

Independent Normal Distribution Variable Sequence

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Problem 1 (Exercise 4.5 of Chapter E)

Background:

A random variable G has a normal $N(0, 1)$ distribution, then for $x > 0$,

$$P(G > x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{1}{2}y^2} dy.$$

Note that this property only is all we need regarding random variable with normal distribution in this problem.

1. Prove that

$$P(G > x) \leq \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

2. Let X_1, X_2, \dots be a sequence of independent $N(0, 1)$ variables. Prove that with probability 1, $L \leq 1$, where

$$L := \limsup \left(\frac{X_n}{\sqrt{2\log n}} \right).$$

3. Prove that

$$P(L = 1) = 1.$$

Proof:

1. This is proven by manipulating the integral.

$$\begin{aligned} P(G > x) &= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{1}{2}y^2} dy \\ &\leq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot \int_x^{\infty} e^{-\frac{1}{2}y^2} \cdot y \cdot dy \\ &= \frac{1}{x\sqrt{2\pi}} \int_{\frac{1}{2}x^2}^{\infty} e^{-y} \cdot dy \\ &= \frac{1}{x\sqrt{2\pi}} (-e^{-y}) \Big|_{\frac{1}{2}x^2}^{\infty} \\ &= \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \end{aligned}$$

2. Let E_n denote the event that $\frac{X_n}{\sqrt{2\log n}} > \sqrt{1+\epsilon}$.

$$\begin{aligned}
\sum_{i \in \mathbb{N}} P(E_i) &= \sum_{i \in \mathbb{N}} P\left(\frac{X_n}{\sqrt{2\log n}} > \sqrt{1+\epsilon}\right) = \sum_{i \in \mathbb{N}} P(X_n > (\sqrt{1+\epsilon})\sqrt{2\log n}) \\
&= \sum_{i \in \mathbb{N}} \frac{1}{\sqrt{2\pi}} \int_{(\sqrt{1+\epsilon})\sqrt{2\log n}}^{\infty} e^{-\frac{1}{2}y^2} dy \\
&\leq \sum_{i \in \mathbb{N}} \frac{1}{(\sqrt{1+\epsilon})\sqrt{2\log n}\sqrt{2\pi}} e^{-\frac{1}{2}((\sqrt{1+\epsilon})\sqrt{2\log n})^2} \\
&= \sum_{i \in \mathbb{N}} \frac{1}{(\sqrt{1+\epsilon})\sqrt{2\log n}\sqrt{2\pi}} e^{-(1+\epsilon)\log n} \\
&< \sum_{i \in \mathbb{N}} \frac{1}{2\sqrt{(1+\epsilon)\pi}} \cdot \frac{1}{n^{1+\epsilon}}.
\end{aligned}$$

$\sum_{i \in \mathbb{N}} \frac{1}{n^{1+\epsilon}}$ converges, thus

$$\sum_{i \in \mathbb{N}} P(E_i) < \infty.$$

By **First Borel-Cantelli Lemma(BC1)**, we have

$$P(E_n, \text{i.o.}) = 0,$$

thus

$$P(E_n^c, \text{ev}) = P\left(\frac{X_n}{\sqrt{2\log n}} \leq \sqrt{1+\epsilon}, \text{ev}\right) = 1.$$

Finally,

$$\begin{aligned}
P(L \leq 1) &= P(\limsup(\frac{X_n}{\sqrt{2\log n}}) \leq 1) = P(\frac{X_n}{\sqrt{2\log n}} \leq 1, \text{ev}) \\
&= \lim_{\epsilon \rightarrow 0} P(\frac{X_n}{\sqrt{2\log n}} \leq \sqrt{1+\epsilon}, \text{ev}) \\
&= 1.
\end{aligned}$$

3. We only need to prove that $P(L \geq 1) = 1$, then $P(L \geq 1) = P(L \leq 1) = 1 \implies P(L = 1) = 1$.
We have

$$P(L \geq 1) = P(\limsup(\frac{X_n}{\sqrt{2\log n}}) \geq 1) = P(\frac{X_n}{\sqrt{2\log n}} \geq 1, \text{i.o.}).$$

Thus we need to prove that $P(\frac{X_n}{\sqrt{2\log n}} \geq 1, \text{i.o.}) = 1$.

Let E_n denote the event that $\frac{X_n}{\sqrt{2\log n}} \geq 1$. It is easy to verify that (E_n) are independent events since (X_n) are independent random variables.

$$\sum_{i \in \mathbb{N}} P(E_i) = \sum_{i \in \mathbb{N}} P\left(\frac{X_n}{\sqrt{2\log n}} \geq 1\right) = \sum_{i \in \mathbb{N}} P(X_n \geq \sqrt{2\log n}).$$

Similarly to (1), we can deduce that

$$P(G \geq x) \geq \frac{1}{\sqrt{2\pi}} \cdot \frac{x}{x^2 + 1} \cdot e^{-\frac{1}{2}x^2}.$$

Thus,

$$\begin{aligned} \sum_{i \in \mathbb{N}} P(E_i) &\geq \sum_{i \in \mathbb{N}} \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\log n}}{(\sqrt{2\log n})^2 + 1} \cdot e^{-\frac{1}{2}(\sqrt{2\log n})^2} \\ &= \sum_{i \in \mathbb{N}} \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\log n}}{2\log n + 1} \cdot e^{-\log n} \\ &\geq \sum_{i \in \mathbb{N}} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{n \cdot \sqrt{2\log n}} = \infty. \end{aligned}$$

Therefore, by **First Borel-Cantelli Lemma(BC2)**, we have

$$P\left(\frac{X_n}{\sqrt{2\log n}} \geq 1, \text{i.o.}\right) = P(E_n, \text{i.o.}) = 1,$$

which finishes the proof.

□