An Application of Lovász Local Lemma: Hypergraph 2-Coloring

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During the lecture, we have established **Lovász Local Lemma**:

Lemma 1 (Erdös,Lovász, 1975). *Let $E_1, E_2, \ldots, E_n$ be a sequence of events on some probability space $(\Omega, \Sigma, \mathcal{P})$ with $\mathcal{P}(E_i) \le p$ for all $i \in [n]$ and all degrees in dependency graph $G = (V, E)$ are at most d. If $4pd \le 1$, then*

$$\mathcal{P}\left(\bigcap_{i=1}^{n} E_i^c\right) > 0.$$

The above lemma provides a deeper insight into many combinatorics problems, in particular to give existence proofs. We learned about k-sat problem in the lecture, and another application is **Hypergraph 2-Coloring**.

Definition 2. *A hypergraph H is a pair $H = (X, E)$ where X is a set of elements called vertices, and E is a set of non-empty subsets of X called hyperedges or edges.*

Intuitively, a hypergraph is a generalization of a graph where an edge consists of any number of vertices. And the Hypergraph 2-Coloring problem is

Question 3. *Given a hypergraph H, can we assign two colors to vertices so that no edge is monochromatic?*

We say H is *2-colorable* if the answer is yes. For example, the hypergraph in Fig. 1(a) is 2-colorable, while Fig. 1(b) is not.

With the help of Lemma 1, we can prove that

Theorem 4. *Let $H = (X, E)$ be a hypergraph in which every edge has at least k vertices and intersects at most d other edges. If $4d \le 2^{k-1}$, then H is 2-colorable.*

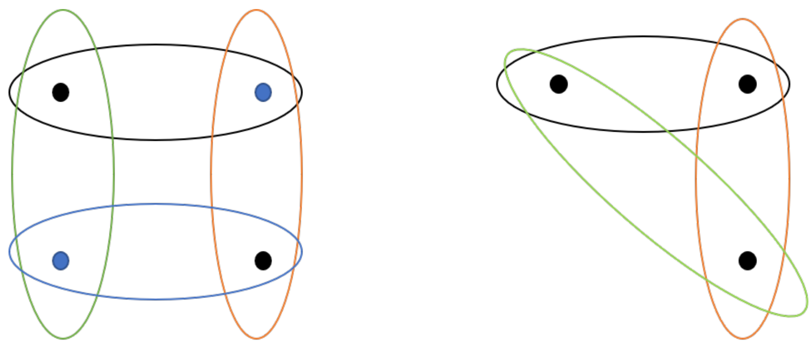

Figure 1: Example of 2-coloring of hypergraph.

Proof. Let $E = \{e_1, e_2, \dots, e_n\}$. We color the vertices randomly and define E_i as the event that edge e_i is monochromatic. Clearly, $\mathcal{P}(E_i) \leq \frac{1}{2^{k-1}}$ for all $i \in [n]$ and the degree in dependency graph is at most d. Take $p = \frac{1}{2^{k-1}}$ and we have $4pd \leq 1$. Hence, by Lemma 1 we get

$$\mathcal{P}\left(\bigcap_{i=1}^{n} E_i^c\right) > 0,$$

which means H is 2-colorable. □

Another application of Lemma 1 is **package routing**. To find the optimal solution is NP-hard, but with the help of *the algorithmic version of the Lovasz Local Lemma*(I did not look into this in detail...), we can design and validate some good schedules [1].

References

[1] Frank Thomson Leighton, Bruce M Maggs, and Satish B Rao. Packet routing and job-shop scheduling ino (congestion+ dilation) steps. *Combinatorica*, 14(2):167–186, 1994.