Measure Space

by 刘成锴

Mainly about σ-algebras, π-systems, and measures

Definitions of algebra, σ-algebra

Algebra on S

A collection Σ_0 of subsets of S is called an ***algebra*** on S if

$$\begin{aligned}&\text{(i) } S \in \Sigma_0\\&\text{(ii) } F \in \Sigma_0 \Rightarrow F^c := S \backslash F \in \Sigma_0\\&\text{(iii) } F, G \in \Sigma_0 \Rightarrow F \cup G \in \Sigma_0\end{aligned}$$

Thus, an algebra on S is a family of subsets of S stable under finitely many set operations.

（对有限次集合运算封闭）

σ-algebra on S

A collection Σ of subsets of S is called a ***σ-algebra*** on S if

1. Σ is an algebra on S
2. if $F_n \in \Sigma(n \in \mathbf{N})$ then $\bigcup_n F_n \in \Sigma$

Thus, a σ-algebra on S is a family of subsets of S 'stable under any countable collection of set operations'.

（对任意多次集合运算封闭）

Measurable space

A pair (S, Σ) is called a ***measurable space***.

Borel σ-algebras, $\mathcal{B}(S), \mathcal{B} = \mathcal{B}(\mathbf{R})$

Definition

$\mathcal{B}(S)$ is the σ-algebra generated by the family of open subsets of S.

$$\mathcal{B}(S) := \sigma(\text{open sets})$$

A standard Shorthand $\mathcal{B} := \mathcal{B}(\mathbf{R})$

A Theorem

The collection $\pi(\mathbf{R}) := \{(-\infty, x] | x \in \mathbf{R}\}$, then

$$\mathcal{B} = \sigma(\pi(\mathbf{R}))$$

The proof is on the book.

additive

Let μ_0 be a non-negative ***set function***: $\mu_0 : \Sigma_0 \rightarrow [0, \infty]$

Then μ_0 is called ***additive*** if $\mu_0(\emptyset) = 0$ and, for $F, G \in \Sigma_0$, $F \cap G = \emptyset$,

$$\mu_0(F \cup G) = \mu_0(F) + \mu_0(G)$$

Measure Space (S, Σ, μ)

μ is a ***measure*** on measurable space (S, Σ) and μ is countably additive.

finite and σ-finite

finite

if $\mu(S) < \infty$

σ-finite

if there is a sequence $(S_n : n \in \mathbf{N})$ of elements of Σ such that

$$\mu(S_n) < \infty (\forall n \in \mathbf{N}) \text{ and } \bigcup S_n = S$$

Probability measure, probability triple 概率测度，概率空间

Measure μ is called a ***probability measure*** if

$$\mu(S) = 1$$

and (S, Σ, μ) is then called a ***probability triple***.

π-system

Definition

$\mathcal{I}$ is a ***π-system*** on S iff

$$I_1, I_2 \in \mathcal{I} \quad \Rightarrow \quad I_1 \cap I_2 \in \mathcal{I}$$

that is, a family of sbsets of S stable under finite intersection.

Uniqueness Lemma

μ_1, μ_2 are measures on (S, Σ) such that $\mu_1(S) = \mu_2(S) < \infty$ and $\mu_1 = \mu_2$ on $\mathcal{I}$, then

$$\mu_1 = \mu_2 \text{ on } \Sigma$$

The proof is in A1.4 on the book.

Corollary

If two probability measures agree on a π-system, then they agree on the σ-algebra generated by that π-system.

Carathéodory's Extension Theorem

Let S be a set, let Σ_0 be an algebra on S, and let

$$\Sigma := \sigma(\Sigma_0)$$

If μ_0 is a *countably additive* map $\mu_0 : \Sigma_0 \to [0, \infty]$, then there exists a measure μ on (S, Σ) such that

$$\mu = \mu_0 \text{ on } \Sigma_0$$

If $\mu_0(S) < \infty$, then by Uniqueness Lemma, this extension is *unique* - an algebra is a π-system.

Lebesgue Measure Leb on ((0, 1], $\mathcal{B}(0, 1]$)

Let $S = (0, 1]$. For $F \subseteq S$, say that $F \in \Sigma_0$ if F may be written as a finite union

$$F = (a_1, b_1] \cup \ldots \cup (a_r, b_r]$$

where $r \in \mathbf{N}, 0 \leq a_1 \leq b_1 \leq \cdots \leq a_r \leq b_r \leq 1$. Then Σ_0 is an algebra on $(0, 1]$ and

$$\Sigma := \sigma(\Sigma_0) = \mathcal{B}(0, 1]$$

Let

$$\mu_0(F) = \sum_{k \leq r} (b_k - a_k)$$