Martingales with Graph Theory

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1 Large Deviations

When we study a Stochastic process we can use the central limit theorem to learn the process in infinity, but usually we concern the situation at a specific point in time. Large Deviations Theory can help us.

Theorem 1. ***(Azuma's Inequality)*** *Let (X_i) be a martingales with respect to filter $(\mathcal{F}_i)$, and let $Y_i = X_i - X_{i-1}$. If the $|Y_i| \leq c_i$ for all i, then*

$$\left.\begin{array}{l}\Pr\left[X_n \geq X_0 + \lambda\right] \\ \Pr\left[X_n \leq X_0 - \lambda\right]\end{array}\right\} \leq \exp\left(-\frac{\lambda^2}{2\sum_{i=1}^{n} c_i^2}\right)$$

In order to proof Azuma's Inequality, we need to indruduce a lemma.

Lemma 2. *Let Y be a random variable such that $Y \in [-1, +1]$ and $\mathrm{E}[Y] = 0$. Then for any $t \geq 0$, we have that $\mathrm{E}\left[e^{tY}\right] \leq e^{t^2/2}$.*

Proof. We know e^{tx} is convexity function, so for $x \in [-1, 1]$, $e^{tx} \leq \frac{1}{2}(1+x)e^t + \frac{1}{2}(1-x)e^{-t}$.

$$\begin{aligned}\mathrm{E}\left[e^{tY}\right] &\leq \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t) \\ &= 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} \dots \\ &= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \leq \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!} = e^{\frac{t^2}{2}}\end{aligned}$$

□

Then we can implement the lemma in **Proof of Azuma's Inequality**. For $t > 0$

$$\Pr\left[X_n - X_0 \geq \lambda\right] = \Pr\left[e^{t(X_n - X_0)} \geq e^{\lambda t}\right]$$

Applying Markov's inequality

$$\Pr\left[e^{t(X_n - X_0)} \geq e^{\lambda t}\right] \leq e^{-\lambda t}\mathrm{E}\left[e^{t(X_n - X_0)}\right] = e^{-\lambda t}\mathrm{E}\left[e^{t(Y_n + X_{n-1} - X_0))}\right]$$

Using Condition Expection trick and Martingales property

$$\mathrm{E}\left[e^{t(Y_n+X_{n-1}-X_0))}\right] = \mathrm{E}\left[\mathrm{E}\left[e^{t(Y_n+X_{n-1}-X_0))}\middle|\mathcal{F}_{n-1}\right]\right] = \mathrm{E}\left[e^{t(X_{n-1}-X_0))}\mathrm{E}\left[e^{tY_n}\middle|\mathcal{F}_{n-1}\right]\right]$$

From above lemma, we know

$$\mathrm{E}\left[e^{tY_n}\middle|\mathcal{F}_{n-1}\right] \leq e^{\frac{t^2c_n^2}{2}}$$

and we can handle $\mathrm{E}[e^{t(X_{n-1}-X_0)}]$ with same way, hence

$$\Pr[X_n - X_0 \geq \lambda] \leq e^{-\lambda t + t^2 \sum_{i=1}^n c_i^2/2}$$

Finally, optimize the inequality by taking $t = \frac{\lambda}{\sum c_i^2}$, which gives

$$\Pr\left[X_n - X_0 \geq \lambda\right] \leq \exp\left(-\frac{\lambda^2}{2\sum_i c_i^2}\right)$$

2 Random Graph

2.1 Clique Number

Before introducing the martingale method, let's see a simple application of probabilistic method.

Theorem 3. *For $G \in \mathcal{G}_{n,p}$, the clique number of $G \sim 2\log_{1/p}(n)$.*

Proof. Consider the X_k be the number of k-clique in G, then $\mathrm{E}X_k = \binom{n}{k}p^{\binom{k}{2}}$. Let

$$c(n) = \max\{k : \mathrm{E}X_k \geq 1\}$$

Note that $k \ll n$, $g(k) \sim \frac{n^k}{k!}p^{\frac{k^2}{2}} \sim (1/p)^{k\log_{1/p} n - k\log_{1/p} k - k^2/2} \sim (1/p)^{k\log_{1/p} n - k^2/2}$. Hence, it gives us $c(n) \sim 2\log_{1/p} n$. ☐

In addition, for $k \sim 2\log_2 n$, we have

$$\frac{g(k+1)}{g(k)} = \frac{n-k}{k+1}2^{-k} \sim \frac{1}{2n\log n} \to 0 \text{ as } n \to \infty$$

For example, we can get $g(c(n)+3) = n^{3+o(1)}$. Let $k(n) = c(n)+3$, we will use it later.

