Lovász Local Lemma and its Application

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Lemma 1. *Let* $(E_1, E_2, \cdots, E_N)$ *be a sequence of events. For each* E_i*, we have* $P(E_i) \leq p$ *and* E_i *depends on at most* d *other events. If* $4dp \leq 1$*, then*

$$P\left(\bigcap_{i=1}^{N} E_i^c\right) \geq (1-2p)^N > 0$$

Claim 2. *Let* $S \subseteq \{1, 2, \cdots, N\}$*, then for every* i*, we have*

$$P\left(E_i \middle| \bigcap_{j \in S} E_j^c\right) \leq 2p$$

Assuming the claim, it's easy to prove the Lovász Local Lemma.

$$P\left(\bigcap_{i=1}^{N} E_i^c\right) = \Pi_{i=1}^{N} P\left(E_i^c \middle| \bigcap_{j<i} E_j^c\right) = \Pi_{i=1}^{N}\left(1 - P\left(E_i \middle| \bigcap_{j<i} E_j^c\right)\right) \geq (1-2p)^N > 0$$

Next we proof the claim.

Proof. We prove the claim by induction on $|S|$. If $|S| = 0$, the claim holds, because

$$P\left(E_i \middle| \bigcap_{j \in S} E_j^c\right) = P(E_i) \leq p \leq 2p$$

Assume that the claim holds when $|S| < s$. We will prove the claim for $|S| = s$. Suppose that D be the set of all j such that E_i depends on E_j. Here are two cases.

case 1. $S \cap D = \emptyset$

$$P\left(E_i \middle| \bigcap_{j\in S} E_j^c\right) = P(E_i) \le p \le 2p$$

case 2. $S \cap D \neq \emptyset$

Let $E_D = \bigcap_{j\in D} E_j^c$ and $E_{S\setminus D} = \bigcap_{j\in S\setminus D} E_j^c$

$$\begin{aligned} P\left(E_i \middle| \cap_{j\in S} E_j^c\right) = & \quad P\left(E_i \middle| \left(E_D \bigcap E_{S\setminus D}\right)\right) \\ = & \quad \frac{P\left(E_i \bigcap E_D \bigcap E_{S\setminus D}\right)}{P\left(E_D \bigcap E_{S\setminus D}\right)} \\ = & \quad \frac{P\left(E_i \bigcap E_D \middle| E_{S\setminus D}\right) P\left(E_{S\setminus D}\right)}{P\left(E_D \middle| E_{S\setminus D}\right) P\left(E_{S\setminus D}\right)} \\ = & \quad \frac{P\left(E_i \bigcap E_D \middle| E_{S\setminus D}\right)}{P\left(E_D \middle| E_{S\setminus D}\right)} \end{aligned} \tag{1}$$

For the numerator, we have

$$P\left(E_i \bigcap E_D \middle| E_{S\setminus D}\right) \le P\left(E_i \middle| E_{S\setminus D}\right) = P\left(E_i\right) \le p \tag{2}$$

Note that $S \cap D \neq \emptyset$, so $|S\setminus D| < |S|$. We can apply the inductive hypothesis, so for all $E_k \in E_D$, $P(E_k|E_{S\setminus D}) \le 2p$. For the denominator, we have

$$\begin{aligned} P\left(E_D | E_{S\setminus D}\right) = & \quad P\left(\bigcap_{k\in D} E_k^c \middle| E_{S\setminus D}\right) \\ = & \quad P\left(\left(\bigcup_{k\in D} E_k\right)^c \middle| E_{S\setminus D}\right) \\ = & \quad 1 - P\left(\bigcup_{k\in D} E_k \middle| E_{S\setminus D}\right) \\ \ge & \quad 1 - \sum_{E_k\in E_D} P\left(E_k \middle| E_{S\setminus D}\right) \\ \ge & \quad 1 - d \times 2p \\ \ge & \quad \tfrac{1}{2} \end{aligned} \tag{3}$$

Combining (2) and (3) , we have $P\left(E_i \mid \cap_{j\in S} E_j^c\right) = \frac{P\left(E_i \bigcap E_D | E_{S\setminus D}\right)}{P\left(E_D | E_{S\setminus D}\right)} \le 2p$. We have proved the claim. □

Example 3. *In a k-SAT formula like* $(x_1 \vee \neg x_3 \vee x_5) \wedge (\neg x_2 \vee x_4) \wedge (x_3 \vee \neg x_4 \vee x_6)$*, every variable* $x_i = \{0, 1\}$ *occurs in at most* $\frac{2^k}{4k}$ *different clauses. Then there exists an assignment satisfies all the clauses in the formula.*

Proof. Let event E_i be that the i-th clause is not satisfied. If the i-th clause has more than

k variables, we only need to care k of them because of the feature of the operation $\vee$. Thus $P(E_i) \leq \frac{1}{2^k}$. Every variable occurs in at most $\frac{2^k}{4k}$ different clauses, so the i-th clause depends on at most other $(\frac{2^k}{4k} - 1) \times k \leq \frac{2^k}{4}$ events. Note that $4 \times \frac{2^k}{4} \times \frac{1}{2^k} = 1$, Lovász Local Lemma implies $P\left(\bigcap_{i=1}^N E_i^c\right) > 0$, which means there exists an assignment satisfies all the clauses in the formula. $\square$

Example 4. *Let $G = (V, E)$ be a graph. For every vertex v, its color is $C_v = \{1, 2, \cdots, k\}$ and its degree is at most m. If $m \leq \frac{k}{8}$, then there must exist an assignment of $\{C_v | v \in V\}$ such that every edge connects two vertexes of different colors.*

Proof. Let event $E_e(e = (u, v) \in E)$ be that e connects two vertexes of the same color. Note that $P(E_e) = \frac{\sum_{C_u=1}^{k} \sum_{C_v=1}^{k} [C_u = C_v]}{k^2} = \frac{1}{k}$. Every vertex's degree is at most m, so the event E_e depends on at most other $2(m-1) \leq 2m \leq \frac{k}{4}$ events. Node that $4 \times \frac{k}{4} \times \frac{1}{k} = 1$, Lovász Local Lemma implies $P\left(\bigcap_{e \in E} E_e^c\right) > 0$, which means there must exist an assignment such that every edge connects two vertexes of different colors. $\square$