Upcrossing Lemma and Martingale Convergence

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1 Doob's Upcrossing Lemma

In this section, I want to show how the Doob's upcrossing lemma is generated and introduced.

First we introduced what is a predictable sequence. Let $\mathcal{F}_n, n \geq 0$ be a filtration. $H_n, n \geq 1$ is said to be a predictable sequence if $H_n \in \mathcal{F}_{n-1}$ for all $n \geq 1$. That is to say, H_n can be predicted from the information available at time $n-1$. We now consider H_n as a gambling system.

Theorem 1. *Let $X_n, n \geq 0$ be a supermartingale. If $H_n \geq 0$ is predictable and each H_n is bounded then $(H \cdot X)_n$ is a supermartingale. (Obviously, the result is also true for martingales and submartingales.)*

Proof. Because the conditional expectation has a property of linearity, and using the fact "If $X \in \mathcal{F}$ and $E|Y|, E|XY| < \infty$ then $E(XY|\mathcal{F}) = XE(Y|\mathcal{F})$". Since $(H \cdot X)_n \in \mathcal{F}_n, H_n \in \mathcal{F}_{n-1}$ we have

$$\begin{aligned} E((H \cdot X)_{n+1}|\mathcal{F}_n) &= (H \cdot X)_n + E(H_{n+1}(X_{n+1} - Xn)|\mathcal{F}_n) \\ &= (H \cdot X)_n + H_{n+1}E((X_{n+1} - Xn)|\mathcal{F}_n) \end{aligned}$$

since $E((X_{n+1} - Xn)|\mathcal{F}_n) \leq 0$ and $H_{n+1} \geq 0$, so we have $E((H \cdot X)_{n+1}|\mathcal{F}_n) \leq (H \cdot X)_n$ □

Now we consider a special gambling system : bet \$1 when $n \leq N$ then stop playing. If we let $H_n = 1_{\{N \geq n\}}$ then $\{N \geq n\} = \{N \leq n-1\}^c \in \mathcal{F}_{n-1}$, so H_n is predictable, and it follows from Theorem 1 that $(H \cdot X)_n = X_{N \wedge n} - X_0$ is a supermartingale. The constant sequence is a supermartingale and the sum of two supermartingales is also a supermartingale, so we have the following result.

Theorem 2. *If X_n is a supermartingale and N is a stopping time, then $x_{N \wedge n}$ is a supermartingale*

Having those two theorems above, now we can introduce word "crossing" and then lead to the Doob's crossing lemma. The number of times that a process passes upwards or downwards through an interval is refered to as the number of upcrossings and respectively the number of downcrossings of the process.

Consider a process X_t whose time index t runs through an index set $\mathbb{T} \subseteq \mathbb{R}$. For real numbers $a < b$, the number of upcrossings of X across the interval $[a, b]$ is the supremum of the nonnegative integers n such that there exists times $s_k, t_k \in \mathbb{T}$ satisfying

$$s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n$$

for which $X_{s_k} \leq a < b \leq X_{t_k}$. $U[a, b]$ means the number of upcrossings, which is either a nonnegative integer or is infinite. Similarly, $D[a, b]$ means the number of downcrossings, is the supremum of the nonnegative integers n such that there are times $s_k, t_k \in \mathbb{T}$ satisfying the above inequality relationship and such that $X_{s_k} \geq b > a \geq X_{t_k}$. Note that between any two upcrossings there is a downcrossing and, similarly, between any two downcrossings there is an upcrossing. It follows that $U[a, b]$ and $D[a, b]$ can differ by at most 1, and they are either both finite or both infinite. Below is a a process with 3 upcrossings of the interval $[a, b]$ [1].

