The σ-Finite Constraint on Uniqueness of Extension Lemma

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Lemma 1 Let I be a π-system on a set S and let $\Sigma = \sigma(I)$. Suppose that μ_1 and μ_2 are two measures on (S, Σ) such that $\mu_1 = \mu_2$ on I. If there is a sequence $A_n \in I$ with $\bigcup_{n=1}^{\infty} A_n = S$ and $\mu_(A_n) < \infty$, then $\mu_1 = \mu_2$.

We will give a proof for this lemma and show the reason that we need to add the σ-finite constraint.

First we need to introduce the definition of a d-system [1].

Definition 2 Let S be a set, and let $\mathcal{D}$ be a collection of subsets of S. Then $\mathcal{D}$ is called a d-system (on S) if

(a) $S \in \mathcal{D}$,

(b) if $A, B \in \mathcal{D}$ and $A \subseteq B$ then $B \setminus A \in \mathcal{D}$,

(c) if $A_1, A_2, A_3, \ldots$ is a sequence of subsets in $\mathcal{D}$ and $A_n \subseteq A_{n+1}$ for all $n \geq 1$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

It's easy to verify the intersection of d-systems is still a d-system. Thus similar

[1] The word "d-system" is an abbreviation of Dynkin System to honor Eugene Dynkin. It is sometimes referred to as λ-system (Dynkin himself used this term).

to the definition of $\sigma(C)$, for $C \subseteq 2^S$ we denote the smallest d-system containing C by $d(C)$. We present Dynkin's lemma here without proof.

Lemma 3 (Dynkin's Lemma) If $\mathcal{I}$ is a π-system, then $d(\mathcal{I}) = \sigma(\mathcal{I})$.

Now we will give a proof for the finite case of Lemma 1: "If $\mu_1(S) = \mu_2(S) < \infty$, then $\mu_1 = \mu_2$". Define

$$\mathcal{D} = \{F \in \Sigma : \mu_1(F) = \mu_2(F)\}$$

Evidently $I \subseteq \mathcal{D}$. Thus we simply need to verify $\mathcal{D}$ is a d-system on S. Then according to Dynkin's Lemma, we can see

$$\Sigma = \sigma(\mathcal{I}) = d(\mathcal{I}) \subseteq \mathcal{D}$$

which implies $\mu_1 = \mu_2$ on Σ.

First, the fact that $S \in \mathcal{D}$ is given. Then, if $A, B \in \mathcal{D}$ and $A \subseteq B$,

$$\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A) \tag{1}$$

so that $B \setminus A \in \mathcal{D}$. Finally, if $A_n \in \mathcal{D}$ and $A_n \subseteq A_{n+1}$. Due to the monotone-convergence properties of measures,

$$\mu_1(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu_1(A_n) = \lim_{n\to\infty} \mu_2(A_n) = \mu_2(\bigcup_{n=1}^{\infty} A_n)$$

Hence, $\mathcal{D}$ is a d-system.

This proof is only valid for the finite case because correctness of Eq. (1) may cause problem. We need to notice whether $\infty - \infty = \infty - \infty$ is uncertain. Thus we can see why Example 4 has two extensions.

Example 4 Let $S = (0, 1]$, and let Σ_0 be the subsets of S which are finite unions of disjoint left-open right-closed intervals. Obviously, Σ_0 is a π-system. Define

$$\mu_0(F) = \begin{cases} 0 & \text{if } F = \emptyset \\ \infty & \text{if } F \neq \emptyset \end{cases}$$

We find two ways to extend μ_0 to $\mathcal{B}(0,1]$: μ_1 and μ_2:

$$\mu_1(F) = \begin{cases} 0 & \text{if } F = \emptyset \\ \infty & \text{if } F \neq \emptyset \end{cases}$$

$$\mu_2(F) = \text{number of elements in } F.$$

It's easy to check they are both correct. However, we notice that

$$\mu_1(\{1\}) = \infty \neq 1 = \mu_2(\{1\})$$

That's because $\{1\} = (0,1] \setminus \bigcup_{n=1}^{\infty}(0, 1 - \frac{1}{2^n}]$, but $\mu(\{1\})$ cannot be determined by $\infty - \infty$.

Now let us consider the σ-finite case. For any arbitrary set $A \in \Sigma$ satisfying $\mu_1(A) = \mu_2(A) < \infty$, it's easy to see $\mathcal{D} = \{F \in \Sigma : \mu_1(F \cap A) = \mu_2(F \cap A)\} = \Sigma$ by repeating the argument above, because for any $F \in \Sigma$, $\mu_i(F \cap A) \leq \mu_i(A) < \infty$.

By assumption, there exists a sequence $A_n \in I$ with $\bigcup_{n=1}^{\infty} A_n = S$. Define $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$. Since B_n is pairwise distinct and $\bigcup_{n=1}^{\infty} B_n = S$, for $F \in \Sigma$,

$$\mu_1(F) = \mu_1(F \cap \bigcup_{n=1}^{\infty} B_n) = \mu_1(\bigcup_{n=1}^{\infty}(F \cap B_n)) = \sum_{n=1}^{\infty} \mu_1(F \cap B_n)$$

$$\mu_2(F) = \mu_2(F \cap \bigcup_{n=1}^{\infty} B_n) = \mu_2(\bigcup_{n=1}^{\infty}(F \cap B_n)) = \sum_{n=1}^{\infty} \mu_2(F \cap B_n)$$

Because $B_n \in \Sigma$, we have

$$\mu_1(B_n) = \mu_1(B_n \cap A_n) = \mu_2(B_n \cap A_n) = \mu_2(B_n) < \infty$$

Now we can see that $\mu_1(F \cap B_n) = \mu_2(F \cap B_n)$. Hence, $\mu_1(F) = \mu_2(F)$ for any $F \in \Sigma$. This finishes the proof.

Notice that the so-called "σ-finite" constraint in this lemma is a stronger constraint than "μ_1 and μ_2 are σ-finite". Here we provide a counterexample for the latter condition.

Example 5 Let $S = \mathbb{R}$, and let $I = \{[0,u]|u \in [0,+\infty)\} \cup \mathbb{R}$. Obviously I is a π-system on $\mathbb{R}$. Furthermore, $\sigma(I) \subset \mathcal{B}(\mathbb{R})$. We use μ_1 to denote Lebesgue measure on $\sigma(I)$. Thus $(\mathbb{R}, \sigma(I), \mu_1)$ is a σ-finite measure space.

We can construct another σ-finite measure $(\mathbb{R}, \sigma(I), \mu_2)$ where $\mu_2(F) = \mathrm{Leb}(F \cap [0,+\infty))$. Clearly $\mu_1 = \mu_2$ on I, but $\mu_1((-\infty, 1]) = \infty \neq 1 = \mathrm{Leb}([0,1]) = \mu_2((-\infty, 1])$.