MATH50003 Numerical Analysis

IV.1 Polynomial Interpolation and Regression

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Course content

I. Calculus on a Computer
   - Integration, differentiation, root finding

II. Representing Numbers
   - Floating point numbers, bounding errors, interval arithmetic

III. Numerical Linear Algebra
   - Structured matrices, LU & QR factorisations, least squares

IV. Linear Algebra Applications
   - Data regression, singular value decomposition, matrix compression

V. Numerical Fourier series
   - Fourier expansions and transforms, convolutions

VI. Orthogonal Polynomials
   - Classical orthogonal polynomials, Gaussian quadrature

Chapter IV

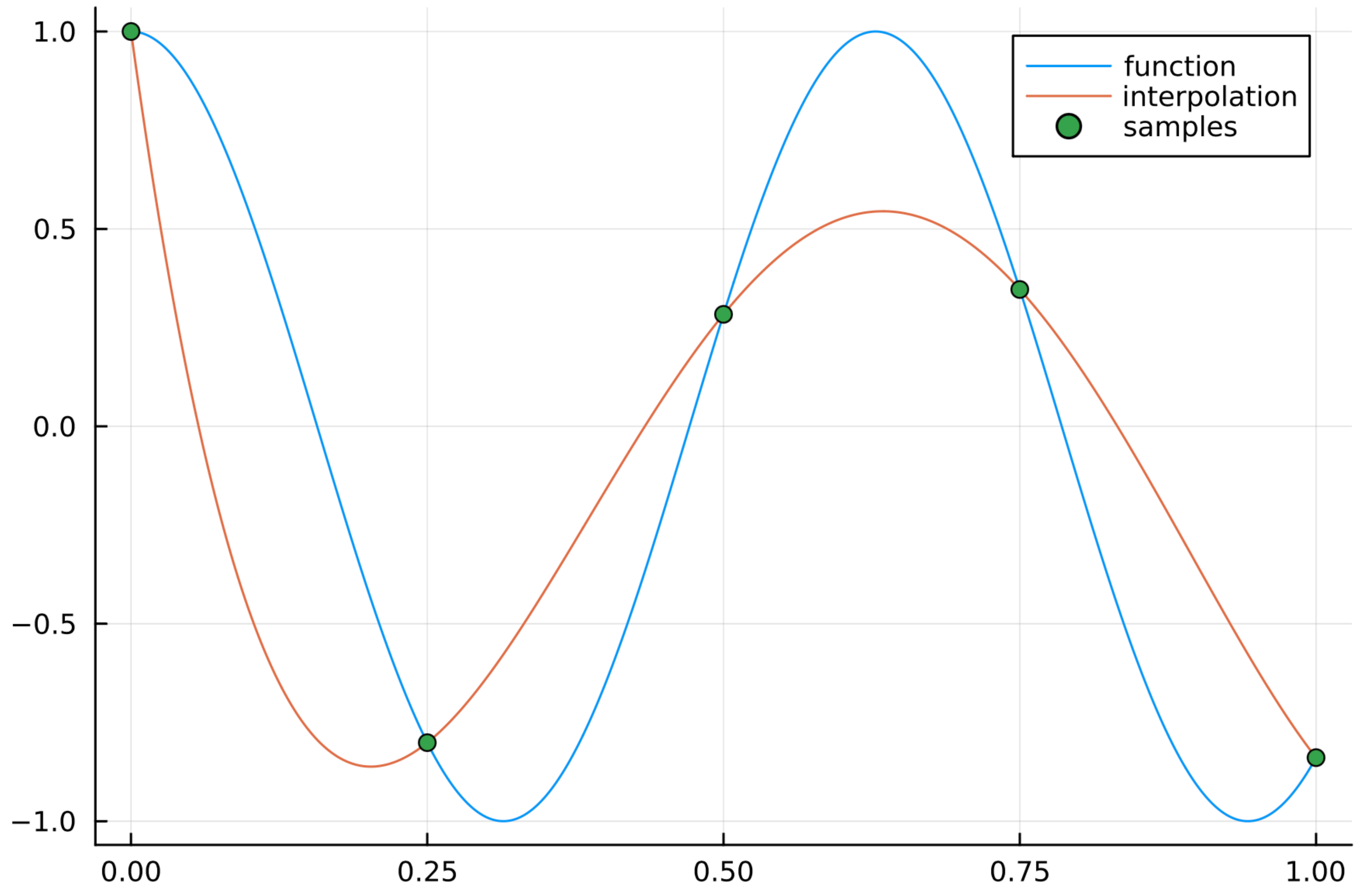
Applications of Linear Algebra

1. Polynomial Interpolation and Regression for approximating data

2. Singular Value Decomposition and matix compression via best low rank approximation

IV.1.1 Polynomial interpolation

