Extended BC1

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For a sequence of events $\{E_n\}_{n\in\mathbb{N}}$ with respect to probability measure p, if there is $p(E_n^c \cap E_{n+1}) < \infty$ and $\lim_{n\to\infty} E_n = 0$, then $p(E_n\ i.o.) = 1$

Proof. Let $G_m = \cup_{n>m} E_n$ and $G'_m = \cup_{n\geq m}(E_n^c \cap E_{n+1})$. Then there is:

$$G'_m = \cup_{n\geq m}(E_n^c \cap E_{n+1}) = (\cup_{n>m} E_n)/(E_m \cap E_{m+1}) = G_m/(E_m \cap E_{m+1})$$

Then use (1.9,b) and (1.10,a) in textbook there is:

$$p(E_n i.o.) = p(\cap_{m\in\mathbb{Z}^+} G_m) \leq p(G_m) \leq p(G'_m) + p(E_m)$$

As $p(G'_m) \leq \sum_{n\geq m}(E_n^c \cap E_{n+1})$, let $m \uparrow \infty$ there is $\lim_{m\to\infty} p(G'_m) = 0$. By assumption there is $\lim_{m\to\infty} p(E_m) = 0$.

Combining the two result, let $m \uparrow \infty$ there is

$$0 \leq p(E_n i.o.) \leq \lim m \to \infty p(G'_m) + p(E_m) = 0$$

□

Now construct a condition that does not fit the condition of normal BC1 but fit this condition:

For a point x on $[0,1]$, E_n is that x is smaller than $\frac{1}{n}$, so $p(E_n) = \frac{1}{n}$(thus $p(E_n) \downarrow 0$) and $p(E_{n-1}^c \cap E_n) = 0$(thus the sum is $0 < \infty$), while $\sum_n p(E_n) = \infty$.

It is obvious that $p(E_n\ i.o.) = 0$(only $x = 0$ makes it infinitely often).