Generating Function And Its Applications

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Probability Theory, 2020 Spring
ACM Class, Shanghai Jiao Tong University

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To reconstruct a, just note that $a_n = \frac{G_a^{(n)}(0)}{n!}$, where $G_a^{(n)}$ denotes the n-th derivative of G_a.

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Theorem. (Uniqueness) If $G_a(s) = G_b(s)$ for all $|s| < R'$ where $0 < R' \leq R$, then $a_n = b_n$ for all n.

R is called *radius of concergence*. However for simplicity we don't discuss it in slides.

Probability Generating Function

Definition.

The probability generating function of a *discrete* random variable X is defined to be the generating function of its probability mass function:

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X should be discrete!

Most of the time we assume that X takes non-negative integers. But actually it can take negative intergers, and then the form of $G_X(s)$ needs to be changed a bit.

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Examples.

- Constant r.v.: $\Pr(X = c) = 1 \longrightarrow G_X(s) = \mathbb{E}(s^X) = s^c$
- Bernoulli r.v.: $\Pr(X = 1) = p, \Pr(X = 0) = 1 - p$

  $\longrightarrow G_X(s) = \mathbb{E}(s^X) = (1-p)s^0 + ps^1 = (1-p) + ps$
- Geometric r.v.: $\Pr(X = k) = p(1-p)^{k-1}$ for $0 < p \leq 1$ and $k \geq 1$

  $\longrightarrow G_X(s) = \mathbb{E}(s^X) = \sum_{k=1}^{\infty} p(1-p)^{k-1} s^k = \frac{ps}{1-s(1-p)}$

Probability Generating Function

Probability generating functions have many important and interesting properties.

Properties.

(1) Expectations: $\mathbb{E}[X(X-1)\ldots(X-k+1)] = G_X^{(k)}(1)$

More specifically, $\mathbb{E}(X) = G'_X(1)$.

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(3) Generating function of the sum of two independent r.v.:

$G_{X+Y}(s) = G_X(s)G_Y(s)$ if X and Y are independent.

More over, if $S = X_1 + X_2 + \ldots + X_n$ with (X_i) being independent, then $G_S = G_{X_1}G_{X_2}\dots G_{X_n}$.

Probability Generating Function

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(4) Convolution of sequences:

Given two sequences (a_n) and (b_n), the convolution sequence (c_n) is defined as

$$c_n = \sum_{i=0}^{n} a_i b_{n-i},$$

and we denote the convolution by $c = a * b$.

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$G_c(s) = G_a(s)G_b(s)$!

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The joint probability generating function can help us characterize "independence of random variables"......

Probability Generating Function

Theorem. X_1 and X_2 are independent *if and only if*

$$G_{X_1,X_2}(s_1, s_2) = G_{X_1}(s_1)G_{X_2}(s_2) \text{ for all } s_1 \text{ and } s_2.$$

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Proof.

X_1 and X_2 are independent.

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$\Pr[X_1 = n, X_2 = m] = \Pr[X_1 = n]\Pr[X_2 = m]$ for all n, m.

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Example: Matching Envelope Problem

Here is an example of applying generating functions:

Example. An absent-minded secretary prepares n letters and n matching envelopes to send to n different people, but then but then randomly stuffs the letters into the envelopes. What is the probability p_n that all letters are inserted into the wrong envelopes.

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Let E denote that "all letters are mismatched", and then we have

$$p_n = \Pr(E) = \Pr[E \mid F]\Pr(F).$$

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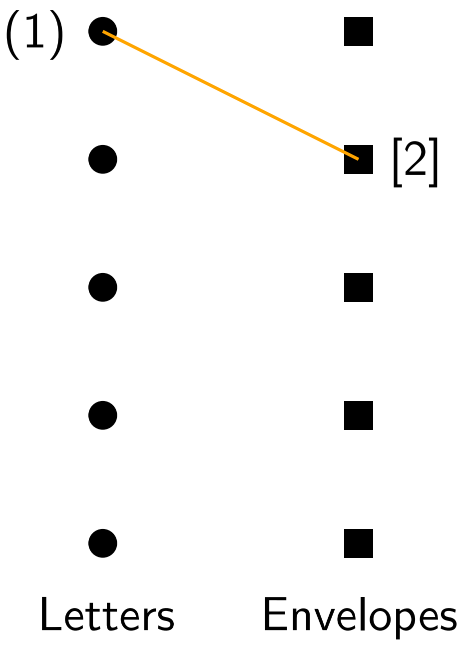

