The relationship between RN theorem and Doob's theorem

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In class, we have learnt the Doob's Martingale Convergence Theorem(Doob's 'Forward' Convergence Theorem) and Radon Nikodym Theorem. In fact, the prove of Radon Nikodym Theorem uses the result of Doob's Martingale Convergence Theorem in our book. So to show the relationship bewteen these two theorem, we uses Radon Nikodym Theorem to prove the Martingale Convergence Theorem with the help of some materials. And then we can show that these two conclusion are essentially the same thing.

First of all, let's review these two theorems.

Theorem 1. $\mathbf{Doob's\ 'Forward'\ Convergence\ Theorem}$: Let X be a supermartingale bounded in $\mathcal{L}^1$: $sup_n\mathbf{E}(|X_n|) < \infty$. Then, almost surely, $X_\infty\ := \ limX_n$ exsits and is finite. For defineness, we $define$ $X_\infty(\omega)\ := \lim\ \sup X_n(\omega),\ \forall\omega$, so that X_∞ is $\mathcal{F}_\infty$ measurable and $X_\infty\ = \lim X_n,$ **a.s.**.

Theorem 2. $\mathbf{The\ Radon-Nikodym\ Theorem}$. Suppose that $(\Omega,\ \mathcal{F},\ P)$ is a probability triple and suppose that Q is a finite measure on $(\Omega,\ \mathcal{F})$ which is absolutely continuous relative to P in that

$$for\ F\in\mathcal{F},\ P(F)\ =\ 0\longrightarrow\ Q(F)\ =\ 0.$$

Then there exists X in $\mathcal{L}^1(\Omega,\ \mathcal{F},\ P)$ such that $Q\ =\ XP$ in that

$$Q(F)\ =\ \int_F X\,dP\ =\ E(X;\ F),\ \ \forall F\in\mathcal{F}.$$

The variable X is called a version of the Radon-Nikodym derivative of Q relative to P on $(\Omega,\ \mathcal{F})$. Two such version agree **a.s.**. We write

$$\frac{dQ}{dP}\ =\ X\ on\ \mathcal{F},\ \ \mathbf{a.s.}$$

In our book, we can find that we can use the Martingale Convergence Theorem to prove The Radon-Nikodym Theorem.(We use this to prove that $X_\infty\ := \lim X_n$ exists.) So now let's prove the Martingale Convergence Theorem by using the Martingale Convergence Theorem.

Proof of Radon-Nikodym Theorem. Let's assume that the Radon-Nikodym Theorem holds. So for the given supermartingale $(X_n,\ \mathcal{F}_n),\ n \geq 1$, we have $sup\ E(|X_n|) < \infty$. We set the set function

$$q_n(F) \ = \ \int_F X_n \, dP, \ \ \forall F \ \in \ \mathcal{F}.$$

Since $sup(E(X_n)) < \infty$, we set $sup(E(X_n)) = M$, so q_n is bounded. Since X_n is a supermartingale, we have q_n is monotonic decreasing. Since q_n is bounded and monotonic decresing, q_n is convergence. So we let

$$Q(F) \ = \ lim\ q_n(F), \ \ \forall F \ \in \ \mathcal{F}.$$

Then q_{n+1} can be seen as generated by q_n from $\mathcal{F}_n$ to $\mathcal{F}_{n+1}$, we can define the $Q(F)$ on the $\mathcal{F}_0 \ := \ \cup_{n=1}^{\infty} \mathcal{F}_n \ \subseteq \ \mathcal{F}$.
So the set function Q is additive. So one idea is that we can decomposite Q into two part Q_1 and Q_2 to make Q_1 $\sigma-$additive and absolutely continuous relative to P in order to use the Radon-Nikodym Theorem. And we need to ensure that Q_2 is small enough to have a slight effect on the result. So I find following lemma which is found by Hewitt-Yosida.

Lemma 1. Hewitt – Yosida′s Theorem [1] *If Q is finitely additive, then we can divide Q into two part: Q_1 and Q_2 such that Q_1 is $\sigma-$additive and Q_2 is purely finitely additive, that is,*
whenever $\mu : \mathcal{F} \ \longrightarrow \ R^+$ is conutably additive and $0 < \mu < Q_2$ *on $\mathcal{F}$, then* $\mu \ = \ 0.$

We can use this to divide Q into two set functions Q_1 and Q_2 which Q_1 is conutably additive and Q_2 is purely finitely additive. But how to prove that Q_2 is small? The following lemma will show this point.

Lemma 2. *If m is a purely finitely additive set function,and we have a random given countably additive set function p, then for $\forall\ A \in \mathcal{F}$ and $\forall\ \epsilon > 0$, there exists a set A_0, $A_0 \in \mathcal{F}$, $A_0 \subseteq A$, s.t. $m(A_0) + p(A_0' \cap A) < \epsilon$. ($A'$ is the complementary set of A in Ω)*

