Traditional Approaches to Machine Learning

A brief overview of different Machine Learning Algorithms

Tilman Roeder

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Today

Naive Bayes Classifiers

Support Vector Machines

Decision Trees

Section 1

Naive Bayes Classifiers

(Super Brief) Primer in Bayesian Statistic

Define conditional probabilities

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In Bayesian Statistics probabilities quantify our 'degree of believe'.
Bayes Theorem allows us to 'update' our believes (prior) based on new evidence (data) to obtain a new believe (posterior).

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- We seek $P(X|Y)$
- Apply Bayes Theorem to compute posterior probability
- Conditional probabilities are based on simple (naive) heuristics

Example: Spam Filtering

Given a message M consisting of words $(W_1, W_2, \ldots, W_n)$, and assuming words occurrences are independent we have:

$$P(S|W) = \frac{P(W|S)P(S)}{P(W)} = \frac{P(S)\prod_i P(W_i|S)}{\sum_{s\in\{S,\neg S\}} P(s)\prod_i P(W_i|s)}, \quad (3)$$

where $P(S)$ is the probability of the message being spam.

Example: Spam Filtering

Note that the previous expression is prone to numeric instability. We can fix this by computing the probabilities in *log* space:

$$\ln\left(\frac{1}{P(S|W)} - 1\right) = \sum_i \ln\left(P(W_i|\neg S)\right) - \ln\left(P(W_i|S)\right) \\ + \ln(P(\neg S)) - \ln(P(S)) \tag{4}$$

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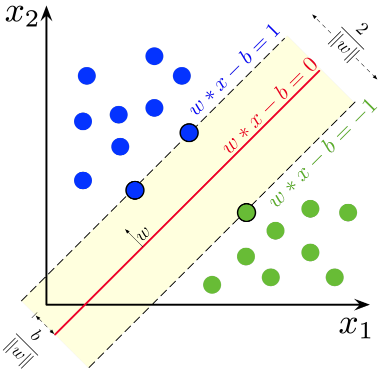
These are very cheap to compute, and one of the oldest spam filters. Bayesian Classifiers are not as good as more advanced techniques. But for applications where speed and simplicity are important, they can be very competitive.

Section 2

Support Vector Machines

Linear Separable Data



(graphic by ZackWeinberg)

- SVMs classify data with binary labels
- Classification is achieved by drawing a hyperplane between the data

Support Vector Machine Classifier

A hyperplane P in $\mathbb{R}^n$ is defined by

$$\vec{\omega} \cdot \vec{x} - b = 0, \tag{5}$$

with $\vec{\omega}, \vec{x} \in \mathbb{R}^n$, $b \in \mathbb{R}$ and where $\cdot$ denotes an inner product.

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Note that the classifier has $n + 1$ free parameters.

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Notice that the parameters $\vec{\omega}, b$ are completely determined by those $\vec{x}_i$ which are closest to P. These are called the 'support vectors'.

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We can use a soft-margin classifier instead. This is done by modifying the loss function:

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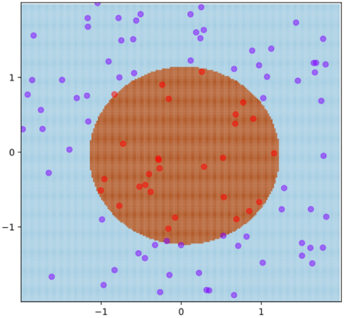
We now find $\vec{\omega}, b$ by minimizing L.

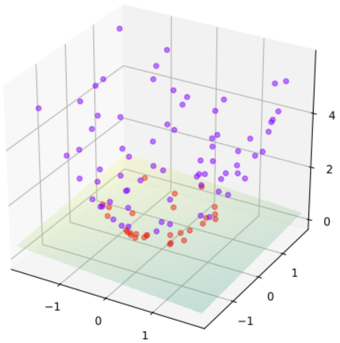
The Kernel Trick



(graphic by Shiyu Ji)

If the data is not linearly separable, a soft-margin SVM performs poorly.

The Kernel Trick



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However, there may be a map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ to a higher dimensional space $\mathbb{R}^{m>n}$, in which the data is linearly separable.

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We further define $k : \mathbb{R}^2 \to \mathbb{R}$ such that

$$k(\vec{x}, \vec{y}) = \phi(\vec{x}) \cdot \phi(\vec{y}) = \vec{x} \cdot \vec{y} + |\vec{x}|^2 |\vec{y}|^2. \tag{12}$$

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k is called the 'Kernel' and allows us to easily compute inner products, without having to explicitly map $\vec{x}_i$ to the higher dimensional space.

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The answer is as follows: assume $\vec{\omega}$ lies in the span of $\phi(\vec{x}_i)$, then we may write

$$\vec{\omega} \cdot \phi(\vec{x}) = \sum_i c_i y_i \phi(\vec{x}_i) \cdot \phi(\vec{x}), \tag{13}$$

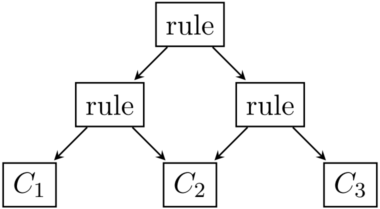
with $c_i \in \mathbb{R}$ and where y_i is included for mathematical convenience.

Section 3

Decision Trees

What is a Decision Tree

A decision tree is a **directed**, **acyclic graph**. The nodes in the graph represent decision points and all paths lead to a classification $C_i \in C$.



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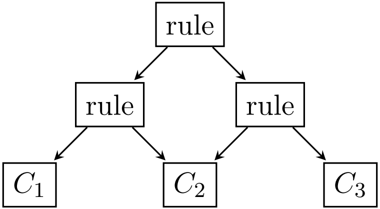

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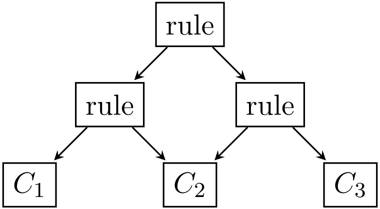

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Note that the decision nodes can use simple rules (like $\vec{x}_i > \vec{y}_i$ for some i), or more complex rules (e.g. a SVM or Bayes Filter).

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- Write the rules by hand
- Select a set of parameterized rules, and construct tree by choosing best separation achieved at each node

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In practice, random forests are good and fast predictors, that are competitive with neural networks (depending on specific task).