The continuity of distribution function

Guo Linsong 518030910419

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Question 1. *Construct an example to show that the distribution function of a random variable may not be left-continuous.*

Theorem 2. *Let X be a random variable.The distribution function $F_X(x) = \mathcal{L}_X(-\infty, x]$ is right-continuous.*

Proof. Let $\{a_n = \frac{1}{n}, n \in \mathbb{N}\}$ be a countable sequence of numbers.Let $A_n = (-\infty, x + a_n]$ and $A = (-\infty, x]$.Then $A_n, A \in \mathcal{B}$ and $A_n \downarrow A$.$\mathcal{L}_x$ is a probability measure on $(R, \mathcal{B})$.According to lemma 1.10(b) in textbook, we have

$$\mathcal{L}_X(A_n) \downarrow \mathcal{L}_X(A)$$

Thus

$$\mathcal{L}_X(-\infty, x] = \lim_{n\to\infty} \mathcal{L}_X(-\infty, x + \frac{1}{n}]$$

Together with the monotonicity of $\mathcal{L}_X$, we have $\mathcal{L}_X(-\infty, x] = \mathcal{L}_X(-\infty, x^+]$.Thus $F_X(x) = F_X(x^+)$,the distribution function $F_X(x) = \mathcal{L}_X(-\infty, x]$ is right-continuous. □

However,The distribution function may not left-continuous.Considering two consecutive coin tosses,let X be the number of heads.

$$F_X(x) = \mathcal{L}_X(-\infty, x] = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \\ 1/4, & \text{if } x \in [0, 1) \\ 3/4, & \text{if } x \in [1, 2) \\ 1, & \text{if } x \in [2, +\infty) \end{cases}$$

We have $F_X(0^-) \neq F_X(0) = F_X(0^+), F_X(1^-) \neq F_X(1) = F_X(1^+)$ and $F_X(2^-) \neq F_X(2) = F_X(2^+)$.Therefore, $F_X(x)$ is not left-continuous.