Financial Engineering & Risk Management

The Multi-Period Binomial Model

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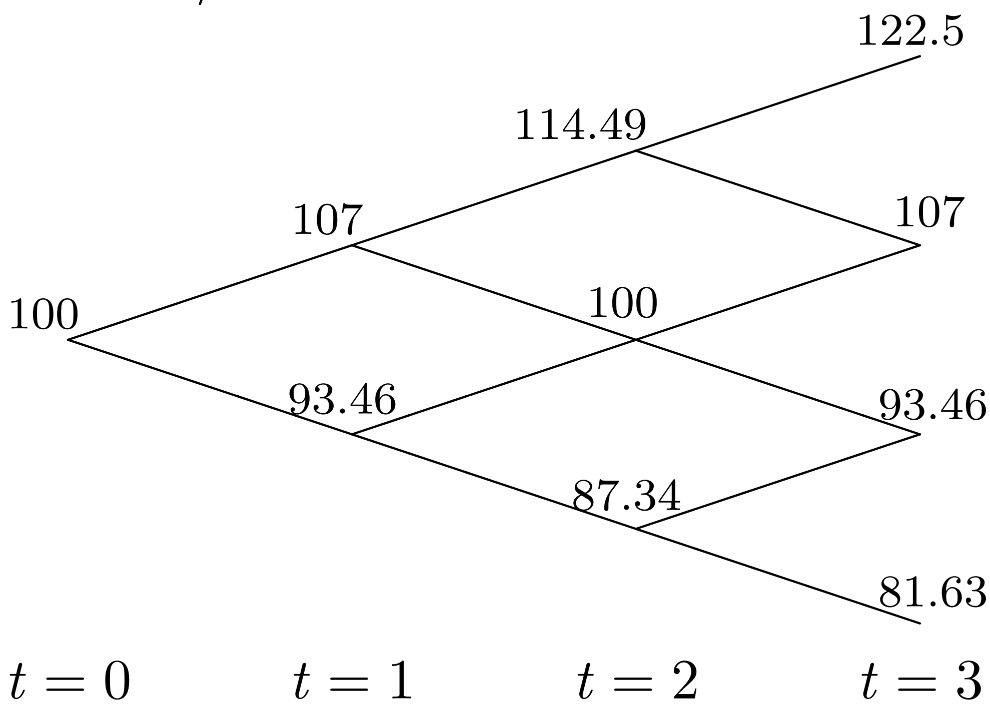
A 3-period Binomial Model

Recall $R = 1.01$ and $u = 1/d = 1.07$.

