Σ-measurable ramp function

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Exercise 9. *Let* (S, Σ) *be a measurable space and take* $h \in \mathbb{R}^S$*.Let* $h^+ = max(h, 0)$ *and* $h^- = max(-h, 0)$*. Show that* $h \in m\Sigma$ *if and only if* $h^+, h^- \in m\Sigma$*.*

Solution. Observe that

$$h^+ = \begin{cases} 0 & h < 0 \\ h & h \geq 0 \end{cases}$$

$$h^- = \begin{cases} -h & h < 0 \\ 0 & h \geq 0 \end{cases}$$

So we have

$$h = h^+ - h^-$$

Since $m\Sigma$ is closed under taking sum and scalar multiplication, if $h^+, h^- \in m\Sigma$, $h \in m\Sigma$.

Then we'll focus on another side. Assume $h \in m\Sigma$. Consider

$$\{h^+ \leq c\} = \begin{cases} \emptyset & c < 0 \\ \{h \leq c\} & c \geq 0 \end{cases}$$

By the definition of σ-algebra, $\emptyset \in \Sigma$. $\{h \leq c\} = h^{-1}(-\infty, c] \in \Sigma$. So $\{h^+ \leq c\} \in \Sigma$ $(\forall c \in \mathbb{R})$. We can derive that $h^+ \in m\Sigma$.

$h^- \in m\Sigma$ can be derived similarly.

In conclusion, $h \in m\Sigma$ if and only if $h^+, h^- \in m\Sigma$. $\square$