Notes on the Proof of Lovasz Local Lemma

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Lemma 1 (**Lovasz Local Lemma** [1]). *For events $E_1, \ldots, E_n$ with a dependency graph G, suppose,*

(1) $\forall i \exists p \in (0,1), P(E_i) \leq p$

(2) $\max \deg_G(V) \leq d$

(3) $4dp \leq 1$

then,

$$P(\bigcap E_i^C) > 0$$

Proof. To prove the lemma, we need to first introduce two statements.

For $s = 0, 1, \ldots, N-1$, $\forall |S| \leq s$,

(a)

$$P(\bigcap_{j \in S} E_j^C) > 0$$

(b)

$$\forall k \in [N] \backslash S, P(E_k \cap \bigcap_{j \in S} E_j^C) \leq 2pP(\bigcap_{j \in S} E_j^C)$$

We first prove two statements by induction.

$s = 0$ **:** When $s = 0$, we have $S = \emptyset$, so,

(a)

$$P(\bigcap_{j \in S = \emptyset} E_j^C) = 1 > 0$$

(b)

$$\frac{P(E_k \cap \bigcap_{j \in S} E_j^C)}{P(\bigcap_{j \in S} E_j^C)} = P(E_k) \leq p \leq 2p$$

$s > 0$ **:** Suppose two statements holds for $0, \dots, s-1$.
For s,

(a)

$$P(\bigcap_{j \in S} E_j^C) = \frac{P(\bigcap_{j \in [s]} E_j^C)}{P(\bigcap_{j \in [s-1]} E_j^c)} \times \frac{P(\bigcap_{j \in [s-1]} E_j^c)}{P(\bigcap_{j \in [s-2]} E_j^c)} \times \cdots \times \frac{P(\bigcap_{j \in [2]} E_j^c)}{P(\bigcap_{j \in [1]} E_j^c)} \times \frac{P(\bigcap_{j \in [1]} E_j^c)}{1} \tag{1}$$

Since statements (b) holds for $0, \dots, s-1$, $\forall 1 \le s' \le s$,

$$\frac{P(\bigcap_{j \in [s']} E_j^C)}{P(\bigcap_{j \in [s'-1]} E_j^C)} \ge 1 - 2p$$

So,

$$P(\bigcap_{j \in S} E_j^C) \ge (1-2p)^n$$

Since we have $4dp \le 1$, we can derive $2p \le \frac{1}{2d} \le \frac{1}{2}$, so, $1 - 2p \ge \frac{1}{2}$.
Thus,

$$P(\bigcap_{j \in S} E_j^C) > 0$$

(b) We need to prove

$$\frac{P(E_k \cap \bigcap_{j \in S} E_j^C)}{P(\bigcap_{j \in S} E_j^C)} = P(E_k \mid \bigcap_{j \in S} E_j^C) \le 2p$$

Let $S_1 = j \in S : j \sim k$ in $G, S_2 = S \backslash S_1$.
When $S_1 = \emptyset$, $P(E_k \mid \bigcap_{j \in S} E_j^C) = P(E_k) \le p < 2p$.
Otherwise, $S_1 \ne \emptyset, |S_2| < S$.
Let $F_{S_1} := \bigcap_{j \in S_1} E_j^C, F_{S_2} := \bigcap_{j \in S_2} E_j^C$, we have,

$$\begin{aligned} P(E_k \cap F_{S_1} \mid F_{S_2}) &\le P(E_k \mid F_{S_2}) \\ &= P(E_k) \le p \end{aligned}$$

and,

$$\begin{aligned} P(F_{S_1} \mid F_{S_2}) &= P(\bigcap_{i \in S_1} E_i^C \mid \bigcap_{j \in S_2} E_j^C) \\ &\ge 1 - \sum_{i \in S_1} P(E_i \mid \bigcap_{j \in S_2} E_j^C) \\ &\ge 1 - 2pd \ge \frac{1}{2} \end{aligned}$$

So,

$$
\begin{aligned}
P(E_k \mid \bigcap_{j \in S} E_j^C) &= P(E_k \mid F_{S_1} \cap F_{S_2}) \\
&= \frac{P(E_k \cap F_{S_1} \mid F_{S_2})}{P(F_{S_1} \mid F_{S_2})} \\
&\le \frac{p}{\frac{1}{2}} \le 2p
\end{aligned}
$$

Now we have both statements (a) and (b) stand, we substitute N into s in Eq. (1). We now can derive that all the denominators on the right hand side is larger than 0, and,

$$P(\bigcap E_i^C) = P(\bigcap_{i \in [N]} E_i^C) \ge (1 - 2p)^N \ge \left(\frac{1}{2}\right)^N > 0$$

□

Here is an application of Lemma 1.

Definition 2 (Conjunctive Normal Form)**.** *A formula is said to be in Conjunctive Normal Form (CNF) if it consists of AND's of several clause. For instance,* $(x \vee y) \wedge (y \vee \neg z \vee w)$ *is a CNF formula.*

Definition 3 (k-SAT)**.** *A k-SAT problem is that, given a CNF formula* f*, in which each clause has exactly* K *literals, decide whether there is an assignment satisfying all the clauses in* f*.*

Theorem 4. *Suppose variables appear in a CNF formula* f *are* $\{x_1, \ldots, x_n\}$*, and they can only be assigned by* 0 *or* 1 *with equal probability. If every variable appears in at most* $\frac{2^k}{4k}$ *clauses, there exists an assignment satisfying all the clauses in* f*.*

proof of Theorem 4. Let E_i be the event that the i-th clauses is wrong. Since every variable is assigned with equal probability, we have,

$$P(E_i) \le \frac{1}{2^k} =: p \text{ (in the Lovasz Local Lemma)}$$

Consider the clauses as the vertices in a dependency graph, and two vertices have an edge only if they share a same variable. So in this dependency graph, we have,

$$\deg E_i \le (\frac{2^k}{4k} - 1)k \le \frac{2^k}{4} =: d \text{ (in the Lovasz Local Lemma)}$$

Note that,

$$4dp = 4\frac{2^k}{4}\frac{1}{2^k} \le 1$$

So by Lemma 1, we have $P(\bigcap E_i^C) > 0$.

Thus, there exists an assignment satisfying all the clauses in f. □

References

[1] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. 1975.