Numerical Analysis MATH50003 (2025–26) Problem Sheet 6

We explore simple rotations, reflections, and the properties of orthogonal/unitary matrices. We also see how reflections can be used to introduce zeros into a matrix. This leads to an algorithm for computing the QR factorisation of a matrix.

We begin with some simple examples of 2×2 rotations:

Problem 1 What simple rotation matrices $Q_1, Q_2 \in SO(2)$ have the property that:

$$Q_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}, Q_2 \begin{bmatrix} \sqrt{5} \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

We now look for specific examples of Householder reflections. These have the important property that they introduce zeros below the top entry, a feature that will be used in the development of an algorithm for the QR factorisation:

Problem 2 Consider the vectors

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2\mathrm{i} \\ 2 \end{bmatrix}.$$

Use reflections to determine the entries of orthogonal/unitary matrices Q_1, Q_2, Q_3 such that

$$Q_1 \boldsymbol{a} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, Q_2 \boldsymbol{a} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, Q_3 \boldsymbol{b} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

In the notes we discussed properties of orthogonal/unitary matrices. Here we prove these properties, which will be used in subsequent proofs. These problems also introduce an important concept of a normal matrix: one which commutes with its adjoint. Both symmetric and orthogonal/unitary matrices are normal. Note that normal matrices are diagonalisable via unitary matrices (an example of *the spectral theorem*).

Problem 3(a) Show for a unitary matrix $Q \in U(n)$ and a vector $\boldsymbol{x} \in \mathbb{C}^n$ that multiplication by Q preserve the 2-norm: $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$.

Problem 3(b) Show that the eigenvalues λ of a unitary matrix Q are on the unit circle: $|\lambda| = 1$. Hint: recall for any eigenvalue λ that there exists a unit eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ (satisfying $\|\boldsymbol{v}\| = 1$).

Problem 3(c) Show for an orthogonal matrix $Q \in O(n)$ that $\det Q = \pm 1$. Give an example of $Q \in U(n)$ such that $\det Q \neq \pm 1$. Hint: recall for any real matrices A and B that $\det A = \det A^\top$ and $\det(AB) = \det A \det B$.

Problem 3(d) A normal matrix commutes with its adjoint. Show that $Q \in U(n)$ is normal.

Problem 3(e) The spectral theorem states that any normal matrix is unitarily diagonalisable: if A is normal then $A = V\Lambda V^\star$ where $V \in U(n)$ and Λ is diagonal. Use this to show that $Q \in U(n)$ is equal to I if and only if all its eigenvalues are 1.

We now turn to QR factorisation beginning with an example that can be done by hand.

(This example was very delicately chosen; so this problem is not examinable but does help to understand how the QR factorisation works and its relationship to the reduced QR factorisation.)

Problem 4 Use Householder reflections to compute QR and reduce QR factorisations of the matrix

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 4 \\ 2 & 3 \end{bmatrix}.$$

Our last two problems concern the uniqueness of the QR factorisation.

Problem 5(a) Show that every matrix $A \in \mathbb{R}^{m \times n}$ has a QR factorisation such that the diagonal of R is non-negative. Make sure to include the case of more columns than rows (i.e. $m < n$).

Problem 5(b) Show that the QR factorisation of a square invertible matrix $A \in \mathbb{R}^{n \times n}$ is unique, provided that the diagonal of R is positive.
