Usual length function is countably additive

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Let $S = (0, 1]$, let Σ_0 be the subsets of S which are finite unions of disjoint left-open right-closed intervals. It is clear that Σ_0 is an algebra.
Let μ_0 be the usual length function on Σ_0. Show that μ_0 is countably additive.

Proof. Consider the disjoint sets $A_1, A_2, \cdots, A_n, \cdots$ with $\forall i \in \mathbb{N}_+, A_i \in \Sigma_0$ and $A = \bigcup_{i\in\mathbb{N}_+} A_i \in \Sigma_0$.

We would like that to show that

$$\mu_0(A) = \sum_{i\in\mathbb{N}_+} \mu_0(A_i).$$

By the definition of Σ_0, we have $A_i = \bigcup_{j=1}^{n_i} I_{ij}$ where I_{ij} are left-open right-closed intervals and are disjoint.

Then $A = \bigcup_{i\in\mathbb{N}_+} A_i = \bigcup_{i\in\mathbb{N}_+} \bigcup_{j=1}^{n_i} I_{ij}$, that is a countable union of finite union of sets, which is also countable. By renaming, let $A = \bigcup_{i\in\mathbb{N}_+} I_i$, where I_i are left-open right-closed and disjoint.

μ_0 satisfies finite additivity. Thus for all $N \in \mathbb{N}_+$,

$$\sum_{i=1}^{N} \mu_0(I_i) = \mu_0(\bigcup_{i=1}^{N} I_i).$$

And we have $\mu_0(\bigcup_{i=1}^{N+1} I_i) = \mu_0(\bigcup_{i=1}^{N} I_i) = \mu_0(I_{N+1}) \geq \mu_0(\bigcup_{i=1}^{N} I_i)$, this implies

$$\sum_{i=1}^{\infty} \mu_0(I_i) \leq \mu_0(A).$$

On the other hand, note that $\forall I_i \neq \varnothing, \mu_0(I_i) = \mu_0((a_i, b_i]) = b_i - a_i > 0$. That is,

$$\forall \varepsilon > 0, \exists I_i' := (a_i, b_i + \frac{\varepsilon}{2}) \in \Sigma_0, I_i \subseteq I_i', \mu_0(I_i') - \mu_0(I_i) = \frac{\varepsilon}{2} < \varepsilon.$$

Also, we have $A \in \Sigma_0$, which means $A = \bigcup_{t=1}^{r} A_t$. Similarly, for a fixed $\varepsilon > 0$, for each A_t, we can have $[l_t, r_t] \subseteq A_t$ with $\mu_0(A_t) - \mu_0([l_t, r_t]) < \frac{\varepsilon}{r}$.

From this we can get a infinite open cover of $[l_t, r_t]$. By the property of $\mathbb{R}$, there exists a finite open cover $I'_{tk_1}, I'_{tk_2}, \cdots I'_{tk_M}$ such that $A_t \subseteq \bigcup_{i=1}^{M} I'_{tk_i}$. Then by the elementary inequality,

$$\begin{aligned}
\mu_0(A) = \sum_{t=1}^{r} \mu_0(A_t) < \varepsilon + \sum_{t=1}^{r} \mu_0([l_t, r_t]) &= \varepsilon + \sum_{t=1}^{r} \mu_0(\bigcup_{i=1}^{M} I_{tk_i}) \\
&\leq \varepsilon + \sum_{t=1}^{r} \sum_{i=1}^{M} \mu_0(I'_{tk_i}) \\
&\leq \varepsilon + \sum_{t=1}^{r} \sum_{i=1}^{M} (\mu_0(I_{tk_i}) + \varepsilon_i) \\
&\leq \varepsilon + \sum_{i=1}^{\infty} \mu_0(I_i) + r \sum_{i=1}^{M} \varepsilon_i.
\end{aligned}$$

Let $\varepsilon_i = \frac{\varepsilon}{r2^i}$, finally We get $\mu_0(A) \leq \sum_{i=1}^{\infty} \mu_0(I_i) + 2\varepsilon$ for all $\varepsilon > 0$.
Combine the two sides, resulting in

$$\mu_0(A) = \sum_{i \in \mathbb{N}_+} \mu_0(A_i).$$

That finished the proof for the countable additivity of μ_0.

$\square$