Bijection between $[0,1]$ and $\mathbb{R}$

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Theorem 1. *Find an explicit bijection from* $[0,1]$ *to* $\mathbb{R}$.

Fact 2. *There exists a bijection between* $(0,1)$ *and* $\mathbb{R}$.

证明. Function $f : y = \tan(\pi x - \frac{\pi}{2})$ is obviously a bijection between $(0,1)$ and $\mathbb{R}$. □

Fact 3. *There exists a bijection between* $(0,1)$ *and* $[0,1]$.

证明. Choose an infinite sequence $(x_n)_{n\geq 1}$ of distinct elements of $(0,1)$. Let $X = \{x_n \mid n \geq 1\}$, hence $X \subset (0,1)$. Let $x_0 = 1$. Define $f(x_n) = x_{n+1}$ for every $n \geq 0$ and $f(x) = x$ for every x in $(0,1)\backslash X$. Then f is defined on $(0,1]$ and the map $f : (0,1] \to (0,1)$ is bijective.

Similarly, we can find a bijection between $(0,1]$ and $[0,1]$. Thus there exists a bijection between $(0,1)$ and $[0,1]$. □

Combine Fact 2 and Fact 3 and we get a bijection between $[0,1]$ and $\mathbb{R}$.