Independence in infty

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If sub-σ-algebras $\mathcal{G}_1, \mathcal{G}_2 \dots$ are independent, then whenever $G_i \in \mathcal{G}_i (i \in \mathbb{N})$ and $C \subseteq \mathbb{N}$ is a countable set, there is

$$P(\cap_{c_i \in C} G_{c_i}) = \prod_{c_i \in C} P(G_{c_i})$$

Proof.

$$\begin{aligned} P(\cap_{c_i \in C} G_{c_i}) &= P(\cap_{i \in \mathbb{N}} G_{c_i}) \\ &= P(\cup_{t \in \mathbb{N}} \cap_{0 \le i \le t} G_{c_i}) \end{aligned}$$

Let $E_i = \cap_{0 \le i \le t} G_{c_i}$, and $E = \cup_{t \in \mathbb{N}} \cap_{0 \le i \le t} G_{c_i}$, then there is $E_i \downarrow E$. Since $P(E_i) \in [0, 1)$, there is $P(E_i) \downarrow P(E)$.

By definition of independence, there is

$$P(E_i) = P(\cap_{0 \le i \le t} G_{c_i}) = \prod_{0 \le i \le t} P(G_{c_i})$$

So there is:

$$P(E) = \lim_{t \to \infty} P(E_i) = \lim_{t \to \infty} \prod_{0 \le i \le t} P(G_{c_i}) = \prod_{c_i \in C} P(G_{c_i})$$

□

But things are different in the uncountable case. To consider this problem, we define independence in uncountable set be that all countable subsets are independent:

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Consider sub-σ-algebras $(\mathcal{G}_\alpha : \alpha \in A)$ where A is uncountable, even if for all countable $N \subseteq A$, $(\mathcal{G}_i : i \in N)$ are independent(so they are independent in our definition),

$$P(\cap_{\gamma \in C} G_\gamma) = \prod_{c_i \in C} P(G_{c_i})$$

does not hold for some $G_\alpha \in \mathcal{G}_\alpha (\alpha \in A)$ and uncountable $C \subseteq A$.

Proof. Consider σ-algebras in probability triple $(\Omega, \mathcal{F}, P)$ be $([0,1), \mathcal{B}[0,1), \mathrm{Leb})$:

$$\mathcal{G}_\alpha = \{\varnothing, \{\alpha\}, \Omega \setminus \{\alpha\}, \Omega\} (\alpha \in \Omega)$$

For all countable $N \subseteq A$, for $i \in N$:

If $G_i = \varnothing$ or $G_i = \{i\}$, and $P(G_i) = 0$. We have

$$\bigcap_{i \in N} G_i \subseteq G_i \implies P(\bigcap_{i \in N} G_i) = 0.$$

If $G_i = \Omega \setminus \{i\}$ or $G_i = \Omega$, and $P(G_i) = 1$ for all $i \in N$.

$$\Omega \setminus N \subseteq \bigcap_{i \in N} G_i \implies P(\bigcap_{i \in N} G_i) = 1.$$

This shows that for every countable subset N of A, $(\mathcal{G}_i : i \in N)$ are independent.

But if we take $A = \Omega$, then consider $(\mathcal{G}_\alpha : \alpha \in A)$ and the intersection of $(G_\alpha : \alpha \in A)$ where $G_\alpha \in \mathcal{G}_\alpha$.

Let $G_\alpha = \Omega \setminus \{\alpha\}$, we have $P(G_\alpha) = 1$, but

$$\bigcap_{\alpha \in A} G_\alpha = \Omega \setminus A = \varnothing \implies P(\bigcap_{\alpha \in A} G_\alpha) = 0 \neq 1 = \prod_{\alpha \in A} P(G_\alpha)$$

This implies that even if every countable subset of $(\mathcal{G}_\alpha : \alpha \in A)$ are independent, we cannot say that for all $B \subseteq A$, $G_\alpha \in \mathcal{G}_\alpha$:

$$P(\bigcap_{\alpha \in B} G_\alpha) = \prod_{\alpha \in B} P(G_\alpha).$$

□