Reals Allowing Rational Approximation of Too High Order Are Negligible

Zhiyang Xun

March 17, 2020

Definition 1 A_ϕ is the set of reals x in $(0,1]$ such that

$$\left|x-\frac{p}{q}\right|<\frac{1}{q^2\phi(q)}$$

has infinitely many irreducible rational solutions (p,q), that is, $(p,q)\in\mathbb{Z}^2$, $q>0$ and $\gcd(p,q)=1$.

Theorem 2 Suppose that ϕ is positive. If $\sum_q \frac{1}{q\phi(q)}<\infty$, then $P(A_\phi)=0$.

Proof. For each $(p,q)\in\mathbb{Z}^+\times\mathbb{Z}^+$ satisfying $p\le q$, we define interval $I_{(p,q)}=[\frac{p}{q}-\frac{1}{q^2\phi(q)},\frac{p}{q}+\frac{1}{q^2\phi(q)}]$. Clearly, $A_\phi\subseteq\bigcup I_{(p,q)}$. Because each element $x\in A_\phi$ has infinitely many solutions to $\left|x-\frac{p}{q}\right|<\frac{1}{q^2\phi(q)}$, and for each $k\ge 1$, only finite (p,q) satisfying $p\le q<k$, we can see that

$$A_\phi\subseteq\bigcup_{q=k}^{\infty}\bigcup_{p=1}^{q}I_{(p,q)}$$

Now we only need to prove for each $\epsilon>0$, there is a positive integer k satisfying $\sum_{q=k}^{\infty}\sum_{p=1}^{q}\left|I_{(p,q)}\right|<\epsilon$ to illustrate $P(A_\phi)=0$.

We can see

$$
\begin{aligned}
&\sum_{q=k}^{\infty}\sum_{p=1}^{q}\left|I_{(p,q)}\right| \\
=&\sum_{q=k}^{\infty}\sum_{p=1}^{q}\frac{2}{q^2\phi(q)} \\
=&\sum_{q=k}^{\infty}\frac{2}{q\phi(q)} \\
=&2\sum_{q=k}^{\infty}\frac{1}{q\phi(q)}
\end{aligned}
$$

Since $\sum_q \frac{1}{q\phi(q)} < \infty$, for each $\epsilon > 0$, there is a $k \geq 1$ satisfying $\sum_{q=k}^{\infty} \frac{1}{q\phi(q)} < \frac{\epsilon}{2}$. It implies $\sum_{q=k}^{\infty}\sum_{p=1}^{q}\left|I_{(p,q)}\right| = 2\sum_{q=k}^{\infty}\frac{1}{q\phi(q)} < \epsilon$.

Hence, $P(A_\phi) = 0$.

$\square$