Kullback–Leibler divergence ≥ 0

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Proof. We shall prove that $D_{KL}(P||Q) \geq 0$, here D_{KL} means Kullback–Leibler divergence and $D(P||Q) = -\sum_x P(x) \ln \frac{Q(x)}{P(x)}$.

$$
\begin{aligned}
-D(P||Q) &= \sum_x P(x) \ln \frac{Q(x)}{P(x)} && (1) \\
&\leq \sum_x P(x) \left(\frac{Q(x)}{P(x)} - 1 \right) && (2) \\
&= \sum_x Q(x) - \sum_x P(x) && (3) \\
&= 1 - 1 && (4) \\
&= 0 && (5)
\end{aligned}
$$

In the (2) step we use the inequality $\ln x \leq x - 1$. □

proof 2. I recall Jensen's inequality in our textbook when I write the first proof. We can use it to get another proof.

$$
\begin{aligned}
-D(P||Q) &= \sum_x P(x) \ln \frac{Q(x)}{P(x)} && (6) \\
&\le \ln \sum_x P(x) \frac{Q(x)}{P(x)} && (7) \\
&= \ln \sum_x Q(x) && (8) \\
&= \ln 1 && (9) \\
&= 0 && (10)
\end{aligned}
$$

In the (7) step we use Jensen's inequality. $\square$