Proof of $|C| = |R| = |[0, 1]|$

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Question 1. ***Let C be the Cantor set. Show that $|C| = |R| = |[0, 1]|$.***

Definition 2. ***Cantor Set.***

Let $C_0 = [0, 1]$. For each positive integer n, let C_n be obtained from C_{n-1} by dividing each interval of C_{n-1} into three intervals of equal length and then removing the middle open interval from each of the intervals from C_{n-1}. The Cantor set is defined to be $\bigcap_{n\geq 0} C_n$.

Fact 3. ***The Cantor set can be represented as***

$$C = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} (\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}})$$

Fact 4. ***The numbers in C have only 0s and 2s in their ternary(base 3) representation.***

Considering some intervals in the form of $(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}})$ are removed from $[0, 1]$, Fact 4 maybe obvious.But there're some special numbers which don't seem to satisfy the fact, such as $\frac{1}{3} = 0.1_3$ and $\frac{7}{9} = 0.21_3$.These numbers are the left endpoint of intervals $(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}})$.However, $\frac{1}{3}$ can be written as $0.0222222\cdots_3$.Similarly, $\frac{7}{9}$ can be written as $0.020222222\cdots_3$ and all the special numbers can be represented in this way.

Lemma 5. ***There's a surjective mapping from C to $[0, 1]$.***

The mapping can be defined by taking the ternary numbers that consist of 0s and 2s, replacing all the 2s by 1s, and interpreting the sequence as a binary representation of a real number in $[0, 1]$. In a formula,

$$f(\sum_{k\in\mathbb{N}^+} a_k 3^{-k}) = \sum_{k\in\mathbb{N}^+} \frac{a_k}{2} 2^{-k} (a_k \in \{0, 2\})$$

For example, $f(\frac{2}{9}) = f(0.02_3) = 0.01_2 = \frac{1}{4}$.

As the set $\{\sum_{k\in\mathbb{N}^+} \frac{a_k}{2} 2^{-k}\}$ is actually $[0, 1]$,f is surjective.

Lemma 6. $|C| = |[0,1]|$

Identity mapping is an injective mapping from C to $[0,1]$, so we have $|C| \leq |[0,1]|$.And according to Lemma 5, $|C| \geq |[0,1]|$.Therefore, we can conclude that $|C| = |[0,1]|$.

Lemma 7. $|(0,1)| = |\mathbb{R}|$

We define a mapping g from $(0,1)$ to $\mathbb{R}$:

$$g(x) = tan(\pi x - \frac{\pi}{2})$$

Obviously, g is a bijection, which implies that $|(0,1)| = |\mathbb{R}|$.The conclusion maybe beautiful, but we expect to get $|[0,1]| = |\mathbb{R}|$. So I looked up some papers and found an amazing proposition $|(0,1)| = |[0,1)|$.

Proposition 8. $|(0,1)| = |[0,1)|$

Let $b_n = \frac{1}{n+1}$ for $n \in \mathbb{N}^+$ and $B = \{b_n | n \in \mathbb{N}^+\}$.We define a mapping h from B to $B \bigcup \{0\}$:

$$h(x) = \begin{cases} 0 & x = b_1 \\ b_{n-1} & x = b_n (n \geq 2) \end{cases} \tag{1}$$

Assume $h(b_i) = h(b_j) = y$.If $y = 0$, then $b_i = b_j = b_1$.Otherwise $y = b_k$, then $b_i = b_j = b_{k+1}$.Hence h is injective.For any b_n,$h(b_{n+1}) = b_n$ and $h(b_1) = 0$, so h is surjective.Therefore,h is bijective.Next we can define identify mapping on $(0,1) - B$(clearly a bijection).Therefore, we can conclude that $|(0,1)| = |[0,1)|$.

Similarly, we can prove that $|[0,1)| = |[0,1]|$. Therefore, we can conclude that

$$R = |(0,1)| = |[0,1)| = |[0,1]|$$

Conclusion 9. $|C| = |R| = |[0,1]|$

We have proved that $|C| = |R|$ and $|R| = |[0,1]|$, so we can get the conclusion.

Reference

https://www.math.ubc.ca/ gor/Math220_2016/cardinality_workshop.pdf