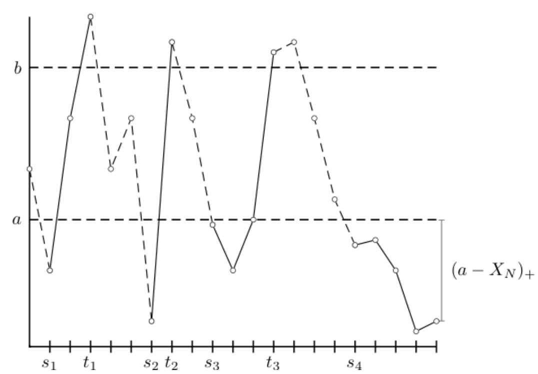


Figure 1: a process with 3 upcrossings of the interval

Lemma 1. Doob's upcrossing lemma *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability space. Let (X_n) be a supermartingale. For $(a, b) \subset \mathbb{R}$ and $N \in \mathbb{N}$, let $U_N(a, b)$ be the number of upcrossings of $(X_n)_{n \leq N}$ in (a, b). Then*

$$(b - a)\mathbb{E}[U_N(a, b)] \leq \mathbb{E}[(X_N - a)^-]$$

Proof. Define a predictable process H_n by

$$H_1 = 1_{\{X_0 \leq a\}} \quad H_{n+1} = H_n 1\{X_n \leq b\} + (1 - H_n) 1\{X_n \leq a\}$$

Then

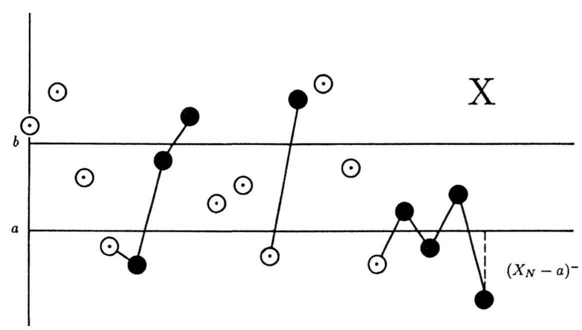
$$(b-a)U_N(a,b) \le (H \cdot X)_N + (X_N - a)^-$$

According to Theorem 1, we know $(H \cdot X)_n$ is a supermartingale, then

$$(b-a)\mathbb{E}[U_N(a,b)] \le \mathbb{E}[(X_N - a)^-]$$

□

This proof seemed to be complicated, but the idea behind it is really simple : consider (X_n) as points in a trail, n means time. An upcross means move from less than a to larger than b, this will move at least $(b-a)$ one time. Then the number of upcrossing multiplies with $(b-a)$ will not exceed the distance of upcrossing + the final remain. Then we follow this idea we get the doob's equality. This idea can be shown in the below picture.



2 Martingale convergence

In last section, we introduced the doob's upcrossing inequality and proved it. Once we get this powerful lemma, we can use it to prove various convergence theorem of martingale. Following is Doob's supermartingale convergence.

Theorem 3. Doob's forward convergence Theorem *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability space. Let (X_n) be a supermartingale such that $\sup_n \mathbb{E}[|X_n|] < \infty$ Then $\exists X \in \mathcal{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$ such that*

$$X_n \longrightarrow X a.s.$$

Proof. Define

$$\begin{aligned} E &= \{\omega \in \Omega : X_n(\omega) \not\longrightarrow X(\omega)\} \\ &= \{\omega \in \Omega : \liminf_{n\to\infty} X_n(\omega) < \limsup_{n\to\infty} X(\omega)\} \\ &= \bigcup_{a,b\in\mathbb{Q} a<b} \{\omega \in \Omega : \liminf_{n\to\infty} X_n(\omega) < a < b < \limsup_{n\to\infty} X(\omega)\} \end{aligned}$$

And we can find that

$$\{\omega \in \Omega : \liminf_{n\to\infty} X_n(\omega) < a < b < \limsup_{n\to\infty} X(\omega)\} \subseteq \{\omega \in \Omega : U_\infty(a,b)(w) = \infty\}$$

Then according to upcrossing lemma, we have

$$(b-a)\mathbb{E}[U_N(a,b)] \leq \mathbb{E}[(X_N - a)^-] \leq \sup_n \mathbb{E}[|X_n|] + |a| \leq \infty$$

Then take $N \to \infty$, we have

$$(b-a)\mathbb{E}[U_\infty(a,b)] \leq \infty \Longrightarrow \mathbb{P}[U_\infty(a,b) = \infty] = 0$$

Thus, we obtain $\mathbb{P}[E] = 0$ □

We can also have the theorem with martingale, submartingale. This proof is really beautiful, as it consider the "not convergence" part as an event, and decompose this into several countable event, then using doob's upcrossing lemma to prove every event has a probability of zero.

