It will be seen that, by starting with depression i. *applied to its utmost extent* as Step I., only four *separate* steps of depression have been necessary, and the *ascending* steps (which are the most difficult) have involved the solution of only *three* (first order) differential equations and *two* simple integrations.

DIFFERENTIAL EQUATION OF CONIC. OTHER DEPRESSIONS.

Inasmuch as this differential equation is clear of x, y, y', it may be depressed one order by depression i., and will still be clear of the new variables x, y, and will still be multiply homogeneous. It may therefore be depressed (Art. 16) two orders by a double application of depression ii. without destroying the homogeneity; moreover the depression for homogeneity of order ∞ may be effected before or after the second of depressions ii. (Art. 9).

Hence the 5 depressions required in all may be effected in any of the following orders:—

Course 1°. Depressions i., ii., ii., iii. (ν arbitrary), iii. ($\nu = 1$).

„ 2°. „ i., ii., ii., iv., iii. ($\nu = -1$).

„ 3°. „ i., ii., iv., ii., iii. ($\nu = 2$).

But this procedure is of course not so easy as that before shewn in abstract in Tables VIII. and IX., especially in the ascending steps, the reversal of Step II. not being so easy as in the former procedure.

It is proposed to show the detail in a further paper.

AN EXTENSION OF A CERTAIN THEOREM IN INEQUALITIES.

By *L. J. Rogers.*

§ 1. I PROPOSE in the following pages to show how, by a slight extension of the well-known theorem in inequalities concerning the arithmetic and geometrical means of n positive quantities, we can deduce many others, including those usually given in the text-books.

The theorem is as follows:

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be all positive quantities, then

$$\left(\frac{a_1b_1 + a_2b_2 + \ldots a_nb_n}{a_1 + a_2 + \ldots a_n}\right)^{a_1+a_2+\ldots a_n} > b_1^{a_1}b_2^{a_2}\ldots b_n^{a_n} \ldots\ldots (1).$$

Firstly, let $a_1, a_2, \ldots a_n$ be integers.

Then we merely have a particular case of the well-known theorem, wherein we have a_1 quantities, each equal to b_1, &c., the whole number of them being $a_1 + a_2 + \ldots a_n$.

Secondly, let $a_1, a_2, \ldots$ be fractional.

Let N be the least common measure of their denominators and let $Na_1 = A_1$, $Na_2 = A_2$, &c., then we get by what is proved above

$$\left(\frac{A_1 b_1 + A_2 b_2 + \ldots}{A_1 + A_2 + \ldots}\right)^{A_1+A_2+\ldots} > b_1^{A_1} b_2^{A_2} \ldots .$$

Taking the real positive N^{th} root of each side we get after reducing the bracketted fraction, the inequality (1).

Thirdly, let the a's be incommensurable.

Then we may substitute for each of these quantities fractions, which may differ from them by less than any assigned quantities, and since the theorem is true for the substituted fractions, we may assume it also true for the given incommensurables.

Hence we may consider (1) as established.

It will be found conveniently brief to write s_r for $a_1^r + a_2^r + \ldots$, as we shall do henceforth.

Let $b_r = \dfrac{1}{a_r}$ for all values of r from 1 to n, then from (1)

we get $$\left(\frac{s_0}{s_1}\right)^{s_1} > \frac{1}{a_1^{a_1}} \frac{1}{a_2^{a_2}} \ldots,$$

i.e. $$\left(\frac{a_1 + a_2 + \ldots a_n}{n}\right)^{a_1+a_2+\ldots a_n} < a_1^{a_1} a_2^{a_2} \quad \ldots\ldots\ldots\ldots (2),$$

a well known result.

Write a_1^r for a_1, a_2^r for a_2, &c., and let $b_1 = a_1^{m-r}$, $b_2 = a_2^{m-r}$, where $m > r$. Then (1) gives

$$\left(\frac{s_m}{s_r}\right)^{s_r} > (a_1^{a_1^r} a_2^{a_2^r} \ldots)^{m-r}.$$

Again, let $b_1 = a_1^{t-r}$ where $t < r$.

