Liouville Number

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2020 年 3 月 17 日

Exercise *1) Show that* $\sum_{j\in\mathbb{N}} \frac{1}{2^{j!}}$ *is a Liouville number.*

2) *Show that every Liouville number must be transcendental.*

Solution:

1) I present here a constructive proof.

   Since $\sum_{j\in\mathbb{N}} \frac{1}{2^{j!}}$ is a sum of infinite series, we can choose, for every n, $\frac{p_n}{q_n}$ to be the prefix sum of $\frac{1}{2^{j!}}$.

   With this intuition, we let
$$\frac{p_n}{q_n} = \sum_{j=1}^{n} \frac{1}{2^{j!}}$$

   Then,
$$\begin{aligned}
|\sum_{j\in\mathbb{N}} \frac{1}{2^{j!}} - \frac{p_n}{q_n}| &= \sum_{j\in\mathbb{N}} \frac{1}{2^{j!}} - \sum_{j=1}^{n} \frac{1}{2^{j!}} \\
&= \sum_{j=n+1}^{\infty} \frac{1}{2^{j!}} \\
&< \sum_{j=(n+1)!}^{\infty} \frac{1}{2^{j}} = \frac{1}{2^{(n+1)!-1}} \\
&< \frac{1}{(2^{n!})^n} \quad (n! \cdot n < n! \cdot (n+1) < (n+1)!)
\end{aligned}$$

   Thus, we choose q_n to be $2^{n!}$, and p_n to be $q_n * \sum_{j=1}^{n} \frac{1}{2^{j!}} = 2^{n!} * \sum_{j=1}^{n} \frac{1}{2^{j!}}$

2) We prove by contradiction.

   Assume that there is a Liouville number z that is not transcendental, then it must be a algebraic number of degree n. We know from the *lecture notes* that, there exists a positive integer

M such that for all integers p and q,

$$|z - \frac{p}{q}| > \frac{1}{M \cdot q^n} \tag{1}$$

It also holds for Liouville number z that for every integer n there exists integers p and q such that

$$|z - \frac{p}{q}| < \frac{1}{q^n} \tag{2}$$

Notice that *(1)* and *(2)* are inequalities of opposite signs, in order to induce a contradiction, it must be the case that for some n'(not to be confused with the degree n),

$$\frac{1}{q^{n'}} < \frac{1}{M \cdot q^n} \iff M \cdot q^n < q^{n'}$$

This could easily be achieved by setting

$$n' = \lceil n + \log_q M \rceil$$

Thus a contradiction!
Therefore, every Liouville number must be transcendental.

$\square$