Independence of $\pi-$system

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In the text book $E4.1$, the author put forward a theorem as follows:

1 E4.1 in textbook

Theorem 1. *$\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ are three π-system that satisfy:*
(1) $\mathcal{I}_k \subseteq \mathcal{F}(k=1,2,3)$;
(2) $\Omega \in \mathcal{I}_k(k=1,2,3)$.
If

$$\forall I_i \in \mathcal{I}_i, P(I_1 \cap I_2 \cap I_3) = P(I_1)P(I_2)P(I_3), \tag{1}$$

then $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), \sigma(\mathcal{I}_3)$ are independent.

Proof. Define $\mathcal{J}_i := \sigma(\mathcal{I}_i)$. Fix $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. Consider maps

$$J_3 \mapsto P(I_1 \cap I_2 \cap J_3) \text{ and } J_3 \mapsto P(I_1)P(I_2)P(J_3),$$

then two mapping agree on $\mathcal{I}_3$. Also when $I_3 = \Omega$ in equation (1), we can conclude that

$$P(I_1 \cap I_2) = P(I_1 \cap I_2 \cap \Omega) = P(I_1)P(I_2)P(\Omega) = P(I_1)P(I_2),$$

which means, two mapping have the same total mass.Thus we can conclude that $P(I_1 \cap I_2 \cap J_3) = P(I_1)P(I_2)P(J_3)$ holds in the space $(\Omega, \mathcal{J}_3)$.

Similarly, we can conclude the conclusion on both $\sigma(\mathcal{I}_1)$ and $\sigma(\mathcal{I}_2)$. Therefore, $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), \sigma(\mathcal{I}_3)$ are independent. □

Question WHY we need the condition "$\Omega \in \mathcal{I}_i$"?

Solution. In the lemma 1.6 in the textbook, one of the premises is that μ_1 and μ_2 has the same mass on S. Then this condition guarantees that each pair of mapping in our prove has the same mass. □

2 Further Discussion

Actually, we can strengthen this theorem:

Theorem 2. *$\mathcal{I}_i (i = 1, 2, 3 \ldots, n)$ are independent $\pi-$system, then $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), \ldots, \sigma(\mathcal{I}_n)$ are independent.*

Proof. We define $\mathcal{J}_i := \sigma(\mathcal{I}_i), \Omega \in \mathcal{I}_n (k = 1, 2, \ldots, n)$.

(1)For fixed $I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \ldots, I_{n-1} \in \mathcal{I}_{n-1}$, consider maps

$$J_n \mapsto P(I_1 \cap I_2 \cap \cdots \cap I_{n-1} \cap J_n) \text{ and } J_n \mapsto P(I_1)P(I_2)\ldots P(I_{n-1})P(J_n),$$

with the same mass $P(I_1)P(I_2)\ldots P(I_{n-1})$, and agree on $\mathcal{I}_n$. According to Lemma 1.6, they agree on $\sigma(\mathcal{I}_n)$, i.e.$\mathcal{J}_n$.

(2)For fixed $I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \ldots, I_{n-2} \in \mathcal{I}_{n-2}, J_n \in \mathcal{J}_n$, consider maps

$$J_{n-1} \mapsto P(I_1 \cap I_2 \cap \cdots \cap I_{n-2} \cap J_{n-1} \cap J_n) \text{ and } I_{n-2} \cap J_{n-1} \mapsto P(I_1)P(I_2)\ldots P(I_{n-2})P(J_{n-1})P(J_n),$$

with the same mass $P(I_1)P(I_2)\ldots P(I_{n-2})P(J_n)$, and agree on $\mathcal{I}_{n-1}$. According to Lemma 1.6, they agree on $\sigma(\mathcal{I}_{n-1})$, i.e.$\mathcal{J}_{n-1}$.

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(n)For fixed $J_2 \in \mathcal{J}_2, J_3 \in \mathcal{I}_3, \ldots, J_n \in \mathcal{J}_n$, consider maps

$$J_1 \mapsto P(J_2 \cap J_3 \cap \cdots \cap J_n) \text{ and } J_1 \mapsto P(J_2)P(J_3)\ldots P(J_n),$$

with the same mass $P(J_2)P(J_3)\ldots P(J_n)$, and agree on $\mathcal{I}_1$. According to Lemma 1.6, they agree on $\sigma(\mathcal{I}_1)$, i.e.$\mathcal{J}_1$.

Then we finish our proof.

$\square$