An important special case of Theorem 3 is the next, we can get this easily follows from Fatous's lemma.

Theorem 4. *If $X_n \geq 0$ is a supermartingale, then as $n \longrightarrow \infty, X_n \longrightarrow X a.s$ and $EX \leq EX_0$*

There are two counterexamples in Probability-Theory and Examples [4], I want to show them here since they can really help the understand the convergence of martingale.

Example 1. *This example shows that the assumptions of Theorem 3 (or 4) do not guarantee convergence in L^1. Let S_n be a symmetric simple random walk with $S_0 = 1, i.e., S_n = S_{n-1} + \xi_n$ where ξ_1, ξ_2 are i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Let $N = inf\{n : S_n = 0\}$ and let $X_n = S_{N\wedge n}$. Theorem 2 implies that X_n is a nonnegative martingale. Theorem 4 implies X_n converges to a limit $X_\infty < \infty$ that must be $\equiv 0$, since convergence to $k > 0$ is impossible. (If $X_n = k > 0$ then $X_{n+1} = k \pm 1$.) Since $EX_n = EX_0 = 1$ for all n and $X_\infty = 0$, convergence can't occur in L^1.*

Example 2. *Now give an example of a martingale with $X_k \longrightarrow 0$ inprobability but not a.s. Let$X_0 = 0$. When$X_{k-1} = 0$,let$X_k = 1$or -1 with probability 1/2k and = 0 with probability 1 − 1/k. When $X_{k-1} \neq 0$, let $X_k = kX_{k-1}$ with probability 1/k and = 0 with probability 1 - 1/k. From the construction, $P(X_k = 0) = 1 - 1/k$ so $X_k \longrightarrow 0$ in probability. On the other hand, the second Borel-Cantelli lemma implies $P(X_k = 0$ for $k \geq K) = 0$, and values in $(-1,1) - \{0\}$ are impossible, so X_k does not converge to 0 a.s.*

Also, as an example application of the martingale convergence theorem, it is easy to show that a standard random walk started started at 0 will visit every level with probability one.

Corollary 1. *Let $(X_n)_{n\in\mathbb{N}}$ be a standard random walk. That is, $X_1 = 0$ and*

$$\mathbb{P}(X_{n+1} = X_n + 1 \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = X_n - 1 \mid \mathcal{F}_n) = 1/2.$$

Then, for every integer a, with probability one $X_n = a$ for some n.

Proof. Without loss of generality, suppose that$a \leq 0$. Let $T : \Omega \to \mathbb{N} \cup \{\infty\}$ be the first time n for which $X_n = a$. It is easy to see that the stopped process X_n^T defined by $X_n^T = X_{\min(n,T)}$ is a martingale and $X^T - a$ is non-negative. Therefore, by the martingale convergence theorem, the limit $X_\infty^T = \lim_{n\to\infty} X_n^T$ exists and is finite (almost surely). In particular, $|X_{n+1}^T - X_n^T|$ converges to 0 and must be less than 1 for large n. However, $|X_{n+1}^T - X_n^T| = 1$ whenever $n < T$, so we have $T < \infty$ and therefore $X_n = a$ for some n. □

Corollary 2. *Let $(X_n)_{n=1,2,\ldots}$ be a uniformly integrable martingale and let $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$ be the sigma-algebra at infinity. Then, there is an almost-surely unique and integrable $\mathcal{F}_\infty$-measurable random variable X_∞ such that $X_n = \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ for each n. Furthermore, $X_n \to X_\infty$ as $n \to \infty$, with probability one.*

Proof. First, by Doob's forward convergence theorem, we can set $X_\infty = \lim_n X_n$, which exists with probability one. By uniform integrability, this converges in L^1 and it follows that $X_n = \mathbb{E}[X_m \mid \mathcal{F}_n]$ converges to $\mathbb{E}[X_\infty \mid \mathcal{F}_n]$ as $m \to \infty$. So, X_∞ satisfies the required properties.

