Discussion about symmetric and exchangability

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1 Introduction

In the class we've introduced Hewitt-Savage 0-1 law, which is about symmetric random variables, without proof. Here I combine with some reference and write two proofs.

2 Two proofs of Hewitt-Savage 0-1 law

Theorem 1. ***Hewitt-Savage 0-1 law*** *If $X_1, X_2, \ldots$ are independent and identically distributed and $A \in \mathcal{E}$, then $P(A) \in \{0, 1\}$.*

Proof. Let $\mathcal{T}_n = \sigma(X_1, X_2, \ldots, X_n)$. Let $B_n = E(A|\mathcal{T}_n)$. Then by the Lévy Upwards Theorem B_n converges in $\mathcal{L}^1$ to A. Now define $\mathcal{T}_n = \sigma(X_n, X_{n+1}, \ldots)$. Let $C_n = E(A|\mathcal{T}_n)$, then by the Lévy Upwards Theorem $D = E(A|\mathcal{T}) = \lim_{n\to\infty} C_n$ in $\mathcal{L}^1$. By another application of the Lévy Downwards Theorem $E(B_n|\mathcal{T}_m)$ converges to $E(B_n|\mathcal{T})$ as m increases.

Now fix $\epsilon > 0$. Then for all $n > N, m > M$ and $n > m$ we have

$$E(|B_n - A|) < \epsilon/3$$

$$E(|C_m - D|) < \epsilon/3$$

$$E(|B_n - E(B_n|\mathcal{T}_m)|) < \epsilon/3$$

Let ρ be a permutation that exchanges $1, \ldots, n$ with $m, \ldots, n+m-1$. Then we get

$$B_n = E(A|\mathcal{T}_n) = E((A \circ \rho)|\mathcal{T}_n)$$

Now we are conditioning on variables that are in the slots $m, \ldots, m+n-1$ and we take $E(A|\rho(X_m, \ldots, X_{m+n-1}))$ instead by invoking Fubinis theorem. But this $\sigma-$algebra is a subset of $\mathcal{T}_m$. Thus

$$E(B_n|\mathcal{T}_m)E(E((A \circ \rho)|\mathcal{T}_n)|\mathcal{T}_m) = E(A|\mathcal{T}_m)$$

Meanwhile we have $E(|B_n - C_m|) < \epsilon/3$, and by the e triangle inequality $E(|A - D|) < \epsilon/3$. □

The Lévy Upwards Theorem is a powerful tool for reconstructing a random variable from various observations of it. But we can also prove without it and make more use of symmetry.

Proof. Let $A \in \mathcal{E}$. As the proof of Kolmogorov's 0-1 Law, we will show that A is independent of itself, i.e. $P(A) = P(A \cap A) = P(A)P(A)$ so that

$$P(A) \in \{0, 1\} \tag{1}$$

We mark $(A - B) \cup (B - A)$ as $A\Delta B$, which is the symmetric difference.
Let $A_n \in \sigma(X_1, \ldots, X_n)$ so that

$$P(A_n \Delta A) \to 0$$

A_n can be written as $\{\omega : (\omega_1, \ldots, \omega_n) \in B_n\}$ with $B_n \in \mathcal{S}^n$. Let

$$\pi(j) = \begin{cases} j + n & \text{if } 1 \leq j \leq n \\ j - n & \text{if } n + 1 \leq j \leq 2n \\ j & \text{if } j \geq 2n + 1 \end{cases}$$

Since π^2 is identity, we have $\pi^{-1} = \pi$. Therefore we don't need to worry about write π^{-1} or π. Since the coordinates are independent and identically distributed, so are the permuted coordinates. Thus

$$P(\omega : \omega \in A_n \Delta A) = P(\omega : \pi\omega \in A_n \Delta A) \tag{2}$$

Now we have $\{\omega : \pi\omega \in A\} = \{\omega : \omega \in A\}$, since A is permutable, and

$$\{\omega : \pi\omega \in A_n\} = \{\omega : (\omega_{n+1}, \ldots, \omega_{2n}) \in B_n\}$$

We use A'_n to denote the last event then we get

$$\{\omega : \pi\omega \in A_n \Delta A\} = \{\omega : \omega \in A'_n \Delta A\} \tag{3}$$

