Poset $\mathbb{R}^3$ Cannot Be Stuffed into $\mathbb{R}^2$

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Problem 1. *View* $\mathbb{R}^k$ *as a poset which* $(x_1, x_2, \ldots x_k) \leq (y_1, y_2, \ldots, y_k)$ *if and only if* $y - x \geq 0$ *namely* $\forall i, y_i \geq x_i$*. Is there a subposet of* $\mathbb{R}^2$ *poset which is isomorphic with* $\mathbb{R}^3$*?*

The answer is no.

1 A Naïve Discomfort

When we look at a cube drawn on a plane, we always feel there is something wrong. In Figure 1, we can see A and B are not comparable in $\mathbb{R}^3$ poset. However, as we draw them on a plane, it **seems** as if they were perfectly comparable. We'll prove this discomfort actually makes sense.

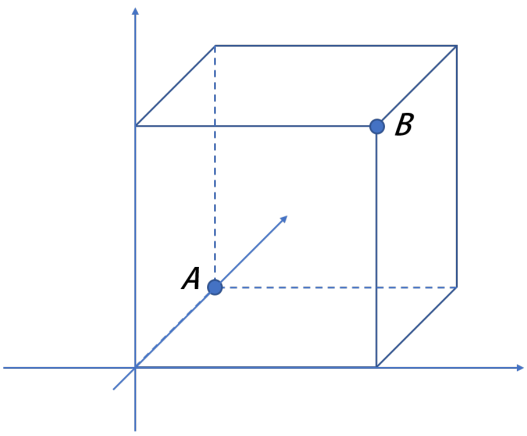

Figure 1: Something goes wrong

2 Proof

The proof is actually a formalization of the discomfort above.

Assume there is an isomorphism between poset $\mathbb{R}^3$ and a subposet of $\mathbb{R}^2$. We denote this bijection as φ. Now we introduce the following notations:

$$
\begin{aligned}
(1,0,0) = a &\xrightarrow{\varphi} a' = (x_a, y_a) \\
(0,1,1) = b &\xrightarrow{\varphi} b' = (x_b, y_b) \\
(0,0,1) = c &\xrightarrow{\varphi} c' = (x_c, y_c) \\
(1,1,0) = d &\xrightarrow{\varphi} d' = (x_d, y_d) \\
(1,0,1) = e &\xrightarrow{\varphi} e' = (x_e, y_e) \\
(0,1,0) = f &\xrightarrow{\varphi} f' = (x_f, y_f)
\end{aligned}
$$

Without loss of generality, we assume $x_a \geq x_b$ (the proof of the $x_a \leq x_b$ case is almost the same). Because a and b are not comparable, a' and b' are not as well. Thus we know $y_a \leq y_b$.

From $c \leq b$, we can see $x_c \leq x_b$. Similarly, $x_a \leq x_d$. Thus, $x_c \leq x_b \leq x_a \leq x_d$. Since c' and d' are not comparable, $y_c \geq y_d$.

From $e \geq a$, we know $e' \geq a'$. Thus $x_e \geq x_a \geq x_b$. Also, $e \geq c$ and we can conclude $y_e \geq y_c \geq y_d$. In addition, $f \leq b$ indicates $x_f \leq x_b$ and $f \leq d$ leads to $y_f \leq y_d$.

Here is the discomfort. We've proved $x_f \leq x_b \leq x_e$ and $y_f \leq y_d \leq y_e$. It immediately leads to $f' \leq e'$! However, we know e and f are not comparable. Our assumption that such bijection exists leads to a contradiction.

Hence, a subposet of $\mathbb{R}^2$ cannot be isomorphic with $\mathbb{R}^3$.

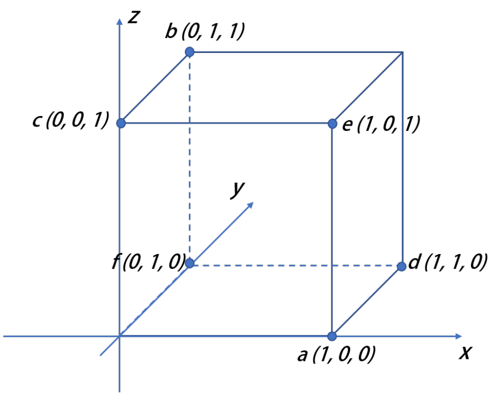

Figure 2: Some points on a cube