

Hölder's inequality and Rogers' inequality

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Before starting, the base of these inequalities is the famous inequality between arithmetic means and geometric means:

┌ **1: AM-GM inequality**

Let x_i be positive for all $i \in [n]$, then

$$\frac{\sum x_i}{n} \geq (\prod x_i)^{\frac{1}{n}}.$$

And the well-known version of Hölder's inequality:

┌ **2: The well-known Hölder's inequality**

Let a_i, b_i be positive for all $i \in [n]$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum a_i b_i \leq (\sum a_i^p)^{\frac{1}{p}} (\sum b_i^q)^{\frac{1}{q}}.$$

1 Rogers and his inequality



Leonard James Rogers (30 March 1862, Oxford, England, –12 September 1933, Oxford, England) was a British mathematician who was the first to discover the Rogers–Ramanujan identity and Hölder's inequality, and who introduced Rogers polynomials. The Rogers–Szegő polynomials are named after him.

Rogers published his result in "*An extension of a certain theorem in inequalities*" in 1888 starting with the extended AM-GM inequality:

3: Extended AM-GM inequality

Let a_i, b_i be positive for all $i \in [n]$, then

$$\left(\frac{\sum a_i b_i}{\sum a_i}\right)^{\sum a_i} \geq \prod b_i^{a_i}.$$

Here is the proof in his paper:

Proof.

- Firstly, let a_i be integers, then it can be considered as 1: AM-GM inequality with each b_i occurring a_i times.
- Then, if a_i are rational numbers, we can also multiply with the least common multiple of denominators of a_i . Then it becomes the first situation.
- Finally, as for real numbers, we can always use fractions to approximate them with arbitrary precision. Thus we can substitute real numbers with fractions and then apply the second case. (*Can this really work? I kind of doubt it.*) Here is the original text:

Then we may substitute for each of these quantities fractions, which may differ from them by less than any assigned quantities, and since the theorem is true for the substituted fractions, we may assume it also true for the given incommensurables.

□

Then, with the help with this inequality, Rogers proposed:

4: Rogers' inequality

If $0 < t < r < m$ and a_i, b_i are positive for all $i \in [n]$, then

$$(\sum a_i b_i^m)^{r-t} (\sum a_i b_i^r)^{m-r} \geq (\sum a_i b_i^r)^{m-t}.$$

Proof. Let $S_k = \sum a_i b_i^k$, $0 < t < r < m$, and substitute a_i in 3: Extended AM-GM inequality with $a_i b_i^r$, b_i with b_i^{m-r} :

$$\left(\frac{S_m}{S_r}\right)^{S_r} \geq \prod (b_i^{m-r})^{a_i b_i^r} = \left(\prod b_i^{a_i b_i^r}\right)^{m-r}.$$

Then we substitute a_i with $a_i b_i^r$, while b_i becomes b_i^{t-r} :

$$\left(\frac{S_t}{S_r}\right)^{S_r} \geq \left(\prod b_i^{a_i b_i^r}\right)^{t-r} \iff \left(\frac{S_r}{S_t}\right)^{S_r} \leq \left(\prod b_i^{a_i b_i^r}\right)^{r-t}.$$

Combine the two parts, we can get

$$\left(\frac{S_m}{S_r}\right)^{\frac{S_r}{m-r}} \geq \prod b_i^{a_i b_i^r} \geq \left(\frac{S_r}{S_t}\right)^{\frac{S_r}{r-t}}.$$

And it can be simplified to

$$S_m^{r-t} \cdot S_t^{m-r} \geq S_r^{m-t}.$$

□

2 Hölder and his inequality



Otto Ludwig Hölder (December 22, 1859 – August 29, 1937) was a German mathematician born in Stuttgart.

Hölder first studied at the Polytechnikum (which today is the University of Stuttgart) and then in 1877 went to Berlin where he was a student of Leopold Kronecker, Karl Weierstrass, and Ernst Kummer.

He is noted for many theorems including: Hölder's inequality, the Jordan–Hölder theorem, the theorem stating that every linearly ordered group that satisfies an Archimedean property is isomorphic to a subgroup of the additive group of real numbers, the classification of simple groups of order up to 200, the anomalous outer automorphisms of the symmetric group S_6 , and Hölder's theorem, which implies that the Gamma function satisfies no algebraic differential equation. Another idea related to his name is the Hölder condition (or Hölder continuity) which is used in many areas of analysis, including the theories of partial differential equations and function spaces.

Hölder introduced the inequality in his paper "*Ueber einen Mittelwerthssatz*" in 1889:

Bedeutet nämlich $\phi(x)$ eine Function einer reellen Veränderlichen mit zunehmendem Differentialquotienten, so ist das arithmetische Mittel aus einer beliebigen Zahl von Functionswerthen stets größer als der Functionswerth, welcher dem in derselben Weise gebildeten Mittelwerth der zugehörigen Argumente entspricht. Dabei ist der Begriff des arithmetischen Mittels gleich in der allgemeinen Weise zu nehmen, daß jedem der Werthe, aus welchen dasselbe zu bilden ist, eine beliebige positive GröÙe als Gewicht zugeordnet wird, so daß also die Formel

$$\frac{a_1\phi(x_1) + a_2\phi(x_2) + \cdots + a_n\phi(x_n)}{a_1 + a_2 + \cdots + a_n} > \phi\left(\frac{a_1x_1 + a_2x_2 + \cdots + a_nx_n}{a_1 + a_2 + \cdots + a_n}\right).$$

den Ausdruck des genannten Satzes darstellt.

