MATH50003
Numerical Analysis

II.3 Floating Point Arithmetic

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Part II

Representing Numbers

1. Reals via floating point
2. Floating point arithmetic and bounding errors
3. Interval arithmetic for rigorous computations

Rounding

How does a computer round a real to a float?

Definition 7 (rounding)**.** $\mathrm{fl}^{\mathrm{up}}_{\sigma,Q,S} : \mathbb{R} \to F_{\sigma,Q,S}$

$$\mathrm{fl}^{\mathrm{down}}_{\sigma,Q,S} : \mathbb{R} \to F_{\sigma,Q,S}$$

$$\mathrm{fl}^{\mathrm{nearest}}_{\sigma,Q,S} : \mathbb{R} \to F_{\sigma,Q,S}$$

Arithmetic

Operations are exact up to rounding

$$
\begin{aligned}
x \oplus y &:= \mathrm{fl}(x + y) \\
x \ominus y &:= \mathrm{fl}(x - y) \\
x \otimes y &:= \mathrm{fl}(x * y) \\
x \oslash y &:= \mathrm{fl}(x / y)
\end{aligned}
$$

Example 8 (decimal is not exact)**.**

II.2.1 Bounding errors

Analysis on rounding errors

Definition 8 (machine epsilon/smallest positive normal number/largest normal number)**.**
Machine epsilon is denoted

$$\epsilon_{\mathrm{m},S} := 2^{-S}.$$

Definition 9 (normalised range)**.** The *normalised range* $\mathcal{N}_{\sigma,Q,S} \subset \mathbb{R}$ is the subset of real numbers that lies between the smallest and largest normal floating-point number:

$$\mathcal{N}_{\sigma,Q,S} := \{x : \min |F^{\text{normal}}_{\sigma,Q,S}| \leq |x| \leq \max F^{\text{normal}}_{\sigma,Q,S}\}$$

Proposition 2 (round bound)**.** *If* $x \in \mathcal{N}$ *then*

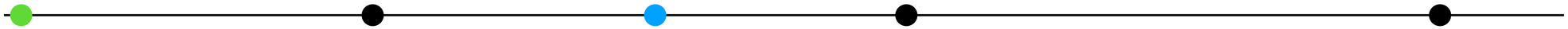
$$\text{fl}^{\text{mode}}(x) = x(1 + \delta_x^{\text{mode}})$$

where the relative error *is bounded by:*

$$|\delta_x^{\text{nearest}}| \leq \frac{\epsilon_{\text{m}}}{2}$$
$$|\delta_x^{\text{up/down}}| < \epsilon_{\text{m}}.$$

(Round Down)

$$x = 2^{q-\sigma}(1.b_1 b_2 \ldots b_S b_{S+1} \ldots)_2$$



$2^{q-\sigma}$

$x_- := \mathrm{fl}^{\mathrm{down}}(x)$

$= 2^{q-\sigma}(1.b_1 b_2 \ldots b_S)_2$

$x_{\mathrm{h}} := \dfrac{x_+ + x_-}{2}$

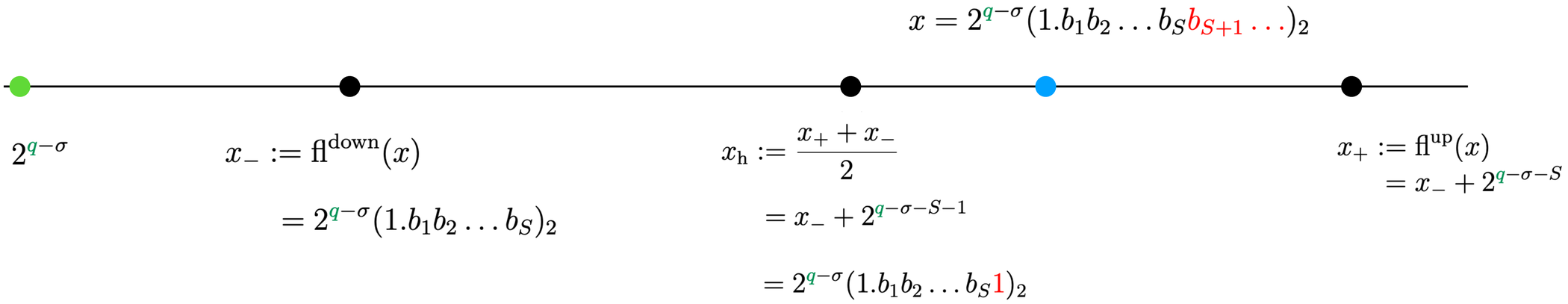
$= x_- + 2^{q-\sigma-S-1}$

$= 2^{q-\sigma}(1.b_1 b_2 \ldots b_S 1)_2$

$x_+ := \mathrm{fl}^{\mathrm{up}}(x)$

$= x_- + 2^{q-\sigma-S}$

(Round Up)


$x = 2^{q-\sigma}(1.b_1b_2\ldots b_S b_{S+1}\ldots)_2$
$2^{q-\sigma}$
$x_- := \mathrm{fl}^{\mathrm{down}}(x)$
$= 2^{q-\sigma}(1.b_1b_2\ldots b_S)_2$
$x_{\mathrm{h}} := \dfrac{x_+ + x_-}{2}$
$= x_- + 2^{q-\sigma-S-1}$
$= 2^{q-\sigma}(1.b_1b_2\ldots b_S 1)_2$
$x_+ := \mathrm{fl}^{\mathrm{up}}(x)$
$= x_- + 2^{q-\sigma-S}$

Example 9 (bounding a simple computation)**.**

II.2.2 Idealised floating point

A simplifed model for analysis

Definition 10 (idealised floating point)**.** An idealised mathematical model of floating point numbers for which the only subnormal number is zero can be defined as:

$$F_{\infty,S} := \{\pm 2^q \times (1.b_1 b_2 b_3 \dots b_S)_2 : q \in \mathbb{Z}\} \cup \{0\}$$

II.2.3 Divided differences floating point error bound

Explain the unexplained error in divided differences

General model of a function implemented in floating point:

$$f(x) = f^{\mathrm{FP}}(x) + \delta_x^f$$

such that

$$|\delta_x^f| \leq c\epsilon_{\mathrm{m}}$$

Theorem 4 (divided difference error bound)**.**

$$\frac{f^{\mathrm{FP}}(x+h) \ominus f^{\mathrm{FP}}(x)}{h} = f'(x) + \delta_{x,h}^{\mathrm{FD}}$$

where

$$|\delta_{x,h}^{\mathrm{FD}}| \leq \frac{|f'(x)|}{2}\epsilon_{\mathrm{m}} + Mh + \frac{4c\epsilon_{\mathrm{m}}}{h}$$

for $M = \sup_{x \leq t \leq x+h} |f''(t)|$.

Corollary 2 (divided differences in practice)**.** *We have*

$$(f^{\mathrm{FP}}(x \oplus h) \ominus f^{\mathrm{FP}}(x)) \oslash h = \frac{f^{\mathrm{FP}}(x+h) \ominus f^{\mathrm{FP}}(x)}{h}$$

whenever $h = 2^{j-n}$ *for* $0 \leq n \leq S$ *and the last binary place of* $x \in F_{\infty,S}$ *is zero, that is* $x = \pm 2^j (1.b_1 \ldots b_{S-1}0)_2$.

Heuristic (divided difference with floating-point step)

Now to Lab 3
To see rounding modes.