Indicator function and Infinite events

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This is just a simple discussion on $Exer.2, Exer.3$ in the handout 0324

Exercise 1. *Show that*

$$\lim_{n\to\infty} sup\boldsymbol{1}_{E_n} = \boldsymbol{1}_{\lim\limits_{n\to\infty} supE_n}$$

$$\lim_{n\to\infty} inf\boldsymbol{1}_{E_n} = \boldsymbol{1}_{\lim\limits_{n\to\infty} infE_n}$$

Proof. To prove this, we need to show the equations keep for any $\omega \in \mathcal{F}$.

For left side, derive the function as defined in 2.5(a) in textbook for real consequences, we have

$$\begin{aligned}\lim_{n\to\infty} sup\mathbf{1}_{E_n}(\omega) &=\downarrow \lim_m \{\sup_{n\geq m} \mathbf{1}_{E_n}(\omega)\} \\ &= \begin{cases} 1 & \forall m \in \mathbb{N}, \exists n \geq m, \omega \in E_n \\ 0 & \forall m \in \mathbb{N}, \forall n \geq m, \omega \notin E_n \end{cases}\end{aligned}$$

Thus, we have

$$left = \begin{cases} 1 & \forall m \in \mathbb{N}, \omega \in \cup_{n\geq m} E_n \Rightarrow \omega \in \cap_{m\in\mathbb{N}} \cup_{n\geq m} E_n \\ 0 & \forall m \in \mathbb{N}, \omega \notin \cup_{n\geq m} E_n, \Rightarrow \omega \notin \cap_{m\in\mathbb{N}} \cup_{n\geq m} E_n \end{cases} \tag{1}$$

For right side, the function can be written as

$$\mathbf{1}_{\lim\limits_{n\to\infty}} supE_n(\omega) = \begin{cases} 1 & \omega \in \cap_{m\in\mathbb{N}} \cup_{n\geq m} E_n \\ 0 & \omega \notin \cap_{m\in\mathbb{N}} \cup_{n\geq m} E_n \end{cases} \tag{2}$$

Based on (1) and (2), we have proved $\lim_{n\to\infty} sup\mathbf{1}_{E_n} = \mathbf{1}_{\lim\limits_{n\to\infty} supE_n}$

For $liminf$, it is a similar proof. □

Exercise 2. *Let* $(y_n)_{n\in\mathbb{N}}$ *be a sequence of reals from [0,1] such that* $\sum_{n\in\mathbb{N}} y_n = \infty$. *Show that* $\prod_{n\in\mathbb{N}}(1-y_n) = 0$

Proof.

$\forall y_n \geq 0, 1-y_n \leq e^{-y_n}$, since $f = 1-x-e^{-x}$, $f(0) = 0, f'(x) = -1+e^{-x} > 0 (x \geq 0)$ So we have

$$0 \leq \prod_{n\in\mathbb{N}}(1-y_n) \leq \prod_{n\in\mathbb{N}} e^{-y_n} \leq e^{-\sum_{n\in\mathbb{N}} y_n} \leq e^{-\infty} = 0$$

Then we have $\prod_{n\in\mathbb{N}}(1-y_n) = 0$.

And this exercise is used in the proof of Borel-Cantelli lemma II □