# Proof of MON

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**Lemma 1** Suppose $A$ is a measurable set and $f_k(k \in N)$ is a nondecreasing sequence of non-negative measurable functions on $S$ such that

$$\lim_k f_k(x) \geq 1$$

for almost all $x \in A$. Then

$$\lim_k \int f_k \, d\mu \geq \mu(A)$$

*Proof.* Let's fix $\varepsilon > 0$ and define the sequence of measurable sets

$$B_k = \{x \in A : f_k(x) \geq 1 - \varepsilon\}$$

By monotonicity of the integral, it follows that for any $k \in N$

$$(1-\varepsilon)\mu(B_k) = \int (1-\varepsilon) 1_{B_k} \, d\mu \leq \int f_k \, d\mu$$

Because almost every $x$ is in $B_k$ for large enough $k$, we have

$$(\bigcup_k B_k) \cup C = A$$

with a set $C$ of measure 0. Thus by countable additivity of $\mu$, and because $B_k$ increases with $k$, we have

$$\mu(A) = \lim_k \mu(B_k) \leq \lim_k (1-\varepsilon)^{-1} \int f_k \, d\mu$$

As this is true for any positive $\varepsilon$, then the result follows.

$\square$

**Theorem 2** (Lebesgue monotone-convergence theorem) If $(f_n)$ is a sequence of elements of $(m\Sigma)^+$ such that $f_n \uparrow f$, then

$$\mu(f_n) = \mu(f) \leq \infty$$

*Proof.* By the monotonicity property of the integral, it is immediate that:

$$\int f \, d\mu \geq \lim_k \int f_k \, d\mu$$

and the limit on the right exists, because the sequence is monotonic. We now prove the inequality in the other direction. That is,

$$\int f d\mu \leq \lim_k \int f_k d\mu$$

It follows from the definition of integral that there is a non-decreasing sequence $(g_n)$ of non-negative simple functions such that $g_n \leq f$ and

$$\lim_n \int g_n \, d\mu = \int f \, d\mu.$$

Therefore, we just need to prove that for all $n \in N$,

$$\int g_n \, d\mu \leq \lim_k \int f_k \, d\mu$$

We know that

$$\lim_k f_k(x) \geq g_n(x)$$

almost everywhere, then we can break up the function $g_n$ into its constant value parts, this reduces to the case in which $g_n$ is the indicator function of a set. So we can use Lemma 1 and get that

$$\lim_k \int f_k \, d\mu \geq \int g_n \, d\mu$$

This result is for all $n \in N$, so we finish the proof.

$\square$

# Reference


[1] Probability with martingales(2014)