$P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1} < \infty)$ $\Rightarrow$ $P(A_n, \text{i.o.}) = 0$

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This note proves a extension of the First Borel-Cantelli Lemma(BC1).

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(A_n : n \in \mathbb{N}) \subseteq \mathcal{F}$ be a sequence of events. We have the following proposition.

Proposition 1. *If* $\lim P(A_n) = 0$ *and* $\sum\limits_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$*, then* $P(A_n, i.o.) = 0$*. [1]*

Proof. For an arbitrary fixed $n \in \mathbb{N}$, we have

$$\begin{aligned}
P(A_n, \text{i.o.}) &= P(\limsup A_n) \\
&= P(\bigcap_n \bigcup_{m \geqslant n} A_m) \\
&\leqslant P(\bigcup_{m \geqslant n} A_n) \\
&= P(A_n \sqcup \bigsqcup_{m>n} (A_m \setminus \bigcup_{n \leqslant i < m} A_i)) \\
&= P(A_n) + \sum_{m>n} P(A_m \setminus \bigcup_{n \leqslant i < m} A_i) \\
&= P(A_n) + \sum_{m>n} P(A_m \cap \bigcap_{n \leqslant i < m} A_i^c) \\
&\leqslant P(A_n) + \sum_{m>n} P(A_m \cap A_{m-1}^c) \\
&= P(A_n) + \sum_{m \geqslant n} P(A_{m+1} \cap A_m^c).
\end{aligned}$$

Since it is given that $\lim P(A_n) = 0$ and $\sum\limits_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$, we know that $\lim\limits_{n\to\infty} \sum\limits_{m \geqslant n} P(A_{m+1} \cap A_m^c) = 0$. If we let $n \to \infty$, immediately we get

$$\begin{aligned}
P(A_n, \text{i.o.}) &\leqslant \lim_{n\to\infty} P(A_n) + \lim_{n\to\infty} \sum_{m \geqslant n} P(A_{m+1} \cap A_m^c) \\
&= 0 + 0 = 0.
\end{aligned}$$

As $P(A_n, \text{i.o.}) \geqslant 0$ always holds, it follows that $P(A_n, \text{i.o.}) = 0$. □

References

[1] Tapas Kumar Chandra. The Borel-Cantelli Lemma. Springer India, India, 2012.