How A_n is constructed in the proof of Poincare' Recurrence Theorem

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The construction of A_n in the proof given by Mr. Wu is really tricky when I take the first look.

$$A_n = \{x \in E \mid x \notin T^{-kn}(E), \forall k\}.$$

And now I try to give some intuition behind the construction. First let us take a look at another version of Poincare' Recurrence Theorem:

Problem 1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let T be a map from Ω to itself with the property that $T(F) \in \mathcal{F}, \forall F \in \mathcal{F}$. Plus $P(T(F)) = P(F)$.*
For $\forall E \in \mathcal{F}$, prove that $P(E \setminus \limsup_{n\to\infty} T^n(E)) = 0$

Proof: First we use a simple trick in set operation

$$E \setminus \bigcap_{n\geq 1} R_n = \bigcup_{n\geq 1} (E \setminus R_n).$$

to simpify the problem. Applying the equation above, our goal as

$$P\left(\bigcup_{k\geq 1}\left(E \setminus \bigcup_{n\geq k} T^n(E)\right)\right) = 0.$$

Otherwise we assume

$$\exists k \geq 1, s.t.\ P\left(E \setminus \bigcup_{n\geq k} T^n(E)\right) > 0.$$

Therefore this problem is equivalent to prove

$$P(A_k) = 0, \text{ in which } A_k = \{x \in E | x \notin \bigcup_{n\geq k} T^n(E)\}.$$

Notice that **this is exactly what the original form has proved**. Here we give a direct proof for it. Since $P(X) \leq \infty$, we have

$$P\left(E \cup \bigcup_{n\geq k} T^n(E)\right) > P\left(\bigcup_{n\geq k} T^n(E)\right).$$

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Applying T to the set above k times, there is

$$
\begin{aligned}
P\left(T^k\left(E\right)\cup\bigcup_{n\geq 2k}T^n\left(E\right)\right) &= P\left(T^k\left(E\cup\bigcup_{n\geq k}T^n\left(E\right)\right)\right)\\
&= P\left(E\cup\bigcup_{n\geq k}T^n\left(E\right)\right)\\
&> P\left(\bigcup_{n\geq k}T^n\left(E\right)\right).
\end{aligned}
$$

This leads to a contradiction since

$$T^k\left(E\right)\cup\bigcup_{n\geq 2k}T^n(E)\subset\bigcup_{n\geq k}T^n(E).$$

Which completes our proof. $\square$

Remark : what this proof tells us is that the construction of A_n in the original proof **does not come from nowhere**. It is exactly something derived from the objective equation, which reveals that the original proof is somewhat natural.