课堂笔记

May. 22, 2020

课程内容 A list of properties of conditional expectations

Martingale,filtration,optional time

Discrete stochastic integral

Exercise $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ $\mathcal{G} \subseteq \mathcal{F}$. If Y is any version of $E(X|\mathcal{G})$ then $EY = EX$

Proof By definition

Linearity $E(a_1X_1 + a_2X_2|\mathcal{G}) = a_1E(X_1|\mathcal{G}) + a_2E(X_2|\mathcal{G})$
证明方法：回到随机变量$X_1|\mathcal{G}, X_2|\mathcal{G}$，利用线性性。

全期望公式(Tower property) If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then $E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$

Proof Y:a version of $E(X|\mathcal{G})$
Z: a version of $E(Y|\mathcal{H})$
$\int_H ZdP = \int_H YdP = \int_H XdP, \forall H \in \mathcal{H}$

Taking out what is known $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P), \mathcal{G} \subseteq \mathcal{F}$.If Z is $\mathcal{G}-measurable$ and bounded, then

$$E[ZX|\mathcal{G}] = ZE[X|\mathcal{G}], a.s.$$

Proof $Z = I_U \rightarrow Z \in SF^+ \rightarrow Z \in (mG)^+$(standard machine)

Independence If $\mathcal{H}$ is independent of $\sigma(X, \mathcal{G})$, then $E[X|\sigma(\mathcal{G}, \mathcal{H})] = E[X|\mathcal{G}], a.s.$

Proof 在$\forall W \in \sigma(\mathcal{G}, \mathcal{H})$上面一样，可以用在$\pi - system$上面一样来延拓。

Chapter 10. Martingale

Filtration $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_n\}_{n\geq 0}$ with $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$
$\mathcal{F}_\infty = \lim \mathcal{F}_n = \bigcup \mathcal{F}_n$

Adapted A process $X = (X_n : n \geq 0)$ is adapted to the filtration $(\mathcal{F}_n)$ if for each n, X_n is $\mathcal{F}_n$-measurable.

Martingale A process X is a martingale relative to $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$ if
(1) X is adapted
(2) $E(|X_n) < \infty$
(3)$E[X_n|\mathcal{F}_{n-1}] = X_{n-1}$
X:每个单位赌注值多少钱
第三条等价于$E[X_n - X_{n-1}|\mathcal{F}_{n-1}] = 0$

Doob-martingale(An Example) $\xi \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$
通过对一列$\mathcal{F}_n$的观察，得到最佳的逼近：

$$M_n = E(\xi|\mathcal{F}_n)$$

$$E(M_n|\mathcal{F}_{n-1}) = E(E(\xi|\mathcal{F}_n)|\mathcal{F}_{n-1}) \overset{\text{Tower}}{=} E(\xi|\mathcal{F}_{n-1}) \overset{\text{def}}{=} M_{n-1}$$

赌博过程 过程 $C = (C_n : n \geq 1)$ is previsible if C_n is $\mathcal{F}_{n-1}$ measurable.
$\int_0^n CdX = \sum_{1\leq k\leq n} C_k(X_k - X_{k-1})$

$$\begin{aligned}
&C_i X_i|_0^n \\
&= \int_0^n CdX + \int_0^n XdC \\
&= \sum_{1\leq k\leq n} C_k(X_k - X_{k-1}) + \sum_{i=0}^{n-1} X_i(C_{i+1} - C_i)
\end{aligned}$$

定义$(C \cdot X)$,$(C \cdot X) = \int_0^n CdX$ If C is a bounded previsible process and X is a martingale, then $(C \cdot X)$ is a martingale null at 0.

Proof 记$(C \cdot X)$为Y,$Y_n = \int_0^n CdX$

$$\begin{aligned}
&E[Y_n - Y_{n-1}|\mathcal{F}_{n-1}] \\
&= E[C_n(X_n - X_{n-1})|\mathcal{F}_{n-1}] \\
&= C_n E(X_n - X_{n-1}|\mathcal{F}_{n-1}) = 0
\end{aligned}$$