Probability, Week 3, exerciese 2

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0. By the definition of $\pi([a,b])$ and $\pi(\mathbb{R})$, $\pi([a,b])$ can be written as:

$$\pi([a,b]) = \{[a,x] \mid x \in [a,b]\}$$

.

1. Now prove that $\sigma(\pi([a,b])) \subseteq \mathcal{B}[a,b]$.

As all open subsets of $[a,b]$ are in $\mathcal{B}[a,b]$, all closed subsets of $[a,b]$ are also in $\mathcal{B}[a,b]$, so $\forall x \in [a,b], [a,x] \in \mathcal{B}[a,b]$.

So there is $\pi([a,b]) \subset \mathcal{B}[a,b]$. As $\mathcal{B}[a,b]$ is also a $\sigma-$algebra, $\sigma(\pi([a,b])) \subseteq \mathcal{B}[a,b]$.

2. Now prove that $\mathcal{B}[a,b] \subseteq \sigma(\pi([a,b]))$.

Since $\mathcal{B}[a,b]$ is generated by all open subsets of $[a,b]$(also open subsets of $\mathbb{R}$), each one(said s), can is a countably union of open intervals on $\mathbb{R}$, denoted by $I_1, I_2, \cdots$. So there is $s = \cap_i I_i$

Now Let $I'_i = I_i \cap [a,b]$, it is obvious that $s = \cap_i I'_i$. Otherwise, there are some elements in I_j but not I'_j for some j, so it is not in $[a,b]$. But $s \subset [a,b]$.

Now we prove that for each i, $I'_i \in \sigma(\pi([a,b]))$:

1)I'_i can only be form like $[a,b], [a,x), (x,b], (x,y)$ where $x, y \in (a,b)$;

2)$[a,b] \in \pi([a,b])$, so $[a,b] \in \sigma(\pi([a,b]))$;

3)as $x < b$, we have $\exists n \to \forall i > n, x + 2^{-i} < b$. So $[a,x) = \cap_{i=n+1}^{\infty}[a, x+2^{-i}]$ and $[a, x+2^{-i}] \in \pi([a,b])$. So $[a,x) \in \sigma(\pi([a,b]))$

4)as $[a,x], [a,b] \in \pi([a,b])$, there is $(x,b] = [a,b]/[a,x] \in \pi([a,b]) \subseteq \sigma(\pi([a,b]))$;

5)By 3) and 4) $[a,y), (x,b] \in \sigma(\pi([a,b]))$, there is $(x,y) = [a,y) \cap (x,b] \in \sigma(\pi([a,b]))$.