Events

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Model for experiment: $(\Omega, \mathcal{F}, P)$

A probability triple $(\Omega, \mathcal{F}, P)$.

Sample space Ω

Ω is a set called the *sample space*.

Sample point ω

A point ω of Ω is called *a sample point*.

Event

The σ-algebra $\mathcal{F}$ on Ω is called the family of events（事件类）, so that an *event* is an element of $\mathcal{F}$, that is, an $\mathcal{F}$-measurable（$\mathcal{F}$-可测） subset of Ω.

P is a probability measure on $(\Omega, \mathcal{F})$

For $F \in \mathcal{F}$, $P(F)$ is the probability that ω is in F.

Almost surely (a.s.)

A statement S about outcomes is said to be true *almost surely (a.s.)*, or *with probability* 1 *(w.p.1)*, if

$$F := \{\omega : S(\omega) \text{ is true }\} \in \mathcal{F} \text{ and } \mathbf{P}(F) = 1$$

Some definitions

Let $(x_n : n \in \mathbf{N})$ be a sequence of real numbers.

(a)

$$\limsup x_n := \inf_m \left\{\sup_{n \geq m} x_n\right\} =\downarrow \lim_m \left\{\sup_{n \geq m} x_n\right\} \in [-\infty, \infty]$$

(b)

$$\liminf x_n := \sup_m \left\{\inf_{n \geq m} x_n\right\} =\uparrow \lim_m \left\{\inf_{n \geq m} x_n\right\} \in [-\infty, \infty]$$

(c)

$$x_n \text{ converges in } [-\infty, \infty] \Longleftrightarrow \limsup x_n = \liminf x_n$$

and then $\lim x_n = \limsup x_n = \liminf x_n$

Definition. $\limsup E_n, (E_n, \text{ i.o. })$

If E is an event, then

$$E = \omega : \omega \in E$$

Suppose now that $(E_n : n \in \mathbf{N})$ is a sequence of events.

(a) We define

$$\begin{aligned}(E_n, \text{ i.o. }) &:= (E_n \text{ infinitely often }) \\ &:= \limsup E_n := \bigcap_m \bigcup_{n \geq m} E_n \\ &= \left\{\omega : \text{ for every } m, \quad \exists n(\omega) \geq m \text{ such that } \omega \in E_{n(\omega)}\right\} \\ &= \{\omega : \omega \in E_n \text{ for infinitely many } n\}\end{aligned}$$

(b) (**Reverse Fatou Lemma** - needs *FINITENESS* of P)

$$P(\limsup E_n) \geq \limsup P(E_n)$$

First Borel-Cantelli lemma (BC1)

Let $(E_n : n \in \mathbf{N})$ be a sequence of events such that $\sum_n P(E_n) < \infty$. Then

$$P(\limsup E_n) = P(E_n, i.o.) = 0$$

Definitions. $\liminf E_n$, (E_n, ev)

Suppose that $(E_n : n \in \mathbf{N})$ is a sequence of events.

(a) We define

$$\begin{aligned}(E_n, \mathrm{ev}) &:= (E_n \text{ eventually }) \\ &= \liminf E_n := \bigcup_m \bigcap_{n \geq m} E_n \\ &= \{\omega : \text{ for some } m(\omega), \quad \omega \in E_n, \forall n \geq m(\omega)\} \\ &= \{\omega : \omega \in E_n \text{ for all large } n\}\end{aligned}$$

(b) Note that $(E_n, \mathrm{ev})^c = (E_n^c, \text{ i.o. })$.

(c) (**Fatou's Lemma for sets** - true for *ALL* measure spaces)

$$P(\liminf E_n) \leq \liminf P(E_n)$$