Σ-measurable Functions and Simple Functions

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April 5, 2020

Exercise 1. *Let (S, Σ) be a measurable space. Take $f \in m\Sigma$. Let $f^+ = \max(f, 0)$ and $f^- = \max(-h, 0)$. A function $g \in \mathbb{R}^S$ is a simple function with respect to (S, Σ) provided it falls into the linear subspace of $\mathbb{R}^S$ spanned by $\{\mathbf{1}_A \mid A \in \Sigma\}$. For each positive integer n, define the dyadic function $d_n \in \mathbb{R}^{\mathbb{R}}$ to be*

$$\sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n}]} + n\mathbf{1}_{[n,+\infty]}.$$

For each $n \in \mathbb{N}$, show that $f_n = d_n \circ f^+ - d_n \circ f^-$ is a simple function with respect to (S, Σ). Then illustrate that f is the limit of a sequence of simple functions.

Proof. Let $x \in \mathbb{R}$, $x \geqslant 0$. We can rewrite the definition of d_n as a step function

$$d_n(x) = \begin{cases} 0, & \frac{0}{2^n} \leqslant x < \frac{1}{2^n} \\ \frac{1}{2^n}, & \frac{1}{2^n} \leqslant x < \frac{2}{2^n} \\ \vdots \\ \frac{n \cdot 2^n - 1}{2^n}, & \frac{n \cdot 2^n - 1}{2^n} \leqslant x < \frac{n \cdot 2^n}{2^n} \\ \frac{n \cdot 2^n}{2^n}, & x \geqslant \frac{n \cdot 2^n}{2^n}. \end{cases}$$

Then we can write down $d_n \circ f^+(s)$ explicitly:

$$d_n \circ f^+(s) = \begin{cases} 0, & s \in f^{-1}((-\infty, \frac{1}{2^n})) \\ \frac{1}{2^n}, & s \in f^{-1}([\frac{1}{2^n}, \frac{2}{2^n})) \\ \vdots \\ \frac{n \cdot 2^n - 1}{2^n}, & s \in f^{-1}([\frac{n \cdot 2^n - 1}{2^n}, \frac{n \cdot 2^n}{2^n})) \\ \frac{n \cdot 2^n}{2^n}, & s \in f^{-1}([\frac{n \cdot 2^n}{2^n}, +\infty)). \end{cases}$$

Let $A_0 = (-\infty, \frac{1}{2^n}), A_1 = [\frac{1}{2^n}, \frac{2}{2^n}), \ldots, A_{n \cdot 2^n - 1} = [\frac{n \cdot 2^n - 1}{2^n}, \frac{n \cdot 2^n}{2^n}), A_{n \cdot 2^n} = [\frac{n \cdot 2^n}{2^n}, +\infty)$. We now can write $d_n \circ f^+$ as

$$d_n \circ f^+ = 0\mathbf{1}_{A_0} + \frac{1}{2^n}\mathbf{1}_{A_1} + \ldots + \frac{n \cdot 2^n - 1}{2^n}\mathbf{1}_{n \cdot 2^n - 1} + \frac{n \cdot 2^n}{2^n}\mathbf{1}_{n \cdot 2^n}.$$

By the definition of simple function, it follows that $d_n \circ f^+$ is a simple function. Similarly, $d_n \circ f^-$ is also a simple function. And then we have $f_n = d_n \circ f^+ - d_n \circ f^-$ is a simple function. Moreover, we have

$$f_n(s) = d_n \circ f^+(s) - d_n \circ f^-(s) = \begin{cases} -\frac{n\cdot 2^n}{2^n}, & s \in f^{-1}((-\infty, \frac{n\cdot 2^n}{2^n})) \\ -\frac{n\cdot 2^n - 1}{2^n}, & s \in f^{-1}(-\frac{n\cdot 2^n}{2^n}, -\frac{n\cdot 2^n-1}{2^n}] \\ \vdots \\ -\frac{1}{2^n}, & s \in f^{-1}((-\frac{2}{2^n}, \frac{1}{2^n}]) \\ 0, & s \in f^{-1}((-\frac{1}{2^n}, \frac{1}{2^n})) \\ \frac{1}{2^n}, & s \in f^{-1}([\frac{1}{2^n}, \frac{2}{2^n})) \\ \vdots \\ \frac{n\cdot 2^n-1}{2^n}, & s \in f^{-1}([\frac{n\cdot 2^n-1}{2^n}, \frac{n\cdot 2^n}{2^n})) \\ \frac{n\cdot 2^n}{2^n}, & s \in f^{-1}([\frac{n\cdot 2^n}{2^n}, +\infty)). \end{cases}$$

So for any $s \in S$ and any $\varepsilon > 0$, let $N = \lceil \max\{|f(s)|, \log_2 \varepsilon\} \rceil$. For any n such that $n > N$, we have $|f(s) - f_n(s)| < \varepsilon$. So we can say that f is the limit of a sequence of simple functions. $\square$