Suppose X'_∞ is any other random variable satisfying the required properties. Then, $\mathbb{E}[1_A X_\infty] = \mathbb{E}[1_A X'_\infty] = \mathbb{E}[1_A X_n]$ for any n and $A \in \mathcal{F}_n$. By the monotone class theorem, this extends to all $A \in \mathcal{F}_\infty$. As X_∞, X'_∞ are $\mathcal{F}_\infty$-measurable, it follows that they are almost surely equal.

The condition that X_n is L^1-bounded in Doob' s forward convergence theorem is automatically satisfied in many cases. In particular, if X is a non-negative supermartingale then $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] \leq \mathbb{E}[X_1]$ for all $n \geq 1$, so $\mathbb{E}[|X_n|]$ is bounded, giving the following corollary.

□

3 Applications of Martingale Convergence Theorem

3.1 Convergence in L^p

Theorem 5. *If X_n is a submartingale then for $1 < p < \infty$,*

$$E(\bar{X}_n^p) \leq (\frac{p}{p-1})^p E(X_n^+)^p$$

Consequently, if Y_n is a martingale and $Y_n^* = max_{0\le m\le n}|Y_m|$,

$$E|Y_n^*|^p \le (\frac{p}{p-1})^p E(|Y_n|^p)$$

Theorem 6. L^p convergence theorem *If $_n$ is a martingale with $\sup E|X_n|^p < \infty$ where $p > 1$, then $X_n \longrightarrow X$ a.s. and in L^p.*

Proof. $(EX_n^+)^p \le (E|X_n|)^p \le E|X_n|^p$, so it follows from the martingale convergence theorem that $X_n \longrightarrow X$ a.s. The second conclusion in Theorem 5 implies

$$E(\sup_{0\le m\le n}|X_m|)^p \le (\frac{p}{p-1})^p E(|X_n|^p)$$

Let $n \longrightarrow \infty$ and use the monotone convergence theorem implies $\sup|Xn| \in L^p$. Since $|X_n - X|^p \le (2\sup|X_n|)^p$, it follows from the dominated convergence theorem, that $E|X_n - X|^p \longrightarrow 0$. □

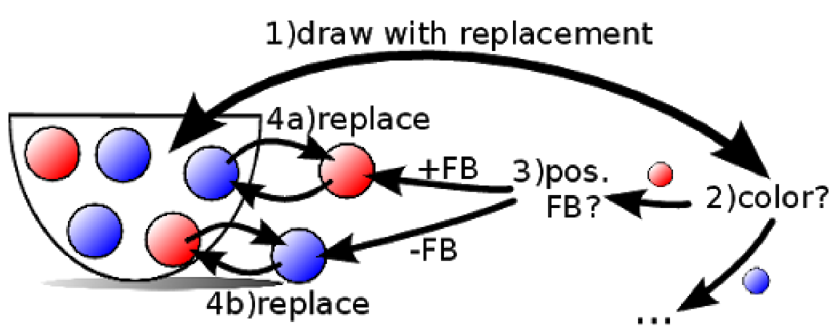
The most important special case of the results in this section occurs when $p = 2$. There are two corollary in Probability-Theory and Examples.

Corollary 3. Orthogonality of martingale increments *Let X_n be a martingale with $EX_n^2 < \infty$ for all n. If $m \le n$ and $Y \in \mathcal{F}_m$ has $EY^2 < \infty$ then $E((X_n - X_m)Y) = 0$ and hence if $l < m < n$, $E((X_n - X_m)(X_m - X_l) = 0$*

Corollary 4. Conditional variance formula *If X_n is a martingale with $EX_n^2 < \infty$ for all n, $E((X_n - X_m)^2|\mathcal{F}_m) = E(X_n^2|\mathcal{F}_m) - X_m^2$*

3.2 Polya's Urn Scheme

An urn contains 1 red ball and 1 green ball (in order to simplify this, we discuss just one ball for each color). At each time we draw a ball out, then put it back with an extra ball of the same color. Let R_n be the number of red balls after the n-th draw, and G_n be the number of green balls after the n-th draw.



