Independent of Tail Sigma Algebra

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Problem 1 (Exercise 4.9 of *Chapter E*)
Let $Y_0, Y_1, Y_2, \dots$ *be independent random variables with*

$$P(Y_n = +1) = P(Y_n = -1) = \frac{1}{2}, \ \ \forall n$$

For n $n \in \mathbb{N}$*, define*

$$X_n := Y_0 Y_1 \dots Y_n$$

Prove that the variables $X_1, X_2, \dots$ *are independent. define*

$$\mathcal{Y} := \sigma(Y_1, Y_2, \dots), \ \ \mathcal{T}_n := \sigma(X_r : r > n)$$

Prove that

$$\mathcal{L} := \bigcup_n \sigma(\mathcal{Y}, \mathcal{T}_n) \neq \sigma\left(\mathcal{Y}, \bigcup_n \mathcal{T}_n\right) =: \mathcal{R}$$

Proof: (a)

For $i < j$, let $Y_{ij} = Y_{i+1}Y_{i+2}\dots Y_j$, then we can know $P(Y_{ij} = +1) = P(Y_{ij} = -1) = \frac{1}{2}$ by symmetry.

And we can easily get for $j > i$, X_i, Y_{ij} be independent variable. Hence

$$P(X_i = 1, X_j = 1) = P(X_i = 1, Y_{in} = 1) = P(X_i = 1)P(Y_{in} = 1) = \frac{1}{4}$$
$$P(X_i = 1, X_j = -1) = P(X_i = 1, Y_{in} = -1) = P(X_i = 1)P(Y_{in} = -1) = \frac{1}{4}$$
$$P(X_i = -1, X_j = 1) = P(X_i = -1, Y_{in} = -1) = P(X_i = -1)P(Y_{in} = -1) = \frac{1}{4}$$
$$P(X_i = -1, X_j = -1) = P(X_i = -1, Y_{in} = 1) = P(X_i = -1)P(Y_{in} = 1) = \frac{1}{4}$$

Therefore X_i and X_j are independent for all $i \neq j$.

(b) Obviously, Y_0 is independent of $\mathcal{Y}$. $X_0 = Y_0$, so Y_0 is independent of X_n for $n > 0$. Therefore Y_0 is independent of $\mathcal{T}_n$ for all n. So Y_0 is independent of $\mathcal{R}$.

Then we can proof $Y_0 \in m\mathcal{L}$, it means Y_0 is measurable in $\mathcal{L}$. And we know that $Y_0 = X_{n+1}/Y_{1,n+1}$, X_{n+1} is measurable in $\mathcal{T}_n$, and $Y_{1,n+1}$ is measurable in $\mathcal{Y}$. It means that X_{n+1} and $Y_1, Y_2, \ldots, Y_{n+1}$ enables us to solve Y_0. Hence Y_0 ignorespaces measurable in $\sigma(\mathcal{F}, \mathcal{T}_n)$ for all n. It implies that Y_0 is measurable in $\mathcal{L}$.

Because Y_0 is independent of $\mathcal{R}$ and Y_0 is measurable in $\mathcal{L}$, we conlude that $\mathcal{L} \neq \mathcal{R}$. $\square$