The application of positive and negative part in constructing simple functions

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Exercise 9. *Let* (S, Σ) *be a measurable space and take* $h \in \mathbb{R}^S$*.Let* $h^+ = max(h, 0)$ *and* $h^- = max(-h, 0)$*. Show that* $h \in m\Sigma$ *if and only if* $h^+, h^- \in m\Sigma$*.*

Solution. Observe that

$$h^+ = \begin{cases} 0 & h < 0 \\ h & h \geq 0 \end{cases}$$

$$h^- = \begin{cases} -h & h < 0 \\ 0 & h \geq 0 \end{cases}$$

So we have

$$h = h^+ - h^-$$

Since $m\Sigma$ is closed under taking sum and scalar multiplication, if $h^+, h^- \in m\Sigma$, $h \in m\Sigma$.

Then we'll focus on another side. Assume $h \in m\Sigma$. Consider

$$\{h^+ \leq c\} = \begin{cases} \emptyset & c < 0 \\ \{h \leq c\} & c \geq 0 \end{cases}$$

By the definition of σ-algebra, $\emptyset \in \Sigma$. $\{h \leq c\} = h^{-1}(-\infty, c] \in \Sigma$. So $\{h^+ \leq c\} \in \Sigma$ $(\forall c \in \mathbb{R})$. We can derive that $h^+ \in m\Sigma$.

$h^- \in m\Sigma$ can be derived similarly.

In conclusion, $h \in m\Sigma$ if and only if $h^+, h^- \in m\Sigma$. □

Definition 1. *Let* (S, Σ) *be a measurable space. A function* $f \in \mathbb{R}^S$ *is a simple function with respect to* (S, Σ) *provided it falls into the linear subspace of* $\mathbb{R}^S$ *spanned by* $\{\mathbf{1}_A : A \in \Sigma\}$*. Note that every simple function is* Σ*-measurable. For each positive integer* n*, define the dyadic function* $d_n \in \mathbb{R}^{\mathbb{R}}$ *to be*

$$\sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})} + n\mathbf{1}_{[n,\infty)}$$

Exercise 10. *Take* $f \in m\Sigma$*. For each* $n \in \mathbb{N}$*, show that* $f_n = d_n \circ f^+ - d_n \circ f^-$ *is a simple function with respect to* (S, Σ)*. Then illustrate that* f *is the limit of a sequence of simple functions.*

Solution. Observe that

$$f_n(s) = \begin{cases} \frac{k-1}{2^n} & f(s) \in [\frac{k-1}{2^n}, \frac{k}{2^n}), k \in \mathbb{N}, 1 \leq k \leq n2^n \\ n & f(s) \in [n, +\infty) \\ -n & f(s) \in (-\infty, -n] \\ -\frac{k-1}{2^n} & f(s) \in (-\frac{k}{2^n}, -\frac{k-1}{2^n}], k \in \mathbb{N}, 1 \leq k \leq n2^n \end{cases}$$

which is equal to:

$$f_n(s) = \begin{cases} \frac{k-1}{2^n} & s \in f^{-1}[\frac{k-1}{2^n}, \frac{k}{2^n}), k \in \mathbb{N}, 1 \leq k \leq n2^n \\ n & s \in f^{-1}[n, +\infty) \\ -n & s \in f^{-1}(-\infty, -n] \\ -\frac{k-1}{2^n} & s \in f^{-1}(-\frac{k}{2^n}, -\frac{k-1}{2^n}], k \in \mathbb{N}, 1 \leq k \leq n2^n \end{cases}$$

Now we construct

$$f_n = n\mathbf{1}_{f^{-1}[n,+\infty)} - n\mathbf{1}_{f^{-1}(-\infty,-n]} + \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{f^{-1}[\frac{k-1}{2^n}, \frac{k}{2^n})} - \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{f^{-1}(-\frac{k}{2^n}, -\frac{k-1}{2^n}]}$$

Since

$$\begin{aligned} f^{-1}[n, +\infty) &\in \Sigma \\ f^{-1}(-\infty, -n] &\in \Sigma \\ f^{-1}[\frac{k-1}{2^n}, \frac{k}{2^n}) &\in \Sigma \\ f^{-1}(-\frac{k}{2^n}, -\frac{k-1}{2^n}] &\in \Sigma \end{aligned}$$

we can derive that f_n is a simple function with respect to (S, Σ).

Without loss of generality, assume $f \geq 0$. Suppose $f < 2^n$, then there exists $k \in \mathbb{N}$ and $1 \leq k \leq n2^n$ such that $f \in [\frac{k-1}{2^n}, \frac{k}{2^n})$. We can then find that

$$f_n = \frac{k-1}{2^n}$$
$$\forall \epsilon > 0, \exists n_0 = \lceil max(\log_2 f, \log_2 \frac{1}{\epsilon}) \rceil + 1 > 0, \forall n > n_0, |f_n - f| < \frac{k}{2^k} - \frac{k-1}{2^n} = \frac{1}{2^n} < \epsilon$$

So f is the limit of a sequence of simple functions f_n.

□