Trival Proofs of Inclusion-exclusion Principle

Fu Lingyue

March 20, 2020

Theorem 1.

Here I will give the second way to proof it. Another proof has been given by Chen Tong.

Proof. Choose an arbitrary point $x \in \bigcup_{k=1}^{n} A_k$, and let $A_{l_1}, A_{l_2}, \ldots, A_{l_t}$ $(t < n)$ be the subsets that $x \in A_{l_k}$ $(1 \leq k \leq t)$.

x is counted for one time on the left hand side of the equation. And the number that x be counted on the right hand side is

$$\begin{aligned}\text{Num\_right} &= \Sigma_{k=1}^{n}(-1)^{k+1}|\{\bigcap_{p=1}^{k} A_{m_p} : 1 \leq m_1 \leq m_2 \ldots m_k \leq t\} \\ &= \Sigma_{k=1}^{n}(-1)^{k+1}C(t,k) \\ &= 1. \qquad \text{(by Binomial Theorem)}\end{aligned}$$

Thus x is counted for equal times on left and right. Therefore, lhs=rhs.

□

Theorem 2. *Let (X, μ) be a finite measure space. For any finite numbers of measurable sets $A_1, A_2, \ldots, A_n \subseteq X$, we have*

$$\mu(\bigcup_{k=1}^{n} A_k) = \Sigma_{\varnothing \neq S \subseteq 1,2,\ldots,n}^{n}(-1)^{|S|-1}\mu(\bigcap_{k \in S} A_k)$$

Proof. Let A denote the union $\bigcup_{k=1}^{n} A_k$. We have to verify the identity

$$1_A = \Sigma_{k=1}^{n}(-1)^{k-1}\Sigma_{I \subseteq \{1,2,\ldots,n\},|I|=k} 1_{A_I}. \tag{1}$$

Here we denote $A_I = \bigcap_{k \in I} A_k$.

We can write an equation

$$(1_A - 1_{A_1})(1_A - 1_{A_2}) \ldots (1_A - 1_{A_n}) = 0. \tag{2}$$

If $x \notin A$, then all the factors are $0 - 0 = 0$; If $x \in A$, then x must in some subset A_j, thus the factor $1_A - 1_{A_j} = 0$. Thus the equation (2) holds. Expand the left hand side of the equation(2), we get the equation (1).

Using equation (1), we have proved the theorem.

□

A question Why the Inclusion-exclusion Principle need the measure space finite?