Independent Normal Distribution Variable Sequence

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Problem 1 (Exercise 4.5 of *Chapter E*)
Background:

A random variable G has a normal $N(0,1)$ distribution, then for $x > 0$,

$$P(G > x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}y^2} dy.$$

Note that this property only is all we need regarding random variable with normal distribution in this problem.

1. *Prove that*
$$P(G > x) \leqslant \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$
2. *Let $X_1, X_2, \ldots$ be a sequence of independent $N(0,1)$ variables. Prove that with probability 1, $L \leqslant 1$, where*
$$L := \limsup\left(\frac{X_n}{\sqrt{2 \log n}}\right).$$
3. *Prove that*
$$P(L = 1) = 1.$$

Proof:

1. This is proven by manipulating the integral.
$$\begin{aligned} P(G > x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}y^2} dy \\ &\leqslant \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot \int_x^\infty e^{-\frac{1}{2}y^2} \cdot y \cdot dy \\ &= \frac{1}{x\sqrt{2\pi}} \int_{\frac{1}{2}x^2}^\infty e^{-y} \cdot dy \\ &= \frac{1}{x\sqrt{2\pi}} (-e^{-y}) \Big|_{\frac{1}{2}x^2}^\infty \\ &= \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \end{aligned}$$

2. Let E_n denote the event that $\frac{X_n}{\sqrt{2\log n}} > \sqrt{1+\epsilon}$.

$$\begin{aligned}
\sum_{i\in\mathbb{N}} P(E_i) &= \sum_{i\in\mathbb{N}} P(\frac{X_n}{\sqrt{2\log n}} > \sqrt{1+\epsilon}) = \sum_{i\in\mathbb{N}} P(X_n > (\sqrt{1+\epsilon})\sqrt{2\log n}) \\
&= \sum_{i\in\mathbb{N}} \frac{1}{\sqrt{2\pi}} \int_{(\sqrt{1+\epsilon})\sqrt{2\log n}}^{\infty} e^{-\frac{1}{2}y^2} dy \\
&\leqslant \sum_{i\in\mathbb{N}} \frac{1}{(\sqrt{1+\epsilon})\sqrt{2\log n}\sqrt{2\pi}} e^{-\frac{1}{2}((\sqrt{1+\epsilon})\sqrt{2\log n})^2} \\
&= \sum_{i\in\mathbb{N}} \frac{1}{(\sqrt{1+\epsilon})\sqrt{2\log n}\sqrt{2\pi}} e^{-(1+\epsilon)\log n} \\
&< \sum_{i\in\mathbb{N}} \frac{1}{2\sqrt{(1+\epsilon)\pi}} \cdot \frac{1}{n^{1+\epsilon}}.
\end{aligned}$$

$\sum_{i\in\mathbb{N}} \frac{1}{n^{1+\epsilon}}$ converges, thus

$$\sum_{i\in\mathbb{N}} P(E_i) < \infty.$$

By **First Borel-Cantelli Lemma(BC1)**, we have

$$P(E_n, \text{i.o.}) = 0,$$

thus

$$P(E_n^c, ev) = P(\frac{X_n}{\sqrt{2\log n}} \leqslant \sqrt{1+\epsilon}, ev) = 1.$$

Finally,

$$\begin{aligned}
P(L \leqslant 1) &= P(\limsup(\frac{X_n}{\sqrt{2\log n}}) \leqslant 1) = P(\frac{X_n}{\sqrt{2\log n}} \leqslant 1, ev) \\
&= \lim_{\epsilon\to 0} P(\frac{X_n}{\sqrt{2\log n}} \leqslant \sqrt{1+\epsilon}, ev) \\
&= 1.
\end{aligned}$$

3. We have already proven that $P(L \leqslant 1) = 1$. Since $P(L \geqslant 1) = P(L \leqslant 1) = 1$ implies $P(L = 1) = 1$, we only need to prove that $P(L \geqslant 1) = 1$.

   We have

$$P(L \geqslant 1) = P(\limsup(\frac{X_n}{\sqrt{2\log n}}) \geqslant 1) = P(\frac{X_n}{\sqrt{2\log n}} \geqslant 1, \text{i.o.}).$$

   Thus we need to prove that $P(\frac{X_n}{\sqrt{2\log n}} \geqslant 1, \text{i.o.}) = 1$.

Let E_n denote the event that $\frac{X_n}{\sqrt{2\log n}} \geqslant 1$. It is easy to verify that (E_n) are independent events since (X_n) are independent random variables.

$$\sum_{i\in\mathbb{N}} P(E_i) = \sum_{i\in\mathbb{N}} P(\frac{X_n}{\sqrt{2\log n}} \geqslant 1) = \sum_{i\in\mathbb{N}} P(X_n \geqslant \sqrt{2\log n}).$$

Similarly to *(1),(2)*, we first show that

$$P(G \geqslant x) \geqslant \frac{1}{\sqrt{2\pi}} \cdot \frac{x}{x^2+1} \cdot e^{-\frac{1}{2}x^2}.$$

Thus,

$$\begin{aligned}
\sum_{i\in\mathbb{N}} P(E_i) &\geqslant \sum_{i\in\mathbb{N}} \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\log n}}{(\sqrt{2\log n})^2+1} \cdot e^{-\frac{1}{2}(\sqrt{2\log n})^2} \\
&= \sum_{i\in\mathbb{N}} \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\log n}}{2\log n+1} \cdot e^{-\log n} \\
&\geqslant \sum_{i\in\mathbb{N}} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{n \cdot \sqrt{2\log n}} = \infty.
\end{aligned}$$

Therefore, by **Second Borel-Cantelli Lemma(BC2)**, we have

$$P(\frac{X_n}{\sqrt{2\log n}} \geqslant 1, \text{i.o.}) = P(E_n, \text{i.o.}) = 1,$$

which finishes the proof.

$\square$