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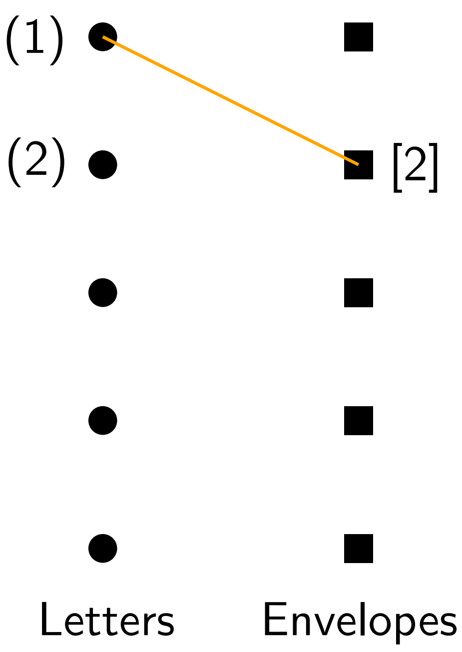

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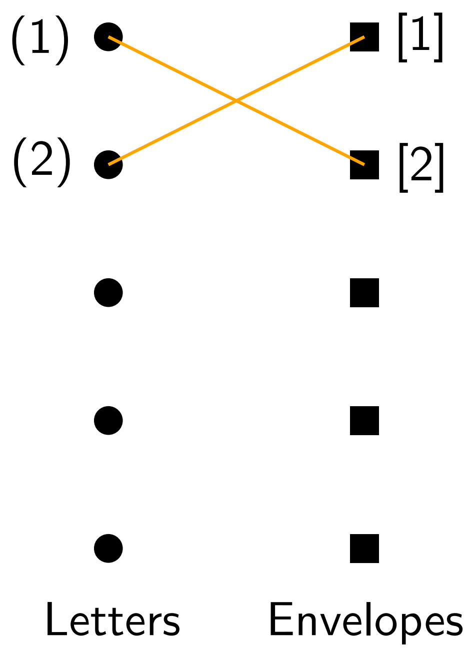

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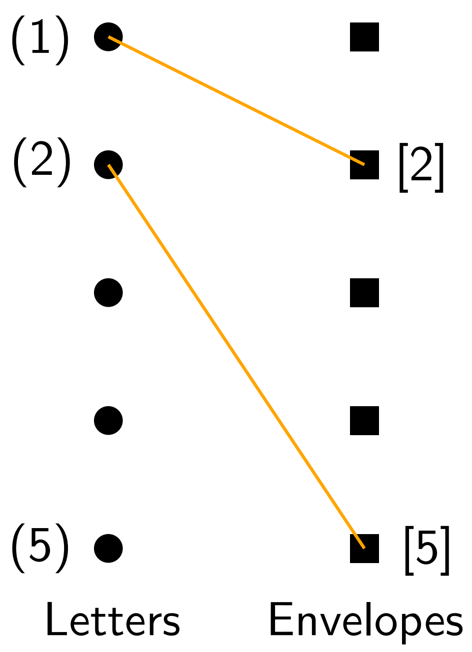

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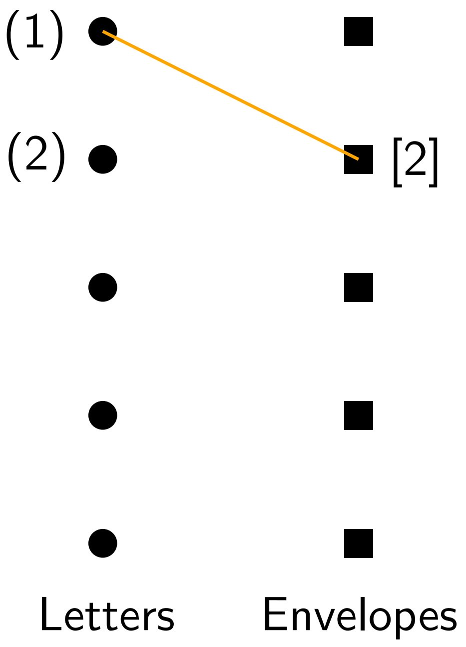

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Thus, $\alpha_n = \frac{1}{n-1} p_{n-2} + \frac{n-2}{n-1} \alpha_{n-1}$.

