How Brownian Motion Relate to a Martingale?

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Brownian motion, known as the random motion of particles suspended in a fluid which was first noticed by Robert Brown in 1827, is also an important stochastic process described as Wiener process in mathematics. Along with the Bernoulli trials process and the Poisson process, the Brownian motion process is of central importance in probability. The Brownian Motion process owns a lot of properties which are worth discussing, **self-similarity, temporally and spatially homogeneous, Irregularity, strong Markov property**... Among all of them, what we choose to focus is primarily **the martingale property of Brownian Motion**, where a lot of application is derived from. Is Brownian Motion also a martingale? And what can we do by its martingale property?

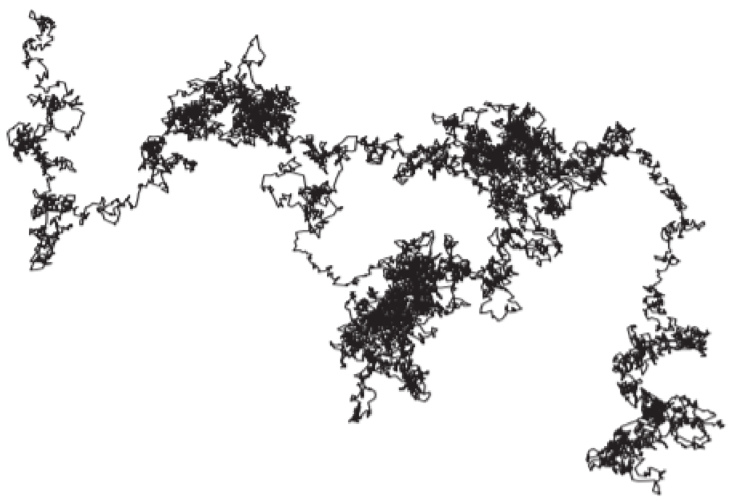

Figure 1: The range $\{B(t) : 0 \le t \le 1\}$ of a planar Brownian motion

1 Generalization to continuous-time

To derive the martingale property of Brownian motion, we should first realize that a standard Brownian motion is a continuous-time process. So the definitions of martingale and related theorem should first be generalized from discrete to continuous time.

Definition 1 (Filtration). *A filtration is a family* $\{\mathcal{F}(t) : t \ge 0\}$ *of sub-*σ*-fields such that* $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ *for all* $s \le t$

Definition 2 (**Continuous-time martingale**). *A real-valued stochastic process*$\{X(t)\}_{t\geq 0}$ *is a continuous martingale with respect to a filtration* $\{\mathcal{F}(t)\}$ *if it is adapted, that is,* $X(t) \in \mathcal{F}(t)$ *for all* $t \geq 0$, *if* $E|X(t)| < +\infty$ *for all* $t \geq 0$, *and if*

$$\mathbb{E}[X(t) \mid \mathcal{F}(s)] = X(s), \text{ almost surely, for all } 0 \leq s \leq t$$

Definition 3 (**Stopping time**). *A RV T with values in* $[0, +\infty]$ *is a stopping time with respect to the filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$ *if for all* $t \geq 0$

$$\{T \leq t\} \in \mathcal{F}(t)$$

We have already learned in class that the optional stopping theorem enables the defining equation for martingales to extend from fixed times $0 \leq s \leq t$ to stopping times $0 \leq S \leq T$ in discrete-time situation. Correspondingly, the optional stopping theorem can be extended to continuous-time by approximation. The continuous-time optional stopping theorem is very useful in many situations, e.g. in the analysis of chapter 4.

