Depicting Independence of Random Variables by Their Distribution Functions

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April 6, 2020

First we give a definition for distribution for multiple random variables

Definition 1 (Joint distribution function) *For n random variables $X_1, \cdots, X_n$, the joint cumulative distribution function $F_{X_1, X_2 \cdots, X_n}$ is given by*

$$F_{X_1, \cdots, X_n} = P\left(X_1 \leq x_1, \cdots, X_n \leq x_n\right).$$

Interpreting the n random variables as a random vector $\boldsymbol{X} = \left(X_1, \cdots, X_n\right)^\top$ yields a shorter notation

$$F_{\boldsymbol{X}}\left(\boldsymbol{x}\right) = P\left(X_1 \leq x_1, \cdots, X_n \leq x_n\right).$$

And we also need the lemma for independence of sigma algebra, which is proved in class. Here I give a generalization form of it.

Lemma 1 *Suppose $\mathcal{A}_1, \cdots, \mathcal{A}_n$ are independent and each $\mathcal{A}_i$ is a π-system, then $\sigma\left(\mathcal{A}_1\right), \cdots, \sigma\left(\mathcal{A}_n\right)$ are independent.*

The essence is the same with the case of 2 sigma algebras.

With the preparation done, we can depict the independence of random variables by their distribution functions as follows

Theorem 1 *Random variables $X_1, X_2, \cdots, X_n$ are independent, if and only if $\forall \boldsymbol{x} = \left(x_1, \cdots, x_n\right) \in \mathbb{R}^n$*

$$P\left(X_1 \leq x_1, \cdots, X_n \leq x_n\right) = \prod_{i=1}^{n} F_i\left(x_i\right).$$

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Proof: Using the definition of joint distribution function, the formula above is equivalent with

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{n} F_i(x_i).$$

Which is clean and beautiful. First we prove $\Rightarrow$, the result is obvious by definition. Since

$$\forall i \le n, B_i := \{X_i | X_i \le x_i\} \subset \mathcal{B}.$$

By definition, we obtain that

$$P\left(\bigcap_{i\in[n]} B_i\right) = \prod_{i=1}^{n} P(B_i).$$

Now we prove $\Leftarrow$. Consider $\mathcal{A}_i$ be the set of form $\{X_i \le x_i\}$. Hence by

$$\{X_i \le r\} \cap \{X_i \le s\} = \{X_i \le \min(r, s)\}.$$

We know that $\mathcal{A}_i$ is a π-system plus we can write it as $X^{-1}(\pi(\mathbb{R}))$, and they are independent (from the definition of independence). Moreover by the conclusion (the second half) proved by Ruihang Lai, we have

$$\sigma(\mathcal{A}_i) = \sigma\left(X_i^{-1}(\pi(\mathbb{R}))\right) = X_i^{-1}(\mathcal{B}) = \sigma(X_i).$$

It's obvious that the independence of random variables is equivalent with the independence of the sigma algebra they generated. So by **Lemma 1** we know that $\sigma(\mathcal{A}_1), \cdots, \sigma(\mathcal{A}_n)$ are independent, which means $\sigma(X_1), \cdots, \sigma(X_n)$ are independent, hence $X_1, \cdots, X_n$ are independent. $\square$