Distrubution Function May Not be Left-Continuous
&
$h \in m\Sigma$ iff $h^+, h^- \in m\Sigma$

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Exercise 1 (Distrubution Function May Not be Left-Continuous)**.** *Construct an example to show that the distribution function of a random variable may not be left-continuous.*

Solution. Consider the probability space $(\Omega, \mathcal{F}, P) = ([0,1], B[0,1], Leb)$. And let $X(\omega)$ be a random variable which always takes 0 for any real $\omega \in [0,1]$. Now we try to calculate the distribution function $F_X(c)$ for $c \in \mathbb{R}$. There are three cases:

1. If $c < 0$, then $F_X(c) = P(X \leqslant c) = P(\{\omega | X(\omega) \leqslant c\}) = P(\emptyset) = 0$ since there is no such ω with $X(\omega) < 0$.
2. If $c = 0$, then $F_X(c) = F_X(0) = P(X \leqslant 0) = P(\{\omega | X(\omega) \leqslant 0\}) = P([0,1]) = 1$ since for any $\omega \in [0,1]$, $X(\omega) = 1$.
3. If $c > 0$, then $F_X(c) = P(X \leqslant c) = P(\{\omega | X(\omega) \leqslant c\}) = P([0,1]) = 1$ since for any $\omega \in [0,1]$, $X(\omega) = 1$.

Hence we have

$$F_X(c) = \begin{cases} 0, c < 0, \\ 1, c \geqslant 0. \end{cases}$$

It is clear that $F_X(c)$ is right-continuous but not left-continuous at $c = 0$, which finishes our proof. $\square$

Exercise 2 ($h \in m\Sigma$ iff $h^+, h^- \in m\Sigma$)**.** *Let* (S, Σ) *be a measurable space and take* $h \in \mathbb{R}^S$*. Let* $h^+ = \max(h, 0)$ *and* $h^- = \max(-h, 0)$*. Show that* $h \in m\Sigma$ *if and only if* $h^+, h^- \in m\Sigma$*.*

Proof. We first prove from left to right. To show that $h^+, h^- \in m\Sigma$, we only need to show that $\{h^+ \leqslant c_1\} \in \Sigma$ for $\forall c_1 \in \mathbb{R}$ and $\{h^- \geqslant c_2\} \in \Sigma$ for $\forall c_2 \in \mathbb{R}$. There are two cases for each:

1. If $c_1 < 0$, $\{h^+ \leqslant c_1\} = \emptyset \in \Sigma$ since $h^+ \geqslant 0$ always holds.
2. If $c_1 \geqslant 0$, $\{h^+ \leqslant c_1\} = \{h \leqslant c_1\} \in \Sigma$ since $h \leqslant c_1$ implies $h^+ = \max(h, 0) \leqslant c_1$.
3. If $c_2 < 0$, $\{h^- \geqslant c_2\} = \emptyset \in \Sigma$ since $h^+ \geqslant 0$ always holds, too.
4. If $c_2 \geqslant 0$, $\{h^- \geqslant c_2\} = \{h \geqslant -c_2\} \in \Sigma$ since $h \geqslant -c_2$ implies $h^- = \max(-h, 0) \leqslant c_2$.

Hence we have $\{h^+ \leqslant c_1\} \in \Sigma$ for $\forall c_1 \in \mathbb{R}$ and $\{h^- \geqslant c_2\} \in \Sigma$ for $\forall c_2 \in \mathbb{R}$. So from $h \in m\Sigma$ we can know that $h^+, h^- \in m\Sigma$.

Then we prove from right to left. Observe that:

$$h \geqslant 0 \Rightarrow h^+ = h, h^- = 0, \text{and}$$
$$h < 0 \Rightarrow h^+ = 0, h^- = -h.$$

So $h = h^+ - h^-$. Since $h^+, h^- \in m\Sigma$, we can conclude that $h \in m\Sigma$.

Therefore, $h \in m\Sigma$ if and only if $h^+, h^- \in m\Sigma$. $\square$