Independence of multiple π-systems and their conditions

Lu Jiaxin
Student ID: 518030910412

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Here is an lemma in *Probability with Martingales.*

This lemma shows us the independence of two π-systems and its approach of proof is also used below.

Lemma 1 (LEMMA 4.2(a))**.** *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are sub-σ-algebras of $\mathcal{F}$ and that $\mathcal{I}$ and $\mathcal{J}$ are π-systems with*

$$\sigma(\mathcal{I}) = \mathcal{G}, \quad \sigma(\mathcal{J}) = \mathcal{H}$$

Then $\mathcal{G}$ and $\mathcal{H}$ are independent if and only if $\mathcal{I}$ and $\mathcal{J}$ are independent in that

$$P(I \cap J) = P(I)P(J), \quad I \in \mathcal{I}, J \in \mathcal{J}.$$

Proof. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are independent. For fixed I in $\mathcal{I}$,

$$H \mapsto P(I \cap H) \text{ and } H \mapsto P(I)P(H)$$

are measures(since they are maps $\mathcal{F} \to [0, \infty]$ on $(\Omega, \mathcal{F})$ and by definition they are measures) on $(\Omega, \mathcal{H})$ have the same total mass $P(I)$, and agree on $\mathcal{J}$. By the uniqueness of extension(see in the same book **Lemma 1.6**), they therefore agree on $\sigma(\mathcal{J}) = \mathcal{H}$. Hence,

$$P(I \cap H) = P(I)P(H), \quad I \in \mathcal{I}, H \in \mathcal{H}.$$

Thus, for fixed H in $\mathcal{H}$, the measures

$$G \mapsto P(G \cap H) \text{ and } G \mapsto P(G)P(H)$$

on $(\Omega, \mathcal{G})$ have the same total mass $P(H)$, and agree on $\mathcal{I}$. They therefore agree on $\sigma(\mathcal{I}) = \mathcal{G}$.

Thus we finish our proof. $\square$

Now consider the case where there are three π-systems. This is the case in **Exercise 4.1**.

This exercise requires us to prove the independence of three π-systems. We can prove it with the approach above. But we'll have to face a problem that to complete the proof, we must add a condition.

I'll explain the reason in the remark below.

Exercise 2 (E4.1)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability triple. Let $\mathcal{I}_1, \mathcal{I}_2 and \mathcal{I}_3$ be three π-systems on Ω such that, for $k = 1, 2, 3$,*

$$\mathcal{I}_k \subseteq \mathcal{F}$$

and

$$\Omega \in \mathcal{I}_k$$

Prove that if

$$P(I_1 \cap I_2 \cap I_3) = P(I_1)P(I_2)P(I_3)$$

whenever $I_k \in \mathcal{I}_k$ $(k = 1, 2, 3)$ then $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), \sigma(\mathcal{I}_3)$ are independent.

Proof. Let $\sigma(\mathcal{I}_3) = \mathcal{J}_3$. Fix $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. Consider the maps

$$J_3 \mapsto P(I_1 \cap I_2 \cap J_3) \text{ and } J_3 \mapsto P(I_1)P(I_2)P(I_3),$$

for $J_3 \in \sigma(\mathcal{I}_3)$. We can verify that these are measures on the measure space $(\Omega, \sigma(\mathcal{I}_3))$.

Note that since $\Omega \in \mathcal{I}_3$, we derive

$$P(I_1 \cap I_2) = P(I_1 \cap I_2 \cap \Omega) = P(I_1)P(I_2)P(\Omega) = P(I_1)P(I_2), \tag{1}$$

so these measures both have a total mass of

$$P(I_1 \cap I_2) \text{ and } P(I_1)P(I_2),$$

which are exactly the same, and agree on $\mathcal{I}_3$.

By the uniqueness of extension(see in the same book **Lemma 1.6**), they therefore agree on $\sigma(\mathcal{I}_3) = \mathcal{J}_3$. Using this approach on $k = 1, 2$ we naturally have $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), \sigma(\mathcal{I}_3)$ are independent. $\square$

Remark 3. *Although the approachs used in both proofs look the exactly the same, the conditions of two statements are different.*

Exercise 2 requires the condition $\Omega \in \mathcal{I}_k$ while Lemma 1 doesn't. Here's my thought.

In the proof of Lemma 1, the sentence 'have the same total mass $P(I)$' comes naturally since when there are two π-systems,

$$P(I \cap \Omega) = P(I)P(\Omega)$$

holds vacuously. So including the hypothesis would be unnecessary.

While in the proof of Exercise 2, we have to use Eq. (1) *to prove that*

$$P(I_1 \cap I_2) = P(I_1)P(I_2)$$

under the condition $\Omega \in \mathcal{I}_k$. Because if $\Omega \notin \mathcal{I}_k$, we can't derive $P(I_1 \cap I_2) = P(I_1)P(I_2)$ from $P(I_1 \cap I_2 \cap I_3) = P(I_1)P(I_2)P(I_3)$.

This condition is also required if we want to extend this theorem to arbitary $n \in \mathbb{N}(n \geq 3)$ π-systems. Otherwise, we won't be able to apply the uniqueness of extension(see in the same book ***Lemma 1.6****) to verify two measures we create agree on $\sigma(\mathcal{I}_k)$.*