Theorem 1 (**Optional stopping theorem**). *Suppose* $\{X(t) : t \geq 0\}$ *is a continuous martingale, and* $0 \leq S \leq \mathrm{T}$ *are stopping times. If the process* $\{X(t \wedge T) : t \geq 0\}$ *is dominated by an integrable random variable* X, *i.e.* $|X(t \wedge T)| \leq X$ *almost surely, for all* $t \geq 0$, *then*

$$\mathbb{E}[X(T) \mid \mathcal{F}(S)] = X(S), \text{ almost surely.}$$

Proof. We first consider the discrete-time MG and then extend the discrete-time MG to continuous-time. Fix $N \in \mathbb{N}$ and with respect to the filtration $(\mathcal{G}(n) : n \in \mathbb{N})$ given by $\mathcal{G}(n) = \mathcal{F}\left(n2^{-N}\right)$, we can define a discrete time martingale

$$X_n = X\left(T \wedge n2^{-N}\right)$$

and stopping times

$$S' = \left\lfloor 2^N S \right\rfloor + 1$$

and

$$T' = \left\lfloor 2^N T \right\rfloor + 1.$$

Obviously X_n is dominated by an integrable random variable. From the discrete-time optional stopping theorem, we have

$$\mathbb{E}\left[X_{T'} \mid \mathcal{G}\left(S'\right)\right] = X_{S'}$$

which can be translated as

$$\begin{aligned} \mathbb{E}\left[X\left(T \wedge 2^{-N} T_N'\right) \mid \mathcal{F}\left(2^{-N} S_N'\right)\right] &= \mathbb{E}\left[X(T) \mid \mathcal{F}\left(2^{-N} S_N'\right)\right] \\ &= X\left(T \wedge 2^{-N} S_N'\right). \end{aligned}$$

Letting $N \uparrow \infty$ and using dominated convergence for conditional expectations, then we have for $A \in \mathcal{F}(S) \subseteq \mathcal{F}\left(2^{-N} S_N'\right)$,

$$\begin{aligned}
\mathbb{E}[X(T); A] &= \lim_N \mathbb{E}\left[\mathbb{E}\left[X(T) \mid \mathcal{F}\left(2^{-N} S_N'\right)\right]; A\right] \\
&= \mathbb{E}\left[\lim_N X\left(T \wedge 2^{-N} S_N'\right); A\right] \\
&= \mathbb{E}[X(S); A].
\end{aligned}$$

Hence it is proved. □

2 The first martingale

We first focus on one-dimensional, or linear Brownian motion, since it simplifies the analysis and has a lot of applications.

Definition 4 ((linear) Brownian Motion). *A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a (linear) Brownian motion with start in $x \in \mathbb{R}$ if the following holds:*

- $B(0) = x$,
- *The process has **independent increments**, i.e. for all times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ the increments $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_2) - B(t_1)$ are independent random variables*
- *The process has **stationary increments**. That is, for all $t \geq 0$ and $h > 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation zero and variance h*
- *Almost surely, the function $t \mapsto B(t)$ is continuous.*

*We say that $\{B(t) : t \geq 0\}$ is a **standard linear Brownian motion** if $x = 0$.*

As usual, Let $\mathcal{F}_t = \sigma\{B(s) : 0 \leq s \leq t\}$ for $t \in [0, \infty)$, so that $\mathfrak{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$ is the natural filtration for a standard Brownian Motion $\boldsymbol{B}$. Our first result is that the standard BM itself is a martingale.

Theorem 2. *Standard BM$\{B(t) : t \geq 0\}$ is a martingale, i.e., BM$\{B(t) : t \geq 0\}$ is a martingale respect to itself.*

Proof. By the stationary, independent increments and 0 means, since B_s is measurable with respect to $\mathcal{F}_s$ and $B(t) - B(s)$ is independent of $\mathcal{F}_s$, we have

$$\begin{aligned}
\mathbb{E}[B(t) \mid \mathcal{F}_s] &= \mathbb{E}[B(t) - B(s) + B(s) \mid \mathcal{F}_s] \\
&= \mathbb{E}[B(t) - B(s) \mid \mathcal{F}_s] + \mathbb{E}[B(s) \mid \mathcal{F}_s] \\
&= \mathbb{E}[B(t) - B(s)] + \mathbb{E}[B(s) \mid \mathcal{F}_s] \\
&= 0 + B(s) \\
&= B(s)
\end{aligned}$$

