Converse to SLLN

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April 12, 2020

Problem 1 (Exercise 4.6 of *Chapter E*)
Let Z be a non-negative RV. Let Y be the integer part of Z . Show that

$$Y = \sum_{n \in \mathbb{N}} I_{\{Z \geqslant n\}}$$

and deduce that

$$\sum_{n \in \mathbb{N}} P[Z \geqslant n] \leqslant E(Z) \leqslant 1 + \sum_{n \in \mathbb{N}} P[Z \geqslant n]$$

Let (X_n) be a sequence of IID RVs (independent, identically distributed random variables) with $E(|X_n|) = \infty, \forall n$. *Prove that*

$$\sum_n P[|X_n| > kn] = \infty (k \in \mathbb{N}) \text{ and } \limsup \frac{|X_n|}{n} = \infty (a.s.)$$

Deduce that if $S_n = X_1 + X_2 + \cdots + X_n$, then

$$\limsup \frac{|S_n|}{n} = \infty (a.s.)$$

Proof: The proof of the first equality is trivial，and

$$\begin{aligned} E(Z) &= E(Y) + E(Z - Y) \\ &= E\left(\sum_{n \in \mathbb{N}} I_{\{Z \geqslant n\}}\right) + E(Z - Y) \\ &= \sum_{n \in \mathbb{N}} P[Z \geqslant n] + E(Z - Y) \\ &= \sum_{n \in \mathbb{N}} P[Z \geqslant n] + \frac{1}{2} \end{aligned}$$

So

$$\sum_{n \in \mathbb{N}} P[Z \geqslant n] \leqslant E(Z) \leqslant 1 + \sum_{n \in \mathbb{N}} P[Z \geqslant n]$$

Let $E_n^k = \left\{\frac{X_n}{n} > k\right\}$. Then

$$\begin{aligned}
\sum_n P(E_n^k) &\geqslant \sum_n P\left(\frac{|X_n|}{n} \geqslant k+1\right) \\
&= \sum_n P\left(\frac{|X_n|}{k+1} \geqslant n\right) \\
&\geqslant E\left(\frac{|X_n|}{k+1}\right) - 1 \\
&= \infty
\end{aligned}$$

by BC2

$$P(E_n^k, \text{i.o.}) = P(\limsup E_n^k) = 1$$

Hence

$$P(\limsup \frac{|X_n|}{n} = \infty) = 1 \Rightarrow \limsup \frac{|X_n|}{n} = \infty(a.s.)$$

At last, consider

$$\frac{X_n}{n} = \frac{S_n}{n} - \left(\frac{n-1}{n}\right)\frac{S_{n-1}}{n-1}$$

we can know

$$\frac{|X_n|}{n} \leqslant \frac{|S_n|}{n} + \left(\frac{n-1}{n}\right)\frac{|S_{n-1}|}{n-1}$$

then

$$\limsup \frac{|X_n|}{n} \leqslant 2 \limsup \frac{|S_n|}{n}$$

Hence

$$P(\limsup \frac{|S_n|}{n} = \infty) = 1 \Rightarrow \limsup \frac{|S_n|}{n} = \infty(a.s.)$$

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