Proof of Extended BC1

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Theorem 1 If $\lim_{n\to\infty} A_n = 0$ and $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$, then $P(A_n\ i.o.) = 0$

Proof. Let $G_m = \cup_{n>m} A_n$ and $G_m \downarrow G$, where $G := limsup A_n$ By the proof of BC1, we know that:

$$P(A_n i.o.) = P(G) \leq \lim_{m\to\infty} P(G_m)$$

Because $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$, we know that $\lim_{m\to\infty} \sum_{m}^{\infty} P(A_n^c \cap A_{n+1}) = 0$. And by $\cup_{n\geq m}(A_n^c \cap A_{n+1}) = G_m/(A_m \cap A_{m+1})$, we can get that:

$$\lim_{m\to\infty} P(G_m) \leq \lim_{m\to\infty} P(\cup_{n\geq m}(A_n^c \cap A_{n+1})) + \lim_{n\to\infty} A_n \leq \lim_{m\to\infty} \sum_{m}^{\infty} P(A_n^c \cap A_{n+1}) = 0$$

So $P(A_n i.o.) = 0$. $\square$