BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022 (PRACTICE)

This paper is also taken for the relevant examination for the Associateship.

XXX

XXX (Solutions)

Setter's signature	Checker's signature	Editor's signature
....	....	....

1. (a) (i) Using the geometric series we have

$$(2^{-4} + 2^{-5} + 2^{-6})\sum_{k=0}^{\infty} 2^{-6k} = \frac{7}{64}\frac{1}{1-2^{-6}} = \frac{7}{64}\frac{64}{63} = \frac{1}{9}.$$

The exponent is $-4 = 11 - 15$ hence the bits are:

$$0\ 01011\ 1100011100$$

(ii) We have

$$(1/9 * (1 + \delta_1) - 1)(1 + \delta_2) = -8/9 + \underbrace{\delta_1/9 - 8/9\delta_2 + \delta_1\delta_2/9}_{\delta}$$

where

$$|\delta| \leq \epsilon_m/2(1/9 + 8/9 + 1/9) \leq \epsilon_m$$

I.e. $c = 1$.

(b) (i) We have

$$f(x + h) = f(x) + f'(x)h + \frac{f''(t_1)}{2}h^2$$

$$f(x - h) = f(x) - f'(x)h + \frac{f''(t_2)}{2}h^2$$

where $t_1, t_2 \in [-h, h]$. Thus we have

$$\frac{f(x + h) + f(x) - 2f(x - h)}{3h} = f'(x) + \frac{(f''(t_1)/2 - f''(t_2))h^2}{3h}$$

hence the error is bounded by

$$Mh(1/2 + 1)/3 = Mh/2.$$

(ii) We have

$$g(h)\oplus g(0) = (f(h)+f(0)+\delta_h+\delta_0)(1+\delta^1) = f(h)+f(0)+\underbrace{(f(h) + f(0))\delta^1 + (\delta_h + \delta_0)(1+\delta^1)}_{\delta^2}$$

where $|\delta^2| \leq (N + 4c)\epsilon_m$. Then we have

$$(g(h) \oplus g(0)) \ominus 2g(-h) = (f(h) + f(0) - 2f(-h) + \delta^2 - 2\delta_{-h})(1 + \delta_3)$$
$$= f(h) + f(0) - 2f(-h)$$
$$+ \underbrace{(f(h) + f(0) - 2f(-h))\delta_3 + (\delta^2 - 2\delta_{-h})(1 + \delta_3)}_{\delta^3}$$

where $|\delta^3| \leq (4N + 8c)\epsilon_m$. Putting everything together we have

$$\frac{(g(h) \oplus g(0)) \ominus 2g(-h)}{3h} = \frac{f(h) + f(0) - 2f(-h)}{3h} + \frac{\delta^3}{3h} = f'(0) + Mh/2 + \frac{\delta^3}{3h}$$

hence $A = M/2$ and $B = 4N/3 + 8c/3$.

(c) (i)

$$\exp(\cos(1/2+\epsilon)) = \exp(\cos(1/2)-\sin(1/2)\epsilon) = \exp\cos(1/2)-\sin(1/2)\exp(\cos 1/2)\epsilon$$

hence the derivative is $-\sin(1/2)\exp(\cos 1/2)$.

(ii) A dual extension is a definition that is consistent with differentiation. Hence we have

$$\text{erf}(a + b\epsilon) := \text{erf}(a) + \frac{2}{\sqrt{\pi}}b\exp(-a^2)\epsilon.$$

2. (a) Let $K \in \mathbb{R}^{m \times (m-n)}$ be a matrix with orthonormal columns so that

$$U = [K|Q] \in \mathbb{R}^{m \times m}$$

is orthogonal. Then

$$A = QL = U \begin{bmatrix} 0_{(m-n) \times n} \\ L \end{bmatrix}}_{\tilde{L}}$$

Thus

$$\|A\mathbf{x} - \mathbf{b}\|^2 = \|U\tilde{L}\mathbf{x} - \mathbf{b}\|^2 = \|\tilde{L}\mathbf{x} - U^\top \mathbf{b}\|^2 = \|L\mathbf{x} - Q^\top \mathbf{b}\|^2 + \text{const}$$

where the constant term is independent of $\mathbf{x}$. The first term is minimised when it is 0, and thus $\mathbf{x} = L^{-1}Q^\top \mathbf{b}$.

