Solution to Exercise 1 of March 13

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Exercise 1. *1) Show that the Cantor Set C is nowhere dense in* $[0,1]$.

2) Find a meager set T in $\mathbb{R}$ such that $T+T=\mathbb{R}$.

3) Show that every subset of the real line $\mathbb{R}$ can be partitioned into two sets, one being of first category and the other being negligible.

Proof. 1) Recall the the explicit closed formulas for the Cantor set are:

$$C=[0,1]\setminus\bigcup_{n=0}^{\infty}\bigcup_{k=0}^{3^n-1}\left(\frac{3k+1}{3^{n+1}},\frac{3k+2}{3^{n+1}}\right)$$

It's obvious that $(\frac{3k+1}{3^{n+1}},\frac{3k+2}{3^{n+1}})$ is an open set. Since the union of open sets is an open set, the complement of C is also open, which leads to the fact that C is a closed set. So $\bar{C}=C$.

Note that Cantor set can also be characterized as the set of all number in $[0,1]$ whose base-3 expansion doesn't contain any 1s. Assume there exists an interior point x of C, then there exists a ball B, with radius $r>0$, which is contained in C. Since $C\subseteq[0,1]$, B is an interval, whose length is $2r$. Arbitarily choose $x_1\in B$ and $x_2\in B$ such that $x_1<x_2$ and $x_2-x_1\geq r$. Let $n=\lceil\log_{\frac{1}{3}}r\rceil+2$.Consider $x_3=x_1+(\frac{1}{3})^n, x_4=x_1+2\cdot(\frac{1}{3})^n$. Since $(\frac{1}{3})^n<(\frac{1}{3})^{\log_{\frac{1}{3}}r+1}=\frac{1}{3}r$, $x_3,x_4\in B$. It' s obvious that one of them conatins 1 in its base-3 expansion, which means one of them doesn't belong to Cantor set. This leads to a contradiction, or equivalently, that there is no interior point of C.

Now we can safely conclude that C is nowhere dense.

2)Let

$$T(n)=\{a+n|a\in C\}$$
$$T=\bigcup_{n\in\mathbb{Z}}T(n)$$

First We will prove T is a meager set. Since $T(n)$ is constructed by translation of all points in C by n, $T(n)$ is also nowhere dense. This leads to the fact that T is a meager set, because $\mathbb{Z}$ is countable.

Then we prove $T+T=\mathbb{R}$. From $C+C=[0,2]$, we can similarly get $T(n)+T(n)=[2n,2n+2]$. Note that:

$$\bigcup_{n\in\mathbb{Z}}(T(n)+T(n))\subseteq T+T\subseteq\mathbb{R}$$

However,

$$\bigcup_{n\in\mathbb{Z}}(T(n)+T(n)) = \bigcup_{n\in\mathbb{Z}}[2n, 2n+2] = \mathbb{R}$$

Now we conclude that $T+T=\mathbb{R}$.

3) Pick an enumeration of $\mathbb{Q}$, named q_n. Let:

$$I_{i,j} = (q_i - \frac{1}{2^{i+j+2}}, q_i + \frac{1}{2^{i+j+2}})$$
$$A_j = \bigcup_{i\in\mathbb{N}} I_{i,j}$$
$$B = \bigcap_{j\in\mathbb{N}} A_j$$

For any $\epsilon > 0$, exist $j_0 > 0$ such that $|\frac{1}{2^{j_0}}| < \epsilon$. So we have :

$$|A_j| \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+j+1}} = \frac{1}{2^j}$$
$$|B| \leq |A_{j_0}| \leq \frac{1}{2^{j_0}} < \epsilon$$

This leads to that B is negligible. Then we prove B^C is a meager set.

$$B^C = \bigcup_{j\in\mathbb{N}} A_j^C$$

Since $\mathbb{Q} \subseteq A_j$ and $\mathbb{Q}$ is dense, we can know that A_j is dense. A_j is also open because A_j is the union of open intervals. These 2 properties of A_j can lead to that A_j^C is nowhere dense. So B^C is a meager set, which finishes the proof. $\square$