Find a polynomial equal to data at a grid



Definition 25 (interpolatory polynomial)**.** Given *distinct* points $\boldsymbol{x} = [x_1, \ldots, x_n]^\top \in \mathbb{C}^n$ and *data* $\boldsymbol{f} = [f_1, \ldots, f_n]^\top \in \mathbb{C}^n$, a degree $n-1$ *interpolatory polynomial* $p(x)$ satisfies

$$p(x_j) = f_j$$

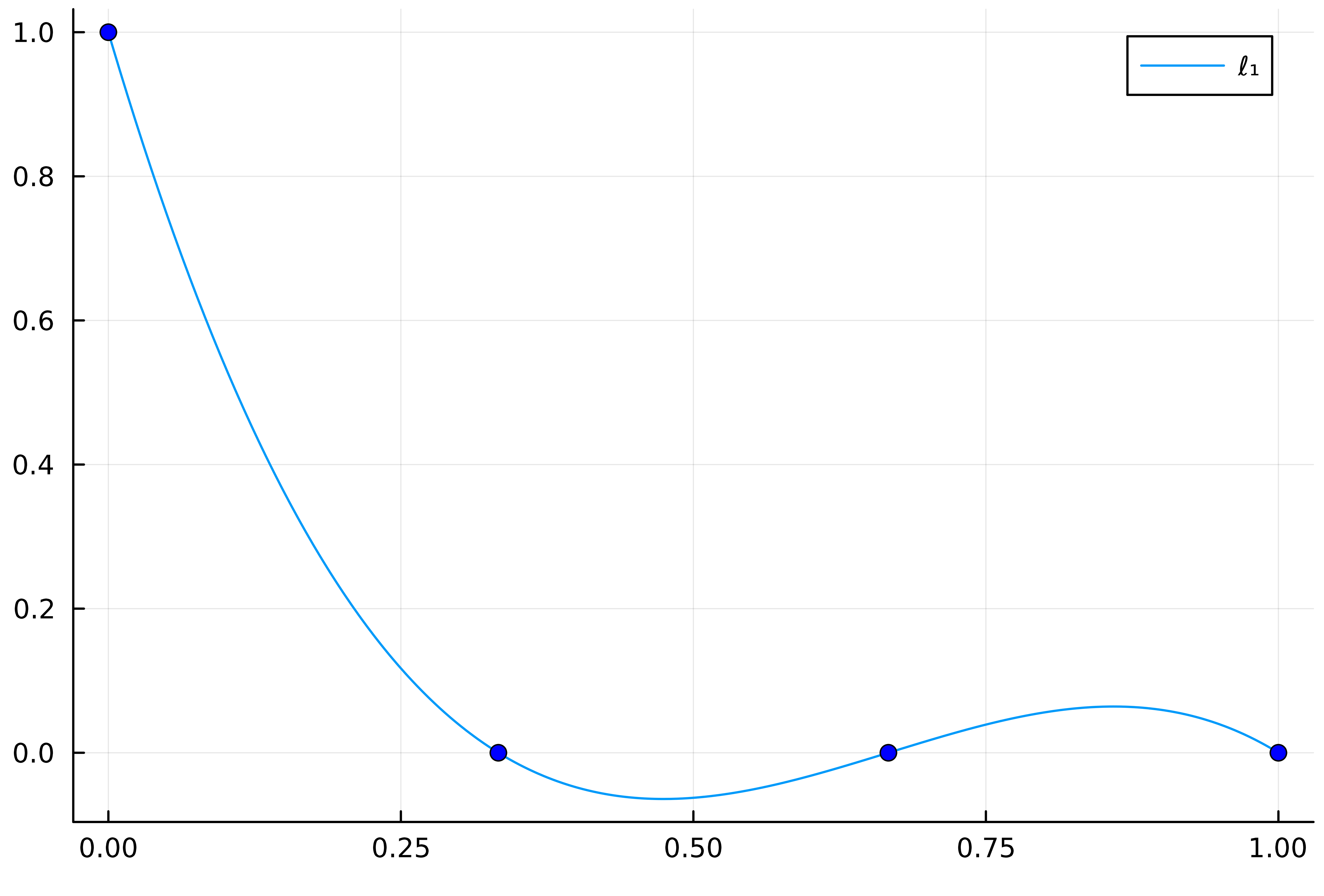
Definition 26 (Vandermonde)**.** The *Vandermonde matrix* associated with $\boldsymbol{x} \in \mathbb{C}^m$ is the matrix

