Product of A Divergent Series

An application of Borel-Cantelli Theorem

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Problem *Let* $(y_n)_{n\in\mathbb{N}}$ *be a sequence of reals from* $[0,1]$ *such that* $\sum_{n\in\mathbb{N}} y_n = \infty$. *Show that* $\prod_{n\in\mathbb{N}}(1-y_n) = 0$.

Proof: Recall **the second Borel-Cantelli Lemma(BC2)** we learned in class:

Lemma *If the events* E_n *are pairwisely independent, then*

$$\sum_n P(E_n) = \infty \Longrightarrow P(\mathrm{limsup} E_n) = 1$$

We observe that this problem is very similar to this Lemma, in particular y_n is analogous to $P(E_n)$. Suppose we can find for each y_n an event E_n such that $y_n = P(E_n)$, then $\sum_n P(E_n) = \infty$.

Then by the BC2 lemma,

$$\begin{aligned}
P(\limsup E_n) &= P(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geqslant m} E_n) = 1 \\
\Longrightarrow P(\bigcup_{n\geqslant 1} E_n) &= 1 \\
\Longrightarrow P((\bigcup_{n\geqslant 1} E_n)^c) &= P(\bigcap_{n\geqslant 1} E_n^c) = 0
\end{aligned}$$

Now we can prove the statement in the problem,

$$\begin{aligned}
\prod_{n\in\mathbb{N}}(1-y_n) &= \prod_{n\in\mathbb{N}} P(E_n^c) \\
&= P(\bigcap_{n\geqslant 1} E_n^c) \\
&= 0
\end{aligned}$$

The only thing left is an explicit expression for E_n such that $P(E_n) = y_n$. Consider a probaility space $([0,1], \mathcal{B}, Leb)$. Then we can simply let

$$E_n = \{\omega \mid \omega \in [0, y_n]\}$$

We have $P(E_n) = y_n$.

Thus, we have shown that $\prod_{n \in \mathbb{N}} (1 - y_n) = 0$. $\square$