Let $\mathcal{F}_n = \sigma(R_0, G_0, R_1, G_1, \ldots, R_n, G_n$. Define M_n to be the fraction of green balls. Then

$$\begin{aligned}
\mathbb{E}[M_n|\mathcal{F}_{n-1}] &= \frac{R_{n-1}}{G_{n-1}+R_{n-1}} \frac{G_{n-1}}{G_{n-1}+R_{n-1}+1} + \frac{G_{n-1}}{G_{n-1}+R_{n-1}} \frac{G_{n-1}}{G_{n-1}+R_{n-1}+1} \\
&= \frac{G_{n-1}}{G_{n-1}+R_{n-1}} \\
&= M_{n-1}
\end{aligned}$$

Since $M_n \geq 0$ and is a martingale, according to Theorem 4, we have $M_n \longrightarrow M_\infty a.s.$ To compute the distribution of the limit, we observe (a) the probability of getting green on the first m draws then red on the next $l = n - m$ draws is

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{1+(m-1)}{2+(m-1)} \cdot \frac{1}{2+m} \cdots \frac{1+(l-1)}{2+(n-1)}$$

and (b) any other outcome of the first n draws with m green balls drawn and l red balls drawn has the same probability since the denominator remains the same and the numerator is permuted. It follows from (a) and (b) that

$$\mathbb{P}[G_n = m+1] = \binom{n}{m} \frac{m!(n-m)!}{(n+1)!} = \frac{1}{n+1}$$

so that

$$\mathbb{P}[M_n \leq x] = \frac{\lfloor x(n+2)-1 \rfloor}{n+1} \longrightarrow x$$

3.3 Radon-Nikodym Derivatives

Martingale convergence theorem can be applied to prove the Radon-Nikodym theoren, which states that if μ and ν are $\sigma - finite$ measures on a measurable space $(\Omega, \mathcal{F})$ and ν is absolutely continuous with respect to μ then there exists a non-negstive and measurable $f : \Omega \longrightarrow \mathbb{R}$ such that $\mu(A) = \int_A f d\mu$ for all measurable sets A.

As any σ-finite measure is equivalent to a probability measure, it is enough to prove the result in the case where μ and ν are probability measures. Furthermore, by the Jordan decomposition, the result generalizes to the case where $\nu$is a signed measure. So, we just need to prove the following.

Theorem 7. Radon-Nikodym *Let P and Q be probability measures on the measurable space $(\Omega, \mathcal{F})$, such that Q is absolutely continuous with respect to P. Then, there exists a non-negative random variable X such that $E_p[X] = 1$ and $Q(A) = E_p[1_A X]$ for every $A \in \mathcal{F}$*

First, the easy case. For a finite σ-algebra, the Radon-Nikodym derivative can be written out explicitly.

Lemma 2. *If $\mathcal{G}$ is a finite sub-σ-algebra of $\mathcal{F}$ then the X_G exists.*

Next, martingale convergence is used to prove the existence of the Radon-Nikodym derivative in the case where the *sigma*-algebra $\mathcal{G}$ is separable. By separable, we mean that there is a countable sequence of sets $A_1, A_2, \dots$ generating $\mathcal{G}$.

Lemma 3. *If $\mathcal{G}$ be a separable sub-σ-algebra of $\mathcal{F}$ then the Radon-Nikodym derivative X_G exists.If furthermore, G_n is an increasing sequence of finite $\sigma-algebras$ satisfying $\mathcal{G} = \sigma(\bigcup_n \mathcal{G}_n)$ then $E_p[|X_G - X_{G_n}|] \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof. Set $X_n \equiv X_{\mathcal{G}_n}$. If$m < n$ then the conditional expectation $\mathbb{E}_p[X_n|\mathcal{G}_m]$ is $\mathcal{G}_m$-measurable, and for every $A \in \mathcal{G}_m$,

$$\mathbb{E}_p[1_A \mathbb{E}_p[X_n|\mathcal{G}_m]] = \mathbb{E}_p[1_A X_n] = \mathbb{Q}(A)$$

