Events in tail σ-algebra

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$X_1, X_2, \cdots$ are random variables. Define

$$\mathcal{T}_n \doteq \sigma(X_{n+1}, X_{n+2}, \cdots), \mathcal{T} \doteq \bigcup_n \mathcal{T}_n.$$

σ-algebra $\mathcal{T}$ is called the tail σ-algebra of sequence$(X_n : n \in \mathbb{N})$.
$\mathcal{T}$ contains many important events, such as:

$$F_1 \doteq (\lim X_k \text{ exists}) \doteq \{\omega : \lim_k X_k(\omega) \text{ exists}\}. \tag{1}$$
$$F_2 \doteq \left(\sum X_k \text{ converges}\right). \tag{2}$$
$$F_3 \doteq \left(\lim \frac{X_1 + X_2 + \cdots + X_k}{k} \text{ exists}\right). \tag{3}$$

Prove F_1, F_2 and $F_3 \in \mathcal{T}$.

Proof: For an arbitrary $n \in \mathbb{N}$, $\mathcal{T}_n \doteq \sigma(X_{n+1}, X_{n+2}, \cdots)$.
As $X_{n+1}, X_{n+2}, \cdots$ are all $\mathcal{T}_n$-measurable, $\{\omega : \lim_{k\to\infty \wedge k>n} X_k(\omega) \text{ exists}\} \in \mathcal{T}_n$.
In other words, $\{\omega : \lim_{k\to\infty} X_k(\omega) \text{ exists}\} \in \mathcal{T}_n$.
Thus $\{\omega : \lim_{k\to\infty} X_k(\omega) \text{ exists}\} \in \bigcap_n \mathcal{T}_n = \mathcal{T}$ and $F_1 \in \mathcal{T}$ follows.
Also, for an arbitrary $n \in \mathbb{N}$, define

$$A_n \doteq \{\omega : \sum_{k>n} X_k(\omega) \text{ converges}\}.$$

Define $S_{n+1} = X_{n+1}$ and $S_{n+k} = S_{n+k-1} + X_{n+k}$ for $k > 1$.
Notice that $A_n = \{\omega : \lim_{k>n} S_k(\omega) \text{ exists}\}$
$X_{n+1}, X_{n+2}, \cdots$ are $\mathcal{T}_n$-measurable $\Rightarrow S_{n+1}, S_{n+2}, \cdots$ are $\mathcal{T}_n$-measurable.
$\Rightarrow \{\omega : \lim_{k>n} S_k(\omega) \text{ exists}\} \in \mathcal{T}_n \Rightarrow A_n \in \mathcal{T}_n$.

Now look back at A_n, we find that $A_0 = A_n$ because the sum from X_1 to X_n are finite.

So $A_0 = A_n \in \mathcal{T}_n \Rightarrow A_0 \in \bigcap \mathcal{T}_n = \mathcal{T}$, and $F_2 = A_0 \in \mathcal{T}$ follows.

Again, for a fixed $n \in \mathbb{N}$, define

$$B_n \doteq \{\omega : \lim_{k>n} \frac{X_{n+1}(\omega) + X_{n+2}(\omega) + \cdots + X_k(\omega)}{k} \text{ exists}\}.$$

As $\frac{X_{n+1}}{n+1}, \frac{X_{n+1}+X_{n+2}}{n+2}, \cdots$ are $\mathcal{T}_n$-measurable, $B_n \in \mathcal{T}_n$.

$\omega \in B_0 \Leftrightarrow \lim_{k>0} \frac{X_1(\omega)+X_2(\omega)+\cdots+X_k(\omega)}{k}$ exists

$\Leftrightarrow \lim_{k>0} \frac{X_1(\omega)+X_2(\omega)+\cdots+X_k(\omega)}{k} - \lim_{k>0} \frac{X_1(\omega)+X_2(\omega)+\cdots+X_n(\omega)}{k}$ exists

$\Leftrightarrow \lim_{k>0} \frac{X_{n+1}(\omega)+X_{n+2}(\omega)+\cdots+X_k(\omega)}{k}$ exists $\Leftrightarrow w \in B_n$.

So $B_0 = B_n \in \mathcal{T}_n \Rightarrow B_0 \in \bigcap \mathcal{T}_n = \mathcal{T}$, and $F_3 = B_0 \in \mathcal{T}$ follows. $\square$