$P(A_k, i.o.)$ of Throw a Coin(E4.4)

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Throw a coin is a basic problem of probability, but mathematicians are not satisfied with the case where p is equal to 1/2. In the textbook, the author puts forward a problem as follows.

Theorem 1. *Suppose that a coin with probability p of heads is tossed repeatedly. Let A_k be the event that a sequence of k (or more) consecutive heads occurs amongst tosses numbered $2^k, 2^{k+1}, \ldots, 2^{k+1}-1$. Prove that*

$$P(A_k, i.o) = \begin{cases} 1, & \text{if } p \geq 1/2, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Proof. 1) When $p < \frac{1}{2}$, we use BC1 to prove. Let E_n be the event that there are k heads starting from nth toss. Thus

$$A_k = \bigcup_{n=2^k}^{2^{k+1}-k} B_n.$$

In this way, we obtain(by inclusion-exclusion principle)

$$P(A_k) \leq \Sigma_{n=2^k}^{2^{k+1}-k} P(B_n) \leq 2^k p^k.$$

For $p < 1/2$,

$$\Sigma_k P(A_k) \leq \frac{2p}{1-2p} \leq \infty.$$

According BC1, we get $P(A_k, i.o.) = 0$ when $p < 1/2$.

2) When $p \geq 1/2$, use BC2 to prove. According to hint, we can firstly let E_i^k be the event that there are k consecutive heads beginning at toss numbered $2^k + (i-1)k$. Then i is between 1 and $2^k/k$. That is, the beginning of k consecutive heads are $2^k, 2^k + k, \ldots, 2^k + 2^k - k(i.e., 2^{k+1} - k)$. These events E_i^k are independent, and it is clear that

$$\{E_i^k, i.o.\} \Rightarrow \{A^k, i.o.\}.$$

For we have

$$\begin{aligned} \Sigma_k \Sigma_{i=1}^{2^k/k} P(E_i^k) &\geq \Sigma_k (2^k/k - 1)p^k \\ &= \Sigma_k \frac{1}{k}\frac{1}{2}^{k-1} - \frac{p}{1-p} \\ &\geq \Sigma_k \frac{1}{k} - \frac{p}{1-p} = +\infty \end{aligned} \tag{2}$$

According BC2, we get $P(E_i^k, i.o.) = 1$. Thus $P(A_k, i.o.) = 1$ when $p \geq 1/2$. $\square$