Brownian motion and Lévy's martingale characterization

Zhuang YongHao, 518030910440

June 30, 2020

This note is strongly based on [1] [2].

1 Brownian motion

We define Brownian motion(also called Wiener process) by:

Definition 1.1 (Brownian motion). *A stochastic process $B(t)$ is a Brownian motion if:*

- $B_0 = 0$ *a.s.*
- $P(B_t \text{ is continuous}) = 1$
- *for all* $0 < s < t$, $(B_t - B_s) \sim N(0, t-s)$
- $B_{t_i} - B_{s_i}$ *is independent with* $B_{t_j} - B_{s_j}$ *if* $[s_i, t_i]$ *and* $[s_j, t_j]$ *do not intersect.*

The first assumption is trivial, simply requires it to start from zero; The second assumption is to let it continuous(however, not differentiable), which lets many analysis possible.

The third and fouth assumptions are non-trivial, requiring the progress to follow Gaussian distribution locally and be independent.

The Brownian motion is widely discussed in stochastic process, since it can describe some useful models. A Brownian motion in 1-dimension space is like below, and it is close to the price of stokes, futures and options. Later we will introduce Itô process, which is more close to these price patterns.

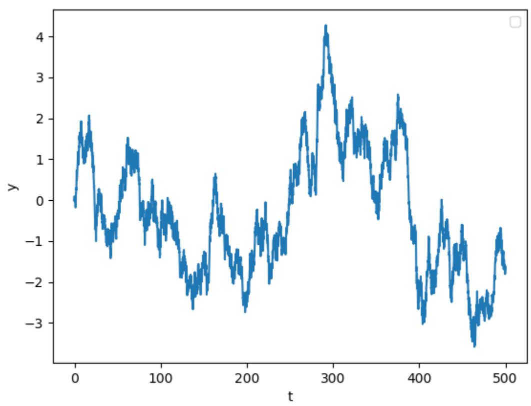

Figure 1: Brownian motion in 1-dimension space

More close to our course, there is a theorem describing another definition of the Brownian motion:

Theorem 1.2 (Lévy's martingale characterization). *A stochastic process $\{M_t\}$ is a Brownian motion if:*

- $P(M_0 = 0) = 1$
- *M_t is a continuous martingale*
- *The quadratic varitation $< M_t >= \lim_{|\Delta|\to 0} \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2 = t$ a.s. in L^2(Δ ranges over all partitions of $[0,t]$ and $|\Delta|$ is the length of maximum subinterval of Δ)*

This theorem gives another definition of Brownian motion. The first and second assumptions are trivial, corresponding to the first two assumptions in definition1.1, except for a martingale assumption. The third assumption is non-trivial, corresponding to the Gaussian distribution assumption.

For convenience, from now on we denote $P_{[a,b]}$ be the set of all partitions over $[a,b]$, denote $Var_p(f;\Delta)$ by

$$Var_p(f;\Delta) = |f(0)|^p + \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p$$

And let $Var_p(f;[a,b]) = \sup_{\Delta\in P_{[0,t]}} Var_p(f;\Delta)$

To prove this theorem, the Itô's Lemma is required, which is our next topic.

2 Itô's Lemma

The Brownian motion is a continuous martingale, but our tool to analyze continuous function—Riemann integration—is not useful for Brownian motion. We will show this in the following part.

Recall that our final goal is to prove theorem 1.2. Although some part of the proof is not clear now, we can still do some other part, which is helpful for the next part.

Proposition 2.1 (The quadratic variation of Brownian motion). *Let $\{B_t\}$ be a Brownian motion, the quadratic varitation $< B_t >= \lim_{|\Delta|\to 0} \sum_{k=1}^{n} (B_{t_k} - B_{t_{k-1}})^2 = t$ a.s. in L^2, where Δ ranges over $P[a,b]$.*

Proof. For all $0 \le r \le s \le u \le v$, by the definition of Brownian motion and property of Gaussian distribution, there is:

$$\mathbb{E}\left[((B_s - B_r)^2 - (s-r))^2\right] = 2(s-r)^2$$

By the independence of Brownian distribution there is also

$$\mathbb{E}\left[((B_s - B_r)^2 - (s-r))((B_v - B_u)^2 - (v-u))\right] = 0$$

So for any partition $\Delta \in P_{[0,t]}$ there is:

$$\mathbb{E}\left[(Var_2(B;\Delta) - t)^2\right] = 2Var_2(I;\Delta)$$

where I is the identity function $I(x) = x$.

Notice that $Var_2(I;\Delta) = \sum_{i=1}^{n} (t_i - t_{i-1})^2 \le t|\Delta|$, so the proposition follows as $n\to\infty, |\Delta| \to 0$. $\square$

Corollary 2.2. *Let $\{B_t\}$ a Brownian motion, then there is:*

$$\forall t > 0, Var_1(B; [0, t]) = \infty, a.s.$$

Proof. For a partition $\Delta = \{t_0, t_1 \dots t_k\} \in P_{[0,t]}$, by definition of Var_p there is

$$Var_2(B; \Delta) \leq Var_1(B; \Delta)\delta(B; \Delta)$$

where $\delta(B; \Delta) = \sup_{1 \leq i \leq n} |B_{t_i} - B_{t_{i-1}}|$.

