Proof of Radon-Nikodym theorem

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Theorem . *If μ and λ are σ-finite measures on (S, Σ) such that for all $F \in \Sigma$ with $\mu(F) = 0$, $\lambda(F) = 0$, then $\lambda = f\mu$ for some $f \in (m\Sigma)^+$.*

Proof. Let's first consider a simplified version of Radon–Nikodym theorem: Let μ and λ be finite measures on (S, Σ) such that for all $F \in \Sigma$ with $\mu(F) = 0$, $\lambda(F) = 0$, then there exists a μ-nullset D and $f \in (m\Sigma)^+$ such that $\lambda(A) = \lambda(A \cap D) + f\mu(A)$ for all $A \in \Sigma$.

Let $H = \{h \in (m\Sigma)^+ : h\mu(A) \leq \lambda(A) \text{for all } A \in \Sigma\}$. Then $H \neq \emptyset$, because it contains at least the zero function. Now suppose $h_1, h_2 \in H$. Let $A_1 = \{x \in A : h_1(x) \leq h_2(x)\}$, $A_2 = \{x \in A : h_1(x) > h_2(x)\}$. Then

$$\begin{aligned}
\max\{h_1, h_2\}\mu(A) &= \int_S \max\{h_1, h_2\} 1_A d\mu \\
&= \int_S \max\{h_1, h_2\} 1_{A_1} d\mu + \int_S \max\{h_1, h_2\} 1_{A_2} d\mu \\
&= \int_S h_2 1_{A_1} d\mu + \int_S h_1 1_{A_2} d\mu \\
&= h_2\mu(A_1) + h_1\mu(A_2) \\
&< \lambda(A_1) + \lambda(A_2) \\
&= \lambda(A)
\end{aligned}$$

So $\max h_1, h_2 \in H$.

Now let h_n be a sequence such that $\lim\limits_{n\to\infty} h_n\mu(A) = \sup\limits_{h\in H} h\mu(A)$. Since H is closed by taking the maximum, we can simply replace h_n by $\max\limits_{i=1}^{n} h_i$, and this makes h_n an increasing sequnece. Assume $h_n \uparrow f$, then we have $h_n\mu(A) \uparrow f\mu(A)$ by MON. So $f\mu(A) = \sup\limits_{h\in H} h\mu(A) \leq \lambda(A)$.

Let $\tau(A) = \lambda(A) - f\mu(A)$. It's obviously a non-negative function. Define $\mathcal{E}_n = \{E \in \Sigma : \mu(E) > n\tau(E)\}$.

Claim 1. There exists a countable disjoiont subfamily $\mathcal{F}_n$ of $\mathcal{E}_n$ such that $S \setminus \bigcup\limits_{F\in\mathcal{F}_n} F$ does not contain any member of $\mathcal{E}_n$.

Proof. By Zorn's lemma, we know there's a maximal element in disjoint subfamilies of $\mathcal{E}_n$. Let $\mathcal{F}_n$ be such a subfamily. If $S \setminus \bigcup\limits_{F\in\mathcal{F}_n} F$ contains $E \in \mathcal{E}_n$, then we can enlarge $\mathcal{F}_n$ to be $\mathcal{F}_n \cup \{E\}$, because E is disjoint with all members in $\mathcal{F}_n$. This violates that $\mathcal{F}_n$ is the maximal element. So $S \setminus \bigcup\limits_{F\in\mathcal{F}_n} F$ does not contain any member of $\mathcal{E}_n$.

Since $\mu(S) < \infty$ and $\bigcup\limits_{F \in \mathcal{F}_n} F \subseteq S$, we can know that $\forall k \in \mathbb{N}, B_k := \{F \in \mathcal{F}_n : \mu(F) > \frac{1}{k}\}$ has finite elements. $\mathcal{F}_n = \{F \in \mathcal{F}_n : \mu(F) > 0\} = \bigcup_{k=1}^{\infty} B_k$, which means that $\mathcal{F}_n$ is at most countable.

