Independence of coin tossing events

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In probability triple $(\Omega, \mathcal{F}, P) = ([0,1), \mathcal{B}[0,1), \text{Leb})$. Let event

$$A_n = [0, \frac{1}{2^n}) \cup [\frac{2}{2^n}, \frac{3}{2^n}) \cup \cdots \cup [\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}).$$

Then $(A_n : n \in \mathbb{N}_+)$ are independent.

Proof. By definition, we should show that

$$\mathcal{A}_i := \{\varnothing, A_n, \Omega \backslash A_n, \Omega\}, i \in \mathbb{N}_+.$$

are independent σ-algebras.

Consider any $B_i \in \mathcal{A}_i$ and distinct $i_1, i_2, \cdots, i_n$. For some $l \in [n]$,

- If $B_{i_l} = \varnothing$, then $\bigcap_{k=1}^n B_{i_k} = \varnothing$ and $P(B_i) = 0$ thus $\prod_{k=1}^n P(B_{i_k}) = 0$. The equation holds for any other B_{i_k}.
- If $B_{i_l} = \Omega$, then $\bigcap_{k\in[n]} B_{i_k} = \bigcap_{k\in[n]\backslash\{l\}} B_{i_k}$ and $P(B_{i_l}) = 1$. Thus

$$P(\bigcap_{k\in[n]} B_{i_k}) = \prod_{k\in[n]} P(B_{i_k}) \iff P(\bigcap_{k\in[n]\backslash\{l\}} B_{i_k}) = \prod_{k\in[n]\backslash\{l\}} P(B_{i_k}).$$

So we can assume that either $B_{i_k} = A_{i_k}$ or $B_{i_k} = \Omega \backslash A_{i_k}$.

Let $N = \max_{k\in[n]} i_k$ and $L_i = [\frac{i}{2^N}, \frac{i+1}{2^N})$. (L_i) are disjoint. Then

$$\begin{aligned}
A_{i_k} &= \bigcup_{s=0}^{2^{i_k}-1} [\frac{2s}{2^{i_k}}, \frac{2s+1}{2^{i_k}}) \\
&= \bigcup_{s=0}^{2^{i_k}-1} \bigcup_{i=2^{N-i_k}\cdot 2s}^{2^{N-i_k}(2s+1)-1} L_i \\
&= \bigcup_{\{i\in\{0,1,\cdots,2^N-1\}: \left\lfloor \frac{i}{2^{i_k}} \right\rfloor\} \equiv 0 (\bmod 2)\}} L_i.
\end{aligned}$$

Similarly, $\Omega \setminus A_{i_k} = \bigcup_{\{i\in\{0,1,\cdots,2^N-1\}:\left\lfloor \frac{i}{2^{i_k}} \right\rfloor\}\equiv 1(\bmod 2)\}} L_i$.

Thus

$$P(\bigcap_{k\in[n]} B_{i_k}) = P(\bigcup_{\{i\in\{0,1,\cdots,2^N-1\}:\forall k\in[n],\left\lfloor \frac{i}{2^{i_k}} \right\rfloor\equiv a_k(\bmod 2)\}} L_i) = 2^{-k}$$

,where $a_k = \begin{cases} 0, B_{i_k} = A_{i_k} \\ 1, B_{i_k} = \Omega \setminus A_{i_k} \end{cases}$.

We also have $\forall k, P(\Omega \setminus A_{i_k}) = P(A_{i_k}) = 2^{i_k-1} \cdot 2^{N-i_k} \cdot 2^{-N} = 2^{-1}$. Therefore

$$P(\bigcap_{k\in[n]} B_{i_k}) = \prod_{k\in[n]} P(B_{i_k}) = 2^{-k}.$$

This implies that $(\mathcal{A}_i)$ are independent σ-algebras, and then (A_n) are independent events.

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