E9.2 Symmetry Conditional Probability

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E9.2 Suppose $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, and $E(X|Y) = Y, a.s.$, $E(Y|X) = X, a.x.$. To proof $P(X = Y) = 1$

Solution As the hint said,

$$E((X-Y)I_{Y\le c}) = E((X-Y)I_{Y\le c}I_{X\le c}) + E((X-Y)I_{Y\le c}I_{X>c})$$
$$E((X-Y)I_{X\le c}) = E((X-Y)I_{Y\le c}I_{X\le c}) + E((X-Y)I_{X\le c}I_{Y>c})$$

At the same time,

$$\begin{aligned} E((X-Y)I_{X\le c}) &= E((X - E(X|Y))I_{X\le c}) \\ &= E(XI_{X\le c}) - E((X|Y)I_{X\le c}) \\ &= E(XI_{X\le c}) - E(XI_{X\le c}) = 0 \end{aligned}$$

So that

$$E((X-Y)I_{Y\le c}I_{X>c}) = E((X-Y)I_{X\le c}I_{Y>c})$$

Because the left side is non-negative and the right side is non-positive, both of them equal to 0. Then we can present the following events:

$$(X < Y) = \bigcup_{c\in\mathbb{Q}} (X \le c) \cap (Y > c)$$
$$(X > Y) = \bigcup_{c\in\mathbb{Q}} (X > c) \cap (Y \le c)$$

They are all null events, which leads to the event$(X \ne Y)$ is a null event. Then we got $P(X = Y) = 1$.

Question Is the union of infinite null-set is also a null-set?

Another way to proof

$$E(X^2) = E(XE(Y|X)) = E(E(XY|X)) = E(XY)$$
$$E(Y^2) = E(YE(X|Y)) = E(E(XY|Y)) = E(XY)$$

So that

$$E((X-Y)^2) = E(X^2 + Y^2 - 2XY) = 0$$

Which means X and Y is different on a null-set.