Consecutive heads in coin tossing

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Suppose that a coin with probability p of heads is tossed repeatedly.
Let A_k be the event that a sequence of k (or more) consecutive heads occurs amongst tosses numbered $2^k, 2^k+1, 2^k+2, ..., 2^{k+1}-1$. Prove that

$$P(A_k, \text{ i.o.}) = \begin{cases} 1, \text{ if } p \geq \frac{1}{2} \\ 0, \text{ if } p < \frac{1}{2} \end{cases}.$$

Proof. Let E_{ki} be the event that there are k consecutive heads beginning at toss numbered 2^k+i. There is $P(E_{ki}) = p^k$ for all i. We have

$$A_k = \bigcup_{i=0}^{2^k-k} E_{ki}.$$

By inclusion-exclusion formula,

$$P(A_k) \leq \sum_{i=0}^{2^k-k} P(E_{ki}) = (2^k-k+1)p^k.$$

And if we consider the disjoint blocks from 2^k+ik to $2^k+(i+1)k-1$for $i = 0, 1, \cdots, \left\lfloor \frac{2^k}{k} \right\rfloor$, namely $E_{k0}, E_{kk}, \cdots$.

We have $\bigcup_{j=0}^{\left\lfloor \frac{2^k}{k} \right\rfloor} E_{k(jk)} \subseteq A_k$, thus

$$P(A_k^c) \leq P((\bigcup_{j=0}^{\left\lfloor \frac{2^k}{k} \right\rfloor} E_{k(jk)})^c) = (1-p^k)^{\left\lfloor \frac{2^k}{k} \right\rfloor}.$$

Therefore,

$$1-(1-p^k)^{\left\lfloor \frac{2^k}{k} \right\rfloor} \leq P(A_k) \leq (2^k-k+1)p^k.$$

If $p < \frac{1}{2}$, $\sum_k P(A_k) \le \sum_k (2p)^k < \infty$. By BC1, we know that $P(A_k, \text{ i.o.}) = 0$.

If $p \ge \frac{1}{2}$,

$$\begin{aligned}
\sum_k P(A_k) &\ge \sum_k (1 - (1 - p^k)^{\frac{2^k}{k}}) \\
&= \sum_k (1 - \exp(\frac{2^k}{k} \ln(1 - p^k))) \\
&= \sum_k (1 - \exp(-\frac{2^k}{k} p^k)) \\
&\ge \sum_k (1 - \exp(-\frac{1}{k})) \\
&\ge \sum_k \frac{1}{k} \to \infty.
\end{aligned}$$

And apparently all events $A_k, k \in \mathbb{N}$ are independent. By BC2, $P(A_k, \text{ i.o.}) = 1$.

As a conclusion,

$$P(A_k, \text{ i.o.}) = \begin{cases} 1, \text{ if } p \ge \frac{1}{2} \\ 0, \text{ if } p < \frac{1}{2} \end{cases}.$$

$\square$