$\mathcal{B}([a,b]) = ?$

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Let $(\mathbb{R},\tau)$ be the usual topology of $\mathbb{R}$.

According to the definition of Borel σ-algebra, $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by all sets in τ.

But consider interval $[a,b]$. We want to ask what is $\mathcal{B}([a,b])$?

We need to know the definition of **Subspace Topology**:

Given the usual topology space $(\mathbb{R},\tau)$ and a subset S of $\mathbb{R}$, the subspace topology on S is defined by:

$$\tau_S = \{S \cap U \mid U \in \tau\} \tag{1}$$

So according to the definition above, the subspace topology on $[a,b]$ is:

$$\tau_{[a,b]} = \{[a,b] \cap U \mid U \in \tau\} \tag{2}$$

Therefore, let $U_1 := (a-1,b+1), U_2 := (a-1,x), U_3 := (y,b+1)$ with $x,y \in (a,b)$. Obviously $U_1, U_2, U_3 \in \tau$. Then we have:

$$\begin{aligned} [a,b] &= [a,b] \cap U_1 \Rightarrow [a,b] \in \tau_{[a,b]} \\ [a,x) &= [a,b] \cap U_2 \Rightarrow [a,x) \in \tau_{[a,b]} \\ (y,b] &= [a,b] \cap U_3 \Rightarrow (y,b] \in \tau_{[a,b]}. \end{aligned}$$

It means that we can regard $[a,b], [a,x), (y,b]$ as "open sets" on $[a,b]$. And $\mathcal{B}([a,b])$ is the σ-algebra generated by all sets in $\tau_{[a,b]}$, which includes $[a,b], [a,x), (y,b]$ as well.