A Discussion About Pólya Urn

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The following exercise is E10.1 of the textbook.
At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn and is replaced together with a new ball of the same colour. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n + 1)/(n+2)$, the proportion of black balls in the urn just after time n.

Question 1. *Prove that (relative to a natural filtration which you should specify)M is a martingale.*

Let $\mathcal{F}_n := \sigma(B_i, 0 \le i \le n)$ be the natural filtration, then $M = (M_n : n \ge 0)$ is adapted and integrable. Next we prove that $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}, \text{a.s.}(n \ge 1)$.

$$\begin{aligned}
& \mathbb{E}\left[M_n|\mathcal{F}_{n-1}\right] \\
= & \mathbb{E}\left[\tfrac{B_n+1}{n+2}\middle|\mathcal{F}_{n-1}\right] \\
= & \tfrac{1}{n+2}\mathbb{E}\left[B_n+1\middle|\mathcal{F}_{n-1}\right] \\
= & \tfrac{1}{n+2}\mathbb{E}\left[B_{n-1}+\tfrac{B_{n-1}+1}{n+1}+1\middle|\mathcal{F}_{n-1}\right] \\
= & \mathbb{E}\left[\tfrac{B_{n-1}+1}{(n-1)+2}\middle|\mathcal{F}_{n-1}\right] \\
= & M_{n-1}
\end{aligned}$$

Therefore, M is a martingale relative to $\{\{\mathcal{F}_n\}, \mathbf{P}\}$.

Question 2. *Prove that* $\mathbf{P}(B_n = k) = (n+1)^{-1}$ *for* $0 \le k \le n$.

$$\begin{aligned}
& \mathbf{P}(B_n = k) \\
= & \tbinom{n}{k}\tfrac{(1\times\cdots\times k)\times(1\times\cdots\times(n-k))}{2\times 3\times\cdots\times(n+1)} \\
= & \tfrac{n!}{k!(n-k)!}\cdot\tfrac{k!(n-k)!}{(n+1)!} \\
= & \tfrac{1}{n+1}
\end{aligned}$$

Question 3. *What is the distribution of* Θ*, where* $\Theta := \lim M_n$?

We have known that $P(B_n = k) = \frac{1}{n+1}$, so M_n is uniform in $\{\frac{1}{n+2}, \frac{2}{n+2}, \cdots, \frac{n+2}{n+2}\}$. Thus we have

$$\begin{aligned} & P(\Theta \le t)(t \in [0,1]) \\ = \; & P(\lim_{n\to\infty} M_n \le t) \\ = \; & \lim_{n\to\infty} P(M_n \le t) \\ = \; & t \end{aligned}$$

Therefore, Θ satisfies the uniform distribution.