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$$\begin{aligned} p_n &= \Pr(E) = \Pr[E \mid F]\Pr(F) \\ &= \left(1 - \tfrac{1}{n}\right)\alpha_n \\ &= \left(1 - \tfrac{1}{n}\right)\left(\tfrac{1}{n-1} p_{n-2} + \left(1 - \tfrac{1}{n-1}\right)\alpha_{n-1}\right) \end{aligned}$$

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$p_n = (1 - \frac{1}{n})\alpha_n$, so $p_{n-1} = (1 - \frac{1}{n-1})\alpha_{n-1}$.

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Let $G_p(s) = \sum_{n=1}^{\infty} p_n s^n$. By multiplying the equation above by ns^{n-1} and then taking sum, we get

$$\sum_{n=3}^{\infty} ns^{n-1}p_n = s\sum_{n=3}^{\infty} s^{n-2}p_{n-2} + s\sum_{n=3}^{\infty}(n-1)s^{n-2}p_{n-1}.$$

Using generating function, we have

$$(1-s)G'(s) = sG(s) + s.$$

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$$\Big\downarrow \text{ solve this equation}$$

$G(s) = \frac{1}{1-s}e^{-s} - 1$

$$\Big\downarrow \text{ expand as a power series}$$

$$p_n = 1 + \frac{(-1)}{1!} + \ldots + \frac{(-1)^n}{n!}$$

$\blacksquare$

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We can use generating functions to study random walks.

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Since the process between two consecutive are pariwise independent, we can only focus on the time that the man returns the origin *for the first time*.

Random Walks

Let $p_0(n) := \Pr[S_n = 0]$ be the probability that he returns the origin at timestamp n. Let $f_0(n) := \Pr[S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0]$ be the probability that he returns the origin for the first time at timestamp n.

And write down the generating function of p_0 and f_0:

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Let T_0 be the r.v. denoting the time of the first return.
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Then we have:

(1) $P_0(s) = (1 - 4p(1-p)s^2)^{-\frac{1}{2}}$,
(2) $P_0(s) = 1 + P_0(s)F_0(s)$.

From (1), (2), it directly follows that $F_0(s) = 1 - (1 - 4p(1-p)s^2)^{\frac{1}{2}}$.

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multiply by s^n
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$$\sum_{n=1}^{\infty} p_0(n) s^n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} f_0(k) p_0(n-k) \right) s^n$$

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$$\underbrace{\sum_{n=1}^{\infty} p_0(n) s^n}_{= P_0(s) - 1} = \underbrace{\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} f_0(k) p_0(n-k) \right) s^n}_{= F_0(s)P_0(s) \text{ by convolution}}$$

$\blacksquare$

Random Walks

We can get two corollary from (3), which are consistent with our intuition:

- The man will return the origin at least once with probability

$$\sum_{n=1}^{\infty} f_0(n) = F_0(1) = 1 - \sqrt{1 - 4p(1-p)} = 1 - |2p - 1|.$$

  Specially, he almost surely returns the origin *if and only if* $p = \frac{1}{2}$.

- If he returns the origin almost surely (i.e., $p = \frac{1}{2}$), the expected time of the first return is

$$\sum_{n=1}^{\infty} n f_0(n) = F_0'(1) = \infty.$$

Random Walks

There is a relation between f_0 and p_0:

Proposition. For a symmetric random walk ("symmetric" means that $p = 1 - p = 1/2$), we have $2kf_0(2k) = p_0(2k-2)$.

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Proof.

Note that a man must have walked even steps if he returns the origin. So $f_0(k) = 0$ if k is odd.

Consider the generating function of LHS:

$$\sum_{k=1}^{\infty} 2kf_0(2k)s^{2k} = s\sum_{k=1}^{\infty} f_0(2k)(2ks^{2k-1}) = sF_0'(s) = \frac{s^2}{\sqrt{1-s^2}}$$

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Meanwhile, from RHS we have:

$$\sum_{k=1}^{\infty} p_0(2k-2)s^{2k} = s^2\sum_{k=1}^{\infty} p_0(2k-2)s^{2k-2} = s^2P_0(s) = \frac{s^2}{\sqrt{1-s^2}}$$

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Therefore $2kf_0(2k) = p_0(2k-2)$.

