An Extended Super-Martingale Convergence Theorem with its Proof

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When searching information for the final project of this course, I find an extended super-martingale convergence theorem, which I think is worth sharing. The proof of this theorem also uses the martingale convergence theorem itself.

Theorem 1 *Let $Y_t, X_t, Z_t, t = 1, 2, 3, \ldots$ be three sequences of random variables and let $\mathcal{F}_t$ be sets of random variables such that $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all t, suppose that:*

1. *The random variables Y_t, X_t, Z_t are non-negative and are functions of the random variables in $\mathcal{F}_t$*
2. *For each t we have $E\left[Y_{t+1} \mid \mathcal{F}_t\right] \leq Y_t - X_t + Z_t$*
3. $\sum_{t=0}^{\infty} Z_t < \infty$

Then we have $\sum_{t=0}^{\infty} X_t < \infty$ and there exists a non-negative random variable Y such that $Y_t \to Y$ with probability 1.

Proof: First we let $R_t := Y_t + \sum_{i=1}^{t-1} X_i - \sum_{i=1}^{t-1} Z_i$, it can be easily noticed that it's a super-martingale. Since

$$R_{t+1} - R_t = Y_{t+1} - Y_t + X_t - Z_t,$$

we then have

$$E\left(R_{t+1} - R_t \mid \mathcal{F}_t\right) = E\left(Y_{t+1} \mid \mathcal{F}_t\right) - Y_t + X_t - Z_t.$$

By condition(2), it is less then or equal to 0 with probability 1.
Since we don't have a fixed lower bound for the super-martingale R, we can't apply the convergence theorem directly. However, for any $a > 0$, consider the stopping time

$$\tau_a = \inf\left\{t : \sum_{i=1}^{t} Z_i > a\right\},$$

with $\tau_a = \infty$ if $\sum_{i=1}^{t} Z_i \leq a$ for all t.
We can then define

$$R^{(a)}(t) := R\left(t \wedge \tau_a\right) = \begin{cases} R_t & \text{if } t < \tau_a \\ R_{\tau_a} & \text{if } t \geq \tau_a \end{cases}$$

$R^{(a)}$ is also a super-martingale for any a, and $R^{(a)}(t)$ is bounded below by $-a$. Then we can use the martingale convergence theorem. For any given $a, R^{(a)}(t)$ converges to some finite limit with probability 1. By countable additive, we get that with probability $1, R^{(a)}(t)$ converges to a finite limit for all $a \in Z$.

But if $\sum_{i=0}^{\infty} Z_i < \infty$, which from (c) we assume happens with probability 1, then for all large enough $a \in Z$, we have $\tau_a = \infty$, and so $R^{(a)}(t) = R(t)$ for all t. since we know $R^{(a)}(t)$ converges, we also get that $R(t)$ converges.

Finally, since $R(t)$ converges and $\sum_{i=0}^{t-1} Z_i$ converges, we also have that $Y_t + \sum_{i=1}^{t-1} X_i$ converges. since $\sum_{i=1}^{t-1} X_i$ is non-decreasing in t, and Y_t is non-negative for all t, the only way this can happen is if Y_t and $\sum_{i=1}^{t-1} X_i$ both converge, as required.