(b) Assuming $\alpha > 0$ write

$$K_n^\alpha := \begin{bmatrix} \alpha & 1 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha} & \\ \frac{\mathbf{e}_1}{\sqrt{\alpha}} & I \end{bmatrix} \begin{bmatrix} 1 & \\ & A_{n-1} - \frac{\mathbf{e}_1 \mathbf{e}_1^\top}{\alpha} \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & \frac{\mathbf{e}_1^\top}{\sqrt{\alpha}} \\ & I \end{bmatrix}$$

But note that $A_{n-1} - \frac{\mathbf{e}_1 \mathbf{e}_1^\top}{\alpha} = K_{n-1}^{3-1/\alpha}$. Note if $\alpha > 1$ then $3 - 1/\alpha > 1$. Hence by induction and the fact that $A_n = K_n^3$ we conclude that the matrix has a Cholesky decomposition and hence is symmetric positive definite.

(c) (i)

$$\|A\|_1 = \max_j \|A\mathbf{e}_j\|_1 = \max(3, 9, 7) = 9$$

$$\|A\|_\infty = \max_k \|\mathbf{e}_k^\top A\|_1 = \max(3, 9, 7) = 9$$

(ii) Compute the PLU decomposition of,

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

We begin by permuting the first and second row as —2— ¿ —0—, hence,

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_1 A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

We then choose,

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}, \qquad L_1 P_1 A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & \frac{9}{2} \end{bmatrix}$$

There is no need to permute at this stage, so $P_2 = I_3$, the $3-$dimensional identity. Then we can choose,

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad L_2 P_2 L_1 P_1 A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 11/2 \end{bmatrix} =: U$$

Since $P_2 = I_3$, this reduces to $L_2 L_1 P_1 A = U \Rightarrow A = P_1^\top L_1^{-1} L_2^{-1} U$. Since P_1 simply permutes two rows, it is its own inverse, and $L_1^{-1} L_2^{-1}$ is simply,

$$L := L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix}$$

Hence, we have $A = P^\top L U$, where,

$$P^\top = P_1^\top = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 11/2 \end{bmatrix}$$

<div style="text-align: right;">

4, B

</div>

(d) We have the system:

<div style="text-align: right;">

unseen ⇓

</div>

$$u_1 = 1$$

$$\frac{u_2 - u_1}{h} = at_1$$

$$\frac{u_3 - u_1}{2h} = at_1$$

$$\frac{u_4 - u_2}{2h} = at_2$$

$$\vdots$$

$$\frac{u_n - u_{n-2}}{2h} = at_{n-2}$$

This has the lower bidiagonal discretisation

$$\begin{bmatrix} 1 & & & & \\ -1/h & 1/h & & & \\ -1/(2h) & 0 & 1/(2h) & & \\ & \ddots & \ddots & \ddots & \\ & & -1/(2h) & 0 & 1/(2h) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 1 \\ at_1 \\ at_1 \\ \vdots \\ at_{n-2} \end{bmatrix}$$

<div style="text-align: right;">

4, C

</div>

3. (a) We have

$$\sin 3\theta = e^{3i\theta}/(2i) - e^{-3i\theta}/(2i)$$

hence $\hat{f}_{-3} = -1/(2i)$ and $\hat{f}_3 = 1/(2i)$, and $\hat{f}_k = 0$ otherwise.
We use these to deduce the following:

$$\hat{f}_k^1 = \hat{f}_{-3} + \hat{f}_3 = 0$$
$$\hat{f}_{2p}^2 = 0, \hat{f}_{1+2p}^2 = \hat{f}_{-3} + \hat{f}_3 = 0$$
$$\hat{f}_{3p}^3 = \hat{f}_{-3} + \hat{f}_3 = 0, \hat{f}_k^3 = 0 \text{ otherwise}$$
$$\hat{f}_{4p}^4 = 0, \hat{f}_{4p+1}^4 = \hat{f}_{-3} = -1/(2i), \hat{f}_{4p+2}^4 = 0, \hat{f}_{4p+3}^3 = \hat{f}_3 = 1/(2i)$$
$$\hat{f}_{5p}^5 = 0, \hat{f}_{5p+1}^5 = 0, \hat{f}_{5p+2}^5 = \hat{f}_{-3} = -1/(2i), \hat{f}_{5p+3}^5 = \hat{f}_3 = 1/(2i), \hat{f}_{5p+4}^5 = 0,$$
$$\hat{f}_{3+6p}^6 = \hat{f}_{-3} + \hat{f}_3 = 0, \hat{f}_k^6 = 0.$$

For $n \geq 7$ we then have

$$\hat{f}_{-3+np}^n = \hat{f}_{-3}, \hat{f}_{3+np}^n = \hat{f}_3, \hat{f}_k^n = \hat{f}_k \text{ otherwise.}$$

(b) (i) Note since $w(x) = w(-x)$ we know that $a_n = 0$ in the 3-term recurrence.

