Financial Engineering & Risk Management

Including Dividends

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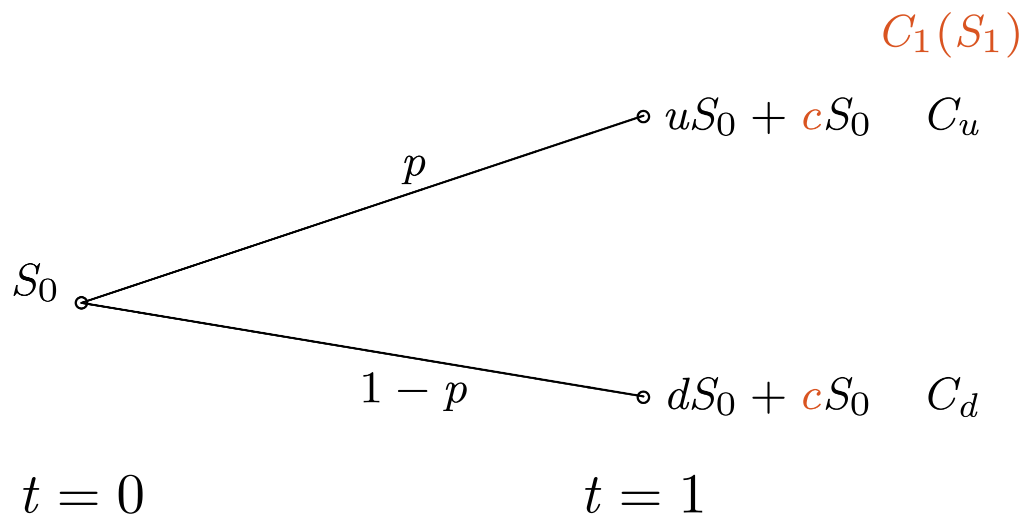
Including Dividends



- Consider again 1-period model and assume stock pays a proportional dividend of cS_0 at $t = 1$.
- No-arbitrage conditions are now $d + c < R < u + c$.
- Can use same replicating portfolio argument to find price, C_0, of any derivative security with payoff function, $C_1(S_1)$, at time $t = 1$.
- Set up replicating portfolio as before:

$$
\begin{aligned}
uS_0x + cS_0x + Ry &= C_u \\
dS_0x + cS_0x + Ry &= C_d
\end{aligned}
$$

Derivative Security Pricing with Dividends

- Solve for x and y as before and then must have $C_0 = xS_0 + y$.
- Obtain

$$
\begin{aligned}
C_0 &= \frac{1}{R}\left[\frac{R-d-c}{u-d}C_u + \frac{u+c-R}{u-d}C_d\right] \\
&= \frac{1}{R}\left[qC_u + (1-q)C_d\right] \\
&= \frac{1}{R}\mathsf{E}_0^{\mathbb{Q}}[C_1].
\end{aligned} \tag{5}
$$

- Again, can price any derivative security in this 1-period model.
- Multi-period binomial model assumes a proportional dividend in each period
  - so dividend of cS_i is paid at $t = i+1$ for each i.
- Then each embedded 1-period model has identical risk-neutral probabilities
  - and derivative securities priced as before.
- In practice dividends are not paid in every period
  - and are therefore just a little more awkward to handle.

The Binomial Model with Dividends

- Suppose the underlying security does not pay dividends. Then

$$S_0 \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n}{R^n}\right] \tag{6}$$

  – this is just risk-neutral pricing of European call option with $K = 0$.

- Suppose now underlying security pays dividends in each time period.
- Then can check (6) no longer holds.
- Instead have

$$S_0 \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n}{R^n} + \sum_{i=1}^{n} \frac{D_i}{R^i}\right] \tag{7}$$

  - D_i is the dividend at time i
  - and S_n is the ex-dividend security price at time n.

- Don't need any new theory to prove (7)
  - it follows from risk-neutral pricing and observing that dividends and S_n may be viewed as a portfolio of securities.

Viewing a Dividend-Paying Security as a Portfolio

- To see this, we can view the i^{th} dividend as a separate security with value

$$P_i \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{D_i}{R^i}\right].$$

- Then owner of underlying security owns a "portfolio" of securities at time 0
  - value of this "portfolio" is $\sum_{i=1}^{n} P_i \;+\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n}{R^n}\right]$.
- But value of underlying security is S_0.
- Therefore must have

$$S_0 \;=\; \sum_{i=1}^{n} P_i \;+\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n}{R^n}\right]$$

which is (7).

