Integration

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Notation $\mu(f) :=: \int_s f \mathrm{d}\mu,\ \mu(f; A)$

Let (S, Σ, μ) be a measure space.

The (Lebesgue) integral of f (elements of $m\Sigma$) with respect to μ. Here are the notations :

$$\mu(f) :=: \int_s f(s)\mu(\mathrm{d}s) :=: \int_s f \mathrm{d}\mu$$

The equivalences for $A \in \Sigma$:

$$\int_A f(s)\mu(\mathrm{d}s) :=: \int_A f \mathrm{d}\mu :=: \mu(f; A) := \mu(f I_A)$$

Summation is a special type of integration. If $(a_n : n \in \mathbf{N})$ is a sequence of real numbers, then with $S = \mathbf{N}$, $\Sigma = \mathcal{P}(\mathbf{N})$, and μ the measure on (S, Σ) with $\mu(\{k\}) = 1$ for every k in $\mathbf{N}$, then $s \mapsto a_s$ is μ-integrable if and only if $\sum |a_n| < \infty$, and then

$$\sum a_n = \int_s a_s \mu(\mathrm{d}s) = \int_s a \mathrm{d}\mu$$

Definition of $\mu(f),\ f \in (m\Sigma)^+$

For $f \in (m\Sigma)^+$ we define

$$\mu(f) := \sup\{\mu_0(h) : h \in SF^+, h \leq f\} \leq \infty$$

Clearly, for $f \in SF^+$, we have $\mu(f) = \mu_0(f)$.

LEMMA

If $f \in (m\Sigma)^+$ and $\mu(f) = 0$, then

$$\mu(\{f > 0\}) = 0$$

Monotone-Convergence Theorem (MON)

If (f_n) is a sequence of elements of $(m\Sigma)^+$ such that $f_n \uparrow f$, then

$$\mu(f_n) \uparrow \mu(f) \leq \infty$$

or, in other notation,

$$\int_S f_n(s)\mu(ds) \uparrow \int_S f(s)\mu(ds)$$

It is proved in A5.4.

The Fatou Lemmas for functions

FATOU

For a sequence (f_n) in $(m\Sigma)^+$,

$$\mu(\liminf f_n) \leq \liminf \mu(f_n)$$

Reverse Fatou Lemma

If (f_n) is a sequence in $(m\Sigma)^+$ such that for some g in $(m\Sigma)^+$, we have $f_n \leq g$, $\forall n$, and $\mu(g) < \infty$, then

$$\mu(\limsup f_n) \geq \limsup \mu(f_n)$$

Integrable function, $\mathcal{L}^1(S, \Sigma, \mu)$

For $f \in m\Sigma$, we say that f is μ-integrable, and we write

$$f \in \mathcal{L}^1(S, \Sigma, \mu)$$

if

$$\mu(|f|) = \mu(f^+) + \mu(f^-) < \infty$$

and then we define

$$\int f du := \mu(f) := \mu(f^+) - \mu(f^-)$$

Note that, for $f \in \mathcal{L}^1(S, \Sigma, \mu)$

$$|\mu(f)| \leq \mu(|f|)$$

This is *the modulus of the integral is less that or equal to the integral of the modulus.*

Linearity

For $\alpha, \beta \in \mathbf{R}$ and $f, g \in \mathcal{L}^1(S, \Sigma, \mu)$,

$$\alpha f + \beta g \in L^1(S, \Sigma, \mu)$$

and

$$\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$$

Dominated-Convergence Theorem (DOM)

Suppose that $f_n, f \in m\Sigma$, that $f_n() \to f(s)$ for every s in S and that the sequence (f_n) is **dominated** by an element g of $\mathcal{L}^1(S, \Sigma, \mu)^+$:

$$|f_n(s)| \leq g(s), \quad \forall s \in S, \forall n \in \mathbf{N},$$

where $\mu(g) < \infty$. Then

$$f_n \to f \text{ in } \mathcal{L}^1(S, \Sigma, \mu) \text{: that is, } \mu(|f_n - f|) \to 0,$$

whence

$$\mu(f_n) \to \mu(f).$$

Scheffé's Lemma (SCHEFFÉ)

Suppose that $f_n, f \in \mathcal{L}^1(S, \Sigma, \mu)^+$; in particular, f_n and f are non-negative. Suppose that $f_n \to f$ (a.e.). Then

$$\mu(|f_n - f|) \to 0 \iff \mu(f_n) \to \mu(f)$$

Standard machine

The standard machine is a much cruder alternative to the Monotone-Class Therorem.

The idea is that to prove that a 'linear' result is true for all functions h in a space such as $\mathcal{L}^1(S, \Sigma, \mu)$,

- first, we show the result is true for the case when h is an indicator function - which it normally is by definition;
- then, we use linearity to obtain the result for h in SF^+;
- next, we use (MON) to obtain the result for $h \in (m\Sigma)^+$, integrability conditions on h usually being superfluous at this stage;
- finally, we show, nny writing $h = h^+ - h^-$ and using linearity, that the claimed result is true.