Martingale Formulation of Bellman's Optimality Principle

Wang Yi - 518030910413

June 15, 2020

Martingale Formulation of Bellman's Optimality Principle(Ex10.2)

Problem 1

Statement Your winnings per unit stake on game n are ε_n, where the ε_n are IID RVs with

$$P(\epsilon_n = +1) = p, P(\epsilon_n = -1) = q, \text{where } \frac{1}{2} < p = 1 - q < 1$$

Your stake C_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n-1$. Your object is to maximize the expected 'interest rate' $E \log(Z_N/Z_0)$, where N is a given integer representing the length of the game, and Z_0, your fortune at time 0 is a given constant. Let $\mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$ be your 'history' up to time n. Show that if C is any previsible strategy, the $\log Z_n - n\alpha$ is a supermartingale, where α denotes the 'entropy'

$$\alpha = p \log p + q \log q + \log 2,$$

so that $E \log(Z_n/Z_0) \leq N\alpha$, but that, for a certain strategy, $\log Z_n - n\alpha$ is a martingale. What is the best strategy?

Solution

Proof. We can get $Z_n = Z_{n-1} + C_n \cdot \epsilon_n$ by definition of this game. Then we will get

$$\begin{aligned}
E[(\log Z_n - n\alpha) - (\log Z_{n-1} - (n-1)\alpha)|\mathcal{F}_{n-1}] &= E[\log Z_n - \log Z_{n-1}|\mathcal{F}_{n-1}] - \alpha \\
&= E[\log(1 + \frac{C_n \cdot \epsilon_n}{Z_{n-1}})|\mathcal{F}_{n-1}] - \alpha \\
&= p \log(1 + \frac{C_n}{Z_{n-1}}) + q \log(1 - \frac{C_n}{Z_{n-1}}) - \alpha
\end{aligned}$$

$p \log(1 + \frac{C_n}{Z_{n-1}}) + q \log(1 - \frac{C_n}{Z_{n-1}}) - \alpha$ reaches the maximum α, when $\frac{C_n}{Z_{n-1}} = \frac{p-q}{p+q} = p - q$. The maximium is $p \log(1 + p - q) + q \log(1 - (p - q)) - \alpha = \alpha - \alpha = 0$. Thus

$$E[(\log Z_n - n\alpha) - (\log Z_{n-1} - (n-1)\alpha)|\mathcal{F}_{n-1}] \leq 0.$$

Therefore $\log Z_n - n\alpha$ is a supermartingale.
When $C_n = (p - q)Z_{n-1}$, it is a martingale, so the best strategy is $C_n = (p - q)Z_{n-1}$. □