A martingale in a finance problem option pricing

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In the semester, I found probability theory very hard to understand. The reason behind it ,in my opinion, is that it's too abstract and there're not enough examples for those lacking the ability to corelate abstract math concepts with something more concrete. Therefore, I tried to use probability theory learned in class in concrete real life problems in this final report, option strategy in particular.

Keywords: martingale, finance, option pricing(fiance)

1 INTRODUCTION

Martingale [MAN09] is an important part in our probability class, it is originally referred to a class of betting strategies popular back in 18th-century France. Paul Lévy introduced the concept of martingale into probability in 1934, since then, with a lot of successful work developed by many top mathematicians like Joseph Leo Doob, martingale theory has been a basis in modern probability.

Finance theory focuses on decision making under uncertain constraints. Take stock pricing, option pricing as an example. The parameters like expected dividend and expected profits are all uncertain. Since 1970s, economists have been using martingale theory in describing price fluctuation. With martingale, finance analysis becomes more concise and easier to compute with modern numeral calculation tools.

The reason why martingale can be used in finance is obvious. Just like these TV programs in finance channel, investors always predict future stock market based on previous information, and they use expectations to express the process of price fluctuation.

On the other side, martingale is based on conditional expectations, much similar to the traditional finance analysis. It's this similarity that makes martingale popular in finance.

2 Martingale in Option (finance)

2.1 Option(finance)

In finance, an option [MM98] is a contract which gives the buyer (the owner or holder of the option) the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price prior to or on a specified date, depending on the form of the option. There're several types of option according to different classification rule.

type	*reason*
call option	holder has right but not obligation to buy at a specific price for a specific time period
put option	holder has right but not obligation to sell at a specific price for a specific time period

Table 2.1: According to option rights

type	*reason*
American option	an option may be exercised on any trading day on or before expiration
European option	an option may only be exercised on expiry

Table 2.2: According to option styles

Option has a nonlinear profit-loss structure, which alleviates investors of constructing different portfolio investment. Basically, If you buy a call option and it reaches the price you expected before the end of the contract, you can sell it and get profit after deducting the option price. Therefore, how to set the price of an option is something we need to take care of.

2.2 European option

Since european option must be exercised at the end of it, so it's much similar to discrete martingale we learned in class. So we focus on european option. Here are some notations.
Suppose investor buy option at time 0, and at the expiration day T, the stock price is $S(T)$, the strike price(at which the underlying transaction will occur upon exercise) of an option is K. Then the profit of a european call option is:

$$P = (S(T) - K)^* = \begin{cases} S(T) - K, & S(T) > K \\ 0, & S(T) \le K \end{cases}$$

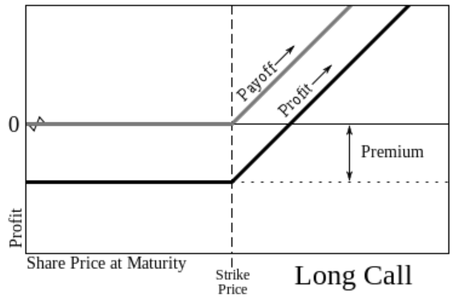

Figure 2.1: Payoff from buying a call option

2.3 1-period model

In the simple one-period pricing model [Chu79], we assume it is arbitrage-free, that is, there're no portfolio that ensures profit. The sample space Ω consists of two elements H, T, such that $P\{H\} = p, P\{T\} = 1 - p, p \in (0, 1)$. Let S_0(a constant random variable: $S_0(H) = S_0(T)$) be the initial price of stock. If $\omega = H$, then $S_1(H) = uS_0$, $S_1(H)$ stands for the price of stock at expiry. While $\omega = T, S_1(H) = dS_0$. Here, H, T means two different conditions, H means the stock will profit while T means the stock loss. Let r be the rate of return of riskless security. Based on this intuition, we have

$$0 < d < 1 + r < u \tag{2.1}$$

And the rate of return can be written as

$$g(\omega) = \frac{S_1 - S_0}{S_0} = \frac{S_1(\omega) - S_0}{S_0} = \begin{cases} u - 1, & \omega = H, \\ d - 1, & \omega = T \end{cases} \tag{2.2}$$

Rewrite (2.1) as

$$d - 1 < r < u - 1$$

We can see that if r satisfies 2.1, it is a meaningful option where we gain more profit than investing stock only.