Note that

$$\|R_0\|^2 = \int_{-1}^1 (1 - x^2)\mathrm{d}x = 4/3$$

Thus we have

$$xR_0(x) = R_1(x)$$

hence $R_1(x) = x$. Then

$$xR_1(x) = c_0 R_0(x) + R_2(x)$$

where

$$c_0 = \langle xR_1, R_0 \rangle / \|R_0\|^2 = \frac{3}{4} \int_{-1}^1 (1 - x^2)x^2 \mathrm{d}x = 1/5.$$

Thus

$$R_2(x) = xR_1(x) - c_0 R_0(x) = x^2 - 1/5.$$

(ii) Denote

$$p_n(x) = \frac{(n+2)!}{(2(n+1))!} \frac{(-)^n}{1 - x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1 - x^2)^{n+1}].$$

We want to show (1) graded polynomials, (2) orthogonal to all lower degree polynomials and (3) right normalisation constant. To show (1) note that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1 - x^2)^{n+1}]$$

is a degree $2(n+1) - n = n+2$ polynomial that vanishes at ± 1. Thus dividing by $1 - x^2$ we have a degree n polynomial. To show (2) we simple integrate by parts: let r_m be a degree $m < n$ polynomial, then

$$\int_{-1}^1 p_n(x)r_m(x)(1 - x^2)\mathrm{d}x = \text{const.} \int_{-1}^1 \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1 - x^2)^{n+1}]r_m(x)\mathrm{d}x$$

$$= \underbrace{\cdots}_{\text{integrate by parts } n \text{ times}}$$

$$= \text{const.}(-)^n(1 - x^2)^{n+1}r_m^{(n)}(x)\mathrm{d}x = 0.$$

For (3) consider

$$(1-x^2)p_n(x) = \frac{(n+2)!}{(2(n+1))!}(-)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(-)^{n+1}x^{2n+2} + O(x^{2n+1})]$$

$$= -\frac{(n+2)!}{(2(n+1))!}(2n+2)\cdots(n+3)x^{n+2} + O(x^{n+1}) = -x^{n+2} + O(x^{n+1})$$

We note by uniqueness that $p_n(x) = cR_n(x)$ so we just need to show that $c = 1$. But we have

$$(1-x^2)p_n(x) = c(1-x^2)R_n(x) = -cx^{n+2} + O(x^{n+1})$$

which shows that $c = 1$ and $p_n(x) = R_n(x)$.

(c) The two quadrature points are the roots of $R_2(x)$, that is:

$$x_1 = -1/\sqrt{5}, x_2 = 1/\sqrt{5}.$$

To determine the weights we compute the integral of the Lagrange interpolating polynomial (noting that $\int_{-1}^{1} w(x)x\mathrm{d}x = 0$ by symmetry to simplify the computation):

$$w_1 = \int_{-1}^{1} w(x)\ell_1(x)\mathrm{d}x = \int_{-1}^{1}(1-x^2)\frac{x-x_2}{x_1-x_2}\mathrm{d}x = \frac{1}{2}\int_{-1}^{1}(1-x^2)\mathrm{d}x = \frac{2}{3}.$$

Again by symmetry $w_2 = w_1$. Thus we get the quadrature rule:

$$\frac{2}{3}(p(-1/\sqrt{5}) + p(1/\sqrt{5}))$$

Hint: double check its correct. For $p(x) = 1$ we have

$$\frac{2}{3}(p(-1/\sqrt{5}) + p(1/\sqrt{5})) = \frac{4}{3} = \int_{-1}^{1}(1-x^2)\mathrm{d}x$$

For $p(x) = x$ the integral is zero. For $p(x) = x^2$ we have:

$$\frac{2}{3}(p(-1/\sqrt{5}) + p(1/\sqrt{5})) = \frac{4}{15} = \int_{-1}^{1}(1-x^2)x^2\mathrm{d}x$$

For $p(x) = x^3$ the integral is also zero.

Review of mark distribution:

Total A marks: 19 of 32 marks

Total B marks: 17 of 20 marks

Total C marks: 9 of 12 marks

Total D marks: 15 of 16 marks

Total marks: 60 of 80 marks

Total Mastery marks: 0 of 20 marks