Can an event space be a countably infinite set?

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Exercise 2. Can an event space be a countably infinite set?

Solution First of all, event space in measure theory based probability, it is required to be a sigma-algebra. Then we need to proof a sigma-algebra that contains infinitely many sets must be uncountable.
Assume we have a set $\mathbb{X}$, and an infinite sigma-algebra $\mathbb{S}$ on it. I want to proof that $\mathbb{S}$ is uncountable by contradiction.

Assumption $\mathbb{S} = \{A_i\}_{i=1}^{\infty}$,$B_x = \cap_{x \in A_i} A_i$. Due to $\mathbb{S}$ is countable, B_x is made of countable intersection, which means it belongs to $\mathbb{S}$.

Lemma $c \in B_x \cap B_y \to B_x = B_y$

Proof If $x \notin B_c$, $B_x \backslash B_c \subset S$ with $x \in B_x \backslash B_c$. But B_x is the intersection of all the intervals containing x. Therefore $B_x \backslash B_c = B_x$, which means,$B_x = B_c$.
Analogouly, we have $B_y = B_c$. So we've got $B_x = B_y$ when $c \in B_x \cap B_y$.

If there are finite sets of the form B_x, then: $\mathbb{S}$ is a union of a finite number of disjoint sets, which leads to $\mathbb{S}$ is finite.

If there are countable-infinite sets of the form B_x, then suppose $G = \{B_x\}_{x \in X}$.By taking all the possible disjoint unions from G you can form $\|P(G)\|$ new different sets(P means power set), hense an uncountable number of different sets.(G is countably infinite, then P(G) is uncountable)

Notice that every possible union of sets in G is a set that belongs to S, since $B_x \in S$ and S is a sigma-algebra. This means that S should be uncountable in order to contain this uncountable number of all possible different unions of the sets in the family G.

In conclusion, an event space can't be a countably infinite set.