1-dimensional Random Walk

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1 Introduction

Random walk is a random process, that describes a path that consists of a succession of random steps on some mathematical space such as the integers [1]. For example, when a direction idiot struggles to find his destination, the path he pass is a random walk, which is based on the model of graph theory.

The graph model is rather complicated. To simplify, another somehow abstract example is a gambler gambling on a tossed coin, which is known as the Gambler's Ruin Problem. A gamble comes to a casino with x dollars. He tosses a coin again and again, winning 1 dollar for each head and losing 1 dollar for each tail. He is so greedy that he will leave only if he has already owned A dollars, or lost all his money.

The problem can be modeled as a random walk on $\mathbb{Z}$, or more precisely $[0, A] \cap \mathbb{Z}$. Winning 1 dollar means stepping towards the positive end, and losing stands for moveing to the negative end. Here we are interested in two problems, which end will he finally reach (which means whether he wins or loses), and how many steps will he make until he reaches an end (it will be terrible if the gamble lasts for several continuous days).

Back to the direction idiot, let's help him but suppose the city he lives in only has one street, thus can be modeled as $\mathbb{Z}$. We will discuss whether he can reach his destination, and how long he will spend getting to his destination.

2 Simple 1-dimensional random walk

First of all, let's formalize the so-called random walk.

Definition 1. *Suppose $(X_n : n \in \mathbb{N})$ is a sequence of IID RVs, each of which having the same distribution $X \in \{-1, 1\}$. Define $S_0 := x \in \mathbb{Z}$ and $S_n = S_0 + X_1 + \cdots + X_n$, and define the distribution of X as F. A random walk on $\mathbb{Z}$ with common strp distribution F and initial state $x \in \mathbb{Z}$ is the sequence S_n.*

Recall the Gambler's Ruin Problem mentioned in the introduction. Supposer the coin is fair, i.e. the RV X here satisfies

$$P(X = 1) = P(X = -1) = \frac{1}{2}.$$

Let

$$T := \inf\{n : S_n = 0 \vee S_n = A\},$$

and suppose

$$\mathcal{F}_n = \sigma(X_1, \dots, X_n) = \sigma(S_0, \dots, S_n).$$

Then S is adapted, so that T is a stopping time.

Before we continue, we claim that

Proposition 2. *$T < \infty$ w.p.1.*

Actually it is simple at first glance, as the game will definitely end if A consecutive heads or A consecutive tails. Here is a more formal prove.

Proof. Define

$$S'_i = X_{Ai+1} + \cdots + X_{A(i+1)},$$

and

$$T' := \inf\{n : S'_n = -A \vee S'_n = A\}.$$

We have $T < T'$, and $P(S'_n = -A \vee S'_n = A) > 0$. Using Kolmogorov's 0-1 law, it is clear that $T' < \infty$ w.p.1. So $T < \infty$ w.p.1. $\square$

Back to the Gambler's Ruin Problem, we prove a simple feature first.

Theorem 3. *Starting with x dollars, the gambler has a possibility of x/A to reach A dollars, winning his gamble.*

Proof. Let $u(x)$ donate the probability to win the gamble starting with x dollars. For $x \in (0, A) \cap \mathbb{Z}$, $u(x) = \frac{1}{2}u(x-1) + \frac{1}{2}u(x+1)$, which is based on the result of the first step. Thus, $u(x)$ is linear on $[0, A] \cap \mathbb{Z}$. As we have $u(0) = 0$ and $u(A) = 1$, we conclude that $u(x) = x/A, x \in [0, A] \cap \mathbb{Z}$. $\square$

Inspired by theorem 3, we may find some features in the transformation equation of the expected stopping time.

Theorem 4. *Starting with x dollars, the expected stopping time $E(T)$ is $x(A - x)$.*

Proof. Let $v(x)$ donate the expected stopping time starting with x dollars. For $x \in (0, A) \cap \mathbb{Z}$, $v(x) = 1 + \frac{1}{2}v(x-1) + \frac{1}{2}v(x+1)$.

To solve this equation, we define $d(x) = v(x+1) - v(x)$, and we have

$$d(x-1) - d(x) = 2, \quad \forall x \in (0, A-1) \cap \mathbb{Z}.$$

Then we conclude that $d(x) = d(0) - 2x$, and $v(A) = v(0) + \sum_{i=0}^{A-1} d(i) = v(0) + (A-1)d(0) - A(A-1)$. Given that $v(0) = v(A) = 0$, we have $d(0) = A$, and $v(x) = v(0) + \sum_{i=0}^{x-1} d(i) = x(A-x)$. Thus we concluded that $E(T) = x(A-x)$. $\square$

We may notice that not all casinos are kind, so we suppose there is an unfair coin, with possibility p to be head when tossed, and q to be tail. Here it becomes a p–q 1-dimensional random walk, which has a weighted possibility, rather than a uniform distribution. Formally, the RV X here satisfies $P(X = 1) = p$ and $P(X = -1) = q$, where $p + q = 1$.

