The Convergence of Summation and Product

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Theorem 1. *Let* $(y_n)_{n\in\mathbb{N}}$ *be a sequence of reals from [0,1] such that*

$$\sum_{n\in\mathbb{N}} y_n = \infty.$$

Show that

$$\prod_{n\in\mathbb{N}} (1 - y_n) = 0.$$

Lemma 1. $e^{-x} \geq 1 - x$ *for every* $x \in \mathbb{R}$.

Proof of 1. Let $f(x) = e^{-x} - (1 - x)$. Then, $f'(x) = -e^{-x} + 1$, and $f'(x) = 0$ iff $x = 0$. It is easy to check that $f(x) \geq f(0) = 0$. □

Proof of theorem 1. Using lemma 1, we have

$$\begin{aligned}
&\prod_{n\in\mathbb{N}} (1 - y_n)) \\
\leq &\prod_{n\in\mathbb{N}} e^{-y_n} \\
\leq &e^{-\sum_{n\in\mathbb{N}} y_n} \\
= &0
\end{aligned}$$

□

Similarly, we can also prove the following complementary theorem.

Theorem 2. *Let* $(y_n)_{n\in\mathbb{N}}$ *be a sequence of reals from [0,1) such that*

$$\sum_{n\in\mathbb{N}} y_n < \infty.$$

Show that

$$\prod_{n\in\mathbb{N}} (1 - y_n) > 0.$$

Lemma 2. *$x \geq -\ln(1-x)$ for each $x \in [0,1)$.*

It is easy to check the above lemma. Just do what we did in the proof of lemma 1.

Proof of theorem 2. Using lemma 2, we have

$$\infty > \sum_{n\in\mathbb{N}} y_n \geq -\sum_{n\in\mathbb{N}} \ln(1-y_n).$$

It means $\sum_{n\in\mathbb{N}} \ln(1-y_n)$ converges. Let $\sum_{n\in\mathbb{N}} \ln(1-y_n) = C$.

Then,

$$\prod_{n\in\mathbb{N}} (1-y_n) = e^{\sum_{n\in\mathbb{N}} \ln(1-y_n)} = e^C > 0.$$

$\square$