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Pricing Forwards and Futures

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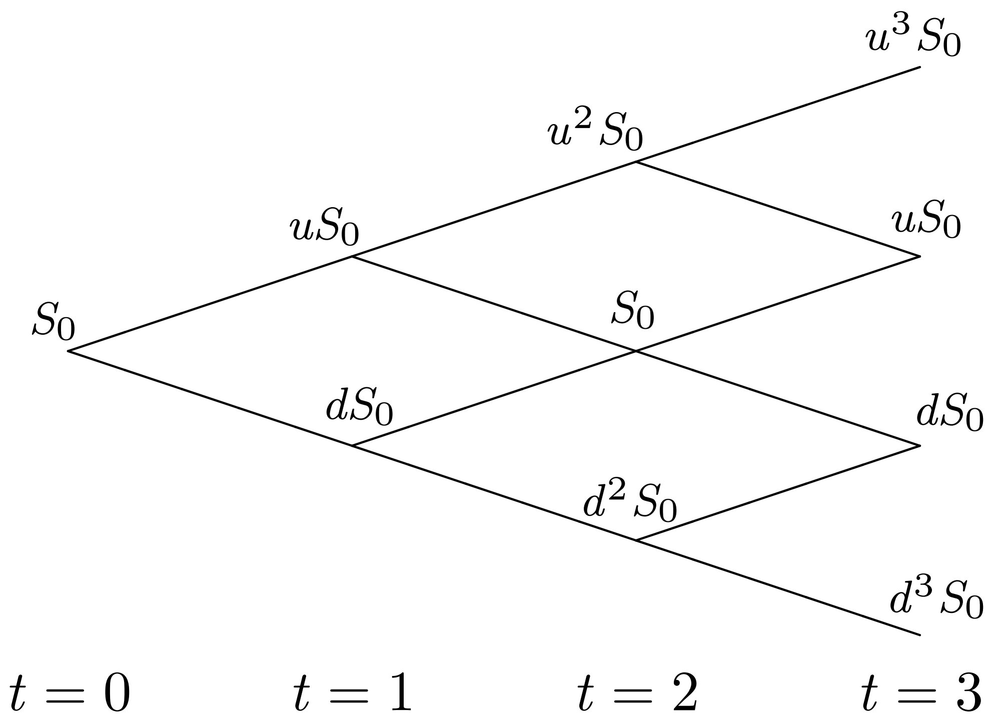
Pricing Forwards in the Binomial Model

- Have an n-period binomial model with $u = 1/d$.



- Consider now a forward contract on the stock that expires after n periods.
- Let G_0 denote date $t = 0$ "price" of the contract.
- Recall G_0 is chosen so that contract is initially worth zero.

Pricing Forwards in the Binomial Model

- Therefore obtain

$$0 = \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n - G_0}{R^n}\right]$$

so that

$$G_0 = \mathsf{E}_0^{\mathbb{Q}}\left[S_n\right]. \tag{8}$$

- Again, (8) holds whether the underlying security pays dividends or not.

What is a Futures "Price"?

- Consider now a futures contract on the stock that expires after n periods.
- Let F_t be the date t "price" of the futures contract for $0 \leq t \leq n$.
- Then $F_n = S_n$. Why?
- A common misconception is that:
  - (i) F_t is how much you must pay at time t to buy one contract
  - (ii) or how much you receive if you sell one contract

  This is **false!**
- A futures contract always costs nothing.
- The "price", F_t is only used to determine the cash-flow associated with holding the contract
  - so that $\pm(F_t - F_{t-1})$ is the payoff received at time t from a long or short position of one contract held between $t-1$ and t.
- In fact a futures contract can be characterized as a security that:
  - (i) is always worth zero
  - (ii) and that pays a dividend of $(F_t - F_{t-1})$ at each time t.

Pricing Futures in the Binomial Model

- Can compute time $t = n-1$ futures price, F_{n-1}, by solving

$$0 \quad = \quad \mathsf{E}^{\mathbb{Q}}_{n-1}\left[\frac{F_n - F_{n-1}}{R}\right]$$

  to obtain $F_{n-1} = \mathsf{E}^{\mathbb{Q}}_{n-1}[F_n]$.
- In general we have $F_t = \mathsf{E}^{\mathbb{Q}}_t[F_{t+1}]$ for $0 \leq t < n$ so that

$$\begin{aligned} F_t &= \mathsf{E}^{\mathbb{Q}}_k[F_{t+1}] \\ &= \mathsf{E}^{\mathbb{Q}}_t[\mathsf{E}^{\mathbb{Q}}_{t+1}[F_{t+2}]] \\ \vdots &\qquad \vdots \\ &= \mathsf{E}^{\mathbb{Q}}_t[\mathsf{E}^{\mathbb{Q}}_{t+1}[\ \cdots\ \mathsf{E}^{\mathbb{Q}}_{n-1}[F_n]]]. \end{aligned}$$