Combine (2) and (3), we have

$$P(A_n \Delta A) = P(A'_n \Delta A) \tag{4}$$

It's obvious that

$$|P(B) - P(C)| \leq |P(B \Delta C)|$$

thus (4) implies $P(A_n)$, $P(A'_n) \to P(A)$. Now $A - C \subset (A - B) \sup(B - C)$ and with a similar inequality for $C - A$ impiles $A\Delta C \subset (A\Delta B) \cup (B\Delta C)$.
The last inequality, (4), and (1) imply

$$P(A_n \Delta A'_n) \leq P(A_n \Delta A) + P(A \Delta A'_n) \to 0$$

The last result implies

$$\begin{aligned} 0 &\leq P(A_n) - P(A_n \cap A'_n) \\ &\leq P(A_n \cup A'_n) - P(A_n \cap A'_n) \\ &= P(A_n \Delta A'_n) \to 0 \end{aligned}$$

so $P(A_n \cap A'_n) \to P(A)$. But A_n and A'_n are independent, so

$$P(A_n \cap A'_n) = P(A_n)P(A'_n) \to P(A)^2$$

This shows $P(A) = P(A)^2$, which finishes the proof of Hewitt-Savage 0-1 law. $\square$

3 de Finettis Theorem

Another theorem about exchangability is de Finettis Theorem. Suppose we have a sequence of exchangeable Bernoulli random variables, that is variables that are either 0 or 1. de Finettis theorem tells us that they are conditionally independent given the tail algebra.

Theorem 2. ***de Finettis Theorem*** *Let $X_1, X_2, \ldots$ be a sequence of exchangable random variables in $\mathcal{L}^1$. Let $\mathcal{T}$ be the tail algebra. Then $E(X_1|\mathcal{T})$, $E(X_2|\mathcal{T}), \ldots$ is a independent identically distributed sequence of random variables.*

Proof. Let $\mathcal{T}_m = \sigma(X_m, X_{m+1}, \ldots)$. Then $\mathcal{T} := \bigcap_m \mathcal{T}_m$. Since the X_i are exchangable, $E(X_i|\mathcal{T})$ is an exchangable sequence.

Independence of $X_1, \ldots, X_n$ is equivalent to $X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n$ being independent events. Let $f_n(x)$ be the indicator function of $x \leq x_n$. Then we need to show that

$$E(f_1(X_1) \ldots f_n(X_n)|\mathcal{T}) = E(f_1(X_1)|\mathcal{T}) \ldots E(f_n(X_n)|\mathcal{T})$$

Consider $S_m^k = f_k(X_1) + f_k(X_2) + \cdots + f_k(X_m)$. By exchangability and a symmetry argment we have $E(f_k(X_i)|\mathcal{T}) = S_m^k/m$.

Then we apply the Lévy Downward Theorem we have that

$$E(f_k(X_i)|\mathcal{T}) = \lim_{m \to \infty} S_m^k/m$$

Then we have

$$E(f_1(X_1)|\mathcal{T}) \ldots E(f_n(X_n)|\mathcal{T}) = \lim_{m \to \infty} \prod_{k=1}^{n} S_m^k/m$$

Now we have

$$m^{-n} \prod_{k=1}^{n} S_m^k = m^{-n} \sum_{1 \leq m_1, \ldots, m_n \leq n} \prod f_{m_i}(X_i)$$

Each term of the sum is either 0 or 1, and there are m^{n-1} terms that have two or more indices the same. As we take the limit the contributions of these terms vanishes since we are dividing by m^n and we are left with

$$\begin{aligned} &\lim_{m \to \infty} \prod_{k=1}^{n} S_m^k/m \\ &= \lim_{m \to \infty} \frac{1}{m(m-1) \ldots (m-n+1)} \sum_{m_i nonequal} \prod_{i=1}^{n} f_i(X_{m_i}) \\ &= \lim_{m \to \infty} E(f_1(X_1) f_2(X_2) \ldots f_n(X_n)|\mathcal{T}) \end{aligned}$$

□

4 reference

Probability, Theory and Examples by Rick Durrett
Some Applications of Martingales to Probablity Theory by Watson Ladd