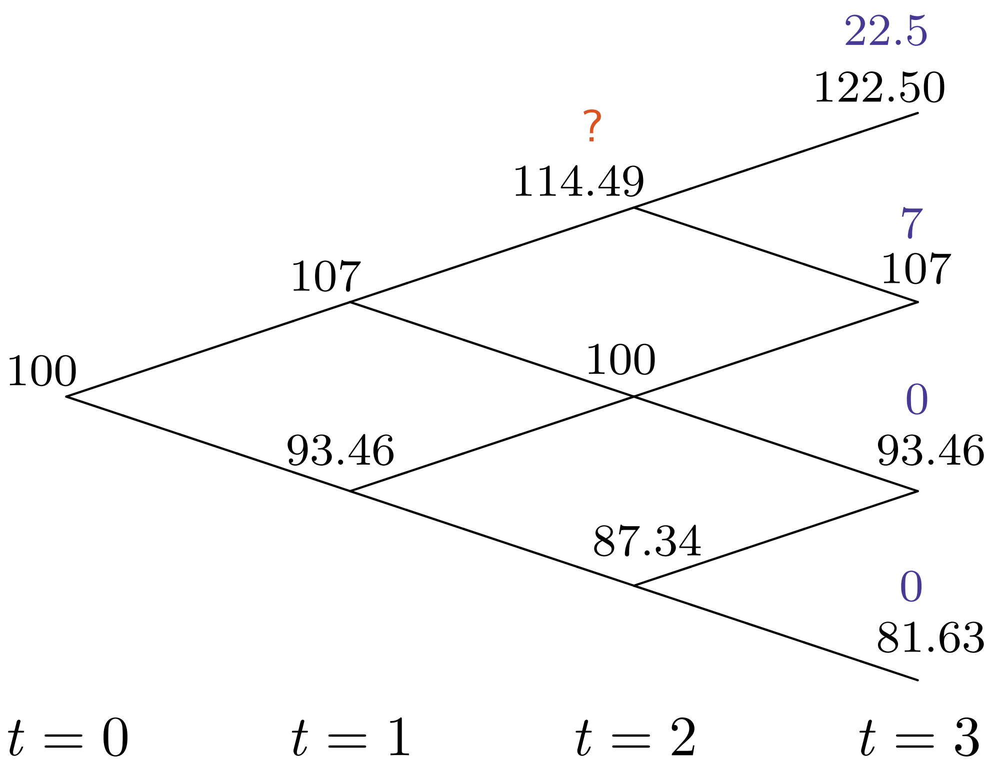


Just a series of 1-period models spliced together!

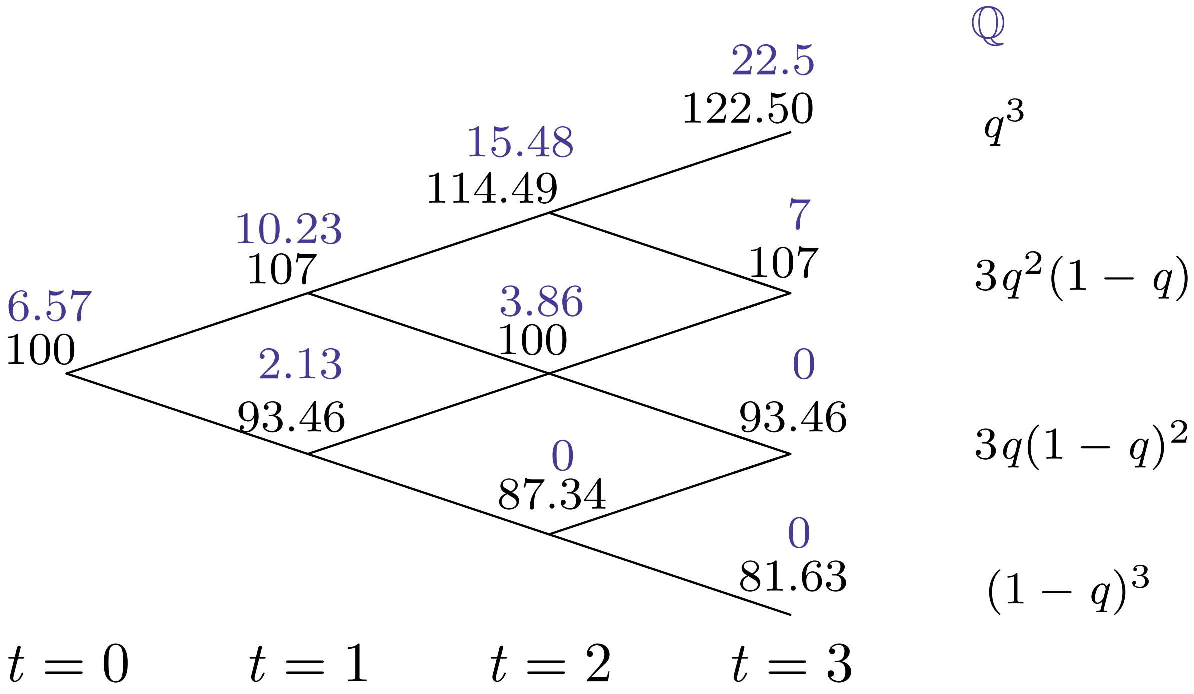
- all the results from the 1-period model apply
- just need to multiply 1-period probabilities along branches to get probabilities in multi-period model.

Pricing a European Call Option

Assumptions: expiration at $t = 3$, strike $= \$100$ and $R = 1.01$.



Pricing a European Call Option



- We can also calculate the price as

$$C_0 = \frac{1}{R^3} \mathsf{E}_0^{\mathbb{Q}} \left[\max(S_T - 100,\ 0) \right] \tag{1}$$

  - this is risk-neutral pricing in the binomial model
  - avoids having to calculate the price at every node.
- How would you find a replicating strategy?
  - to be defined and discussed in another module.

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What's Going On?