2.2 Chromatic Number

Here is a simple application of Azuma's inequality in chromatic number.

Example 4. *Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V chosen uniformly from among all $2^{|V|}$ subsets of V. Let $H = G[U]$ be the induced subgraph of G on U. Prove that*

$$\Pr[\chi(H) \leq 400] < 1/100$$

Proof. Because of $\chi(G) = 1000$, so we can fix a partiton $V = V_1 \cup V_2 \cup \cdots \cup V_{1000}$, and every V_i is a independent set in G. Let $U_i' = U \cap V_i$ and $U_r = \cup_{i=1}^{r} U_i'$. Consider the $\chi(U_i)$, easy to find each U_i' contribute at most 1, so we let $Z_i = \bigcup_{v \in U_i} Y_i$, Y_i be an indicator of whether the vertex v is present in the graph, and $X_i = \mathrm{E}(\chi(H)|Z_i)$ be Doob martingales.

Note that $\chi(G[V/U])$ has the same distribution as $\chi(U)$, and $\chi(H) + \chi(G[V/U]) \geq 1000$. So we know $\mathrm{E}(\chi(H))$ at least 500. Hence we can use the Azuma's inequality.

$$\Pr(\chi(H) \leq 400) \leq \Pr(\chi(H) - \mathrm{E}(\chi(H)) \leq -100) \leq \exp(-\frac{100^2}{2 \cdot 1000}) < 1/100$$

□

In this example, we see the useful of Azuma's Inequality in analyzing the distribution of R.V. There are two commonly used martingales in random graph theory.

edge exposure martingale *In the $\mathcal{G}_{n,p}$ setting, let Z_i be an indicator of whether the i^{th} possible edge is present in the graph. Let $A = f(Z_1 \ldots Z_{\binom{n}{2}})$ be any graph property. Then Doob martingale $X_i = \mathrm{E}[A \mid Z_1 \ldots Z_i]$ is edge exposure martingale.*

vertex exposure martingale *Let $Z_i \in \{0,1\}^{n-i}$ be a vector of indicators of whether edges between vertex i and vertices $j > i$ are present. For any graph property $A = f(Z_1 \ldots Z_n)$ the corresponding martingale $X_i = \mathrm{E}[A \mid Z_1 \ldots Z_i]$ is called a vertex exposure martingale.*

Theorem 5. ***(Shamir and Spencer)*** *Let X be the chromatic number of $G \in \mathcal{G}_{n,\frac{1}{2}}$. Then*

$$\Pr[|X - \mathrm{E}[X]| \geq \lambda] \leq 2\exp\left(-\frac{\lambda^2}{2n}\right)$$

Proof. We just to use the vertex exposure martingale $X_i = \mathrm{E}(\chi(G)|Z_1, \ldots, Z_i)$. It is obvious that $|X_i - X_{i-1}| < 1$, so we can conconlude the result by applying Azuma's Inequality. □

Theorem 6. *For $G \in \mathcal{G}_{n,\frac{1}{2}}$, we have $\mathrm{E}[\chi(\mathrm{G})] \sim \frac{n}{2\log_2 n}$*

Proof. Note that every vertices with the same color must be a independent set, and the independent set in G correspond to clique in $\overline{G}$. So the $\alpha(G) \sim 2\log_{1/(1-p)} n$ *a.s*, then $\mathrm{E}[\chi(G)]$ lower bound be the $\frac{n}{2\log_{1/(1-p)} n}$. Hence we just need to prove the upper bound.

To prove the upper bound, we indruduce a new lemma will be proved later

Lemma 7. *Let $G \in \mathcal{G}_{n,\frac{1}{2}}$, event E is G contains no independent set of size larger than $k(n)$ then* $\Pr[E] \leq \exp\left(-n^{2-o(1)}\right)$

Let S be an arbitrary subset of G, with $m = |S| = \frac{n}{(\log_2 n)^2}$. $G[S]$ be a induced subgraph. From above lemma, we know probability of $G[S]$ contains an independent set of size $c(m) \sim 2\log_2 m \sim 2\log_2 n$, and $1 - \exp\left(-m^{2-o(1)}\right) = 1 - \exp\left(-n^{2-o(1)}\right)$.