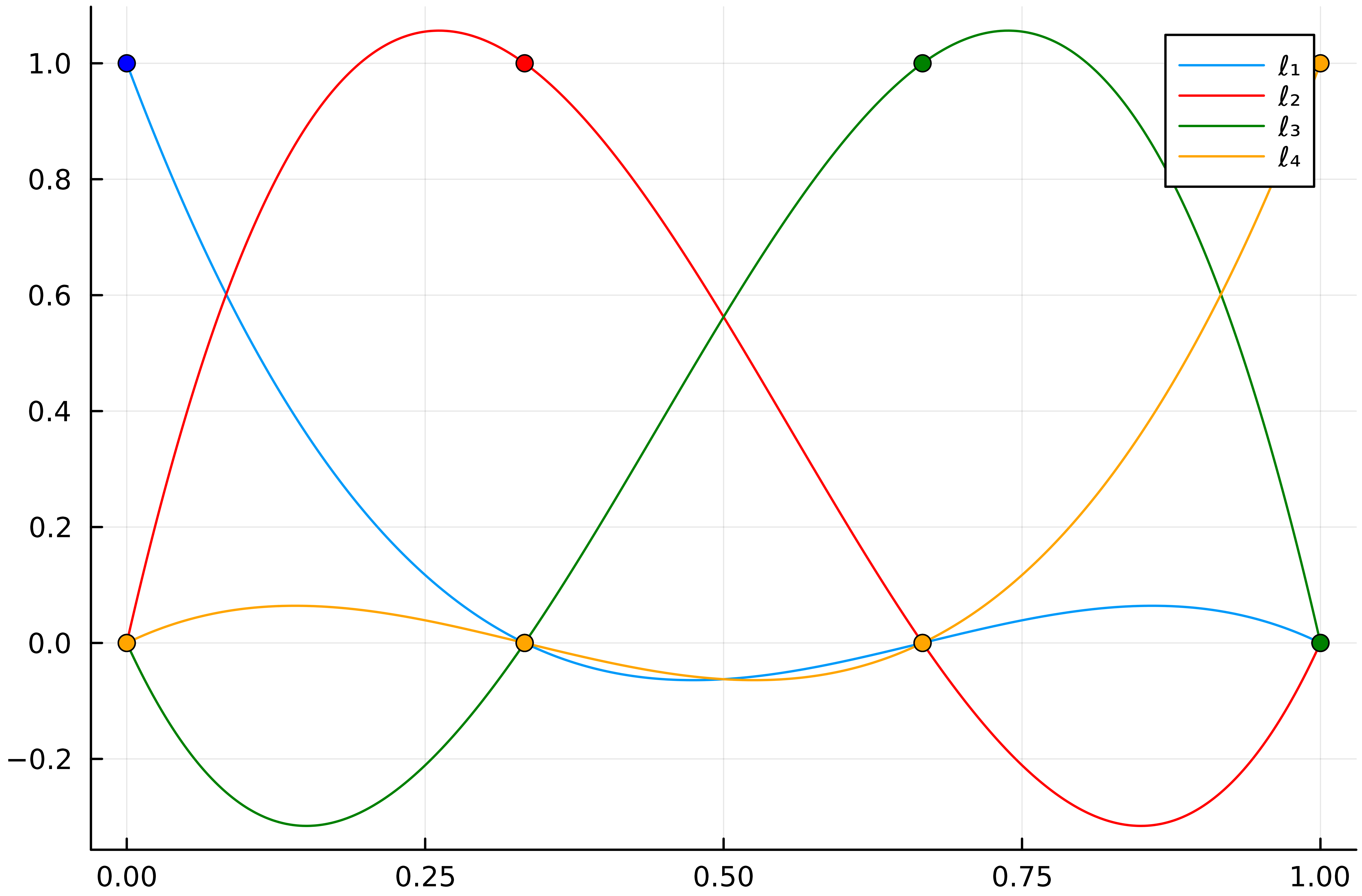
$$V_{\boldsymbol{x},n} := \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x_m^{n-1} \end{bmatrix} \in \mathbb{C}^{m \times n}.$$

