Show that $\mathcal{B}([a,b]) = \sigma(\pi[a,b])$

赖睿航 518030910422

March 30, 2020

Proof. According to the definition of $\pi(\mathbb{R})$ and $\pi[a,b]$, we have:

$$\begin{aligned}
\pi[a,b] &= \{A \cap [a,b] \mid A \in \pi(\mathbb{R})\} \\
&= \{(-\infty, x] \cap [a,b] \mid x \in \mathbb{R}\} \\
&= \{[a,x] \mid x \in [a,b]\}.
\end{aligned}$$

Let τ be all the open sets on $\mathbb{R}$. By the definition of subspace topology, we have

$$\begin{aligned}
\mathcal{B}([a,b]) &= \sigma(\{\text{all open sets on } [a,b]\}) \\
&= \sigma(\{[a,b] \cap U \mid U \in \tau\}).
\end{aligned}$$

Hence intervals such as $[a,b], [a,x), (x,b], x \in (a,b)$ are all open intervals on $[a,b]$.

We first show that $\sigma(\pi[a,b]) \subseteq \mathcal{B}([a,b])$. By definition of σ-algebra, we only need to show that $\pi[a,b] \subseteq \mathcal{B}([a,b])$. For any interval $[a,x] \in \pi[a,b]$ with $x \in [a,b]$, there are two cases:

1. If $x \in [a,b)$, then $[a,x] = \bigcap_{n\in\mathbb{N}} [a, x + \frac{b-x}{2n}) \in \mathcal{B}([a,b])$.
2. If $x = b$, then $[a,x] = [a,b] \in \mathcal{B}([a,b])$ since $[a,b]$ is an open set.

Thus $\pi[a,b] \subseteq \mathcal{B}([a,b])$.

Then we show that $\mathcal{B}([a,b]) \subseteq \sigma(\pi[a,b])$. Equivalently we show that every open set on $[a,b]$ is contained in $\sigma(\pi[a,b])$. Every open set on $\sigma(\pi[a,b])$ is a countable union of open intervals. So we only need to show that every open interval on $[a,b]$ belongs to $\sigma(\pi[a,b])$. For any open interval I, there are four cases:

1. If $I = [a,b]$, then $I = [a,b] \in \sigma(\pi[a,b])$ is trivial.
2. If $I = [a,y), y \in (a,b]$, then $I = [a,y) = \bigcup_{n\in\mathbb{N}} [a, y - \frac{y-a}{2n}] \in \sigma(\pi[a,b])$.
3. If $I = (x,b], x \in [a,b)$, then $I = (x,b] = [a,b] \setminus [a,x] \in \sigma(\pi[a,b])$.
4. If $I = (x,y), a \leqslant x < y \leqslant b$, then $I = (x,y) = [a,y) \setminus [a,x] = (\bigcup_{n\in\mathbb{N}} [a, y - \frac{y-a}{2n}]) \setminus [a,x] \in \sigma(\pi[a,b])$.

Hence every open interval on $[a,b]$ belongs to $\sigma(\pi[a,b])$. So every open set on $[a,b]$ belongs to $\sigma(\pi[a,b])$. And therefore $\mathcal{B}([a,b]) = \sigma(\pi[a,b])$. $\square$