s} \\ &= e^{\mu s} X_t \end{aligned}$$

– so the **expected growth rate** of X_t is μ .

Sample Paths of Geometric Brownian Motion



Geometric Brownian Motion

The following properties of GBM follow immediately from the definition of BM:

1. Fix t_1, t_2, \dots, t_n . Then $\frac{X_{t_2}}{X_{t_1}}, \frac{X_{t_3}}{X_{t_2}}, \dots, \frac{X_{t_n}}{X_{t_{n-1}}}$ are mutually independent.
(For a period of time t , consider $0 < t_1 < t_2 < t_3 < t_4 \dots t_n < t$)
2. Paths of X_t are continuous as a function of t , i.e., they do not jump.
3. For $s > 0$, $\log \left(\frac{X_{t+s}}{X_t} \right) \sim N \left(\left(\mu - \frac{\sigma^2}{2} \right) s, \sigma^2 s \right)$.

Modeling Stock Prices as GBM

Suppose $X_t \sim \text{GBM}(\mu, \sigma)$. Then clear that:

1. If $X_t > 0$, then X_{t+s} is always positive for any $s > 0$.
 - so **limited liability** of stock price is not violated.
2. The distribution of X_{t+s}/X_t only depends on s and not on X_t

These properties suggest that GBM might be a reasonable model for stock prices.

Indeed it is the underlying model for the famous **Black-Scholes** option formula.

Financial Engineering and Risk Management

Review of vectors

Martin Haugh

Garud Iyengar

Columbia University

Industrial Engineering and Operations Research

Reals numbers and vectors

- We will denote the set of real numbers by \mathbb{R}
- Vectors are finite collections of real numbers
- Vectors come in two varieties
 - Row vectors: $\mathbf{v} = [v_1 \quad v_2 \quad \dots \quad v_n]$
 - Column vectors $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$
 - By default, vectors are column vectors
- The set of all vectors with n components is denoted by \mathbb{R}^n

Linear independence

- A vector \mathbf{w} is **linearly dependent** on $\mathbf{v}_1, \mathbf{v}_2$ if

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \text{ for some } \alpha_1, \alpha_2 \in \mathbb{R}$$

Example:

$$\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- Other names: linear combination, linear span
- A set $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ are **linearly independent** if **no** \mathbf{v}_i is linearly dependent on the others, $\{\mathbf{v}_j : j \neq i\}$

Basis

- Every $\mathbf{w} \in \mathbb{R}^n$ is a linear combination of the linearly independent set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \quad \mathbf{w} = w_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{e}_1} + w_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{e}_2} + \dots + w_n \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{e}_n}$$

- Basis \equiv any linearly independent set that spans the entire space
- Any basis for \mathbb{R}^n has exactly n elements

Norms

- A function $\rho(\mathbf{v})$ of a vector \mathbf{v} is called a **norm** if
 - $\rho(\mathbf{v}) \geq 0$ and $\rho(\mathbf{v}) = 0$ implies $\mathbf{v} = \mathbf{0}$
 - $\rho(\alpha\mathbf{v}) = |\alpha| \rho(\mathbf{v})$ for all $\alpha \in \mathbb{R}$
 - $\rho(\mathbf{v}_1 + \mathbf{v}_2) \leq \rho(\mathbf{v}_1) + \rho(\mathbf{v}_2)$ (**triangle inequality**)
- ρ generalizes the notion of “length”

- **Examples:**

- ℓ_2 norm: $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$... usual length
- ℓ_1 norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$
- ℓ_∞ norm: $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$
- ℓ_p norm, $1 \leq p < \infty$: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$

Inner product

- The **inner-product** or **dot-product** of two vector $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v_i w_i$$

- The ℓ_2 norm $\|\mathbf{v}\|_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
- The angle θ between two vectors \mathbf{v} and \mathbf{w} is given by

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|_2 \|\mathbf{w}\|_2}$$

- Will show later: $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^\top \mathbf{w} =$ **product** of \mathbf{v} **transpose** and \mathbf{w}