Gamblers Ruin

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Exercise 1. *Gambler's Ruin. Suppose that* $X_1, X_2, ...$ *are IID RVs with* $P[X = +1] = p$, $P[X = -1] = q$, *where* $0 < p = 1 - q < 1$, *and* $p \neq q$. *Suppose that* a *and* b *are integers with* $0 < a < b$. *Define* $S_n = a + X_1 + X_2 + ... + X_n$, $T = inf\,\{n \mid S_n = 0 \text{ or } S_n = b\}$. *Deduce the values of* $P(S_T = 0)$ *and* $E(S_T)$.

This is an interesting problem about a gambler who starts with an initial fortune of $\$a$ and then on each successive gamble either wins \$1 or loses \$1 independent of the past. The gambler's goal is to get $\$b$.

We denote $P_i = P(S_T = b)$ to be the probability that the gambler wins when $a = i$. Clearly $P_0 = 0$ and $P_b = 1$ by definition, and we next proceed to compute $P_i, 1 \leq i \leq b - 1$

We now focus on a certain gamble our gambler makes when the gambler has money i. If $X = 1$, then the gambler will own one more doller, so by Markov property, the gambler now can win with probability P_{i+1}. Similarly, if $X = -1$, the gambler can win with probability P_{i-1}. Thus we can get the following recursion.

$$P_i = pP_{i+1} + qP_{i-1}$$

The recursion can be rewritten as $pP_i + qP_i = pP_{i+1} + qP_{i-1}$, thus we can get

$$P_{i+1} - P_i = \frac{q}{p}\,(P_i - P_{i-1})$$

And then we can easily derive

$$P_{i+1} - P_i = (\frac{q}{p})^i(P_1 - P_0), 0 < i < b$$

Thus

$$\begin{aligned}
P_{i+1} &= P_{i+1} - P_0 \\
&= \sum_{k=0}^{i} (P_{k+1} - P_k) \\
&= \sum_{k=0}^{i} \left(\frac{q}{p}\right)^k (P_1 - P_0) \\
&= \sum_{k=0}^{i} \left(\frac{q}{p}\right)^k P_1 \\
&= P_1 \frac{1 - (\frac{q}{p})^{i+1}}{1 - (\frac{q}{p})}
\end{aligned}$$

Now we let $i = b - 1$, then we have

$$1 = P_b = P_1 \frac{1 - (\frac{q}{p})^b}{1 - (\frac{q}{p})}$$

Thus

$$P_1 = \frac{1 - \frac{q}{p}}{1 - (\frac{q}{p})^b}$$

And that leads us to

$$P_i = \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})^b}$$

Thus

$$P(S_T = 0) = 1 - P_a = \frac{(\frac{q}{p})^a - (\frac{q}{p})^b}{1 - (\frac{q}{p})^b}$$

$$E(S_T) = b\ \frac{1 - (\frac{q}{p})^a}{1 - (\frac{q}{p})^b}$$

Reference: http://www.columbia.edu/ ks20/FE-Notes/4700-07-Notes-GR.pdf