Then $$\left(\frac{s_t}{s_r}\right)^{s_r} > (a_1^{a_1^r} a_2^{a_2^r} \ldots)^{-(r-t)},$$

i.e. $$\left(\frac{s_r}{s_t}\right)^{s_r} < (a_1^{a_1^r} \ldots)^{r-t}.$$

Combining these results we get

$$\left(\frac{s_m}{s_r}\right)^{\frac{s_r}{m-r}} > \left(\frac{s_r}{s_t}\right)^{\frac{s_r}{r-t}}.$$

Taking the s_rth root and reducing we get

$$s_m^{r-t} s_t^{m-r} > s_r^{m-t} \text{ .... (3),}$$

provided $m > r > t$.

Let $t = 0$.

Then $$\left(\frac{s_m}{s_0}\right)^r > \left(\frac{s_r}{s_0}\right)^m \text{ .... (4),}$$

which for $r = 1$, so that $m > 1$, we get

$$\frac{a_1^m + a_2^m + \dots a_n^m}{n} > \left(\frac{a_1 + a_2 + \dots a_n}{n}\right)^m,$$

a well known relation, if $m > 1$.

If $m = 1$, $t = 0$, then $\left(\frac{s_1}{s_0}\right)^r > \frac{s_1}{s_0}$, where r is a positive proper fraction.

If $m = 0$, $t = -1$, $\left(\frac{s_{-1}}{s_0}\right)^r < \frac{s_{-1}}{s_0}$,

and if $m = 0$, $r = -1$, then $\left(\frac{s_{-1}}{s_0}\right)^t < \frac{s_{-1}}{s_0}$.

Again, from (3) we may deduce many other results.

If $t > u$, we also have

$$s_r^{t-u} s_m^{r-t} > s_t^{r-u},$$

and if $m + u = r + t$, we get on multiplication

$$(s_m s_u)^{r-t} > (s_r s_t)^{r-t},$$

or $$s_m s_u > s_r s_t \text{ .... (5),}$$

provided $m > r > t > u$, and $m + u = r + t$.

The result (5) can be obtained without using (1).

If we multiply the two arrays,

$$\left|\begin{matrix} a_1^{t-u}, & a_2^{t-u}, & \dots, & a_n^{t-u} \\ 1, & 1, & \dots, & 1 \end{matrix}\right|$$

and $$\left|\begin{matrix} a_1^r, & a_2^r, & \dots, & a_n^r \\ a_1^t, & a_2^t, & \dots, & a_n^t \end{matrix}\right|,$$

we get the relation

$$\left|\begin{matrix} s_m, & s_r \\ s_t, & s_u \end{matrix}\right| = \Sigma a_1^t a_2^t (a_1^{t-u} - a_2^{t-u})(a_1^{r-t} - a_2^{r-t}),$$

the right side of which contains only positive terms.

Therefore $$s_m s_u > s_r s_t.$$

Let $u=0$ in (5), so that $s_{r+t}s_0 > s_r s_t$, we see then that in the same way

$$s_{\alpha+\beta+\gamma}s_0 > s_\alpha s_{\beta+\gamma},$$

$$s_{\alpha+\beta+\gamma}s_0^2 > s_\alpha s_{\beta+\gamma}s_0 > s_\alpha s_\beta s_\gamma,$$

and so on, or as we may better write it,

$$\frac{s_{\alpha+\beta+\gamma}}{s_0^3} > \frac{s_\alpha}{s_0}\,\frac{s_\beta}{s_0}\,\frac{s_\gamma}{s_0} \;\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

a result which admits of easy extension to n suffixes $\alpha, \beta, \gamma, \ldots$.

We shall now pass on to applications of the above results to Integral Calculus.

§ 2. In the inequality § 1 (3) let

$$a_1 = f(x_1+h), \quad a_2 = f(x_1+2h), \quad \ldots, \quad a_n = f(x_1+nh),$$

where $$x_1 + nh = x_2.$$

We then get, after multipyling s_m, s_r, s_t by h, and putting $f(x)=y$, and making h decrease indefinitely,

$$\{\textstyle\int y^m dx\}^{r-t}\{\int y^t dx\}^{m-r} > \{\int y^r dx\}^{m-t} \ldots\ldots\ldots\ldots (1),$$

where $m > r > t$, and the limits are such that y^m, y^r, and y^t remain finite and positive for all values between these limits.

As an example of this we may put $y \equiv x$, whence after changing $m+1$ to m, &c., it follows that

$$\left(\frac{r}{t}\right)^m \left(\frac{t}{m}\right)^r \left(\frac{m}{t}\right)^t > 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $m > r > t$.

Here we take for limits 1 and 0.