From the proof above, is is obvious that the third step is derived from independent increment and the fourth step is derived from stationary increment and 0 mean. **So the martingale property of standard BW can be directly derived from only three of the local property of BW** while the continuity of $B(t)$ is not necessary. $\square$

3 More interesting martingales

Apart from the BM itself, we could construct a few more interesting martingales by using appropriate function of BM. Those martingales play an essential role in applications. We introduce two of them,$\{B(t)^2 - t : t \geq 0\}$ and $\left\{e^{\theta B(t) - \theta^2 t/2} : t \geq 0\right\}$, which can be directly used in chapter 4.

Theorem 3. *The stochastic process* $\{B(t)^2 - t : t \geq 0\}$ *is a martingale with respect to BM*$\{B(t) : t \geq 0\}$*, i.e., with respect to* $\mathfrak{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$*, where* $\mathcal{F}_t = \sigma\{B(s) : 0 \leq s \leq t\}$*.*

Proof. Let $s, t \in [0, \infty)$ with $s < t$. We use normal trick of writing $B(t) = B(s) + (B(t) - B(s))$ in order to take advantage of the stationary and independent increments properties of Brownian motion. Then we have

$$\begin{aligned} X(t) :&= B(t)^2 - t \\ &= (B(s) + (B(t) - B(s)))^2 - t \\ &= B(s)^2 + 2B(s)\left(B(t) - B(s)\right) + \left(B(t) - B(s)\right)^2 - t. \end{aligned}$$

Since $B(s)$ is measurable with respect to $\mathcal{F}_s$ and $B(t) - B(s)$ is independent of $\mathcal{F}_s$, we get

$$\mathbb{E}(X(t) \mid \mathcal{F}_s) = B(s)^2 + 2B(s)\mathbb{E}[B(t) - B(s)] + \mathbb{E}[(B(t) - B(s))^2] - t.$$

Since $B(t) - B(s)$ is a stationary increment with expectation 0 and variance $t - s$, i.e.,

$$\mathbb{E}[B(t) - B(s)] = 0,$$

$$\mathbb{E}[(B(t) - B(s))^2] = \text{var}(B(t) - B(s)) = t - s.$$

Hence we get the result,

$$\mathbb{E}[X(t) \mid \mathcal{F}_s] = B(s)^2 - s = X(s)$$

$\square$

Theorem 4. *The stochastic process* $\left\{e^{\theta B(t) - \theta^2 t/2} : t \geq 0\right\}$ *is a martingale with respect to BM*$\{B(t) : t \geq 0\}$*, i.e., with respect to* $\mathfrak{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$*, where* $\mathcal{F}_t = \sigma\{B(s) : 0 \leq s \leq t\}$*.*

Proof. Let $s, t \in [0, \infty)$ with $s < t$. Again, we use normal trick of writing $B(t) = B(s) + (B(t) - B(s))$ in order to take advantage of the stationary and independent increments properties of Brownian motion. Then we have

$$X(t) := e^{\theta B(t) - \theta^2 t/2} = e^{-\frac{\theta^2}{2}t + \theta B(s) + \theta(B(t) - B(s))}.$$

Since $Z(s)$ is measurable with respect to $\mathcal{F}_s$ and $Z(t) - Z(s)$ is independent of $\mathcal{F}_s$, we get

$$\mathbb{E}\left[X(t) \mid \mathcal{F}_s\right] = e^{-\frac{\theta^2}{2}t + \theta B(s)} \mathbb{E}\left[e^{\theta(B(t) - B(s))}\right].$$

By the property of BM, $B(t) - B(s)$ has the normal distribution with mean 0 and variance $t - s$. So we have

$$\mathbb{E}[e^{\theta(B(t) - B(s))}] = e^{\frac{\theta^2}{2}(t-s)}$$

and

$$\mathbb{E}\left[X(t) \mid \mathcal{F}_s\right] = e^{-\frac{\theta^2}{2}s + \theta B(s)} = X(s).$$

Hence is proved. $\square$

The proof of the Theorem 4 can be simplified from Geometric Brownian Motion perspective, the sketch of this innovative proof is described as below.