Hence,

$$
\begin{aligned}
\Pr[\exists S \text{s.t. } G[S] \text{ contains no independent set of size } k(m)] &\le \binom{n}{m} \exp\left(-n^{2-o(1)}\right) \\
&\le 2^n \exp\left(-n^{2-o(1)}\right) \\
&= o(1)
\end{aligned}
$$

Now consider the following method for coloring G :

while $\exists$ more than m uncolored vertices in G **do**
  pick an arbitrary uncolored subset $S \subseteq V(G)$ of size m
  pick a new color and apply this to a largest independent set in S
color each remaining vertex of G with a different new color

We know that every iteration of the while-loop colors at least $k(m)$ vertices. Hence the number of colors used by the above at most

$$
\frac{n}{k(m)} + m = \frac{n}{2\log_2 n}(1 + o(1))
$$

Hence we proves the upper bound on $\chi(G)$. $\square$

Now, let's use martingales to prove the lemma.

Let $G \in \mathcal{G}_{n,\frac{1}{2}}$, event E is Gcontains no independent set of size larger than $k(n)$, then $\Pr[E] \le \exp\left(-n^{2-o(1)}\right)$

Proof. Let $Y = f(Z_1, Z_2, \ldots, Z_{\binom{n}{2}})$ be the size of a maximal family of edge-disjoint $k(n)$-cliques in G. Let X_i is the edge exposure martingale of G, $X_i = \mathrm{E}(Y|Z_1, Z_2, \ldots, Z_i)$, and the Z_i is the edge exposure process. Note that we required edge-disjoint. So f is 1-Lipschitz! So we can use Azuma's Inequality in Y.

Claim 8.

$$
\mathrm{E}[Y] \ge \frac{n^2}{2k(n)^4}(1 + o(1))
$$

Define

$$
\begin{aligned}
&K = \{H \in G : |H| = k(n), H \textit{ is a clique }\} \\
&P = \{\{A, B\} : A, B \in K, |A \cap B| > 1\} \\
&\mu = E[|K|]
\end{aligned}
$$

By second moment method Now consider the $\mathrm{Var}[X_k]$*,* X_S *is a indicator R.V. of the event* S *is a clique in* $\mathcal{G}_{n,1/2}$*,note that*

$$
\begin{aligned}
\mathrm{Var}(X_k) &= \mathbb{E}\left[X_k^2\right] - (\mathbb{E}[X_k])^2 \\
&= \mathbb{E}\left[\sum_S \sum_T X_S X_T\right] - \left(\mathbb{E}\left[\sum_S X_S\right]\right)^2 \\
&= \sum_S \sum_T (\mathbb{E}[X_S X_T] - \mathbb{E}[X_S]\mathbb{E}[X_T])
\end{aligned}
$$

Consider the intersection size $\ell = |S \cap T|$, if the $\ell \leq 1$, X_S and X_T are independent, then $\mathbb{E}[X_S X_T] - \mathbb{E}[X_S]\mathbb{E}[X_T]$ will be zero.

$$\begin{aligned}
\operatorname{Var}(X_k) &= \sum_{\ell=2}^{k} \sum_{S} \sum_{T:|S\cap T|=\ell} \left(\mathbb{E}[X_S X_T] - \mathbb{E}[X_S]\mathbb{E}[X_T]\right) \\
&\leq \sum_{\ell=2}^{k} \sum_{S} \sum_{T:|S\cap T|=\ell} \mathbb{E}[X_S X_T] \\
&\leq \sum_{\ell=2}^{k} \binom{n}{k}\binom{k}{\ell}\binom{n-k}{k-\ell} \mathbb{E}[X_S X_T] \\
&\leq \sum_{\ell=2}^{k} \binom{n}{k}\binom{k}{\ell}\binom{n-k}{k-\ell} 2^{-2\binom{k}{2}+\binom{\ell}{2}}
\end{aligned}$$

From above analysis, know $\mathrm{E}[|P|] = \operatorname{Var}(X_k)$

$$\begin{aligned}
\frac{\mathrm{E}[|P|]}{\mu^2} &= \frac{\mathrm{Var(X_k)}}{(E(X_k)^2)} \\
&\sim \sum_{\ell=2}^{k} \frac{2\binom{k}{\ell}\binom{n-k}{k-\ell}\binom{\ell}{2}}{\binom{n}{k}} \\
&\sim \sum_{\ell=2}^{k} \frac{k^{2\ell}}{(n-k)^{\ell}} 2^{\ell^2/2}
\end{aligned}$$