Let V_0, V_1 be the values of a European option at time 0 and 1, respectively. Consider such an investment environment. We can only invest 3 following assets: the option, its underlying stock and the riskless security, namely the bond. Therefore, a portfolio is a triple (α, β, γ), where α is the number of options, β the number of stock, γ the number of bond. For bond, we also define B_0, B_1 as the initial, terminal bond price. Then an successful investment is just the following

1. $\alpha V_0 + \beta S_0 + \gamma B_0 = 0$
2. $\alpha V_1 + \beta S_1 + \gamma B_1 \geq 0$
3. $E(\alpha V_1 + \beta S_1 + \gamma B_1) > 0$

Lemma 2.1. *Let* $\Omega = \{H, T\}$*,There exists* $(\alpha_0, \beta_0, \gamma_0)$ *such that*

$$\begin{cases} \alpha_0 = 0 \\ \beta_0 S_1(\omega) + \gamma_0 B_1(\omega) = g(\omega), \forall \omega \in \Omega \end{cases}$$

Proof. Since $\Omega = \{H, T\}$, β_0, γ_0 must satisfy:

$$\beta_0 S_1(H) + \gamma_0 B_1(H) = g(H)$$
$$\beta_0 S_1(T) + \gamma_0 B_1(T) = g(T)$$

With r, u, d, we can have

$$\beta_0 = \frac{g(H) - g(T)}{(u-d)S_0}$$
$$\gamma_0 = \frac{1}{(1+r)B_0}\frac{ug(T) - dg(H)}{u-d}$$

Here we calculate β_0, γ_0 out, and the lemma naturally holds. □

With this, we can calculate the time-0 price of a European option on a stock with price S and payoff g at time 1. We claim:

$$V_0 = \frac{1}{1+r}(pg(H) + (1-p)g(T)) \tag{2.3}$$

where

$$p = \frac{1+r-d}{u-d} \tag{2.4}$$

Proof. We first show $V_0 = \beta_0 S_0 + \gamma_0 B_0$, β_0, γ_0 following 2.1.
Suppose $\epsilon = V_0 - (\beta_0 S_0 + \gamma_0 B_0) > 0$, take $(-1, \beta_0, \gamma_0 + \frac{\epsilon}{B_0})$ into consideration. We have

$$W_0 = -V_0 + \beta_0 S_0 + (\gamma_0 + \frac{\epsilon}{B_0})B_0 = 0$$
$$W_1 = -V_1 + \beta_0 S_0 + \gamma_0 B_1 + \epsilon\frac{B_1}{B_0}$$

Let $V_1 = g(\omega)$, we have

$$-V_1(\omega) + \beta_0 S_1(\omega) + \gamma_0 B_1(\omega) = 0, \forall \omega \in \Omega$$

Since B_0, B_1 both positive, $W_1(\omega) = \epsilon\frac{B_1}{B_0} > 0, \forall \omega \in \Omega$. Which means $(-1, \beta_0, \gamma_0 + \frac{\epsilon}{B_0})$ is arbitrage, contradiction to the assumption. So $\epsilon = V_0 - (\beta_0 S_0 + \gamma_0 B_0) > 0$ impossible. Similarly, we can prove $\epsilon = V_0 - (\beta_0 S_0 + \gamma_0 B_0) < 0$ impossible. Therefore,

$$V_0 = \beta_0 S_0 + \gamma_0 B_0 \tag{2.5}$$

Take r, u, d into 2.5, we have

$$\begin{aligned} V_0 &= \beta_0 S_0 + \gamma_0 B_0 \\ &= \frac{g(H) - g(T)}{u-d} + \frac{1}{1+r}\frac{ug(T) - dg(H)}{u-d} \\ &= \frac{1}{1+r}(\frac{1+r-d}{u-d}g(H) + \frac{u-(1+r)}{u-d}g(T)) \end{aligned}$$

