

E4.2: A distribution on \mathbb{N} related to zeta function

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Let $s > 1$, and define $\zeta(s) := \sum_{n \in \mathbb{N}} \frac{1}{n^s}$, as usual. Let X and Y be independent \mathbb{N} -valued random variables with

$$\mathcal{P}(X = n) = \mathcal{P}(Y = n) = \frac{n^{-s}}{\zeta(s)}.$$

Define events $E_n := \{X \text{ is divisible by } n\}$. One can easily see that

$$\mathcal{P}(E_n) = \sum_{i \in \mathbb{N}} \mathcal{P}(X = ni) = \sum_{i \in \mathbb{N}} \frac{(ni)^{-s}}{\zeta(s)} = \frac{n^{-s}}{\zeta(s)} \sum_{i \in \mathbb{N}} i^{-s} = n^{-s}. \quad (1)$$

Everything is set up now, and we shall explore some interesting properties of X and Y .

Proposition 1. *$(E_p : p \in \mathbb{P})$ are independent with \mathbb{P} denoting all prime numbers.*

Proof. Let $S := \{p_1, p_2, \dots, p_m\}$ be a finite subset of \mathbb{P} . By definition of E_n and Eq. (1), we immediately have

$$\mathcal{P}\left(\bigcap_{i=1}^m E_i\right) = \mathcal{P}(E_{p_1 p_2 \cdots p_m}) = (p_1 p_2 \cdots p_m)^{-s} = \prod_{i=1}^m \mathcal{P}(E_i),$$

which implies $(E_p : p \in \mathbb{P})$ are independent. □

We can use this observation to cast light on **Euler's formula**

$$\frac{1}{\zeta(s)} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right). \quad (2)$$

The LHS of Eq. (2) is obviously $\mathcal{P}(X = 1)$. The event that $X = 1$ can also be described as $E := \{p \nmid X, \forall p \in \mathbb{P}\} = \bigcap_{p \in \mathbb{P}} E_p^c$. Combining Eq. (1) and Proposition 1, we have $\mathcal{P}(E) = \prod_{p \in \mathbb{P}} \mathcal{P}(E_p^c) = \prod_{p \in \mathbb{P}} (1 - \frac{1}{p^s})$.

Clearly, $(E_{p^2}, p \in \mathbb{P})$ is independent as well (the argument is similar to Proposition 1). Hence,

$$\mathcal{P}(\text{no square other than 1 divides } X) = \mathcal{P}\left(\bigcap_{p \in \mathbb{P}} E_{p^2}^c\right) = \prod_{p \in \mathbb{P}} (1 - \frac{1}{p^{2s}}) = \frac{1}{\zeta(2s)},$$

where the last equality follows from Eq. (2).

Define r.v. $H := \gcd(X, Y)$, and we shall show that

Proposition 2. $\mathcal{P}(H = n) = \frac{n^{-2s}}{\zeta(2s)}$.

Proof. We need the concept of conditional probability from elementary theory. Note that

$$\mathcal{P}(X/n = k | E_n) = \frac{(nk)^{-s}}{\sum_{i \geq 1} (ni)^{-s}} = \frac{k^{-s}}{\zeta(s)}.$$

Hence, $(X/n | E_n)$ has the same distribution as X . Let F_n be the event that n divides Y , then $(Y/n | E_n)$ also has the same distribution. Thus,

$$\begin{aligned} \mathcal{P}(H = n) &= \mathcal{P}\left(E_n \cap F_n \cap \gcd\left(\frac{X}{n}, \frac{Y}{n}\right) = 1\right) \\ &= \mathcal{P}(E_n) \mathcal{P}(F_n) \mathcal{P}\left(\gcd\left(\frac{X}{n}, \frac{Y}{n}\right) = 1 | E_n \cap F_n\right) \\ &= n^{-s} \cdot n^{-s} \cdot \mathcal{P}(\gcd(X, Y) = 1). \end{aligned} \tag{3}$$

Since $(E_p, F_p : p \in \mathbb{P})$ is independent,

$$\mathcal{P}(\gcd(X, Y) = 1) = \mathcal{P}\left(\bigcap_{p \in \mathbb{P}} (E_p \cap F_p)^c\right) = \prod_{p \in \mathbb{P}} (1 - \frac{1}{p^{2s}}) = \frac{1}{\zeta(2s)}.$$

Plugging this into Eq. (3), we get the desired result. \square

However, we have not rigorously defined conditional probability yet. Can we avoid using conditional probability? The answer is yes, but we need a useful tool, **Möbius function**, which is defined as

$$\begin{aligned} \mu : \mathbb{N} &\rightarrow \{-1, 0, 1\} \\ n &\mapsto \begin{cases} (-1)^m, & \text{if } n = p_1 p_2 \cdots p_m, \text{ where } (p_i)_{i \in [m]} \text{ are distinct prime numbers;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Here we list two facts about μ without giving detailed proof ¹.

Proposition 3. (i) $\sum_{d|n} \mu(d) = [n = 1]$.

(ii) $\sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{(1-p^s)} = \frac{1}{\zeta(s)}$.

Proof. (i) can be proved by counting the number of -1 and $+1$ among all $\mu(d)$. (ii) is simply expansion of the product. \square

Now we are well-equipped to give the coup de grace.

Another proof of Proposition 2. Let G_n be the event that X, Y are both divisible by n , which is equivalent to $n|H$. Clearly, $\mathcal{P}(G_n) = n^{-2s}$. By Proposition 3 (i),

$$\mathcal{P}(H = n) = \sum_{k=1}^{\infty} \mathcal{P}(H = kn) \sum_{d|k} \mu(d) = \sum_{d=1}^{\infty} \mu(d) \sum_{i=1}^{\infty} \mathcal{P}(H = idn). \quad (*)$$

Note that $\sum_{i=1}^{\infty} \mathcal{P}(H = in) = \mathcal{P}(G_n)$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} (*) &= \sum_{d=1}^{\infty} \mu(d) \mathcal{P}(G_{dn}) = \sum_{d=1}^{\infty} \mu(d) (dn)^{-2s} \\ &= n^{-2s} \sum_{d=1}^{\infty} \mu(d) d^{-2s} \\ &= \frac{n^{-2s}}{\zeta(2s)}, \end{aligned}$$

where the last equality follows from Proposition 3 (ii). \square

Remark. If we take $s = 1$ (this makes no sense) in Proposition 2, we get the probability that X, Y are relatively prime is $\frac{1}{\zeta(2)}$. In fact, the **natural density** of relatively prime pairs among \mathbb{N}^2 is $\frac{1}{\zeta(2)}$! Formally,

$$\lim_{N \rightarrow \infty} \frac{\sum_{i,j \in [N]} [\gcd(i,j) = 1]}{N^2} = \frac{1}{\zeta(2)}.$$

This is not just a coincidence. Let A be a subset of \mathbb{N} , the **analytic density** of A is defined as

$$\mathbf{d}(A) = \lim_{s \rightarrow 1^+} \frac{1}{\zeta(s)} \sum_{n \in A} n^{-s}.$$

If natural density exists, then analytic density exists and they are equal.

¹This kind of technique is called **Möbius inversion**. More information can be found at <https://math.berkeley.edu/~stankova/MathCircle/Multiplicative.pdf>