BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022 (PRACTICE)

This paper is also taken for the relevant examination for the Associateship.

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XXX (Solutions)

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1. (a) (i) Using the geometric series we have

sim. seen ⇓

$$(2^{-4}+2^{-5}+2^{-6})\sum_{k=0}^{\infty}2^{-6k} = \frac{7}{64}\frac{1}{1-2^{-6}} = \frac{7}{64}\frac{64}{63} = \frac{1}{9}.$$

The exponent is $-4 = 11 - 15$ hence the bits are:

$$0\ 01011\ 1100011100$$

3, A

(ii) We have

meth seen ⇓

$$(1/9*(1+\delta_1)-1)(1+\delta_2) = -8/9 + \underbrace{\delta_1/9 - 8/9\delta_2 + \delta_1\delta_2/9}_{\delta}$$

where

$$|\delta| \leq \epsilon_{\mathrm{m}}/2(1/9+8/9+1/9) \leq \epsilon_{\mathrm{m}}$$

I.e. $c = 1$.

4, A

(b) (i) We have

sim. seen ⇓

$$f(x+h) = f(x) + f'(x)h + \frac{f''(t_1)}{2}h^2$$
$$f(x-h) = f(x) - f'(x)h + \frac{f''(t_2)}{2}h^2$$

where $t_1, t_2 \in [-h, h]$. Thus we have

$$\frac{f(x+h)+f(x)-2f(x-h)}{3h} = f'(x) + \frac{(f''(t_1)/2 - f''(t_2))h^2}{3h}$$

hence the error is bounded by

$$Mh(1/2+1)/3 = Mh/2.$$

3, B

(ii) We have

unseen ⇓

$$g(h)\oplus g(0) = (f(h)+f(0)+\delta_h+\delta_0)(1+\delta^1) = f(h)+f(0)+\underbrace{(f(h)+f(0))\delta^1 + (\delta_h+\delta_0)(1+\delta^1)}_{\delta^2}$$

where $|\delta^2| \leq (N+4c)\epsilon_m$. Then we have

$$(g(h)\oplus g(0))\ominus 2g(-h) = f(h)+f(0)-2f(-h) + \underbrace{\delta^2 - 2\delta_{-h}}_{\delta^3}$$

where $|\delta^3| \leq (N+6c)\epsilon_m$. Putting everything together we have

$$\frac{(g(h)\oplus g(0))\ominus 2g(-h)}{3h} = \frac{f(h)+f(0)-2f(-h)}{3h} + \frac{\delta^3}{3h} = f'(0) + Mh/2 + \frac{\delta^3}{3h}$$

hence $A = M/2$ and $B = N/3 + 2c$.

5, D

(c) (i)

sim. seen ⇓

$$\exp(\cos(1/2+\epsilon)) = \exp(\cos(1/2) - \sin(1/2)\epsilon) = \exp\cos(1/2) - \sin(1/2)\exp(\cos 1/2)\epsilon$$

hence the derivative is $-\sin(1/2)\exp(\cos 1/2)$.

2, A

(ii) A dual extension is a definition that is consistent with differentiation. Hence we have

unseen ⇓

$$\operatorname{erf}(a+b\epsilon) := \operatorname{erf}(a) + \frac{2}{\sqrt{\pi}}b\exp(-a^2)\epsilon.$$

3, A

2. (a) Let $K \in \mathbb{R}^{m\times(m-n)}$ be a matrix with orthonormal columns so that

$$U = [K|Q] \in \mathbb{R}^{m\times m}$$

is orthogonal. Then

$$A = QL = U \underbrace{\begin{bmatrix} 0_{(m-n)\times n} \\ L \end{bmatrix}}_{\tilde{L}}$$

Thus

$$\|A\mathbf{x} - \mathbf{b}\|^2 = \|U\tilde{L}\mathbf{x} - \mathbf{b}\|^2 = \|\tilde{L}\mathbf{x} - U^\top\mathbf{b}\|^2 = \|L\mathbf{x} - Q^\top\mathbf{b}\|^2 + \text{const}$$

where the constant term is independent of $\mathbf{x}$. The first term is minimised when it is 0, and thus $\mathbf{x} = L^{-1}Q^\top\mathbf{b}$.

unseen ⇓

5, C

(b) Assuming $\alpha > 0$ write

sim. seen ⇓

$$K_n^\alpha := \begin{bmatrix} \alpha & 1 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha} & \\ \frac{\mathbf{e}_1}{\sqrt{\alpha}} & I \end{bmatrix} \begin{bmatrix} 1 & \\ & A_{n-1} - \frac{\mathbf{e}_1\mathbf{e}_1^\top}{\alpha} \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & \frac{\mathbf{e}_1^\top}{\sqrt{\alpha}} \\ & I \end{bmatrix}$$

But note that $A_{n-1} - \frac{\mathbf{e}_1\mathbf{e}_1^\top}{\alpha} = K_{n-1}^{3-1/\alpha}$. Note if $\alpha > 1$ then $3 - 1/\alpha > 1$. Hence by induction and the fact that $A_n = K_n^3$ we conclude that the matrix has a Cholesky decomposition and hence is symmetric positive definite.

