Borel set on [a,b] is generated by π[a,b]

Ji Jiabao

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Proof.

$\Leftarrow$: prove $\sigma(\pi[a,b]) \subseteq \mathcal{B}[a,b]$

$$\pi[a,b] = \{A \cap [a,b] : A \in \pi(\mathbb{R})\} = \{(-\infty, x] \cap [a,b] : x \in \mathbb{R}\} = \{[a,x] : x \in [a,b]\}$$

Simlilar to the proof in class, we need to show that any $[a,x]$ is a union of countable open sets. Observing that $[a,x] = \cap_{n\in\mathbb{N}} = (a - \frac{1}{n}, x + \frac{1}{n})$, $[a,x] \in \mathcal{B}[a,b]$, which leads to $\sigma(\pi[a,b]) \subseteq \mathcal{B}[a,b]$

$\Rightarrow$: prove $\mathcal{B}([a,b]) \subseteq \sigma(\pi[a,b])$

Similar to the proof in class, we need to show that any open set of $[a,b]$ is contained in $\sigma(\pi[a,b])$. Still similar to the proof, each of these open sets is a countably union of open intervals, we need to show that every $s = (x,y) \cap [a,b] \in \sigma(\pi([a,b]))$.

We can see such s is in the four classes below, we prove $s \in \sigma(\pi([a,b]))$ for each case .

1. $s = [a,y)$, $s = \cap_{n > \frac{1}{b-y}, n\in\mathbb{N}}[a, y + \frac{1}{n}]$ and $[a, y + \frac{1}{n}] \in \pi([a,b])$, so $s \in \sigma(\pi([a,b]))$
2. $s = (x,b]$, $s = \cap_{n > \frac{1}{x-a}, n\in\mathbb{N}}[x - \frac{1}{n}, b]$ and $[x - \frac{1}{n}, b] \in \pi([a,b])$, so $s \in \sigma(\pi([a,b]))$
3. $s = [a,b]$, $[a,b] \in \pi([a,b])$ so $s \in \sigma(\pi([a,b]))$
4. $s = (x,y)$, $(x,y) = (x,b] \cap [a,y) \in \sigma(\pi([a,b]))$

Therefore, $\mathcal{B}([a,b]) \subseteq \sigma(\pi[a,b])$ $\square$