Proof. First we should realize that $\left\{e^{\theta B(t) - \theta^2 t/2} : t \geq 0\right\}$ is a special case of **Geometric Brownian Motion** $\left\{e^{\theta B(t) - \theta^2 t/2 + \mu t} : t \geq 0\right\}$ where the draft parameter $\mu = 0$. Let $X(t) := e^{\theta B(t) - \theta^2 t/2}$. The Geometric BM has an important property that it satisfies the stochastic differential equation

$$dX(t) = \mu X(t)dt + \theta X(t)dB(t).$$

In our case with $\mu = 0$, we have $dX(t) = \theta X(t)dB(t)$ and therefore

$$X(t) = 1 + \theta \int_0^t X(s)dB(s), \quad t \geq 0$$

The process associated with a stochastic integral is always a martingale, assuming the usual assumptions on the integrand process (which are satisfied here). $\square$

4 An application: Gambler's Ruin Problem

The martingale property derived from BM is very useful in many situations. Here I give an example in the analysis of continuous-time Gambler's Ruin Game.

4.1 Background

We first define $X(t) := \sigma B(t) + \mu t$, for $t \geq 0$. By convention, we call this process the Brownian motion with drift μ and diffusion or variance coefficient σ^2. Then $X(t)$ has the distribution of $N(\mu t, \sigma^2 t)$ for each $t \geq 0$.

Now we start with an initial money w and gamble following the stochastic process X, i.e., our wealth at time t is given by

$$W(t) = w + X(t) = w + \sigma B(t) + \mu t, \quad t \geq 0$$

Let T be the first time that we either win x or lose w, i.e., $T := \inf\{t \geq 0 : W(t) = 0 \quad \text{or} \quad w + x\} = \inf\{t \geq 0 : X(t) = -w \quad \text{or} \quad +x\}$. We mainly focus on two questions in this situation,

1. The probability p we win x (reach wealth $w + x$) before we lose our initial wealth w (reach wealth 0). i.e. $p := \mathbb{P}[W(T) = w + x] = \mathbb{P}[X(T) = x]$
2. The expected time before the "game" ends, i.e. $\mathbb{E}[T]$.

We categorize the problem into two cases,

1. No drift ($\mu = 0$)
2. Consider drift ($\mu \neq 0$)

In both cases we utilize **Martingale** and **Optional Stopping Theorem** as our main method.

4.2 The case without drift

Consider standard Brownian motion, which starts at $B(0) = 0$. For the first step, we let T be the first time that standard Brownian motion first hits either $+a$ or $-b$. We apply the Optional Stopping Theorem to get,

$$\mathbb{E}[B(T)] = \mathbb{E}[B(0)] = 0.$$

Since $B(T)$ should be either +a or -b. Let p be the probability that $B(T) = a$. Then the equation above can be translated as

$$\mathbb{E}[B(T)] = pa + (1 - p)(-b) = 0$$

$$p = \frac{b}{(a + b)}. \tag{1}$$

With the quadratic martingale By the same reasoning, we get

$$\mathbb{E}\left[B(T)^2 - T\right] = \mathbb{E}\left[B(0)^2 - 0\right] = 0$$

which implies that

$$\mathbb{E}[T] = \mathbb{E}\left[B(T)^2\right] = pa^2 + (1-p)b^2 = \frac{ab(a+b)}{a+b} = ab$$

