Construction of 4 sets to satisfy 3 rules

马浩博 518030910428

March 9, 2020

Exercise 8

Construct three sets A, B, C and a set E $\subseteq A \times B \times C$ such that the following hold: 1) For every (a, b, c) $\in A \times B \times C$, E has a matching which saturates a, b, c; 2) E has an (A$\cup$B)-saturated matching, a (B$\cup$C)-saturated matching, and an (A$\cup$C)-saturated matching; 3) E does not have any (A$\cup$B$\cup$C)-saturated matching.

Answer. Let me first give out the construction:

$$A = \{1, 0, 4, 7\}$$

$$B = \{2, 0, 4, 7\}$$

$$C = \{3, 0, 4, 7\}$$

$$E = \bigcup_{i=0}^{5} \{e_i\}$$

$$e_0 = (1, 7, 4)$$

$$e_1 = (4, 2, 7)$$

$$e_2 = (7, 4, 3)$$

$$e_3 = (0, 7, 4)$$

$$e_4 = (4, 0, 7)$$

$$e_5 = (7, 4, 0)$$

First, we shall prove it satisfies 3). Assume there is such matching E'. Then e_0, e_1, e_2 must belong to E' because only they have the elements 1,2,3. Then we know $e_3 \notin E'$ because it's not disjoint with e_0, and same as e_4, e_5. Thus there it no element which can cover 0. So such E' doesn't exist.

Next, we shall prove it satisfies 2). For $A \cup B$, we just need to let $\{e_0, e_1, e_5\}$ be the matching. Beause of symmetry, we can know there are also $(B \cup C)$-saturated matching and $(C \cup A)$-saturated matching.

Finally, we need to prove 1). If the a in the choosen (a, b, c) doesn't equal to 1, then we can just use the $(B \cup C)$-saturated matching. Same way if $b \neq 2$ or $c \neq 3$. So we just need to concern $(1, 2, 3)$. To saturate this, we can use the macthing $\{e_0, e_1, e_2\}$.

Hence, the construction is right. I thought for a long time and tried many constructions. This should be the most concise one.

□