Fair Dice

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1   Problem

骸子的六个面上分别有数字 1,...,6。假设你有办法对骸子内部添东西使得抛掷后各个面出现的概率分布为给定分布。是否可以造出两个骸子，同时抛出后得到的两个数之和在 2,...,12 之间等概率分布？

2   Solution

We have utilizfe the tool of Generative Function to represent the distribution of each roll.

$$P(x) = p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5 + p_6 x^6$$

Here, q_i represent the probability of the dice roll being i.

With 2 dices, we can simply multiply two generative functions.

$$
\begin{aligned}
P(x) &= p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5 + p_6 x^6 \\
Q(x) &= q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 + q_5 x^5 + q_6 x^6 \\
\therefore T(x) &= P(x)Q(x) \\
&= (p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5 + p_6 x^6)(q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 + q_5 x^5 + q_6 x^6) \\
&= t_2 x^2 + ... + t_{12} x^{12}
\end{aligned}
$$

The problem asks for a equal distribution between 2 and 12. Thus, we have the following con-

straints:

$$t_2 = t_3 = \dots = t_{12} = \frac{1}{11} \tag{1}$$

$$\sum_{i=1}^{6} p_i = 1 \tag{2}$$

$$\sum_{i=1}^{6} q_i = 1 \tag{3}$$

The is a set of equations. So I tried using mathematica to solve it but it only yields complex number answers, which is not accpetable.

```
Solve[
解方程
 p1 * q1 == 1 / 11 && p1 * q2 + p2 * q1 == 1 / 11 && p1 * q3 + p2 * q2 + p3 * q1 == 1 / 11 &&
  p1 * q4 + p2 * q3 + p3 * q2 + p4 * q1 == 1 / 11 && p1 * q5 + p2 * q4 + p3 * q3 + p4 * q2 + p5 * q1 == 1 / 11 &&
  p1 * q6 + p2 * q5 + p3 * q4 + p4 * q3 + p5 * q2 + p6 * q1 == 1 / 11 &&
  p2 * q6 + p3 * q5 + p4 * q4 + p5 * q3 + p6 * q2 == 1 / 11 && p3 * q6 + p4 * q5 + p5 * q4 + p6 * q3 == 1 / 11 &&
  p4 * q6 + p5 * q5 + p6 * q4 == 1 / 11 && p5 * q6 + p6 * q5 == 1 / 11 &&
  p1 + p2 + p3 + p4 + p5 + p6 |== 1 && q1 + q2 + q3 + q4 + q5 + q6 == 1,
 {p1, p2, p3, p4, p5, p6, q1, q2, q3, q4, q5, q6}]
```

图 1: Mathematica sucks

So we have to further consider these constrains.

$$P(x)Q(x) = \frac{1}{11} * (x^2 + \dots + x^{12})$$
$$= \frac{1}{11} * x^2 * \frac{x^{11} - 1}{x - 1}$$

The problem is really asking if there are any factorization of $x^2 * \frac{x^{11}-1}{x-1} \in \mathbb{R}[x]$.

But there is none since the only root $x \in \mathbb{R}$ is 1. This result is confirmed using Mathematica, see *factor.png* in the folder.

3   Conclusion

I used the above method to show that no solution exists. Futher explorations still interest me whether there is a pattern of possible distribution feasible under this rule.