By proposition2.1, as $n \to \infty, |\Delta| \to 0$, $\delta(B; \Delta) \to t$; by the continuity, there is $n \to \infty, \delta(B; \Delta) \to 0$.

Applying the two to the inequality above we get the result. □

The quadratic variation of Brownian motion can show that the Riemann integration is useless. Consider the Taylor's series of a continuous (and smooth enough) function f:

$$\Delta f(x) = f(x + \Delta x) - f(x) = f'(x)\Delta x + \frac{f''(x)}{2}(\Delta x)^2 + \dots$$

If $(\Delta x)^2 << \Delta x$, there is $\Delta f(x) = f'(x)\Delta x$, however, by the quadratic variation, there is $(\Delta B_t)^2 = \Delta t$, so the second term is not ignorable, and the equality turns to be:

$$\mathrm{d}f(B_t) = f'(B_t)\,\mathrm{d}B_t + \frac{1}{2}f''(B_t)\,\mathrm{d}t$$

which is a simplified form of the Itô's lemma.

More formally, we use the following theorem to show that the Riemann integration needs to be developed to analysis Brownian motion. [4]

Theorem 2.3. *For a unique $t > 0$, the Brownian motion is not differentiable at t a.s.*

Proof. Let $\{B_t\}_{t \in [0,\infty)}$ a Brownian motion. Then let $X_t = tB_{\frac{1}{t}}$, which is also a Brownian motion. There is:

$$\limsup_{t \downarrow 0} \frac{B_t - B_0}{t} \geq \limsup_{n \to \infty} \frac{B_{1/n} - B_0}{1/n} \geq \limsup_{n \to \infty} \sqrt{n}B_{1/n} = \limsup_{n \to \infty} \frac{X_n}{\sqrt{n}}$$

For $K \in (0, \infty)$, there is:

$$\begin{aligned}
\mathbb{P}[X_n > K\sqrt{n}, i.o.] &= \mathbb{P}[\cap_{n \in \mathbb{N}} \cup_{m \geq n} \{X_m > K\sqrt{m}\}] \\
&= \lim_{n \to \infty} \mathbb{P}[\cup_{m \geq n}\{X_m > K\sqrt{m}\}] \\
&\geq \lim_{n \to \infty} \mathbb{P}[X_n > K\sqrt{n}]
\end{aligned}$$

By independence of Brownian motion, there is $\mathbb{P}[X_n > K\sqrt{n}] = \mathbb{P}[X_1 > K] > 0$. By Fatou's lemma, there is $\mathbb{P}[X_n > K\sqrt{n}, i.o.] \geq \limsup_{n \to \infty} \mathbb{P}[X_n > K\sqrt{n}] > 0$.

Using Kolmogorov's 0-1 law there is:

$\mathbb{P}[X_n > K\sqrt{n}, i.o.] = 1$, so $\limsup \frac{X_n}{\sqrt{n}} \geq K, a.s.$, for all $K > 0$.

So $\limsup_{t \downarrow 0} \frac{B_t - B_0}{t} = \infty$, and likewise there is $\limsup_{t \uparrow 0} \frac{B_t - B_0}{t} = -\infty$. So when $t = 0$ the Brownian motion is not differentiable. As $B_{s+t} - B_t$ is also a Brownian motion(easy to check), in any unique $t > 0$, B_t is not differentiable. □

The corollary can be also described as:

$$\mathbb{P}[B_t \text{ is bounded variation}] = 0$$

It is clear that the Riemann integration is not enough, so we introduce the Itô integration:

Definition 2.4 (Itô integration). *For a stochastic process X and a Brownian motion B, the Itô integration is:*

$$\int_0^t X \, \mathrm{d}B = \lim_{n \to \infty} \sum_{i=0}^{n-1} X_{t_{i-1}} (B_{t_i} - B_{t_{i-1}})$$

The definition is very close to Riemann integration, but in the Itô process, X is required to take the value in left($X_{t_{i-1}}$). This is very close to a transform over continuous time martingale.

Now we define Itô process:

Definition 2.5 (Itô process). *A stochastic process $\{\xi_t\}_{t\in[0,\infty)}$ is an Itô process if it has continuous paths a.s. and can be represented as*

$$\xi_T = \xi_0 + \int_0^T a(t, B_t) \, \mathrm{d}t + \int_0^T b(t, B_t) \, \mathrm{d}B_t$$

which equals

$$\mathrm{d}\xi_t = a(t, B_t) \, \mathrm{d}t + b(t, B_t) \, \mathrm{d}B_t$$

where $\{B_t\}_{t\in[0,\infty)}$ is a Brownian motion.

It is obvious that, a Brownian motion is an Itô process. The Itô process is more close to a price pattern, since the $a \, \mathrm{d}t$ part is a global trend(called drift) and $b \, \mathrm{d}B_t$ is a local perturbation(called diffuse).

