My Notes on Black Scholes formula

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摘要

I read section 15.1 and 15.2 in our textbook "Probability with Martingales" and learned something about Black-Scholes formula. I rewrite the textbook's context and add my understanding as myself's handout. I also add more details to the 2 proofs.

1 A trivial martingale-representation result

This part I will introduce an important martingale-representation which is useful in proof of Black-Scholes formula.

Let $S = \{-1, 1\}$, Σ be the subset of S, and μ be the probability measure on (S, Σ) with $\mu(\{1\}) = p = 1 - \mu(\{-1\})$.

Define $(\Omega, \mathcal{F}, P) = (S, \Sigma, \mu)^N$. A typical element of Ω is

$$\omega = (\omega_1, \omega_2, \dots, \omega_N), \omega_k \in \{-1, 1\}$$

. Define $\epsilon_k : \Omega \to \mathbb{R}$ by $\epsilon_k(\omega) = \omega_k$. Notice that $\epsilon_1, \epsilon_2, \dots, \epsilon_N$ are IID RVs(Independent and identically distributed random variables).

For $0 \leq n \leq N$, define

$$Z_n := \sum_{k=1}^{n} (\epsilon_k - 2p + 1)$$
$$\mathcal{F}_n := \sigma(Z_0, Z1, \dots, Z_n) = \sigma(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$$

Note that $E(\epsilon_k) = 1 \cdot p + (-1) \cdot (1 - p) = 2p - 1$. So $E(Z_n) = 0$ and $Z = (Z_n : 0 \leq n \leq N)$ is a martingale relative to $(\mathcal{F}_n)$.

Lemma 1. *If $M = (M_n : 0 \leq n \leq N)$ is a martingale (relative to $(\{\mathcal{F}_n\}, P)$), then there exists a unique previsible process H such that*

$$M = M_0 + H \bullet Z, \text{ that is, } M_n = M_0 + \sum_{k=1}^{n} H_k (Z_k - Z_{k-1})$$

proof. I simply construct H explicitly.

As M_n is $\mathcal{F}_n$-measurable, let $M_n(\omega) = M_n(\omega_1, \omega_2, \ldots, \omega_N) = f_n(\omega_1, \omega_2, \ldots, \omega_n)$ for some function $f : \{-1, 1\}^n \to \mathbb{R}$.

Since M is a martingale, we have

$$\begin{aligned} 0 &= E(M_n - M_{n-1}|\mathcal{F}_{n-1})(\omega) \\ &= p \cdot f_n(\omega_1, \ldots, \omega_{n-1}, 1) + (1-p) \cdot f_n(\omega_1, \ldots, \omega_{n-1}, -1) - f_{n-1}(\omega_1, \ldots, \omega_{n-1}) \end{aligned}$$

. Hence

$$\frac{f_n(\omega_1, \ldots, \omega_{n-1}, 1) - f_{n-1}(\omega_1, \ldots, \omega_{n-1})}{2(1-p)} = \frac{f_{n-1}(\omega_1, \ldots, \omega_{n-1}) - f_n(\omega_1, \ldots, \omega_{n-1}, -1)}{2p} \tag{1}$$

Define $H_n(\omega)$ to be their common value and H is obviously previsible. We can easily check the correctness by checking $M_n - M_{n-1} = H_n(Z_n - Z_{n-1})$:

For those $\omega_n = 1$, $H_n(Z_n - Z_{n-1}) = \frac{f_n(\omega_1, \ldots, \omega_{n-1}, 1) - f_{n-1}(\omega_1, \ldots, \omega_{n-1})}{2(1-p)}(1 - 2p + 1) = f_n(\omega_1, \ldots, \omega_{n-1}, 1) - f_{n-1}(\omega_1, \ldots, \omega_{n-1}) = M_n - M_{n-1}$.

For those $\omega_n = -1$, $H_n(Z_n - Z_{n-1}) = \frac{f_{n-1}(\omega_1, \ldots, \omega_{n-1}) - f_n(\omega_1, \ldots, \omega_{n-1}, -1)}{2p}(-1 - 2p + 1) = f_n(\omega_1, \ldots, \omega_{n-1}, -1) - f_{n-1}(\omega_1, \ldots, \omega_{n-1}) = M_n - M_{n-1}$.

