Proof of Discrete Poincaré's Recurrence Theorem

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Theorem 1 (Discrete Poincaré's Recurrence Theorem)**.** *Let T be a measure-preserving transformation on $(\Omega, \mathcal{F}, P)$. Then, for any $E \in \mathcal{F}$ with $P(E) > 0$, almost all points of E returns to E infinitely often under positive iterations by T.*

Proof. For all $n \geqslant 1$, let

$$\begin{aligned} A_n &:= \{x \in E | x \notin T^{-kn}(E), \forall k \geqslant 1\} \\ &= E \backslash \bigcup_{k \geqslant 1} T^{-kn}(E). \end{aligned}$$

Since T is a measure-preserving transformation and $E \in \mathcal{F}$, it is obvious that $A_n \in \mathcal{F}$ and hence A_n is an event.

Consider a sequence of events $\{A_n, T^{-n}(A_n), T^{-2n}(A_n), \ldots, T^{-kn}(A_n), \ldots\}$. Assume that for two integers p, q with $0 \leqslant p < q$, $T^{-pn}(A_n) \cap T^{-qn}(A_n) \neq \emptyset$. Then we have $A_n \cap T^{-(q-p)n} \neq \emptyset$, which is contrary to $A_n = E \backslash \bigcup_{k \geqslant 1} T^{-kn}(E)$. Therefore, the events $A_n, T^{-n}(A_n), T^{-2n}(A_n)$, $\ldots, T^{-kn}(A_n), \ldots$ are pairwise distinct. Since T is measure-preserving, we know that $P(A_n) = P(T^{-n}(A_n)) = P(T^{-2n}(A_n)) = \ldots$, and hence $P(A_n) = 0 < \infty$ for all $n \geqslant 1$.

By the First Borel-Cantelli Lemma we discussed in class, immediately we have $P(\limsup A_n) = 0$, and $P(E \backslash \limsup A_n) = P(E) > 0$.

So by definition of A_n, for any x,

$$\begin{aligned} x \in E \backslash \limsup A_n &\Longrightarrow x \in E \text{ and } x \notin \limsup A_n \\ &\Longrightarrow x \in E \text{ and } x \notin \bigcap_{m \in \mathbb{N}} \bigcup_{n \geqslant m} A_n \\ &\Longrightarrow \text{There exist infinite numbers of increasing} \\ &\qquad \text{positive integers } \{n_k\} \text{ such that } T^{n_k}(x) \in E. \\ &\Longrightarrow x \text{ returns to } E \text{ infinitely often under positive} \\ &\qquad \text{iterations by } T. \end{aligned}$$

Since $P(\limsup A_n) = 0$, we can say that almost all points of E returns to E infinitely often under positive iterations by T. $\square$