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Therefore $p_0(2n) = a_n$. ∎

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We have

(1) $F_r(s) = [F_1(s)]^r$ for $r \geq 1$,
(2) $F_1(s) = [1 - (1 - 4p(1-p)s^2)^{\frac{1}{2}}]/(2(1-p)s)$.

Random Walks

Proof of (1).

$$f_r(n) = \sum_{k=1}^{n-1} f_1(k) f_{r-1}(n-k) \text{ for } r > 1.$$

Then the generating function is

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By this definition, T_n may be ∞, but that's not a big deal.

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$$\begin{aligned} f_1(n) = \ \Pr[T_1 = n] &= \Pr[T_1 = n \mid X_1 = 1]p + \Pr[T_1 = n \mid X_1 = -1](1-p) \\ &= 0 \cdot p + \Pr[T_2 = n-1](1-p) \\ &= (1-p) f_2(n-1), \text{ for } n > 1. \end{aligned}$$

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$f_1(n) = (1-p)f_2(n-1)$ for $n > 1$

Again by multiplying and summing, we have

$$(1-p)sF_1(s)^2 - F_1(s) + ps = 0.$$

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Since $F_1(0) = 0$, by solving the equation above we get

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(1) $F_r(s) = [F_r(s)]^r$ for $r \geq 1$,
(2) $F_1(s) = [1 - (1 - 4p(1-p)s^2)^{\frac{1}{2}}]/(2(1-p)s)$.

Similar to "returning to the origin", we can also conclude a corollary:

Corollary. The man will arrive at position 1 at least once with probability

$$F_1(1) = \frac{1-|2p-1|}{2(1-p)} = \min\{\frac{p}{1-p}, 1\}.$$

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Definition. Let $X_1, X_2, \ldots$ be i.i.d. random variables taking values in integers and $S_n := X_1 + \ldots + X_n$. A random walk is called *right-continuous* if $\Pr[X_i > 1] = 0$.

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In this discrete setting, "continuous" means that "the difference is no more than 1", i.e., the step distribution puts no mass on integers ≥ 2.

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For generality, assume $\Pr[X_i = 1] > 0$. We want to prove the following theorem.

Theorem. (Hitting time theorem) Let S be a right-continuous random walk and T_r be the time the man first arrives at position r. Then

$$\Pr[T_r = n] = \tfrac{r}{n}\Pr[S_n = r] \text{ for } n, r \geq 1.$$

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Quite an interesting statement!

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Proof.

Define $G(z) := \mathbb{E}(z^{-X_1}) = \sum_{n=-1}^{\infty} \Pr[X_1 = -n]z^n$, $F_r(z) := \mathbb{E}(z^{T_r}) = \sum_{n=1}^{\infty} \Pr[T_r = n]z^n$. $\quad (z \in \mathbb{C})$

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We first focus on $F_1(z)$:

At timestamp 1, the man is at position X_1.

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$$\begin{aligned} F_1(z) = \mathbb{E}(z^{T_1}) &= \mathbb{E}(z^{1+T_J}), \text{ where } J = 1 - X_1 \text{ for shorthand.} \\ &= z\mathbb{E}(z^{T_J}) = z\mathbb{E}(F_J(z)) = z\mathbb{E}(F_{1-X_1}(z)) \\ &= z\mathbb{E}(F_1(z)^{1-X_1}) \end{aligned}$$

Random Walks

Theorem. (Hitting time theorem) Let S be a right-continuous random walk and T_r be the time the man first arrives at position r. Then

$$\Pr[T_r = n] = \frac{r}{n}\Pr[S_n = r] \text{ for } n, r \geq 1.$$

Proof.

Define $G(z) := \mathbb{E}(z^{-X_1}) = \sum_{n=-1}^{\infty} \Pr[X_1 = -n]z^n,\ F_r(z) := \mathbb{E}(z^{T_r}) = \sum_{n=1}^{\infty} \Pr[T_r = n]z^n. \qquad (z \in \mathbb{C})$

We first focus on $F_1(z)$:

At timestamp 1, the man is at position X_1.

