Events and Basic Properties

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1 Events

We have a probability space $(\Omega, \mathcal{F}, P)$.

Ω The origin set, called sample space.

$\omega \in \Omega$ ω is the element in sample space, called sample.

Event $\mathcal{F}$ is the events on the space Ω. Every event $E \in \mathcal{F}$ is a $\mathcal{F}$-measurable subset of Ω.

1.1 Intuitive Understanding

Ω is the biggest set, we choose an arbitrary element ω in it. Then $P(\mathcal{F})$ is the probability that ω is in $\mathcal{F}$. That is to say, we can consider Ω as the all possible cases, and $\mathcal{F}$ is the set of those cases we have to calculate their probability. The way to calculate probability is the mapping $P : \mathcal{F} \mapsto [0, +\infty)$.

2 Monotone convergence of measure

$F_n \uparrow F$ Let $\{F_n\}$ as a sequence of sets which satisfy

$$F_n \subseteq F_{n+1} (\forall n \in \mathbb{N}), \bigcup F_n = F.$$

$f_n \uparrow f$ Let f_n as a sequence of real numbers which satisfy

$$f_n \leq f_{n+1} (\forall n \in \mathbb{N}), \lim_{n \to \infty} f_n = f.$$

Theorem 1. *(a) If $F_n \in \Sigma (n \in \mathbb{N})$ and $F_n \uparrow F$, then $\mu(F_n) \uparrow \mu(F)$.*
(b) If $G_n \in \Sigma (n \in \mathbb{N}), \mu(G_k) < \infty$ for some k and $G_n \downarrow G$, then $\mu(G_n) \downarrow \mu(G)$.

Proof. (a) Denote $G_1 := F_1, G_n := F_n \setminus F_{n-1} (n \geq 2)$, then G_n mutually disjoint. And we have

$$\mu(F_n) = \mu(G_1 \cup G_2 \cup \cdots \cup G_n) = \Sigma_{k\leq n}\mu(G_n)$$

Consider the sequence $\{\Sigma_{k\leq n}\mu(G_k)\}$, we have

$$\Sigma_{k\leq n}\mu(G_k) \leq \Sigma_{k\leq n+1}\mu(G_k), \text{ and } \lim_{n\to\infty} \Sigma_{k\leq n}\mu(G_k) = \Sigma_{k<\infty}\mu(G_k).$$

Thus we can get

$$\mu(F_n) = \Sigma_{k\leq n}\mu(G_n) \uparrow \Sigma_{k<\infty}\mu(G_k) = \mu(F).$$

(b) Define $F_n = G_k \setminus G_{k+n}$, then use (a) to prove it. □

Corollary 1. ***(Continuity from below)*** *If $F_n \in \Sigma$, $F_n \subseteq F_{n+1}, n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \mu(F_n) = \mu(\bigcup_{n\to\infty} F_n) = \mu(\lim_{n\to\infty} F_n).$$

Corollary 2. ***(Continuity from above)*** *If $G_n \in \Sigma$, $G_n \supseteq G_{n+1}, n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \mu(G_n) = \mu(\bigcap_{n\to\infty} G_n) = \mu(\lim_{n\to\infty} G_n)$$

provided there exists a k with $\mu(G_k) < \infty$.

Theorem 2. *If $\mu(F_n) = 0$, then $\mu(\bigcup_{n=1}^{\infty} F_n) = 0$.*

2.1 lim,sup/inf and ↓,↑

Discuss the real number sequence $\{x_n\}$ here.

Denote

$$\limsup x_n := \inf_m\{\sup_{n\geq m} x_n\} =\downarrow \lim_m\{\sup_{n\geq m} x_n\} \in [-\infty, \infty].$$

It is clear that the sequence $\sup_{n\geq m} x_n$ satisfy:

$$\sup_{n\geq m} x_n \geq \sup_{n\geq m+1} x_n.$$

Thus the limit of it exists.

Similarly, denote

$$\liminf x_n := \sup_m\{\inf_{n\geq m} x_n\} =\uparrow \lim_m\{\inf_{n\geq m} x_n\} \in [-\infty, \infty].$$

Theorem 3. *x_n converge to $[-\infty, \infty]$* $\Leftrightarrow \limsup x_n = \liminf x_n$.

In order to describe probability in the future, we discuss the relation between a real number z with $\limsup x_n$.

1. If $z > \limsup x_n$, then x_n will **eventually** less than z, i.e., $x_n < z$ when $n > N$.
2. If $z < \limsup .x_n$, then x_n will **infinitely often** times greater than z, i.e., $x_n > z$ for infinite n.