Proof of Lemma. Let $g \ = \ inf_{A_0 \in \mathcal{F}, A_0 \subseteq A}(m(A_0) + p(A_0' \cap A))$. For any set function p_0 that $0 \ \leq \ p_0 \ \leq \ p$, we find any infinite sequence $\{A_n\ |n \ \geq \ 1\}$ of set in $\mathcal{F}$ in descending order that $\cap_{n=1}^{\infty} A_n \ = \ 0$. So we can get that $0 \leq p_0(A_i) \leq p(A_i),\ for\ all\ i$. Because p is countably additive, we have $\lim_{\infty} p(A_i) = p(\cap_{n=1}^{\infty} A_n) = 0$. So we can get $\lim_{\infty} p_0(A_i) = 0$, which tells us that p_0 is also conutably additive.
So for any set function $0 \leq p_0 \leq p$, p_0 is a conutably function. If $p_0 \leq m$, because m is

purely finitely additive, we can know that $p_0 = 0$.
Then we can define a partial order in the space of finitely additive set function. For a set function h, $h \geq 0$ if $\forall T \in \mathcal{F}, h(T) \geq 0$. For two function h_1 and h_2, $h_1 \leq h_2$ if $h_2 - h_1 \geq 0$. So the space is a lattice in this partial order.
Now we can use the fact that the space of finitely additive set function is a lattice to show that $g = 0$. So for $\forall\ A \in \mathcal{F}$ and $\forall\ \epsilon > 0$, there exists a set $A_0 \in \mathcal{F}$, $A_0 \subseteq A$, s.t. $m(A_0) + p(A_0' \cap A) < \epsilon$. We have proven the lemma.

Now we can back to the original question. Because Q_1 is conutably additive and absolutely continuous relative to P, we can use the Dadon-Nikodym Theorem to show that there exists $Y \in \mathcal{L}^1(\Omega,\ \mathcal{F}_\infty,\ P)(\mathcal{F}_\infty$ is generated by $\mathcal{F}_0)$, s.t. $Q_1(F) = \int_F Y\,dP$. We set Q_{1_n} as the restriction of Q_1 on $\mathcal{F}_n$ and Y_n as the $E(Y;\mathcal{F}_n)$. We can get $Q_{1_n}(F) = \int_F Y_n\,dP$ and Q_{2_n}, which is the restriction of Q_2 on $\mathcal{F}_n$ can be written as $Q_{2_n}(F) = \int_F W_n\,dP$. So $X_n = Y_n + Z_n$. All we have left to do is to prove that $\lim Y_n$ and $\lim Z_n$ exist.
For $\lim Y_n$, because Y is $\mathcal{F}_\infty$ measurable, for $\forall\ \epsilon > 0$ we can find a $\mathcal{F}_0$ measurable set function f s.t. $\|Y - f\|_1 \leq \epsilon$. So

$$\begin{aligned} |Y_m - Y_n| &= |E(Y;\mathcal{F}_m) - E(Y;\mathcal{F}_n| \\ &\leq |E(f;\mathcal{F}_m) - E(f;\mathcal{F}_n)| + |E(Y-f;\mathcal{F}_m) - E(Y-f;\mathcal{F}_n)| \\ &\leq |E(f;\mathcal{F}_m) - E(f;\mathcal{F}_n)| + |E(Y-f;\mathcal{F}_m)| + |E(Y-f;\mathcal{F}_n)| \\ &\leq |E(f;\mathcal{F}_m) - E(f;\mathcal{F}_n)| + 2sup_n E(|Y-f;\mathcal{F}_n|). \end{aligned}$$

So for $\forall \epsilon_0$, we set $\mu_0 = 2\frac{\epsilon}{\epsilon_0}$ we can get

$$\begin{aligned} P(limsup_{m,n\longrightarrow\infty}|Y_m - Y_n| \geq \epsilon_0) &\leq P(2sup_n E(|Y-f;\mathcal{F}_n|) \geq \epsilon_0) \\ &\leq 2\,\|Y-f\|_1/\epsilon_0\ (By\ Doob's\ Submartingale\ inequation) \\ &\leq 2\frac{\epsilon}{\epsilon_0} \\ &= \mu_0. \end{aligned}$$

Because μ_0 can take any value, we get $P(limsup_{m,n\longrightarrow\infty}|Y_m - Y_n| \geq \epsilon_0) = 0$, so $\lim Y_n$ exist.
For $\lim Z_n$, we can know from above lemma that Z is small enough, so we guess that $\lim Z_n = 0$. We can get from the lemma2 that for $\forall \epsilon > 0$ we can find a set $T \in \mathcal{F}_0$, s.t. $Q_2(T) + P(T') < \epsilon$. So for $\forall \epsilon_0 > 0$, we set μ_0 as $2\frac{\epsilon}{\epsilon_0}$, then we can use Doob's Submartingale inequation:

$$\begin{aligned} &P(T; sup_n|Z_n| \geq \epsilon_0) < sup_n \int_T |Z_n|\,dP/\epsilon_0 < Q_2(T)/\epsilon_0. \\ &P(T'; sup_n|Z_n| \geq \epsilon_0) < P(T') < \epsilon. \end{aligned}$$

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\begin{aligned}
P(sup_n|Z_n| \geq \epsilon_0) = & P(T; sup_n|Z_n| \geq \epsilon_0) + P(T'; sup_n|Z_n| \geq \epsilon_0) \\
& \leq Q_2(T)/\epsilon_0 + \epsilon \\
& < \mu_0/2 + \epsilon \\
& < \mu_0.
\end{aligned}
$$

So we can get $\lim Z_n = 0$. So we can get $\lim X_n$ exists. So we finish the prove.

Reference.

[1] C.Swartz, *Measure, integration and function spaces*. World Scientific, 1994, 49-50.