From (1) we may observe that it is impossible that

$$\int_b^a u^m dx \times \int_{b'}^{a'} v^m dx = 1 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where the limits are independent of m and taken so that the functions u, v should remain positive between their respective limits.

For let $$\int_b^a u^m dx = \phi(m).$$

Then, by (1),

$$\{\phi(m)\}^{r-t}\{\phi(t)\}^{m-r} > \{\phi(r)\}^{m-t}.$$

But if (3) were true we should have the reverse inequality also true, which is impossible.

Hence (3) cannot hold good.

As an example we have

$$\int_0^1 x^m dx = \frac{1}{m+1},$$

so that no function u can be found such that

$$\int_b^a u^m dx = m+1,$$

where a, b are subject to the afore-stated conditions.

As we deduced (1) from § 1 (3), so may we draw from § 1 (4) that

$$(x_1 - x_2)^{m-r} \left\{\int_{x_2}^{x_1} y^m dx\right\}^r > \left\{\int_{x_2}^{x_1} y^r dx\right\}^m \text{..........(4)},$$

where $m > r$, and the limits are under conditions as before.

If $y \equiv x$, we get

$$(r+1)^m > (m+1)^r \text{....................(5)},$$

where $m > r$.

§ 3. If we treat the inequality § 1 (3) in the same way as we deduced § 1 (1) from the well-known A.M. and G.M. relation, we shall get

$$(\Sigma ab^m)^{r-t} (\Sigma ab^t)^{m-r} > (\Sigma ab^r)^{m-t} \text{.............(1)}.$$

From § 1 (5) we get

$$\Sigma ab^m . \Sigma ab^n > \Sigma ab^r . \Sigma ab^t \text{................(2)},$$

and from § 1 (6)

$$\frac{\Sigma ab^{\alpha+\beta+\gamma+\dots}}{\Sigma a} > \frac{\Sigma ab^\alpha}{\Sigma a} \frac{\Sigma ab^\beta}{\Sigma b} \text{..............(3)}.$$

These give results similar to § 2 (1), viz.

$$(\int yv^m dx)^{r-t} (\int yv^t dx)^{m-r} > (\int yv^r dx)^{m-t} \text{..........(4)},$$

$$\int yv^m dx \int yv^n dx > \int yv^r dx \int yv^t dx \text{.............(5)},$$

$$\frac{\int yv^{\alpha+\beta+\dots} dx}{\int y dx} > \frac{\int yv^\alpha dx}{\int y dx} \frac{\int yv^\beta dx}{\int y dx} \text{.............(6)},$$

where y, v are functions of x, and the limits are under the same conditions as before.

In (4) put $y=\epsilon^{-x}$, $v=x$, then

$$(\Gamma m)^{r-t}(\Gamma t)^{m-r} > (\Gamma r)^{m-t}.$$

If $y=x^{l-1}$, $v=1-x$,

$$B(l.m)^{r-t}B(l.t)^{m-r} > B(l.r)^{m-t}.$$

If $u=\dfrac{1}{1+x^q}$, $v=x$,

$$\sin^{r-t}\frac{m\pi}{q}\sin^{m-r}\frac{t\pi}{q} < \sin^{m-t}\frac{r\pi}{q},$$

provided the sines are all positive.

We may also get similar inequalities from (5) and (6).

As in § 2 (3), we may also show that if

$$\int_b^a yv^m dx = \phi(m),$$

where a, b, y, v are subject to the same conditions, then we cannot have

$$\int_{b'}^{a'} uz^m dx = \frac{1}{\phi(m)} \quad\text{.....................(7)},$$

with conditions as before.

The following equations are therefore absurd:

$$\int_{x_2}^{x_1} uz^m dx = \frac{1}{\Gamma m}, \text{ or } \frac{1}{B(l,m)}, \text{ or } \sin m\pi,$$

if uz^m is always positive between $x=x_1$ and $x=x_2$, and x_1, x_2 are independent of m.

§ 4. We may also obtain a few inequalities from taking logarithms in § 1 (1), whence

$$\log\frac{\Sigma ab}{\Sigma a} > \frac{\Sigma a\log b}{\Sigma a}.$$

From this may be deduced

$$\log\frac{\int vy\,dx}{\int v\,dx} > \frac{\int v\log y\,dx}{\int v\,dx},$$

with restrictions as before as to limits.

These last inequalities do not appear to lead to very interesting results.

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