Principle of inclusion-exclusion

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Theorem 1. *(Principle of inclusion-exclusion) Let $A_1, \ldots, A_k$ be k finite sets. We have*

$$|\cup_{i=1}^{k} A_i| = \sum_{i} |A_i| - \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^{k-1}|A_1 \cap \cdots \cap A_k|$$

证明. We use induction.

While n=2, we have $|A \cup B| = |A| + |B| - |A \cap B|$. Obviously.

Suppose while $n = s(s \geq 2, s \in \mathbb{N})$, the theorem is right.

Then while $n = s+1$,

$|\cup_{i=1}^{n} A_i| = |\cup_{i=1}^{s+1} A_i| = |(\cup_{i=1}^{s} A_i) \cup A_{s+1}|$

$= |\cup_{i=1}^{s} A_i| + |A_{s+1}| - |(\cup_{i=1}^{s} A_i) \cap A_{s+1}|$

$= |\cup_{i=1}^{s} A_i| + |A_{s+1}| - |\cup_{i=1}^{s} (A_i \cap A_{s+1})|$

$= \sum_{1 \leq i \leq s+1} |A_i| + \sum_{k=2}^{s} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| + \sum_{k=1}^{s-1} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} \cap A_{s+1}| + (-1)^s |A_1 \cap A_2 \cap \cdots \cap A_s \cap A_{s+1}|$

$= \sum_{1 \leq i \leq s+1} |A_i| + \sum_{k=2}^{s} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s+1} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| + (-1)^s |A_1 \cap A_2 \cap \cdots \cap A_s \cap A_{s+1}|$

$= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s+1} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$ $\square$