More to say on Scheffe's Lemma

Xinyu Mao

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Scheffe's Lemma is proved in our textbook(see section 5.10):

Lemma 1 (Scheffe). *Suppose that* $f_n, f \in \mathcal{L}^1(S, \Sigma, \mu)$ *and* $f_n \to f_n$ *(a.e.), then*

$$\mu(|f_n|) \to \mu(|f|) \iff \mu(|f_n - f|) \to 0.$$

Actually, we can get a more accurate result:

Theorem 2. *Suppose that* $f_n, f \in \mathcal{L}^1(S, \Sigma, \mu)^+$ *and* $f_n \to f_n$ *(a.e.), then*

$$\mu(f_n) - \mu(f) - \mu(|f_n - f|) \to 0, \text{ as } n \to \infty.$$

Proof. Let $g_n := \min(f_n, f), h_n := \max(f_n, f)$. Clearly, $g_n, h_n \in \mathcal{L}^1(S, \Sigma, \mu)^+$. Since $|f - f_n| = h_n - g_n, f_n = h_n + g_n - f$, we have

$$\begin{aligned} \mu(f_n) - \mu(f) - \mu(|f_n - f|) &= \mu(h_n + g_n - f) - \mu(f) - \mu(h_n - g_n) \\ &= 2[\mu(g_n) - \mu(f)]. \end{aligned} \tag{1}$$

Note that $g_n \leq f, g_n \to f$, and thus $\mu(g_n) \to \mu(f)$ by DOM, that is, $\mu(g_n) - \mu(f) \to 0$. On plugging this into Eq. (1) we get what we set out to prove. $\square$

Of course we also have the second part:

Theorem 3. *Suppose that* $f_n, f \in \mathcal{L}^1(S, \Sigma, \mu)$ *and* $f_n \to f_n$ *(a.e.), then*

$$\mu(|f_n|) - \mu(|f|) - \mu(|f_n - f|) \to 0, \text{ as } n \to \infty.$$

Proof. Applying Theorem 2 to $|f_n|, |f|$ yields

$$\mu(|f_n|) - \mu(|f|) - \mu(||f_n| - |f||) \to 0. \tag{2}$$

Note that $f_n^+ \to f^+, f_n^- \to f^-$, and by Theorem 2

$$\mu(f_n^\pm) - \mu(f^\pm) - \mu(|f_n^\pm - f^\pm|) \to 0. \tag{3}$$

Rewrite $|f|, |f_n|$ as $f^+ + f^-$ and $f_n^+ + f_n^-$, and by Eq. (3) we have

$$\mu(|f_n|) - \mu(|f|) - [\mu(|f_n^+ - f^+) + \mu(|f_n^- - f^-|)] \to 0. \tag{4}$$

Since $||f_n| - |f|| \leq |f_n - f| \leq |f_n^+ - f^+| + |f_n^- - f^-|$, and the theorem follows from Eq. (2) and Eq. (4). $\square$

Remark. Lemma 1 immediately follows from the theorem above. Informally, the theorem says some mass is missing when taking limit and the loss (i.e. difference between $\mu(\lim_{n\to\infty} f_n)$ and $\lim_{n\to\infty} \mu(f_n)$) can be measured by $\lim_{n\to\infty} \mu(|f - f_n|)$. I learned about Theorem 2 while glancing over [1](see Exercise 1.4.48).

References

[1] Terence Tao. *An introduction to measure theory*, volume 126. American Mathematical Society Providence, RI, 2011.