A Non-Lebesgue-Measurable Set in $\mathbb{R}^n$

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Our goal here is to construct a non-Lebesgue-measurable set in $\mathbb{R}^n$ for $n \in \mathbb{Z}_+$. I have to say that the desired set constructed here is quite similar to Vitali set on $\mathbb{R}$.

Let's consider the collection of cosets $A := \mathbb{R}^n/\mathbb{Q}^n$. And we limit the elements in A on $[0,1]^n$, that is, we define $B := \{S \cap [0,1]^n : S \in A\}$.

Using axiom of choice, we can obtain a choice function $f : B \to [0,1]^n$, and we define $V := \{f(S) : S \in B\}$. Undoubtedly, $V \subseteq [0,1]^n$.

Furthermore, we construct a new set W by translating V in all directions with some limitations, namely,

$$W := \bigcup_{v \in \mathbb{Q}^n \cap [-1,1]^n} (V + v) \tag{1}$$

Lemma 1. *For $u, v \in \mathbb{Q}^n \cap [-1,1]^n$ with $u \neq v$, $(V+v) \cap (V+u) = \emptyset$.*

Proof of lemma 1. Assume not, say, $z \in (V+v) \cap (V+u)$. Then for some $x, y \in V$, we have

$$x + v = y + u = z$$

which means

$$x - y = u - v.$$

As $u - v \in \mathbb{Q}^n$, we have $x - y \in \mathbb{Q}^n$, which means $x, y \in \mathbb{Q}^n + t$ for some $t \in \mathbb{R}^n$. Therefore, by the definition of V, it is impossible for both x and y to be in V. It contracts our assumption. $\square$

Lemma 2. $[0,1]^n \subseteq W$.

Proof of lemma 2. For each $x \in [0,1]^n$, we know that $x \in \mathbb{Q}^n + t$ for some $t \in \mathbb{R}$ as $\mathbb{Q}^n$ is a subgroup of $\mathbb{R}^n$. Consider the choice on $\mathbb{Q}^n + t$, say, $y = f(\mathbb{Q}^n + t) \in \mathbb{Q}^n + t$. It is clear that $y - x \in \mathbb{Q}^n$. And since $x, y \in [0,1]^n$, $y - x \in \mathbb{Q}^n \cap [-1,1]^n$, which completes the proof. $\square$

Lemma 3. $W \subseteq [-1,2]^n$.

Proof of lemma 3. This is quite obvious as we have $x \in [0,1]^n$ for each $x \in V$, and of course, $x + v \in [-1,2]^n$ since $v \in [-1,1]^n$. $\square$

The following corollary is a direct result by combining lemma 2 and lemma 3.

Corollary 1. $[0,1]^n \subseteq W \subseteq [-1,2]^n$

We assume all elements in $\mathbb{R}^n$ is Lebesgue-measurable. By the properties of measure and corollary 1, we have

$$\mathrm{Leb}([0,1]^n) \leq \mathrm{Leb}(W) \leq \mathrm{Leb}([-1,2]^n) \tag{2}$$

Combining Eq.1 and lemma 1, we have

$$\mathrm{Leb}(W) = \mathrm{Leb}(\bigcup_{v\in\mathbb{Q}^n\cap[-1,1]^n} (V+v)) = \sum_{\mathbb{Q}^n\cap[-1,1]^n} \mathrm{Leb}(V+v)$$

.

According to the translation invariant of Lebesgue measure [1], we know that $\mathrm{Leb}(V+v) = \mathrm{Leb}(V+u)$ for $u, v \in \mathbb{Q}^n \cap [-1,1]^n$.

Now we pick an arbitrary $v \in \mathbb{Q}^n \cap [-1,1]^n$. If $\mathrm{Leb}(V+v) = 0$, then $\mathrm{Leb}(W) = 0$, too. And by Eq.2, we have $\mathrm{Leb}([0,1]^n) = 0$, which is not Lebesgue measure means to do. On the other hand, if $\mathrm{Leb}(V+v) > 0$, it implies $\mathrm{Leb}(W) = \infty$, which means $\mathrm{Leb}([-1,2]^n) = \infty$. This also contradicts the definition of Lebesgue measure.

In conclusion, W is not a Lebesgue-measurable set in $\mathbb{R}^n$.

[1] Lebesgue measure can also be obtained by limiting the Lebesgue outer measure on Lebesgue $\sigma-$algebra, and we can easily tell from the definition of Lebesgue outer measure that it has the property of translation invariant.