Measurable function is the limit of a sequence of simple function

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Define the dyadic function $d_n \in \mathbb{R}^{\mathbb{R}}$

$$d_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})} + n\mathbf{1}_{[n,\infty)}$$

$f \in \mathbb{R}^S$, define $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$.

1. Take $f \in m\Sigma$. For each $n \in \mathbb{N}$, show that $f_n \doteq d_n \circ f^+ - d_n \circ f^-$ is a simple function with respect to (S, Σ).

Proof: If $f(s) \geq 0$, $f^+(s) = f(s) \geq 0$ and $f^-(s) = 0$. So $f_n(s) = d_n \circ f(s)$.

$$f_n(s) = \begin{cases} \frac{k-1}{2^n}, & f(s) \in [\frac{k-1}{2^n}, \frac{k}{2^n}), k \in [1, n2^n] \cap \mathbb{Z} \\ n, & f(s) \in [n, \infty) \end{cases}$$

f is a measurable function, $\{[\frac{k-1}{2^n}, \frac{k}{2^n}), [n, \infty)\} \subseteq \mathcal{B}$($\mathcal{B}$ is Borel set).

We can induce $\{f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n})), f^{-1}([n, \infty))\} \subseteq S$. Then change the form of $f_n(s)$:

$$f_n(s) = \begin{cases} \frac{k-1}{2^n}, & s \in f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n})), k \in [1, n2^n] \cap \mathbb{Z} \\ n, & s \in f^{-1}([n, \infty)) \end{cases}$$

So $f_n(s) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n}))} + n\mathbf{1}_{f^{-1}([n,\infty))}$ for $f(s) > 0$.

If $f(s) \leq 0$, $f^+(s) = 0$ and $f^-(s) = -f(s) \geq 0$. So $f_n(s) = -d_n \circ f^-(s) = -d_n \circ -f(s)$. In this case:

$$f_n(s) = \begin{cases} -\frac{k-1}{2^n}, & -f(s) \in [\frac{k-1}{2^n}, \frac{k}{2^n}), k \in [1, n2^n] \cap \mathbb{Z} \\ -n, & -f(s) \in [n, \infty) \end{cases}$$

$$= \begin{cases} \frac{k+1}{2^n}, & f(s) \in (\frac{k}{2^n}, \frac{k+1}{2^n}], k \in [-n2^n, -1] \cap \mathbb{Z} \\ -n, & f(s) \in (-\infty, -n] \end{cases}$$

$$= \begin{cases} \frac{k+1}{2^n}, & s \in f^{-1}((\frac{k}{2^n}, \frac{k+1}{2^n}]), k \in [-n2^n, -1] \cap \mathbb{Z} \\ -n, & s \in f^{-1}((-\infty, -n]) \end{cases}$$

Above all, we can conclude that

$$f_n(s) = \begin{cases} n, & s \in f^{-1}([n, \infty)) \\ \frac{k-1}{2^n}, & s \in f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n})), k \in [1, n2^n] \cap \mathbb{Z} \\ \frac{k+1}{2^n}, & s \in f^{-1}((\frac{k}{2^n}, \frac{k+1}{2^n}]), k \in [-n2^n, -1] \cap \mathbb{Z} \\ -n, & s \in f^{-1}((-\infty, -n]) \end{cases}$$

Notice that when k = 1 or -1, $f_n(s) = 0$. Above all, we can rewrite f_n to a simple function.

$$f_n = \sum_{i=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n}))} + \sum_{i=-n2^n}^{-1} \frac{k+1}{2^n} \mathbf{1}_{f^{-1}((\frac{k}{2^n}, \frac{k+1}{2^n}])} + n\mathbf{1}_{[n,\infty)} - n\mathbf{1}_{(-\infty,-n]}$$

$\square$

2. $f_n \uparrow f$

Proof: We need to prove for every $s \in S$, $\lim_{n\to\infty} f_n(s) = f(s)$.

Assume $f(s) \geq 0$. There exists an n that $f(s) < n$. Also, there exists a k that $f(s) \in [\frac{k-1}{2^n}, \frac{k}{2^n})$.

As $f(s) \in [\frac{k-1}{2^n}, \frac{k}{2^n})$, we get $f_n(s) = \frac{k-1}{2^n}$ from the deduction above.

We have $0 \leq f(s) - f_n(s) < \frac{k}{2^n} - \frac{k-1}{2^n} = \frac{1}{2^n}$ and $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

By Squeeze Theorem, $\lim_{n\to\infty} f(s) - f_n(s) = 0 \Rightarrow \lim_{n\to\infty} f_n(s) = f(s)$

The proof of the other case $f(s) < 0$ is similar. $\square$