If we instead consider the stochastic process, $X(t) = \sigma B(t)$, then we can easily get that the appropriate quadratic martingale is instead $\{X(t)^2 - \sigma^2 t : t \geq 0\}$. Hence we get

$$\mathbb{E}\left[X(T)^2 - \sigma^2 T\right] = \mathbb{E}\left[B(0)^2 - \sigma^2 0\right] = 0,$$

which implies that

$$\mathbb{E}\left[\sigma^2 T\right] = \sigma^2 \mathbb{E}[T] = \mathbb{E}\left[X(T)^2\right] = pa^2 + (1-p)b^2 = \frac{ab(a+b)}{a+b} = ab$$

$$\mathbb{E}[T] = \frac{ab}{\sigma^2} \tag{2}$$

The equation above can be reduced to our situation when we let $X(t) = \sigma B(t), a = x$ and $b = w$, hence we get the result

$$p = \frac{w}{x+w}$$
$$\mathbb{E}[T] = \frac{xw}{\sigma^2}$$

4.3 The case with drift

Similarly to the case without drift, by applying Optional Stopping Theorem with exponential martingale $\left\{e^{\theta B(t) - \theta^2 t/2} : t \geq 0\right\}$, we have

$$\mathbb{E}\left[e^{\theta B(T) - \theta^2 T/2}\right] = \mathbb{E}\left[e^{\theta B(0) - \theta^2 0/2}\right] = e^0 = 1.$$

Thus, upon expressing $B(t)$ in terms of $X(t)$, we have

$$\mathbb{E}\left[e^{\theta[(X(T) - \mu T)/\sigma] - \theta^2 T/2}\right] = \mathbb{E}\left[e^{(\theta X(T)/\sigma) - (\theta \mu T/\sigma) - \theta^2 T/2}\right] = 1.$$

Now let $\theta = \frac{-2\mu}{\sigma}$, we get

$$\mathbb{E}\left[e^{\left(-2\mu X(T)/\sigma^2\right)}\right] = 1.$$

Similarly to the case without drift, since $X(t) = x$ with probability p and $X(t) = -w$ with probability $1-p$, we have the equation

$$pe^{-2/4x/\sigma^2} + (1-p)e^{2\mu w/\sigma^2} = 1$$

$$p = \frac{e^{+2\mu w/\sigma^2} - 1}{e^{+2\mu w/\sigma^2} - e^{-2\mu x/\sigma^2}}. \tag{3}$$

After knowing p, it is then convenient to use the linear martingale to find $\mathbb{E}[T]$, we already have

$$\mathbb{E}[B(T)] = \mathbb{E}[(X(T) - \mu T)/\sigma] = 0,$$

thus

$$\mathbb{E}[T] = \frac{\mathbb{E}[X(T)]}{\mu} = \frac{(px - (1-p)w]}{\mu}, \tag{4}$$

where the p is already known in equation(3).

5 Deeper insight into the relationship between BM and MG

It has already been shown that from a Brownian Motion, we can construct many martingales and there is lots of applications. However, what is surprising is that we can also construct a Brownian Motion from a martingale, i.e., a martingale which satisfies certain constraints can be a Brownian Motion. This implicit relationship is known as **Levy's Characterization of Brownian Motion**

Theorem 5. *If a stochastic process $\{M_t\}$ defined on probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with filtration $\{\mathscr{F}_t\}_{t\in\mathbb{R}^+}$ satisfies the following conditions:*

1. $\mathbb{P}[M_0 = 0] = 1$
2. *M_t is a continuous martingale w.r.t. the filtration $\{\mathscr{F}_t\}_{t\in\mathbb{R}^+}$ under $\mathbb{P}$*
3. *The quadratic variation $< M_t >= t$ almost surely w.r.t. $\mathbb{P}$*

Then

the stochastic process $\{M_t\}$ is a Brownian Motion.

The proof of it must introduce some other concepts such as quadratic variation. Due to the fact that it's not the focus of this article, we do not cover it now.

6 Conclusion

As is shown above, there are still a lot of things to discuss about the Brownian Motion from the Martingale perspective. All in all, The Brownian Motion is closely related with continuous-time martingale in both application filed and pure mathematics field. Perhaps that's why the Brownian Motion give rise to the study of continuous-time martingale.

References

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