Random Variables

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Let (S, Σ) be a measurable space, so that Σ is a σ-algebra on S.

Definitions. Σ-measurable function, $m\Sigma$, $(m\Sigma)^+$, $b\Sigma$

Suppose that $h : S \to \mathbf{R}$. For $A \in \mathbf{R}$, define

$$h^{-1}(A) := s \in S : h(s) \in A$$

Then h is called Σ-*measurable* if $h^{-1} : \mathcal{B} \to \Sigma$, that is, $h^{-1}(A) \in \Sigma, \forall A \in \mathcal{B}$.

Here is a picture of a Σ-measurable function h:

$$\begin{aligned} S &\xrightarrow{\text{h}} \mathbf{R} \\ \Sigma &\xleftarrow{\text{h}^{-1}} \mathcal{B} \end{aligned}$$

$m\Sigma$: the class of Σ-measurable functions on S

$(m\Sigma)^+$: the class of non-negative elements in $m\Sigma$

$b\Sigma$: the class of bounded Σ-measurable functions on S

Borel function

A function h from a topological space S to $\mathbf{R}$ is called **Borel** if h is $\mathcal{B}(S)$-measurable.

The most important case is when S itself is $\mathbf{R}$.

Elementary Propositions on measurability

(a) The map h^{-1} preserves all set operations:

$h^{-1}\left(\cup_\alpha A_\alpha\right) = U_\alpha h^{-1}\left(A_\alpha\right), \quad h^{-1}\left(A^c\right) = \left(h^{-1}(A)\right)^c$, etc.

(b) $\text{If } \mathcal{C} \subseteq \mathcal{B} \text{ and } \sigma(\mathcal{C}) = \mathcal{B}, \text{ then } h^{-1} : \mathcal{C} \to \Sigma \quad \Rightarrow \quad h \in \mathrm{m}\Sigma$

(c) If S is *topological* and $h : S \to \mathbf{R}$ is *continuous*, then h is ***Borel***.

(d) For any measurable space (S, Σ), a function $h : S \to \mathbf{R}$ is Σ-measurable if

$\{h \leq c\} := \{s \in S : h(s) \leq c\} \in \Sigma \quad (\forall c \in \mathbf{R})$

Lemma. Sums and products of measurable functions are measurable

$m\Sigma$ is an algebra over $\mathbf{R}$, that is,

if $\lambda \in \mathbf{R}$ and $h, h_1, h_2 \in m\Sigma$, then

$$h1 + h2 \in m\Sigma, \quad h_1 h_2 \in m\Sigma, \quad \lambda h \in m\Sigma$$

Composition Lemma 复合函数可测性引理

If $h \in m\Sigma$ and $f \in m\mathcal{B}$, then $f \ \circ \ h \in m\Sigma$.

Proof.

$$\begin{array}{c} S \xrightarrow{\text{h}} \mathbf{R} \xrightarrow{\text{f}} \mathbf{R} \\ \Sigma \xleftarrow{\text{h}^{-1}} \mathcal{B} \xleftarrow{\text{f}^{-1}} \mathcal{B} \end{array}$$

Definition. Random Variable

Let $(\Omega, \mathcal{F})$ be our (sample space, family of events). A *random variable* is an element of $m\mathcal{F}$. Thus,

$$X : \Omega \to \mathbf{R}, \quad X^{-1} : \mathcal{B} \to \mathcal{F}$$