

Zero-One Law For First Order Logic

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Definition 1. *Erdős–Rényi random graph $G(n, p)$ model is a model where a n -vertex graph is constructed by connecting nodes randomly. Each edge is included in the graph with probability p (a function of n) independent from every other edge.*

Problem 1. Suppose we want to know which values of p are likely to produce graphs with isolated vertices and which are not when $n \rightarrow \infty$.

Solution. Define X to be the number of isolated vertices in a n -vertex graph. Obviously X is a random variable.

Let's introduce the first moment method: For a non-negative, integer-valued random variable X , if $E[X]$ tends to 0, then the value of X tends to 0. This can be directly derived from Markov's inequality:

$$P[X \geq a] \leq \frac{E[X]}{a} \quad (1)$$

Let $a = 1$, then $P[X > 0] = P[X \geq 1] \leq E[X]$. So if $E[X] \rightarrow 0$, then $X = 0$ almost surely.

Now focus on the problem.

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] \quad (2)$$

where X_i is 1 if node i is isolated, and 0 otherwise. So

$$\begin{aligned} E[X_i] &= P(\text{node } i \text{ is isolated}) = (1 - p)^{n-1} \\ E[X] &= n(1 - p)^{n-1} < ne^{-p(n-1)} \end{aligned}$$

If $\forall \epsilon > 0, p > \frac{(1+\epsilon)\log(n)}{n}$, then $E[X] \rightarrow 0$. $P[\text{Graph has no isolated vertex}] = 1 - P[X > 0] = 1$

To deal with the opposite part where graph has isolated vertex almost surely, we'll use the second moment method. The Chebyshev Inequality tells us:

$$P[|X - E[X]| \geq c] \leq \frac{\text{Var}[X]}{c^2} \quad (3)$$

Let $c = E[X]$, then

$$P[X = 0] \leq P[|X - E[X]| \geq E[X]] \leq \frac{\text{Var}[X]}{E[X]^2} \quad (4)$$

Now we compute $Var[X]$ and $E[X]^2$.

$$\begin{aligned} E[X]^2 &= n^2(1-p)^{2n-2} \\ Var[X] &= E[X^2] - E[X]^2 = E[X^2] - n^2(1-p)^{2n-2} \end{aligned}$$

Then expand $X = \sum_{i=1}^n X_i$.

$$\begin{aligned} E[X^2] &= E\left[\sum_{i=1}^n X_i^2\right] + E\left[\sum_{i \neq j} X_i X_j\right] \\ &= \sum_{i=1}^n E[X_i^2] + \sum_{i \neq j} E[X_i X_j] \end{aligned}$$

We find that X_i are indicators (only have value of 0 and 1), so $X_i^2 = X_i$. So it can simplified to

$$\begin{aligned} E[X^2] &= \sum_{i=1}^n E[X_i] + \sum_{i \neq j} E[X_i X_j] \\ &= n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3} \end{aligned}$$

Now we can get

$$\begin{aligned} Var[X] &= n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3} - n^2(1-p)^{2n-2} \\ \frac{Var[X]}{E[X]^2} &= n^{-1}(1-p)^{-n+1} + \frac{n-1}{n}(1-p)^{-1} - 1 \end{aligned}$$

If $\forall \epsilon > 0, p < \frac{(1-\epsilon)\log(n)}{n}$, then $\frac{Var[X]}{E[X]^2} = o(1)$. Then $P[\text{Graph has isolated vertex}] = 1 - P[X = 0] = 1$ \square

Now we discover an interesting fact that the probability that $G(n, p)$ has isolated vertices is asymptotically either 0 or 1. (not exactly because we haven't discussed the situation of $p = \frac{\log n}{n}$, but that situation is much harder). Actually many properties of graph have similar phenomenon.

To study this phenomenon in general, we'll introduce sentences about graphs in first order logic. We have studied what a sentence in first order logic is in Mathematical Logic. For graphs, similarly, a sentence uses existential and universal quantifiers over variables, but the only relation is (u, v) , which means that there is an edge between u and v .

Example. the condition that a graph does not have any isolated vertices may be expressed by the sentence $\forall u \exists v e(u, v)$. Now we state the formal Zero One Law for First Order Logic.

Theorem 1. let S be a fixed first-order sentence, and choose a random n -vertex graph G_n uniformly at random among all graphs on a set of n labeled vertices. Then in the limit as n tends to infinity the probability that G_n models S will tend either to zero or to one

Proof. The idea is to define a set of "basic sentences" that "spans" the space of all first-order sentences.

Define $\phi_{k,l}$ as: for every set of k vertices and every set of l vertices, there is some other vertex adjacent to the k vertices but not the l vertices. Or in math language,

$$\forall x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_l \exists z (e(x_1, z) \vee e(x_2, z) \vee \dots \vee e(x_k, z)) \wedge \neg(e(y_1, z) \vee e(y_2, z) \vee \dots \vee e(y_l, z))$$

Claim $P(G_n \models \phi_{k,l}) \rightarrow 1$. The proof is as follows: We have $\binom{n}{k} \binom{n-k}{n-k-l}$ choices to select these 2 sets. For each selection and each vertex z , the probability of z not satisfying the condition is $1 - p^k(1-p)^l$. So the total probability of $\phi_{k,l}$ being false is

$$\binom{n}{k} \binom{n-k}{n-k-l} (1 - p^k(1-p)^l)^{n-k-l} \leq \binom{n}{k} \binom{n-k}{n-k-l} \left(1 - \left(\frac{k}{k+l}\right)^k \left(1 - \frac{k}{k+l}\right)^l\right)^{n-k-l} \rightarrow 0$$

We now finish the proof of the claim.

Let $\Phi = \{\phi_{k,l} : k, l \in \mathbb{N}\}$. Then $\Phi^\models = \{\phi : \Phi \models \phi\}$ is a theory in perspective of logic. To prove it, we just need to prove that Φ is consistent. From graph theory, we know the Rado Graph is the unique countable model that models Φ . So Φ^\models is consistent because it's satisfiable. Furthermore, we'll prove Φ^\models is complete. Assume Φ^\models is not complete, which means there exists ψ that neither ψ nor $\neg\psi$ is the consequence of Φ . This is equivalent to $\Phi \cup \{\psi\}$ and $\Phi \cup \{\neg\psi\}$ are consistent. If we extend Φ by adding ψ and $\neg\psi$ respectively, both new theories are countable. Then by Lowenheim-Skolem Theorem, they have countable models. Since the new models is still the model of Φ , they can only be the Rado Graph. This yields a contradiction.

Now given a property ϕ that can be expressed in first order logic. Then either ϕ or $\neg\phi$ is in Φ^\models . Without loss of generality, suppose ϕ is in Φ^\models . Then $\Phi \vdash \phi$. Since there are only finite assumptions in sequent calculus, we only have finite $\phi_{k,l}$ as assumptions. We've proved that $P(G_n \models \phi_{k,l}) \rightarrow 1$, so $P(G_n \models \phi) \rightarrow 1$. If $\neg\phi$ is in Φ^\models , we can similarly derive that $P(G_n \models \neg\phi) \rightarrow 1$. This finishes the proof.

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