A proof of a μ-integrable lemma and $h(f\mu) = (hf)\mu$ using the standard machine

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Lemma 1. *If $f \in (\mathrm{m}\Sigma)^+$ and $h \in (\mathrm{m}\Sigma)$, then $h \in \mathcal{L}^1(S, \Sigma, f\mu)$ iff $fh \in \mathcal{L}^1(S, \Sigma, \mu)$ and then $(f\mu)(h) = \mu(fh)$.*

To that end, let's consider the following lemma.

Lemma 2. *If $f \in (\mathrm{m}\Sigma)^+$ and $h \in (\mathrm{m}\Sigma)^+$, then $(f\mu)(h) = \mu(fh)$.*

Proof of lemma 2. Let's prove it using the standard machine.

First, we should show it holds when h is an indicator function. Suppose $h = \mathbf{1}_A$ with $A \in \Sigma$, and then, by definition, we have $(f\mu)(\mathbf{1}_A) = (f\mu)(A) = \mu(f \cdot \mathbf{1}_A)$.

Then, consider the situation where $h \in \mathrm{SF}^+$. Assume $h = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$. By the linearity, the left hand side equals $(f\mu)(\sum_{k=1}^n a_k \mathbf{1}_{A_k}) = \sum_{k=1}^n a_k (f\mu)(\mathbf{1}_{A_k}) = \sum_{k=1}^n a_k \cdot \mu(f \cdot \mathbf{1}_{A_k}) = \mu(f \cdot \sum_{k=1}^n a_k \mathbf{1}_{A_k}) = \mu(fh)$.

When $h \in \mathrm{m}\Sigma^+$, we can easily construct a sequence of non-negative simple functions (h_n) such that $h_n \uparrow h$, and obviously $fh_n \uparrow fh$ as well. As we already have $(f\mu)(h_n) = \mu(fh_n)$ for each $n \in \mathbb{N}$, by MON and letting $n \to \infty$ on both sides, we obtain $(f\mu)(h) = \mu(fh)$.

$\square$

Now we can prove the first part of the lemma 1, that is,

$$h \in \mathcal{L}^1(S, \Sigma, f\mu) \text{ iff } fh \in \mathcal{L}^1(S, \Sigma, \mu).$$

By definition, $h \in \mathcal{L}^1(S, \Sigma, f\mu)$ if $(f\mu)(|h|) < \infty$. Since $|h| \in \mathrm{m}\Sigma^+$, we have $(f\mu)(|h|) = \mu(f|h|) = \mu(|fh|) < \infty$, which means $fh \in \mathcal{L}^1(S, \Sigma, \mu)$, and it holds vice versa.

When $h \in \mathcal{L}^1(S, \Sigma, f\mu)$, by linearity, $(f\mu)(h) = (f\mu)(h^+ - h^-) = (f\mu)(h^+) - (f\mu)(h^-) = \mu(fh^+) - \mu(fh^-) = \mu(fh)$.

Remark 1. *This lemma implies something about "division" and integration by substitution because it shows that*

$$\int_S h \, \mathrm{d}(f\mu) = (f\mu)(h) = \mu(fh) = \int_S hf \, \mathrm{d}\mu$$

which to some degree means $\mathrm{d}(f\mu) = f \, \mathrm{d}\mu$*, and furthermore,* $f = \frac{\mathrm{d}(f\mu)}{\mathrm{d}\mu}$*. That looks like a division and the method of substitution on integration. Let* λ *denote* $f\mu$*, then we have* $f = \frac{\mathrm{d}\lambda}{\mu}$ *here, and we say that* λ *has* ***density*** f *relative to* μ*.*

Remark 2. *A trivial but common corollary goes here. Let* $F \in \Sigma$*, then* $\mu(F) = 0$ *implies* $\lambda(F) = 0$ *because* $\lambda(F) = (f\mu)(F) = \mu(f \cdot \mathbf{1}_F) = 0$*. That means* μ*-null sets have nothing to do with the result of integration.*