dual Cantor-Bernstein theorem

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1. (Cantor-Bernstein theorem) Let $f \in Y^X$ and $g \in X^Y$ be two injective maps. Then there is a bijection $h \in Y^X$ such that $h \subseteq f \cup g^{-1}$.

Proof: Let $C_0 = X \backslash g(Y)$, $C_{n+1} = g(f(C_n))$. And

$$C = \bigcup_{n=0}^{\infty} C_n$$

For every $x \in X$, define

$$h(x) = \begin{cases} f(x), & x \in C \\ g^{-1}(x), & x \notin C \end{cases}$$

We can easily see $h \subseteq f \cup g^{-1}$ from the definition. Next, prove h is bijective.

h is injective: Assume $a \neq b \wedge h(a) = h(b)$(h is not injective). If $a \in C \wedge b \in C$, $h(a) = f(a) \neq f(b) = h(b)$. If $a \notin C \wedge b \notin C$, $h(a) = g^{-1}(a) \neq g^{-1}(b) = h(b)$. If $a \in C \wedge b \notin C$, $g^{-1}(b) = f(a) \Rightarrow b = g(f(a)) \Rightarrow b \in C$, contradicting to $b \notin C$. Otherwise is the same. All cases contradict to the premise. So h is injective.

h is surjective: For any $y \in Y$, If $g(y) \in C$, there is a certion $n \geq 1$ such that $g(y) \in C_n$. Also, there is a $x \in C_{n-1} \subseteq X$ that $h(x) = f(x) = y$. If $g(y) \notin C$, there is $g(y) \in X$ that $h(g(y)) = g^{-1}(g(y)) = y$. Thus h is surjective.

Above all, $h \in Y^X$ is a bijection and $h \subseteq f \cup g^{-1}$. □

2. (dual Cantor-Bernstein theorem) Let $f \in Y^X$ and $g \in X^Y$ be two surjectve maps. Assuming AC, show that there is a bijection $h \in Y^X$ such that $h \subseteq f \cup g^{-1}$.

Proof: Given any surjections $f \in Y^X$ and $g \in X^Y$, by AC (we can get a Right inverse of a surjective map) there are injections $u \subseteq f^{-1}$ and $v \subseteq g^{-1}$.

By Cantor-Bernstein Theorem, there exits a bijection $h \in Y^X$ such that $h \subseteq u^{-1} \cup v$. As $u^{-1} \subseteq f$ and $v \subseteq g^{-1}$, $h \subseteq f \cup g^{-1}$. $\square$