

Usual length function is countably additive

董海辰 518030910417

March 20, 2020

┌ 1

Let $S = (0, 1]$, let Σ_0 be the subsets of S which are finite unions of disjoint left-open right-closed intervals. It is clear that Σ_0 is an algebra.

Let μ_0 be the usual length function on Σ_0 . Show that μ_0 is countably additive. ┐

Proof. Consider the disjoint sets $A_1, A_2, \dots, A_n, \dots$ with $\forall i \in \mathbb{N}_+, A_i \in \Sigma_0$ and $A = \bigcup_{i \in \mathbb{N}_+} A_i \in \Sigma_0$.

We would like that to show that

$$\mu_0(A) = \sum_{i \in \mathbb{N}_+} \mu_0(A_i).$$

By the definition of Σ_0 , we have $A_i = \bigcup_{j=1}^{n_i} I_{ij}$ where I_{ij} are left-open right-closed intervals and are disjoint.

Then $A = \bigcup_{i \in \mathbb{N}_+} A_i = \bigcup_{i \in \mathbb{N}_+} \bigcup_{j=1}^{n_i} I_{ij}$, that is a countable union of finite union of sets, which is also countable. By renaming, let $A = \bigcup_{i \in \mathbb{N}_+} I_i$, where I_i are left-open right-closed and disjoint.

μ_0 satisfies finite additivity. Thus for all $N \in \mathbb{N}_+$,

$$\sum_{i=1}^N \mu_0(I_i) = \mu_0\left(\bigcup_{i=1}^N I_i\right).$$

And we have $\mu_0(\bigcup_{i=1}^{N+1} I_i) = \mu_0(\bigcup_{i=1}^N I_i) + \mu_0(I_{N+1}) \geq \mu_0(\bigcup_{i=1}^N I_i)$, this implies

$$\sum_{i=1}^{\infty} \mu_0(I_i) \leq \mu_0(A).$$

On the other hand, note that $\forall I_i \neq \emptyset, \mu_0(I_i) = \mu_0((a_i, b_i]) = b_i - a_i > 0$. That is,

$$\forall \varepsilon > 0, \exists I'_i := (a_i, b_i + \frac{\varepsilon}{2}) \in \Sigma_0, I_i \subseteq I'_i, \mu_0(I'_i) - \mu_0(I_i) = \frac{\varepsilon}{2} < \varepsilon.$$

Also, we have $A \in \Sigma_0$, similarly, for a fixed $\varepsilon > 0$, we can have $[l, r] \subseteq A$ with $\mu_0(A) - \mu_0([l, r]) < \varepsilon$.

From this we can get a infinite open cover of $[l, r]$. By the property of \mathbb{R} , there exists a finite open cover $I'_{k_1}, I'_{k_2}, \dots, I'_{k_M}$ such that $M \subseteq \bigcup_{i=1}^M I'_{k_i}$. Then by the elementary inequality,

$$\begin{aligned} \mu_0(A) - \varepsilon &< \mu_0([l, r]) = \mu_0\left(\bigcup_{i=1}^M I_{k_i}\right) \\ &\leq \sum_{i=1}^M \mu_0(I'_{k_i}) \\ &\leq \sum_{i=1}^M (\mu_0(I_{k_i}) + \varepsilon_i) \\ &\leq \sum_{i=1}^{\infty} \mu_0(I_i) + \sum_{i=1}^M \varepsilon_i. \end{aligned}$$

Let $\varepsilon_i = \frac{\varepsilon}{2^i}$, finally We get $\mu_0(A) \leq \sum_{i=1}^{\infty} \mu_0(I_i) + 2\varepsilon$ for all $\varepsilon > 0$.

Combine the two sides, resulting in

$$\mu_0(A) = \sum_{i \in \mathbb{N}_+} \mu_0(A_i).$$

That finished the proof for the countable additivity of μ_0 .

□