E9. Conditional Expectation

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Problem 1 *Prove that if* $\mathcal{G}$ *is a sub-*σ*-algebra of* $\mathcal{F}$ *and if* $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbf{P})$ *and if* $Y \in \mathcal{L}^1(\Omega, \mathcal{G}, \mathbf{P})$ *and*

$$E(X; G) = E(Y; G) \tag{1}$$

for every G *in a* π*-system which contains* Ω *and generates* $\mathcal{G}$*, then* (1) *holds for every* G *in* $\mathcal{G}$*.*

Solution: Denote the π-system as I. And WLOG, we assume that $X \geqslant 0$, which is neccesary in the construction of the measures.

We consider two measures:

$$\mu_0 : G \mapsto E(X; G)$$
$$\mu_1 : G \mapsto E(Y; G).$$

Then μ_0 and μ_1 agree on the π-system I, and therefore must agree on on the σ-algebra it generates, namely $\sigma(I) = \mathcal{G}$.

Thus, for every $G \in \mathcal{G}$,

$$E(X; G) = E(Y; G),$$

which finishes the proof. □

Problem 2 *Suppose that* $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbf{P})$ *and that*

$$E(X|Y) = Y, \quad a.s.,$$
$$E(Y|X) = X, \quad a.s.$$

Prove that $P(X = Y) = 1$.

Solution: We first consider $E(X - Y; Y \leqslant c) = E(X; Y \leqslant c) - E(Y; Y \leqslant c)$.

Since $E(X|Y) = Y$, a.s., for every $G \in \sigma(Y)$,

$$E(X; G) = E(Y; G)$$

and therefore

$$\begin{aligned} &E(X; Y \leqslant c) = E(Y; Y \leqslant c) \\ \Longrightarrow &E(X - Y; Y \leqslant c) = 0. \end{aligned}$$

Since

$$E(X - Y; Y \leqslant c) = E(X - Y; X > c, Y \leqslant c) + E(X - Y; X \leqslant c, Y \leqslant c),$$

and because

$$E(X - Y; X > c, Y \leqslant c) \geqslant 0,$$

we have

$$E(X - Y; X \leqslant c, Y \leqslant c) \leqslant 0.$$

Similarly, we have

$$E(Y - X; X \leqslant c, Y \leqslant c) \leqslant 0.$$

Thus,

$$\begin{aligned} & E(X - Y; X \leqslant c, Y \leqslant c) = E(Y - X; X \leqslant c, Y \leqslant c) = 0 \\ \Longrightarrow \quad & E(X - Y; X > c, Y \leqslant c) = 0 = E((X - Y)I_{\{X > c, Y \leqslant c\}}) \\ \Longrightarrow \quad & E(I_{\{X > c, Y \leqslant c\}}) = 0 \end{aligned}$$

Since $X > Y \Longleftrightarrow \exists c \in \mathbb{Q}\ X > c, Y \leqslant c$, and by countably additivity of P,

$$P(X > Y) = P(\bigcup_{c \in \mathbb{Q}} \{X > c\} \cap \{Y \leqslant c\}) \leqslant \sum_{c \in \mathbb{Q}} P(X > c, Y \leqslant c) = \sum E(I_{\{X > c, Y \leqslant c\}}) = 0.$$

Similarly $P(X < Y) = 0$ and thus,

$$P(X = Y) = 1.$$

$\square$