Numerical Analysis MATH50003 (2024–25) Problem Sheet 5

Problem 1 Compute the LU decomposition of the following matrices:

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Problem 2 Compute the PLU decomposition for the following matrices:

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 1 \\ -3 & -5 & 6 & 1 \\ -1 & 2 & 8 & -2 \end{bmatrix}$$

Problem 3 Prove that every invertible matrix as a UPL factorisation

$$A = UPL$$

where P is a permutation matrix, U is upper triangular, and L is lower triangular.

Problem 1 By computing the Cholesky factorisation, determine which of the following matrices are symmetric positive definite:

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

Problem 2 Show that a matrix $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite if and only if it has a *reverse* Cholesky factorisation of the form

$$A = UU^\top$$

where U is upper triangular with positive entries on the diagonal.

Problem 3(a) Use the Cholesky factorisation to prove that the following $n \times n$ matrix is symmetric positive definite for any n:

$$\Delta_n := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Hint: consider a matrix $K_n^{(\alpha)}$ that equals Δ_n apart from the top left entry which is $\alpha > 1$ and use a proof by induction.

Problem 3(b) Deduce its Cholesky factorisations: $\Delta_n = L_n L_n^\top$ where L_n is lower triangular.