MATH50003 Numerical Analysis

I.3 Dual Numbers

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Part I

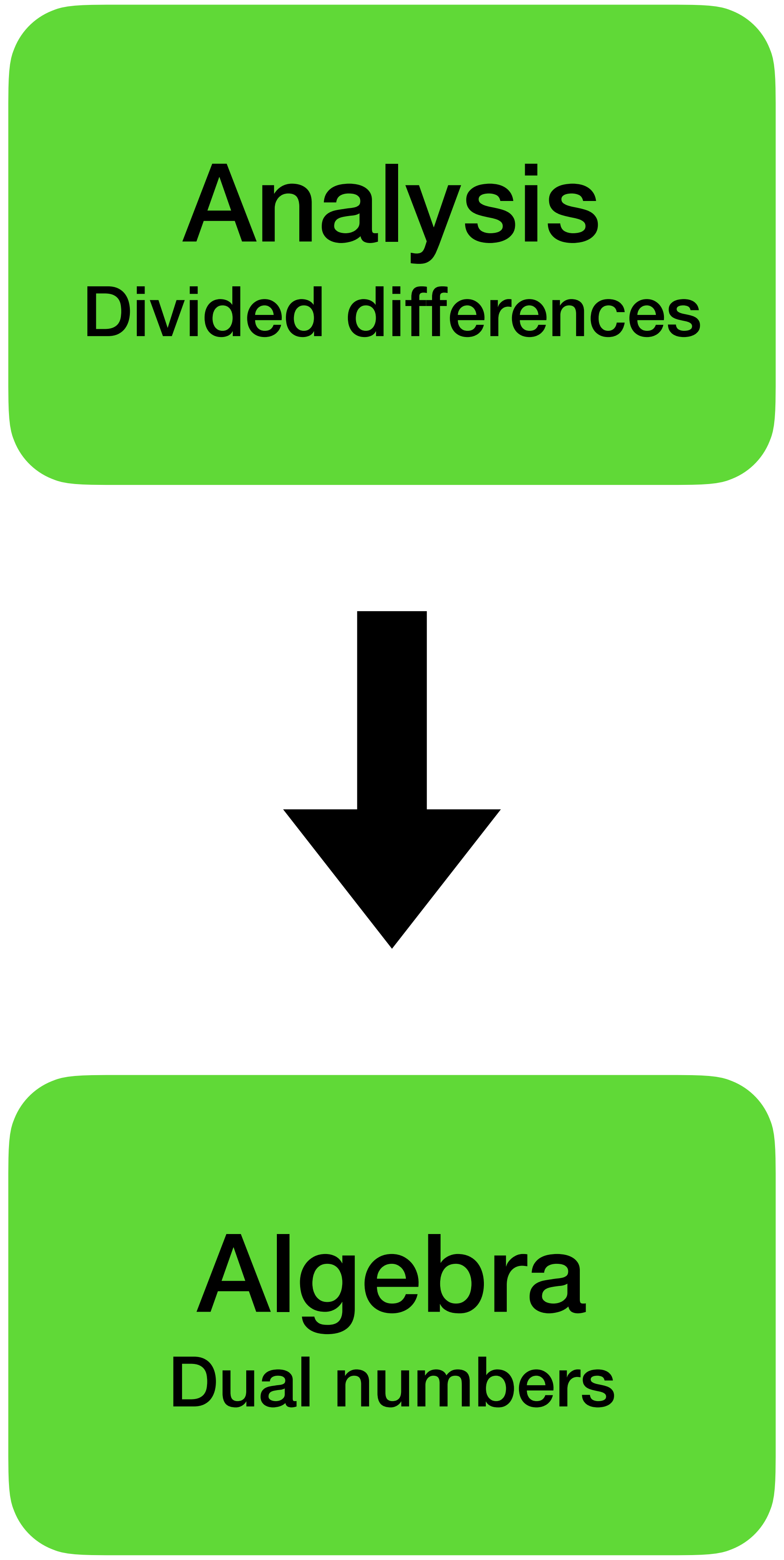
Calculus on a Computer

1. Rectangular rules for integration
2. Divided differences for differentiation
3. Dual numbers for differentiation
4. Newton’s method for root finding

Divided differences can have large errors.

