A Problem of Indicator Function and lim

Lu Jiaxin
Student ID: 518030910412

April 22, 2020

Definition 1 (**Indicator Function**). *The indicator function of a subset A of a set X is a function*

$$1_A : X \to \{0,1\}$$

defined as

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Exercise 2. *Show that*

$$\limsup_{n\to\infty} 1_{E_n} = 1_{\limsup_{n\to\infty} E_n} \tag{1}$$

$$\liminf_{n\to\infty} 1_{E_n} = 1_{\liminf_{n\to\infty} E_n} \tag{2}$$

Proof. (1) By definition $\limsup E_n = \bigcap m \in \mathbb{N} \bigcup l \geq m E_l$. In other words, $x \in \limsup E_n$ if and only if for all $m \in \mathbb{N}$ such that $x \in E_l$ for some $l \geq m$.

Thus $1_{\limsup_{n\to\infty} E_n}(x) = 1$ if and only if for all $m \in \mathbb{N}$ such that for some $l \geq m$ we have $x \in E_l$.

Now consider a sequence of $(x_n \mid n \in \mathbb{N})$ of real numbers x_n. By definition, we have $\limsup_{n\to\infty} x_n = \inf_{m\in\mathbb{N}} \sup_{l\geq m} x_l$. So $\limsup_{n\to\infty} 1_{E_n}(x) = 1$ if and only if for all $m \in \mathbb{N}$ such that there is some $l \geq m$ we have that $\sup_{l\geq m} 1_{E_n}(x) = 1$. This also holds if and only if for all $m \in \mathbb{N}$ such that there is some $l \geq m$ we have that $x \in E_l$.

By these, we have $1_{\limsup_{n\to\infty} E_n}(x) = 1$ if and only if $\limsup_{n\to\infty} 1_{E_n}(x) = 1$. And these also yields $1_{\limsup_{n\to\infty} E_n}(x) = 0$ if and only if $\limsup_{n\to\infty} 1_{E_n}(x) = 0$ and consequently $1_{\limsup_{n\to\infty} E_n}(x) = \limsup_{n\to\infty} 1_{E_n}(x)$.

Thus, $\limsup_{n\to\infty} 1_{E_n} = 1_{\limsup_{n\to\infty} E_n}$ holds.

(2) By definition $\liminf E_n = \bigcup_{m\in\mathbb{N}} \bigcap_{l\geq m} E_l$. In other words, $x \in \liminf E_n$ if and only if there is an $m \in \mathbb{N}$ such that $x \in E_l$ for all $l \geq m$.

Thus, $1_{\liminf_{n\to\infty} E_n}(x) = 1$ if and only if there is some $m \in \mathbb{N}$ such that for all $l \geq m$ we have $x \in E_l$.

Also, from (1) we can have $\liminf_{n\to\infty} x_n = \sup_{m\in\mathbb{N}} \inf_{l\geq m} x_l$. So $\liminf_{n\to\infty} 1_{E_n}(x) = 1$ if and only if there is some $m \in \mathbb{N}$ such that for all $l \geq m$ we have that $\inf_{l\geq m} 1_{E_n}(x) = 1$. This also holds if and only if there is some $m \in \mathbb{N}$ such that for all $l \geq m$ we have that $x \in E_l$.

By these, we have $1_{\liminf_{n\to\infty} E_n}(x) = 1$ if and only if $\liminf_{n\to\infty} 1_{E_n}(x) = 1$. And these also yields $1_{\liminf_{n\to\infty} E_n}(x) = 0$ if and only if $\liminf_{n\to\infty} 1_{E_n}(x) = 0$ and consequently $1_{\liminf_{n\to\infty} E_n}(x) = \liminf_{n\to\infty} 1_{E_n}(x)$.

Thus, $\liminf_{n\to\infty} 1_{E_n} = 1_{\liminf_{n\to\infty} E_n}$ holds.

$\square$