Proposition 11 (interpolatory polynomial uniqueness)**.** *Interpolatory polynomials are unique and therefore square Vandermonde matrices are invertible.*

Definition 27 (Lagrange basis polynomial)**.** The *Lagrange basis polynomial* is defined as

$$\ell_k(x) := \prod_{j \neq k} \frac{x - x_j}{x_k - x_j} = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$





1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.00
0.25
0.50
0.75
1.00
ℓ_1
ℓ_2
ℓ_3
ℓ_4

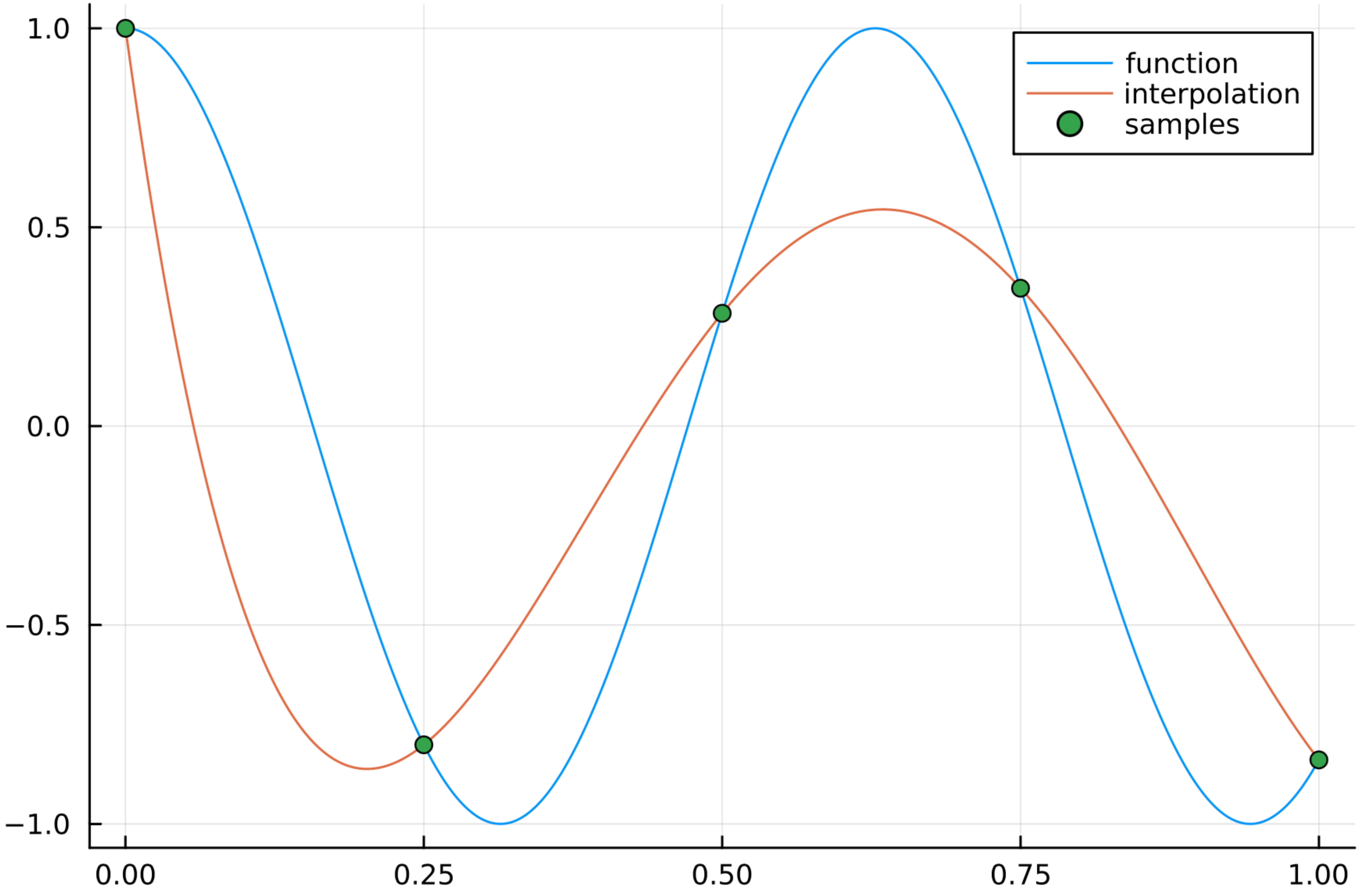
Theorem 9 (Lagrange interpolation)**.** *The unique interpolation polynomial is:*

$$p(x) = f_1\ell_1(x) + \cdots + f_n\ell_n(x)$$

Example 18 (interpolating an exponential).

IV.1.2 Interpolatory quadrature rules

Interpolate by polynomials and integrate exactly



$$\int_a^b f(x)w(x)\mathrm{d}x \approx \int_a^b p(x)w(x)\mathrm{d}x$$

Definition 28 (interpolatory quadrature rule)**.** Given a set of points $\boldsymbol{x} = [x_1, \ldots, x_n]^\top$ the interpolatory quadrature rule is:

