The Distribution Function of X1 Divided by X2

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Exercise. *Given* $(X_1, X_2)^T \sim \mathcal{N}(\mu, V)$ *where*

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then what's the distribution function of $\frac{X_1}{X_2}$*?*

Let's first recall the definition of multivariate normal distribution.

Definition 1. *A random vector* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T$ *is called a normal random vector, which means* $\boldsymbol{X} \sim \mathcal{N}(\mu, V)$ *for some* μ, V*, if it has the probability density function for* $x \in \mathbb{R}^n$

$$f(x) = \frac{\sqrt{|V^{-1}|}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}(x-\mu)^T V^{-1}(x-\mu)}$$

It remains to check some important properties like $\int_{x \in \mathbb{R}^n} f(x) \, \mathrm{d}x = 1$ and so on. However, the following definition is equivalent to the above, but the proof of equivalence seems to be complicated.

Definition 2. *A random vector* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T$ *is called a normal random vector if* $\sum_{i=1}^n r_i X_i$ *is normal for any* $r_i \in \mathbb{R}$.

Now let's get back to the exercise. Consider the following general theorem.

Theorem 1. *Let* $(X, Y)^T$ *be a random vector with probability density function* $f(x, y)$*. Then the probability density function of* $\frac{Y}{X}$ *is given by*

$$f_{Y/X}(z) = \int_{-\infty}^{+\infty} |x| f(x, xz) \, \mathrm{d}x.$$

Proof of theorem 1. Let $G = \{(x,y) : \frac{y}{x} \le z\}$ for some z. And then the distribution function of $\frac{Y}{X}$ is

$$\begin{aligned} F_{Y/X}(z) = P\{Y/X \le z\} &= \iint_G f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= (\int_{y/x\le z, x<0} + \int_{y/x \le x, x>0}) f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{0} \left[\int_{xz}^{\infty} f(x,y)\,\mathrm{d}y\right] \mathrm{d}x + \int_{0}^{\infty} \left[\int_{-\infty}^{xz} f(x,y)\,\mathrm{d}y\right] \mathrm{d}x \\ &= \int_{-\infty}^{z} \left[\int_{-\infty}^{\infty} |x| f(x, xw)\right] \mathrm{d}w \end{aligned}$$

Since typing formulas in LaTeX is tough and despairing, I omit some intermediate step and give the final expression directly. By the definition of pdf, we conclude

$$f_{Y/X}(z) = \int_{-\infty}^{+\infty} |x| f(x, xz)\,\mathrm{d}x.$$

□

Suppose $(X_1, X_2)^T \sim \mathcal{N}(\mu, V)$ where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By definition 1, it has probability density function

$$f(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{x_1^2 + x_2^2}{2}}$$

Applying theorem 1, all we need to do is computing

$$
\begin{aligned}
f_{X_1/X_2}(z) &= \int_{-\infty}^{+\infty} |x| f(xz, x)\,\mathrm{d}x \\
&= 2\int_{0}^{+\infty} x f(xz, x)\,\mathrm{d}x \\
&= \int_{0}^{+\infty} f(xz, x)\,\mathrm{d}x^2 \\
&= \frac{1}{2\pi}\int_{0}^{+\infty} e^{-\frac{x^2(1+z^2)}{2}}\,\mathrm{d}x^2 \\
&= -\frac{1}{\pi(1+z^2)}\, e^{-\frac{t(1+z^2)}{2}}\Bigg|_{0}^{+\infty} \\
&= \frac{1}{\pi(1+z^2)}
\end{aligned}
$$

Thus, the distribution function of $\frac{X_1}{X_2}$ is

$$
F_{X_1/X_2}(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+z^2)}\,\mathrm{d}z = \frac{\arctan(x)}{\pi} + \frac{1}{2}
$$