MATH50003
Numerical Analysis

III.1 Structured Matrices

Dr Sheehan Olver

Course content

I. Calculus on a Computer
    - Integration, differentiation, root finding

II. Representing Numbers
    - Floating point numbers, bounding errors, interval arithmetic

III. Numerical Linear Algebra
    - Structured matrices, LU & QR factorisations, least squares

IV. Linear Algebra Applications
    - Data regression, differential equations

V. Numerical Fourier series
    - Fourier expansions and transforms, differential equations with periodic conditions

VI. Orthogonal Polynomials
    - Classical orthogonal polynomials, Gaussian quadrature

Chapter III

Numerical Linear Algebra

1. Structured matrices such as banded
2. LU and PLU factorisations for solving linear systems
3. Cholesky factorisation for symmetric positive definite
4. Orthogonal matrices such as Householder reflections
5. QR factorisation for solving least squares

III.1.1 Dense matrices

And their usage in matrix multiplication

Consider a matrix $A \in \mathbb{F}^{m \times n}$ where $\mathbb{F}$ is a field ($\mathbb{R}$ or $\mathbb{C}$)

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

And a vector $\mathbf{x} \in \mathbb{F}^n$. We have (“multiplication by rows”)

$$A\boldsymbol{x} := \begin{bmatrix} \sum_{j=1}^n a_{1j} x_j \\ \vdots \\ \sum_{j=1}^n a_{mj} x_j \end{bmatrix}$$

Or with floats: $A\boldsymbol{x} \approx \begin{bmatrix} \bigoplus_{j=1}^n (a_{1j} \otimes x_j) \\ \vdots \\ \bigoplus_{j=1}^n (a_{mj} \otimes x_j) \end{bmatrix}$

We can also write a matrix in terms of its columns:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \left[\boldsymbol{a}_1 | \cdots | \boldsymbol{a}_n \right]$$

Alternative formula for multiplication (“multiplication by columns”):

$$A\boldsymbol{x} = x_1 \boldsymbol{a}_1 + \cdots + x_n \boldsymbol{a}_n$$

Computational complexity

How many floating point operations? Count the number of $\oplus$, $\otimes$, $\oslash$, $\ominus$

$$A\boldsymbol{x} := \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix} \approx \begin{bmatrix} \bigoplus_{j=1}^{n}(a_{1j} \otimes x_j) \\ \vdots \\ \bigoplus_{j=1}^{n}(a_{mj} \otimes x_j) \end{bmatrix}$$

III.1.2 Triangular Matrices

Exploiting zero structure in a matrix

$$U = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ & \ddots & \vdots \\ & & u_{nn} \end{bmatrix}, \qquad L = \begin{bmatrix} \ell_{11} & & \\ \vdots & \ddots & \\ \ell_{n1} & \cdots & \ell_{nn} \end{bmatrix}$$

Multiplication takes roughly half the operations (still same complexity):

Can invert via back/forward substitution:

III.1.3 Banded Matrices

Matrices that are only non-zero near the diagonal

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1,u+1} & & & \\ \vdots & a_{22} & \ddots & a_{2,u+2} & & \\ a_{1+l,1} & \ddots & \ddots & \ddots & \ddots & \\ & a_{2+l,2} & \ddots & \ddots & \ddots & a_{n-u,n} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & a_{n,n-l} & \cdots & a_{nn} \end{bmatrix}$$

Definition 13 (Bidiagonal)**.** If a square matrix has bandwidths $(l, u) = (1, 0)$ it is *lower-bidiagonal* and if it has bandwidths $(l, u) = (0, 1)$ it is *upper-bidiagonal*.

$$L = \begin{bmatrix} \ell_{11} & & & \\ \ell_{21} & \ell_{22} & & \\ & \ddots & \ddots & \\ & & \ell_{n,n-1} & \ell_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & & \\ & u_{22} & \ddots & \\ & & \ddots & u_{n-1,n} \\ & & & u_{nn} \end{bmatrix}$$

Multiplication is linear complexity:

Back/forward substitution are also linear complexity:

Definition 14 (Tridiagonal)**.** If a square matrix has bandwidths $l = u = 1$ it is *tridiagonal.*

$$A = \begin{bmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{nn} \end{bmatrix}$$

Multiplication is linear complexity.
We will see later inversion is also linear complexity.