The Uniqueness Theorem of Measure Expansion from $(X, \boldsymbol{A})$ to $(X, \sigma(\boldsymbol{A} \cap \{C\}))$

Mar. 08, 2020

Labels In the following proof, we suppose

$$\mu^*(C) = inf\{\mu(A) : C \subset A, \boldsymbol{A}\} \text{ (outer measure)}$$
$$\mu_*(C) = sup\{\mu(A) : C \supset A, \boldsymbol{A}\} \text{ (inner measure)}$$

Theorem Suppose μ is a sigma-finite measure on $(X, \boldsymbol{A})$, $C \subset X$. If μ_1 and μ_2 are two expanding measure of μ from $(X, \boldsymbol{A})$ to $(X, \sigma(\boldsymbol{A} \cap \{C\}))$, which also satisfing

$$\mu_i(C) = \mu^*(C) < +\infty, i = 1, 2$$

Then $\mu_1 = \mu_2$ on $\sigma(\boldsymbol{A} \cap \{C\})$

Proof If v is any expanding measure of μ from $(X, \boldsymbol{A})$ to $(X, \sigma(\boldsymbol{A} \cap \{C\}))$, and $v(C) = \mu^*(C) < +\infty$, there will be a $C_1 \in \boldsymbol{A}$, with both $C \subset C_1$, and

$$\forall A \in \boldsymbol{A}, v(A \cap C) = v(A \cap C_1) \tag{1}$$

Due to μ is finite on $\boldsymbol{A}$, there exists a measurable cover of C. Now we select any one of them, take it as C_1. In this way, $C \subset C_1, C_1 \in \boldsymbol{A}$, and also $(C_1 \setminus C) = 0$. At the same time, refering to v is an expanding measure of μ, $v = \mu$ on $\boldsymbol{A}$. Then we get to know

$$\begin{aligned} v(C_1) = \mu(C_1) &= \mu_*(C \cup (C_1 \setminus C)) \\ &\leq \mu^*(C) + \mu_*(C_1 \setminus C) = \mu^*(C) \\ &\leq \mu(C_1) = v(C_1) \end{aligned}$$

So we've got $\mu^*(C) = v(C_1)$ and so that $v(C) = v(C_1)$
Considering $v(C) = \mu^*(C) < +\infty$, we now know that

$$\begin{aligned} v(A \cap C) &\leq v(A \cap C_1) = v(A \cap C) + v(A \cap (C_1 \subset C)) \\ &\leq v(A \cap C) + v(C_1 \subset C) = v(A \cap C) \end{aligned}$$

So formula (1) is correct.

Due to μ_1 and μ_2 are two expanding measure of μ from $(X, \boldsymbol{A})$ to $(X, \sigma(\boldsymbol{A} \cap \{C\}))$ which satisfying $\mu_i(C) = \mu^*(C) < +\infty, i = 1, 2$, we can get the following formula :

$$\forall A \in \boldsymbol{A}, \mu_1(A \cap C) = \mu_1(A \cap C_1) \tag{2}$$

$$\forall A \in \boldsymbol{A}, \mu_2(A \cap C) = \mu_2(A \cap C_1) \tag{3}$$

Because of $\mu_1 = \mu_2 = \mu$ on $\boldsymbol{A}$, $A \cap C_1 \in \boldsymbol{A}$

$$\mu_1(A \cap C_1) = \mu_2(A \cap C_1) \tag{4}$$

According formula (2),(3) and (4), we've got

$$\mu_1(A \cap C) = \mu_2(A \cap C) \tag{5}$$

Besides that, we also know

$$\mu_1(A \cap C) \leq \mu_1(C) = \mu^*(C) < +\infty, i = 1, 2$$

So that,

$$\mu_1(A \cap C^c) = \mu_2(A \cap C^c) \tag{6}$$

So there exists $A_1, A_2 \in \boldsymbol{A}$, such that

$$A = (A_1 \cap C) \cup (A_2 \cap C^c)$$

By considering formula (5) and (6), we know that

$$\begin{aligned} \mu_1(A) &= \mu_1[(A_1 \cap C) \cup (A_2 \cap C^c)] \\ &= \mu_1(A_1 \cap C) + \mu_2(A_2 \cap C^c) \\ &= \mu_2(A_1 \cap C) + \mu_2(A_2 \cap C^c) \\ &= \mu_2[(A_1 \cap C) \cup (A_2 \cap C^c)] \\ &= \mu_2(A) \end{aligned}$$

Which means now we have proved that $\mu_1 = \mu_2$ on $\sigma(\boldsymbol{A} \cap \{C\})$