Expection

by 刘成锴

Definition of expectation

For a random variable $X \in \mathcal{L}^1 = L^1(\Omega, \mathcal{F}, P)$, we define the $expectation$ $E(X)$ of X by

$$E(X) := \int_\Omega X dP = \int_\Omega X(\omega) P(d\omega)$$

We also define $E(X)(\leq \infty)$ for $X \in (m\mathcal{F})^+$. In short, $E(X) = P(X)$.

Convergence theorems

Suppose that (X_n) is a sequence of RVs, that X is a RV, and that $X_n \to X$ almost surely:

$$P(X_n \to X) = 1$$

The notation $E(X; F)$

For $X \in \mathcal{L}^1$ or $(m\mathcal{F})^+$ and $F \in \mathcal{F}$, we define

$$E(X; F) := \int_F X(\omega) P(d\omega) := E(X I_F)$$

where,

$$\mathrm{I}_F(\omega) := \begin{cases} 1 & \text{if } \omega \in F \\ 0 & \text{if } \omega \notin F \end{cases}$$

Jensen's inequality for convex functions

A function $c : G \to \mathbf{R}$, where G is an open subinterval of $\mathbf{R}$, is called **convex** on G if its graph lies below any of its chords: for $x, y \in G$ and $0 \leq p = 1 - q \leq 1$,

$$c(px + qy) \leq pc(x) + qc(y)$$

Jensen's inequality

Suppose that $c : G \to \mathbf{R}$ is a convex function on an open subinterval G of $\mathbf{R}$ and that X is a random variable such that

$$E(|X|) < \infty, \quad P(X \in G) = 1, \quad E|c(X)| < \infty$$

Then

$$Ec(X) \geq c(E(X))$$

Monotonicity of $\mathcal{L}^p$ norms

For $1\ leqp < \infty$, we say that $X \in \mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, P)$ if

$$E(|X|^p) < \infty$$

define

$$\|X\|_p := \{\mathbf{E}\left(|X|^p\right)\}^{\frac{1}{p}}$$

Monotonicity

If $1 \leq p \leq r < \infty$ and $Y \in \mathcal{L}^r$, then $Y \in \mathcal{L}^p$ and

$$\|Y\|_p \leq \|Y\|_r$$

Schwarz inequality

If X and Y are in $\mathcal{L}^2$, then $XY \in \mathcal{L}^1$, and

$$|E(XY)| \leq E(|XY|) \leq \|X\|_2 \|Y\|_2$$

Hölder inequality

Suppose that $f, g \in \mathcal{L}^p(S, \Sigma, \mu),\ h \in \mathcal{L}^q(S, \Sigma, \mu)$. Then

Hölder's inequality

$fh \in \mathcal{L}^1(S, \Sigma, \mu)$ and

$$|\mu(fh)| \leq \mu(|fh|) \leq \|f\|_p \|h\|_q$$

Minkowski's inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Completeness of $\mathcal{L}^p (1 \leq p < \infty)$

Let $p \in [1, \infty)$.

If (X_n) is a Cauchy sequence in $\mathcal{L}^p$ in that

$$\sup_{r,s\geq k} \|X_r - X_s\|_p \to 0 \quad (k \to \infty)$$

then there exists X in $\mathcal{L}^p$ such that $X_r \to X$ in $\mathcal{L}^p$:

$$\|X_r - X\| \to 0 \quad (r \to \infty)$$

Orthogonal projection

Theorem

Let $\mathcal{K}$ be a vector subspace of $\mathcal{L}^2$ which is complete in that whenever (V_n) is a sequence in $\mathcal{K}$ which has the Cauchy property that

$$\sup_{r,s\geq k} \|V_r - V_s\| \to 0 \quad (k \to \infty)$$

then there exists a V in $\mathcal{K}$ such that

$$\|V_n - V\| \to 0 \quad (n \to \infty)$$

Then given X in $\mathcal{L}^2$, there exists Y in $\mathcal{K}$ such that

(i) $\|X - Y\| = \Delta := \inf\{\|X - W\| : W \in \mathcal{K}\}$

(ii) $X - Y \perp Z, \quad \forall Z \in \mathcal{K}$

Properties (i) and (ii) of Y in $\mathcal{K}$ are equivalent and if $\tilde{Y}$ shares either property (i) or (ii) with Y, then

$$\|\tilde{Y} - Y\| = 0 \quad (\text{equivalently,} \quad Y = \tilde{Y}, \text{ a.s. })$$

Definition

The random variable Y in the theorem is called a version of the *orthogonal projection* of X onto $\mathcal{K}$. If $\tilde{Y}$ is another version, then $\tilde{Y} = Y$, a.s.

Covariance

Civariance and variance

$$\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mu_X \mu_Y$$

$$\mathrm{Var}(X) := E[(X - \mu_X)^2] = E(X^2) - \mu_X^2 = \mathrm{Cov}(X, X)$$

Inner product, angle

Inner (or scalar) product

$$\langle U, V \rangle := \mathrm{E}(UV)$$

And if $\|U\|_2$ and $\|V\|_2 \neq 0$, we define the cosine of the angle θ between U and V by

$$\cos\theta = \frac{\langle U, V \rangle}{\|U\|_2 \|V\|_2}$$

the correlation ρ of X and Y is $\cos\alpha$ where α is the angle between $\tilde{X}$ and $\tilde{Y}$.