Truth set of the Strong Law for the subsequence α

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Problem 1. Toss a fair coin infinitely often. The outcome of each toss is either head (H) or tail (T) with equal probability. Take $\Omega = \{H, T\}^{\mathbb{N}}$. which is the set of all possible outcomes of our infinitely many tosses. A typical point $\omega \in \Omega$ is a sequence

$$\omega = (\omega_1, \omega_2, \dots), \omega_n \in \{H, T\}.$$

For any increasing sequence (α_n) of positive integers, let

$$F_\alpha = \{\omega : \frac{\#(k \le n : \omega_{\alpha(k)} = H)}{n} \to \frac{1}{2}\}$$

Show that $\bigcap_\alpha F_\alpha = \emptyset$.

Solution.

We can think the α is a subsequence, and the F_α is the truth set of the Strong Law for the subsequence α. If $\bigcap_\alpha F_\alpha \neq \emptyset$, it means that exists a ω, its any subsequence satisfy SLLN. But in fact for any ω, we can construct a α, make it unsatisfactory.

For any

$$\omega = (\omega_1, \omega_2, \dots), \omega_n \in \{H, T\}.$$

Define $A = \{k : \omega_k = H\}$, $B = \{k : \omega_k = T\}$. If A is infinite, we construct α from A. Because A is well-ordered set, so we can let

$$\alpha(n) = \begin{cases} \min(A) & n = 1 \\ \min(A \cap \bigcup_{k<n} \{\alpha(k)\}) & n > 1 \end{cases}$$

Obviously, *alpha* is a increasing sequence, and $\omega_{\alpha(n)} = H$ for all $n > 0$. So

$$\frac{\#(k \leq n : \omega_{\alpha(k)} = H)}{n} \to 1$$

and if A is finite, we must have B is infinite. Construct the function with the same way. We can get

$$\frac{\#(k \leq n : \omega_{\alpha(k)} = H)}{n} \to 0$$

So $\forall \omega, \exists \alpha, s.t. \omega \notin F_\alpha$. If $\bigcap_\alpha F_\alpha \neq \emptyset$, it means $\exists \omega, \forall \alpha \omega \in F_\alpha$. It is contradictory. Therefore

$$\bigcap_\alpha F_\alpha = \emptyset$$

$\square$