Discussions for Exercise 8

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1   A natural thought

Problem 1.1 *View $\mathbb{R}^k$ as a poset which $(x_1, x_2, \cdots x_k) \leq (y_1, y_2, \cdots, y_k)$ if and only if $y - x \geq 0$ namely $\forall i, y_i \geq x_i$. Is there any injective poset homomorphism from $\mathbb{R}^3$ to $\mathbb{R}^2$?*

Proof: First notice that every real number in $(0, 1)$ could be represented as

$$\sum_{i=1}^{\infty} a_i 10^{-i}.$$

Which would be helpful to construct a mapping. Consider $\forall (a, b, c) \in (0, 1)^3$, their representations are

$$\sum_{i=1}^{\infty} a_i 10^{-i}, \sum_{i=1}^{\infty} b_i 10^{-i}, \sum_{i=1}^{\infty} c_i 10^{-i}.$$

We consider a mapping

$$f : (0, 1)^3 \to (0, 1)^2$$
$$(a, b, c) \to (d, e).$$

which satisfies that

$$d = \sum_{i=1}^{\infty} a_i 10^{-2i+1} + \sum_{i=1}^{\infty} b_i 10^{-2i}$$
$$e = c.$$

f is obviously a bijection. Now we prove it is poset homomorphism. We only need to prove that if $(a, b, c) \leq (a', b', c')$ then
$$f(a, b, c) = (d, e) \leq (d', e') = f(a', b', c')$$
It's clear that $e = c \leq c' = e'$. And now consider the relationship between d, d'. Knowing that $a \leq a'$, thus $a = a'$ or $\exists k$ s.t.
$$a_j = a'_j, \forall j < k \quad a_k < a'_k.$$
Similarly for $b \leq b'$ there will be a l. If $a \neq a', b \neq b'$, let $s = \min(k, l)$, then

$$d_j = d'_j, \forall j < s, \quad d_s < d'_s.$$

Hence $d \leq d'$. Otherwise if $a = a'$ or $b = b'$, we can discuss these cases similarly. That is $(d, e) \leq (d', e')$.

Now we extend the mapping to $\mathbb{R}^3 \to \mathbb{R}^2$. Just notice that there are 2 bijections

$$g_1 : (-\infty, \infty) \mapsto (0, \infty)$$
$$x \mapsto \mathrm{e}^x.$$

and

$$g_2 : (0, \infty) \mapsto (0, 1)$$
$$x \mapsto \frac{2}{\pi} \tan^{-1} x.$$

And most importantly, g_1, g_2 are both monotonic functions, also poset homomorphisms.

To sum up, the homomorphism I constructed is

$$g_1 \circ g_2 \circ f \circ g_2^{-1} \circ g_1^{-1}.$$

$\square$

This construction, however, is a typical wrong solution. Since $\frac{1}{2}$ could be represented either as $0.5000\cdots$ or $0.49999\cdots$. Thus it's almost impossible to construct a bijection in this way.

Cantor originally tried this method himself, but Dedekind pointed out the problem of nonunique decimal representations.

Nevertheless, if we always choose the latter representation, i.e. the one that ends with $999\cdots$, we could construct an injection.

In order to constrcut a bijection, **Haichen Dong** has come up with a revision to this method.

2   Bijection between $\mathbb{R}$ and $\mathbb{R}^2$

Theorem 2.1 *There is a bijection f between $\mathbb{R}$ and $\mathbb{R}^2$, which implies $|\mathbb{R}| = |\mathbb{R}^2|$.*

Proof: First we construct a bijective map from $\mathbb{R}$ to $(0, 1)$:

$$g : \mathbb{R} \to (0, 1)$$
$$= x \mapsto \frac{2 \tan^{-1} x}{\pi}.$$

The next step is to find a bijection from $(0, 1)$ to $(0, 1]$, which can be obtained by

$$h : (0, 1) \to (0, 1]$$
$$x \mapsto \begin{cases} 2^{-n-1}, x = 2^{-n} \\ x, \text{ otherwise} \end{cases}.$$

And also the 2-dimensional version g', h'.

This construction will work with any sequence $\{a_1, a_2, \cdots\}$ with $a_1 = 1$ and $\lim_{n \to \infty} a_n = 0$.

The final step is to fix the problem brought by the construction that maps $(0.a_1 a_2 a_3 \cdots, 0.b_1 b_2 b_3 \cdots)$ to $(0.a_1 b_1 a_2 b_2 \cdots)$

That problem exists because finite numbers, for example, $\frac{1}{2}$, can be written as both $0.500\cdots$ and $0.499\cdots$. So we make the choice that always written finite numbers with infinite 9s trailing.

But without further construction, there will be no inverse image for $0.5303030\cdots$. Noticing that there will never be infinite zero string in the resulting number. So we can involve the zero string together with the non-zero digit after it.

For example, $0.53030303\cdots = 0.[5][3][03][03][03]\cdots$, is mapped by $(0.50303\cdots, 0.30303\cdots)$. Call this mapping ϕ, and finally we get the bijection:

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto (g^{-1} \circ h^{-1} \circ \phi \circ h' \circ g')(x).$$

$\square$

3   Future work

By our trivial thought, we could find a injective poset homomorphism. But how about surjections?

Problem 3.1 *View $\mathbb{R}^k$ as a poset which $(x_1, x_2, \cdots x_k) \leq (y_1, y_2, \cdots, y_k)$ if and only if $y - x \geq 0$ namely $\forall i, y_i \geq x_i$. Is there a subposet of $\mathbb{R}^3$ poset which is isomorphic with $\mathbb{R}^2$?*