Numerical Analysis MATH50003 (2024–25) Problem Sheet 2

Problem 1 Using dual number arithmetic, compute the following polynomials evaluated at the dual number $2 + \epsilon$ and use this to deduce their derivative at 2:

$$2x^2 + 3x + 4, (x + 1)(x + 2)(x + 3), (2x + 1)x^3.$$

SOLUTION (a)

$$2(2 + \epsilon)^2 + 3(2 + \epsilon) + 4 = 2(4 + 4\epsilon) + 6 + 3\epsilon + 4 = 18 + 11\epsilon$$

so the derivative is 11.

(b)

$$(3 + \epsilon)(4 + \epsilon)(5 + \epsilon) = (12 + 7\epsilon)(5 + \epsilon) = 60 + 47\epsilon$$

so the derivative is 47.

(c)

$$(2(2 + \epsilon) + 1)(2 + \epsilon)^3 = (5 + 2\epsilon)(4 + 4\epsilon)(2 + \epsilon) = (20 + 28\epsilon)(2 + \epsilon) = 40 + 76\epsilon$$

so the derivative is 76.

END

Problem 2 What should the following functions applied to dual numbers return for $x = a+b\epsilon$:

$$f(x) = x^{100} + 1, g(x) = 1/x, h(x) = \tan x.$$

SOLUTION

$$f(a + b\epsilon) = f(a) + bf'(a)\epsilon = a^{100} + 1 + 100ba^{99}\epsilon$$

valid everywhere.

$$g(a + b\epsilon) = \frac{1}{a} - \frac{b}{a^2}\epsilon$$

valid for $a \neq 0$.

$$h(a + b\epsilon) = \tan a + b \sec^2 a\epsilon$$

valid for $a \notin \{k\pi + \pi/2 : k \in \mathbb{Z}\}$.

END

Problem 3(a) What is the correct definition of division on dual numbers, i.e., for what choice of s and t does the following hold:

$$(a + b\epsilon)/(c + d\epsilon) = s + t\epsilon.$$

SOLUTION

As with complex numbers, division is easiest to understand by first multiplying with the conjugate, that is:

$$\frac{a + b\epsilon}{c + d\epsilon} = \frac{(a + b\epsilon)(c - d\epsilon)}{(c + d\epsilon)(c - d\epsilon)}.$$

Expanding the products and dropping terms with ϵ^2 then leaves us with the definition of division for dual numbers (where the denominator must have non-zero real part):

$$\frac{a}{c} + \frac{bc - ad}{c^2}\epsilon.$$

Thus we have $s = \frac{a}{c}$ and $t = \frac{bc-ad}{c^2}$.

END

Problem 3(b) A *field* is a commutative ring such that $0 \neq 1$ and all nonzero elements have a multiplicative inverse, i.e., there exists a^{-1} such that $aa^{-1} = 1$. Can we use the previous part to define $a^{-1} := 1/a$ to make $\mathbb{D}$ a field? Why or why not?

SOLUTION

An example that doesn't work is $z = 0 + \epsilon$ where the formula is undefined, i.e, it would give:

$$z^{-1} = \infty + \infty\epsilon$$

END

Problem 4 Use dual numbers to compute the derivative of the following functions at $x = 0.1$:

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^{3} \left(\frac{x}{k} - 1\right), \text{ and } f_2^{\mathrm{s}}(x) = 1 + \frac{x-1}{2 + \frac{x-1}{2}}$$

SOLUTION

We now compute the derivatives of the three functions by evaluating for $x = 0.1 + \epsilon$. For the first function we have:

$$\begin{aligned}
&\exp(\exp(0.1+\epsilon)\cos(0.1+\epsilon) + \sin(0.1+\epsilon)) \\
&= \exp((\exp(0.1) + \epsilon\exp(0.1))(\cos(0.1) - \sin(0.1)\epsilon) + \sin(0.1) + \cos(0.1)\epsilon) \\
&= \exp(\exp(0.1)\cos(0.1) + \sin(0.1) + (\exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))\epsilon) \\
&= \exp(\exp(0.1)\cos(0.1) + \sin(0.1)) \\
&\qquad + \exp(\exp(0.1)\cos(0.1) + \sin(0.1))\exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))\epsilon
\end{aligned}$$

therefore the derivative is the dual part

$$\exp(\exp(0.1)\cos(0.1) + \sin(0.1))(\exp(0.1)(\cos(0.1) - \sin(0.1)) + \cos(0.1))$$

For the second function we have:

$$\begin{aligned}
(0.1 + \epsilon - 1)\left(\frac{0.1+\epsilon}{2} - 1\right)\left(\frac{0.1+\epsilon}{3} - 1\right) &= (-0.9+\epsilon)(-0.95+\epsilon/2)(-29/30+\epsilon/3) \\
&= (171/200 - 1.4\epsilon)(-29/30 + \epsilon/3) \\
&= -1653/2000 + 983\epsilon/600
\end{aligned}$$

Thus the derivative is 983/600.

