The Changing of Measure

A journey beginning with Radon-Nikodym Theorem

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1 Introduction

In class, we have learned the famous Radon-Nikodym Theorem. It can be interpreted as the derivative of two measures But I have always wondered where can this theorem be applied.

The main work I have done in this essay is discussing the change of measure which is an extension of the course material by intrepreting the Radon-Nikodym theorem as an elegant way of constructing a new probability measure. Futhermore, I utilize the tool of Brownian Motion and provide a real life application where changing of measure is important.

I think this essay has at least answered my question about Radon-Nikodym Theorem and provides some background knowledge about Mathematical Finance.

For the structure of the essay, I first revisit the Radon-Nikodym Theorem in a bit more depth, and introduce a powerful tool-Brownian Motion. Then I talks about one motivation behind the changing of measure, that is to change the physical measure into the risk-neutral measure. We could accomplish this with the help of Radon-Nikodym Theorem and a more powerful Girsanov's Theorem.

2 Revisiting the Radon-Nikodym Theorem

Theorem 1: Radon-Nikodym Theorem

Given two equivalent(defined later) probability measures $\mathbb{P}$ and $\mathbb{Q}$ constructed on the measurable space $(\Omega, \mathcal{F})$, there exists a nonnegative-valued random variable X such that

$$\mathbb{Q}(A) = E^{\mathbb{P}}[X; A], \quad \forall A \in \mathcal{F}.$$

And the random-variable X is called the derivative of $\mathbb{P}$ and $\mathbb{Q}$, often written as $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

2.1 Constructing a new probability measure

This theorem aleady provides us with an insight of how to construct a new probability measure.

Suppose we are given a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We seek a random variable X with the properties:

1. Non-negative. $\forall \omega \in \Omega \ X(\omega) \geq 0$.
2. $E^{\mathbb{P}}[X; \Omega] = 1$.

We simply set the new measure $\mathbb{Q}$ as the following:

$$\mathbb{Q}(A) := E^{\mathbb{P}}[X; A] = \int_A X d\mathbb{P}, \quad \forall A \in \mathcal{F}.$$

It is not difficult to verify that $\mathbb{Q}$ is indeed a measure.

2.2 Relationship

The Radon-Nikodym Theorem and the above construction looks the same, but is actually different in that

1. Radon-Nikodym Theorem proves the existence of such a random variable X given $\mathbb{P}$ and $\mathbb{Q}$.
2. The construction simply asks for a random variable X and a probability measure $\mathbb{P}$, and a new measure $\mathbb{Q}$ could be constructed.

2.3 Equivalent Measures

Theorem 2: Equivalent Measures

$\mathbb{P}$ and $\mathbb{Q}$ are called equivalent measures if

$$\mathbb{P}(A) = 0 \Longleftrightarrow \mathbb{Q}(A) = 0, \quad \forall A \in \mathcal{F}.$$

This is the existence condition of the Radon-Nikodym derivative, i.e. the density function, for $\mathbb{Q}$ with respect to $\mathbb{P}$.
If only one side holds, e.g. $\mathbb{P}(A) = 0 \Longrightarrow \mathbb{Q}(A) = 0$, we call that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ and vice versa.

If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on a filtration $\{\mathcal{F}_n\}$, then we can a family of densities $X = (X_n)$. Then the following holds:

1. X_n is $\mathcal{F}_n$-measurable, and therefore X_n is an adapted process.
2. X is a martingale.

Proof. We want to prove that

$$E[X_n|\mathcal{F}_{n-1}] = X_{n-1}, \quad a.s.$$

For every $G \in \mathcal{F}_{n-1}$, we have

$$\int_G X_n d\mathbb{P} = \mathbb{Q}(A),$$

and since $G \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$, we also have

$$\mathbb{Q}(A) = \int_G X_{n-1} d\mathbb{P}$$

Therefore, we have X_{n-1} is a version of $E[X_n|\mathcal{F}_{n-1}]$, and therefore X is a martingale. □

3 Some powerful tools

Before we can extend beyond the Radon-Nikodym Theorem, we need some powerful tools about **continuous time**, in which time is represented by $\mathbb{R}^+$ rather than $\mathbb{N}$. This interests us because it could depict an object more precisely than discrete time.

3.1 Brownian Motion

Theorem 3: Brownian Motion

Let $X = \{X_t \mid t \in \mathbb{R}^+\}$. And X is a called a Brownian Motion if the following holds:

1. For a fixed ω, $f(t) = X_t(\omega)$ is a continuous function defined on $\mathbb{R}^+$.
2. For all ω, $X_0(\omega) = 0$.
3. For each $s > 0$, $\{X_t - X_s \mid t \geq s\}$ is independent of $\mathcal{F}_s$.
4. $X_t - X_s$ is $N(0, t-s)$ distributed for $0 \leq s < t$.

Property (1) allows us to define a Brownian Motion's integral over time.

$$Y(\omega) := \int_0^T B_t(\omega) dt$$

Two properties will help us better understand the Brownian Motion and the second one will appear in the Girsanov's Theorem later.

Theorem 4

Brownian motion X is a martingale.

