Properties of Cantor Set

金弘义 518030910333

March 18, 2020

Exercise 1. *1) Show that the Cantor Set C is nowhere dense in* $[0,1]$.

2) Find a meager set T in $\mathbb{R}$ such that $T+T=\mathbb{R}$.

3) Show that every subset of the real line $\mathbb{R}$ can be partitioned into two sets, one being of first category and the other being negligible.

Proof. 1) Recall the the explicit closed formulas for the Cantor set are:

$$C = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} (\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}})$$

It's obvious that $(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}})$ is an open set. Since the union of open sets is an open set, the complement of C is also open, which leads to the fact that C is a closed set. So $\bar{C} = C$.

Note that Cantor set can also be characterized as the set of all number in $[0,1]$ whose base-3 expansion doesn't contain any 1s. Assume there exists an interior point x of C, then there exists a ball B, with radius $r > 0$, which is contained in C. Since $C \subseteq [0,1]$, B is an interval, whose length is $2r$. Arbitarily choose $x_1 \in B$ and $x_2 \in B$ such that $x_1 < x_2$ and $x_2 - x_1 \geq r$. Let $n = \lceil \log_{\frac{1}{3}} r \rceil + 2$.Consider $x_3 = x_1 + (\frac{1}{3})^n, x_4 = x_1 + 2 \cdot (\frac{1}{3})^n$. Since $(\frac{1}{3})^n < (\frac{1}{3})^{\log_{\frac{1}{3}} r+1} = \frac{1}{3}r$, $x_3, x_4 \in B$. It' s obvious that one of them conatins 1 in its base-3 expansion, which means one of them doesn't belong to Cantor set. This leads to a contradiction, or equivalently, that there is no interior point of C.

Now we can safely conclude that C is nowhere dense.

2)Let

$$T(n) = \{a + n | a \in C\}$$
$$T = \bigcup_{n \in \mathbb{Z}} T(n)$$

First We will prove T is a meager set. Since $T(n)$ is constructed by translation of all points in C by n, $T(n)$ is also nowhere dense. This leads to the fact that T is a meager set, because $\mathbb{Z}$ is countable.

Then we prove $T+T = \mathbb{R}$. From $C+C = [0,2]$, we can similarly get $T(n)+T(n) = [2n, 2n+2]$. Note that:

$$\bigcup_{n \in \mathbb{Z}} (T(n) + T(n)) \subseteq T + T \subseteq \mathbb{R}$$

However,

$$\bigcup_{n\in\mathbb{Z}}(T(n)+T(n))=\bigcup_{n\in\mathbb{Z}}[2n,2n+2]=\mathbb{R}$$

Now we conclude that $T+T=\mathbb{R}$.

3) Method 1:

Pick an enumeration of $\mathbb{Q}$, named q_n. Let:

$$I_{i,j}=(q_i-\frac{1}{2^{i+j+2}},q_i+\frac{1}{2^{i+j+2}})$$
$$A_j=\bigcup_{i\in\mathbb{N}}I_{i,j}$$
$$B=\bigcap_{j\in\mathbb{N}}A_j$$

For any $\epsilon>0$, exist $j_0>0$ such that $|\frac{1}{2^{j_0}}|<\epsilon$. So we have :

$$|A_j|\leq\sum_{i=0}^{\infty}\frac{1}{2^{i+j+1}}=\frac{1}{2^j}$$
$$|B|\leq|A_{j_0}|\leq\frac{1}{2^{j_0}}<\epsilon$$

This leads to that B is negligible. Then we prove B^C is a meager set.

$$B^C=\bigcup_{j\in\mathbb{N}}A_j^C$$

Since $\mathbb{Q}\subseteq A_j$ and $\mathbb{Q}$ is dense, we can know that A_j is dense. A_j is also open because A_j is the union of open intervals. These 2 properties of A_j can lead to that A_j^C is nowhere dense. So B^C is a meager set, which finishes the proof.

Method 2: Construct a sequence of fat cantor set by the following steps: To construct $C_n(n\geq 2)$, first remove the middle $\frac{1}{2n}$ from the interval $[0,1]$. Then in the i^{th} step(i starts from 2), remove subintervals of width $\frac{1}{2n\cdot 4^{i-1}}$ from the middle of each of the 2^{i-1} remaining intervals. We can then calculate

$$|C_n|=1-\sum_{i=1}^{\infty}\frac{1}{2n\cdot 4^{i-1}}\cdot 2^{i-1}=1-\frac{1}{n}$$
$$|C_n^C|=\frac{1}{n}$$

For any $\epsilon>0$, there exists $n_0>0$ such that $\frac{1}{n_0}<\epsilon$. Let $C=\bigcup\limits_{n=2}^{\infty}C_n$, then $C^C=\bigcap\limits_{n=2}^{\infty}C_n$. Note that $|C^C|\leq|C_{n_0}^C|=\frac{1}{n_0}<\epsilon$, which leads to that C^C is negligible in $[0,1]$.

By construction,C_n contains no intervals and therefore has empty interior, or equivalently, C_n is nowhere dense in $[0,1]$. So C is a meager set in $[0,1]$.

We can easily extend this conclusion to $\mathbb{R}$. Let $C'(i)=\{i+c|c\in C,i\in\mathbb{Z}\}$, $C'=\bigcup\limits_{i\in\mathbb{Z}}C'(i)$. It's easy to see that C' is a meager set in $\mathbb{R}$ and C'^C is negligible in $\mathbb{R}$. $\square$