Pricing Futures in the Binomial Model

- Law of iterated expectations then implies $F_t = \mathsf{E}_t^{\mathbb{Q}}[F_n]$
  - so the futures price process is a $\mathbb{Q}$-martingale.
- Taking $t = 0$ and using $F_n = S_n$ we also have

$$F_0 = \mathsf{E}_0^{\mathbb{Q}}[S_n]. \tag{9}$$

- Note that (9) holds whether the security pays dividends or not
  - dividends only enter through $\mathbb{Q}$.
- Comparing (8) and (9) and we see that $F_0 = G_0$ in the binomial model
  - not true in general.

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The Black-Scholes Model

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The Black-Scholes Model

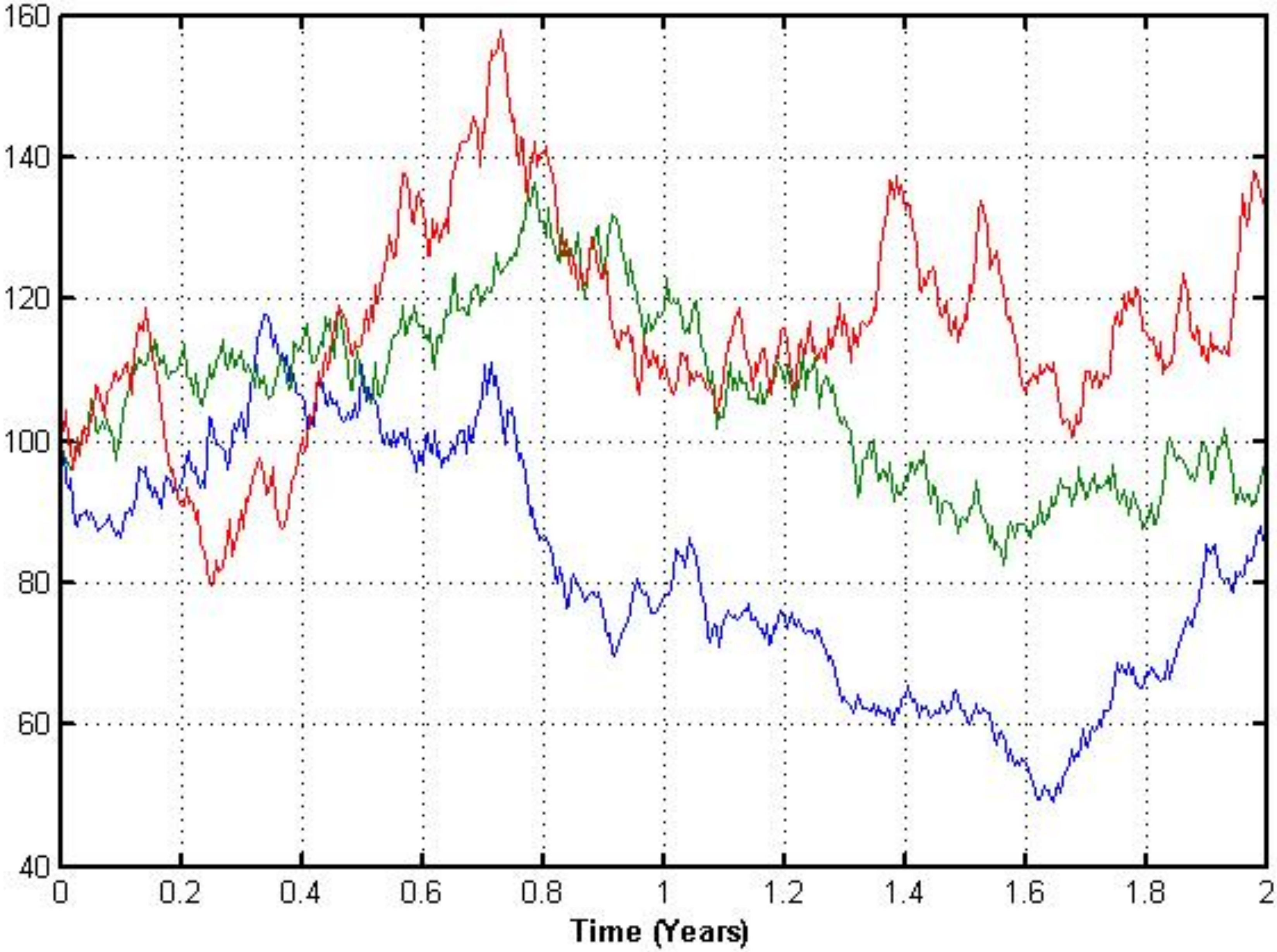
Black and Scholes assumed:

