$\sigma(Y) = Y^{-1}(\mathcal{B})$
&
$\sigma(Y)$ can be generated by $\pi(Y)$

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Exercise 1. *Show that* $\sigma(Y) = Y^{-1}(\mathcal{B}) := (\{\omega\colon Y(\omega) \in B\}\colon B \in \mathcal{B})$.

Proof. Let (S, Σ) be a measurable space. It is already known that

$$\begin{aligned}\sigma(Y) &:= \sigma(Y^{-1}(\mathcal{B}))\\ &= \sigma(Y^{-1}(B)\colon B \in \mathcal{B})\\ &= \sigma(\{\omega\colon Y(\omega) \in B\}\colon B \in \mathcal{B}).\end{aligned}$$

So we only need to show that $\Sigma_0 := Y^{-1}(\mathcal{B}) = \{Y^{-1}(B)\colon B \in \mathcal{B}\}$ itself is a σ-algebra. By definition we need to prove two properties of Σ_0:

1. $S_0 \in \Sigma_0 \Rightarrow S_0^c \in \Sigma_0$, and
2. $(S_i)_{i\in\mathbb{N}} \subseteq \Sigma_0 \Rightarrow \bigcup_{i\in\mathbb{N}} S_i \in \Sigma_0$.

Since Y is a random variable which is a Σ-measurable function by definition, the mapping Y^{-1} satisfies that

$$Y^{-1}(A^c) = (Y^{-1}(A))^c, Y^{-1}(\bigcup_{\alpha} A_\alpha) = \bigcup_{\alpha} Y^{-1}(A_\alpha)$$

where $A, A_\alpha \in \mathcal{B}$. Thus we have

$$\begin{aligned}S_0 \in \Sigma_0 &\Rightarrow S_0 \in \{Y^{-1}(B)\colon B \in \mathcal{B}\}\\ &\Rightarrow \exists B_{S_0} \in \mathcal{B} \text{ s.t. } Y^{-1}(B_{S_0}) = S_0\\ &\Rightarrow Y^{-1}({B_{S_0}}^c) = (Y^{-1}(B_{S_0}))^c = {S_0}^c\\ &\Rightarrow {S_0}^c \in \Sigma_0,\end{aligned}$$

and

$$\begin{aligned}(S_i)_{i\in\mathbb{N}} \subseteq \Sigma_0 &\Rightarrow \exists (B_i)_{i\in\mathbb{N}} \text{ s.t. } \forall i \in \mathbb{N}, Y^{-1}(B_i) = S_i\\ &\Rightarrow Y^{-1}(\bigcup_{i\in\mathbb{N}} B_i) = \bigcup_{i\in\mathbb{N}} Y^{-1}(B_i) = \bigcup_{i\in\mathbb{N}} S_i\\ &\Rightarrow \bigcup_{i\in\mathbb{N}} S_i \in \Sigma_0.\end{aligned}$$

Therefore, $\Sigma_0 = \{Y^{-1}(B)\colon B \in \mathcal{B}\}$ is a σ-algebra, which means $\sigma(Y) = \sigma(Y^{-1}(\mathcal{B})) = Y^{-1}(\mathcal{B})$. $\square$

Exercise 2. *$\sigma(Y)$ can be generated by the π-system*

$$\pi(Y) := (\{\omega\colon Y(\omega) \leqslant x\}\colon x \in \mathbb{R}) = Y^{-1}(\pi(\mathbb{R})).$$

Proof. Let (S, Σ) be a measurable space. We know that

$$\begin{aligned}\sigma(Y) &= Y^{-1}(\mathcal{B}) = (\{\omega \in S\colon Y(\omega) \in B\}\colon B \in \mathcal{B}),\\ \sigma(\pi(Y)) &= \sigma(Y^{-1}(\pi(\mathbb{R}))) = \sigma(\{\omega \in S\colon Y(\omega) \leqslant x\}\colon x \in \mathbb{R}).\end{aligned}$$

From the Σ-measurability of Y, we know that Y^{-1} preverves all set operations, which is so powerful for us to finish our proof.

Since $\pi(\mathbb{R}) \subseteq \mathcal{B}$ and $\mathcal{B} = \sigma(\pi(\mathbb{R}))$, it follows that $Y^{-1}(\pi(\mathbb{R})) \subseteq Y^{-1}(\mathcal{B})$. And from the result of the previous exercise we know that $Y^{-1}(\mathcal{B})$ is a σ-algebra. So the σ-algebra generated by $Y^{-1}(\pi(\mathbb{R}))$ is a sub-σ-algebra of $Y^{-1}(\mathcal{B})$, i.e., $\sigma(\pi(Y)) = \sigma(Y^{-1}(\pi(\mathbb{R}))) \subseteq \sigma(Y)$.

Next we show that $Y^{-1}(\mathcal{B}) \subseteq \sigma(Y^{-1}(\pi(\mathbb{R})))$. Equivalently we show that for $\forall B \in \mathcal{B}, Y^{-1}(B) = \{\omega \in S\colon Y(\omega) \in B\} \in \sigma(Y^{-1}(\pi(\mathbb{R}))) = \sigma(\{\omega \in S\colon Y(\omega) \leqslant x\}\colon x \in \mathbb{R})$. In the next several steps we use the Σ-measurability of Y implicitly or explicitly.

Note that any Borel set can be obtained by a set of countable open sets of the usual topology on $\mathbb{R}$. And every open set on $\mathbb{R}$ is a countable union of open intervals. Any open interval $I = (a, b)$ can be written as $\bigcup_{n\in\mathbb{N}}(a, b - \frac{b-a}{2n}]$. Then we have

$$\begin{aligned}Y^{-1}(I) &= Y^{-1}(\bigcup_{n\in\mathbb{N}}(a, b - \frac{b-a}{2n}])\\ &= \bigcup_{n\in\mathbb{N}} Y^{-1}((a, b - \frac{b-a}{2n}])\\ &= \bigcup_{n\in\mathbb{N}} Y^{-1}((-\infty, b - \frac{b-a}{2n}] \setminus (-\infty, a])\\ &= \bigcup_{n\in\mathbb{N}} (Y^{-1}((-\infty, b - \frac{b-a}{2n}]) \setminus Y^{-1}((-\infty, a]))\\ &\in \sigma(Y^{-1}(\pi(\mathbb{R}))).\end{aligned}$$

Hence every open interval belongs to $\sigma(Y^{-1}(\pi(\mathbb{R})))$, which equivalently means $Y^{-1}(\mathcal{B}) \subseteq \sigma(Y^{-1}(\pi(\mathbb{R})))$ by our previous analysis.

Therefore, $\sigma(Y) = \sigma(\pi(Y))$, i.e., $\sigma(Y)$ can be generated by $\pi(Y)$. $\square$