MATH50003
Numerical Analysis

III.2 LU and PLU factorisations

Dr Sheehan Olver

Chapter III

Numerical Linear Algebra

1. Structured matrices such as banded
2. LU and PLU factorisations for solving linear systems
3. Cholesky factorisation for symmetric positive definite
4. Orthogonal matrices such as Householder reflections
5. QR factorisation for solving least squares

LU factorisation:

$$A = LU$$

PLU factorisation:

$$A = P^{\top} LU$$

III.2.1 Outer products

Definition 15 (outer product)**.** Given $\boldsymbol{x} \in \mathbb{F}^m$ and $\boldsymbol{y} \in \mathbb{F}^n$ the *outer product* is:

$$\boldsymbol{x}\boldsymbol{y}^\top := [\boldsymbol{x}y_1 | \cdots | \boldsymbol{x}y_n] = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{bmatrix} \in \mathbb{F}^{m \times n}.$$

Proposition 4 (rank-1)**.** *A matrix $A \in \mathbb{F}^{m\times n}$ has rank 1 if and only if there exists $\boldsymbol{x} \in \mathbb{F}^m$ and $\boldsymbol{y} \in \mathbb{F}^n$ such that*

$$A = \boldsymbol{x}\boldsymbol{y}^\top.$$

III.2.2 LU factorisation $A = LU$

Gaussian elimination w/o pivoting computes an LU factorisation

Example 12 (LU by-hand).

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 1 & 4 & 9 \end{bmatrix}$$

III.2.3 PLU factorisation $A = P^\top LU$

Gaussian elimination w/ pivoting is a PLU factorisation

Permutation matrices:

Theorem 5 (PLU)**.** *A matrix $A \in \mathbb{C}^{n\times n}$ is invertible if and only if it has a PLU decomposition:*

$$A = P^\top LU$$

where the diagonal of L are all equal to 1 and the diagonal of U are all non-zero, and P is a permutation matrix.

Example 13 (PLU by-hand). Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$