Independence

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability triple.

Definitions of independence

Independent σ-algebras

Sub-σ-algebras $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of $\mathcal{F}$ are called *independent* if, whenever $G_i \in \mathcal{G}_i (i \in \mathbf{N})$ and $i_1, \ldots, i_n$ are distinct, then

$$P(G_{i_1} \cap \cdots \cap G_{i_n} = \prod_{k=1}^{n} P(G_{i_k}))$$

Indenpendent random variables

Random variables $X_1, X_2, \ldots$ are called *independent* if the σ-algebras

$$\sigma(X_1), \sigma(X_2), \ldots$$

are independent.

Independent events

Events $E_1, E_2, \ldots$ are called *independent* if the σ-algebras $\mathcal{E}_1, \mathcal{E}_2, \ldots$ are independent, where

$$\mathcal{E}_n \text{ is the } \sigma\text{-algebra } \{\emptyset, E_n, \Omega \backslash E_n, \Omega\}$$

Since $\mathcal{E}_n = \sigma(I_{E_n})$, it follows that

event $E_1, E_2, \ldots$ are independent if and only if the random variables $I_{E_1}, I_{E_2}, \ldots$ are independent.

The π-system Lemma; and the more familiar definitions

We study independence via π-systems rather than σ-algebras.

(a) **LEMMA.** Suppose that $\mathcal{G}$ and $\mathcal{H}$ are sub-σ-algebras of $\mathcal{F}$, and that $\mathcal{I}$ and $\mathcal{J}$ are π-systems with

$$\sigma(\mathcal{I}) = \mathcal{G}, \quad \sigma(\mathcal{J}) = \mathcal{H}$$

Then $\mathcal{G}$ and $\mathcal{H}$ are *independent* if and only if $\mathcal{I}$ and $\mathcal{J}$ are independent in that

$$P(I \cap J) = P(I)P(J), \qquad I \in \mathcal{I}, \quad J \in \mathcal{J}$$

(b)

Suppose that X and Y are two random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ such that, whenever $x, y \in \mathbf{R}$,

$$P(X \leq x; Y \leq y) = P(X \leq x) P(Y \leq y)$$

The π-systems $\pi(X)$ and $\pi(Y)$ are independent. Hence $\sigma(X)$ and $\sigma(Y)$ are independent.

Second Borel-Cantelli Lemma (BC2)

If $E_n : n \in \mathbf{N}$ is a sequence of **independent** events, then

$$\sum P(E_n) = \infty \Rightarrow P(E_n, \text{ i.o.}) = P(\limsup E_n) = 1$$

Definitions. Tail σ-algebras

Let $X_1, X_2, \ldots$ be random variables. Define

$$\mathcal{T}_n := \sigma(X_{n+1}, X_{n+2}, \ldots), \quad \mathcal{T} := \bigcap_n \mathcal{T}_n$$

The σ-algebra $\mathcal{T}$ is called the *tail σ-algebra* of the sequence $(X_n : n \in \mathbf{N})$.

Theorem. Kolmogorov's 0-1 Law

Let $(X_n : n \in \mathbf{N})$ be a sequence of **independent** random variables, and let $\mathcal{T}$ be the tail σ-algebra of $(X_n : n \in \mathbf{N})$. Then $\mathcal{T}$ is P-trivial, that is

(i) $F \in \mathcal{T} \Rightarrow P(F) = 0 \text{ or } P(F) = 1$

(ii) if ξ is a $\mathcal{T}$-measurable random variable, then, ξ is almost deterministic in that for some constant c in $[-\infty, +\infty]$,

$$P(\xi = c) = 1$$

We allow $\xi = \pm\infty$ at (ii) for obvious reasons.