MATH50003
Numerical Analysis

IV.2 Singular Value Decomposition and Matrix Compression

Dr Sheehan Olver

Chapter IV

Applications of Linear Algebra

1. Polynomial Interpolation and Regression for approximating data

2. Singular Value Decomposition and matix compression via best low rank approximation

Motivation: low rank approximation

Can we approximate a matrix by one of lower rank?

28×28

$\approx$

Rank 5

$=$

SVD

Definition 29 (singular value decomposition)**.** For $A \in \mathbb{C}^{m \times n}$ with rank $r > 0$, the *(reduced) singular value decomposition (SVD)* is

$$A = U \Sigma V^\star$$

where $U \in \mathbb{C}^{m \times r}$ and $V \in \mathbb{C}^{n \times r}$ have orthonormal columns and $\Sigma \in \mathbb{R}^{r \times r}$ is diagonal whose diagonal entries, which which we call *singular values*, are all positive and non-increasing: $\sigma_1 \geq \cdots \geq \sigma_r > 0$.

IV.2.1 Existence

We relate the SVD to the eigendecomposition of the Gram matrix

Proposition 13 (Gram matrix kernel). *The kernel of A equals the kernel of $A^\star A$.*

Proposition 14 (Gram matrix diagonalisation)**.** *The Gram-matrix satisfies*

$$A^\star A = Q\Lambda Q^\star \in \mathbb{C}^{n\times n}$$

is a Hermitian matrix where $Q \in U(n)$ *and the eigenvalues* λ_k *are real and non-negative. If* $A \in \mathbb{R}^{m\times n}$ *then* $Q \in O(n)$*.*

Theorem 10 (SVD existence). *Every $A \in \mathbb{C}^{m \times n}$ has an SVD.*

IV.2.2 2-norm and SVD

The 2-norm is given by largest singular value

Proposition 15 (diagonal/orthogonal 2-norms)**.** *If Λ is diagonal with entries λ_k then $\|\Lambda\| = \max_k |\lambda_k|$. If Q is orthogonal then $\|Q\| = 1$.*

Corollary 3 (singular values and norm)**.**

$$\|A\| = \sigma_1$$

and if $A \in \mathbb{C}^{n \times n}$ *is invertible, then*

$$\|A^{-1}\| = \sigma_n^{-1}$$

IV.2.3 Best rank-k approximation and compression

Use the SVD to compress matrices

Theorem 11 (best low rank approximation). *The matrix*

$$A_k := \underbrace{\begin{bmatrix} \boldsymbol{u}_1 | \cdots | \boldsymbol{u}_k \end{bmatrix}}_{=:U_k \in \mathbb{C}^{m\times k}} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix}}_{=:\Sigma_k \in \mathbb{C}^{k\times k}} \underbrace{\begin{bmatrix} \boldsymbol{v}_1 | \cdots | \boldsymbol{v}_k \end{bmatrix}^\star}_{=:V_k^\star \in \mathbb{C}^{k\times n}}$$

is the best 2-norm approximation of A by a rank k matrix, that is, for all rank-k matrices B, we have $\|A - A_k\| \leq \|A - B\|$.