Integral of Positive Simple Functions(SF^+)

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April 17, 2020

Let (S, Σ, μ) be a measure space. A function $f \in (m\Sigma)^+$ is called *simple* provided f can be written as a finite sum

$$f = \sum_{k=1}^{m} a_k \mathbf{1}_{A_k},$$

where $a_k \in [0, \infty]$, $A_k \in \Sigma$. We write $f \in SF^+$. Then we define the *Lebesgue integral* of f as

$$\mu(f) = \sum_{k=1}^{m} a_k \mu(A_k) \leqslant \infty.$$

1 Lebesgue Integral of Simple Function is *Well-Defined*

We first show that the $\mu(f)$ is well-defined.

Property 1 (Well-definition). *Let $\sum_{i=1}^{n} a_i \mathbf{1}_{A_i} = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j}$ be two representations of the same simple function $f \in SF^+$ with $A_i, B_j \in [0, \infty]$. Then*

$$\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{j=1}^{m} b_j \mu(B_j).$$

Proof. Note that for $p, q \in [n]$ with $p \neq q$, A_p and A_q may have *non-empty intersection*, which is not what we want. So we construct some *pairwise disjoint* sets from (A_i): let $(r_{t_1} r_{t_2} \dots r_{t_n})_2$ be the binary representation of $t \in \{0, 1, \dots 2^n - 1\}$, where r_{t_k} is either 0 or 1. Then we define set C_t as

$$C_t := \bigcap_{k=1}^{n} R_{t_k}, \text{ where } R_{t_k} = \begin{cases} A_k, & \text{if } r_{t_k} = 1 \\ A_k^c, & \text{if } r_{t_k} = 0 \end{cases}$$

For example let $n = 3$, and then we have

$$\begin{aligned} C_0 &= A_1^c \cap A_2^c \cap A_3^c,\ C_1 = A_1 \cap A_2^c \cap A_3^c \\ C_2 &= A_1^c \cap A_2 \cap A_3^c,\ C_3 = A_1 \cap A_2 \cap A_3^c \\ C_4 &= A_1^c \cap A_2^c \cap A_3,\ C_5 = A_1 \cap A_2^c \cap A_3 \\ C_6 &= A_1^c \cap A_2 \cap A_3,\ C_7 = A_1 \cap A_2 \cap A_3. \end{aligned}$$

Obviously the sets in (C_t) are pairwise disjoint. Then let

$$c_i := \sum_{j=1}^{n} [A_j \cap C_i \neq \emptyset] a_j$$

where $[x] = 1$ provided x is true, and $[x] = 0$ otherwise. Hence we have

$$f = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i} = \sum_{i=0}^{2^n-1} c_i \mathbf{1}_{C_i}, \ \sum_{i=1}^{n} a_i \mu(A_i) = \sum_{i=0}^{2^n-1} c_i \mu(C_i).$$

Note that for some $i \in \{0, 1, \ldots 2^n - 1\}$, c_i may be 0, which means $c_i \mathbf{1}_{C_i} = 0$. We remove such terms from the representation of f and finally we will obtain

$$f = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i} = \sum_{i=0}^{N} c_i \mathbf{1}_{C_i}, \ \sum_{i=1}^{n} a_i \mu(A_i) = \sum_{i=0}^{N} c_i \mu(C_i)$$

where $N \leqslant 2^n - 1$, $c_i > 0$, and sets in (C_i) are pairwise disjoint. Similarly we also have

$$f = \sum_{i=1}^{m} b_i \mathbf{1}_{B_i} = \sum_{i=0}^{M} d_i \mathbf{1}_{D_i}, \ \sum_{j=1}^{m} b_i \mu(B_i) = \sum_{j=0}^{M} d_j \mu(D_j).$$

So we only need to show that

$$\sum_{i=0}^{N} c_i \mu(C_i) = \sum_{j=0}^{M} d_j \mu(D_j).$$

with factors c_i, d_j being positive and sets in both $(C_i), (D_j)$ being pairwise disjoint.

Claim that $\bigsqcup_{i=0}^{N} C_i = \bigsqcup_{j=0}^{M} D_j$: if $x \in \bigsqcup_{i=0}^{N} C_i$, then $f(x) > 0$ since $c_i > 0$ always holds, which means $x \in \bigsqcup_{j=0}^{M} D_j$ as well, and vice versa.

Then for every C_i, we have $C_i \subseteq \bigsqcup_{i=0}^{N} C_i = \bigsqcup_{j=0}^{M} D_j$, so $C_i = \bigsqcup_{j=0}^{M} (D_j \cap C_i)$. Then we have

$$\begin{aligned}
\sum_{i=0}^{N} c_i \mathbf{1}_{C_i} &= \sum_{i=0}^{N} c_i \mathbf{1}_{\bigsqcup_{j=0}^{M} (D_j \cap C_i)} \\
&= \sum_{i=0}^{N} c_i \sum_{j=0}^{M} \mathbf{1}_{D_j \cap C_i} \\
&= \sum_{i=0}^{N} \sum_{j=0}^{M} c_i \mathbf{1}_{C_i \cap D_j}.
\end{aligned}$$

Similarly, we also have

$$\sum_{j=0}^{M} d_j \mathbf{1}_{D_j} = \sum_{i=0}^{N} \sum_{j=0}^{M} d_j \mathbf{1}_{C_i \cap D_j}.$$

So for any i, j, if $C_i \cap D_j \neq \emptyset$, then $c_i = d_j$ must be true.

We perform the same operations to $\sum_{i=0}^{N} c_i\mu(C_i)$ and $\sum_{j=0}^{M} d_i\mu(D_i)$:

$$\begin{aligned}
\sum_{i=0}^{N} c_i\mu(C_i) &= \sum_{i=0}^{N} c_i\mu(\bigsqcup_{j=0}^{M}(D_j \cap C_i)) \\
&= \sum_{i=0}^{N} c_i \sum_{j=0}^{M} \mu(D_j \cap C_i) \\
&= \sum_{i=0}^{N} \sum_{j=0}^{M} c_i\mu(C_i \cap D_j), \\
\sum_{j=0}^{M} d_j\mu(D_j) &= \sum_{i=0}^{N} \sum_{j=0}^{M} d_j\mu(C_i \cap D_j).
\end{aligned}$$

Observe that if $C_i \cap D_j \neq \emptyset$, then we know that $c_i = d_j$. Otherwise if $C_i \cap D_j = \emptyset$, then $\mu(C_i \cap D_j) = 0$. So for any i, j,

$$c_i\mu(C_i \cap D_j) = d_j\mu(C_i \cap D_j).$$

Thus we have $\sum_{i=0}^{N} \sum_{j=0}^{M} c_i\mu(C_i \cap D_j) = \sum_{i=0}^{N} \sum_{j=0}^{M} d_j\mu(C_i \cap D_j)$, i.e.,

$$\sum_{i=0}^{N} c_i\mu(C_i) = \sum_{j=0}^{M} d_j\mu(D_j).$$

$\square$