

Relationships between Convergences

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Problem 1 *Almost sure convergence \implies Convergence in Probability.*

Assume that $\mu : \Sigma \mapsto [0, \infty)$ and that we have $(f_n) \in m\Sigma$, $f \in m\Sigma$ such that

$$f_n \xrightarrow{a.s.} f,$$

i.e.

$$\mu(f_n \not\rightarrow f) = \mu(\{\omega \in S \mid f_n(\omega) \not\rightarrow f(\omega)\}) = 0.$$

Prove that for any $\epsilon > 0$,

$$\mu(|f_n - f| > \epsilon) \rightarrow 0.$$

Proof:

$$\begin{aligned} f_n \not\rightarrow f &= \{\omega \in S \mid f_n(\omega) \not\rightarrow f(\omega)\} \\ &= \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega \in S \mid |f_n(\omega) - f(\omega)| > \frac{1}{k}\} \\ &= \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} (|f_n - f| > \frac{1}{k}). \end{aligned}$$

For any $\epsilon > 0$, since $\mu(f_n \not\rightarrow f) = 0$, we can deduce that

$$\mu\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} |f_n - f| > \epsilon\right) = 0,$$

thus

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mu(|f_n - f| > \epsilon) &\leq \lim_{m \rightarrow \infty} \mu\left(\bigcup_{n=m}^{\infty} |f_n - f| > \epsilon\right) \\
&= \mu\left(\lim_{m \rightarrow \infty} \bigcup_{n=m}^{\infty} |f_n - f| > \epsilon\right) \quad (\text{MON of Measure}) \\
&= \mu\left(\bigcap_m \bigcup_{n=m}^{\infty} |f_n - f| > \epsilon\right) \\
&= 0.
\end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \mu(|f_n - f| > \epsilon) = 0,$$

i.e. (f_n) converges to f in probability. □

Problem 2 *Convergence in Probability \implies Almost sure convergence of a subsequence.*
We have $(f_n) \in m\Sigma$, $f \in m\Sigma$ such that for any $\epsilon > 0$,

$$\mu(|f_n - f| > \epsilon) \rightarrow 0$$

Prove that there exists an increasing subsequence (n_k) such that

$$f_{n_k} \xrightarrow{\text{a.s.}} f,$$

i.e.

$$\mu(f_{n_k} \not\rightarrow f) = \mu(\{\omega \in S \mid f_{n_k}(\omega) \not\rightarrow f(\omega)\}) = 0.$$

Proof: As a premise, we have

$$\mu(|f_n - f| > \epsilon) \rightarrow 0.$$

For an arbitrarily fixed $\epsilon > 0$, we will have an increasing sequence (n_k) such that

$$\mu(|f_{n_k} - f| > \epsilon) < \frac{1}{2^k}.$$

Taking the infinite sum on both sides yields,

$$\sum_{k=1}^{\infty} \mu(|f_{n_k} - f| > \epsilon) < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$$

By the *First Borel-Cantelli Lemma on general measure space*, we have

$$\mu(|f_{n_k} - f| > \epsilon, \text{i.o.}) = 0.$$

Based on the arbitrariness of ϵ , we have proven the statement,

$$\mu(f_{n_k} \not\rightarrow f) = 0.$$

□

Remark *The almost sure convergence might not hold for the whole sequence. The construction I resort to is relatively complicated. Let $(S, \Sigma, \mu) = ([0, 1], \text{Leb}[0, 1], \text{Leb})$. Define f_n and f as following: For every $x \in [0, 1]$*

$$\begin{aligned} f_1(x) &= x + I_{[0,1]} \cdot x \\ f_2(x) &= x + I_{[0, \frac{1}{2}]} \cdot x, \quad f_3(x) = x + I_{[\frac{1}{2}, 1]} \cdot x \\ f_4(x) &= x + I_{[0, \frac{1}{3}]} \cdot x, \quad f_5(x) = x + I_{[\frac{1}{3}, \frac{2}{3}]} \cdot x, \quad f_6(x) = x + I_{[\frac{2}{3}, 1]} \cdot x \end{aligned}$$

and

$$f(x) = x.$$

Then we have

$$\mu(|f_n - f| > \epsilon) < \frac{1}{\sqrt{2n} + 1} \rightarrow 0,$$

however

$$\mu(f_n \not\rightarrow f) = \text{Leb}([0, 1]) \neq 0.$$

Therefore, given convergence in probability, only a subsequence of (f_n) converges almost surely.

Problem 3 *Convergence in Probability of Monotone $(f_n) \implies$ Almost sure convergence. We require (f_n) to be **monotone** in the sense that*

$$\forall \omega \in S \quad f_{n+1}(\omega) > f_n(\omega)$$

And same with last problem, We have $(f_n) \in m\Sigma$, $f \in m\Sigma$ such that for any $\epsilon > 0$,

$$\mu(|f_n - f| > \epsilon) \rightarrow 0$$

Prove that such that

$$f_n \xrightarrow{\text{a.s.}} f,$$

i.e.

$$\mu(f_n \not\rightarrow f) = \mu(\{\omega \in S \mid f_n(\omega) \not\rightarrow f(\omega)\}) = 0.$$

Proof: Using the result of *Problem 2*, we can obtain a subsequence (n_k) such that

$$\mu(f_{n_k} \not\rightarrow f) = \mu(\{\omega \in S \mid f_{n_k}(\omega) \not\rightarrow f(\omega)\}) = 0.$$

Consider any ω such that $f_{n_k}(\omega) \rightarrow f(\omega)$.

We prove that with monotonicity of (f_n) , $f_n(\omega) \rightarrow f(\omega)$.

$$\begin{aligned} f_{n_k}(\omega) \rightarrow f(\omega) &\implies \forall \epsilon > 0 \exists K \forall k > K |f_{n_k}(\omega) - f(\omega)| < \epsilon \\ &\xRightarrow{\text{Monotone}} \forall \epsilon > 0 \exists K \forall n > n_K |f_n(\omega) - f(\omega)| < \epsilon \\ &\implies f_n(\omega) \rightarrow f(\omega) \end{aligned}$$

Therefore,

$$\mu(f_n \not\rightarrow f) \leq \mu(f_{n_k} \not\rightarrow f) = 0,$$

thus

$$f_n \xrightarrow{\text{a.s.}} f.$$

□