More Discussions About Random Walk

WU Runzhe
Student ID : 518030910432

1 Introduction

In the probability theory class this semester, Prof. Wu once talked about an interesting mathematical object—random walk—in which I've been interested since I was a freshman student. I learned about the random walk for the first time in a lecture of the course *Some "Little" Stories Behind Scientific Ideas*[1], at which Prof. Wang Weike gave a brief introduction about it. At that time, I was only able to do some basic research through elementary methods. Therefore, in the past year, I was always eager to gain more knowledge so that I could look at this problem from different aspects and found more extensions of it. And now I believe I can really do a little more work on it!

Figure 1. The Quantum Cloud: a sculpture designed by a computer using a random walk algorithm [2]

[1] 《科学思想背后的“小”故事》王维克教授

[2] https://commons.wikimedia.org/wiki/File:Antony_Gormley_Quantum_Cloud_2000.jpg

2 Classic Model

2.1 Description

There are various models of the random walk. However, let's focus on the most concise one. A variant of the random walk will be discussed in later sections.

Definition 1. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be our probability triple. Set* $X_1, X_2, \ldots$ *to be IID RVs with* $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = \frac{1}{2}$. *And set* $S_0 = 0$ *and* $S_n = \sum_{i=1}^{n} X_i$ *for* $n > 0$. *Then the series* $\{S_n\}$ *is called* symmetric random walk on $\mathbb{Z}$.

2.2 Two Proofs of Recurrence

Definition 2. *We say that a random walk is* recurrent *if it visits its starting position infinitely often with probability 1, i.e.,*

$$\mathbf{P}\left(\sum_{n\geq 0} \mathbf{1}_{\{S_n = S_0\}} = \infty\right) = 1.$$

And we call it transient *if the above conditions are not met.*

Theorem 1. *The random walk described in Definition 1 is* recurrent.

We will give two proofs of Theorem 1 using Hewitt-Savage 0-1 law and Doob's 'forward' convergence theorem respectively.

2.2.1 Proof by Hewitt-Savage 0-1 law

Lemma 1. $\mathbf{P}(\limsup S_n \in B) =$ *0 or 1 for any* $B \in \mathcal{B}$.

Proof of lemma 1. To apply Hewitt-Savage 0-1 law, we only need to prove $\limsup S_n$ is permutation invariant, or equivalently, $\mathbf{1}_{\limsup S_n}$ is permutation invariant. Anyway, it is trivial since $\mathbf{1}_{\limsup S_n}$ has nothing to do with the order of first finite random variables. □

Using the same technique in the second part of the proof of Kolmogorov's 0-1 law (in 4.11 of our textbook), we obtain the following corollary.

Corollary 1. $\limsup S_n = c$ *a.s. for some* $c \in [-\infty, +\infty]$.

However, we can make it more precise.

Theorem 2. $\limsup S_n = +\infty, \liminf S_n = -\infty$ *a.s.*.

Proof of theorem 2. First off, we need to show that the c in corollary 1 cannot be finite.

Suppose c is finite. We conclude that $\limsup S_n = c$ a.s., and equivalently, $lim \sup S_{n+1} = c$ a.s.. And obviously we also have $\limsup(S_{n+1} - X_1) = c$ a.s. , and thus, $\limsup(S_{n+1}) = c + X_1$ a.s.. Combining these two equations, we get $c = c + X_1$ a.s.. That is, $X_1 = 0$ a.s., which contradicts the setting $\mathbf{P}(X_1 = 0) = 0$.

As $\mathbf{E}(X_i) = 0$, the case $\limsup S_n = -\infty$ a.s. can never happen. Analogously, it must be true that $\liminf S_n = -\infty$ a.s.. Hence the only possibility for $\limsup S_n$ and $\liminf S_n$ is that almost surely

$$\limsup S_n = +\infty, \liminf S_n = -\infty.$$

$\square$

Theorem 2 tells us a lot. For $\omega \in \Omega$, if $\limsup S_n(\omega) = +\infty$ and $\liminf S_n(\omega) = -\infty$, then ω must walk back and forth between $\mathbb{Z}_+$ and $\mathbb{Z}_-$, which means ω has no choice but to pass through the origin infinitely often. Then we are done.

