Pointwise convergence and convergence in probability

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In measure space (S, Σ, μ), consider function sequence $(f_n) \subseteq m\Sigma$ and $f \in m\Sigma$:

- (f_n) almost surely(pointwise) converges to f_n if:

$$\mu(f_n \not\to f) = 0.$$

  In probability space $(\Omega, \mathcal{F}, P)$, we can also say that $P(f_n \to f) = 1$.
- (f_n) converges to f_n in probability if:

$$\forall \varepsilon > 0, \mu(|f_n - f| > \varepsilon) \to 0.$$

To make the equations more readable, in the following discussion, let

$$A_{n,\varepsilon} = (|f_n - f| > \varepsilon).$$

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If $f_n \to f$ almost surely, then $f_n \to f$ in probability.

Proof. $f_n \to f$ almost surely, thus

$$\begin{aligned}
\mu(f_n \not\to f) &= \mu(\{x : \exists \varepsilon > 0, \forall m \in \mathbb{N}_+, \exists n \geq m, |f_n(x) - f(x)| > \varepsilon\}) \\
&= \mu(\{x : \exists k \in \mathbb{N}_+, \forall m \in \mathbb{N}_+, \exists n \geq m, |f_n(x) - f(x)| > \frac{1}{k}\}) \\
&= \mu(\bigcup_{k=1}^{\infty} \bigcap_m \bigcup_{n=m}^{\infty} A_{n,\frac{1}{k}}) = 0.
\end{aligned}$$

Then $\forall k > 0, \mu(\bigcap_m \bigcup_{n=m} A_{n,\frac{1}{k}}) = 0$ and therefore

$$\forall \varepsilon > 0, \mu(\bigcap_m \bigcup_{n=m} A_{n,\varepsilon}) = 0.$$

Then by Reversed Fatou Lemma,

$$\forall \varepsilon > 0, \lim_{n\to\infty} \mu(A_{n,\varepsilon}) \leq \bigcap_m \bigcup_{n=m} \mu(A_{n,\varepsilon}) \leq \mu(\bigcap_m \bigcup_{n=m} A_{n,\varepsilon}) = 0.$$

Therefore, $f_n \to f$ in probability. □

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If $f_n \to f$ in probability, it is not necessary that $f_n \to f$ almost surely.

Counter-example. Let $(S, \mathcal{F}, \mu) = ([0,1), \mathcal{B}([0,1), \mathrm{Leb}), n = \sum_{j=1}^{i} j + d (1 \leq d \leq i)$,

$$f_n(x) = \begin{cases} 1, x \in [\frac{d-1}{i}, \frac{d}{i}) \\ 0, \text{ otherwise} \end{cases} .$$

Then, $\forall \varepsilon > 0, \mu(A_{n,\varepsilon}) \leq \frac{1}{i} \to 0$.
But $\forall x \in [0,1)$, $f_n(x)$ does not converge, thus $\mu(f_n \not\to f) = 1 \neq 0$. □

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If $f_n \to f$ in probability, then there is a subsequence (f_{n_i}) such that $f_{n_i} \to f$ almost surely.

Proof 1. $f_n \to f$ in probability, thus

$$\forall k, \exists n_k, \mu(A_{n_k, \frac{1}{k}}) < \frac{1}{k^2}.$$

In this case, $\sum_k \mu(A_{n_k,\frac{1}{k}})$ converges. By BC1, we know that

$$\mu(\bigcap_l \bigcup_{i=l} A_{n_i,\frac{1}{i}}) = 0.$$

Assume that $x \in (f_n \not\to f)$, i.e., $\forall \varepsilon > 0, x \in A_{n_i,\varepsilon}$ infinitely often.
Let $n_{i_0} = 0$, and at every step we let $\varepsilon_j = \frac{1}{j}$, and choose an n_{i_j} such that $n_{i_j} > \max(j, n_{i_{j-1}})$ and $x \in A_{n_{i_j},\varepsilon_j} \subseteq A_{n_{i_j},\frac{1}{i_j}}$, i.e., $x \in A_{n_i,\frac{1}{i}}$ infinitely often.
Thus $(f_{n_i} \not\to f) \subseteq \bigcap_l \bigcup_{i=l} A_{n_i,\frac{1}{i}}$.And we have $\mu(\bigcap_l \bigcup_{i=l} A_{n_i,\frac{1}{i}}) = 0$, so $\mu(f_{n_i} \not\to f) = 0$, $f_{n_i} \to f$ almost surely.

□

Proof 2. To show that there is a subsequence (f_{n_k}) such that $f_{n_k} \to f$ almost surely, i.e.,

$$\forall \varepsilon > 0, \mu(\bigcap_l \bigcup_{i=l} A_{n_i,\varepsilon}) = 0.$$

An idea is to get this via BC1, which needs us to prove that

$$\forall \varepsilon > 0, \sum_{i=1}^{\infty} \mu(A_{n_i, \frac{1}{k}}) < \infty.$$

And we know that $f_n \to f$ in probability. Let $n_0 = 0$, $\forall k$, there exists $n_k > n_{k-1}$, such that $\mu(A_{n_i, \frac{1}{i}}) < \frac{1}{i^2}$. Therefore:

$$\begin{aligned}
\forall \varepsilon > 0, \sum_{i=1}^{\infty} \mu(A_{n_i, \frac{1}{k}}) &= (\sum_{i=1}^{\lceil \frac{1}{\varepsilon} \rceil} + \sum_{i=\lceil \frac{1}{\varepsilon} \rceil + 1}^{\infty})\mu(A_{n_i, \varepsilon}) \\
&< \sum_{i=1}^{\lceil \frac{1}{\varepsilon} \rceil} \mu(A_{n_i, \varepsilon}) + \sum_{i=\lceil \frac{1}{\varepsilon} \rceil + 1}^{\infty} \mu(A_{n_i, \frac{1}{i}}) \\
&< \sum_{i=1}^{\lceil \frac{1}{\varepsilon} \rceil} \mu(A_{n_i, \varepsilon}) + \sum_{i=\lceil \frac{1}{\varepsilon} \rceil + 1}^{\infty} \frac{1}{i^2} < \infty.
\end{aligned}$$

Therefore, $f_{n_k} \to f$ almost surely. $\square$