MATH50003 Numerical Analysis

III.2 Cholesky factorisation

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Chapter III

Numerical Linear Algebra

1. Structured matrices such as banded
2. LU and PLU factorisations for solving linear systems
3. Cholesky factorisation for symmetric positive definite
4. Orthogonal matrices such as Householder reflections
5. QR factorisation for solving least squares

LU factorisation:

$$A = LU$$

PLU factorisation:

$$A = P^\top LU$$

Cholesky factorisation:

$$A = LL^\top$$

III.2.4 Cholesky factorisations $A = LL^\top$

Symmetric positive definite matrices have Cholesky factorisations

Definition 16 (positive definite)**.** A square matrix $A \in \mathbb{R}^{n \times n}$ is *positive definite* if for all $\boldsymbol{x} \in \mathbb{R}^n, x \neq 0$ we have

$$\boldsymbol{x}^\top A \boldsymbol{x} > 0$$

Proposition 5 (conjugating positive definite)**.** *If* $A \in \mathbb{R}^{n\times n}$ *is positive definite and* $V \in \mathbb{R}^{n\times n}$ *is non-singular then*

$$V^\top A V$$

is positive definite.

Proposition 6 (diag positivity)**.** *If $A \in \mathbb{R}^{n\times n}$ is positive definite then its diagonal entries are positive: $a_{kk} > 0$.*

Lemma 4 (subslice positive definite)**.** *If $A \in \mathbb{R}^{n\times n}$ is positive definite then $A[2:n, 2:n] \in \mathbb{R}^{(n-1)\times(n-1)}$ is also positive definite.*

Theorem 6 (Cholesky and SPD). *A matrix A is symmetric positive definite if and only if it has a Cholesky factorisation*

$$A = LL^\top$$

where L is lower triangular with positive diagonal entries.

Example 14 (Cholesky by hand)**.**

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$