# Bijection between $[0, 1]$ and $(0, 1)$ and $\mathbb{R}$

Ji Jiabao

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The problem i encountered is that what the image for the endpoint 0,1 should be, since [0,1] is a closed interval while $\mathbb{R}$ is an open interval. But we know there is a simple bijection between (0, 1) and $\mathbb{R}$ with the help of $tan$. And from the reading of Was Cantor Surprised, we can take advantage of the injection Cantor constructed between [0,1] and [0, 1).

First we give the simple bijection $f = tan(\frac{\pi}{2}x - \frac{\pi}{2})(x \in (0,1))$ between $(0, 1)$ and $\mathbb{R}$. Next construct the bijection between $(0,1)$ and $[0, 1]$.
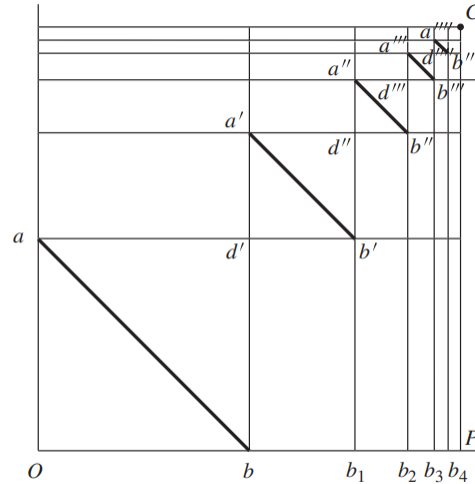


**Figure 1.** Cantor's function from [0, 1] to (0, 1).

图 1: Cantor's bijection

The domain has been divided by a geometric progression, so $b = 1/2$, $b_1 = 3/4$, and so on; $a = (0, 1/2)$, $a' = (1/2, 3/4)$, etc. The point $C$ is (1, 1). The points

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$d' = (1/2, 1/2)$, $d'' = (3/4, 3/4)$, etc. give the corresponding subdivision of the main diagonal.

I can't explain the curve in english clearly, so i quote the discription in Was Cantor Surprised [1] here. We can slightly change this bijection by ignoring the endpoint C, by doing so, we get the desired bijection $g$.

Also, in a $'modern'$ way(as Cantor would say), we can construct the bijection like this.

$$g(x) = \begin{cases} \dfrac{1}{2}, & x = 0 \\ \dfrac{1}{n+2}, & (x \in \{\dfrac{1}{n}, n \in \mathbb{N}\} \\ x, & (x \notin \{\dfrac{1}{n}, n \in \mathbb{N}\}) \end{cases} \tag{1}$$

By $g$, we map $0, 1, \frac{1}{2}...$ to $\frac{1}{2}, \frac{1}{3}...$, since we have infinite $\frac{1}{n}$ in $(0, 1)$, this can be done.

To sum up, the bijection between $[0, 1]$ and $\mathbb{R}$ is : $f \circ g$

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[1] I also add my notes on this article, see `WasCantorSurprised_AfterReading.md`