

## Numerical Analysis MATH50003 (2025-26) Problem Sheet 1

In the lectures/notes we saw some basic *analysis* of the errors of simple integration and differentiation rules: the right-sided rectangular rule for integration and divided difference for differentiation. This problem sheets explores the analysis of some other simple rules that may have *better* convergence rates: they can calculate integrals/derivatives more accurately with the same amount of data as the rules discussed in lectures. In the lab the practical implementation of these methods is explored.

The first example is the left-sided rectangular rule, which has the same linear ( $O(h)$ ) convergence rate as the right-sided rectangular rule:

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**Problem 1** Assuming  $f$  is differentiable on  $[a, b]$  and its derivative is integrable, prove the left-point Rectangular rule error formula

$$\int_a^b f(x)dx = h \sum_{j=0}^{n-1} f(x_j) + \delta$$

where  $|\delta| \leq M(b-a)h$  for  $M = \sup_{a \leq x \leq b} |f'(x)|$ ,  $h = (b-a)/n$  and  $x_j = a + jh$ .

### SOLUTION

This proof is very similar to the right-point rule, the only difference is we use a different constant in the indefinite integration in the integration-by-parts. First we need to adapt

**Lemma 1 (Rect. rule error on one panel):**

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^b (x-b)'f(x)dx = [(x-b)f(x)]_a^b - \int_a^b (x-b)f'(x)dx \\ &= (b-a)f(a) + \underbrace{\left(-\int_a^b (x-b)f'(x)dx\right)}_{\varepsilon}. \end{aligned}$$

where

$$|\varepsilon| \leq (b-a) \sup_{a \leq x \leq b} |(x-b)f'(x)| \leq M(b-a)^2$$

Applying this result on  $[x_{j-1}, x_j]$  we get

$$\int_{x_{j-1}}^{x_j} f(x)dx = hf(x_{j-1}) + \delta_j$$

where  $|\delta_j| \leq Mh^2$ . Splitting the integral into a sum of smaller integrals:

$$\int_a^b f(x)dx = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x)dx = h \sum_{j=1}^n f(x_{j-1}) + \underbrace{\sum_{j=1}^n \delta_j}_{\delta}$$

where using the triangular inequality we have

$$|\delta| = \left| \sum_{j=1}^n \delta_j \right| \leq \sum_{j=1}^n |\delta_j| \leq Mnh^2 = M(b-a)h.$$

**END**

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We now turn our attention to the Trapezium rule, which will have a faster quadratic ( $O(h^2)$ ) convergence rate using the same number of samples. That is to say: we can get much more accurate results with the exact same amount of work! We begin with a simple one-panel error result showing cubic ( $O(h^3)$ ) error:

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**Problem 2(a)** Assuming  $f$  is twice-differentiable on  $[a, b]$  and its second derivative is integrable, prove a one-panel Trapezium rule error bound:

$$\int_a^b f(x)dx = (b-a)\frac{f(a)+f(b)}{2} + \delta$$

where  $|\delta| \leq M(b-a)^3$  for  $M = \sup_{a \leq x \leq b} |f''(x)|$ .

*Hint:* Recall from the notes

$$\int_a^b \frac{(b-x)f(a) + (x-a)f(b)}{b-a} dx = (b-a)\frac{f(a)+f(b)}{2}$$

and you may need to use Taylor's theorem. Note that the bound is not sharp and so you may arrive at something sharper like  $|\delta| \leq 3(b-a)^3 M/4$ . The sharpest bound is  $|\delta| \leq (b-a)^3 M/12$  but that would be a significantly harder challenge to show!

### SOLUTION

Recall from the notes:

$$\int_a^b \frac{(b-x)f(a) + (x-a)f(b)}{b-a} dx = (b-a)\frac{f(a)+f(b)}{2}$$

Thus we can find by integration by parts twice (noting that the integrand vanishes at  $a$  and  $b$ ):

$$\begin{aligned} \delta &= \int_a^b \left[ f(x) - \frac{(b-x)f(a) + (x-a)f(b)}{b-a} \right] dx \\ &= - \int_a^b (x-b) \left[ f'(x) - \frac{f(b)-f(a)}{b-a} \right] dx \\ &= \frac{(b-a)^2}{2} \left[ f'(a) - \frac{f(b)-f(a)}{b-a} \right] + \int_a^b \frac{(x-b)^2}{2} f''(x) dx \end{aligned}$$

Applying **Proposition 1** we know

$$\left| f'(a) - \frac{f(b)-f(a)}{b-a} \right| \leq M(b-a)/2$$

Further we have

$$\left| \int_a^b \frac{(x-b)^2}{2} f''(x) dx \right| \leq \frac{(b-a)^3}{2} M$$

Thus we have the bound

$$|\delta| \leq \frac{(b-a)^2}{2} M(b-a)/2 + \frac{(b-a)^3}{2} M \leq \frac{3(b-a)^3}{4} M \leq (b-a)^3 M.$$

For the sharper  $1/12$  constant check out the [Euler–Maclaurin formula](#).