For the third function we have:

$$\begin{aligned}
1 + \frac{0.1+\epsilon-1}{2 + \frac{0.1+\epsilon-1}{2}} &= 1 + \frac{-0.9+\epsilon}{1.55+\epsilon/2} \\
&= 1 - 18/31 + 2\epsilon/1.55^2
\end{aligned}$$

Thus the derivative is $2/1.55^2$.

END

Consider a 2D analogue of dual numbers $a + b\epsilon_x + c\epsilon_y$ defined by the relationship $\epsilon_x\epsilon_y = \epsilon_x^2 = \epsilon_y^2 = 0$.

Problem 5(a) Derive the formula for writing the product of two 2D dual numbers $(a + a_x\epsilon_x + a_y\epsilon_y)(b + b_x\epsilon_x + b_y\epsilon_y)$ where $a, a_x, a_y, b, b_x, b_y \in \mathbb{R}$ as a 2D dual number.

SOLUTION

$$(a + a_x\epsilon_x + a_y\epsilon_y)(b + b_x\epsilon_x + b_y\epsilon_y) = ab + (ba_x + ab_x)\epsilon_x + (ba_y + ab_y)\epsilon_y$$

END

Problem 5(b) Show for all 2D polynomials

$$p(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} c_{kj}x^k y^j$$

that

$$p(x + a\epsilon_x, y + b\epsilon_y) = p(x,y) + a\frac{\partial p}{\partial x}\epsilon_x + b\frac{\partial p}{\partial y}\epsilon_y.$$

SOLUTION By linearity it suffices to consider monomials $x^k y^j$. Assume it is true for all lower degree polynomials with the degree 0 case holding trivially. If $j = 0$ we have:

$$(x + a\epsilon_x)^k = (x + a\epsilon_x)(x + a\epsilon_x)^{k-1} = (x + a\epsilon_x)(x^{k-1} + a(k-1)x^{k-2}\epsilon_x) = x^k + akx^{k-1}\epsilon_x$$

and similarly for $k = 0$. For $k, j \neq 0$ we can use the previous cases to get:

$$(x + a\epsilon_x)^k(y + b\epsilon_y)^j = (x^k + kax^{k-1}\epsilon_x)(y^j + jby^{j-1}\epsilon_y) = x^k y^j + kax^{k-1}y^j\epsilon_x + bjx^k y^{j-1}$$

END

Problem 5(c) Use 2D dual numbers to compute the gradient of $p(x,y) = (1+x+3xy)(1+y)$ at $x = 1$ and $y = 2$.

SOLUTION

$$\begin{aligned} p(1+\epsilon_x, 2+\epsilon_y) &= (2 + \epsilon_x + 3(1+\epsilon_x)(2+\epsilon_y))(3+\epsilon_y) = (2 + \epsilon_x + 3(2 + 2\epsilon_x + \epsilon_y))(3+\epsilon_y) \\ &= (8 + 7\epsilon_x + 3\epsilon_y)(3+\epsilon_y) = 24 + 21\epsilon_x + 17\epsilon_y \end{aligned}$$

hence the gradient is $[21, 17]^\top$. **END**

Problem 6 Suppose f is twice-differentiable in a neighbourhood of B of r such that $f(r) = f'(r) = 0$, where f'' does in B. Show that the error of the k-th Newton iteration $\varepsilon_k := x - x_k$ satisfies

$$|\varepsilon_{k+1}| \leq \tilde{M}|\varepsilon_k|$$

where

$$M \tilde{=} \frac{1}{2}\sup_{y\in B}|f''(y)|\sup_{y\in B}\frac{1}{|f''(y)|}.$$

SOLUTION

Note that

$$f'(x_k) = f'(x) + f''(\tau)\varepsilon_k = f''(\tau)\varepsilon_k$$

for some τ between x and x_k. Thus we get

$$\varepsilon_{k+1} = -\frac{f''(t)}{2f'(x_k)}\varepsilon_k^2 = -\frac{f''(t)}{2f''(\tau)}\varepsilon_k.$$

Taking absolute values of each side gives the result.

END