Proof.

$$\begin{aligned} E[X_n|\mathcal{F}_{n-1}] &= E[X_{n-1} + (X_n - X_{n-1})|\mathcal{F}_{n-1}] \\ &= E[X_{n-1}|\mathcal{F}_{n-1}] \\ &= X_{n-1}. \end{aligned}$$

Therefore, X is a martingale. □

Theorem 5

For any $\theta \in \mathbb{R}$, the process $Y_t = e^{\theta X_t - \frac{1}{2}\theta^2 t}$ is a martingale.

Proof.

$$\begin{aligned} E[Y_n|\mathcal{F}_{n-1}] &= E[e^{\theta X_n - \frac{1}{2}\theta^2 n}|\mathcal{F}_{n-1}] \\ &= E[Y_{n-1} \cdot e^{\theta(X_n - X_{n-1}) - \frac{1}{2}\theta^2)}|\mathcal{F}_{n-1}] \\ &= Y_{n-1} \cdot E[e^{\theta(X_n - X_{n-1}) - \frac{1}{2}\theta^2)}|\mathcal{F}_{n-1}] \end{aligned}$$

And since $X_n - X_{n-1}$ is $N(0,1)$ distributed, we have

$$\begin{aligned} E[Y_n|\mathcal{F}_{n-1}] &= Y_{n-1} \cdot E[e^{\theta(X_n - X_{n-1}) - \frac{1}{2}\theta^2)}|\mathcal{F}_{n-1}] \\ &= Y_{n-1} \end{aligned}$$

Therefore, Y is a martingale. □

4 Girsanov's Theorem - A change of measure

4.1 Risk-Neutral Measure

Unlike the natural measure, risk-neutral measure aims to be an objective measure of the asset price. This facilitates the process of representing the value of an asset in a model.

Example 1

Suppose $\mathbb{P}$ is the physical measure and $\mathbb{Q}$ is the risk-neutral measure. Then the *current* price of an asset that pays 1 dollar at time T if event A occurs will be

$$\mathbb{Q}(A) \cdot e^{-rT}.$$

However, if we were to use measure $\mathbb{P}$, we couldn't easily calculate the current price.

Formally, given the asset's price H_T at time T, the today's fair value of an asset H_0 is

$$\begin{aligned} H_0 &= D(0,T) \cdot E^{\mathbb{Q}}[H_T] \\ &= D(0,T) \cdot E^{P}[\frac{d\mathbb{Q}}{d\mathbb{P}} H_T], \end{aligned}$$

where $D(0,T)$ is the discount factor between 0 and T.

It is exactly for this reason that the risk-neutral lies in the very heart of asset pricing theory.

Theorem 6: Fundamental theorem of arbitrage-free pricing

If there is just one risk-neutral measure, then it's *unique*, and thus every asset will have a unique arbitrage-free price.
But if there are more than one risk-neutral measure, then no arbitrage is possible in an interval of prices for each asset. If no such risk-neutral measure exists, then arbitrage is possible, which is generally assumed to be not possible.

Since the risk-neutral measure is so fascinating, we want to change our physical measure to the risk-neutral measure. And this is where the Girsanov's Theorem come into place.

4.2 Girsanov's Theorem

Theorem 7: Girsanov's Theorem

Let $\gamma = \{\gamma_t \mid t \in [0, T]\}$ be an adapted process such that

$$E^{\mathbb{P}}[e^{\frac{1}{2}\int_0^T \gamma_t^2 dt}] < \infty. \quad \text{(Novikov condition)}$$

Then there exists a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$.
2. $$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt}.$$
   However, due to limited length of the essay, Itô's integral will not be defined explicitly here. A not rigorous enough definition could help the understanding:
   $$\int_0^T h_t(\omega) dB_t(\omega) := \sum_{i=0}^{n-1} h_{t_i}(\omega)(B_{t_{i+1}}(\omega) - B_{t_i}(\omega)),$$
   and then taking the limit of the right hand side.
3. Let $W^{\mathbb{P}}$ be a Brownian motion, then the process $\tilde{W^{\mathbb{Q}}}$ defined by
   $$\tilde{W^{\mathbb{Q}}}_t = W_t^{\mathbb{P}} + \int_0^t \gamma_t^2 dt$$
   is a Brownian Motion over the measure $\mathbb{Q}$.

This is what we want to change the physical measure to a risk-neutral measure. Different models yield different $\frac{d\mathbb{Q}}{d\mathbb{P}}$. In the Black-Scholes model, we have

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \epsilon(\int_0 \frac{r-\mu}{\sigma} dW_s),$$

in which r, μ and σ depicts the asset and ϵ is the Doléans-Dade exponential.

But what if the risk-neutral measure is not unique? Then we will have a bunch of different pricings and we call that an incomplete market.

5 Conclusion

The changing into the risk-neutral measure plays an important role in mathematical finance. Arbitrage-free pricing theory is the foundation for many modern finance models, including the Black-Scholes model.

Some of these theories that I have encountered are fascinating, but due to the limited time, I don't quite have the opportunity to fully comprehend many of them. But in the meantime, I have come to realize that martingale theory is actually very important and if time permits, I am interested in continuing to explore the field of mathematical finance.

6 References

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