# Poincarés Recurrence Theorem for Incompressible Transformations

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**Problem 1** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $T$ be an incompressible transformation on it, namely it is measurable and there is no $F \in \mathcal{F}$ such that $F \subset T^{-1}(F)$ and $P(F) < P\left(T^{-1}(F)\right)$. Prove that $\forall E \in \mathcal{F}$ with $P(E) > 0$, almost all points of $E$ return to $E$ infinitely often under positive iterations by $T$.*

**Proof:** This problem is almost the same as the last one with some slight changes. Our goal is to prove

$$P\left(E \setminus \limsup_{n\to\infty} T^{-n}(E)\right) = 0.$$

The same as the proof of another version of PR theorem, the above is equivalent to prove

$$P(A_k) = 0, \text{ in which } A_k = E \setminus \bigcup_{n\geq k} T^{-n}(E).$$

We will prove by making contradiction. Assume $P(A_k) > 0$, we have

$$P(B_k) = P\left(E \cup \bigcup_{n\geq k} T^{-n}(E)\right) > P\left(\bigcup_{n\geq k} T^{-n}(E)\right) = P(C_k).$$

Here $B_k, C_k$ are notations for simplicity. Notice that the definition of incompressible transformation is exactly

$$F \subset T^{-1}(F) \Rightarrow P(F) = P(T^{-1}(F)).$$

Hence by $T^{-1}(C_{k-1}) = C_k$ we have

$$P(C_k) = P(C_{k-1}).$$

Similarly, as $T^{-1}(B_k) \subset C_1$ we derive that

$$P(B_k) = P(C_1) = P(C_k).$$

Which leads to a contradiction. $\square$

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