5, D

(c) (i)

sim. seen ⇓

$$\|A\|_1 = \max_j \|A\mathbf{e}_j\|_1 = \max(3, 9, 7) = 9$$
$$\|A\|_\infty = \max_k \|\mathbf{e}_k^\top A\|_1 = \max(3, 9, 7) = 9$$

2, A

(ii) Compute the PLU decomposition of,

sim. seen ⇓

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

We begin by permuting the first and second row as —2— ¿ —0—, hence,

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_1A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

We then choose,

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}, \qquad L_1P_1A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & \frac{9}{2} \end{bmatrix}$$

There is no need to permute at this stage, so $P_2 = I_3$, the $3-$dimensional identity. Then we can choose,

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad L_2 P_2 L_1 P_1 A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 11/2 \end{bmatrix} =: U$$

Since $P_2 = I_3$, this reduces to $L_2 L_1 P_1 A = U \Rightarrow A = P_1^\top L_1^{-1} L_2^{-1} U$. Since P_1 simply permutes two rows, it is its own inverse, and $L_1^{-1} L_2^{-1}$ is simply,

$$L := L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix}$$

Hence, we have $A = P^\top L U$, where,

$$P^\top = P_1^\top = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 11/2 \end{bmatrix}$$

4, B

(d) We have the system:

unseen ⇓

$$\begin{aligned} u_1 &= 1 \\ \frac{u_2 - u_1}{h} &= a t_1 \\ \frac{u_3 - u_1}{2h} &= a t_1 \\ \frac{u_4 - u_2}{2h} &= a t_2 \\ &\vdots \\ \frac{u_n - u_{n-2}}{2h} &= a t_{n-2} \end{aligned}$$

This has the lower bidiagonal discretisation

$$\begin{bmatrix} 1 & & & & \\ -1/h & 1/h & & & \\ -1/(2h) & 0 & 1/(2h) & & \\ & \ddots & \ddots & \ddots & \\ & & -1/(2h) & 0 & 1/(2h) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 1 \\ a t_1 \\ a t_1 \\ \vdots \\ a t_{n-2} \end{bmatrix}$$

4, C

3. (a) We have

sim. seen ⇓

$$\sin 3\theta = \mathrm{e}^{3\mathrm{i}\theta}/(2\mathrm{i}) - \mathrm{e}^{-3\mathrm{i}\theta}/(2\mathrm{i})$$

hence $\hat{f}_{-3} = -1/(2\mathrm{i})$ and $\hat{f}_3 = 1/(2\mathrm{i})$, and $\hat{f}_k = 0$ otherwise.
We use these to deduce the following:

$$\begin{aligned}
\hat{f}^1_k &= \hat{f}_{-3} + \hat{f}_3 = 0\\
\hat{f}^2_{2p} &= 0, \hat{f}^2_{1+2p} = \hat{f}_{-3} + \hat{f}_3 = 0\\
\hat{f}^3_{3p} &= \hat{f}_{-3} + \hat{f}_3 = 0, \hat{f}^3_k = 0 \text{ otherwise}\\
\hat{f}^4_{4p} &= 0, \hat{f}^4_{4p+1} = \hat{f}_{-3} = -1/(2\mathrm{i}), \hat{f}^4_{4p+2} = 0, \hat{f}^3_{4p+3} = \hat{f}_3 = 1/(2\mathrm{i})\\
\hat{f}^5_{5p} &= 0, \hat{f}^5_{5p+1} = 0, \hat{f}^5_{5p+2} = \hat{f}_{-3} = -1/(2\mathrm{i}), \hat{f}^5_{5p+3} = \hat{f}_3 = 1/(2\mathrm{i}), \hat{f}^5_{5p+4} = 0,\\
\hat{f}^6_{3+6p} &= \hat{f}_{-3} + \hat{f}_3 = 0, \hat{f}^6_k = 0.
\end{aligned}$$

For $n \geq 7$ we then have

$$\hat{f}^n_{-3+np} = \hat{f}_{-3}, \hat{f}^n_{3+np} = \hat{f}_3, \hat{f}^n_k = \hat{f}_k \text{ otherwise.}$$