Let's suppose our casino is not so bad (or so kind) by letting $p, q > 0$. Actually if any of them is zero, the problem becomes trivial.

Theorem 5. *Given a p–q coin ($p \neq q$) and starting from x dollars, the gambler has a possibility of $\dfrac{1-(\frac{q}{p})^x}{1-(\frac{q}{p})^A}$ to reach A dollars, winning his gamble, and the expected stopping time $E(T)$, whether the gambler wins or loses, is $A\dfrac{1-(\frac{q}{p})^x}{1-(\frac{q}{p})^A}$.*

Proof. Similar to former, we define the corresponding $u(x)$, and have $u(x) = q \cdot u(x-1) + p \cdot u(x+1)$ for $x \in (0, A) \cap \mathbb{Z}$. We define $d(x) = u(x+1) - u(x)$ and have $d(x) = \dfrac{q}{p} d(x-1)$, and then $d(x) = (\dfrac{q}{p})^x d(0)$. With $u(0) = 0$, $u(A) = 1$ and $u(A) = u(0) + \sum_{i=0}^{A-1} d(i)$, we may find $d(0) = \dfrac{1-\frac{q}{p}}{1-(\frac{q}{p})^A}$, and finally $u(x) = \dfrac{1-(\frac{q}{p})^x}{1-(\frac{q}{p})^A}$. The calculation for $E(T) = v(x) = 1 + q \cdot v(x-1) + p \cdot v(x+1)$ is somehow complicated. Let $c(x) = v(x+1) - v(x)$, and

$$q \cdot c(x-1) = 1 + p \cdot c(x), \quad \forall x \in (0, A-1) \cap \mathbb{Z},$$

which we transform into

$$c(x) + \frac{1}{p-q} = \frac{q}{p}(c(x-1) + \frac{1}{p-q}).$$

Suppose $c = c(0) + \dfrac{1}{p-q}$, and we have

$$v(A) = v(0) + \sum_{i=0}^{A-1} c(i) = v(0) + c \cdot \frac{1-(\frac{q}{p})^A}{1-\frac{q}{p}} - \frac{A}{p-q}.$$

As $v(0) = v(A) = 0$,

$$c = \frac{A}{p-q} \frac{1-\frac{q}{p}}{1-(\frac{q}{p})^A}.$$

And finally,

$$v(x) = v(0) + \sum_{i=0}^{x-1} c(i) = \frac{A}{p-q} \frac{1-(\frac{q}{p})^x}{1-(\frac{q}{p})^A} - \frac{x}{p-q}.$$

i.e.

$$E(T) = \frac{A}{p-q} \frac{1-(\frac{q}{p})^x}{1-(\frac{q}{p})^A} - \frac{x}{p-q}.$$

□

3 Coverage of 1-dimensional random walk

The random walk we discussed in last section is actually on a segment rather than on a line. Back to the direction idiot, we have the following theorem.

Theorem 6. *If X satisfies $P(X = 1) = P(X = -1) = \dfrac{1}{2}$, (or $P(X = 1) = p$ and $P(X = -1) = q$, where $p + q = 1$), $P(\exists n : S_n = 0) = 1, \forall x \in \mathbb{Z}$, where S_n is defined the same as in definition 1.*

Proof. For an arbitary $A > x$, by theorem 3,

$$P^* := P(\exists n : S_n = 0 \wedge S_i \neq A, \forall i \in \{0, \dots, n\}) = 1 - x/A > 0.$$

As $P(\exists n : S_n = 0) \geq P^* > 0$, by Kolmogorov 0-1 law,

$$P(\exists n : S_n = 0) = 1.$$

□

By reversing and moving the axis, we may find out that

Proposition 7. *$P(\exists n : S_n = y) = 1$ for all $x, y \in \mathbb{Z}$.*

Here we can say the direction idiot can find his destination with possibility one. Moreover, if he cannot realize that he has arrived on his first reach, we can prove that he will arrive at his destination infinitely often as long as he keeps searching.

Before proving this, we will prove a commonly used proverty first.

Proposition 8 (Strong Markov Property)**.** *If T is a stopping time $\inf\{n : S_n = y\}$ for a random walk $\{S_n\}_{n\geq 0}$, then the post-T sequence $\{S_{T+j}\}_{j\geq 0}$ is also a random walk, with the same step distribution, started at S_T, and is independent of the random path $\{S_j\}_{j\leq T}$.*

Proof. What we want to prove is actually

$$\begin{aligned} &P(X_i = a_i, \forall i \leq m \wedge T = k) \\ &\qquad\qquad = P(X_i = a_i, \forall i \leq k \wedge T = k)P(X_i = a_i, \forall i \in (k, m]) \end{aligned}$$

for any $k < m \in \mathbb{N}$ and $(a_i)_{i\in\{0,\dots,m\}} \subseteq \{-1, 1\}$.

We may prove this by reduction on m, after noticing that the distribution of each step is independent.

