

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022 (PRACTICE)

This paper is also taken for the relevant examination for the Associateship of the
Royal College of Science.

Numerical Analysis

Date: ??

Time: ??

Time Allowed: 1.5 Hours

This paper has 3 *Questions*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) (i) Show that

$$\frac{1}{9} = (0.00011100011100111000111 \dots)_2$$

and deduce the bits of $1/9$ in half-precision (F_{16}) floating point arithmetic (rounded to the nearest floating point number). (3 marks)

- (ii) For the floating point computation

$$(1 \oslash 9) \ominus 1 = \frac{1}{9} - 1 + \delta$$

find an explicit constant c such that $|\delta| \leq c\epsilon_m$ holds, where ϵ_m is machine epsilon. You may assume that all computations involve normal numbers. (4 marks)

- (b) (i) Use Taylor series to derive a bound for the three-term finite-difference approximation

$$\left| f'(x) - \frac{f(x+h) + f(x) - 2f(x-h)}{3h} \right|$$

in terms of $M = \sup_{x-h \leq \chi \leq x+h} |f''(\chi)|$ (assuming exact arithmetic). (3 marks)

- (ii) Find an error bound of the form

$$\left| f'(0) - \frac{(g(h) \oplus g(0)) \ominus 2g(-h)}{3h} \right| \leq \frac{|f'(0)|}{2} \epsilon_m + Ah + \frac{B\epsilon_m}{h}$$

where $g(x) = f(x) + \delta_x$ for $|\delta_x| \leq c\epsilon_m$ for some constant c , A depends on M and B depends on $N = \sup_{x-h \leq \chi \leq x+h} |f(\chi)|$. You may assume that all computations involve normal numbers, and the division by $3h$ is done exactly (without error). (5 marks)

- (c) (i) Use dual numbers— $a + b\epsilon$ such that $\epsilon^2 = 0$ —to differentiate

$$\exp(\cos(x))$$

at the point $x = 1/2$. (2 marks)

- (ii) The error function is defined as:

$$\operatorname{erf} x := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Write down a *dual extension* of the definition to dual numbers $\operatorname{erf}(a + b\epsilon)$ in terms of $\operatorname{erf}(a)$ and elementary functions. (3 marks)

(Total: 20 marks)

2. (a) Consider a QL decomposition for $A \in \mathbb{R}^{m \times n}$, $m \geq n$ with full column rank:

$$A = QL$$

where $Q \in \mathbb{R}^{m \times n}$ has orthogonal columns and $L \in \mathbb{R}^{n \times n}$ is (square) lower triangular. Show that $\mathbf{x} := L^{-1}Q^\top \mathbf{b}$ minimises the 2-norm error $\|A\mathbf{x} - \mathbf{b}\|$. (5 marks)

- (b) Prove that

$$A_n = \begin{bmatrix} 3 & 1 & & \\ 1 & 3 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 3 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

is positive definite for all n . (5 marks)

- (c) For the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

- (i) Deduce the 1 and ∞ norms for the matrix. (2 marks)
(ii) Deduce a decomposition of the form $A = P^\top LU$ where P is a permutation matrix, L lower triangular with ones on the diagonal and U is upper triangular. (4 marks)
(d) For an evenly spaced grid $t_k = (k-1)h$, for $k = 1, \dots, n$ and $h = 1/(n-1)$, use the approximation

$$u'(t_1) \approx \frac{u_2 - u_1}{h}$$

$$u'(t_k) \approx \frac{u_{k+2} - u_k}{2h}$$

to write down a *lower-tridiagonal* linear system to approximate the solution to

$$u(0) = 1$$

$$u'(t) = at$$

(4 marks)

(Total: 20 marks)

3. (a) For the function $f(\theta) = \sin 3\theta$, state explicit formulae for its Fourier coefficients

$$\hat{f}_k := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta$$

and their discrete approximation:

$$\hat{f}_k^n := \frac{1}{n} \sum_{j=0}^{n-1} f(\theta_j) e^{-ik\theta_j}.$$

for *all* integers k , $n = 1, 2, \dots$ and $\theta_j = 2\pi j/n$. (5 marks)

- (b) Consider monic orthogonal polynomials

$$R_0(x) = 1, \quad R_n(x) = x^n + O(x^{n-1})$$

as $x \rightarrow \infty$ and $n = 1, 2, \dots$, orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)w(x)dx, \quad w(x) = 1 - x^2.$$

- (i) Construct $R_0(x)$, $R_1(x)$ and hence show that $R_2(x) = x^2 - 1/5$. (5 marks)
(ii) Show for all $n = 0, 1, 2, \dots$ the Rodriguez formula

$$R_n(x) = \frac{(n+2)!}{(2(n+1))!} \frac{(-)^n}{1-x^2} \frac{d^n}{dx^n} [(1-x^2)^{n+1}].$$

(5 marks)

- (c) Compute the 2-point Gauss quadrature rule for $w(x) = 1 - x^2$, that exactly integrates all polynomials p up to degree 3:

$$\int_{-1}^1 p(x)w(x)dx = w_1p(x_1) + w_2p(x_2).$$

You may use the formula for $R_2(x)$ from part (b). (5 marks)

(Total: 20 marks)