Now we give the Itô's Lemma:

Lemma 2.6 (The Itô's Lemma). *Let ξ_t a Itô process, let $f(t,x)$ a real-valued function with continuous partial derivatives. Then $f(t,\xi_t)$ is an Itô process such that*

$$\mathrm{d}f = (\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi_t} a + \frac{1}{2} \frac{\partial^2 f}{\partial \xi_t^2} b^2) \, \mathrm{d}t + \frac{\partial f}{\partial \xi_t} b \, \mathrm{d}W_t$$

The idea of the proof is close to the proof of existance of conditional expectation: we first prove a subset(in existance of conditional expectation, it is random variables in $\mathcal{L}^2$, in Itô's lemma, it is $x^i f(x)$ where $f(x)$ satisfies), then we use the construct a sequence to approach the rest. The proof is long and I'm sure that it has no need to appear in a minipaper showing my work, since I do nothing more than reading and understanding.

A simplified version is that:

$$\mathrm{d}f = \frac{\partial f}{\partial t} \, \mathrm{d}t + \frac{\partial f}{\partial \xi_t} \, \mathrm{d}\xi_t + \frac{1}{2} \frac{\partial^2 f}{\partial \xi_t^2} (\mathrm{d}\xi_t)^2$$

3 Proof of Lévy's martingale characterization

Now we can prove 1.2.

Proof. 1. We first prove that by definition1.1, all assumptions in Lévy's martingale characterization are satisfied:

The two definition shares the almost-sure-start-at-zero assumption, and the third assumption(quadratic variation) is already proved. So we only need to prove the second one, that a Brownian motion is a continuous martingale:

$$\begin{aligned}\mathbb{E}[B_i|\mathcal{F}_j] &= \mathbb{E}[B_j + (B_i - B_j)|\mathcal{F}_j] \\ &= B_j + \mathbb{E}[(B_i - B_j)|\mathcal{F}_j] = B_j\end{aligned}$$

Which is trivial.

2. Now we prove that a stochastic process $\{M_t\}$ satisfying assumptions in Lévy's martingale characterization is a Brownian motion. Still, the almost-sure-start-at-zero assumption is shared, so no need to prove. The continuous assumption is satisfied since $\{M_t\}$ is a continuous martingale.

Now we prove the Gaussian distribution and independence assumption. Use the property of CF(characteristic function), we equivalently prove that $\forall s, t, \alpha \in \mathbb{R}, \mathbb{E}[e^{i\alpha(M_s - M_r)}|\mathcal{F}_r] = e^{-\frac{1}{2}\alpha^2(s-r)}$.

Observe that by rearranging, we only need to prove that $\mathbb{E}[e^{i\alpha M_s + \frac{1}{2}\alpha^2 s}|\mathcal{F}_r] = e^{i\alpha M_r + \frac{1}{2}\alpha^2 r}$, which is to prove that $X_t = e^{i\alpha M_t + \frac{1}{2}\alpha^2 t}$ is a martingale with respect to filtration $\{\mathcal{F}_t\}$

Now apply the Itô Lemma to $X(t, M_t) = e^{i\alpha M_t + \frac{1}{2}\alpha^2 t}$, use $(\mathrm{d}M_t)^2 = \mathrm{d} < M_t >= \mathrm{d}t, a = 0, b = 1$ we get:

$$\begin{aligned}\mathrm{d}X(t, M_t) &= \frac{\partial X}{\partial t}\,\mathrm{d}t + \frac{\partial X}{\partial M_t}\,\mathrm{d}M_t + \frac{1}{2}\frac{\partial^2 X}{\partial M_t^2}(\mathrm{d}M_t)^2 \\ &= \frac{1}{2}\alpha^2 X\,\mathrm{d}t + i\alpha X\,\mathrm{d}M_t + \frac{1}{2}(-\alpha^2 X)\,\mathrm{d}t \\ &= i\alpha X\,\mathrm{d}M_t\end{aligned}$$

Since $\{M_t\}$ is a martingale with respect to $\{\mathcal{F}_t\}$, $\{X_t\}$ is an exponential martingale with respect to the same filtration. And that's what we want to prove. □

Another approach towards the theorem is in [3], which is more complex but requires less knowledge.

4 Related Work

The Brownian motion is only a path to approach stochastic process, which is an area with adequate result:

If we introduce local martingale, the Lévy's martingale characterization can be in form of local martingale, which is more widely used.

In the proof above, we can see that the Itô Lemma can be used to judge if a random process is a martingale, though in this example we use exponential martingale, which makes the method not very universal.

However, actually the Itô Lemma can used to judge whether a random process is a local martingale. By a little more judgement(whether bounded in L^2 or not), a local martingale turns to a martingale. So this is a mechanical method.

In the proof, some properties of CF is used, but it may take much time to write the path towards them, making the note complex and hard to show main idea, so I decide not to mention them.

References

[1] Jean-François Le Gall. *Brownian Motion, Martingales, and Stochastic Calculus.* Springer, 2016.

[2] P. Mörters and Y. Peres. *Brownian Motion.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.

[3] David Pollard. *Lecture Notes on Advanced Stochastic Processes.*

[4] Sebastien Roch. *Lecture Notes on Measure-theoretic Probability Theory.*