□

Now we calculate out the value of one-period option, but where is the martingale? Define probability P on $\Omega = \{H, T\}$, which is called pricing probability in finance. $P(H) = p, P(T) = 1-p$, Then we have:

$$\begin{aligned} E[S_1|S_0] &= puS_0 + (1-p)dS_0 \\ &= \frac{1+r-d}{u-d}uS_0 + \frac{u-(1+r)}{u-d}dS_0 \\ &= (1+r)S_0 \end{aligned}$$

Let $Y_k = \frac{1}{(1+r)^k}S_k, k = 0, 1$, Then we have

$$E[Y_1|Y_0] = Y_0 \tag{2.6}$$

So the discounted stock price $\{Y_k\}_{k=0}^{1}$ is a martingale in one-period model.

2.4 N-period model

After considering one-period model, we turn to more general case of N-periods model. It's still an arbitrage-free market. The sample space $\Omega = \{H, T\}$. A sample point is of the form $(\omega_1, \omega_2 ..., \omega_N)$. Similar to 1-period model, we have:

$$S_n(\overline{\omega}_n) = \begin{cases} uS_{n-1}(\overline{\omega}_{n-1}, 1), & \text{with probability } p \\ dS_{n-1}(\overline{\omega}_{n-1}, 0), & \text{with probability } 1-p \end{cases}$$

The probability P on Ω is:

$$P\{(\omega_1, ...\omega_N)\} = p^{\sum_{i=1}^{N}\omega_i}(1-p)^{N-\sum_{i=1}^{N}\omega_i}$$

At time $t_0, t_1 ..., t_N$, let $V_0, V_1 ..., V_N$ be the values of a European option with underlying stock prices $S_0, S_1, ...S_N$ and strike price is K, similar to 1-period model, we have:

$$V_N = \begin{cases} max(S_N - K, 0) & \text{for a call} \\ max(K - S_N, 0) & \text{for a put} \end{cases}$$

Expand S_N with u, d, we have:

$$V_N = \begin{cases} max(u^{\sum_{i=1}^{N}\omega_i}d^{N-\sum_{i=1}^{N}\omega_i}S_0 - K, 0) & \text{for a call} \\ max(K - u^{\sum_{i=1}^{N}\omega_i}d^{N-\sum_{i=1}^{N}\omega_i}S_0, 0) & \text{for a put} \end{cases}$$

To determine the option values, we have to calculate it recursively. Use V_N to calculate V_{N-1}, then V_{N-2}, finally V_0.
We also need to define portfolio in N-period model, it's a sequence of triples $\{(\alpha_i, \beta_i, \gamma_i), i = 0, 1, ..., N\}$, $\alpha_i, \beta_i, \gamma_i \in \Omega$ Similar to the proof in 1-period model, we have following lemma:

Lemma 2.2. *At time n, there exists portfolio $(\alpha_{n-1}(\omega_{n-1}), \beta_{n-1}(\omega_{n-1}), \gamma_{n-1}(\omega_{n-1}))$, such that*

$$\begin{cases} \alpha_{n-1}(\omega_{n-1}) & = 0 \\ \beta_{n-1}(\omega_{n-1})S_n(\omega_{n-1}, \omega_n) + \gamma_{n-1}(\omega_{n-1})B_n(\omega_{n-1}, \omega_n) & = V_n(\omega_{n-1}, \omega_n), \omega_n \in \Omega \end{cases}$$

Proof. Still similar to 2.1, take $\omega_n = H, T$ into the equations. Then we can solve out $\beta_{n-1}, \gamma_{n-1}$

$$\beta_{n-1} = \frac{V_n(\omega_{n-1}, 0) - V_n(\omega_{n-1}, 1)}{(u-d)S_{n-1}(\omega_{n-1})}$$
$$\gamma_{n-1} = \frac{1}{(1+r)B_{n-1}(\omega_{n-1})} \frac{uV_n(\omega_{n-1}, 1) - dV_n(\omega_{n-1}, 0)}{u-d}$$

