Hewitt-Savage Zero-One Law and Random Walk

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April 11, 2020

1 Hewitt-Savage zero-one law

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be our probability triple.

For a sequence of IID RVs $(X_n)_{n\in\mathbb{N}}$, let $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$, i.e., the σ-algebra generated by the first n random variables, and $\mathcal{F}_\infty = \lim_{n\to\infty} \mathcal{F}_n$.

By *Doob-Dynkin Lemma* (or *3.13. (d)*), for any $f \in m\mathcal{F}_\infty$ (namely, $\sigma(f) \subseteq \sigma(X_1, X_2, \dots)$), there exists $Y : \mathbb{R}^\infty \to \mathbb{R}$ such that $f = Y \circ X$ where $X = (X_1, X_2, \dots)$.

By a *finite permutation* of $\mathbb{N}$ we mean a bijection map $p : \mathbb{N} \to \mathbb{N}$ such that $p = n$ for all but finitely many n. We say f is *invariant under finite permutation* or *permutation invariant* or *permutable* if $f = f \circ p$ for every finite permutation p where $f \circ p = Y \circ X \circ p = Y \circ (X_1, X_2, \dots) \circ p \coloneqq Y \circ (X_{p_1}, X_{p_2}, \dots)$.

We say an event A is permutation invariant if $\mathbf{1}_A$ is permutation invariant.

Theorem 1. *Suppose that $(X_n)_{n\in\mathbb{N}}$ is a sequence of IID RVs. Then every permutation invariant event has probability 0 or 1.*

2 Random Walk (trichotomy)

Let $X_1, X_2, \dots$ be IID RVs, and put $S_n = X_1 + X_2 + \dots$ where $\mathbf{P}(X_n = 0) < 1$. Undoubtedly S_n is also a random variable. Furthermore, $\limsup S_n$ and $\liminf S_n$ are also random variables. By *Hewitt-Savage zero-one law*, the following result is easy to validate.

Lemma 1. $\mathbf{P}(\limsup S_n \in B) =$ *0 or 1 for any* $B \in \mathcal{B}$.

Proof of lemma 1. We only need to prove $\limsup S_n$ is permutation invariant, or equivalently, $\mathbf{1}_{\limsup S_n}$ is permutation invariant. However, it is trivial since $\mathbf{1}_{\limsup S_n}$ has nothing to do with the order of first finite random variables. □

Using the same method in the second part of the proof of Kolmogorov's zero-one law (in 4.11. of our textbook), we obtain the following result.

Corollary 1. $\limsup S_n = c$ *a.s. for some* $c \in [-\infty, +\infty]$.

We call random walk a trichotomy because there are only three possibilities for $\lim S_n$.

Theorem 2. *One of the followings happens a.s.:*

(1) $\lim S_n = +\infty$.

(2) $\lim S_n = -\infty$.

(3) $\limsup S_n = +\infty, \liminf S_n = -\infty$.

Proof of theorem 2. To finish the proof, we just need to show that the c in corollary 1 cannot be finite, namely, $c \in \{-\infty, +\infty\}$.

Suppose c is finite. We know $\limsup S_n = c$ a.s., and equivalently, $\limsup S_{n+1} = c$ a.s..

However, obviously we also have $\limsup(S_{n+1} - X_1) = c$ a.s. , and thus, $\limsup(S_{n+1}) = c + X_1$ a.s..

Combining these two results, we have $c = c + X_1$ a.s. — that is, $X_1 = 0$ a.s., which contradicts our premise $\mathbf{P}(X_n = 0) < 1$.

□