I do not know much about German, but with the help of Google translation, the main idea may be:

┌ **5: Inequality**

Let ϕ be a convex function and $a_i > 0, x_i > 0$ for all $i \in [n]$, then there holds the inequality:

$$\frac{\sum a_i \phi(x_i)}{\sum a_i} \geq \phi\left(\frac{\sum a_i x_i}{\sum a_i}\right).$$

The original proof is omitted because it was totally written in German. I would like to consider it as an extended version of Jensen's inequality. (But note that Jensen's inequality was proposed in 1906, later than this paper.)

In section 7, 8, 9, 10, Hölder gave some applications with 5: Inequality.

Let $\phi(x) = x^m (m > 1)$, we have $\phi''(x) = m(m-1)x^{m-2} > 0$. Thus

┌ **6: Inequality**

Let $a_i > 0, x_i > 0$ for all $i \in [n]$ and $m > 1$, then:

$$(\sum a_i)^{m-1} \cdot (\sum a_i x_i^m) \geq (\sum a_i x_i)^m.$$

This is actually 4: Rogers' inequality when $r = 1, t = 0$.

We can also let $\phi(x) = e^x, y_i = e^{x_i}$, resulting in

$$\left(\frac{\sum a_i y_i}{\sum a_i}\right)^{\sum a_i} \geq \prod y_i^{a_i}.$$

This is exactly the result('Ausgangspunkt', starting point) in Rogers' paper. Hölder even mentioned this result of Rogers' here!

Dies ist der Satz vom arithmetischen und geometrischen Mittel in seiner allgemeinsten Gestalt, wie ihn Herr Rogers zum Ausgangspunkt wählt, 1(1).

3 Relationship between the inequalities

The inequalities above all have similar forms, so I think they might be equivalent.

┌ **7: Theorem**

2: The well-known Hölder's inequality \iff 6: Inequality.

Proof.

- \Leftarrow : Note that the second term in 6: Inequality have a_i as well as x_i , so we can try to let $a_i x_i = a_i^s \cdot (a_i^{1-s} x_i)$ in order to eliminate a_i .

$$\begin{aligned} \sum a_i x_i &= \sum a_i^s (a_i^{1-s} x_i) \\ (\text{by 6: Inequality}) &\leq (\sum a_i^s)^{\frac{m-1}{m}} (\sum a_i^s \cdot a_i^{m(1-s)} x_i^m)^{\frac{1}{m}}. \end{aligned}$$

Solve $s + m(1-s) = 0$ and we have $s = \frac{m}{m-1}$. Thus

$$\sum a_i x_i \leq (\sum a_i^{\frac{m}{m-1}})^{\frac{m-1}{m}} (\sum x_i^m)^{\frac{1}{m}}.$$

In this case, $p = \frac{m}{m-1}$, $q = m$ and every pair (p, q) can be reached with $m = q > 1$.

- \Rightarrow : Similarly,

$$\begin{aligned} (\sum a_i x_i)^m &= (\sum a_i^s a_i^{1-s} x_i)^m \\ (\text{by 2: The well-known Hölder's inequality}) &\leq (\sum a_i^{ps})^{\frac{m}{p}} (\sum (a_i^{1-s} x_i)^{\frac{p}{p-1}})^{\frac{m(p-1)}{p}}. \end{aligned}$$

Let $s = \frac{1}{p}$, $p = \frac{m}{m-1}$, then

$$(\sum a_i x_i)^m \leq (\sum a_i)^{m-1} (\sum a_i x_i^m).$$

Every $m > 1$ can be reached by letting $p = \frac{m}{m-1} > 1$.

□

4: Rogers' inequality and 6: Inequality seems similar, but more complicated. I think 4: Rogers' inequality and 2: The well-known Hölder's inequality might be also equivalent, and that's why we can say Rogers and Hölder actually proposed the same inequality.

4 Stigler's law of eponymy

Like Hölder's inequality, it is not rare for a theorem not to be named after its first discoverer. Actually, Stephen Stigler in his 1980 publication proposed *Stigler's law of eponymy*¹, which states that *no scientific discovery is named after its original discoverer*. As Mark Twain said,

It takes a thousand men to invent a telegraph, or a steam engine, or a phonograph, or a photograph, or a telephone or any other important thing —and the last man gets the credit and we forget the others. He added his little mite —that is all he did. These object lessons should teach us that ninety-nine parts of all things that proceed from the intellect are plagiarisms, pure and simple; and the lesson ought to make us modest. But nothing can do that.

¹from Greek *epōnumos* "given as a name, giving one's name to something," as a plural noun (short for *eponymoi* heroes) denoting founders (legendary or real) of tribes, cities, etc.; from combining form of *epi* "upon, (called) after," + *onyma*, Aeolic dialectal variant of *onoma* "name".

Famous theorems including *Chebyshev's inequality*, *Burnside's lemma*, *Cayley–Hamilton theorem*, *L'Hôpital's rule*, *Zorn's lemma* are also misnamed, see https://en.wikipedia.org/wiki/List_of_misnamed_theorems and https://en.wikipedia.org/wiki/List_of_examples_of_Stigler%27s_law.

5 Conclusion

Rogers introduced the inequality in 1888. Hölder, knowing the work of Rogers', introduced another (strengthened?) version of it in 1889. This result is now known as Hölder's inequality.

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