σ-algebra Generated By Stopping Time

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Definition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_n : n \geq 0)$, $X = (X_n : n \geq 0)$ be a process adapted to $(\mathcal{F}_n)$ and T be a stopping time. We define σ-algebra generated by T:*

$$\mathcal{F}_T = \sigma\{A \in \mathcal{F}, A \cap \{T \leq n\} \in \mathcal{F}_n, \forall n\}$$

Next we prove that $\mathcal{F}_T$ is a σ-algebra.

Proof. 1. $\emptyset \cap \{T \leq n\} = \emptyset \in \mathcal{F}_n$ implies $\emptyset \in \mathcal{F}_T$.

2. $\Omega \cap \{T \leq n\} = \{T \leq n\} \in \mathcal{F}_n$ implies $\Omega \in \mathcal{F}_T$.

3. If $A \in \mathcal{F}_T$, then $A \cap \{T \leq n\} \in \mathcal{F}_n$ for every n.
   Thus $A^c \cap \{T \leq n\} = (A^c \cap \{T \leq n\})^c \cap \{T \leq n\} \in \mathcal{F}_n$ implies $A^c \in \mathcal{F}_T$.

4. If $A_i \in \mathcal{F}_T$ for every i, then $(A_i \cap \{T \leq n\}) \in \mathcal{F}_n$ for every n.
   Thus $(\bigcup_i A_i) \cap \{T \leq n\} = \bigcup_i (A_i \cap \{T \leq n\}) \in \mathcal{F}_n$ implies $\bigcup_i A_i \in \mathcal{F}_n$.

Hence $\mathcal{F}_T$ is a σ-algebra. $\square$

Lemma 2. *If S and T are stopping times such that $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.*

Proof. For every $A \in \mathcal{F}_S$, we have

$$A \cap \{S \leq n\} \in \mathcal{F}_n, \forall n$$

Thus we have

$$A \cap \{T \leq n\} = \{A \cap \{S \leq n\}\} \cap \{T \leq n\} \in \mathcal{F}_n, \forall n$$

So $A \in \mathcal{F}_T$. This implies $\mathcal{F}_S \subset \mathcal{F}_T$. $\square$

Lemma 3. *If S and T are **bounded** stopping times such that $S \leq T$ and $X = (X_n : n \geq 0)$ is a martingale, then $\mathbb{E}[X_T|\mathcal{F}_S] = X_S$,a.s.*

Proof. As S and T are bounded, there exists $N \in \mathbb{N}$ such that $S \leq T \leq N$.

Firstly we prove that $\mathbb{E}[X_N \mathbb{1}_A] = \mathbb{E}\left[\mathbb{E}\left[X_N|\mathcal{F}_S\right]\mathbb{1}_A\right]$ for every $A \in \mathcal{F}_S$. By the definition of the conditional expectation, we have

$$\int_A X_N dP = \int_A \mathbb{E}[X_N|\mathcal{F}_S]dP$$

Hence

$$\mathbb{E}[X_N; A] = \mathbb{E}\left[\mathbb{E}\left[X_N|\mathcal{F}_S\right]; A\right]$$

This implies $\mathbb{E}[X_N \mathbb{1}_A] = \mathbb{E}\left[\mathbb{E}\left[X_N|\mathcal{F}_S\right]\mathbb{1}_A\right]$.

For every $A \in \mathcal{F}_S$, we have

$$\begin{aligned} \mathbb{E}[X_N \mathbb{1}_A] = & \ \mathbb{E}\left[\mathbb{E}\left[X_N|\mathcal{F}_S\right]\mathbb{1}_A\right] \\ = & \ \sum_{i=1}^{N} \mathbb{E}\left[\mathbb{E}\left[X_N|\mathcal{F}_i\right]\mathbb{1}_A \mathbb{1}_{S=i}\right] \\ = & \ \sum_{i=1}^{N} \mathbb{E}\left[X_i \mathbb{1}_A \mathbb{1}_{S=i}\right] \\ = & \ \mathbb{E}\left[X_S \mathbb{1}_A\right] \end{aligned}$$

The above equation implies that $\int_A X_N dP = \int_A X_S dP$ for every $A \in \mathcal{F}_S$. Thus $\mathbb{E}[X_N|\mathcal{F}_S] = X_S$. In the similar way, we have $\mathbb{E}[X_N|\mathcal{F}_T] = X_T$. Thus we can conclude that

$$\mathbb{E}[X_T|\mathcal{F}_S] = \mathbb{E}\left[\mathbb{E}\left[X_N|\mathcal{F}_T\right]|\mathcal{F}_S\right] = \mathbb{E}[X_N|\mathcal{F}_S] = X_S$$

□