This equality just uses the definition of the conditional expectation and then the definition of X_n as the Radon-Nikodym derivative restricted to $\mathcal{G}_n$. So, $\mathbb{E}_P[X_n|\mathcal{G}_m]$ is the Radon-Nikodym derivative restricted to $\mathcal{G}_m$ and equals X_m a.s.Therefore, X_n is a martingale and the martingale convergence theorem implies that the limit $X_{\mathcal{G}} = \lim_{n\to\infty} X_n$ exists almost surely. We now show that the sequence X_n is uniformly integrable. Choose any $\epsilon > 0$. As $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, there exists a $\delta > 0$ such that $\mathbb{Q}(A) < \epsilon$ whenever $\mathbb{P}(A) < \delta$. Using

$$\mathbb{P}(X_n > K) = \mathbb{E}_P[1_{\{X_n > K\}}] \leq \mathbb{E}_P[\frac{X_n}{K}] = \frac{1}{K}$$

since $\mathbb{P}(X_n > K)$ whenever $K > \delta^{-1}$ and therefore, $\mathbb{Q}(X_n > K) < \epsilon$. So $\mathbb{E}_P[X_n 1_{\{X_n > K\}}] = \mathbb{Q}(X_n > K) \leq \epsilon$ for every n, showing that X_n is a uniformly integrable sequence with respect to $\mathbb{P}$. Therefore, convergence in $X_{\mathcal{G}} = \lim_{n\to\infty} X_n$ is in L^1, and $\mathbb{E}_P[|x_n - X_{\mathcal{G}}|] \longrightarrow 0$ as $n \longrightarrow \infty$. So, for any $A \in \bigcup_n \mathcal{G}_n$,

$$\mathbb{E}_P[X_{\mathcal{G}} 1_A] = \lim_{m\to\infty} \mathbb{E}_P[X_m 1_A] = \mathbb{Q}(A)$$

By linearity and the monotone convergence theorem, the collection of sets A satisfying the above is a Dynkin system containing the π-system $\bigcup_n \mathcal{G}_n$ so by Dynkin's lemma, is satisfied for every $A \in \sigma(\bigcup_n \mathcal{G}_n) = \mathcal{G}$and by definition: $X_{\mathcal{G}}$ is the Radon-Nikodym derivative restricted to $\mathcal{G}$. □

Finally, by approximating by finite σ-algebras we can prove the Radon-Nikodym theorem for arbitrary inseparable σ-algebras $\mathcal{F}$. For most σ-algebra we met is separable, the proof of the final general Radon-Nikodym won't be shown here, more details can be seen at [2] [5], which is the reference of this part.

4 About Joseph L. Doob

Joseph Leo "Joe" Doob (February 27, 1910 –June 7, 2004) was an American mathematician, specializing in analysis and probability theory.The theory of martingales was developed by Doob.

Doob was born in Cincinnati, Ohio, February 27, 1910, the son of a Jewish couple, Leo Doob and Mollie Doerfler Doob. The family moved to New York City before he was three years old. The parents felt that he was underachieving in grade school and placed him in the Ethical Culture School, from which he graduated in 1926. He then went on to Harvard where he received a BA in 1930, an MA in 1931, and a PhD (Boundary Values of Analytic Functions, advisor Joseph L. Walsh) in 1932. After postdoctoral research at Columbia and Princeton, he joined the Department of Mathematics of the University of Illinois in 1935 and served until his retirement in 1978. He was a member of the Urbana campus's Center for Advanced Study from its beginning in 1959. During the Second World War, he worked in Washington, D. C. and Guam as a civilian consultant to the Navy from 1942 to 1945; he was at the Institute for Advanced Study for the academic year 1941–1942 [1] when Oswald Veblen approached him to work on mine warfare for the Navy. [3]

Why was martingale be translated in Chinese "鞅"?

它是一开始就造出来的指马鞍和某种游戏的专有名词，是 Doob 等数学家的恶趣味，让马缰绳与「控制平衡、公平」等性质的数学结构产生联系。但无论如何，中文「鞅」，取的的确是马缰绳之类的玩意儿的意思，和英文/法文的马缰绳对上了，同时，「夫焉得不感天之仁爱，阴使中外和会，救黄人将亡之种以脱独夫民贼之鞅轭乎？(谭嗣同《仁学》三五)」这里的鞅和轭都是马驾具，比喻「控制」。至于赌博，还能有「赌马」产生千丝万缕的联系⋯⋯语言嘛，有巧合就是好的、可爱的！

这里就只是作为整个 easay 的一个小插曲哈哈。

References

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