$P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$ $\Rightarrow$ $P(A_n, \text{i.o.}) = 0$

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This note proves a extension of the First Borel-Cantelli Lemma(BC1).

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(A_n : n \in \mathbb{N}) \subseteq \mathcal{F}$ be a sequence of events. Recall BC1 which was discussed in class.

Theorem 1 (The First Borel-Cantelli Lemma)**.** *If* $\sum_{n\in\mathbb{N}} P(A_n) < \infty$*, then*

$$P(\limsup A_n) = P(A_n, i.o.) = 0.$$

And we have the following extension proposition of BC1.

Proposition 2. *If* $\lim P(A_n) = 0$ *and* $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$*, then* $P(A_n, i.o.) = 0$*. [1]*

Proof. For an arbitrary fixed $n \in \mathbb{N}$, we have

$$\begin{aligned}
P(A_n, \text{i.o.}) &= P(\limsup A_n) \\
&= P(\bigcap_n \bigcup_{m\geqslant n} A_m) \\
&\leqslant P(\bigcup_{m\geqslant n} A_n) \\
&= P(A_n \sqcup \bigsqcup_{m>n} (A_m \setminus \bigcup_{n\leqslant i<m} A_i)) \\
&= P(A_n) + \sum_{m>n} P(A_m \setminus \bigcup_{n\leqslant i<m} A_i) \\
&= P(A_n) + \sum_{m>n} P(A_m \cap \bigcap_{n\leqslant i<m} A_i^c) \\
&\leqslant P(A_n) + \sum_{m>n} P(A_m \cap A_{m-1}^c) \\
&= P(A_n) + \sum_{m\geqslant n} P(A_{m+1} \cap A_m^c).
\end{aligned}$$

Since it is given that $\lim P(A_n) = 0$ and $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$, we know that $\lim_{n\to\infty} \sum_{m\geqslant n} P(A_{m+1} \cap A_m^c) = 0$. By letting $n \to \infty$, immediately we get

$$\begin{aligned} P(A_n, \text{i.o.}) &\leqslant \lim_{n\to\infty} P(A_n) + \lim_{n\to\infty} \sum_{m\geqslant n} P(A_{m+1} \cap A_m^c) \\ &= 0 + 0 = 0. \end{aligned}$$

As $P(A_n, \text{i.o.}) \geqslant 0$ always holds, it follows that $P(A_n, \text{i.o.}) = 0$. □

Here is an example of a sequence of events to which BC1 cannot be applied but the extension proposition can be applied.

Example 3. *Let the probability space* $(\Omega, \mathcal{F}, P)$ *be* $([0,1], \mathcal{B}[0,1], Leb)$ *and* $A_n := (0, \frac{1}{n}) \in \mathcal{B}[0,1]$.

Obviously

$$\sum_{n\in\mathbb{N}} P(A_n) = \sum_{n\in\mathbb{N}} \frac{1}{n}$$

is divergent. So BC1 cannot be applied to this example. However, observe that $A_n^c \cap A_{n+1} = [\frac{1}{n}, 1] \cap (0, \frac{1}{n+1}) = \emptyset$. So $P(A_n^c \cap A_{n+1}) = 0$ hence $\sum_{n\in\mathbb{N}} P(A_n^c \cap A_{n+1}) = 0 < \infty$. By the extension proposition of BC1 above, it follows that $P(A_n, \text{i.o.}) = 0$.

Therefore, in the above example, the extension proposition can be applied while BC1 cannot. But can we conclude that "the usage of this proposition is wider than BC1"? ~~It depends on whether we can show that the premise of BC1 implies the premise of the extension proposition. I'm thinking this problem, and I hope someone can help me.~~ Here is the proof that the premise of BC1 implies the premise of the proposition.

Proof. Let $(A_n : n \in \mathbb{N})$ be a sequence of events which satisfies the premise of BC1, i.e., $\sum_{n\in\mathbb{N}} P(A_n) < \infty$. By one of the basic properties of convergent series, $\lim_{n\to\infty} P(A_n) = 0$ holds. And

$$\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \sum_{n=1}^{\infty} P(A_{n+1}) < \infty.$$

Thus the premise of the extension proposition holds, which means the extension has a wider usage than BC1. □

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References

[1] T. K. Chandra. The Borel-Cantelli Lemma. Springer India, India, 2012.