Lebesgue Covering Dimension of One-dimensional Euclidean Space Equals to One

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Problem 1 Suppose real line is covered by a family of open intervals (I_k). Is there always a family of open intervals (J_k) satisfying $I_k \supseteq J_k$ such that every point on the real line is covered once or twice?

The answer is yes. We can prove a stronger result: We can always find a family of open intervals (J_k) that covers every point once or twice. In addition, for each J_k, either $J_k = I_k$ or $J_k = \varnothing$.

For simplicity, we will regard a family of open intervals as a set consisting of open intervals. For example, $(J_k) \subseteq (I_k) \cup \{\varnothing\}$.

Suppose F_j is a family of open intervals that covers the real line. Let $F_0 := (I_k)$. When $j > 0$, $F_j \subseteq F_{j-1}$ and every point in $[-j, +j]$ is covered by F_j once or twice.

If for each $j \in \mathbb{N}$, such F_j exists, then it is easy to check that

$$\bigcap_{j=0}^{\infty} F_j \cup \{\varnothing\}$$

is the (J_k) we are looking for.

To verify the existence of F_j, we can give an inductive proof.

Basis Step: $F_0 = (I_k)$, so F_0 exits.

Inductive Hypothesis: F_{j-1} exists.

Inductive Step: Define

$$S := \{\alpha \in F_{j-1} | \alpha \cap [-j, +j] \neq \emptyset\}$$
$$T := \bigcup_{I \in (F_{j-1} \setminus S)} I.$$

Choose an interval $(l_1, r_1) \in S$ such that

$$\forall p \in (R \setminus T),\ l_1 \leq p.$$

Again, choose an interval $(l_2, r_2) \in S$ such that

$$\forall p \in (R \setminus T),\ r_2 \geq p.$$

Since S covers $[-j, +j]$, it has a finite subcover S_f on $[-j, +j]$. Let

$$S' := S_f \cup \{[l_1, r_1], [l_2, r_2]\}.$$

Obviously S' is also a finite subcover of S on $[-j, +j]$.

Now we are going to remove some "bad intervals" from S'. Each time, we choose a bad interval and eliminate it. Since S' has finite intervals, the elimination process will end after finite steps. We call an interval I_r as a "bad interval" if and only if we can find two other intervals $I_s, I_t \in S'$ such that

$$I_r \subseteq I_s \cup I_t.$$

Denote the S' after the whole elimination process by S^*. Every point in $[-j, +j]$ is covered once or twice by S^*. That's because if one point is covered for more than twice, we can continue our elimination process.

$S^* \cup (F_{j-1} \setminus S)$ is the F_j we want. Since every point in $[-j, +j]$ is only covered by intervals from S^*, every point is covered for no more than twice. From the construction of S^*, we know $S^* \cup (F_{j-1} \setminus S)$ covers the whole real line. Hence, we proved the existence F_j.

This finishes the proof.