Liouville Number and π

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1 Liouville Number

Exercise 1) *Show that* $\sum_{j \in \mathbb{N}} \frac{1}{2^{j!}}$ *is a Liouville number.*

2) *Show that every Liouville number must be transcendental.*

Solution:

1) I present here a constructive proof.
Since $\sum_{j \in \mathbb{N}} \frac{1}{2^{j!}}$ is a sum of infinite series, we can choose, for every n, $\frac{p_n}{q_n}$ to be the prefix sum of $\frac{1}{2^{j!}}$.

With this intuition, we let

$$\frac{p_n}{q_n} = \sum_{j=1}^{n} \frac{1}{2^{j!}}$$

Then,

$$\begin{aligned} |\sum_{j \in \mathbb{N}} \frac{1}{2^{j!}} - \frac{p_n}{q_n}| &= \sum_{j \in \mathbb{N}} \frac{1}{2^{j!}} - \sum_{j=1}^{n} \frac{1}{2^{j!}} \\ &= \sum_{j=n+1}^{\infty} \frac{1}{2^{j!}} \\ &< \sum_{j=(n+1)!}^{\infty} \frac{1}{2^{j}} = \frac{1}{2^{(n+1)!-1}} \\ &< \frac{1}{(2^{n!})^n} \quad (n! \cdot n < n! \cdot (n+1) < (n+1)!) \end{aligned}$$

Thus, we choose q_n to be $2^{n!}$, and p_n to be $q_n * \sum_{j=1}^{n} \frac{1}{2^{j!}} = 2^{n!} * \sum_{j=1}^{n} \frac{1}{2^{j!}}$

2) We prove by contradiction.

Assume that there is a Liouville number z that is not transcendental, then it must be a algebraic number of degree n. We know from the *lecture notes* that, there exists a positive integer M such that for all integers p and q,

$$|z - \frac{p}{q}| > \frac{1}{M \cdot q^n} \tag{1}$$

It also holds for Liouville number z that for every integer n there exists integers p and q such that

$$|z - \frac{p}{q}| < \frac{1}{q^n} \tag{2}$$

Notice that *(1)* and *(2)* are inequalities of opposite signs, in order to induce a contradiction, it must be the case that for some n'(not to be confused with the degree n),

$$\frac{1}{q^{n'}} < \frac{1}{M \cdot q^n} \iff M \cdot q^n < q^{n'}$$

This could easily be achieved by setting

$$n' = \lceil n + \log_q M \rceil$$

Thus a contradiction!

Therefore, every Liouville number must be transcendental.

$\square$

2 Irrationality of π

We define the **irrationality measure** of a number as the smallest μ such that

$$|x - \frac{p}{q}| > \frac{1}{q^{\mu+\epsilon}}$$

holds for any ϵ and all integer p, q sufficiently large.

For any Liouville number L, $\mu(L) = \infty$.

Now we are interested in the value of $\mu(\pi)$. Unfortunately, we cannot say for sure, best proven bounds are as follows:

1. In 2008, Salikhov proved that $\mu(\pi) < 7.606308$ [1]. To the best of my ability, I can only find the russian version.

2. In 2019, Zeilberger and Zudlin inproved the previous bound to $7.10320533413\cdots$ [2]

参考文献

[1] Salikhov, V. Kh. "On the Irrationality Measure of pi." http://www.mathnet.ru/links/2590a72a2a7cbffb697b42a0056ed58a/rm9175.pdf

[2] Zeilberger, D. and Zudlin, W. "The Irrationality Measure of pi is at Most 7.103205334137...." 13 Dec 2019. https://arxiv.org/pdf/1912.06345.pdf