3 a.s. and i.o.

a.s. If

$$F := \{\omega : S(\omega)\text{is true}\} \in \mathcal{F}, P(F) = 1,$$

then the statement S is almost sure.

Lemma 1. *Set S and S^C are both in $\mathcal{F}$, then $P(S) + P(S^C) = 1$.*

Proof. For $P(\mathcal{F}) = 1, S$ and S^C are disjoint, then $P(S) + P(S^C) = P(S \cup S^C) = 1$. □

Theorem 4. *If $F_n \in \mathcal{F}$ and $\forall n P(F_n) = 1$, then $P(\bigcap_n F_n) = 1$.*

Proof. For $P(F_n)^C = 0$, we have

$$P(\bigcap_n F_n) = P((\bigcup_n F_n^C)^C) = 1 - P(\bigcup_n F_n^C) = 1.$$

□

Consider the event sequence E_n.

i.o. We define

$$\begin{aligned}(E_n, \text{i.o.}) &= (\text{events that happen in infinite } E_n) \\ &:= \limsup E_n \\ &:= \bigcap_m \bigcup_{n \geq m} E_n \\ &= \{\omega \mid \forall m, \exists k \geq m, s.t. \omega \in E_k\} \\ &= \{\omega \mid \omega \in E_n \text{ for infinite n}\}\end{aligned}$$

Theorem 5.

$$P(\limsup E_n) \geq \limsup P(E_n).$$

Proof. Define $G_m := \bigcup_{n \geq m} E_n$, and $G_m \downarrow G := \limsup E_n$. According to continuity from below,

$$P(G) =\downarrow \lim_{m\to\infty} P(G_m) =\downarrow \lim_{m\to\infty} P(\bigcup_{n\geq m} E_n).$$

It is clear that $P(\bigcup_{n\geq m} E_n) \geq \sup P(E_n)$, thus

$$P(\limsup E_n) = P(G) \geq \limsup P(E_n).$$

□

ev We write (E_n, ev) as

$$\begin{aligned}(E_n, ev) &:= (E_n \text{ will eventually happens}) \\ &:= \liminf E_n \\ &:= \bigcup_m \bigcap_{n \geq m} E_n \\ &= \{\omega \mid \text{for some m}, \omega \in E_n \text{ for every } n \geq m\}.\end{aligned}$$

Theorem 6.

$$P(\liminf E_n) \leq \liminf P(E_n).$$

Proof. Similar to Theorem 5. □

4 BC1

Theorem 7. ***(BC1)*** *The event sequence E_n satisfies $\Sigma_n P(E_n) < \infty$. Then*

$$P(\limsup E_n) = P(E_n, i.o.) = 0.$$

Proof. Let $G_m = \bigcup_{n \geq m} E_n$ and $G_m \downarrow G = \bigcap_m G_m$. Then we have

$$P(\limsup E_n) = P(G) \leq P(G_m) = P(\bigcup_{n \geq m} E_n) \leq \Sigma_n P(E_n).$$

When $m \to \infty$, $\Sigma_{n \geq m} P(E_n) = 0$. □

4.1 Aplication

Use BC1 to prove Discrete-time Poincaré's Recurence Theorem.

Theorem 8. *Let T be a measure-preserving transformation on a probability space $(\Omega, \mathcal{F}, P)$. Then. for any $E \in \mathcal{F}$ with $P(E) > 0$, almost all points of E returns to E infinitely often under positive iterations by T.*

Proof. For each $n \in \mathbb{N}$, let $A_n := \{x \in E \mid x \notin T^{-kn}(E), \forall k \in \mathbb{N}\} = E \backslash \bigcup_{k \geq 1} T^{-kn}(E)$. $T^{-kn}(E)$ is an event for E is an event and T is measure-preserving. Thus A_n is an event as well.

$A_n, T^{-1}(A_n) \ldots$ are pairwise disjoint. And for T is measure-preserving

$$P(A_n) = P(T^{-1}(A_n) = \cdots = 0.$$

And for $P(\limsup A_n) \leq P(\bigcup A_n) \leq \Sigma P(A_n) = 0$, $P(\limsup A_n)$ obtains 0. It is clear that $\forall x \in E \backslash \limsup A_n, x$ visits E infinitely often under positive iterations by T. For $|(E \backslash \limsup A_n)| = P(E) = 1$, we can conclude that almost surely all points in E are returning to E infinitely often.

□