

# A proof of $|A^A| = |2^A|$ using Zorn's Lemma

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**Theorem 1** Let  $A$  be an infinite set, then  $|A^A| = |2^A|$ .

To validate Theorem 1, we will develop some useful theorems concerning the cardinality of infinite sets.

**Theorem 2** Let  $A$  be an infinite set, then  $|A + A| = |A|$ , where  $A + B := \{(x, 0) : x \in A\} \cup \{(0, y) : y \in B\}$ .

*Proof.* Let

$$\mathcal{F} = \{f \in X^{(X+X)} : X \subseteq A, f \text{ is a bijection}\}.$$

Note that a function from  $A$  to  $B$  can be viewed as a subset of  $A \times B$ , and thus  $\mathcal{F} \subset \mathcal{P}(A \times A \times A)$ , where  $\mathcal{P}(\cdot)$  is the power set of some set. In the following argument, for function  $f$ ,  $\text{range}(f) := \{y : \exists x f(x) = y\}$

Now we check that the poset  $(\mathcal{F}, \subset)$  satisfies the condition of *Zorn's lemma*. First,  $\mathcal{F} \neq \emptyset$  since  $\emptyset \in \mathcal{F}$ , and it remains to show that *all chains in  $\mathcal{F}$  are closed*. Let  $\mathcal{C}$  be a chain in  $\mathcal{F}$ . Clearly,  $\phi := \bigcup_{f \in \mathcal{C}} f$  is also a function. Since every  $f \in \mathcal{C}$  is bijection by definition, say  $Y := \text{range}(\phi) \subseteq A$ ,  $\phi$  is a bijection from  $Y + Y$  to  $Y$ , and thus  $\phi \in \mathcal{F}$ . In another word,  $\phi$  is an upper bound of  $\mathcal{C}$  in  $\mathcal{F}$ .

By *Zorn's lemma*, there is a maximal element in  $\mathcal{F}$ , which is denoted by  $\psi$ . Let  $U = \text{range}(\psi)$ , we shall show that  $A \setminus U$  is finite. Assume that  $A \setminus U$  is infinite, then there is a countable subset of  $A \setminus U$ , say  $V$ . We know that there is a bijection

$\sigma : V + V \rightarrow V$ . By definition,  $\sigma \in \mathcal{F}$ . Note that the domains of  $\psi$  and  $\sigma$  have no intersection, and hence  $\psi \cup \sigma$  is a bijection from  $(U \cup V) + (U \cup V)$  to  $U \cup V$ , which is in contradiction with the maximality of  $\psi$ . Since  $\psi$  is a bijection from  $U + U$  to  $U$ , we conclude that  $|A + A| = |U + U| = |U| = |A|$ .  $\square$

**Theorem 3** Let  $A$  be an infinite set, then  $|A \times A| = |A|$ .

*Proof.* The proof is similar to the proof of Theorem 2. Let

$$\mathcal{F} = \{f \in X^{(X \times X)} : X \subseteq A, f \text{ is a bijection}\}.$$

An analogical argument tells us  $\mathcal{F}$  satisfies the condition of *Zorn's lemma* and thus it contains a maximal element, which is denoted by  $\psi$ . Let  $U = \text{range}(\psi)$  and  $V = A \setminus U$ .

We shall show that  $|V| \leq |U|$ . Assume that  $|V| > |U|$ , then there is an isomorphic copy of  $|U|$  in  $|V|$ , say  $W$ . Clearly, there is also a bijection  $\sigma : W \times W \rightarrow W$  corresponding to  $\psi$ . Since  $W \cap U = \emptyset$ , we can equate  $W + U$  and  $W \cup U$ . With the help of Theorem 2, we have

$$|(U \times U) + (W \times U) + (U \times W)| = |U \times U|.$$

Let  $\tau : (U \times U) + (W \times U) + (U \times W) \rightarrow U \times U$  be a bijection. We define

$$\pi : (W \cup U) \times (W \cup U) \rightarrow (W \times W) \cup (U \times U), x \mapsto \begin{cases} x, & \text{if } x \in W \times W, \\ \tau(x), & \text{otherwise.} \end{cases}$$

Meanwhile,  $\phi := \psi \cup \sigma$  is a bijection from  $(W \times W) \cup (U \times U)$  to  $W \cup U$ . Therefore,  $\phi \circ \pi : (W \cup U) \times (W \cup U) \rightarrow W \cup U$  is also a bijection, which is in contradiction with the maximality of  $\psi$ .

Finally,  $|A| \leq |A \times A|$  is trivial and

$$|A \times A| = |(U + V) \times (U + V)| \leq |(U + U) \times (U + U)| = |U \times U| = |U| \leq |A|,$$

completing the proof.  $\square$

*Proof of Theorem 1.* By Theorem 3 we establish  $|A \times A| = |A|$ , and hence

$$|A^A| \leq |\mathcal{P}(A \times A)| = |\mathcal{P}(A)| = |2^A|.$$

On the other hand, choose  $a, b \in A$  arbitrarily, then an injection from  $2^A$  to  $A^A$  is given by

$$\phi : 2^A \rightarrow A^A, f \mapsto f' \text{ where } f'(x) = \begin{cases} a, & \text{if } f(x) = 0, \\ b, & \text{if } f(x) = 1. \end{cases}$$

Hence,  $|2^A| \leq |A^A|$ , completing the proof.  $\square$

**Remark** The main idea of the first proof comes from [董 88]. I love this proof for it only uses *Zorn's lemma* and the basic conception of set. Other proofs of Theorem 2 and Theorem 3 are based on the rigorous definition of *ordinal* and *cardinality*, such as the one in [李 19].

## Reference

[李 19] 李文威. 代数学方法（第一卷）. 高等教育出版社, 2019.

[董 88] 董延闯. 基础集合论. 北京师范大学出版社, 1988.