Notes of Probability Theory

董海辰 518030910417

with help from 毛昕渝

March 12, 2020

1: Definitions

- From digits(tree paths) to intervals.
- normal numbers
- A subset A of $\mathbb{R}$ is *negligible* if $\forall \varepsilon > 0$, there exists a finite or countable collection $I_1, I_2, \cdots$ of (possibly overlapping) intervals satisfying: $A \subset \bigcup_k I_k$ and $\sum_k |I_k| < \varepsilon$.

Every path of the binary tree is a real number in binary form.

Two numbers are equal if there are no paths between: e.g. $0.010000\cdots$ and $0.0011111\cdots$.

$P(\omega : d(i,\omega) = u_i, i = 1,2,\cdots,n) = 2^{-n}$.

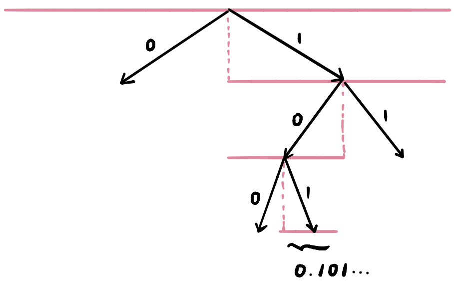

Figure 1: digits to intervals

2: Weak Law of Large Numbers

$$\forall \varepsilon > 0, \lim_{n\to\infty} P(\omega \in \Omega : |\frac{\sum_{i=1}^n d_i(\omega)}{n} - \frac{1}{2}| \geq \varepsilon) = 0.$$

For each n, the P can be calculated by summing up the lengths of every interval that meet the condition.

Let $r_i(\omega) = 2d_i(\omega) - 1$.

> $r_i(\omega)$ are Orthonomal basis, i.e.
>
> $$\int_\Omega r_i(\omega)r_j(\omega)\mathrm{d}w = \delta_{i,j}.$$
>
> If $i \neq j$, say $i < j$, there are equal number if intervals that $r_j(\omega) = 1$ and $r_j(\omega) = -1$, resulting in sum 0.

Let $S_n(\omega) = \sum_{i=1}^n r_i(\omega)$,

$$\begin{aligned} P[\omega : |S_n(\omega)| \geq 2n\varepsilon] &\leq \frac{1}{4n^2\varepsilon^2}\int_0^1 S_n^2(\omega)\mathrm{d}\omega \\ &= \frac{1}{4n^2\varepsilon^2}n = \frac{1}{4n\varepsilon^2} \to 0. \end{aligned}$$

> Here applied the Chebyshev's Inequality:
>
> $$P[|X - E(X)| \geq b] \leq \frac{Var(X)}{b^2}.$$
>
> And note that $r_i(\omega)$ are orthogonal,
>
> $$Var(S_n(\omega)) = \int_\Omega S_n^2(\omega)\mathrm{d}\omega = \int_\Omega (\sum r_i(\omega))^2\mathrm{d}\omega = \int_\Omega r_i^2(\omega)\mathrm{d}\omega = n.$$

3: Strong Law of Large Numbers

Borel's Normal Number Theorem
The complement of the set of normal numbers to base 2 is uncountable but negligible.

$$\mathcal{N} = \{\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = 0\}.$$

$\Omega\backslash\mathcal{N}$ is uncountable, but

$$\forall\varepsilon > 0, \exists I_1, I_2, \cdots I_n \cdots, \Omega\backslash\mathcal{N} \subseteq \bigcup_i I_i, \sum_i |I_i| < \varepsilon.$$

Let $A_n = [\omega : |\frac{S_n(\omega)}{n} \geq \varepsilon_n]$ where $e_n \to 0$. Thus

$$\Omega\backslash\mathcal{N} = \lim_{m\to\infty} \bigcup_{n=m}^{\infty} A_n.$$

> We have
>
> $$\omega \in \mathcal{N} \iff \forall\varepsilon > 0, \exists m > 0, \forall n > m, |\frac{S_n}{n}| < \varepsilon \iff \omega \notin \bigcup_{n=m}^{\infty} A_n.$$
>
> And show $|\Omega\backslash\mathcal{N}| < \sum_{n=m}^{\infty} |A_n| < \varepsilon_n$

$$P[\omega : |\frac{S_n(\omega)}{n}| \geq \varepsilon_n] \leq \frac{1}{n^4\varepsilon_n^4}\int_0^1 S_n^4(\omega)\mathrm{d}\omega$$
$$= \frac{n+3n(n-1)}{n^4\varepsilon_n^4} \leq \frac{3}{n^2\varepsilon_n^4}.$$

> Let $1_{A_n}(\omega) = 1$ for all $\omega \in A_n$ and be 0 otherwise.
> Then $\forall\omega \in \Omega, 1_{A_n}(\omega) \leq \frac{S_n^4(\omega)}{n^4\varepsilon_n^4}$. We can get
>
> $$P[\omega : \omega \in A_n] = \int_\Omega 1_{A_n}(\omega)\mathrm{d}\omega \leq \int_\Omega \frac{S_n^4(\omega)}{n^4\varepsilon_n^4}\mathrm{d}\omega.$$
>
> Similarly, $S_n^4(\omega) = (\sum_{i=1}^n r_i(\omega))^4$. Terms with non-zero values are:
>
> - $r_i^4(\omega) = 1$
> - $r_i^2(\omega)r_j^2(\omega) = 1$
>
> With sum $\binom{n}{1} + \binom{n}{2} \cdot \binom{4}{2} = n + 3n(n-1)$.

$\{e_n\}$ is a sequence with $\lim_{n\to\infty} e_n = 0$, we can let $e_n = n^{-\frac{1}{8}}$, and

$$\sum_{n=1}^{\infty} |A_n| \leq \sum_{n=1}^{\infty} \frac{3}{n^{\frac{3}{2}}}$$

, which is convergent. And intervals of $\bigcup_{n=m}^{\infty} A_n$ is countable. Thus

$$\forall\varepsilon > 0, \exists m > 0, \Omega \backslash \mathcal{N} \subseteq \bigcup_{n=m}^{\infty} A_n, \sum_{n=m}^{\infty} |A_n| < \varepsilon.$$

> If we use the inequality in proof of WLLN: $P[\omega : \omega \in A_n] \leq \frac{1}{n\varepsilon_n^2}$. In this case, whatever $\{e_n\}$ we choose, the sum will never be convergent.