Symmetry Conditional Expectation

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Problem 1 (Exercise 9.2 of *Chapter E*)
Suppose that $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ *and that*

$$E(X|Y) = Y, a.s., \quad E(Y|X) = X, a.s.$$

Prove that $P(X = Y) = 1$.
Hint, consider $E(X - Y; X > c, Y \leqslant c) + E(X - Y; X \leqslant c, Y \leqslant c)$

Proof: Because of $E(X|Y) = Y$, we have $E(X; Y \leqslant c) = E(Y; Y \leqslant c)$ for every c by the definition of conditional expectation. Because $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, so

$$\begin{aligned} 0 &= E(X - Y; Y \leqslant c) \\ &= E(X - Y; X > c, Y \leqslant c) + E(X - Y; X \leqslant c, Y \leqslant c) \end{aligned}$$

Consider $P(X < Y)$, note that event $\{X < Y\} = \bigcup_c \{X < c < Y\}$, and

$$\{X < c - \frac{1}{n} < c + \frac{1}{n} < Y\} \uparrow \{X < c < Y\}$$

If $P(X < Y) > 0$, $\exists c, n$, s.t. $P(X < c - \frac{1}{n} < c + \frac{1}{n} < Y) = p > 0$, hence,

$$E(X - Y; X > c, Y \leqslant c) \geqslant \frac{2}{n} P(X < c - \frac{1}{n} < c + \frac{1}{n} < Y) = \frac{2p}{n} > 0$$

And the $E(X-Y; X \leqslant c, Y \leqslant c)$ must be zero because the symmetric in X, Y, we get $E(X-Y; Y \leqslant c) > 0$, it is impossible. Therefore $P(X < Y) = 0$.
We can prove $P(X > Y) = 0$ from $E(Y|X) = X$ use the same way. Hence $P(X = Y) = 1$.

$\square$