BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022 (PRACTICE)

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Numerical Analysis

Date: ??
Time: ??
Time Allowed: 1.5 Hours
This paper has *3 Questions*.
Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) (i) Show that

$$\frac{1}{9} = (0.000111000111001110001 11\ldots)_2$$

and deduce the bits of $1/9$ in half-precision (F_{16}) floating point arithmetic (rounded to the nearest floating point number). (3 marks)

(ii) For the floating point computation

$$(1 \oslash 9) \ominus 1 = \frac{1}{9} - 1 + \delta$$

find an explicit constant c such that $|\delta| \leq c\epsilon_{\mathrm{m}}$ holds, where ϵ_{m} is machine epsilon. You may assume that all computations involve normal numbers. (4 marks)

(b) (i) Use Taylor series to derive a bound for the three-term finite-difference approximation

$$\left| f'(x) - \frac{f(x+h) + f(x) - 2f(x-h)}{3h} \right|$$

in terms of $M = \sup_{x-h \leq \chi \leq x+h} |f''(\chi)|$ (assuming exact arithmetic). (3 marks)

(ii) Find an error bound of the form

$$\left| f'(0) - \frac{(g(h) \oplus g(0)) \ominus 2g(-h)}{3h} \right| \leq \frac{|f'(0)|}{2}\epsilon_m + Ah + \frac{B\epsilon_m}{h}$$

where $g(x) = f(x) + \delta_x$ for $|\delta_x| \leq c\epsilon_{\mathrm{m}}$ for some constant c, A depends on M and B depends on $N = \sup x - h \leq \chi \leq x + h|f(x)|$. You may assume that all computations involve normal numbers, and the division by $3h$ is done exactly (without error). (5 marks)

(c) (i) Use dual numbers—$a + b\epsilon$ such that $\epsilon^2 = 0$—to differentiate

$$\exp(\cos(x))$$

at the point $x = 1/2$. (2 marks)

(ii) The error function is defined as:

$$\mathrm{erf} x := \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-x^2} \mathrm{d}x$$

Write down a *dual extension* of the definition to dual numbers $\mathrm{erf}(a + b\epsilon)$ in terms of $\mathrm{erf}(a)$ and elementary functions. (3 marks)

(Total: 20 marks)

2. (a) Consider a QL decomposition for $A \in \mathbb{R}^{m\times n}$, $m \geq n$ with full column rank:

$$A = QL$$

where $Q \in \mathbb{R}^{m\times n}$ has orthogonal columns and $L \in \mathbb{R}^{n\times n}$ is (square) lower triangular. Show that $\mathbf{x} := L^{-1}Q^\top b$ minimises the 2-norm error $\|A\mathbf{x} - \mathbf{b}\|$. (5 marks)

(b) Prove that

$$A_n = \begin{bmatrix} 3 & 1 & & \\ 1 & 3 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 3 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

is positive definite for all n. (5 marks)

(c) For the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

(i) Deduce the 1 and ∞ norms for the matrix. (2 marks)

(ii) Deduce a decomposition of the form $A = P^\top LU$ where P is a permutation matrix, L lower triangular with ones on the diagonal and U is upper triangular. (4 marks)

(d) For an evenly spaced grid $t_k = (k-1)h$ use the approximation

$$u'(t_1) \approx \frac{u_{k+1} - u_k}{h}$$
$$u'(t_k) \approx \frac{u_{k+2} - u_k}{2h}$$

to write down a $lower\text{-}tridiagonal$ linear system to approximate the solution to

$$u(0) = 1$$
$$u'(t) = at$$

(4 marks)

(Total: 20 marks)

3. (a) For the function $f(\theta) = \sin 3\theta$, state explicit formulae for its Fourier coefficients

$$\hat{f}_k := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta$$

and their discrete approximation:

$$\hat{f}_k^n := \frac{1}{n} \sum_{j=0}^{n-1} f(\theta_j) \mathrm{e}^{-\mathrm{i}k\theta_j}.$$

for all integers k, $n = 1, 2, \ldots$ and $\theta_j = 2\pi j/n$. (5 marks)

(b) Consider monic orthogonal polynomials

$$R_0(x) = 1, \qquad R_n(x) = x^n + O(x^{n-1})$$

as $x \to \infty$ and $n = 1, 2, \ldots$, orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) w(x) \mathrm{d}x, \qquad w(x) = 1 - x^2.$$

(i) Construct $R_0(x)$, $R_1(x)$ and hence show that $R_2(x) = x^2 - 1/5$. (5 marks)

(ii) Show for all $n = 0, 1, 2, \ldots$ the Rodriguez formula

$$R_n(x) = \frac{(n+2)!}{(2(n+1))!} \frac{(-)^n}{1 - x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} [(1 - x^2)^{n+1}].$$

(5 marks)

(c) Compute the 2-point Gauss quadrature rule for $w(x) = 1 - x^2$, that exactly integrates all polynomials p up to degree 3:

$$\int_{-1}^{1} p(x) w(x) \mathrm{d}x = w_1 p(x_1) + w_2 p(x_2).$$

You may use the formula for $R_2(x)$ from part (b). (5 marks)

(Total: 20 marks)