Numerical Analysis MATH50003 (2024–25) Problem Sheet 7

Problem 1 Use Lagrange interpolation to interpolate the function $\cos x$ by a polynomial at the points $[0, 2, 3, 4]$ and evaluate at $x = 1$.

Problem 2 Compute the LU factorisation of the following transposed Vandermonde matrices:

$$\begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ x^3 & y^3 & z^3 & t^3 \end{bmatrix}$$

Can you spot a pattern? Test your conjecture with a 5×5 Vandermonde matrix.

Problem 3 Compute the interpolatory quadrature rule

$$\int_{-1}^{1} f(x)w(x)\mathrm{d}x \approx \sum_{j=1}^{n} w_j f(x_j)$$

for the points $[x_1, x_2, x_3] = [-1, 1/2, 1]$, for the weights $w(x) = 1$ and $w(x) = \sqrt{1 - x^2}$.

This problem sheet concerns singular value decompositions, pseudo-inverses, and condition numbers.

For the SVD $A = U\Sigma V^\top \in \mathbb{R}^{m\times n}$, where $U \in \mathbb{R}^{m\times r}$ and $V \in \mathbb{R}^{n\times r}$ have orthonormal columns and Σ is a diagonal matrix with non-increasing positive entries, define the *pseudo-inverse*:

$$A^+ := V\Sigma^{-1}U^\top.$$

Problem 4(a) Show that A^+ satisfies the *Moore-Penrose conditions*:

1. $$AA^+A = A$$
2. $$A^+AA^+ = A^+$$
3. $$(AA^+)^\top = AA^+$$

   and $(A^+A)^\top = A^+A$

Problem 4(b) Show for $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and full rank that $\boldsymbol{x} = A^+\boldsymbol{b}$ is the least squares solution, i.e., minimises $\|A\boldsymbol{x} - \boldsymbol{b}\|_2$. Hint: extend U in the SVD to be a square orthogonal matrix.

Problem 4(c) If $A \in \mathbb{R}^{m\times n}$ has a non-empty kernel there are multiple solutions to the least squares problem as we can add any element of the kernel. Show that $\boldsymbol{x} = A^+\boldsymbol{b}$ gives the least squares solution such that $\|\boldsymbol{x}\|_2$ is minimised.
