Solution to Exercises of March 13

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March 15, 2020

Exercise 6 (Union of countable infinite intervals)**.** *1) Let* $(I_i)_{i=0}^{\infty}$ *be a sequence of intervals such that* $\bigcup_{i\in\mathbb{N}} I_i \supseteq I_0$*. Show that*

$$\sum_{i\in\mathbb{N}} |I_i| \geqslant |I_0|$$

.

2) Use 1) to derive again Cantor's Theorem: $[0,1]$ *is uncountable.*

Proof. 1) Let $U_i := \bigcup_{j=1}^{i} I_j$ for $i = 1, 2, ...$ and let $U_0 = \emptyset$. By definition, we have

$$\begin{aligned} U_i &= \bigcup_{j=1}^{i} I_j \\ &= \biguplus_{j=1}^{i}(I_j \backslash (\bigcup_{k=1}^{j-1} I_k)) \\ &= \biguplus_{j=1}^{i}(I_j \backslash U_{j-1}) \end{aligned}$$

where operator $\biguplus$ gets the union of sets which are pairwisely disjoint. Since $\bigcup_{i\in\mathbb{N}} I_i \supseteq I_0$, we have

$$\begin{aligned} |I_0| &\leqslant |\bigcup_{i\in\mathbb{N}} I_i| \\ &= |U_\infty| \\ &= |\biguplus_{i\in\mathbb{N}}(I_i \backslash U_{i-1})| \\ &= \sum_{i=1}^{\infty} |I_i \backslash U_{i-1}| \end{aligned}$$

Obviously, $|U_i| \geqslant 0$. Therefore,

$$\begin{aligned} |I_0| &\leqslant \sum_{i=1}^{\infty} |I_i \backslash U_{i-1}| \\ &\leqslant \sum_{i=1}^{\infty} |I_i| \end{aligned}$$

2) We prove by contradiction.

Assume $I_0 := [0,1]$ is countable. Let $\{a_i\}$ be a sequence of reals in $[0,1]$ where index i starts from 1. For each a_i, let $I_i := [a_i - \frac{1}{2^{i+2}}, a_i + \frac{1}{2^{i+2}}]$. Since for every real number $x \in [0,1]$ there exists a subscript k with $a_k = x$, we have $x \in I_k$. It follows that $\bigcup_{i=1}^{\infty} I_i \supseteq I_0$. By 1) we know that $\sum_{i=1}^{\infty} |I_i| \geqslant |I_0|$.

However, by definition $|I_i| = (a_i + \frac{1}{2^{i+2}}) - (a_i - \frac{1}{2^{i+2}}) = \frac{1}{2^{i+1}}$. Sum up all $|I_i|$, and we get

$$\sum_{i=1}^{\infty} |I_i| = \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2} < 1 = |I_0|$$

. This leads to a contradiction.

Therefore, $[0,1]$ is uncountable. $\square$

Exercise 9 (Baby Vitali Lemma). *Let $A = \{I_1, \dots, I_n\}$ be a family of finite intervals in the real line. Show that there exists a set B of disjoint intervals such that $B \subseteq A$ and $\ell(\bigcup_{i \in B} I) \geqslant \frac{1}{4}\ell(\bigcup_{i \in A} I)$, where ℓ is the usual length function.*

Proof. I write down the proof after referring to some related materials.

Let $N := \{1, 2, \dots, n\}$ be the set of indeces. We construct the set of pairwisely disjoint intervals B each time by choosing an interval whose length is as large as possible. More precisely, let B be a sequence of intervals which is $\emptyset$ in the beginning. In the i-th turn, let $k_i := \arg\max_{j \in N}\{\ell(I_j) | I_j \bigcap I_{k_t} = \emptyset, t = 1, 2, \dots, i-1\}$(if there are multiple choices, choose any). And then we add I_{k_i} to the set B. Repeat choosing intervals from A until no such interval satisfy the condition. Let m be the number of intervals we chose from A and let $M := \{1, 2, \dots, m\}$. Finally we get the set $B = \{I_{k_1}, I_{k_2}, \dots, I_{k_m}\}$.

By construction, it is obvious that the intervals in B are pairwisely disjoint, i.e., for any pair (i,j) with $1 \leqslant i, j \leqslant m, i \neq j$ we have $I_{k_i} \bigcap I_{k_j} = \emptyset$.

Claim that B is what we want, i.e., $\ell(\bigcup_{i \in B} I) \geqslant \frac{1}{4}\ell(\bigcup_{i \in A} I)$.

Now we prove the claim.

Note that for any $i \in N$, there exists a $j \in M$ with $I_i \bigcap I_{k_j} \neq \emptyset$. Otherwise the process of choosing intervals wouldn't stop, which means there are still one or more intervals which can be added to B.

Furthermore, for any $i \in N$, there exists a $j \in M$ with $I_i \bigcap I_{k_j} \neq \emptyset$ and $\ell(I_i) \leqslant \ell(I_{k_j})$. If not(i.e., for every j with $I_i \bigcap I_{k_j} \neq \emptyset$, $\ell(I_i) > \ell(I_{k_j})$ holds), I_i should be added to B before all I_{k_j}s which have nonempty intersection with I_i were added to B according to the construction.

Now, for each $i \in N$, choose any $j \in M$ such that $I_i \bigcap I_{k_j} \neq \emptyset$ and $\ell(I_i) \leqslant \ell(I_{k_j})$. Suppose $I_{k_j} = [a_{k_j}, b_{k_j}]$. We expand the interval I_{k_j} fourfold in length to $I'_{k_j} := [\frac{1}{2}(5a_{k_j} - 3b_{k_j}), \frac{1}{2}(5b_{k_j} - 3a_{k_j})]$. Since $I_i \bigcap I_{k_j} \neq \emptyset$ and $\ell(I_i) \leqslant \ell(I_{k_j})$, it is obvious that $I_i \subseteq I'_{k_j}$.

Hence we have $(\bigcup_{I \in A} I) \subseteq \bigcup_{i=1}^{m} I'_{k_i}$, and thus:

$$\begin{aligned}
\ell(\bigcup_{I \in A} I) &\leqslant \ell(\bigcup_{i=1}^{m} I'_{k_i}) \\
&\leqslant \sum_{i=1}^{m} \ell(I'_{k_i}) = 4\sum_{i=1}^{m} \ell(I_{k_i}) = 4\ell(\bigcup_{i=1}^{m} I_{k_i}) = 4\ell(\bigcup_{I \in B} I)
\end{aligned}$$

Equivalently, we have $\ell(\bigcup_{i \in B} I) \geqslant \frac{1}{4}\ell(\bigcup_{i \in A} I)$. $\square$

Remark of Exercise 9. The conclusion of this lemma could be stronger, since in the last but two paragraph of the proof above, if we expand I_{k_j} threefold to $I''_{k_j} := [2a_{k_j} - b_{k_j}, 2b_{k_j} - a_{k_j}]$, I''_{k_j} also covers I_i.

So we can get a stronger version: there exists a set B of disjoint intervals such that $B \subseteq A$ and $\ell(\bigcup_{i \in B} I) \geqslant \frac{1}{3}\ell(\bigcup_{i \in A} I)$.