Product of A Divergent Series

An application of Borel-Cantelli Theorem

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Problem *Let* $(y_n)_{n\in\mathbb{N}}$ *be a sequence of reals from* $[0,1]$ *such that* $\sum_{n\in\mathbb{N}} y_n = \infty$. *Show that* $\prod_{n\in\mathbb{N}}(1-y_n) = 0$.

Proof: Recall **the second Borel-Cantelli Lemma(BC2)** we learned in class:

Lemma *If the events* E_n *are pairwisely independent, then*

$$\sum_n P(E_n) = \infty \Longrightarrow P(\text{limsup}E_n) = 1$$

We observe that this problem is very similar to this Lemma, in particular y_n is analogous to $P(E_n)$. Suppose we can find for each y_n an event E_n such that $y_n = P(E_n)$, then $\sum_n P(E_n) = \infty$.

Then by the BC2 lemma,

$$\begin{aligned}
P(\limsup E_n) &= P(\bigcap_{m\in\mathbb{N}}\bigcup_{n\geqslant m} E_n) = 1 \\
\Longrightarrow P(\bigcup_{n\geqslant 1} E_n) &= 1 \\
\Longrightarrow P((\bigcup_{n\geqslant 1} E_n)^c) &= P(\bigcap_{n\geqslant 1} E_n^c) = 0
\end{aligned}$$

Now we can prove the statement in the problem,

$$\begin{aligned}
\prod_{n\in\mathbb{N}}(1-y_n) &= \prod_{n\in\mathbb{N}} P(E_n^c) \\
&= P(\bigcap_{n\geqslant 1} E_n^c) \\
&= 0
\end{aligned}$$

The only thing left is an explicit expression for E_n such that $P(E_n) = y_n$.

Define X_n to be a random variable, $X_n : [0,1] \to [0,1]$ with distribution function

$$F_{X_n}(c) = P(\omega \mid X_n(\omega) \leqslant c) = c$$

. Then we can simply let

$$E_n = \{\omega \mid X_n(\omega) \leqslant y_n\}$$

, where E_n are pairwise and mutually independent. We have $P(E_n) = y_n$.

Thus, we have shown that $\prod_{n\in\mathbb{N}}(1-y_n) = 0$. □

Remark 1 *Actually, the proof of BC2 Lemma(with a stronger assumption of mutually independence) is quite simple and similar to the previous proof.*

$$\begin{aligned}
P(\textit{lim sup} E_n) &= P(\bigcap_{m\in\mathbb{N}}\bigcup_{n\geqslant m} E_n) = 1 - P(\bigcup_{m\in\mathbb{N}}\bigcap_{n\geqslant m} E_n^c) \\
&= 1 - \lim_{m\to\infty} P(\bigcap_{n\geqslant m} E_n^c) \quad (\textit{observe that} \bigcap_{n\geqslant m} E_n^c \subseteq \bigcap_{n\geqslant m+1} E_n^c \ \textit{for any } m) \\
&= 1 - \lim_{m\to\infty} \prod_{n\geqslant m}(1 - P(E_n)) \geqslant 1 - \lim_{m\to\infty} e^{-\sum_{n\geqslant m} P(E_n)} = 1 - 0 = 1
\end{aligned}$$

I think that the core of the aforementioned two proofs is the relation

$$(E_n, ev)^c = (E_n^c, \text{i.o.})$$

Remark 2 *Thanks to the almighty 吴润哲 for pointing out loopholes in the proof.*