Further Discussion of Exercise 4

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The exercise 4 put forward that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. After reading some reference books, I'm trying to prove that $|A \times A| = |A|$ for any infinite set A.

Lemma 1. *If A, B are two infinite sets, then*

$$|A| + |B| = max\{|A|, |B|\}.$$

Proof. First, for any two sets, one can be put into one-to-one correspondence with a subset of the other one (Basic Set Theory(2002), A. Shen, page 68, Theorem 25). Thus the notion of "max" is well defined here.

Without loss of generality, assume that $|A| \leq |B|$. Using the propoties of countable sets($\aleph_0$), we have

$$|B| \leq |A| + |B| \leq |B| + |B| \leq |B| \times \aleph_0 = |B|$$

Applying the Canter-Schröder-Berstein Theorem, we can conclude that $|B| = |A| + |B|$ when $|A| \leq |B|$.

□

Proof. First of all, it is clear that the proposition is true when A is countable (the union of a countable family of countable sets is countable, i.e., $\aleph_0 \times \aleph_0 = \aleph_0$). That is to say, we have to prove the theorem when the cardinality of A is equal or greater than $\aleph_1$. We can find a countable $B \subset A$ (maybe because $\aleph_0$ is the smallest, I'm not sure). And for $|B \times B| = |B|$ we can find a bijection $f : B \mapsto B \times B$.

Then we consider the set of all countable $B \subset A$ and its corresponding function f_B. We denote it as

$$Z = \{\langle B, f_B \rangle \mid B \text{ is the countable subset of } A.\}$$

Next, we define a partial order $\leq$ on set Z. $\langle B_1, f_1 \rangle \leq \langle B_2, f_2 \rangle$ when the following two conditions are met:

(1) $B_1 \subset B_2$;

(2) f_1 concides f_2 in the domain B_1, i.e.,

$$f_1(b) = f_2(b) \text{ for every } b \in B_1$$

Here we get a parital order. In order to apply Zorn's lemma, we have to find upperbounds of those chains. Consider an arbitrary chain $C_0 \le C_1 \le \cdots C_n \le \cdots$ (If the chain is finite, then there must be an upperbound C_N). Denote their union U as

$$U = \bigcup_{i=0}^{\infty} C_i$$

And we can also get the combination g of f_i, which is a function from U to $U \times U$ (because of the definition(2) of partial order, there is no conflict in the definition of g).

Claim that g is a bijection on U. We begin our prove here.

If c_i, c_j belong to two different set of the chain, then these two set must be $C_i \le C_j$ or $C_j \le C_i$. That is to say, there must be a set C_k contains both c_i and c_j. In this way, $g(c_i) \ne g(c_j)$ because f_k is bijection. Thus g is injective.

Consider arbitraty $(u, v) \in U \times U$. If $u \in C_i$ and $v \in C_j$, then $u, v \in C_{max(i,j)}$, i.e., $(u, v) \in C_{max(i,j)} \times C_{max(i,j)}$. For every function f_i is bijective, (u, v) must have a preimage in $C_{max(i,j)}$. Thus g is surjective.

Hence, the pair $\langle U, g \rangle$ is the upper bound of the chain. According to **Zorn's Lemma**, Z has a maximal element $\langle M, f_M \rangle$. Note that f_M is the bijection function from M to $M \times M$. Now we have to prove M has the same cardinality with A. We prove by contradiction as follows.

Assume that $|M| < |A|$ ($M \subset A$, then $|M| \le |A|$). Denote $R = A \backslash M$. We can conclude from lemma 1 that $|A| = |M + R| = |M| + |R| = max\{|M|, |R|\}$. For our assumption stipulate $|M| \ne |A|$, then $|R| = |A|$ and $|R| > |M|$. Find a subset R' of R, and R' has the same cardinality as M. Denote $N = M \cup R'$(M and R' are disjoint), and we can visualize the extension as Figure 1 shows. $N \times N$

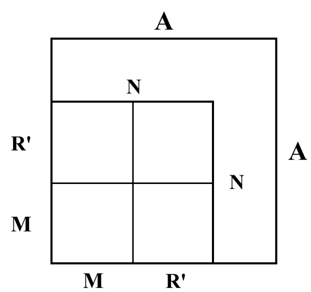

Figure 1: The extension of set M

has four parts, and each part has the same cardinality as $M \times M$. Because of the existence of f_M, each part of $N \times N$ has the same cardinality as M as well. Therefore, we can build a bijection h from R' to $(N \times N) \backslash (M \times M)$ (beacuse $|(N \times N) \backslash (M \times M))| = |M| + |M| + |M| = |M|$). Then we extend the function f_M to $f_N : N \mapsto N \times N$ as follows

$$f_N(n) = \begin{cases} f_M(n) & n \in M, \\ h(n) & n \notin M \end{cases}$$

Then we obtain a bijection function f_N of set N.

Thus we get a bigger set N and its corresponding bijection function f_N. This contradicts to the suppose that M is the maximal set.

Therefore, the assumption fails and $|M| = |A|$. In this case, M, A, $M \times M$ and $A \times A$ has the same cardinality. $\square$