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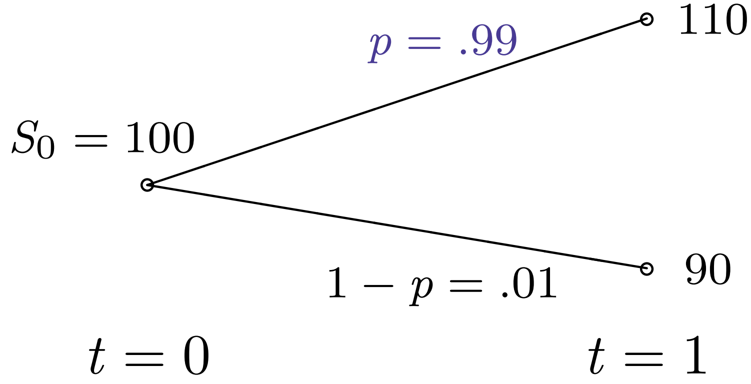
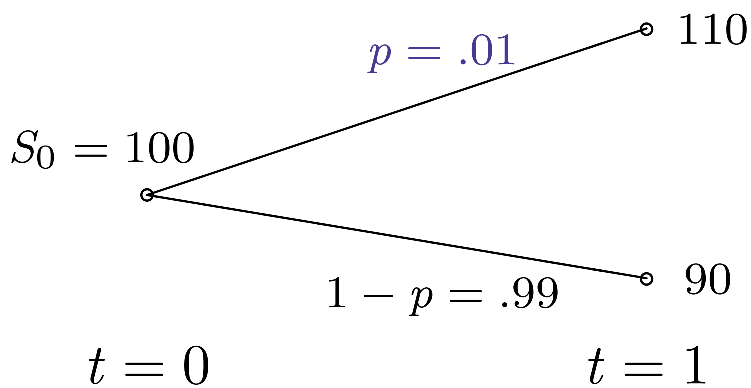
What's Going On?

- Stock ABC



- Stock XYZ



Question: What is the price of a call option on ABC with strike $K = \$100$?

Question: What is the price of a call option on XYZ with strike $K = \$100$?

What's Going On?

- Saw earlier

$$
\begin{aligned}
C_0 &= \frac{1}{R}\left[\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right] \\
&= \frac{1}{R}\left[q\,C_u + (1-q)\,C_d\right] \\
&= \frac{1}{R}\mathsf{E}_0^{\mathbb{Q}}[C_1]
\end{aligned}
$$

- So it appears that p doesn't matter!
- This is true ...
- ... but it only appears surprising because we are asking the wrong question!

Another Surprising Result?

$R = 1.02$

Stock Price

			119.10
		112.36	106.00
	106.00	100.00	94.34
100.00	94.34	89.00	83.96
t=0	t=1	t=2	t=3

European Option Price: K = 95

			24.10
		19.22	11.00
	14.76	7.08	0.00
11.04	4.56	0.00	0.00
t=0	t=1	t=2	t=3

$R = 1.04$

Stock Price

			119.10
		112.36	106.00
	106.00	100.00	94.34
100.00	94.34	89.00	83.96
t=0	t=1	t=2	t=3

European Option Price: K = 95

			24.10
		21.01	11.00
	18.19	8.76	0.00
15.64	6.98	0.00	0.00
t=0	t=1	t=2	t=3

Question: So the option price increases when we increase R. Is this surprising?

(See *"Investment Science"* (OUP) by D. G. Luenberger for additional examples on the binomial model.)

Existence of Risk-Neutral Probabilities $\Leftrightarrow$ No-Arbitrage

Recall our analysis of the binomial model:

- no arbitrage $\Leftrightarrow$ $d < R < u$
- any derivative security with time T payoff, C_T, can be priced using

$$C_0 = \frac{1}{R^n} \mathsf{E}_0^{\mathbb{Q}}[C_T] \tag{2}$$

  where $q > 0$, $1 - q > 0$ and $n = \#$ of periods.

  (If Δt is the length of a period, then $T = n \times \Delta t$.)

In fact for any model if there exists a risk-neutral distribution, $\mathbb{Q}$, such that (2) holds, then arbitrage cannot exist. Why?

Reverse is also true: if there is no arbitrage then a risk-neutral distribution exists.

Together, these last two statements are often called the first fundamental theorem of asset pricing.