Notes of Measure Space

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1 Measure Theory and Topology

Space is often a set with a certain structure or operation, rather than a simple set.

Topology Topology deals with open sets.

We have a set T. T has some open subsets O. If open sets satisfy some requirements in $U = \{O\}$, then (T, U) is a topology space. T is the origin space. U is a subset of 2^T(set consisting of subsets of T).

U should satiafy:

1. T and $\varnothing$ is open set.
2. $\forall O_i \in U, \bigcap_{i=1}^{N} O_i$ is open set (must be finite).
3. $\forall O_i \in U, \bigcup O_i$ is open set (arbitrary number).

We define **continuous function** f on T as

$$\forall G \text{ is an open set, } f^{-1}(G) \text{ is an open set as well.}$$

Measure Space has many similar definitions.

Measurable Space Similarly, measurable space is a dualistic group $(X, \mathcal{F})$. X is our origin discussion space, and $\mathcal{F}$ is a subset of 2^X, which satisfies some conditions as follows:

1. $\varnothing, X \in \mathcal{F}$.
2. If $E \in \mathcal{F}$, then $E^C \in \mathcal{F}$.
3. If $E_i \in \mathcal{F}$, then $E_1 \cup E_2$ as well.

Those elements in $\mathcal{F}$ is measurable. The measurable space $(X, \mathcal{F})$ also can be described as "$\mathcal{F}$ is a $\sigma-$Algebra on X".

Measure Space Measure spaces define a measure m on measurable space. Then measure space is a triple $(X, \mathcal{F}, m)$.

(if $\mathcal{F}$ is too big, we cannot find a measure function m intuitively)

The measure $m : \mathcal{F} \mapsto [0, \infty]$ has to satisfy:

1. $m(\varnothing) = 0$.
2. If $F_i \cap F_j = \varnothing (i \neq j)$ and $F = \cup F_n \in \mathcal{F}$, then
$$m(F) = \Sigma m(F_n).$$

Corollary 1. *(Additive) If $F, G \in \mathcal{F}, F \cap G = \varnothing$, $m(F \cup G) = m(F) + m(G)$.*

2 σ-Algebra

σ-Algebra has three basic rules, and some properties are derived from them.

Corollary 2. *These two corollary is useful!*

1. $\bigcup_{i=1}^n E_i \in \mathcal{F} (n \in \mathbb{N})$
2. $\bigcap_{i=1}^n E_i = (\bigcup_{i=1}^n F_i^C)^C \in \mathcal{F}$.

Thus we conclude that $\sigma-$Algebra is a family of subsets closed to any countable number of operations.

Corollary 3. ***(Principle of Inclusion-exclusion)*** *For all $F_i \in \Sigma$, we have*
$$\mu(\bigcup_{i \leq n} F_i = \Sigma_{i \leq n} \mu(F_i) - \Sigma\Sigma_{i<j\leq n} \mu(F_i \cap F_j) \\ + \cdots + (-1)^{n-1} \mu(F_1 \cap F_2 \cap \cdots \cap F_n).$$

The proof of Corollary 3 has been written in Chen Tong's notes on github.

2.1 Borel σ-Algebra

$$\mathcal{B}(S) := \sigma(\text{open subsets of } S)$$

The most common Borel algebra is $\mathcal{B} := \mathcal{B}(\mathbb{R})$. It consists of the open subsets in $\mathbb{R}$. But it is hard to find a subset of $\mathbb{R}$ but not in $\mathcal{B}$.

Theorem 1. *Define $\pi(\mathbb{R}) := \{(-\infty, x] \mid x \in \mathbb{R}\}$, then*
$$\mathcal{B} = \sigma(\pi(\mathbb{R})).$$

Proof. For all $x \in \mathbb{R}$,. $(-\infty, x] = \bigcap(-\infty, x + 1/n)$. Use the property 2, $(-\infty, x]$ is measurable, i.e. $(-\infty, x] \in \mathcal{B}$.

