Intersection of Uncountable a.s. Events May Be Empty

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Exercise 4

This exercise is going to say that intersection of uncountable events may be empty. We already know that for the countable situation, things are different:

Theorem 1

$$If\ F_n \in \mathcal{F}\ (n \in N)\ and\ P(F_n) = 1,\ \forall n,\ then$$

$$P(\bigcap_n F_n) = 1$$

Now the number of α is obviously uncountable, so the theorem above is not working anymore. And we can get an opposite conclusion that

$$\bigcap_\alpha F_\alpha = \emptyset$$

Proof. For any given ω, we just need to find an α with $\omega \notin F_\alpha$. Then we can easily get the result that $\bigcap\limits_\alpha F_\alpha = \emptyset$. Thus the only thing to concern about is to find the α.

For $\omega = (\omega_1, \omega_2, \dots)$, we can assume without losing generality that

$$\forall m,\ \exists n > m\ \omega_n = H\ (n \in N)$$

If ω doesn't satisify this proposition, then because $\omega = \{H, T\}$, we have $\forall m,\ \exists n > m\ \omega_n = T$. These two cases are just the same, so we only consider the first one.

In this case, we can generate α in this way: For $m = 0$, we can find $a_1 > m$ with $\omega_{a_1} = H$. So we let $\alpha(1) = a_1$. Next, let $m = a_1$, find a_2 in the same way and also let $\alpha(2) = a_2$, and so on ... Eventually, we get a map α with $\alpha(1) < \alpha(2) < \dots$ Obviously,

$$\frac{\#\{k \leq n :\ \omega_{\alpha(k)} = H\}}{n} \to 1$$

Therefore we know that $\omega \notin F_\alpha$ and finish the proof. $\square$