□

The recursive equation of V_{n-1} and V_n is following:

$$V_{n-1}(\omega_{n-1}) = \frac{1}{1+r}(pV_n(\omega_{n-1,0}) + (1-p)V_n(\omega_{n-1}, 1)), 1 \leq n \leq N \tag{2.7}$$

Proof. First we show $V_{n-1} = \beta_{n-1}S_{n-1} + \gamma_{n-1}B_{n-1}$ like one-period model.
Suppose

$$\epsilon = V_{n-1} - (\beta_{n-1}S_{n-1} + \gamma_{n-1}B_{n-1}) > 0$$

Consider portfolio $(-1, \beta_{n-1}, \gamma_{n-1} + \frac{\epsilon}{B_{n-1}})$
At time t_{n-1}, t_n, we have

$$W_{n-1} = -V_{n-1} + \beta_{n-1}S_{n-1} + (\gamma_{n-1} + \frac{\epsilon}{B_{n-1}})B_{n-1} = 0$$
$$W_n = -V_n + \beta_{n-1}S_n + \gamma_{n-1}B_{n-1} + \epsilon\frac{B_n}{B_{n-1}}$$

Take $V_n = g(\omega)$ into it, we have

$$\beta_{n-1}(\omega_{n-1})S_n(\omega_{n-1}, \omega_n) + \gamma_{n-1}(\omega_{n-1})S_n(\omega_{n-1}, \omega_n) = V_n(\omega n - 1, \omega_n) \tag{2.8}$$

Since B_0, B_1 positive, we have $W_n(\omega) = \epsilon(\frac{B_n}{B_{n-1}}) > 0, \forall \omega \in \Omega$, which means portfolio $(-1, \beta_{n-1}, \gamma_{n-1} + \frac{\epsilon}{B_{n-1}})$ is an arbitrage strategy, contradiction.
On the other hand, let portfolio be $(1, -\beta_n, -\gamma_{n-1} - \frac{\epsilon}{B_n})$, we can prove that $\epsilon = V_{n-1} - (\beta_{n-1}S_{n-1} + \gamma_{n-1}B_{n-1}) < 0$ not possible. Therefore:

$$V_{n-1} = \beta_{n-1}S_{n-1} + \gamma_{n-1}B_{n-1} \tag{2.9}$$

Then take d, r into 2.9 we have:

$$V_{n-1}(\omega_{n-1}) = \frac{1}{1+r}(pV_n(\omega_{n-1}, 0) + (1-p)V_n(\omega_{n-1}, 1)) \tag{2.10}$$

□

As for V_0 itself, we have

$$V_0 = \frac{1}{(1+r)^N}E[g|(S_0, B_0)] \tag{2.11}$$

E refers to the expectation under pricing probability P,

Proof. In 1-period model, we have $V_{i-1} = E[\frac{1}{1+r}V_i|(S_{i-1}, B_{i-1})]$ Then

$$\begin{aligned} V_{i-2} &= E[\frac{1}{1+r}V_{i-1}|(S_{i-2}, B_{i-2})] \\ &= E[\frac{1}{(1+r)^2}E[\frac{1}{1+r}V_i|(S_{i-1}, B_{i-1})]|(S_{i-2}, B_{i-2})] \\ &= E[\frac{1}{(1+r)^2}V_i|(S_{i-2}, B_{i-2})] \end{aligned}$$

Then by recursions, we can have $V_0 = \frac{1}{(1+r)^N}E[g|(S_0, B_0)]$ □

3 CONCLUSION

In this report, I found a finance model option which is a martingale in math. And calculated out the initial price of an option of a 1-period model and N-periods model respectively. I read [Chu79] for some help, and thanks to my high school classmate Zhang Tianyu for explaining some finance terms for me.

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