Notes on Week 6

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Definition 1. *Dependency graph for a sequence of events $E_1, \dots, E_n$ is a graph $G = (V, E)$ such that $V = \{1, \dots, N\}$.*

E_i is independent with $\{E_j : i \sim j \notin E\}$ for all $i \in V$

Lovasz Local Lemma Presented in 1975 by Erdös and Lovasz.

For an event sequence $E_1, \dots, E_n$, and G is a dependency graph of it.

Premise

$$(1) \exists p \in (0,1) P(E_i) \le p \qquad \forall i.$$
$$(2) \max deg_G(v) \le d$$
$$(3) 4dp \le 1$$

Conclusion $P\left(\cap E_i^c\right) > 0$

Proof Using inductive method

for $s = 0, 1, \dots, N-1$, $\forall |S| \le s$

$$\begin{cases} (a), & P\left(\bigcap_{j \in S} E_j^c\right) > 0), \\ (b), \quad \forall k \in [n]/S, P(E_k \cap \bigcap_{j \in S} E_j^c) \le 2pP\left(\bigcap_{j \in S} E_j^c\right). \end{cases}$$

Then let's start induction!

s=0 It's easy to verify it.

s>0

For expression a

$$P\left(\bigcap_{j\in[n]} E_j^c\right) = \frac{P\left(\bigcap_{j\in[n]} E_j^c\right)}{P\left(\bigcap_{j\in[n-1]} E_j^c\right)} \times \cdots \times \frac{P\left(\bigcap_{j\in[2]} E_j^c\right)}{P\left(\bigcap_{j\in[1]} E_j^c\right)}$$
$$\geq (1-2p)^n$$

Due to $4dP \leq 1$, $2p \leq \frac{1}{2d}$, so $1-2p \geq \frac{1}{2}$, which means the probability is correct.

For expression b To proof $P\left(E_k \middle| \bigcap_{j\in S} E_j^c\right) \leq 2p$ when $(|S| = s)$
Separate the points in S into two parts:
$S_1 = \{j \in S : j \sim k \text{ in } G\}$
$S_2 = S/S_1$
When S_1 is an empty set, $P(E_k|\bigcap_{j\in S} E_j^c) = P(E_k) \leq p < 2p$
Otherwise, $S_1 \neq \emptyset \to |S_2| < s$
Let $F_{S_1} = \bigcap_{j\in S_1} E_j^c$ $F_{S_2} = \bigcap_{j\in S_2} E_j^c$

$$P\left(E_k \middle| \bigcap_{j\in S} E_j^c\right) = P\left(E_k | F_{S_1} \cap F_{S_2}\right)$$
$$= \frac{P\left(F_{S_1} \cap E_k | F_{S_2}\right)}{P\left(F_{S_1} | F_{S_2}\right)}$$
$$P\left(F_{S_1} \cap E_k | F_{S_2}\right) \leq P\left(E_k | F_{S_2}\right)$$
$$= P\left(E_k\right) \leq p$$
$$P\left(F_{S_1} | F_{S_2}\right) = P\left(\bigcap_{i\in S_1} E_i^c \middle| \bigcap_{j\in S_2} E_j^c\right)$$
$$\geq 1 - \sum_{i\in S_1} P\left(E_i \middle| \bigcap_{j\in S_2} E_j^c\right)$$
$$\geq 1 - 2pd \geq \frac{1}{2}$$

So that $\frac{P\left(F_{S_1} \cap E_k | F_{S_2}\right)}{P\left(F_{S_1} | F_{S_2}\right)} < 2p$ ■