Claim 2. Let $D = \bigcup\limits_{n=1}^{\infty} (S \setminus E_n)$, where $E_n = \bigcup\limits_{F \in \mathcal{F}_n} F$. Then $\tau(S \setminus D) = 0$.

Proof. Using the σ-additivity, we know that $E_n \in \mathcal{E}_n$.

$$\begin{aligned}
\tau(S \setminus D) &\leq \tau(E_n) && (S \setminus D \subseteq E_n) \\
&< \frac{\mu(E_n)}{n} && (E_n \in \mathcal{E}_n) \\
&\leq \frac{\mu(S)}{n} \to 0 && (\mu(S) < \infty)
\end{aligned}$$

Claim 3. $\mu(D) = 0$

Proof. It suffices to show that $\mu(S \setminus E_n) = 0$. Let $B = S \setminus E_n$. If $\mu(B) > 0$, then $(f + \frac{1_B}{n})\mu(S) = f\mu(S) + \frac{\mu(B)}{n} > f\mu(S) = \sup\limits_{h \in H} h\mu(S)$. So $f + \frac{1_B}{n} \notin H$. This means that $\exists A \in \Sigma, (f + \frac{1_B}{n})\mu(A) = f\mu(A) + \frac{\mu(A \cap B)}{n} > \lambda(A)$. Now we have $\mu(A \cap B) = 1_B\mu(A) > n(\lambda(A) - f\mu(A)) = n\tau(A) \geq n\tau(A \cap B)$. So $A \cap B \in \mathcal{E}_n$, which violates claim 1. This finishes the proof of claim 3.

Finally, we gather the 3 claims together,

$$\begin{aligned}
\tau(A) &= \tau(A \cap D) + \tau(A \setminus D) \\
&= \tau(A \cap D) && \text{(by claim 2)} \\
&= \lambda(A \cap D) - f\mu(A \cap D) \\
&= \lambda(A \cap D) && (f1_{A \cap D} = 0 \text{ a.e. by claim 3})
\end{aligned}$$

Now we apply this simplified version: Since $\mu(D) = 0$, then $\lambda(D) = 0$, $\lambda(A \cap D) = 0$. So $\lambda(A) = \lambda(A \cap D) + f\mu(A) = f\mu(A)$.

Since now we have proved that if μ and λ are finite measures on (S, Σ) such that for all $F \in \Sigma$ with $\mu(F) = 0$, $\lambda(F) = 0$, then $\lambda = f\mu$ for some $f \in (m\Sigma)^+$. Then we extend it to σ-finite measures.

Suppose μ, λ is σ-finite. Then there exists a sequence $(A_n) \in \Sigma$ such that $\mu(A_n) < \infty$ and $\bigcup A_n = S$ and a sequence $(B_n) \in \Sigma$ such that $\lambda(B_n) < \infty$ and $\bigcup B_n = S$.

$$A_i = A_i \cap S = A_i \cap \bigcup_j B_j = \bigcup_j A_i \cap B_j$$
$$S = \bigcup_i A_i = \bigcup_{i,j} A_i \cap B_j$$

Let $S_{i,j} = A_i \cap B_j$. Then $S_{i,j}$ is a collection of disjoint sets whose union is S and have finite measures under both μ and λ. For each $S_{i,j}$, there is a $f_{i,j} \in (m\Sigma)^+$ such that $\lambda(A) = f_{i,j}\mu(A)$

for all $A \subseteq S_{i,j}$. Let $f(x) = f_{i,j}(x)$ if $x \in S_{i,j}$.

$$\begin{aligned}
\lambda(A) &= \sum_{i,j} \lambda(A \cap S_{i,j}) \\
&= \sum_{i,j} f_{i,j}\mu(A \cap S_{i,j}) \\
&= \sum_{i,j} f\mu(A \cap S_{i,j}) \\
&= f\mu(A)
\end{aligned}$$

This finishes the proof.

□