I believe it's an overkill to utilize Hewitt-Savage 0-1 law in the proof of recurrence. However, after applying Hewitt-Savage 0-1 law, the rest of the proof is natural.

2.2.2 Proof by Doob's 'Forward' Convergence Theorem

We first show that if we start at 1 and randomly walk on the integer number line, we will visit 0 (the origin) within finite steps with probability 1. To that end. we introduce a new process Z where $Z_0 = 1$ and $Z_n = Z_{n-1} + X_n$. Intuitively, Z just means a random walk starting from 1. It's trivial that Z is a martingale since $\mathbf{E}(Z_{n+1} \mid \mathcal{F}_n) = \mathbf{E}(Z_n + X_{n+1} \mid \mathcal{F}_n) = Z_n + \mathbf{E}(X_{n+1}) = Z_n$ for natural filtration $\{\mathcal{F}_n\}$.

Let's set a stopping time $T \coloneqq \inf\{n > 0 : Z_n = 0\}$. Then $Z_{n\wedge T}$ is also a martingale. Since $Z_{n\wedge T} \geq 0$, we get $\mathbf{E}(|Z_{n\wedge T}|) = \mathbf{E}(Z_{n\wedge T}) = \mathbf{E}(Z_0) < \infty$, $\forall n$.

By Doob's 'Forward' Convergence Theorem, almost surely $Z_{n\wedge T}$ converges. So $\mathbf{P}(T = \infty) \leq \mathbf{P}\left((Z_{n\wedge T}) \text{ does not converge}\right) = 0$, which implies almost surely $T < \infty$. So starting from 1, we will eventually visit 0 with probability 1.

Analogously, if we start at -1, almost surely 0 will be visited at some point in the future. So with probability 1, the random walk starting from 0 is just like: starting from 0, arriving at 1 or -1, going back to 0 after some steps, arriving at 1 or -1 again, going back to 0 after some steps again, arriving at 1 or -1 again, going back to 0 after some steps again ... As we can see, it passes through 0 infinitely often.

3 Variant: Walk on a Cycle

Now let's consider the random walk on a cycle of length n (Figure 2). Vertices are numbered 0 (staring position) through $n-1$. For each step, the probability of walking clockwise and counterclockwise are both $\frac{1}{2}$. Since the graph is finite, we are no longer interested in whether it will pass through the origin position infinitely often. Now the question is: for each vertex $i \in [n-1]$, what's the probability that i is the last vertex to be visited?

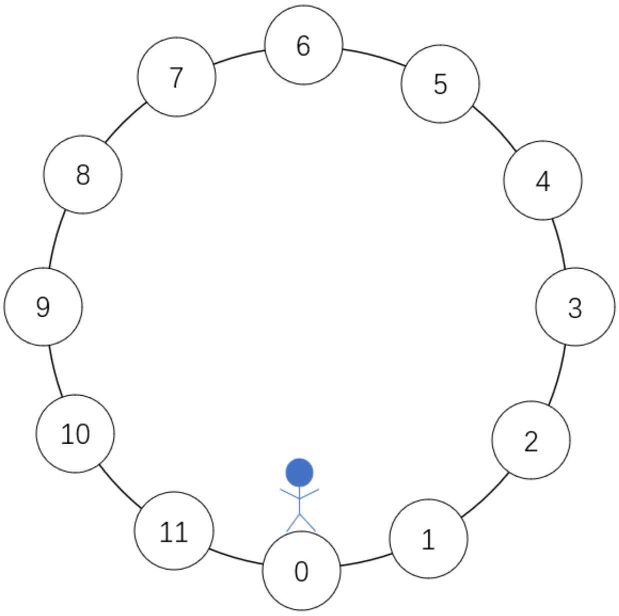

Figure 2. Random Walk on a 12-Cycle

3.1 Intuitive Idea

Let A_i be the event that i is the last vertex to be visited. By symmetry, $\mathbf{P}(A_1) = \mathbf{P}(A_{n-1})$.

