0.012345678910... is simply normal to base 10

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I can only prove 0.123456789..., known as the Champernowne Constant is simply normal to base 10 now [1], below is the proof.

Proof.

Let N_n denotes the number of digits after we write the $a_n = 10^n(n \in \mathbb{Z})$th number in $\mathbb{Z}$.

For N_n, we have a simple equation, by counting digits in different groups divided by the number's length

$$\begin{aligned} N_n &= \sum_{i=1}^{i=n} i(10^i - 10^{i-1}) \\ &= 9\sum_{i=1}^{i=n} i10^{i-1} \\ &= (n - \frac{1}{9})10^n + \frac{1}{9} \end{aligned}$$

For the number of 1s for example, (2...9 is the same as 1) in the number we write, denoted as M_n, we use induction.

Base :

$n = 1$, $M_n = 1$

Induction : After writing N_{n+1} digits

In the first N_n digits, we have m_n 1s. As for the 1s between the $N_n + 1$ digit and N_{n+1} digit, we write the integer number beween $10^n + 1$ and 10^{n+1}. Actually we just write the first 10^n integer for another 9 times and add $1, 2...9$ to the highest digit.So we have.

$$M_{n+1} = 10M_n + 10^n$$

Based on the induction above, we can write the general formula of M_n using high-school maths.

$$M_n = n10^{n-1}$$

[1] I also found the original thesis for Champernowne Constant, but i totally don't understand it after some reading

Based on above, we get

$$\lim_{n=N_n\to\infty} \frac{|x_1x_2...x_n|_d}{n} = \frac{M_n}{N_n}$$

$$= \frac{n10^{n-1}}{(n-\frac{1}{9})10^n+\frac{1}{9}}$$

$$= \frac{1}{10}$$

To finish the proof, we need to prove $\frac{|x_1x_2...x_n|_d}{n}$ decreases with n increases, though not strictly, we can see the trend as m is obviously much much smaller than n.

In all, 0.12345... is simply normal to base 10.

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