Notes on Week 6

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April 7, 2020

1

Definition 1. *Dependency graph for a sequence of events $E_1, \dots, E_n$ is a graph $G = (V, E)$ such that $V = \{1, \dots, N\}$.*

E_i is independent with $\{E_j : i \sim j \notin E\}$ for all $i \in V$

Lovasz Local Lemma Presented in 1975 by Erdös and Lovasz.

For an event sequence $E_1, \dots, E_n$, and G is a dependency graph of it.

Premise

$$
\begin{aligned}
&(1) \exists p \in (0,1) P(E_i) \leq p \qquad \forall i. \\
&(2) \max deg_G(v) \leq d \\
&(3) 4dp \leq 1
\end{aligned}
$$

Conclusion $\quad P\left(\cap E_i^c\right) > 0$

Proof Using inductive method

for $s = 0, 1, \dots, N-1$, $\forall |S| \leq s$

$$
\begin{cases}
(a), & P\left(\bigcap_{j \in S} E_j^c\right) > 0), \\
(b), \quad \forall k \in [n] \setminus S, P(E_k \cap \bigcap_{j \in S} E_j^c) \leq 2pP\left(\bigcap_{j \in S} E_j^c\right). &
\end{cases}
$$

Then let's start induction!

s=0 It's easy to verify it.

s>0

For expression a

$$P\left(\bigcap_{j\in[n]} E_j^c\right) = \frac{P\left(\bigcap_{j\in[n]} E_j^c\right)}{P\left(\bigcap_{j\in[n-1]} E_j^c\right)} \times \cdots \times \frac{P\left(\bigcap_{j\in[1]} E_j^c\right)}{P\left(\bigcap_{j\in[0]} E_j^c\right)}$$
$$\geq (1-2p)^n$$

Due to $4dp \leq 1$, $2p \leq \frac{1}{2d}$, so $1-2p \geq \frac{1}{2}$, which means the probability is correct.

For expression b To prove $P\left(E_k \middle| \bigcap_{j\in S} E_j^c\right) \leq 2p$ when $(|S| = s)$

Separate the points in S into two parts:

$S_1 = \{j \in S : j \sim k \text{ in } G\}$

$S_2 = S \setminus S_1$

When S_1 is an empty set, $P(E_k|\bigcap_{j\in S} E_j^c) = P(E_k) \leq p < 2p$

Otherwise, $S_1 \neq \emptyset \to |S_2| < s$

Let $F_{S_1} = \bigcap_{j\in S_1} E_j^c$ $F_{S_2} = \bigcap_{j\in S_2} E_j^c$

$$P\left(E_k \middle| \bigcap_{j\in S} E_j^c\right) = P\left(E_k|F_{S_1} \cap F_{S_2}\right)$$
$$= \frac{P\left(F_{S_1} \cap E_k|F_{S_2}\right)}{P\left(F_{S_1}|F_{S_2}\right)}$$
$$P\left(F_{S_1} \cap E_k|F_{S_2}\right) \leq P\left(E_k|F_{S_2}\right)$$
$$= P\left(E_k\right) \leq p$$
$$P\left(F_{S_1}|F_{S_2}\right) = P\left(\bigcap_{i\in S_1} E_i^c \middle| \bigcap_{j\in S_2} E_j^c\right)$$
$$\geq 1 - \sum_{i\in S_1} P\left(E_i \middle| \bigcap_{j\in S_2} E_j^c\right)$$
$$\geq 1 - 2pd \geq \frac{1}{2}$$

So that $\frac{P\left(F_{S_1} \cap E_k|F_{S_2}\right)}{P\left(F_{S_1}|F_{S_2}\right)} < 2p$ ∎