A simple way of understanding the construction of simple functions converging to a measurable function

张志成 518030910439

April 4, 2020

Definition:
Given function f, we define

$$f^+(x) = max(f(x), 0)$$
$$f^-(x) = max(-f(x), 0)$$

f is a simple function with respect to (S, Σ) provided it falls into the linear subspace of $\mathbb{R}^S$ spanned by $\{\mathbf{1}_A \mid A \in \Sigma\}$.

Theorem *Let*

$$d_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})} + n\mathbf{1}_{[n,\infty)} \in \mathbb{R}^{\mathbb{R}}$$

Then $f_n = d_n \circ f^+ - d_n \circ f^-$ *is a simple function and* $\{f_n\}$ *converges to* f.

Proof:

For simplicity, it suffices to prove the case when f is nonnegative. Because $f = f^+ - f^-$, the same proof would also work if applied to f^+ and f^-.

Then

$$\begin{aligned} f_n &= d_n \circ f \\ &= \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n}))} + n\mathbf{1}_{f^{-1}([n,\infty))} \in \mathbb{R}^{\mathbb{R}} \end{aligned}$$

f_n is obviously a simple function because it falls into the linear space of $\mathbb{R}^{\mathbb{R}}$ spanned by $\{\mathbf{1}_{f^{-1}([\frac{k-1}{2^n},\frac{k}{2^n})} \mid k \leqslant n2^n\} \cup \mathbf{1}_{[n,\infty)}$.

What this construction does intuitively is that it partitions $[0,\infty)$ to $n2^n$ intervals of length $\frac{1}{2^n}$ and another interval of $[n,\infty)$, which together constitutes $[0,\infty)$.

Explicitly,

$$f_n(x) = \begin{cases} \frac{k-1}{2^n}, & f(x) \in [\frac{k-1}{2^n}, \frac{k}{2^n}) \\ n, & f(x) \in [n, \infty) \end{cases}$$

This construction of f_n makes sure that $\forall x \; f_n(x) \leqslant f(x)$. It also makes sure that $f_n(x)$ is monotonically increasing,

$$[\frac{k-1}{2^n}, \frac{k}{2^n}) = [\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}) \cup [\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}})$$

Thus, if $f(x) \in [\frac{k-1}{2^n}, \frac{k}{2^n})$,

$$f_{n+1}(x) \geqslant \min(\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}) = \frac{2k-2}{2^{n+1}} = f_n(x)$$

Finally, convergence to f is guaranteed by

$$f_n(x) - f(x) \leqslant \frac{k}{2^n} - \frac{k-1}{2^n} = \frac{1}{2^n}$$

Therefore, we have proved that $f_n \uparrow f$. □

Remark *In fact, the essense of this construction is that for any interval* $(I_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n}))$*, it satisfies:*

1. $f_n(x) = \mathit{inf}\,\{f(x) \mid f(x) \in I_{n,k}\} \leqslant f(x)$
2. $\exists t \; I_{n+1,k} \subset I_{n,t}$

These are not difficult to achieve, for example we can partition $[0,\infty)$ *to intervals of length* $\frac{1}{n}$*,*

$$f_n = \sum_{k=1}^{n^2-1} \frac{k}{n} \mathbf{1}_{f^{-1}([\frac{k}{n}, \frac{k+1}{n}))} + n\mathbf{1}_{f^{-1}([n,\infty))}$$