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The distance between the man and position 1 at timestamp 1.

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$$z = \frac{1}{G(w)}, \quad (w = w(z) := F_1(z))$$

Random Walks

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To proceed with our proof, we need to introcude Lagrange's inversion theorem:

Theorem. (Lagrange inversion theorem) Let $z = \frac{w}{f(w)}$ with $\frac{w}{f(w)}$ being an analytical function of w on a neighborhood of the origin. Suppose that g is infinite differentiable, then

$$g(w(z)) = g(0) + \sum_{n=1}^{\infty} \frac{1}{n!} z^n \left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}u^{n-1}} \left[g'(u) f(u)^n \right] \right]_{u=0}.$$

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Now let $f(w) := wG(w)$ so that $z = \frac{1}{G(w)} = \frac{w}{f(w)}$.

Random Walks

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Now let $f(w) := wG(w)$ so that $z = \frac{1}{G(w)} = \frac{w}{f(w)}$.

Not too hard to verify that it is analytic on the neighborhood of the origin.

Random Walks

Proof.

$z = \frac{1}{G(w)} = \frac{w}{f(w)}$, $w = w(z) = F_1(z)$

Let $g(w) := w^r = F_1(z)^r = F_r(z)$, and the inversion theorem tells us that

$$g(w(z)) = F_r(z) = g(0) + \sum_{n=1}^{\infty} \frac{1}{n!} z^n D_n,$$

where $D_n = \left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}u^{n-1}} \left[r u^{r-1} u^n G(u)^n\right]\right]_{u=0}$.

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So $\Pr[T_r = n]$ is the coefficient of z^n: $\Pr[T_r = n] = \frac{1}{n!} D_n$.

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∎

Random Walks

Here is an exercise which will using the hitting time theorem to prove:

Exercise. Let $\{S_n : n \geq 0\}$ be a simple symmetric random walk with $S_0 = 0$. And let $T = \min\{n > 0 : S_n = 0\}$. Show that

$$\mathbb{E}\left(T \wedge 2m\right) = 2\mathbb{E}|S_{2m}| = 4m\Pr[S_{2m} = 0]$$

for $m \geq 0$.

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for $m \geq 0$.

Proof.

(a), (b) and (c) can be regarded as sequences related to m.

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Proof.

(a), (b) and (c) can be regarded as sequences related to m.

So we can consider their generating functions respectively.

Random Walks

Proof.

(a): $\mathbb{E}(T \wedge 2m)$

Random Walks

Proof.

(a): $\mathbb{E}(T \wedge 2m) = \sum_{k=1}^{m} f_0(2k) \cdot (2k) + (2m) \cdot \Pr[T > 2m]$

Random Walks

Proof.

(a): $\mathbb{E}(T \wedge 2m) = \underbrace{\sum_{k=1}^{m} f_0(2k) \cdot (2k)}_{T \leq 2m} + \underbrace{(2m) \cdot \Pr[T > 2m]}_{T > 2m}$

Random Walks

Proof.

(a): $\mathbb{E}(T \wedge 2m) = \sum_{k=1}^{m} f_0(2k) \cdot (2k) + (2m) \cdot \Pr[T > 2m]$

by previous propositions

$= \sum_{k=1}^{m} p_0(2k-2) + 2mp_0(2m)$

Random Walks

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by previous propositions

$= \sum_{k=1}^{m} p_0(2k-2) + 2mp_0(2m)$

Therefore

$$\sum_{m=0}^{\infty} \mathbb{E}(T \wedge 2m) s^{2m} = \sum_{m=0}^{\infty} \sum_{k=1}^{m} p_0(2k-2) s^{2m} + \sum_{m=0}^{\infty} 2mp_0(2m) s^{2m}$$

$$= \sum_{k=1}^{\infty} p_0(2k-2) \sum_{m=k}^{\infty} s^{2m} + s \sum_{m=0}^{\infty} 2mp_0(2m) s^{2m-1}$$

$$= \sum_{k=1}^{\infty} p_0(2k-2) \tfrac{s^{2k}}{1-s^2} + sP_0'(s)$$

$$= \tfrac{s^2}{1-s^2} P_0(s) + sP_0'(s) = \frac{2s^2}{(1-s^2)^{\frac{3}{2}}}.$$

Random Walks

Proof.

