MATH50003 Numerical Analysis

II.3 Interval Arithmetic

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Chapter II

Representing Numbers

1. Reals via floating point
2. Floating point arithmetic and bounding errors
3. Interval arithmetic for rigorous computations

II.4 Interval Arithmetic

Use set operations with rounding to prove rigorous bounds

For sets $X, Y \subseteq \mathbb{R}$ consider the set operations

$$
\begin{aligned}
X + Y &:= \{x + y : x \in X, y \in Y\}, \\
XY &:= \{xy : x \in X, y \in Y\}, \\
X/Y &:= \{x/y : x \in X, y \in Y\}
\end{aligned}
$$

We will use floating point arithmetic to define operations so that

$$
\begin{aligned}
X + Y &\subseteq X \oplus Y, \\
XY &\subseteq X \otimes Y, \\
X/Y &\subseteq X \oslash Y
\end{aligned}
$$

Proposition 3 (interval bounds). *For intervals $X = [a, b]$ and $Y = [c, d]$ satisfying $0 < a \leq b$ and $0 < c \leq d$, and $n > 0$, we have:*

$$\begin{aligned} X + Y &= [a + c, b + d] \\ X/n &= [a/n, b/n] \\ XY &= [ac, bd] \end{aligned}$$

Definition 14 (floating point interval arithmetic)**.** For intervals $A = [a, b]$ and $B = [c, d]$ satisfying $0 < a \leq b$ and $0 < c \leq d$, and $n > 0$, define:

$$\begin{aligned}
[a, b] \oplus [c, d] &:= [\mathrm{fl}^{\mathrm{down}}(a + c), \mathrm{fl}^{\mathrm{up}}(b + d)] \\
[a, b] \ominus [c, d] &:= [\mathrm{fl}^{\mathrm{down}}(a - d), \mathrm{fl}^{\mathrm{up}}(b - c)] \\
[a, b] \oslash n &:= [\mathrm{fl}^{\mathrm{down}}(a/n), \mathrm{fl}^{\mathrm{up}}(b/n)] \\
[a, b] \otimes [c, d] &:= [\mathrm{fl}^{\mathrm{down}}(ac), \mathrm{fl}^{\mathrm{up}}(bd)]
\end{aligned}$$

Example 10 (small sum).

Example 11 (exponential with intervals).

$$\exp(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + \underbrace{\exp(t)\frac{x^{n+1}}{(n+1)!}}_{\delta_{x,n}}$$

$$\exp(X) \subseteq \left(\bigoplus_{k=0}^{n} X \bigcirc\!\!\!\!\wedge\; k \oslash k! \right) \oplus \left[\mathrm{fl}^{\mathrm{down}}\left(-\frac{3}{(n+1)!} \right), \mathrm{fl}^{\mathrm{up}}\left(\frac{3}{(n+1)!} \right) \right]$$

Let’s implement Interval arithmetic in Lab 4.