Numerical Analysis MATH50003 (2023–24) Problem Sheet 2

Problem 1 Using dual number arithmetic, compute the following polynomials evaluated at the dual number $2+\epsilon$ and use this to deduce their derivative at 2:

$$2x^2 + 3x + 4, (x+1)(x+2)(x+3), (2x+1)x^3$$

Problem 2 What should the following functions applied to dual numbers return for $x = a+b\epsilon$:

$$f(x) = x^{100} + 1, g(x) = 1/x, h(x) = \tan x$$

State the domain where these definitions are valid.

Problem 3(a) What is the correct definition of division on dual numbers, i.e.,

$$(a+b\epsilon)/(c+d\epsilon) = s + t\epsilon$$

for what choice of s and t?

Problem 3(b) A *field* is a commutative ring such that $0 \neq 1$ and all nonzero elements have a multiplicative inverse, i.e., there exists a^{-1} such that $aa^{-1} = 1$. Can we use the previous part to define $a^{-1} := 1/a$ to make $\mathbb{D}$ a field? Why or why not?

Problem 4 Use dual numbers to compute the derivative of the following functions at $x = 0.1$:

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^{3} \left(\frac{x}{k} - 1\right), \text{ and } f_2^{\mathrm{s}}(x) = 1 + \frac{x-1}{2 + \frac{x-1}{2}}$$

Consider a 2D analogue of dual numbers $a + b\epsilon_x + c\epsilon_y$ defined by the relationship $\epsilon_x\epsilon_y = \epsilon_x^2 = \epsilon_y^2 = 0$.

Problem 5(a) Derive the formula for writing the product of two 2D dual numbers $(a + a_x\epsilon_x + a_y\epsilon_y)(b + b_x\epsilon_x + b_y\epsilon_y)$ where $a, a_x, a_y, b, b_x, b_y \in \mathbb{R}$ as a 2D dual number.

Problem 5(b) Show for all 2D polynomials

$$p(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} c_{kj} x^k y^j$$

that

$$p(x + a\epsilon_x, y + b\epsilon_y) = p(x,y) + a\frac{\partial p}{\partial x}\epsilon_x + b\frac{\partial p}{\partial y}\epsilon_y.$$

Problem 5(c) Use 2D dual numbers to compute the gradient of $p(x,y) = (1+x+3xy)(1+y)$ at $x = 1$ and $y = 2$.

Problem 6 Suppose f is twice-differentiable, $f(x) = 0$, $f'(x) = 0$, but f'' does not vanish in a neighbourhood of B. Show that the error of the k-th Newton iteration $\varepsilon_k := x - x_k$ satisfies

$$|\varepsilon_{k+1}| \leq \tilde{M}|\varepsilon_k|$$

where

$$M \tilde{=} \frac{1}{2}\sup_{y\in B}|f''(y)| \sup_{y \in B}\frac{1}{|f''(y)|}.$$