Proof that parallel sets do not exist

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1 Problem

Use AC or any of its equivalents to show that there do not exist parallel sets. (Two sets A and B are called parallel if neither $|A| \leqslant |B|$ nor $|B| \leqslant |A|$ holds.)

2 Proof

Under the assumption of the **Well-order Principle**, every set is well-ordered.

2.1 Initial Segment

First we introduce a definition of **initial segment** of set S. Define

$$S(a) := \{b \in S | b \prec a\}$$

$S(a)$ is the strict lower closure of a. Thus $S(a)$ is called an initial segment.

2.2 Lemma

We now prove that for two well-ordered sets A and B, one of the following must hold:
(1). A is isomorphic to an initial segment of B.
(2). B is isomorphic to an initial segment of A.
Without loss of generality, we prove (1) holds.
We use Transfinite Induction to define a mapping f from A to B to be: for any element $a \in A$,

$$f(a) = min\{b \in B \mid b \neq f(a'), a' \prec a\}$$

We can see that the mapping f is defined for all $a \in A$. Because otherwise there will be a smallest element a such that $f(a)$ is not defined, due to the fact that A is well-ordered. Since $\forall a' \prec a$, $f(a')$ is defined, by the definition of f, we can define $f(a)$ for a.

Now we prove that $f(A) = \{f(a) \mid a \in A\}$ is an initial segment of B. Consider $f(a) \in B$. For every $b \prec f(a)$, the following must hold: $b = f(b') \in f(A)$ and $b' \prec a$. Thus $f(A)$ is a strict lower closure in B.

Since A is obviously isomorphic to $f(A)$, we have established *(1)*, namely A is isomorphic to an initial segment of B.

Finally, we complete the proof by asserting that (1) and (2) cannot both be true unless $A = B$

We prove by contradiction. If both are true, then WLOG,

A is isomorphic to an initial segment of A. which is not possible unless the initial segment is A itself, then in this case $A = B$. $\square$

2.3 Final Proof

By the **Lemma** we just proved, for any two sets A and B, if A is isomorphic to an initial segment of B, then it means there is an injection between A and B, thus $|A| \leqslant |B|$, or $|B| \leqslant |A|$ vice versa.

Therefore, no parallel sets exist under AC.

2.4 About the assumption

Since AC, well-ordering principle, Zorn's Lemma, Tychonoff's Theorem are all equivalent to each other, taking any one of them as the assumption will be feasible. However, here I have taken an easy approach by directly assuming the Well-order principle.

3 Reference

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