If we take a closer look, we will find that A_1 just means to start walking at one of 1's neighbor (0 here) and eventually arrive at 1's other neighbor (2 here) without passing through 1 itself. This interpretation is important.

For each vertex i, $(1 < i < n-1)$, starting from 0, with probability 1, we will arrive at one of i's neighbor $\tilde{i}$ for the first time. Now the situation is quite similar. If i is the last vertex to be visited, the only possibility is that we start walking at $\tilde{i}$(remember it's one of i's neighbor) and then reach i's another neighbor without touching i itself. So obviously $\mathbf{P}(A_i) = \mathbf{P}(A_1)$.

In conclusion, the probability for each vertex to be last visited is uniformly distributed on all vertices $[n-1]$.

This result seems to be a bit counter-intuitive the first time I got it, but it becomes reasonable after deep thinking.

3.2 Formal Proof

The above intuitive idea is "correct" but not "rigorous". So let's give a formal proof using martingale theory. The following lemma is helpful.

Lemma 2. *Consider a simple random walk on the integer line starting from 0 and two positive integer a, b. The probability of reaching a before visiting $-b$ is $\frac{b}{a+b}$ while the probability of arriving at $-b$ before visiting a is $\frac{a}{a+b}$.*

Proof of lemma 2. Let's continue to use the variables $\{X_n\}$ and $\{S_n\}$ as we introduced earlier for random walk in Definition 1.

Set a stopping time

$$T := \inf\{n \geq 0 : S_n = a \text{ or } S_n = -b\}.$$

By Doob's 'Forward' Convergence Theorem again, almost surely $S_{n\wedge T}$ converges. So $\mathbf{P}(T = \infty) \leq \mathbf{P}\left((S_{n\wedge T}) \text{ does not converge}\right) = 0$, which implies $\mathbf{P}(T < \infty) = 1$. Furthermore, it's easy to check that $S_{n\wedge T}$ meets UI property since $\sup\{|S_{n\wedge T}|\} \leq \max\{a, b\}$. By Doob's Optional-Stopping theorem,

$$\mathbf{E}(S_T) = \mathbf{E}(S_0) = 0. \tag{1}$$

But as we know one of $S_T = a$ and $S_T = -b$ must hold. Therefore,

$$\mathbf{E}(S_T) = a\mathbf{P}(S_T - a) - b\mathbf{P}(S_T = -b). \tag{2}$$

Combining (1) and (2) together with $\mathbf{P}(S_T = a) + \mathbf{P}(S_T = -b) = 1$, we obtain the following:

$$\mathbf{P}(S_T = a) = \frac{b}{a+b}, \mathbf{P}(S_T = -b) = \frac{a}{a+b}.$$

□

Since the lemma has been verified, let's get back to the question of randomly walking on a cycle. Suppose i is the last visited vertex on that cycle, and then exactly one of the following happens (let $\tilde{i}$ and $\bar{i}$ be two neighbors of i):

1. Starting from 0, the walk visits $\tilde{i}$ before reaching $\bar{i}$, and then, starting from $\tilde{i}$, it visits $\bar{i}$ without visiting i.

2. Starting from 0, the walk visits $\bar{i}$ before reaching $\tilde{i}$, and then, starting from $\bar{i}$, it visits $\tilde{i}$ without visiting i.

By lemma 2, the probabilities of these two cases are $\frac{i-1}{n-2} \cdot \frac{1}{n-1}$ and $\frac{n-i-1}{n-2} \cdot \frac{1}{n-1}$ respectively. Since they are disjoint, the probability of their union is just the sum of their probabilities, which gives $\frac{1}{n-1}$. So for any vertex $i(i \neq 0)$, the probability of i being the last visited vertex is $\frac{1}{n-1}$.

4 Summary

We discussed the classic symmetric random walk and one of its variant—random walk on a cycle. During the discussion, the martingale theory shows its remarkable ability. When we identify a martingale and a stopping time, and apply appropriate theorems, the answer is about to come out, without cumbersome calculation like series summation, which I considered as a must-do when I first saw this problem.