Notes on Lebesgue Dominated Convergence Theorem

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April 19, 2020

1 Statement of the theorem

Theorem 1. *Lebesgue's Dominated Convergence Theorem. Let* (f_n) *be a sequence of complex-valued measurable functions on a measure space* S, Σ, μ*. Suppose that the sequence converges pointwise to a function* f *and is dominated by some integrable function* g *in the sense that*

$$|f_n(x)| \leq g(x)|f_n(x)| \leq g(x)$$

for all numbers n *in the index set of the sequence and all points* x S*. Then* f *is integrable and*

$$\lim_{n\to\infty} \int_S |f_n - f| \, d\mu = 0 \lim_{n\to\infty} \int_S |f_n - f| \, d\mu = 0$$

which also implies

$$\lim_{n\to\infty} \int_S f_n \, d\mu = \int_S f \, d\mu \lim_{n\to\infty} \int_S f_n \, d\mu = \int_S f \, d\mu$$

2 Proof of the theorem

Proof. Without loss of generality, one can assume that f is real, because one can split f into its real and imaginary parts (remember that a sequence of complex numbers converges if and only if both its real and imaginary counterparts converge) and apply the triangle inequality at the end.

Lebesgue's dominated convergence theorem is a special case of the Fatou-Lebesgue theorem. Below, however, is a direct proof that uses Fatou' s lemma as the essential tool.

Since f is the pointwise limit of the sequence (f_n) of measurable functions that are dominated by g, it is also measurable and dominated by g, hence it is integrable. Furthermore, (these will be needed later),

$$|f-f_n| \leq |f|+|f_n| \leq 2g|f-f_n| \leq |f|+|f_n| \leq 2g$$

for all n and

$$\limsup_{n\to\infty} |f-f_n| = 0. \limsup_{n\to\infty} |f-f_n| = 0.$$

The second of these is trivially true (by the very definition of f). Using linearity and monotonicity of the Lebesgue integral,

$$\left|\int_S f\,d\mu - \int_S f_n\,d\mu\right| = \left|\int_S (f-f_n)\,d\mu\right| \leq \int_S |f-f_n|\,d\mu. \left|\int_S f\,d\mu - \int_S f_n\,d\mu\right| = \left|\int_S (f-f_n)\,d\mu\right| \leq \int_S |f-f_n|\,d\mu.$$

By the reverse Fatou lemma (it is here that we use the fact that |f—fn| is bounded above by an integrable function)

$$\limsup_{n\to\infty} \int_S |f-f_n|\,d\mu \leq \int_S \limsup_{n\to\infty} |f-f_n|\,d\mu = 0, \limsup_{n\to\infty} \int_S |f-f_n|\,d\mu \leq \int_S \limsup_{n\to\infty} |f-f_n|\,d\mu = 0,$$

which implies that the limit exists and vanishes i.e.

$$\lim_{n\to\infty} \int_S |f-f_n|\,d\mu = 0. \lim_{n\to\infty} \int_S |f-f_n|\,d\mu = 0.$$

Finally, since

$$\lim_{n\to\infty} \left|\int_S f d\mu - \int_S f_n d\mu\right| \leq \lim_{n\to\infty} \int_S |f-f_n|\,d\mu = 0. \lim_{n\to\infty} \left|\int_S f d\mu - \int_S f_n d\mu\right| \leq \lim_{n\to\infty} \int_S |f-f_n|\,d\mu = 0.$$

we have that

$$\lim_{n\to\infty} \int_S f_n\,d\mu = \int_S f\,d\mu. \lim_{n\to\infty} \int_S f_n\,d\mu = \int_S f\,d\mu.$$

The theorem now follows.

If the assumptions hold only μ-almost everywhere, then there exists a μ-null set $N \in \Sigma$ such that the functions $f_n \mathbf{1}_{S\setminus N}$ satisfy the assumptions everywhere on S. Then the function $f(x)$ defined as the pointwise limit of $f_n(x)$ for $x \in S \setminus N$ and by $f(x) = 0$ for $x \in N$, is measurable and is the pointwise limit of this modified function sequence. The values of these integrals are not influenced by these changes to the integrands on this μ-null set N, so the theorem continues to hold.

DCT holds even if f_n converges to f in measure (finite measure) and the dominating function is non-negative almost everywhere. □

3 Reference

[1] Wikipedia-Dominated_convergence_theorem : `https://en.wikipedia.org/wiki/Dominated_convergence_theorem`