Independence of $\pi-$system

Fu Lingyue X Tang Ze

April 20, 2020

In the text book $E4.1$, the author put forward a theorem as follows:

1 E4.1 in textbook

Theorem 1. *$\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ are three π-system that satisfy:*
(1) $\mathcal{I}_k \subseteq \mathcal{F}(k = 1, 2, 3)$;
(2) $\Omega \in \mathcal{I}_k(k = 1, 2, 3)$.
If

$$\forall I_i \in \mathcal{I}_i, P(I_1 \cap I_2 \cap I_3) = P(I_1)P(I_2)P(I_3), \tag{1}$$

then $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), \sigma(\mathcal{I}_3)$ are independent.

Proof. Define $\mathcal{J}_i := \sigma(\mathcal{I}_i)$. Fix $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. Consider maps

$$\mathcal{J}_3 \mapsto P(I_1 \cap I_2 \cap J_3) \text{ and } \mathcal{J}_3 \mapsto P(I_1)P(I_2)P(J_3),$$

then two mapping agree on $\mathcal{I}_3$. Also when $I_3 = \Omega$ in equation (1), we can conclude that

$$P(I_1 \cap I_2) = P(I_1 \cap I_2 \cap \Omega) = P(I_1)P(I_2)P(\Omega) = P(I_1)P(I_2),$$

which means, two mapping have the same total mass.Thus we can conclude that $P(I_1 \cap I_2 \cap J_3) = P(I_1)P(I_2)P(J_3)$ holds in the space $(\Omega, \mathcal{J}_3)$.

Similarly, we can conclude the conclusion on both $\sigma(\mathcal{J}_1)$ and $\sigma(\mathcal{J}_2)$. Therefore, $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), \sigma(\mathcal{I}_3)$ are independent. $\square$

Question WHY we need the condition "$\Omega \in \mathcal{I}_i$"?

Solution. In the lemma 1.6 in the textbook, one of the premises is that μ_1 and μ_2 has the same mass on S. Then this condition guarantees that each pair of mapping in our prove has the same mass. $\square$

2 Further Discussion

Actually, we can strengthen this theorem:

Theorem 2. $\mathcal{I}_i (i = 1, 2, 3 \ldots, n)$ *are independent* $\pi-$*system, then* $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), \ldots, \sigma(\mathcal{I}_n)$ *are independent.*

Proof. Simply use the induction.

The textbook has proved $n = 2$, and similar to the $\square$