DOM's dominating function is necessary

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Theorem 1. *DOM(Lebesgue's Dominated Convergence Theorem)*

$f_n \to f$(a.e.) and $|f_n(s)| \le g(s), \forall s \in S, \forall n \in \mathbb{N}$, $\mu(g) < \infty \Rightarrow \mu(|f_n - f|) \to 0$ which also implies $\mu(f_n) \to \mu(f)$.

I will prove the necessity of the dominating function $g \in \mathcal{L}^1(S, \Sigma, \mu)$ by giving a counterexample.

Assume $S = (0,1]$, $\Sigma = \mathcal{B}(S)$, and $\mu = \mathrm{Leb}(S)$. Define $f = 0$ and

$$f_n(s) = \begin{cases} n, s \in (0, \frac{1}{n}] \\ 0, s \in (\frac{1}{n}, 1] \end{cases}$$

Clearly, $f_n \to f$(a.e.). But $\mu(|f_n - f|) = n \times \mu((0, \frac{1}{n}]) = 1 \Rightarrow \mu(|f_n - f|) \to 1$, contradicting to DOM's conclusion $\mu(|f_n - f|) \to 0$.