MATH50003 Numerical Analysis (2022–23)

Problem Sheet 4

This problem sheet concerns matrix multiplication, permutation matrices, and properties of orthogonal/unitary matrices.

Problem 1.1 True or false: `mul_rows(A, x)` and `mul_cols(A, x)` from the lectures/notes will always return the exact same result if the input are floating point? Explain your answer.

SOLUTION

True. Even though the implementations are different the mathematical operations are still performed for each entry in the same way. On the other hand, if we had an implementation that reversed the order the columns/rows are traversed this would change the answer.

END

Problem 1.2 Express the vector that `mul_rows(A, x)` returns in terms of $\bigoplus_{j=1}^n$ and $\otimes$, using the notation $A_{k,j}$ for the k, j entry of A and x_j for the j-th entry of $\mathbf{x}$, where $A \in F_{\sigma,Q,S}^{m \times n}$ and $\mathbf{x} \in F_{\sigma,Q,S}^n$.

SOLUTION

$$\begin{pmatrix} \bigoplus_{j=1}^n A_{1,j} \otimes x_j \\ \vdots \\ \bigoplus_{j=1}^n A_{1,j} \otimes x_j \end{pmatrix}$$

END

Problem 1.3 Show when implemented in floating point arithmetic for $A \in F_{\sigma,Q,S}^{m \times n}$ and $\mathbf{x} \in F_{\sigma,Q,S}^n$ that

$$\backslash\text{tt mul}\backslash\_\text{rows}(A, \mathbf{x}) = A\mathbf{x} + \boldsymbol{\delta}$$

where

$$\|\delta\|_\infty \leq 2\|A\|_\infty \|\mathbf{x}\|_\infty E_{n-1,\epsilon_m/2}$$

and

$$E_{n,\varepsilon} := \frac{n\varepsilon}{1 - n\varepsilon},$$

assuming that $n\epsilon_{\rm m} < 2$. We use the notation (to be discussed in detail later):

$$\|A\|_\infty := \max_k \sum_{j=1}^n |a_{kj}|, \|\mathbf{x}\|_\infty := \max_k |x_k|.$$

You may assume all operations are in the normalised range and use Problem 2.3 from PS2.

SOLUTION We have for the k=th row

$$\bigoplus_{j=1}^n A_{k,j} \otimes x_j = \bigoplus_{j=1}^n A_{k,j} x_j (1 + \delta_j) = \sum_{j=1}^n A_{k,j} x_k (1 + \delta_j) + \sigma_{k,n}$$

where from PS2.2.3 we know $|\sigma_n| \leq M_k E_{n-1,\epsilon_{\rm m}/2}$, where

$$M_k \leq \Sigma_{j=1}^n |A_{k,j}||x_k|(1 + |\delta_j|) \leq 2 \max |x_k| \Sigma_{j=1}^n |A_{k,j}| \leq 2\|\mathbf{x}\|_\infty \|A\|_\infty$$

END

Problem 2 What are the permutation matrices corresponding to the following permutations?

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}.$$

SOLUTION

Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}.$$

There are two ways to construct P_σ and P_τ.

- Column by column:

$$P_\sigma = \left(\mathbf{e}_{\sigma_1^{-1}} \middle| \mathbf{e}_{\sigma_2^{-1}} \middle| \mathbf{e}_{\sigma_3^{-1}}\right) = (\mathbf{e}_3|\mathbf{e}_2|\mathbf{e}_1)$$

$$P_\tau = \left(\mathbf{e}_{\tau_1^{-1}} \middle| \mathbf{e}_{\tau_2^{-1}} \middle| \mathbf{e}_{\tau_3^{-1}} \middle| \mathbf{e}_{\tau_4^{-1}} \middle| \mathbf{e}_{\tau_5^{-1}} \middle| \mathbf{e}_{\tau_6^{-1}}\right) = (\mathbf{e}_2|\mathbf{e}_1|\mathbf{e}_4|\mathbf{e}_3|\mathbf{e}_6|\mathbf{e}_5)$$

- Row by row:

$$P_\sigma = I[\boldsymbol{\sigma}, :] = \begin{pmatrix} I[\sigma_1, :] \\ I[\sigma_2, :] \\ I[\sigma_3, :] \end{pmatrix} = \begin{pmatrix} I[3, :] \\ I[2, :] \\ I[1, :] \end{pmatrix}$$

$$P_\tau = I[\boldsymbol{\tau}, :] = \begin{pmatrix} I[\tau_1, :] \\ I[\tau_2, :] \\ I[\tau_3, :] \\ I[\tau_4, :] \\ I[\tau_5, :] \\ I[\tau_6, :] \end{pmatrix} = \begin{pmatrix} I[2, :] \\ I[1, :] \\ I[4, :] \\ I[3, :] \\ I[6, :] \\ I[5, :] \end{pmatrix}$$

Either way, we have

$$P_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P_\tau = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

END

Problem 3.1 Show for a unitary matrix $Q \in U(n)$ and a vector $\mathbf{x} \in \mathbb{C}^n$ that multiplication by Q preserve the 2-norm:

$$\|Q\mathbf{x}\| = \|\mathbf{x}\|.$$

SOLUTION

$$\|Q\mathbf{x}\|^2 = (Q\mathbf{x})^\star Q\mathbf{x} = \mathbf{x}^\star Q^\star Q\mathbf{x} = \mathbf{x}^\star \mathbf{x} = \|\mathbf{x}\|^2$$

END

Problem 3.2 Show that the eigenvalues λ of a unitary matrix Q are on the unit circle: $|\lambda| = 1$. Hint: recall for any eigenvalue λ that there exists a unit vector $\mathbf{v} \in \mathbb{C}^n$ (satisfying $\|\mathbf{v}\| = 1$). Combine this with Problem 3.1.

SOLUTION Let $\mathbf{v}$ be a unit eigenvector corresponding to λ: $Q\mathbf{v} = \lambda\mathbf{v}$ with $\|\mathbf{v}\| = 1$. Then

$$1 = \|\mathbf{v}\| = \|Q\mathbf{v}\| = \|\lambda\mathbf{v}\| = |\lambda|.$$

END

Problem 3.3 Show for an orthogonal matrix $Q \in O(n)$ that $\det Q = \pm 1$. Give an example of $Q \in U(n)$ such that $\det Q \neq \pm 1$. Hint: recall for any real matrices A and B that $\det A = \det A^\top$ and $\det(AB) = \det A \det B$.

SOLUTION

$$(\det Q)^2 = (\det Q^\top)(\det Q) = \det Q^\top Q = \det I = 1.$$

An example would be a 1 × 1 complex-valued matrix $\exp(\mathrm{i})$.

END

Problem 3.4 Show that $Q \in U(n)$ is a normal matrix (that it commutes with its adjoint).

SOLUTION

$$QQ^\star = I = Q^\star Q$$

END

Problem 3.5 Explain why $Q \in U(n)$ must be equal to I if all its eigenvalues are 1. Hint: use the spectral theorem, which says that any normal matrix is diagonalisable with unitary eigenvectors (see notes on C. Adjoints and Normal Matrices).

SOLUTION

Note that Q is normal and therefore by the spectral theorem for normal matrices we have

$$Q = \tilde{Q} \Lambda \tilde{Q}^\star = \tilde{Q} \tilde{Q}^\star = I$$

since $\tilde{Q}$ is unitary.

END