

Numerical Analysis MATH50003 (2025–26) Problem Sheet 1

In the lectures/notes we saw some basic *analysis* of the errors of simple integration and differentiation rules: the right-sided rectangular rule for integration and divided difference for differentiation. This problem sheets explores the analysis of some other simple rules that may have *better* convergence rates: they can calculate integrals/derivatives more accurately with the same amount of data as the rules discussed in lectures. In the lab the practical implementation of these methods is explored.

The first example is the left-sided rectangular rule, which has the same linear ($O(h)$) convergence rate as the right-sided rectangular rule:

Problem 1 Assuming f is differentiable on $[a, b]$ and its derivative is integrable, prove the left-point Rectangular rule error formula

$$\int_a^b f(x)dx = h \sum_{j=0}^{n-1} f(x_j) + \delta$$

where $|\delta| \leq M(b-a)h$ for $M = \sup_{a \leq x \leq b} |f'(x)|$, $h = (b-a)/n$ and $x_j = a + jh$.

We now turn our attention to the Trapezium rule, which will have a faster quadratic ($O(h^2)$) convergence rate using the same number of samples. That is to say: we can get much more accurate results with the exact same amount of work! We begin with a simple one-panel error result showing cubic ($O(h^3)$) error:

Problem 2(a) Assuming f is twice-differentiable on $[a, b]$ and its second derivative is integrable, prove a one-panel Trapezium rule error bound:

$$\int_a^b f(x)dx = (b-a) \frac{f(a) + f(b)}{2} + \delta$$

where $|\delta| \leq M(b-a)^3$ for $M = \sup_{a \leq x \leq b} |f''(x)|$.

Hint: Recall from the notes

$$\int_a^b \frac{(b-x)f(a) + (x-a)f(b)}{b-a} dx = (b-a) \frac{f(a) + f(b)}{2}$$

and you may need to use Taylor's theorem. Note that the bound is not sharp and so you may arrive at something sharper like $|\delta| \leq 3(b-a)^3 M/4$. The sharpest bound is $|\delta| \leq (b-a)^3 M/12$ but that would be a significantly harder challenge to show!

We can use the previous problem to deduce the *global* error, summing up over all panels. This shows quadratic ($O(h^2)$) error.

Problem 2(b) Assuming f is twice-differentiable on $[a, b]$ and its second derivative is integrable, prove a bound for the Trapezium rule error:

$$\int_a^b f(x)dx = h \left[\frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_j) + \frac{f(b)}{2} \right] + \delta$$

where $|\delta| \leq M(b-a)h^2$ for $M = \sup_{a \leq x \leq b} |f''(x)|$.

We now turn our attention to differentiation, with the first example being a left-sided divided difference, which has a linear ($O(h)$) error just like the right-sided divided difference in lectures:

Problem 3 Assuming f is twice-differentiable in $[x-h, x]$, for the left difference approximation

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \delta,$$

show that $|\delta| \leq Mh/2$ for $M = \sup_{x-h \leq t \leq x} |f''(t)|$.

The next example shows that with a more careful choice of sample points, we can obtain a quadratic ($O(h^2)$) error:

Problem 4 Assuming f is thrice-differentiable in $[x-h, x+h]$, for the central differences approximation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \delta,$$

show that $|\delta| \leq Mh^2/6$ for $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$.

By applying a central difference discretisation twice we can obtain an approximation to the second derivative. The following problem asks to prove that the approximation converges with a linear ($O(h)$) error:

Problem 5 Assuming f is thrice-differentiable in $[x-h, x+h]$, for the second-order derivative approximation

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \delta$$

show that $|\delta| \leq Mh/3$ for $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$.

Note in these problems the computational complexity is independent of h . However, if h is too small the practical implementation has large errors. This phenomena is explored in the lab.