Properties of Cantor Set

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Exercise 1. *1) Show that the Cantor Set C is nowhere dense in* $[0,1]$.

2) Find a meager set T in $\mathbb{R}$ such that $T+T=\mathbb{R}$.

3) Show that every subset of the real line $\mathbb{R}$ can be partitioned into two sets, one being of first category and the other being negligible.

Proof. 1) Recall the the explicit closed formulas for the Cantor set are:

$$C=[0,1]\setminus\bigcup_{n=0}^{\infty}\bigcup_{k=0}^{3^n-1}(\frac{3k+1}{3^{n+1}},\frac{3k+2}{3^{n+1}})$$

It's obvious that $(\frac{3k+1}{3^{n+1}},\frac{3k+2}{3^{n+1}})$ is an open set. Since the union of open sets is an open set, the complement of C is also open, which leads to the fact that C is a closed set. So $\bar{C}=C$.

Note that Cantor set can also be characterized as the set of all number in $[0,1]$ whose base-3 expansion doesn't contain any 1s. Assume there exists an interior point x of C, then there exists a ball B, with radius $r>0$, which is contained in C. Since $C\subseteq[0,1]$, B is an interval, whose length is $2r$. Arbitarily choose $x_1\in B$ and $x_2\in B$ such that $x_1<x_2$ and $x_2-x_1\geq r$. Let $n=\lceil\log_{\frac{1}{3}}r\rceil+2$.Consider $x_3=x_1+(\frac{1}{3})^n, x_4=x_1+2\cdot(\frac{1}{3})^n$. Since $(\frac{1}{3})^n<(\frac{1}{3})^{(\log_{\frac{1}{3}}r)+1}=\frac{1}{3}\cdot(\frac{1}{3})^{(\log_{\frac{1}{3}}r)}=\frac{1}{3}r$, $x_3,x_4\in B$. It' s obvious that one of them conatins 1 in its base-3 expansion, which means one of them doesn't belong to Cantor set. This leads to a contradiction, or equivalently, that there is no interior point of C.

Now we can safely conclude that C is nowhere dense.

2)Let

$$T(n)=\{a+n|a\in C\}$$
$$T=\bigcup_{n\in\mathbb{Z}}T(n)$$

First We will prove T is a meager set. Since $T(n)$ is constructed by translation of all points in C by n, $T(n)$ is also nowhere dense. This leads to the fact that T is a meager set, because $\mathbb{Z}$ is countable.

Then we prove $T+T=\mathbb{R}$. From $C+C=[0,2]$, we can similarly get $T(n)+T(n)=[2n,2n+2]$. Note that:

$$\bigcup_{n\in\mathbb{Z}}(T(n)+T(n))\subseteq T+T\subseteq\mathbb{R}$$

However,

$$\bigcup_{n\in\mathbb{Z}}(T(n)+T(n))=\bigcup_{n\in\mathbb{Z}}[2n,2n+2]=\mathbb{R}$$

Now we conclude that $T+T=\mathbb{R}$.

3) Method 1:

Pick an enumeration of $\mathbb{Q}$, named q_n. Let:

$$\begin{aligned}I_{i,j}&=(q_i-\frac{1}{2^{i+j+2}},q_i+\frac{1}{2^{i+j+2}})\\A_j&=\bigcup_{i\in\mathbb{N}}I_{i,j}\\B&=\bigcap_{j\in\mathbb{N}}A_j\end{aligned}$$

For any $\epsilon>0$, exist $j_0>0$ such that $|\frac{1}{2^{j_0}}|<\epsilon$. So we have :

$$\begin{aligned}|A_j|&\leq\sum_{i=0}^{\infty}\frac{1}{2^{i+j+1}}=\frac{1}{2^j}\\|B|&\leq|A_{j_0}|\leq\frac{1}{2^{j_0}}<\epsilon\end{aligned}$$

This leads to that B is negligible. Then we prove B^C is a meager set.

$$B^C=\bigcup_{j\in\mathbb{N}}A_j^C$$

Since $\mathbb{Q}\subseteq A_j$ and $\mathbb{Q}$ is dense, we can know that A_j is dense. A_j is also open because A_j is the union of open intervals. These 2 properties of A_j can lead to that A_j^C is nowhere dense. So B^C is a meager set, which finishes the proof.

Method 2: Construct a sequence of fat cantor set by the following steps: To construct $C_n(n\geq 2)$, first remove the middle $\frac{1}{2n}$ from the interval $[0,1]$. Then in the i^{th} step(i starts from 2), remove subintervals of width $\frac{1}{2n\cdot 4^{i-1}}$ from the middle of each of the 2^{i-1} remaining intervals. We can then calculate

$$\begin{aligned}|C_n|&=1-\sum_{i=1}^{\infty}\frac{1}{2n\cdot 4^{i-1}}\cdot 2^{i-1}=1-\frac{1}{n}\\|C_n^C|&=\frac{1}{n}\end{aligned}$$

For any $\epsilon>0$, there exists $n_0>0$ such that $\frac{1}{n_0}<\epsilon$. Let $C=\bigcup\limits_{n=2}^{\infty}C_n$, then $C^C=\bigcap\limits_{n=2}^{\infty}C_n$. Note that $|C^C|\leq|C_{n_0}^C|=\frac{1}{n_0}<\epsilon$, which leads to that C^C is negligible in $[0,1]$.

By construction,C_n contains no intervals and therefore has empty interior, or equivalently, C_n is nowhere dense in $[0,1]$. So C is a meager set in $[0,1]$.

We can easily extend this conclusion to $\mathbb{R}$. Let $C'(i)=\{i+c|c\in C,i\in\mathbb{Z}\}$, $C'=\bigcup\limits_{i\in\mathbb{Z}}C'(i)$. It's easy to see that C' is a meager set in $\mathbb{R}$ and C'^C is negligible in $\mathbb{R}$.

Method 3: Let L be the set of Liouville numbers. Recall the first claim of Theorem 3 from notes on March 10, which is: Suppose that ϕ is positive. If $\sum_q \frac{1}{q\phi(q)} < \infty$, then $P(A_\phi) = 0$. A_ϕ is the set of reals x in $(0, 1]$ such that $|x - \frac{p}{q}| < \frac{1}{q^2\phi(q)}$. Let $\phi(q) = q$, then $L \subseteq A_\phi$. Since $\sum_q \frac{1}{q^2} < \infty$, $P(A_\phi) = 0$,which leads to that $P(L) = 0$. Equivalently, L is negligible.

Select an enumeration of $\mathbb{Q}$, named q_n.Let

$$U_n = \bigcup_{i=0}^{\infty}\{x : q_i = \frac{p}{q}, 0 < |x - \frac{p}{q}| < \frac{1}{q^n}\}$$

$$L = \bigcap_{n=1}^{\infty} U_n$$

U_n is open, because it's the union of open sets. Since any open set that contains $\frac{p}{q}$ must intersect with $\{x : 0 < |x - \frac{p}{q}| < \frac{1}{q^n}\}$ and $\mathbb{Q}$ is dense, it's easy to see that U_n is dense according to the definition of "dense". So U_n^C is nowhere dense, which means $L^C = \bigcup\limits_{n=1}^{\infty} U_n^C$ is a meager set. This finishes the proof. $\square$