Incomplete subspace of L2

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Here we give an example of an incomplete subspace of L^2:

First we construct the subspace of L^2 that:

$$l_2 = \{(x_n) : \sum_0^\infty x_n^2 = 0\}$$

Here the function f corresponding to sequence (x_n) is that:

$$f(x) = \begin{cases} x_n & n \leq x < n+1 \\ 0 & x < 0 \end{cases}$$

Then all such functions are in L_2 since they are continuous almost everywhere. And it is obviously a vector space(closed in addition and scalar multiplication).

Now construct a Cauchy sequence (S_N) in the subset that: for each element S_N, let $x_0 = -1$, $x_{N,n} = \frac{1}{N}$ for any $N+1 \leq n \leq 2N$, zero otherwise.

For any two element in the sequence S_N, S_M there is:

$$\|S_N - S_M\|_2^2 \leq \frac{1}{N} + \frac{1}{M} \leq \frac{2}{\min N, M}$$

So it is obvious that this sequence is a Cauchy sequence.

However, there is no $\lim_{n\to\infty} S_N$ in l_2, since it is not a zero-sum sequence.

The main idea of this example is that, the most trivial and intuitive elements in Hilbert space are those like a sequence. Giving the sequence a constraint(in this example is the sum of all elements) can easily form a vector subspace.