$$\Sigma_n^{w,\boldsymbol{x}}[f] := \sum_{j=1}^{n} w_j f(x_j)$$

where

$$w_j := \int_a^b \ell_j(x) w(x) \mathrm{d}x.$$

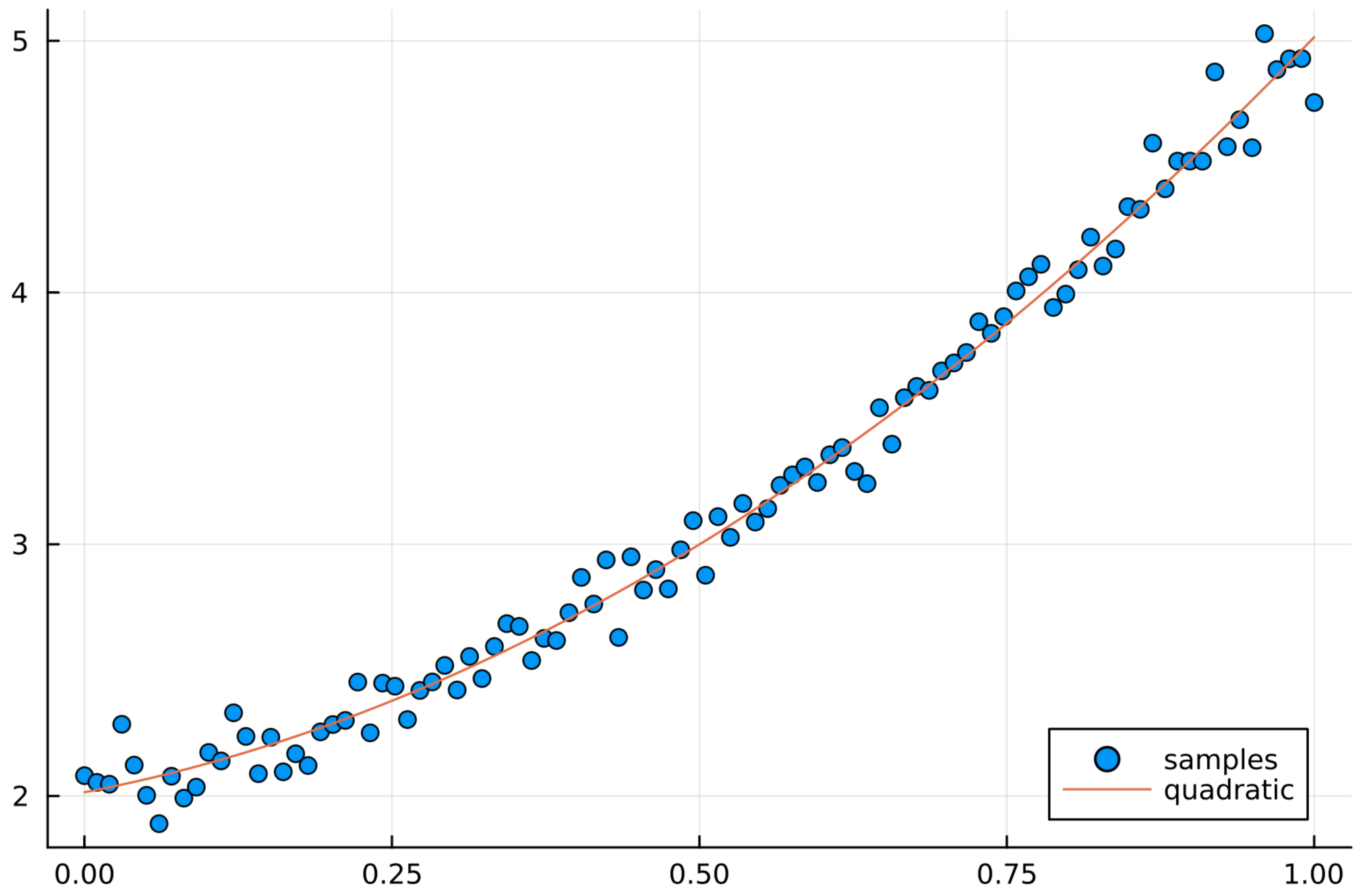
Proposition 12 (interpolatory quadrature is exact for polynomials)**.** *Interpolatory quadrature is exact for all degree $n-1$ polynomials p:*

$$\int_a^b p(x)w(x)\mathrm{d}x = \Sigma_n^{w,\boldsymbol{x}}[p]$$

Example 19 (3-point interpolatory quadrature).

IV.1.3 Polynomial regression

How to fit a polynomial to lots of data?



Find a polynomial such that:

$$\begin{bmatrix} p(x_1) \\ \vdots \\ p(x_m) \end{bmatrix} \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{\boldsymbol{f}}.$$