1. A continuously-compounded interest rate of r.
2. Geometric Brownian motion dynamics for the stock price, S_t, so that

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$$

   where W_t is a standard Brownian motion.
3. The stock pays a dividend yield of c.
4. Continuous trading with no transactions costs and short-selling allowed.

Sample Paths of Geometric Brownian Motion



The Black-Scholes Formula

- The Black-Scholes formula for the price of a European call option with strike K and maturity T is given by

$$C_0 = S_0 e^{-cT} N(d_1) - K e^{-rT} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r - c + \sigma^2/2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

and $N(d) = \mathsf{P}(N(0,1) \leq d)$.

- Note that μ does not appear in the Black-Scholes formula
  - just as p is not used in option pricing calculations for the binomial model.
- European put option price, P_0, can be calculated from put-call parity

$$P_0 + S_0 e^{-cT} = C_0 + K e^{-rT}.$$

The Black-Scholes Formula

- Black-Scholes obtained their formula using a similar replicating strategy argument to the one we used for the binomial model.
- In fact, can show that under the Black-Scholes GBM model

$$C_0 \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[e^{-rT}\max(S_T - K,\, 0)\right]$$

where under $\mathbb{Q}$

$$S_t \;=\; S_0\, e^{(r-c-\sigma^2/2)t+\sigma W_t}.$$

Calibrating a Binomial Model

- Often specify a binomial model in terms of Black-Scholes parameters:
  1. r, the continuously compounded interest rate.
  2. σ, the annualized volatility.
- Can convert them into equivalent binomial model parameters:
  1. $R_n = \exp\left(r\frac{T}{n}\right)$, where $n =$ number of periods in binomial model
  2. $R_n - c_n = \exp\left((r-c)\frac{T}{n}\right) \approx 1 + r\frac{T}{n} - c\frac{T}{n}$
  3. $u_n = \exp\left(\sigma\sqrt{\frac{T}{n}}\right)$
  4. $d_n = 1/u_n$

  and now price European and American options, futures etc. as before.
- Then risk-neutral probabilities calculated as

$$q_n = \frac{e^{(r-c)\frac{T}{n}} - d_n}{u_n - d_n}.$$

- Spreadsheet calculates binomial parameters this way
  - binomial model prices converge to Black-Scholes prices as $n \to \infty$.

The Binomial Model as $\Delta t \to 0$

- Consider a binomial model with n periods
  - each period corresponds to time interval of $\Delta t := T/n$.
- Recall that we can calculate European option price with strike K as

$$C_0 = \frac{1}{R^n} \mathsf{E}_0^{\mathbb{Q}} \left[\max(S_T - K,\ 0) \right] \tag{10}$$

- In the binomial model can write (10) as

$$\begin{aligned} C_0 &= \frac{1}{R_n^n} \sum_{j=0}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j} \max(S_0 u_n^j d_n^{n-j} - K,\ 0) \\ &= \frac{S_0}{R_n^n} \sum_{j=\eta}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j}\ u_n^j d_n^{n-j} - \frac{K}{R_n^n} \sum_{j=\eta}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j} \end{aligned}$$

where $\eta := \min\{ j : S_0 u_n^j d_n^{n-j} \geq K \}$.

- Can show that if $n \to \infty$ then C_0 converges to the Black-Scholes formula.

Some History

- Bachelier (1900) perhaps first to model Brownian motion
  - modeled stock prices on the Paris Bourse
  - predated Einstein by 5 years.
- Samuelson (1965) rediscovered the work of Bachelier
  - proposed geometric Brownian motion as a model for security prices
  - succeeded in pricing some kinds of warrants
  - was Merton's doctoral adviser
- Itô (1950's) developed the Itô or stochastic calculus
  - the main mathematical tool in finance
  - Itô's Lemma used later by Black-Scholes-Merton
  - Doeblin (1940) recently credited with independently developing stochastic calculus
- Black-Scholes-Merton (early 1970's) published their papers
- Many other influential figures
  - Thorpe (card-counting and perhaps first to discover Black-Scholes formula?)
  - Cox and Ross
  - Harrison and Kreps
  - . . .