Independent Normal Distribution Variable Sequence

张志成 518030910439

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Problem 1 (Exercise 4.5 of *Chapter E*)
Background:

A random variable G *has a normal* $N(0,1)$ *distribution, then for* $x > 0$*,*

$$P(G > x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}y^2} dy$$

Note that this property only is all we need regarding random variable with normal distribution in this problem.

1. *Prove that*
$$P(G > x) \leqslant \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

2. *Let* $X_1, X_2, \ldots$ *be a sequence of independent* $N(0,1)$ *variables. Prove that with probability* 1*,* $L \leqslant 1$*, where*
$$L := \lim\sup(\frac{X_n}{\sqrt{2 \log n}})$$

Proof:

1. This is proven by manipulating the integral.

$$\begin{aligned}
P(G > x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}y^2} dy \\
&\leqslant \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot \int_x^\infty e^{-\frac{1}{2}y^2} \cdot y \cdot dy \\
&= \frac{1}{x\sqrt{2\pi}} \int_{\frac{1}{2}x^2}^\infty e^{-y} \cdot dy \\
&= \frac{1}{x\sqrt{2\pi}} (-e^{-y}) \Big|_{\frac{1}{2}x^2}^\infty \\
&= \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}
\end{aligned}$$

2. Let E_n denote the event that $\frac{X_n}{\sqrt{2\log n}} > \sqrt{1+\epsilon}$.

$$\begin{aligned}
\sum_{i\in\mathbb{N}} P(E_i) &= \sum_{i\in\mathbb{N}} P(\frac{X_n}{\sqrt{2\log n}} > \sqrt{1+\epsilon}) = \sum_{i\in\mathbb{N}} P(X_n > (\sqrt{1+\epsilon})\sqrt{2\log n}) \\
&= \sum_{i\in\mathbb{N}} \frac{1}{\sqrt{2\pi}} \int_{(\sqrt{1+\epsilon})\sqrt{2\log n}}^{\infty} e^{-\frac{1}{2}y^2} dy \\
&\leqslant \sum_{i\in\mathbb{N}} \frac{1}{(\sqrt{1+\epsilon})\sqrt{2\log n}\sqrt{2\pi}} e^{-\frac{1}{2}((\sqrt{1+\epsilon})\sqrt{2\log n})^2} \\
&= \sum_{i\in\mathbb{N}} \frac{1}{(\sqrt{1+\epsilon})\sqrt{2\log n}\sqrt{2\pi}} e^{-(1+\epsilon)\log n} \\
&< \sum_{i\in\mathbb{N}} \frac{1}{2\sqrt{(1+\epsilon)\pi}} \cdot \frac{1}{n^{1+\epsilon}}
\end{aligned}$$

$\sum_{i\in\mathbb{N}} \frac{1}{n^{1+\epsilon}}$ converges, thus

$$\sum_{i\in\mathbb{N}} P(E_i) < \infty$$

By **First Borel-Cantelli Lemma(BC1)**, we have

$$P(E_n, \text{i.o.}) = 0$$

, thus

$$P(E_n^c, ev) = P(\frac{X_n}{\sqrt{2\log n}} \leqslant \sqrt{1+\epsilon}, ev) = 1$$

Finally,

$$\begin{aligned}
P(L \leqslant 1) &= P(\lim\sup(\frac{X_n}{\sqrt{2\log n}}) \leqslant 1) = P(\frac{X_n}{\sqrt{2\log n}} \leqslant 1, ev) \\
&= \lim_{\epsilon\to 0} P(\frac{X_n}{\sqrt{2\log n}} \leqslant \sqrt{1+\epsilon}, ev) \\
&= 1
\end{aligned}$$

□