Relationships between Convergences

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April 27, 2020

Theorem 1 *Almost sure convergence $\Longrightarrow$ Convergence in Probability.*
Let μ *be a measure* $\Sigma \mapsto [0, \infty)$ *and* $(f_n) \in m\Sigma$, $f \in m\Sigma$.
If

$$f_n \xrightarrow{a.s.} f,$$

i.e.

$$\mu(f_n \not\to f) = \mu(\{\omega \in S \mid f_n(\omega) \not\to f(\omega)\}) = 0,$$

then for any $\epsilon > 0$,

$$\mu(|f_n - f| > \epsilon) \to 0.$$

Proof:

$$\begin{aligned}
(f_n \not\to f) &= \{\omega \in S \mid f_n(\omega) \not\to f(\omega)\} \\
&= \bigcup_{k=1}^{\infty} \bigcap_{m} \bigcup_{n=m}^{\infty} \{\omega \in S \mid |f_n(\omega) - f(\omega)| > \frac{1}{k}\} \\
&= \bigcup_{k=1}^{\infty} \bigcap_{m} \bigcup_{n=m}^{\infty} (|f_n - f| > \frac{1}{k}).
\end{aligned}$$

For any $\epsilon > 0$, since $\mu(f_n \not\to f) = 0$, we can deduce that

$$\mu(\bigcap_{m} \bigcup_{n=m}^{\infty} |f_n - f| > \epsilon) = 0,$$

thus

$$
\begin{aligned}
\lim_{n\to\infty} \mu(|f_n - f| > \epsilon) &\leqslant \lim_{m\to\infty} \mu(\bigcup_{n=m}^{\infty} |f_n - f| > \epsilon) \\
&= \mu(\lim_{m\to\infty} \bigcup_{n=m}^{\infty} |f_n - f| > \epsilon) \quad \text{(MON of Measure)} \\
&= \mu(\bigcap_{m} \bigcup_{n=m}^{\infty} |f_n - f| > \epsilon) \\
&= 0.
\end{aligned}
$$

Therefore

$$\lim_{n\to\infty} \mu(|f_n - f| > \epsilon) = 0,$$

i.e. (f_n) converges to f in probability. $\square$

Theorem 2 *Convergence in Probability $\Longrightarrow$ Almost sure convergence of a subsequence. Let $(f_n) \in m\Sigma$, $f \in m\Sigma$. If for any $\epsilon > 0$,*

$$\mu(|f_n - f| > \epsilon) \to 0,$$

then there exists an increasing subsequence (n_k) such that

$$f_{n_k} \xrightarrow{a.s.} f,$$

i.e.

$$\mu(f_{n_k} \not\to f) = \mu(\{\omega \in S \mid f_{n_k}(\omega) \not\to f(\omega)\}) = 0.$$

Proof: As a premise, we have

$$\mu(|f_n - f| > \epsilon) \to 0.$$

For an arbitrarily fixed $\epsilon > 0$, we will have an increasing sequence (n_k) such that

$$\mu(|f_{n_k} - f| > \epsilon) < \frac{1}{2^k}.$$

Taking the infinite sum on both sides yields,

$$\sum_{k=1}^{\infty} \mu(|f_{n_k} - f| > \epsilon) < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$$

By the *First Borel-Cantelli Lemma on general measure space*, we have

$$\mu(|f_{n_k} - f| > \epsilon, \text{i.o.}) = 0.$$

Based on the arbitrariness of ϵ, we have proven the statement,

$$\mu(f_{n_k} \not\to f) = 0.$$

$\square$

Remark *The almost sure convergence might not hold for the whole sequence.*
The construction I resorted to is relatively complicated. Let $(S, \Sigma, \mu) = ([0,1], \mathrm{Leb}[0,1], \mathrm{Leb})$.
Define f_n *and* f *as following: For every* $x \in [0,1]$,

$$f_1(x) = x + I_{[0,1]} \cdot x$$
$$f_2(x) = x + I_{[0,\frac{1}{2}]} \cdot x,\ f_3(x) = x + I_{[\frac{1}{2},1]} \cdot x$$
$$f_4(x) = x + I_{[0,\frac{1}{3}]} \cdot x,\ \ f_5(x) = x + I_{[\frac{1}{3},\frac{2}{3}]} \cdot x,\ \ f_6(x) = x + I_{[\frac{2}{3},1]} \cdot x,$$

and

$$f(x) = x.$$

Then we have

$$\mu(|f_n - f| > \epsilon) < \frac{1}{\sqrt{2n} + 1} \to 0,$$

however

$$\mu(f_n \not\to f) = \mathrm{Leb}([0,1]) \neq 0.$$

Therefore, given convergence in probability, only a subsequence of (f_n) *converges almost surely.*

Theorem 3 *Convergence in Probability of Monotone* $(f_n) \Longrightarrow$ *Almost sure convergence.*
We say (f_n) *is* ***monotone*** *in the sense that*

$$\forall \omega \in S\ f_{n+1}(\omega) > f_n(\omega).$$

Let (f_n) be a monotone sequence of measurable functions.
If for any $\epsilon > 0$,

$$\mu(|f_n - f| > \epsilon) \to 0,$$

then

$$f_n \xrightarrow{a.s.} f,$$

i.e.

$$\mu(f_n \not\to f) = \mu(\{\omega \in S \mid f_n(\omega) \not\to f(\omega)\}) = 0.$$

Proof: Using the result of *Problem 2*, we can obtain a subsequence (n_k) such that

$$\mu(f_{n_k} \not\to f) = \mu(\{\omega \in S \mid f_{n_k}(\omega) \not\to f(\omega)\}) = 0.$$

Consider any ω such that $f_{n_k}(\omega) \to f(\omega)$.
We prove that with monotonicity of (f_n), $f_n(\omega) \to f(\omega)$.

$$\begin{aligned} f_{n_k}(\omega) \to f(\omega) &\Longrightarrow \forall \epsilon > 0 \ \exists K \ \forall k > K \ |f_{n_k}(\omega) - f(\omega)| < \epsilon \\ &\xRightarrow{\text{Monotone}} \forall \epsilon > 0 \ \exists K \ \forall n > n_K \ |f_n(\omega) - f(\omega)| < \epsilon \\ &\Longrightarrow f_n(\omega) \to f(\omega) \end{aligned}$$

Therefore,

$$\mu(f_n \not\to f) \leqslant \mu(f_{n_k} \not\to f) = 0,$$

thus

$$f_n \xrightarrow{a.s.} f.$$

□