A Trivial Idea of Exercise 4.1

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By imitating the method we used in the proof of Lemma 4.2, we can easily get the proof of this exercise.

By fixing $I_2 \in \mathcal{I}_2$ and $I_3 \in \mathcal{I}_3$, the two measures on $\sigma(\mathcal{I}_1)$ agree on $\mathcal{I}_1$, and they have the same total mass:

$$\begin{aligned}
&\mathbb{P}(I_2 \cap I_3) \\
&= \mathbb{P}(\Omega \cap I_2 \cap I_3) \\
&= \mathbb{P}(\Omega)\mathbb{P}(I_2)\mathbb{P}(I_3) \\
&= \mathbb{P}(I_2)\mathbb{P}(I_3)
\end{aligned}$$

Hence, they agree on $\sigma(\mathcal{I}_1)$

By fixing $H_1 \in \sigma(\mathcal{I}_1)$ and $I_3 \in \mathcal{I}_3$, the two measures on $\sigma(\mathcal{I}_2)$ agree on $\mathcal{I}_2$, and they have the same total mass:

$$\begin{aligned}
&\mathbb{P}(H_1 \cap I_3) \\
&= \mathbb{P}(H_1)\mathbb{P}(I_3)
\end{aligned}$$

Similarly, by fixing $H_1 \in \sigma(\mathcal{I}_1)$ and $H_2 \in \sigma(\mathcal{I}_2)$, the two measures agree on $\sigma(\mathcal{I}_3)$.

Then we conclude that $\sigma(\mathcal{I}_1)$ $\sigma(\mathcal{I}_1)$ $\sigma(\mathcal{I}_3)$ are independent.

We need $\Omega \in \mathcal{I}_k$ because we need the equation for the total mass in each case. Otherwise, consider one π-system being a set of measure 0.