Numerical Analysis MATH50003 (2024–25) Problem Sheet 5

Problem 1 By computing the Cholesky factorisation, determine which of the following matrices are symmetric positive definite:

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

Problem 2 Show that a matrix $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite if and only if it has a *reverse* Cholesky factorisation of the form

$$A = UU^\top$$

where U is upper triangular with positive entries on the diagonal.

Problem 3(a) Use the Cholesky factorisation to prove that the following $n \times n$ matrix is symmetric positive definite for any n:

$$\Delta_n := \begin{bmatrix} 2 & -1 \\ -1 & 2 & -1 \\ & -1 & 2 & \ddots \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Hint: consider a matrix $K_n^{(\alpha)}$ that equals Δ_n apart from the top left entry which is $\alpha > 1$ and use a proof by induction.

Problem 3(b) Deduce its Cholesky factorisations: $\Delta_n = L_n L_n^\top$ where L_n is lower triangular.

Problem 1(a) Show for a unitary matrix $Q \in U(n)$ and a vector $\boldsymbol{x} \in \mathbb{C}^n$ that multiplication by Q preserve the 2-norm: $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$.

Problem 1(b) Show that the eigenvalues λ of a unitary matrix Q are on the unit circle: $|\lambda| = 1$. Hint: recall for any eigenvalue λ that there exists a unit eigenvector $\boldsymbol{v} \in \mathbb{C}^n$ (satisfying $\|\boldsymbol{v}\| = 1$).

Problem 1(c) Show for an orthogonal matrix $Q \in O(n)$ that $\det Q = \pm 1$. Give an example of $Q \in U(n)$ such that $\det Q \neq \pm 1$. Hint: recall for any real matrices A and B that $\det A = \det A^\top$ and $\det(AB) = \det A \det B$.

Problem 1(d) A normal matrix commutes with its adjoint. Show that $Q \in U(n)$ is normal.

Problem 1(e) The spectral theorem states that any normal matrix is unitarily diagonalisable: if A is normal then $A = V\Lambda V^\star$ where $V \in U(n)$ and Λ is diagonal. Use this to show that $Q \in U(n)$ is equal to I if and only if all its eigenvalues are 1.