Notes on the Second Borel–Cantelli lemma

Lu Jiaxin
Student ID: 518030910412

March 24, 2020

Theorem 1. *Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of reals from $[0,1]$ such that $\sum_{n\in\mathbb{N}} y_n = \infty$. Show that $\prod_{n\in\mathbb{N}}(1-y_n) = 0$.*

Fact 2. $\log(1-x) \leq -x$ *for all* $0 \leq x \leq 1$.

Proof. From Fact 2 we have,

$$\begin{aligned}\log(\prod_{n\in\mathbb{N}}(1-y_n)) &= \sum_{n\in\mathbb{N}}\log(1-y_n)\\ &\leq -\sum_{n\in\mathbb{N}} y_n\end{aligned}$$

Thus,

$$\begin{aligned}\prod_{n\in\mathbb{N}}(1-y_n) &= \exp(-\sum_{n\in\mathbb{N}} y_n)\\ &= 0\end{aligned}$$

□

By using Theorem 1, we can prove the second Borel-Cantelli lemma (BC2).

Theorem 3 (Second Borel-Canteli lemma)**.** *Let (E_n) be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$. If the events E_n are pairwisely independent, then $\sum_{n\in\mathbb{N}} P(E_n) = \infty$ implies that $P(\limsup_{n\to\infty} E_n) = 1$.*

Proof. Let $A = \limsup_{n\to\infty} E_n$. We shall prove that $P(A^C) = 0$. Let $B_i = \bigcap_{n=i}^{\infty} E_n^C$. Then $A^C = \bigcup_{i=1}^{\infty} B_i$. So, we shall prove that $P(B_i) = 0$ for all i. Now, for each i and $k > i$,

$$\begin{aligned}P(B_i) &= P(\bigcap_{n=i}^{\infty} E_n^C)\\ &\leq P(\bigcap_{n=i}^{k} E_n^C) = \prod_{n=i}^{k}[1-P(E_n)]\end{aligned}$$

Use Theorem 1, we can derive,

$$P(B_i) = \prod_{n=i}^{k} [1 - P(E_n)] = 0$$

Thus, $P(B_i) = 0$ for all $i \in \mathbb{N}$. $\square$

Comparing two Borel–Cantelli lemmas(BC1 and BC2), we find that for a sequence of pairwisely independent events (E_n), we either have $P(\limsup_{n\to\infty} E_n) = 0$ or $P(\limsup_{n\to\infty} E_n) = 1$, depending on $\sum_{n\in\mathbb{N}} P(E_n)$. From materials I found, this is known as **Zero-one law**, which is widely used in probability theory.