Hence

$$\frac{\mathrm{E}[|P|]}{\mu^2} \sim \frac{k(n)^4}{n^2}$$

Now let K' be a random subset of K obtained by choosing each $k(n)$ -clique with probability q T.B.D. Let P' be the associated set of pairs of cliques from P.

$$\begin{aligned}
&\mathrm{E}[|K'|] = q\mathrm{E}[|K|] = q\mu \\
&\mathrm{E}[|P'|] = q^2\mathrm{E}[|P|] \sim q^2 \tfrac{k(n)^4}{n^2}\mu^2
\end{aligned}$$

Remove from K' one element of each pair in P'. This now gives an edge disjoint family Y of $k(n)$-cliques with

$$\begin{aligned}
\mathrm{E}[|Y|] &\geq \mathrm{E}[|K'|] - \mathrm{E}[|P'|] \\
&\sim q\mu - q^2 \frac{k(n)^4}{n^2}\mu^2
\end{aligned}$$

Taking $q = \frac{n^2}{\mu k(n)^4} < 1$, it gives us the claim.

Now we can finish the proof by the claim,

$$\begin{aligned}
&\Pr\left[G \text{ contains no cliques of size } k(n)\right] \\
\leq &\Pr[Y=0] = \Pr[Y - \mathrm{E}[Y] \leq -\mathrm{E}[Y]] \\
\leq &\exp\left(-\frac{\mathrm{E}[Y]^2}{2\binom{n}{2}}\right) \\
\leq &\exp\left(-\frac{n^2}{2k(n)^8}(1+o(1))\right) \\
= &\exp\left(-n^{2-o(1)}\right)
\end{aligned}$$

□

2.3 Infection Model

Let $G = (V, E)$ is a undirected graph with n vertices and m edges, each vertex of G colored black or white. Consider the random process on G, which is a simple model of infection. At each step, all vertices simultaneously update their colors,

- Do nothing with probability 0.5
- Pick a neighbor vertex uniformly and choose its color.

Remark 9. *Let $\mathcal{F}$ be the σ-field generated by the outcomes of the first t steps of the process, and X_t donate the sum of degrees white vertices. Then (X_t) is a martingale.*

Proof. Assume $X_t \notin \{0, 2m\}$. At time t, W_t, B_t is the set of white vertices and black vertices. Z_u is a indicator function of the event that u changes color.

$$X_{t+1} - X_t = \sum_{u \in B_t} \deg_u Z_u - \sum_{u \in W_t} \deg_u Z_u$$

Define Y_u is the number of neighbors of u with the opposite color to u, then $\mathrm{E}(Z_u) = \frac{Y_u}{2\deg_u}$,

$$\mathrm{E}(X_{t+1} - X_t | \mathcal{F}_t) = \frac{1}{2}\left(\sum_{u \in B_t} Y_u - \sum_{u \in W_t} Y_u\right) = 0$$

Hence, (X_t) is a martingale. □

If all vertices color are white or black, the process will stop, using Optional Stoping Theorem

Theorem 10. *Let (X_i) be a martingale and T be a stopping time with respect to a filter $(\mathcal{F}_i)$, if*

- *T is bounded.*
- *X is bounded.*
- *$\mathrm{E}[X_i I_{\{T>i\}}] \to 0$ as $i \to \infty$.*

Then we have $\mathrm{E}(X_T) = \mathrm{E}(X_0)$.

Then Optional Stoping Theorem gives us $\mathrm{E}(X_T) = \mathrm{E}(X_0) = X_0$. If p is the probability of end in all white vertices, then

$$p \times 2m + (1 - p) \times 0 = X_0$$

Hence, $p = \frac{X_0}{2m}$.

3 Contribution

In this assignment, the main reference book is Chapter 7 of *Probability Methods* (Noga Alon, Joel H. Spencer), *Graph Theory*(Reinhard Diestel), and a paper named "The chromatic number of random graphs" by B. Bollobas. I learned the Large Deviations from *Probability Methods* and learned the Second Moment Methods in *Graph Theory*.

I finished two exercises in *Probability Methods* and *Probability With Martingales* , arranged proofs about chromatic number expectations and added the second moment proof parts which skipped in the paper. The Infection Model(2.3) section is a small exploration of mine.

My main motivation for choosing this topic is that I learned the application of probabilistic methods in graph theory classes, so I want to learn about the application of martingales in graph theory.