Kullback–Leibler divergence ≥ 0

马浩博 518030910428

May 8, 2020

Kullback–Leibler divergence (also called relative entropy) is a measure of how one probability distribution is different from a second, reference probability distribution.

Definition

For discrete probability distributions P and Q defined on the same probability space, $\mathcal{X}$, the Kullback–Leibler divergence from Q to P is defined to be

$$D_{\mathrm{KL}}(P \parallel Q) = \sum_{x \in \mathcal{X}} P(x) \log \left(\frac{P(x)}{Q(x)} \right)$$

For distributions P and Q of a continuous random variable, the Kullback–Leibler divergence is defined to be the integral:

$$D_{\mathrm{KL}}(P \parallel Q) = \int_{-\infty}^{\infty} p(x) \log \left(\frac{p(x)}{q(x)} \right) dx$$

where p and q denote the probability densities of P and Q.

Applications

Applications include characterizing the relative (Shannon) entropy in information systems, randomness in continuous time-series, and information gain when comparing statistical models of inference. In the simple case, a Kullback–Leibler divergence of 0 indicates that the two distributions in question are identical. In simplified terms, it is a measure of surprise, with diverse applications such as applied statistics, fluid mechanics, neuroscience and machine learning.

Proof. We shall prove that $D_{KL}(P||Q) \geq 0$, here D_{KL} means Kullback–Leibler divergence and $D(P||Q) = -\sum_x P(x) \ln \frac{Q(x)}{P(x)}$.

$$\begin{aligned}
-D(P||Q) &= \sum_x P(x) \ln \frac{Q(x)}{P(x)} && (1) \\
&\leq \sum_x P(x) \left(\frac{Q(x)}{P(x)} - 1 \right) && (2) \\
&= \sum_x Q(x) - \sum_x P(x) && (3) \\
&= 1 - 1 && (4) \\
&= 0 && (5)
\end{aligned}$$

In the (2) step we use the inequality $\ln x \leq x - 1$. □

proof 2. I recall Jensen's inequality in our textbook when I write the first proof. We can use it to get another proof.

$$\begin{aligned}
-D(P||Q) &= \sum_x P(x) \ln \frac{Q(x)}{P(x)} && (6) \\
&\leq \ln \sum_x P(x) \frac{Q(x)}{P(x)} && (7) \\
&= \ln \sum_x Q(x) && (8) \\
&= \ln 1 && (9) \\
&= 0 && (10)
\end{aligned}$$

In the (7) step we use Jensen's inequality. □

Reference

[1] https://en.wikipedia.org/wiki/Kullback%E2%80%93Leibler_divergence