5, B

(b) (i) Note since $w(x) = w(-x)$ we know that $a_n = 0$ in the 3-term recurrence. Note that

sim. seen ⇓

$$\|R_0\|^2 = \int_{-1}^1 (1-x^2)\mathrm{d}x = 4/3$$

Thus we have

$$xR_0(x) = R_1(x)$$

hence $R_1(x) = x$. Then

$$xR_1(x) = c_0 R_0(x) + R_2(x)$$

where

$$c_0 = \langle xR_1, R_0\rangle / \|R_0\|^2 = \frac{3}{4}\int_{-1}^1 (1-x^2)x^2\mathrm{d}x = 1/5.$$

Thus

$$R_2(x) = xR_1(x) - c_0 R_0(x) = x^2 - 1/5.$$

5, A

(ii) Denote

unseen ⇓

$$p_n(x) = \frac{(n+2)!}{(2(n+1))!}\frac{(-)^n}{1-x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x^2)^{n+1}].$$

We want to show (1) graded polynomials, (2) orthogonal to all lower degree polynomials and (3) right normalisation constant. To show (1) note that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x^2)^{n+1}]$$

is a degree $2(n+1)-n = n+2$ polynomial that vanishes at ± 1. Thus dividing by $1-x^2$ we have a degree n polynomial. To show (2) we simple integrate by parts: let r_m be a degree $m < n$ polynomial, then

$$\begin{aligned}
\int_{-1}^1 p_n(x) r_m(x)(1-x^2)\mathrm{d}x &= \text{const.}\int_{-1}^1 \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x^2)^{n+1}] r_m(x)\mathrm{d}x\\
&= \underbrace{\cdots}_{\text{integrate by parts } n \text{ times}}\\
&= \text{const.}(-)^n (1-x^2)^{n+1} r_m^{(n)}(x)\mathrm{d}x = 0.
\end{aligned}$$

For (3) consider

$$(1-x^2)p_n(x) = \frac{(n+2)!}{(2(n+1))!}(-)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(-)^{n+1}x^{2n+2} + O(x^{2n+1})]$$
$$= -\frac{(n+2)!}{(2(n+1))!}(2n+2)\cdots(n+3)x^{n+2} + O(x^{n+1}) = -x^{n+2} + O(x^{n+1})$$

We note by uniqueness that $p_n(x) = cR_n(x)$ so we just need to show that $c = 1$. But we have

$$(1-x^2)p_n(x) = c(1-x^2)R_n(x) = -cx^{n+2} + O(x^{n+1})$$

which shows that $c = 1$ and $p_n(x) = R_n(x)$. $\boxed{5, \mathrm{D}}$

(c) The two quadrature points are the roots of $R_2(x)$, that is: $\boxed{\text{sim. seen} \Downarrow}$

$$x_1 = -1/\sqrt{5}, x_2 = 1/\sqrt{5}.$$

To determine the weights we compute the integral of the Lagrange interpolating polynomial (noting that $\int_{-1}^{1} w(x)x\mathrm{d}x = 0$ by symmetry to simplify the computation):

$$w_1 = \int_{-1}^{1} w(x)\ell_1(x)\mathrm{d}x = \int_{-1}^{1}(1-x^2)\frac{x-x_2}{x_1-x_2}\mathrm{d}x = \frac{1}{2}\int_{-1}^{1}(1-x^2)\mathrm{d}x = \frac{2}{3}.$$

Again by symmetry $w_2 = w_1$. Thus we get the quadrature rule:

$$\frac{2}{3}(p(-1/\sqrt{5}) + p(1/\sqrt{5}))$$

Hint: double check its correct. For $p(x) = 1$ we have

$$\frac{2}{3}(p(-1/\sqrt{5}) + p(1/\sqrt{5})) = \frac{4}{3} = \int_{-1}^{1}(1-x^2)\mathrm{d}x$$

For $p(x) = x$ the integral is zero. For $p(x) = x^2$ we have:

$$\frac{2}{3}(p(-1/\sqrt{5}) + p(1/\sqrt{5})) = \frac{4}{15} = \int_{-1}^{1}(1-x^2)x^2\mathrm{d}x$$

For $p(x) = x^3$ the integral is also zero. $\boxed{5, \mathrm{B}}$

Review of mark distribution:
Total A marks: 19 of 32 marks
Total B marks: 17 of 20 marks
Total C marks: 9 of 12 marks
Total D marks: 15 of 16 marks
Total marks: 60 of 80 marks
Total Mastery marks: 0 of 20 marks