(b):
$$\sum_{m=0}^{\infty} 2\mathbb{E}|S_{2m}|s^{2m}$$

Random Walks

Proof.

(b):
$$\sum_{m=0}^{\infty} 2\mathbb{E}|S_{2m}|s^{2m} = \sum_{m=0}^{\infty} 2 \cdot 2 \left(\sum_{k=1}^{m} 2k \Pr[S_{2m} = 2k] \right) s^{2m}$$

by symmetry of random walk

Random Walks

Proof.

(b):

$$\sum_{m=0}^{\infty} 2\mathbb{E}|S_{2m}|s^{2m} = \sum_{m=0}^{\infty} 2\cdot 2\left(\sum_{k=1}^{m} 2k\Pr[S_{2m}=2k]\right)s^{2m}$$

$$= 4\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} 2m f_{2k}(2m)\right)s^{2m}$$

by hitting time theorem

Random Walks

Proof.

(b):

$$\begin{aligned}\sum_{m=0}^{\infty} 2\mathbb{E}|S_{2m}|s^{2m} &= \sum_{m=0}^{\infty} 2\cdot 2\left(\sum_{k=1}^{m} 2k\Pr[S_{2m}=2k]\right)s^{2m} \\ &= 4\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} 2m f_{2k}(2m)\right)s^{2m} \\ &= 4s\left[\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} f_{2k}(2m)\right)s^{2m}\right]'\end{aligned}$$

Random Walks

Proof.

(b):
$$\sum_{m=0}^{\infty} 2\mathbb{E}|S_{2m}|s^{2m} = \sum_{m=0}^{\infty} 2\cdot 2\left(\sum_{k=1}^{m} 2k\Pr[S_{2m}=2k]\right)s^{2m}$$
$$= 4\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} 2mf_{2k}(2m)\right)s^{2m}$$
$$= 4s\left[\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} f_{2k}(2m)\right)s^{2m}\right]'$$

when $m < k$, $f_{2k}(2m) = 0$

$$= 4s\left[\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} f_{2k}(2m)s^{2m}\right]'$$

Random Walks

Proof.

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&= 4\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} 2m f_{2k}(2m)\right)s^{2m}\\
&= 4s\left[\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} f_{2k}(2m)\right)s^{2m}\right]'\\
&= 4s\left[\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} f_{2k}(2m)s^{2m}\right]'\\
&= 4s\left[\sum_{k=1}^{\infty} F_{2k}(s)\right]' = 4s\left[\sum_{k=1}^{\infty} F_1(s)^{2k}\right]' = 4s\left[\tfrac{F_1(s)^2}{1-F_1(s)^2}\right]'\end{aligned}$$

Random Walks

Proof.

(b):

$$\begin{aligned}
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&= 4\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} 2mf_{2k}(2m)\right)s^{2m} \\
&= 4s\left[\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m} f_{2k}(2m)\right)s^{2m}\right]' \\
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&= \tfrac{2s^2}{(1-s^2)^{\frac{3}{2}}}
\end{aligned}$$

Random Walks

Proof.

(c): $$\sum_{m=0}^{\infty} 4mp_0(2m)s^{2m}$$

Random Walks

Proof.

(c):
$$\sum_{m=0}^{\infty} 4mp_0(2m)s^{2m} = 2s \sum_{m=0}^{\infty} p_0(2m)(2m)s^{2m-1}$$

Random Walks

Proof.

(c):

$$\begin{aligned}\sum_{m=0}^{\infty} 4mp_0(2m)s^{2m} &= 2s\sum_{m=0}^{\infty} p_0(2m)(2m)s^{2m-1} \\ &= 2sP_0'(s) \\ &= \frac{2s^2}{(1-s^2)^{\frac{3}{2}}}\end{aligned}$$

Random Walks

Proof.