Is it even possible to algorithmically calculate derivatives to high accuracy?

Yes: if we have access to the code.



Dual numbers are a commutative ring that allow us to differentiate

Definition 1 (dual numbers)

$$\mathbb{D} := \{a + b\epsilon \quad : \quad a, b \in \mathbb{R}, \quad \epsilon^2 = 0\}$$

Compare with complex numbers:

$$\mathbb{C} := \{a + b\mathrm{i} \quad : \quad a, b \in \mathbb{R}, \quad \mathrm{i}^2 = -1\}$$

Addition/multiplication

Follows from simple algebra

Complex

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$$

Dual

$$(a + b\epsilon) + (c + d\epsilon) = (a + c) + (b + d)\epsilon$$

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (bc + ad)\mathrm{i}$$

$$(a + b\epsilon)(c + d\epsilon) = ac + (bc + ad)\epsilon$$

I.3.1 Differentiating polynomials

Addition/multiplication ⇒ dual numbers compute derivatives

Theorem 2 (polynomials on dual numbers). *Suppose p is a polynomial. Then*

$$p(a + b\epsilon) = p(a) + bp'(a)\epsilon$$

Example 1 (differentiating polynomial)**.** Consider computing $p'(2)$ where

$$p(x) = (x-1)(x-2) + x^2.$$

I.3.2 Differentiating other functions

Theorem 1 gives us a rule to extend differentiation via duals

Motivation: consider a Taylor series

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

And assume a is in radius of convergence.

What will $f(a + b\epsilon)$ return?

Definition 2 (dual extension)**.** Suppose a real-valued function $f : \Omega \to \mathbb{R}$ is differentiable in $\Omega \subset \mathbb{R}$. We can construct the *dual extension* $\underline{f} : \Omega + \epsilon\mathbb{R} \to \mathbb{D}$ by defining

$$\underline{f}(a + b\epsilon) := f(a) + bf'(a)\epsilon.$$

Examples:

$$\exp(a + b\epsilon) := \exp(a) + b\exp(a)\epsilon$$

$$\sin(a + b\epsilon) := \sin(a) + b\cos(a)\epsilon$$

$$\cos(a + b\epsilon) := \cos(a) - b\sin(a)\epsilon$$

$$\log(a + b\epsilon) := \log(a) + \frac{b}{a}\epsilon$$

$$\sqrt{a + b\epsilon} := \sqrt{a} + \frac{b}{2\sqrt{a}}\epsilon$$

$$|a + b\epsilon| := |a| + b\,\mathrm{sign}a\,\epsilon$$

Lemma 2 (addition/multiplication). *Suppose $f, g : \Omega \to \mathbb{R}$ are differentiable for $\Omega \subset \mathbb{R}$ and $c \in \mathbb{R}$. Then for $a \in \Omega$ and $b \in \mathbb{R}$ we have*

$$\begin{aligned}
\underline{f+g}(a+b\epsilon) &= \underline{f}(a+b\epsilon) + \underline{g}(a+b\epsilon) \\
\underline{cf}(a+b\epsilon) &= c\underline{f}(a+b\epsilon) \\
\underline{fg}(a+b\epsilon) &= \underline{f}(a+b\epsilon)\underline{g}(a+b\epsilon)
\end{aligned}$$

Lemma 3 (composition)**.** *Suppose $f : \Gamma \to \mathbb{R}$ and $g : \Omega \to \Gamma$ are differentiable in $\Omega, \Gamma \subset \mathbb{R}$. Then*

$$\underline{(f \circ g)}(a + b\epsilon) = \underline{f}(\underline{g}(a + b\epsilon))$$

Example 2 (differentiating non-polynomial)**.** Consider differentiating $f(x) = \exp(x^2 + \cos x)$ at the point $a = 1$, where we automatically use the dual-extension of exp and cos.

How do we implement this on a computer?