A proof of $|A^A| = |2^A|$ using Zorn's Lemma

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Theorem 1 Let A be an infinite set, then $|A^A| = |2^A|$.

To validate Theorem 1, we will develop some useful the theorems concerning the cardinality of infinite sets.

Theorem 2 Let A be an infinite sets, then $|A + A| = |A|$, where $A + B := \{(x, 0) : x \in A\} \cup \{(0, y) : y \in B\}$.

Proof. Let
$$\mathcal{F} = \{f \in X^{(X+X)} : X \subseteq A, f \text{ is a bijection}\}.$$
Note that a function from A to B can be viewed as a subset of $A \times B$, and thus $\mathcal{F} \subset \mathcal{P}(A \times \{0, 1\} \times A)$, where $\mathcal{P}(\cdot)$ is the power set of some set. In the following argument, for function f, $\text{range}(f) := \{y : \exists x f(x) = y\}$

Now we check that the poset $(\mathcal{F}, \subset)$ satisfies the condition of *Zorn's lemma*. First, $\mathcal{F} \neq \emptyset$ since $\emptyset \in \mathcal{F}$, and it remains to show that *all chains in $\mathcal{F}$ are closed*. Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Clearly, $\phi := \bigcup_{f \in \mathcal{C}} f$ is also a function. Since every $f \in \mathcal{C}$ is bijection by definition, say $Y := \text{range}(\phi) \subseteq A$, ϕ is a bijection from $Y + Y$ to Y, and thus $\phi \in \mathcal{F}$. In another word, ϕ is an upper bound of $\mathcal{C}$ in $\mathcal{F}$.

By *Zorn's lemma* ,there is a maximal element in $\mathcal{F}$, which is denoted by ψ. Let $U = \text{range}(\psi)$, we shall show that $A \setminus U$ is finite. Assume that $A \setminus U$ is infinite, then there is a countable subset of $A \setminus U$, say V. We know that there is a bijection

$\sigma : V + V \to V$. By definition, $\sigma \in \mathcal{F}$. Note that the domains of ψ and σ have no intersection, and hence $\psi \cup \sigma$ is a bijection from $(U \cup V) + (U \cup V)$ to $U \cup V$, which is in contradiction with the maximality of ψ. Since ψ is a bijection from $U + U$ to U, we conclude that $|A + A| = |U + U| = |U| = |A|$. $\square$

Theorem 3 Let A be an infinite set, then $|A \times A| = |A|$.

Proof. The proof is similar to the proof of Theorem 2. Let

$$\mathcal{F} = \{f \in X^{(X \times X)} : X \subseteq A, f \text{ is a bijection}\}.$$

An analogical argument tells us $\mathcal{F}$ satisfies the condition of *Zorn's lemma* and thus it contains a maximal element, which is denoted by ψ. Let $U = \text{range}(\psi)$ and $V = A \setminus U$.

We shall show that $|V| \le |U|$. Assume that $|V| > |U|$, then there is an isomorphic copy of $|U|$ in $|V|$, say W. Clearly, there is also a bijection $\sigma : W \times W \to W$ corresponding to ψ. Since $W \cap U = \emptyset$, we can equate $W + U$ and $W \cup U$. With the help of Theorem 2, we have

$$|(U \times U) + (W \times U) + (U \times W)| = |U \times U|.$$

Let $\tau : (U \times U) + (W \times U) + (W \times U) \to U \times U$ be a bijection. We define

$$\pi : (W \cup U) \times (W \cup U) \to (W \times W) \cup (U \times U), x \mapsto \begin{cases} x, \text{if } x \in W \times W, \\ \tau(x), \text{otherwise.} \end{cases}$$

Meanwhile, $\phi := \psi \cup \sigma$ is a bijection from $(W \times W) \cup (U \times U)$ to $W \cup U$. Therefore, $\phi \circ \pi : (W \cup U) \times (W \cup U) \to W \cup U$ is also a bijection, which is in contradiction with the maximality of ψ.

Finally, $|A| \le |A \times A|$ is trivial and

$$|A \times A| = |(U + V) \times (U + V)| \le |(U + U) \times (U + U)| = |U \times U| = |U| \le |A|,$$

completing the proof. $\square$

Proof of Theorem 1. By Theorem 3 we establish $|A \times A| = |A|$, and hence

$$|A^A| \leq |\mathcal{P}(A \times A)| = |\mathcal{P}(A)| = |2^A|.$$

One the other hand, choose $a, b \in A$ arbitrarily, then a injection from 2^A to A^A is given by

$$\phi : 2^A \to A^A, f \mapsto f' \text{ where } f'(x) = \begin{cases} a, & \text{if } f(x) = 0, \\ b, & \text{if } f(x) = 1. \end{cases}$$

Hence, $|2^A| \leq |A^A|$, completing the proof. $\square$

Remark The main idea of the first proof comes from [董 88]. I love this proof for it only uses *Zorn's lemma* and the basic conception of set. Other proofs of Theorem 2 and Theorem 3 is based on the rigorous definition of *ordinal* and *cardinality*, such as the one in [李 19].

Reference

[李 19] 李文威. 代数学方法（第一卷）. 高等教育出版社, 2019.

[董 88] 董延闿. 基础集合论. 北京师范大学出版社, 1988.