(c):

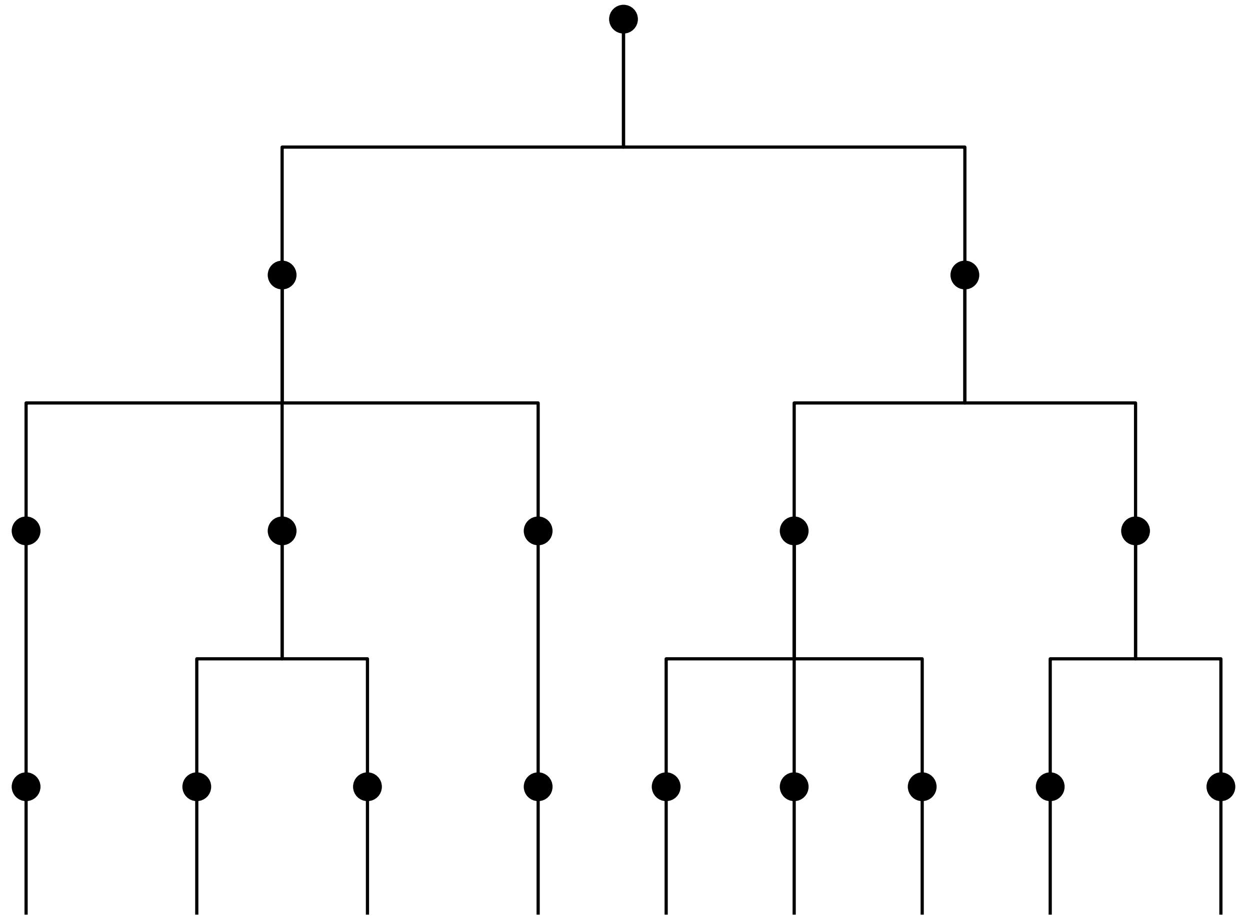
$$\begin{aligned}\sum_{m=0}^{\infty} 4mp_0(2m)s^{2m} &= 2s\sum_{m=0}^{\infty} p_0(2m)(2m)s^{2m-1} \\ &= 2sP_0'(s) \\ &= \frac{2s^2}{(1-s^2)^{\frac{3}{2}}}\end{aligned}$$

We see that $\sum_{m=0}^{\infty} \mathbb{E}(T \wedge 2m)s^{2m} = \sum_{m=0}^{\infty} 2\mathbb{E}|S_{2m}|s^{2m} = \sum_{m=0}^{\infty} 4mp_0(2m)s^{2m}$, and it follows by uniqueness that $\mathbb{E}\left(T \wedge 2m\right) = 2\mathbb{E}|S_{2m}| = 4m\Pr[S_{2m} = 0]$.

