Indicator function for lim sup and lim inf

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Question 1. $(E_n, n \in \mathbb{N})$ ***is a sequence of events.Show that***

$$\forall \omega \in \Omega, \limsup 1_{E_n}(\omega) = 1_{\limsup E_n}(\omega) \ \textbf{\textit{and}} \ \liminf 1_{E_n}(\omega) = 1_{\liminf E_n}(\omega)$$

Proof.
By definition $(2.5(a))$ in textbook, $\limsup 1_{E_n}(\omega) = \downarrow \lim_m \{\sup_{n\geq m} 1_{E_n}(\omega)\}$.
Let us consider $\sup_{n\geq m} 1_{E_n}(\omega)$ firstly.

$$\sup_{n\geq m} 1_{E_n}(\omega) = \begin{cases} 1 & \omega \in \bigcup_{n\geq m} E_n (\text{i.e. } \exists n \geq m \text{ s.t. } \omega \in E_n) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Thus

$$\downarrow \lim_m \{\sup_{n\geq m} 1_{E_n}(\omega)\} = \begin{cases} 1 & \omega \in \bigcap_m \bigcup_{n\geq m} E_n (\text{i.e. } \forall m \exists n \geq m \text{ s.t. } \omega \in E_n) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

According to the definition $(2.6(a))$, $\limsup E_n = \bigcap_m \bigcup_{n\geq m} E_n$

$$\downarrow \lim_m \{\sup_{n\geq m} 1_{E_n}(\omega)\} = \begin{cases} 1 & \omega \in \limsup E_n \\ 0 & \text{otherwise} \end{cases} = 1_{\limsup E_n}(\omega) \tag{3}$$

Therefore

$$\forall \omega \in \Omega, \limsup 1_{E_n}(\omega) = \downarrow \lim_m \{\sup_{n\geq m} 1_{E_n}(\omega)\} = 1_{\limsup E_n}(\omega)$$

□

Next we establish the corresponding result for lim infs.

Proof.
By definition $(2.5(b))$ in textbook, $\liminf 1_{E_n}(\omega) = \uparrow \lim_m \{\inf_{n\geq m} 1_{E_n}(\omega)\}$.

$$\inf_{n\geq m} 1_{E_n}(\omega) = \begin{cases} 1 & \omega \in \bigcap_{n\geq m} E_n (\text{i.e. } \forall n \geq m, \omega \in E_n) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Thus

$$\uparrow \lim_m \{\inf_{n\geq m} 1_{E_n}(\omega)\} = \begin{cases} 1 & \omega \in \bigcup_m \bigcap_{n\geq m} E_n (\text{i.e. } \exists m \forall n \geq m, \omega \in E_n) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

According to the definition $(2.8(a))$, $\liminf E_n = \bigcup_m \bigcap_{n\geq m} E_n$

$$\uparrow \lim_m \{\inf_{n\geq m} 1_{E_n}(\omega)\} = \begin{cases} 1 & \omega \in \liminf E_n \\ 0 & \text{otherwise} \end{cases} = 1_{\liminf E_n}(\omega) \tag{6}$$

Therefore

$$\forall \omega \in \Omega, \liminf 1_{E_n}(\omega) = \uparrow \lim_m \{\inf_{n\geq m} 1_{E_n}(\omega)\} = 1_{\liminf E_n}(\omega)$$

$\square$