Financial Derivatives 3

Three-Period Binomial Model for Option Pricing

1. Introduction

This section extends the one-period binomial model to a three-period model.

The multi-period model allows for more realistic and flexible pricing of options.

2. Model Parameters

• Initial Stock Price (S_0): $100

• Gross Risk-Free Rate (R): 1.01 per period

• Up Factor (u): 1.07

• Down Factor (d): Implied from model (not directly provided)

3. Risk-Neutral Probabilities

• True Probabilities: Denoted as p (up) and 1 - p (down); used for statistical forecasting.

• Risk-Neutral Probabilities: Denoted as q (up) and 1 - q (down); used for pricing derivatives under no-arbitrage condition.

• Formula for Risk-Neutral Probability (q):

q = (R - d) / (u - d)

where R is the risk-free rate, u is the up factor, and d is the down factor.

4. Multi-Period Model Structure

• The three-period model is built from multiple one-period models.

• Each node in the model represents a possible price of the stock at a given time.

• Terminal stock prices at the end of three periods are determined using combinations of up and down movements.

5. Calculating Terminal Probabilities

• The probability of reaching each terminal stock price is given by the binomial distribution:

Probability = C(n, k) * q^k * (1 - q)^(n - k)

where:

- C(n, k) is the binomial coefficient

- n is the number of periods (3 in this case)

- k is the number of up moves

- q is the risk-neutral probability

6. Application in Option Pricing

• Using this framework, the expected payoff of the option under the risk-neutral measure is computed.

• This expected payoff is then discounted at the risk-free rate to get the present value of the option.

• This extends the logic of the one-period model to a multi-period context, allowing for a more robust pricing mechanism.

Pricing a European Call Option Using a Three-Period Binomial Model

1. Introduction

This section focuses on pricing a European call option using the three-period binomial model.

2. Model Parameters

• Strike Price (K): $100

• Stock Price (at t = 3): The stock price outcomes at terminal time (t = 3) are dependent on the up and down movements over three periods.

• Payoffs at t = 3:

- Zero for nodes where stock prices are below $100.

- $7 or $22.5 for nodes where stock prices exceed $100.

3. Backward Calculation Process

• The value of the European call option at time (t = 0) is determined by working backwards through the model.

• Starting from the payoff at time (t = 3), the value is calculated at each node by applying risk-neutral probabilities.

• Steps:

1. Calculate the expected option value at time (t = 2) using the values from time (t = 3).

2. Repeat the process for time (t = 1) using values from time (t = 2).

3. Finally, calculate the value of the option at time (t = 0).

4. Risk-Neutral Probabilities

• To ensure the model is arbitrage-free, risk-neutral probabilities are used.

• These probabilities are calculated based on the risk-free rate and the up/down factors of the stock price.

5. Spreadsheet for Calculation

• A spreadsheet can be provided to allow learners to visualize the calculations at each node and see the formulas used to determine the option's price at each point.

6. Alternative One-Step Method

• An alternative method for calculating the option's price in one step using risk-neutral pricing avoids the need to compute values at every node.

• This approach demonstrates a simplified method for pricing options using the principles of the binomial model.

7. Conclusion

• The three-period binomial model is a powerful tool for option pricing, and by working backwards from the final payoff, the value of a European call option can be determined.

One-Period Binomial Model and Surprising Results in Option Pricing

1. Introduction

This section explores the surprising results regarding option pricing for two identical securities with different probabilities of price movements.

2. Two Securities

• Stock ABC has a 0.99 probability of going up, while Stock XYZ has a 0.01 probability.

• Despite the significant difference in probabilities, both securities have the same option price of approximately $4.8.

3. Option Pricing Theory

• The theory suggests that the fair value of an option is based on risk-neutral probabilities rather than actual probabilities.

• This leads to the conclusion that the option prices remain the same regardless of the differing probabilities.

4. Three-Period Binomial Model

• An example is provided where a European call option is priced under different risk-free rates.

• The example shows that an increase in the interest rate can lead to a higher option price, which contrasts with deterministic models.

5. Risk-Neutral Probabilities

• The existence of risk-neutral probabilities is linked to the concept of no-arbitrage.

• If such probabilities exist, arbitrage opportunities cannot arise, ensuring a fair and consistent pricing model.

6. Conclusion

• The lecture emphasizes the importance of understanding the underlying assumptions in option pricing models.

• It highlights the implications of risk-neutral probabilities and the role they play in option pricing.

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American Options and Early Exercise

1. Introduction

This section focuses on American options, which allow for early exercise, unlike European options.

2. Pricing American Options

• The process involves working backwards through a binomial model, similar to European options, but with the added step of checking if early exercise is optimal at each node.

3. Example of American Put Options

• A put option with a strike price of $100 is analyzed, and the payoff is calculated based on the stock price at expiration.

4. Optimal Exercise

• At each node, the value of continuing versus exercising early is compared.

• The fair value of the American option is determined by this comparison.

