Integrals over subsets

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May 1, 2020

1

In measure space (S, Σ, μ), $A \in \Sigma$. Show that:

1. $(A, \Sigma \cap 2^A)$ is a σ-algebra.
2. $\forall f \in (m\Sigma)^+$,
$$\mu_{\Sigma \cap 2^A}(f) = \mu(f \cdot 1_A).$$

Proof.

1. We have:
   - $A \in \Sigma \implies A \in \Sigma \cap 2^A$.
   - $\forall F \in \Sigma \cap 2^A$,
   $$\begin{aligned} A \setminus F &= (S \setminus (S \setminus A)) \setminus F \\ &= S \setminus (F \cup (S \setminus A)) \in \Sigma \\ &= (S \setminus F) \cap A \in 2^A. \end{aligned}$$
   Thus $A \setminus F \in \Sigma \cap 2^A$.
   - $\forall (F_i)_{i \in \mathbb{N}}$ where $F_i \in \Sigma \cap 2^A \subseteq \Sigma$. Then $\bigcup_i F_i \in \Sigma$, and $\bigcup_i F_i \in 2^A$, thus, $\bigcup_i F_i \in \Sigma \cap 2^A$.

   Therefore, $(A, \Sigma \cup 2^A)$ is a σ-algebra.
2. We will show this with the standard machine.
   - Let $f = 1_B, B \in \Sigma \cap 2^A$. We have $\mu(f \cdot 1_A) = \mu(1_B \cdot 1_A) = \mu(A \cap B)$. Meanwhile,
   $$\mu|_{\Sigma \cap 2^A}(1_B) = \mu|_{\Sigma \cap 2^A}((B \setminus A) \cup (B \cap A)) = \mu|_{\Sigma \cap 2^A}(B \cap A) + 0 = \mu(A \cap B).$$
   Thus $\mu(f \cdot 1_A) = \mu|_{\Sigma \cap 2^A}(f)$ for all $f = 1_B$.

- Then we can let $f \in SF^+$. Let $f = \sum^n b_i 1_{B_i}$. Then

$$\begin{aligned}\mu(f \cdot 1_A) = \mu((\sum^n b_i 1_{B_i}) 1_A) &= \mu(\sum^n b_i 1_{B_i \cap A}) \\ &= \sum^n b_i \mu(1_{B_i \cap A}) \\ &= \sum^n b_i \mu|_{\Sigma \cap 2^A}(1_{B_i}) = \mu|_{\Sigma \cap 2^A}(\sum^n b_i 1_{B_i}).\end{aligned}$$

- Finally we let $f \in (m\Sigma)^+$, $\mu(f) = \sup\{\mu(h) : h \in SF^+, h \leq f\}$.

$$\begin{aligned}\mu(f \cdot 1_A) &= \sup\{\mu(h) : h \in SF^+, h \leq f \cdot 1_A\} \\ &= \sup\{\mu(g \cdot 1_A) : g \in SF^+, g \cdot 1_A \leq f \cdot 1_A\} \\ &= \sup\{\mu|_{\Sigma \cap 2^A}(g) : g \in SF^+, g \cdot 1_A \leq f \cdot 1_A\} \\ &= \sup\{\mu|_{\Sigma \cap 2^A}(g) : g \in SF^+, g \leq f\} = \mu|_{\Sigma \cap 2^A}(f).\end{aligned}$$

Therefore, for all $f \in (m\Sigma)^+$, the result of integral over a subset is equal to the result of the integral in restricted measure space.

$\square$