$\blacksquare$

Branching Processes

Generating function is a powerful tool for studying branching processes.



$Z_0 = 1$

$Z_1 = 2$

$Z_2 = 5$

$Z_3 = 9$

Branching Processes

Branching processes mainly focus on “reproduction”.

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Let Z_n be the number of members of the n-th generation. Assume that $Z_0 = 1$.
Each member of the n-th generation will reproduce some children (may be none), which are its family and the members of the $(n+1)$-th generation.

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The size of a family is a random variable.

We have the assumption:

the family size of each individual is identically independent distributed.

Let G be the generating function of “family size of a individual”.
And let $G_n(s) = \mathbb{E}(s^{Z_n})$ be the generating function of Z_n.

Branching Processes

As proved in textbook *Probability with Martingales*, section 0.3, using simple conditional expectation we can prove an important result for branching process:

Proposition. $G_{n+1}(s) = G_n(G(s))$, and thus $G_n(s) = G(G(\ldots(G(s))\ldots))$ is the n-fold iterate of G.

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Using this we can prove:

Let $\mu := \mathbb{E}(Z_1)$, $\sigma^2 = \mathrm{Var}(Z_1)$, then we have

$$\mathbb{E}(Z_n) = \mu^n, \qquad \mathrm{Var}(Z_n) = \begin{cases} n\sigma^2, & \text{if } \mu = 1, \\ \dfrac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1}, & \text{if } \mu \neq 1. \end{cases}$$

Branching Processes

Proof.

$$G_{n+1}(s) = G_n(G(s))$$

Branching Processes

Proof.

$$G_{n+1}(s) = G_n(G(s))$$

take differentiation,
let $s = 1$

$$\mathbb{E}(Z_n) = \mu \mathbb{E}(Z_{n-1})$$

Branching Processes

Proof.

$$G_{n+1}(s) = G_n(G(s))$$

take two differentiations,
let $s = 1$

$$G_n''(1) = G''(1)G_{n-1}'(1)^2 + G'(1)G_{n-1}''(1)$$

take differentiation,
let $s = 1$

$$\mathbb{E}(Z_n) = \mu\mathbb{E}(Z_{n-1})$$

$$\mathrm{Var}(Z_n) = G_n''(1) + G_n'(1) - G_n'(1)^2$$

(we mentioned it before)

$\blacksquare$

Branching Processes

For branching process, an important topic is about the "ultimate extinction probability", which is already discussed in textbook. So I don't want to discuss more here.

Generating Functions

Not only in probability, generating functions have various applications in many other fields, more in combinatorics.

Generating Functions

Not only in probability, generating functions have various applications in many other fields, more in combinatorics.

We only discussed ordinary generating function there. However, there are other types of generating functions, such as *exponential generating functions*: $E_a(s) = \sum_{i=0}^{\infty} \frac{a_i s^i}{i!}$, *Poisson generating functions*: $P_a(s) = \sum_{i=0}^{\infty} a_i e^{-s} \frac{s^i}{i!} = e^{-x} E_a(x)$, etc. They are all powerful math tools and all have widely usage.

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I knowed a little bit about generating functions before, in high school.

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At the beginning, I have no idea....

Branching process? Seems good...

The material says generating functions are a good tool. Our textbook also mentioned generating functions when dealing with branching process.

I knowed a little bit about generating functions before, in high school.

Okay, generating functions. That's it!

My Contributions

(1) Make this “dynamic slides” so that the contents are more understandable. And maybe it can be used in a lecture.
(2) Fully understand what I make.

(3) Solve two not-too-hard exercises in the book I referenced to.

References

[1] Geoffrey R. Grimmett, David R. Stirzaker, *Probability and Random Processes*, 3rd Edition, Oxford University Press, 2001

[2] Wikipedia Generating Functions

[3] Wikipedia Probability Generating Functions

[4] Hitting time theorem from University of Nebraska–Lincoln

[5] Krishna B. Athreya, Peter E. Ney, *Branching Processes*, Springer-Verlag Berlin Heidelberg, 1972

[6] David Williams, *Probability with Martingales*, Cambridge University Press, 1991

Thank you!