The existence of independent and identically distributed random variables

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Before we prove the theorem, let's first introduce the concept of generalized inverse distribution function.

Definition 1. *the generalized inverse distribution function* $F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}$

Here are some of its properties:

1) F^{-1} is non-decreasing
2) $F^{-1}(F(x)) \leq x$
3) $F(F^{-1}(p)) \geq p$
4) $F^{-1}(p) \leq x$ if and only if $p \leq F(x)$
5) If Y has a $U[0,1]$ distribution then $F^{-1}(Y)$ is distributed as F

Proof. 1.It comes obviously from the fact that F is non-decreasing.

2.$F^{-1}(F(x)) = \inf\{y \in \mathbb{R} : F(y) \geq F(x)\} \leq x$

3.$F(F^{-1}(p)) = F(\inf\{x \in \mathbb{R} : F(x) \geq p\}) \geq p$

4. Impose F on both sides of $F^{-1}(p) \leq x$ and we get $F(F^{-1}(p)) \leq F(x)$. So $p \leq F(F^{-1}(p)) \leq F(x)$ by property 3. It's similar for another side.

5. We need to prove $P(F^{-1}(Y) \leq x) = F(x)$. From property 4, we know $P(F^{-1}(Y) \leq x) = P(Y \leq F(x))$. Since Y is a uniform distribution, $P(Y \leq F(x)) = F(x)$. In conclusion, $P(F^{-1}(Y) \leq x) = F(x)$. □

With the help of Skorokhod representation of random variables and the property 5 of generalized inverse function, we can simplify the problem to finding a countable sequence of independent random variables on $([0,1], \mathcal{B}, Leb)$ which are uniformly distributed.

Theorem 1. *Given a certain distribution function, there exists a countable sequence of independent and identically distributed random variables on* $([0,1], \mathcal{B}, Leb)$.

Proof. $\forall s \in [0,1]$, write it in 2-base (if there are multiple representation, choose the infinite one):

$$s = \sum_{i=1}^{+\infty} B_i (\frac{1}{2})^i \tag{1}$$

Claim $(B_i)_{i\geq 1}$ is independent. This has been proved in previous work "Independence of coin tossing events" by Haichen Dong.

Let p_n be the n^{th} prime number and I_n be $\{p_n^i : i \in \mathbb{N}\}$. It's obvious that (I_n) don't intersect with each other. Denote $\phi_n(i) = p_n^i \in I_n$. Now we define $X_n = \sum_{i=1}^{+\infty} B_{\phi_n(i)} \cdot (\frac{1}{2})^i$. (X_n) is obviously independent.

$$
\begin{aligned}
\forall a \in [0,1], a &= \sum_{i=1}^{+\infty} B'_i (\frac{1}{2})^i. \\
P(X_n \leq a) &= P(B_{\phi_n(1)} < B'_1) + P(B_{\phi_n(1)} = B'_1 \cap B_{\phi_n(2)} < B'_2) + ... \\
&= \sum_{i=1}^{+\infty} P(\bigcap_{j=1}^{i-1} B_{\phi_n(j)} = B'_j \cap B_{\phi_n(i)} < B'_i) \\
&= \sum_{i=1}^{+\infty} (\frac{1}{2})^{i-1} \cdot \frac{1}{2} B'_i \\
&= a
\end{aligned}
$$

So X_n has a uniform distribution on $[0,1]$. We can now derive that $F^{-1}(X_n)$ is an independent sequence on $([0,1], \mathcal{B}, Leb)$ with distribution F. $\square$