And by $H_n = (M_n - M_{n-1})/(Z_n - Z_{n-1})$, in which $Z_n - Z_{n-1} = \omega_n - 2p + 1 \neq 0$, we can see H is unique. □

2 Option pricing; discrete Black-Scholes formula

2.1 Symbol introduction

(In this part, I copy nearly all the context from textbook except the proof because the textbook is so well written and clear enough.)

Consider an economy in which there are two 'securities': bonds of fixed interest rate r, and stock, the value of which fluctuates randomly. Let N be a fixed element of N. We suppose that values of units of stock and of bond units change abruptly at times $1, 2, \ldots, N$. For $n = 0, 1, \ldots, N$, we write

$B_n = (1+r)^n B_0$ for the value of 1 bond unit throughout the open time interval $(n, n+1)$,

S_n for the value of 1 unit of stock throughout the open time interval $(n, n+1)$.

You start just after time 0 with a fortune of value x made up of A_0 units of stock and V_0 of bond, so that

$$A_0 S_0 + V_0 B_0 = x.$$

Between times 0 and 1 you invest this in stocks and bonds, so that just before time 1, you have A_1 units of stock and V_1 of bond so that

$$A_1 S_0 + V_1 B_0 = x.$$

So, (A_1, V_1) represents the portfolio you have as your 'stake on the first game'.

Just after time n-1 (where $n \geq 1$) you have A_{n-1} units of stock and V_{n-1} units of bond with value

$$X_{n-1} = A_{n-1} S_{n-1} + V_{n-1} B_{n-1}.$$

By trading stock for bonds or conversely, you rearrange your portfolio between times n—1 and n so that just before time n, your fortune (still of value X_{n-1} because we assume transaction costs to be zero) is described by

$$X_{n-1} = A_n S_{n-1} + V_n B_{n-1} \ (n \geq 1).$$

Your fortune just after time n is given by

$$X_n = A_n S_n + V_n B_n (n \geq 0) \tag{a}$$

and your change in fortune satisfies

$$X_n - X_{n-1} = A_n (S_n - S_{n-1}) + V_n (B_n - B_{n-1}). \tag{b}$$

Now,

$$B_n - B_{n-1} = r B_n,$$

and

$$S_n - S_{n-1} = R_n S_{n-1},$$

where Rn is the random 'rate of interest of stock at time n'. We may now rewrite (b) as

$$X_n — X_{n-1} = r X_{n-1} + A_n S_{n-1} (R_n - r),$$

so that if we set

$$Y_n = (1+r)^{-n} X_n, \tag{c}$$

then

$$Y_n - Y_{n-1} = (1+r)^{-(n-1)} A_n S_{n-1} (R_n - r). \tag{d}$$

Note that (c) shows Yn to be the discounted value of your fortune at time n, so that the evolution (d) is of primary interest.

Let Ω, $\mathcal{F}$, $\epsilon_n (l \leq n \leq N)$, $Z_n (0 \leq n \leq N)$ and $\mathcal{F}_n (0 \leq n \leq N)$ be as in Section 1.

We build a model in which each R_n takes only values a, b in $(—1, \infty)$, where

$$a < r < b,$$

by setting

$$R_n = \frac{a+b}{2} + \frac{b-a}{2}\epsilon_n. \tag{e}$$

But then

$$R_n - r = \frac{1}{2}(b-a)(\epsilon_n - 2p + 1) = \frac{1}{2}(b-a)(Z_n - Z_{n-1}), \tag{f}$$

where we now choose

$$p := \frac{r-a}{b-a}. \tag{g}$$

2.2 Option pricing

A European option is a contract made just after time 0 which will allow you to buy 1 unit of stock just after time N at a price K; K is the so-called *striking price*. If you have made such a contract, then just after time N, you will exercise the option if $S_N > K$ and will not if $S_N < K$. Thus, the value at time N of such a contract is $(S_N — -K)^+$. *What should you pay for the option at time 0?*

Black and Scholes provide an answer to this question which is based on the concept of a hedging strategy.

