Lecture Note 1 : Proof of WLLN and SLLN

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Definition 1. *Indicator function. For* $S \subset \mathbb{R}$, $\mathbf{1}_S(x) := \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$

Theorem 2 (WLLN)**.** *For all* $\varepsilon > 0$, $\lim_{n\to\infty} P[\omega : |\frac{\sum_{i=1}^n d_i(\omega)}{n} - \frac{1}{2}| \geq \varepsilon] = 0$.

Proof. Let $r_i(\omega) = 2d_i(\omega) - 1$. One can easily check that $\{r_i\}$ are orthogonal functions on $\Omega := (0, 1]$, that is,

$$\int_\Omega r_i(\omega) r_j(\omega) \,\mathrm{d}\,\omega = \delta_{ij}. \tag{1}$$

Let $s_n(\omega) := \sum_{i=1}^n r_i(\omega)$. Note that

$$P\left[\omega : \left|\frac{\sum_{i=1}^n d_i(\omega)}{n} - \frac{1}{2}\right| \geq \varepsilon\right] = P[\omega : |s_n(\omega)| \geq n\varepsilon'] = \int_\Omega \mathbf{1}_{\Omega_2}(\omega) \,\mathrm{d}\,\omega, \tag{2}$$

where $\varepsilon' = 2\varepsilon, \Omega_2 = \{\omega : |s_n(\omega| \geq n\varepsilon'\}$. Since ε is arbitrarily small, we can equate ε and ε' for simplicity. The key observation is that $\mathbf{1}_{\Omega_2}(x) \leq \frac{1}{n^2\varepsilon^2} s_n^2(x)$ and thus

$$\int_\Omega \mathbf{1}_{\Omega_2}(\omega) \,\mathrm{d}\,\omega \leq \frac{1}{n^2\varepsilon^2} \int_0^1 s_n^2(\omega) \,\mathrm{d}\,\omega = \frac{1}{n\varepsilon^2}, \tag{3}$$

where the last equality is obtained by Eq. (1) and the definition of s_n. Combine Eq. (2) and Eq. (3) and we get the desired result. □

Definition 3. *A set* $S \subset R$ *is negligible if for all* $\varepsilon > 0$, *there is a collection of intervals* $\{I_i\}_{i=1}^\infty$ *such that* $S \subset \bigcup_{i=1}^\infty I_i$, *and* $\sum_{i=1}^\infty |I_i| < \varepsilon$, *where* $|\cdot|$ *is the length of the interval.*

Theorem 4 (SLLN)**.** *Let* $\mathcal{N} = \{\omega : \lim_{n\to\infty} \frac{s_n(\omega)}{n} = 0\}$, $\mathcal{N}^c$ *is uncountable but negligible.*

Proof. For $x \in (0,1]$, say $x = 0.b_1b_2\cdots$, let $f(x) = 0.11b_111b_211b_3\cdots$. The set $S := \{f(x) : x \in (0,1]\}$ is uncountable, for f is an injection, and $S \subset \mathcal{N}^c$ since numbers in S clearly violate the property of normal number. Hence, $\mathcal{N}^c$ is uncountable.

By the same method used in the proof of Theorem 2, use s_n^4 instead of s_n^2, we can establish

$$P[\omega : |s_n(\omega)| \geq n\varepsilon] \leq \frac{1}{n^4\varepsilon^4}\int_0^1 s_n^4(\omega)\,\mathrm{d}\,\omega. \tag{4}$$

We rewrite $s_n^4(\omega)$ as

$$s_n^4(\omega) = \sum_{1\leq i,j,k,l\leq n} r_i(\omega)r_j(\omega)r_k(\omega)r_l(\omega). \tag{5}$$

We can figure out the integral value of each term by the orthogonality of $\{r_k\}$, which is shown in Table 1. In s_n^4, n terms are of the first kind, $3n(n-1)$ terms are of the

Term	Interval value on Ω
r_i^4	1
$r_i^2r_j^2$	1
$r_i^2r_jr_k(=r_jr_k)$	0
$r_i^3r_j(=r_ir_j)$	0
$r_ir_jr_kr_l$	0

Table 1: Interval value of terms in s_n^4. In this table, i,j,k,l are pairwise distinct.

second kind, and thus

$$P[\omega : |s_n(\omega)| \geq n\varepsilon] \leq \frac{1}{n^4\varepsilon^4}\int_0^1 s_n^4(\omega)\,\mathrm{d}\,\omega = \frac{1}{n^4\varepsilon^4}[n+3n(n-1)] < \frac{3}{n^2\varepsilon^4}. \tag{6}$$

The key is we can find a decreasing sequence $\{\varepsilon_n\}$ such that

- $\sum_{n=1}^{\infty} \frac{3}{n^2\varepsilon_n^4}$ converges.
- $\lim_{n\to\infty} \varepsilon_n = 0$.

Let $A_n := \{\omega : |s_n(\omega)| \geq \varepsilon_n\}$. Intuitively, numbers in A_n violate the normal property at the n^{th} digit. Note that for all $m > 0$, $\bigcup_{j=m}^{\infty} A_j$ covers $\mathcal{N}^c$, because if $\omega \notin A_j$ for all $j \geq m$, $\omega \in \mathcal{N}$. By Eq. (6) and the proper choice of $\{\varepsilon_n\}$, we have

$$\sum_{n=1}^{\infty} |A_n| < \sum_{n=1}^{\infty} \frac{3}{n^2\varepsilon_n^4} < \infty. \tag{7}$$

Hence, for $\varepsilon > 0$, there is an m such that $\sum_{j=m}^{\infty} |A_j| < \varepsilon$. Meanwhile, $\bigcup_{j=m}^{\infty} A_j$ covers $\mathcal{N}^c$, which implies $\mathcal{N}^c$ is negligible. $\square$