**END**

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We can use the previous problem to deduce the *global* error, summing up over all panels. This shows quadratic ( $O(h^2)$ ) error.

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**Problem 2(b)** Assuming  $f$  is twice-differentiable on  $[a, b]$  and its second derivative is integrable, prove a bound for the Trapezium rule error:

$$\int_a^b f(x)dx = h \left[ \frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_j) + \frac{f(b)}{2} \right] + \delta$$

where  $|\delta| \leq M(b-a)h^2$  for  $M = \sup_{a \leq x \leq b} |f''(x)|$ .

**SOLUTION**

This is very similar to the rectangular rules: applying the preceding result on  $[x_{j-1}, x_j]$  we get

$$\int_{x_{j-1}}^{x_j} f(x)dx = h \frac{f(x_{j-1}) + f(x_j)}{2} + \delta_j$$

where  $|\delta_j| \leq Mh^3$ . Splitting the integral into a sum of smaller integrals:

$$\int_a^b f(x)dx = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x)dx = h \left[ \frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_j) + \frac{f(b)}{2} \right] + \underbrace{\sum_{j=1}^n \delta_j}_{\delta}$$

where using the triangular inequality we have

$$|\delta| = \left| \sum_{j=1}^n \delta_j \right| \leq \sum_{j=1}^n |\delta_j| \leq Mnh^3 = M(b-a)h^2.$$

**0.1 END**

We now turn our attention to differentiation, with the first example being a left-sided divided difference, which has a linear ( $O(h)$ ) error just like the right-sided divided difference in lectures:

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**Problem 3** Assuming  $f$  is twice-differentiable in  $[x-h, x]$ , for the left difference approximation

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \delta,$$

show that  $|\delta| \leq Mh/2$  for  $M = \sup_{x-h \leq t \leq x} |f''(t)|$ .

**SOLUTION**

Almost identical to the right-difference. Use Taylor series to write:

$$f(x-h) = f(x) + f'(x)(-h) + \frac{f''(t)}{2}h^2$$

where  $t \in [x - h, x]$ , so that

$$f'(x) = \frac{f(x) - f(x - h)}{h} + \underbrace{f''(t)/2h}_{\delta}$$

The bound follows immediately:

$$|\delta| \leq |f''(t)/2h| \leq Mh/2.$$

**END**

The next example shows that with a more careful choice of sample points, we can obtain a quadratic ( $O(h^2)$ ) error:

**Problem 4** Assuming  $f$  is thrice-differentiable in  $[x - h, x + h]$ , for the central differences approximation

$$f'(x) = \frac{f(x + h) - f(x - h)}{2h} + \delta,$$

show that  $|\delta| \leq Mh^2/6$  for  $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$ .

**SOLUTION**

By Taylor's theorem, the approximation around  $x + h$  is

$$f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(t_1)}{6}h^3,$$

for some  $t_1 \in (x, x + h)$  and similarly  $f(x - h) = f(x) + f'(x)(-h) + \frac{f''(x)}{2}h^2 - \frac{f'''(t_2)}{6}h^3$ , for some  $t_2 \in (x - h, x)$ .

Subtracting the second expression from the first we obtain  $f(x + h) - f(x - h) = f'(x)(2h) + \frac{f'''(t_1) + f'''(t_2)}{6}h^3$ . Hence,

$$\frac{f(x + h) - f(x - h)}{2h} = f'(x) + \underbrace{\frac{f'''(t_1) + f'''(t_2)}{12}h^2}_{\delta}.$$

Thus, the error can be bounded by  $|\delta| \leq \frac{M}{6}h^2$ .

**0.2 END**

By applying a central difference discretisation twice we can obtain an approximation to the second derivative. The following problem asks to prove that the approximation converges with a linear ( $O(h)$ ) error:

**Problem 5** Assuming  $f$  is thrice-differentiable in  $[x - h, x + h]$ , for the second-order derivative approximation

$$\frac{f(x + h) - 2f(x) + f(x - h))}{h^2} = f''(x) + \delta$$

show that  $|\delta| \leq Mh/3$  for  $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$ .

**SOLUTION** Using the same two formulas as in the previous problem we have  $f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(t_1)}{6}h^3$ , for some  $t_1 \in (x, x+h)$  and  $f(x-h) = f(x) + f'(x)(-h) + \frac{f''(x)}{2}h^2 - \frac{f'''(t_2)}{6}h^3$ , for some  $t_2 \in (x-h, x)$ .

Summing the two we obtain  $f(x+h) + f(x-h) = 2f(x) + f''(x)h^2 + \frac{f'''(t_1)}{6}h^3 - \frac{f'''(t_2)}{6}h^3$ .

Thus,  $f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + \frac{f'''(t_2) - f'''(t_1)}{6}h$ .

Hence, the error is

$$|\delta| = \left| f''(x) - \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \right| = \left| \frac{f'''(t_2) - f'''(t_1)}{6}h \right| \leq \frac{Mh}{3}.$$

**END**

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Note in these problems the computational complexity is independent of  $h$ . However, if  $h$  is too small the practical implementation has large errors. This phenomena is explored in the lab.