A hedging strategy with initial value x for the described option is a portfolio management scheme $\{(A_n, V_n) : 1 \leq n \leq N\}$ where the processes A and V are previsible relative to $\{\mathcal{F}_n\}$ and where, with X satisfying (a) and (b), we have for every ω,

$$\begin{aligned}
&X_0(\omega) = x, && (h1)\\
&X_n(\omega) \geq 0 (0 \leq n \leq N), && (h2)\\
&X_N(\omega) = (S_N(\omega) - K)^+. && (h3)
\end{aligned}$$

Anyone employing a hedging strategy will by appropriate portfolio management, and without going bankrupt, exactly duplicate the value of the option at time N.

The existence of hedging strategy seems incredible to me. Because you need to find a scheme$\{(A_n, V_n)\}$ that $X_N(\omega)$ equals exactly 0 or $(S_N(\omega) - K)$. So the key point is under waht circumstances this strategy exists?

Theorem 2. *A hedging strategy with initial value x exists if and only if*

$$x = x_0 := E[(1+r)^{-N}(S_N - K)^+],$$

where E is the expectation for the measure P of Section 1 with p as at (g). There is a unique hedging strategy with initial value x_0, and it involves no short-selling: A is never negative.

proof. Suppose now that a hedging strategy with initial value x exists, and let A, V, X, Y denote the associated processes. From (d) and (f),

$$Y = Y_0 + F \bullet Z,$$

where F is the previsible process with

$$F_n = \frac{1}{2}(b-a)(1+r)^{-(n-1)} A_n S_{n-1}.$$

F is bounded because there are only finitely combinations. Thus Y is a martingale under the P measure, since Z is; and since $Y_0 = x$ and $Y_N = (1+r)^{-N}(S_n - K)^+$ by (c) and the definition of hedging strategy, we obtain

$$x = x_0.$$

In order to prove A is non-negative, now reconsider the problem and define

$$Y_n := E((1+r)^{-N}(S_N - K)^+ | \mathcal{F}_n).$$

Then Y is a martingale, and by combining (f) with the martingale-representation result in Section 1, we see that for some unique previsible process A, (d) holds.

Define

$$X_n := (1+r)^n Y_n, V_n := (X_n - A_n S_n)/B_n.$$

Then (a) and (b) hold.

Now by (1) we know the previsible $F_n = \frac{1}{2}(b-a)(1+r)^{-(n-1)} A_n S_{n-1} = (E[(1+r)^{-N}(S_N - K)^+ | S_{n-1}, S_n = (1+b)S_{n-1}] - E[(1+r)^{-N}(S_N-K)^+|S_{n-1}])/2(1-p)$.

As $S > 0$, we can prove $A \geq 0$ by simply prove $E[(S_N - K)^+ | S_{n-1}, S_n = (1+b)S_{n-1}] \geq E[(S_N - K)^+ | S_{n-1}]$.

Define $m = N - n + 1$. For some $\omega' = \{(\omega_1, \omega_2, \ldots, \omega_{n-1}, \ldots)\} \in \sigma(S_{n-1})$, consider the all 2^m situations for the stoke's value up or down.

$E(S_N, \omega') = \sum_{k=0}^{m} S_{n-1}(\omega') p^k (1+b)^k (1-p)^{m-k} (1+a)^{m-k} \binom{m}{k} = pE(S_N, \omega' \cap \{\omega_n = 1\}) + (1-p)E(S_N, \omega' \cap \{\omega_n = -1\})$.

$E(S_N, \omega' \cap \{\omega_n = 1\}) = (1+b) \sum_{k=0}^{m-1} S_{n-1}(\omega') p^k (1+b)^k (1-p)^{m-1-k} (1+a)^{m-1-k} \binom{m-1}{k}$

$E(S_N, \omega' \cap \{\omega_n = -1\}) = (1+a) \sum_{k=0}^{m-1} S_{n-1}(\omega') p^k (1+b)^k (1-p)^{m-1-k} (1+a)^{m-1-k} \binom{m-1}{k}$

From the coefficient above we can infer that $E(S_N, \omega' \cap \{\omega_n = 1\}) \geq E(S_N, \omega') \geq E(S_N, \omega' \cap \{\omega_n = -1\})$. Thus $E[(S_N - K)^+ | S_{n-1}, S_n = (1+b)S_{n-1}] \geq E[(S_N-K)^+|S_{n-1}] \Rightarrow$ A is non-negative. $\square$