Prove $\mathbb{R}$ Is Uncountable Without Using AC_ω

Zhiyang Xun

March 16, 2020

Theorem $\mathbb{R}$ is an uncountable set and it doesn't rely on axiom of countable choice.

Proof. Let (a_i) be a sequence of reals. Let $b_1 = a_1 + 1$, $c_1 = a_1 + 2$, so $a_1 \notin [b_1, c_1]$. Then if $a_2 < \frac{1}{3}b + \frac{2}{3}c$, let $b_2 = b_1$ and $c_2 = \frac{2}{3}b_1 + \frac{1}{3}c_1$; else we let $b_2 = \frac{1}{3}b_1 + \frac{2}{3}c_1$ and $c_2 = c_1$. In this way we guaranteed both $a_2 \notin [b_2, c_2]$ and $[b_2, c_2] \in [b_1, c_1]$. Repeat this step and we get countable intervals $[b_i, c_i]$, satifying $a_i \notin [b_i, c_i]$ and $[b_{i+1}, c_{i+1}] \subseteq [b_i, c_i]$. Finally, choose a point $a \in \cap_i [b_i, c_i]$ and we see that $a \notin \{a_i : i \in \mathbb{N}\}$. $\square$

This proof doesn't rely on the axiom of countable choice because we explictly gave a scheme to choose each $[a_i, b_i]$.