

## Numerical Analysis MATH50003 (2025–26) Problem Sheet 1

In the lectures/notes we saw some basic *analysis* of the errors of simple integration and differentiation rules: the right-sided rectangular rule for integration and divided difference for differentiation. This problem sheets explores the analysis of some other simple rules that may have *better* convergence rates: they can calculate integrals/derivatives more accurately with the same amount of data as the rules discussed in lectures. In the lab the practical implementation of these methods is explored.

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The first example is the left-sided rectangular rule, which has the same linear ( $O(h)$ ) convergence rate as the right-sided rectangular rule:

**Problem 1** Assuming  $f$  is differentiable on  $[a, b]$  and its derivative is integrable, prove the left-point Rectangular rule error formula

$$\int_a^b f(x)dx = h \sum_{j=0}^{n-1} f(x_j) + \delta$$

where  $|\delta| \leq M(b-a)h$  for  $M = \sup_{a \leq x \leq b} |f'(x)|$ ,  $h = (b-a)/n$  and  $x_j = a + jh$ .

We now turn our attention to the Trapezium rule, which will have a faster quadratic convergence rate using the same number of samples. That is to say: we can get much more accurate results with the exact same amount of work! We begin with a simple one-panel error result showing cubic ( $O(h^3)$ ) error:

**Problem 2(a)** Assuming  $f$  is twice-differentiable on  $[a, b]$  and its second derivative is integrable, prove a one-panel Trapezium rule error bound:

$$\int_a^b f(x)dx = (b-a) \frac{f(a) + f(b)}{2} + \delta$$

where  $|\delta| \leq M(b-a)^3$  for  $M = \sup_{a \leq x \leq b} |f''(x)|$ .

*Hint:* Recall from the notes

$$\int_a^b \frac{(b-x)f(a) + (x-a)f(b)}{b-a} dx = (b-a) \frac{f(a) + f(b)}{2}$$

and you may need to use Taylor's theorem. Note that the bound is not sharp and so you may arrive at something sharper like  $|\delta| \leq 3(b-a)^3 M/4$ . The sharpest bound is  $|\delta| \leq (b-a)^3 M/12$  but that would be a significantly harder challenge to show!

We can use the previous problem to deduce the *global* error, summing up over all panels. This shows quadratic ( $O(h^2)$ ) error.

**Problem 2(b)** Assuming  $f$  is twice-differentiable on  $[a, b]$  and its second derivative is integrable, prove a bound for the Trapezium rule error:

$$\int_a^b f(x)dx = h \left[ \frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_j) + \frac{f(b)}{2} \right] + \delta$$

where  $|\delta| \leq M(b-a)h^2$  for  $M = \sup_{a \leq x \leq b} |f''(x)|$ .

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We now turn our attention to differentiation, with the first example being a left-sided divided

difference, which has a linear ( $O(h)$ ) error just like the right-sided divided difference in lectures:

**Problem 3** Assuming  $f$  is twice-differentiable in  $[x-h, x]$ , for the left difference approximation

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \delta,$$

show that  $|\delta| \leq Mh/2$  for  $M = \sup_{x-h \leq t \leq x} |f''(t)|$ .

The next example shows that with a more careful choice of sample points, we can obtain a quadratic ( $O(h^2)$ ) error:

**Problem 4** Assuming  $f$  is thrice-differentiable in  $[x-h, x+h]$ , for the central differences approximation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \delta,$$

show that  $|\delta| \leq Mh^2/6$  for  $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$ .

By applying a central difference discretisation twice we can obtain an approximation to the second derivative. The following problem asks to prove that the approximation converges with a linear ( $O(h)$ ) error:

**Problem 5** Assuming  $f$  is thrice-differentiable in  $[x-h, x+h]$ , for the second-order derivative approximation

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \delta$$

show that  $|\delta| \leq Mh/3$  for  $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$ .

Note in these problems the computational complexity is independent of  $h$ . However, if  $h$  is too small the practical implementation has large errors.