The induction base is

$$\begin{aligned} &P(X_i = a_i, \forall i \leq k + 1 \wedge T = k) \\ &\qquad\qquad = P(X_i = a_i, \forall i \leq k \wedge T = k)P(X_{k+1} = a_{k+1}), \quad (1) \end{aligned}$$

and by induction hypothesis

$$\begin{aligned}
&P(X_i = a_i, \forall i \le k' + 1 \wedge T = k) \\
&\qquad = P(X_i = a_i, \forall i \le k' \wedge T = k)P(X_{k'+1} = a_{k'+1}) \\
&\quad = P(X_i = a_i, \forall i \le k \wedge T = k)P(X_i = a_i, \forall i \in (k, k'])P(X_{k'+1} = a_{k'+1}) \\
&\qquad = P(X_i = a_i, \forall i \le k' \wedge T = k)P(X_i = a_i \forall i \in (k, k' + 1]). \quad (2)
\end{aligned}$$

□

Thus, after a visit on y, it just becomes the start of another walk starting from y, and x will be visited by the new walk w.p.1. Repeating this procedure by swapping x and y, we finally get

Theorem 9. *1-dimensional random walk on $\mathbb{Z}$ starting from x will visit any state y infinitely often w.p.1.*

4 Stopping time of 1-dimensional random walk

To simplify, in this section, we fix the start point at 0.

Formally, $(X_n)_{n\in\mathbb{N}}$ is a sequence of IID RVs, each of which having the smae distribution X, where

$$P(X = 1) = P(X = -1) = \frac{1}{2}$$

Define $S_0 := 0$, $S_n := X_1 + \cdots + X_n$, and let

$$T(m) := \{n : S_n = m\}.$$

Specially, we define $T := T(1)$. What we will discuss in this section is the property of $T(m)$.

Suppose $m > 0$. If $m > 1$, the random walk must reach 1 before reaching one, i.e. $T(m) \ge T$. By proposition 8, we may consider the walk after reaching one as a new walk. The additional time needed to reach m is the same as $T(m-1)$, and is independent of T. Thus, $T(m)$ is the sum of m independent copies of T.

We look at a probability generating function

$$F(z) := E(z^T) = \sum_{n=1}^{\infty} z^n P(T = n).$$

For the first step, if $S_1 = 1$, $T = 1$; otherwise $S_1 = -1$, the random walk must return to 0 before reaching 1. The time it takes to reach 0 from -1 has the same distribution as T, and by proposition 8, it is independent of the former and latter part of the walk. Therefore,

$$F(z) = \frac{z}{2} + \frac{z}{2}E(z^{T'+T''}),$$

where T' and T'' are independent random variable having the same distribution as T and independent of T. Notice that a sum of independent variables is the product of their generating function, so

$$F(z) = \frac{1}{2}(z + zF(z)^2).$$

I am not expert at calculas, so using Mathematica, the solutions are $F(z) = \dfrac{1 \pm \sqrt{1-z^2}}{z}$, but only one of these satisfy $F(z) \in (0,1), \forall z \in (0,1)$, so

$$F(z) = \frac{1 - \sqrt{1-z^2}}{z}.$$

Note that the distribution of T, what we want to find, correspond to the coefficient of $F(z)$. As $\sqrt{1-z^2} = \sum_{i=0}^{\infty} \binom{1/2}{n} (-z^2)^n$, the coefficient of z^{2n-1} is $(-1)^{n+1}\binom{1/2}{n}$, resulting in our final result

Theorem 10. $P(T = 2n-1) = (-1)^{n+1}\binom{1/2}{n}$.

The possibility of T being even is apparently zero, as it must take odd steps moving from 0 to 1.

[2] provides another proof, which uses martingale and do not use strong Markov property.

Proof. Let $\theta \in \mathbb{R}$, $E(\exp(\theta x)) = \cosh\theta$, so

$$E[(\operatorname{sech}\theta)\exp(\theta X_n)] = 1, \quad \forall n.$$

Then

$$M_n^\theta = (\operatorname{sech}\theta)^n \exp(\theta S_n)$$

is a martingale. Together with T being a stopping time

$$E(M_{T\wedge n}^\theta) = E[(\operatorname{sech}\theta)^{T\wedge n}\exp(\theta S_{T\wedge n})] = 1.$$

From bounded converage theorem, let $n \to \infty$,

$$E(M_T^\theta) = 1 = E[(\operatorname{sech}\theta)^T \exp(\theta)].$$

Let $\alpha = \operatorname{sech}\theta$, we have

$$E(\alpha^T) = \sum \alpha^n P(T = n) = \exp(-\theta) = \frac{1-\sqrt{1-\alpha^2}}{\alpha},$$

and similar to the previous proof,

$$P(T = 2m-1) = (-1)^{m+1}\binom{1/2}{m}.$$

$\square$

References

[1] Random walk, Jun 2020.

[2] David Williams. *Probability with martingales.* Cambridge Univ. Pr., 2018.

[3] Steven Lally. One-dimensional random walk, 2016.