Further Discussion of Exercise 4

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The exercise 4 put forward that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. After reading some reference books, I'm trying to prove that $|A \times A| = |A|$ for any infinite set A.

Proof. First of all, it is clear that the proposition is true when A is countable, for the exercise 4 has been proved and every countable A is isomorphism to $\mathbb{R}$. For A is infinite, we can find a countable $B \subseteq A$ (maybe because $\aleph_0$ is the smallest, I'm not sure). And for $|B \times B| = |B|$ we can find a bijection $f : B \mapsto B \times B$.

Then we consider the set of all countable $B \subseteq A$ and its corresponding function f_B. We denote it as

$$Z = \{\langle B, f_B \rangle \mid B \text{ is the countable subset of } A.\}$$

Next, we define a partial order $<$ on set Z. $\langle B_1, f_1 \rangle \leq \langle B_2, f_2 \rangle$ when the following two conditions are met:

(1) $B_1 \subset B_2$;

(2) For every $x \in f_1(B_1)$, $x \in f_2(B_2)$ as well.

(3) f_1 concides f_2 in the domain B_1, i.e.,

$$f_1(b) = f_2(b) \text{ for every } b \in B_1$$

Here we get a parital order. In order to apply Zorn's lemma, we have to find upperbounds of those chains. Assume here we have a chain $C_0 \leq C_1 \leq \ldots C_n \leq \ldots$. Denote their union U as

$$U = \bigcup_{i=0}^{\infty} C_i$$

And we can also get the combination g of f_i, which is a function from U to $U \times U$.

Claim that g is a bijection on U. We begin our prove here.

If $c_i c_j$ belong to two different set of the chain, then these two set must be $C_i \leq C_j$ or $C_j \leq C_i$. In this way, $g(c_i) \neq g(c_j)$ because of the definition(3) of $\leq$. Thus g is injective.

Consider arbitraty $(u, v) \in U \times U$. If $u \in C_i$ and $v \in C_j$, then $u, v \in C_{max(i,j)}$, i.e., $(u, v) \in C_{max(i,j)} \times C_{max(i,j)}$. For every function f_i is bijective, (u, v) must have a preimage in $C_{max(i,j)}$. Thus g is surjective.

Hence, union set U is the upper bound of the chain. According to **Zorn's Lemma**, Z has a maximal element $\langle M, f_M \rangle$. f_M is the bijection function from M to $M \times M$. Now we have to prove M has the same cardinality with A. We prove by contradiction as follows.

Lemma 1. *If A, B are two infinite sets, then*

$$| A | + | B |= max\{| A |, | B |\}.$$

Proof. It can be proved by Cantor-Schröder-Bernstein Theorem, maybe I will finish it in the future. ☐

Assume that $| M |<| A |$ ($M \subset A$, then $| M |\leq| A |$). Denote $R = A \backslash M$. We can conclude from lemma 1 that $| A |=| M | + | R |= max\{| M |, | R |\}$. For our assumption stipulate $| M |\neq| A |$, then $| R |=| A |$ and $| R |>| M |$. Find a subset R' of R, and R' has the same cardinality as M. Denote $N = M \cup R'$(M and R' are disjoint), then we obtain

$$| N |=| M | + | R' |=| M |$$

Then we extend the function f_M to $f_N : N \mapsto N \times N$ as follows

$$f_N(n) = \begin{cases} f_M(n) & n \in M, \\ (x, y) \text{ where } (x, y) \in (N \times N)\backslash(M \times M) & n \notin M \end{cases}$$

Then we obtain a bijection function f_N of set N. (The existence of f_N can be proved by the same cardinary of M and R') We can visualize the extension as Figure **??** shows. Thus we get a bigger

Figure 1: The extension of set M

set N and its corresponding bijection function f_N. This contradicts to the suppose that M is the maximal set.

Therefore, the assumption fails and $| M |=| A |$. In this case, M, A, $M \times M$ and $A \times A$ has the same cardinality. ☐