A Proof of Doob-Dynkin Lemma

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> **Doob-Dynkin Lemma.** Let $X, Y : S \to \mathbb{R}$ be random variables. Then $\sigma(Y) \subseteq \sigma(X)$ implies there exists a $\mathcal{B}/\mathcal{B}$-measurable function f such that $f \circ X = Y$.

It seems that there is a common technology to prove the validity of a certain property for some functions that breaks proof to four easier steps as follows

(1) Prove the property for indicator functions.

(2) Using linearity, extend the proof to simple functions.[1]

(3) Using monotone approximation, extend the property to non-negative functions.

(4) Extend the property to some real-valued functions f by writing $f = f^+ - f^-$.

It's called *Standard Machine*. Anyway, the following proof will follow this idea.

Let's do the first step—show the validity of Doob-Dynkin Lemma for indicator functions.

Lemma 1. *Doob-Dynkin Lemma holds when Y is an indicator function.*

Proof of Lemma 1. Suppose $Y = \mathbf{1}_A$ where $A \subseteq S$. Then we have $\sigma(Y) = \{\emptyset, S, A, A^C\} \subseteq \sigma(X)$. Clearly, $f = \mathbf{1}_B$ where $X^{-1}(B) = A$ will suffice, and trivially, f is $\mathcal{B}/\mathcal{B}$-measurable. □

Then we extend it to some simple functions.

Lemma 2. *Doob-Dynkin Lemma holds when Y is a finite linear combination of disjoint indicator functions in $\mathbb{R}$. Formally,*

$$Y = \sum_{i \in I} \lambda_i \mathbf{1}_{A_i}$$

where $\lambda_i \in \mathbb{R}$ and $A_i \cap A_j = \emptyset$ for any $i, j \in I$ with $i \neq j$.

Proof of Lemma 2. Without loss of generality, we assume all λ_i are pairwise distinct. It's easy to see that $\sigma(\mathbf{1}_{A_i}) \subseteq \sigma(Y) \subseteq \sigma(X)$, so by *Lemma 1*, we can obtain some measurable functions $f_i, i \in I$. And $f = \sum_{i \in I} \lambda_i f_i$ will suffice. □

[1] A simple function is a finite linear combination of indicator functions of measurable sets.

Remark 1. *Note that Lemma 2 is true only for* ***finite*** *combination of indicator functions. Otherwise, the measurability of f won't be guaranteed.*

Let's define the dyadic function $d_n \in \mathbb{R}^{\mathbb{R}}$ for $n \in \mathbb{N}$ as follows.

$$d_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})} + n\mathbf{1}_{[n,+\infty)}$$

Lemma 3 (Monotone Approximation)**.** *Let f be a non-negative measurable function. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of non-negative simple functions such that $f_n \uparrow f$.*

Proof of Lemma 3. Let $f_n = d_n \circ f$. The proof of why $f_n \uparrow f$ has been shown in previous work by other students (maybe in folder *week 4 or 5* in our course website), so I won't repeat it here. □

By letting $f = f^+ - f^-$, we obtain the following corollary.

Corollary 1 (Approximation)**.** *Let f be a measurable function. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions such that $f_n \to f$.*

Now we can prove the Doob-Dynkin Lemma. Let X, Y be random variables, and by *corollary 1*, there exists a sequence of simple function $\{Y_n\}_{n=1}^{\infty}$ where $Y_n = d_n \circ Y^+ - d_n \circ Y^-$ such that $Y_n \to Y$. Now we should prove $\sigma(Y_n) \subseteq \sigma(Y)$ in order to use *Lemma 2*. By definition, if $A \in \sigma(Y_n)$, then there exists $B \in \mathcal{B}$ such that $A = Y_n^{-1}B$. As Y_n is a simple function, we can somehow extend B by finite number of subset in $\mathbb{R}$ to B', formally, $B' = B \cup B_1 \cup B_2 \cup \cdots \cup B_k$ such that $A = Y^{-1}B'$. That means $\sigma(Y_n) \subseteq \sigma(Y) \subseteq \sigma(X)$. By *Lemma 2*, each Y_n has one associated f_n to satisfy the Doob-Dynkin Lemma. Then $f = \limsup f_n$ is again measurable by property of measurable functions. And $f \circ X = (\limsup f_n) \circ X = \limsup(f_n \circ X) = \limsup Y_n = \lim Y_n = Y$.