5. Die Throwing Game

• An example illustrates optimal stopping problems, where players decide whether to continue throwing a die or stop to take the current value.

6. Conclusion

• The lecture emphasizes the importance of understanding the conditions under which it is optimal to exercise American options.

• It also discusses how to calculate their value using a binomial model.

Replicating Strategies in the Binomial Model

1. Introduction

This module focuses on replicating strategies in the binomial model and how they can replicate the payoff of options.

2. Replicating Strategies

• These strategies replicate the payoff of options in both one-period and multi-period binomial models.

3. Cash Account

• The cash account is introduced as a security, denoted as (B_t), which grows at a risk-free rate (R).

4. Portfolio Definition

• The portfolio at time (t) is defined as ($\Theta_t = (x_t, y_t)$), where (x_t) is the number of shares held and (y_t) is the cash account units.

5. Self-Financing Trading Strategies

• A self-financing strategy means that changes in the portfolio's value are due solely to trading gains or losses, not from adding or withdrawing cash.

6. Value Process

• The value of the portfolio is calculated as ($V_t = x_t S_t + y_t B_t$).

7. Conclusion

• This module clarifies how replicating strategies and self-financing strategies can be applied in pricing options.

Risk-Neutral Prices and Replicating Strategies

1. Introduction

This lecture focuses on risk-neutral prices of securities and their relationship with replicating strategies in option pricing.

2. Risk-Neutral Pricing

• The prices derived from constructing replicating strategies are equivalent to risk-neutral prices.

3. Replicating Portfolio

• Initially, options were priced using a replicating portfolio without defining risk-neutral probabilities.

• This method is also applicable in multi-period models through dynamic replication.

4. Self-Financing Strategy

• A self-financing trading strategy replicates the option's payoff, ensuring that the initial cost equals the option's value to avoid arbitrage opportunities.

5. Dynamic Replication

• The lecture illustrates how to construct a trading strategy that adjusts holdings in stock and cash to replicate the option's payoff at maturity.

6. Example

• The pricing of a European option is discussed, showing how to compute option prices at different time nodes using one-period models and risk-neutral probabilities.

7. Conclusion

• This lecture emphasizes the importance of understanding both the theoretical and practical aspects of option pricing in financial engineering.

Incorporating Dividends into the Binomial Model for Option Pricing

1. Introduction

This lecture focuses on incorporating dividends into the binomial model for option pricing.

2. One-Period Binomial Model

• The model starts with an initial stock price (S_0) and considers the probabilities of the stock moving up or down.

• If the stock goes up, its value becomes (uS_0); if it goes down, it becomes (dS_0).

3. Dividends

• A proportional dividend ($C \times S_0$) is paid at time ($t=1$), regardless of the stock's movement.

• This dividend is included in the calculations for both up and down states.

4. No-Arbitrage Conditions

• The conditions ensure that the risk-free rate (R) must be greater than or equal to the total returns from the stock, including dividends, to prevent arbitrage opportunities.

5. Replicating Strategy

• A portfolio is constructed to replicate the payoff of a derivative security, leading to equations that help determine the fair value of the derivative.

6. Multi-Period Model

• In a multi-period setting, dividends can be paid in each period, and the pricing of derivatives can still be calculated using risk-neutral probabilities.

7. Conclusion

• This lecture emphasizes the importance of dividends in pricing derivatives and how they can be integrated into the binomial model effectively.

Pricing Forwards and Futures Using the Binomial Model

1. Forward Contracts

• The forward price (G_0) is set so that the contract's initial value is zero, meaning no money changes hands at the start.

• The payoff at maturity is ($S_n - G_0$).

• Using risk-neutral pricing, the expected value of the payoff discounted to the present equals zero, leading to the determination of (G_0).

2. Futures Contracts

• Similar to forwards, the futures price (F_t) is also set to zero at the start.

• The payoff at maturity is defined as ($F_n = S_n$).

• The relationship between future prices at different times can be derived using risk-neutral probabilities.

3. Comparison

• In the binomial model, the prices of forwards and futures are identical, although this is not generally true in other models, especially when interest rates are random.

4. Conclusion

• This lecture emphasizes the mechanics of pricing these contracts and the underlying principles of risk-neutral valuation.

The Black-Scholes Formula

1. Assumptions

• Continuously compounded interest rate (r).

• Stock price follows Geometric Brownian motion.

• Dividend yield (c) is considered.

• Continuous trading with no transaction costs and allowance for short-selling.

2. Formula Derivation

• The formula for a European call option is derived, noting that the drift (Mu) does not appear in the final formula, similar to the binomial model.

• European put option prices can be calculated using put-call parity.

3. Calibration

• The binomial model can approximate the Black-Scholes model, requiring calibration of parameters like r and sigma.

4. Historical Context

• The module briefly mentions influential figures in the development of stochastic calculus and option pricing.

5. Conclusion

• This summary encapsulates the main concepts and applications of the Black-Scholes formula in financial engineering.