Then we just need to prove that $\forall a, b \in \mathbb{R}$, $(a, b) \in \sigma(\pi(\mathbb{R}))$. First, (a, b) can be represented as $\bigcup_i^\infty (a, b - \epsilon/n]$, where $\epsilon = (b - a)/2$ (in facet it can be arbitrary small). Then we have
$$(a, k] = (-\infty, k] \cap (a, \infty) = (-\infty, k] \cap (-\infty, a]^C \in \mathcal{B},$$
thus $(a, b - \epsilon/n] \in \mathcal{B}$. In this way, $(a, b) = \bigcup_i^\infty (a, b - \epsilon/n] \in \mathcal{B}$. $\square$

2.2 Finite and σ-finite

We have a measure space (S, Σ, μ).

Finite The measure space is finite iff $\mu(S) < \infty$.

$\sigma-$finite The measure space is $\sigma-$finite iff there exists a sequence $\{S_n\}(S_i \in \Sigma)$,s.t.

$$\mu(S_n) < \infty (\forall n \in \mathbb{N}) \text{ and } \bigcup S_n = S.$$

2.3 Minimum $\sigma-$algebra

Here we introduce a signal $\sigma(A)$, which denote the minimum σ-algebra including A.

3 π-System

We have a set S and $\mathcal{I} \subseteq 2^S$. $(S, \mathcal{I})$ is a π-System iff

$$I_1, I_2 \in \mathcal{I} \Rightarrow I_1 \cap I_2 \in \mathcal{I}$$

π-System is easier for us to research, for it is just closed on $\cap$ while σ-algebra is closed on both $\cap$ and $\cup$(in my opinion like a group to a ring in abstract algebra).

Theorem 2. *Define $\Sigma := \sigma(\mathcal{I})$. If there exists two measures μ_1, μ_2 on (S, Σ) satisfy:*

$$\mu_1(S) = \mu_2(S) < \infty$$

and

$$\mu_1(x) = \mu_2(x), \forall x \in \mathcal{I},$$

then

$$\mu_1(x) = \mu_2(x), \forall x \in \Sigma.$$

Proof. The proof is in Appendix in our textbook, using Dynkin lemma. □

Theorem 3. ***(Carathéodory Expansion Theorem)*** *S is a set, and Σ_0 is an algebra on S(Σ_0 is a π-system as well). Define*

$$\Sigma := \sigma(\Sigma_0).$$

(Existence) *If $\mu_0 : \Sigma_0 \mapsto [0, \infty]$ is a mapping satisfies the rule of measure function, then there exists a μ on Σ and satisfies*

$$\mu(x) = \mu_0(x), \forall x \in \Sigma_0.$$

(Uniqueness) *Futhermore, if $\mu_0(S) < \infty$, then according to Theorem 2, the expansion μ is unique.*

3.1 Lebesǵue Measure

Let $S = (0, 1]$. Define

$$F = (a_1, b_1] \cup \cdots \cup (a_r, b_r] \subseteq S, \text{ where } 0 \leq a_1 \leq b_1 \leq \cdots \leq a_r \leq b_r.$$

And the union of F is defined as π-System Σ_0. According to 2.3, we can define a minimum σ-algebra

$$\Sigma := \sigma(\Sigma_0) = (0, 1].$$

Define the measure μ_0 on Σ_0 as

$$\mu_0(F) = \Sigma_{k \leq r}(b_k - a_k).$$

We can prove that μ_0 satisfy the premise of theorem 3, thus we conclude that there exists a unique measure μ on Σ. We call this measure μ as Lebegue measure on $(0, 1]$(Leb.$([0, 1], \mathcal{B}[0, 1])$).

* Note that we can simply define $\mu(\{0\}) = 0$ to obtain

$$\text{Leb.}([0, 1], \mathcal{B}[0, 1]) = \text{Leb.}((0, 1], \mathcal{B}(0, 1]).$$

4 Probability space

A probability space is a special measure space where $\mu(S) = 1$. Then μ corresponds to the probability that we're familiar with. This concept will be used in the future.