MATH50003 Numerical Analysis (2022–23)

Problem Sheet 5

This problem sheet concerns Given's rotations, Householder reflections and QR factorisations.

Problem 1.1 Consider the vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2\mathrm{i} \\ 2 \end{bmatrix}.$$

Use reflections to determine the entries of orthogonal/unitary matrices Q_1, Q_2, Q_3 such that

$$Q_1\mathbf{a} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, Q_2\mathbf{a} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, Q_3\mathbf{b} = \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix}$$

Problem 1.2 What simple rotation matrices $Q_1, Q_2 \in SO(2)$ have the property that:

$$Q_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}, Q_2 \begin{bmatrix} \sqrt{5} \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Problem 1.3 Find an orthogonal matrix that is a product of two simple rotations but acting on two different subspaces:

$$Q = \underbrace{\begin{bmatrix} \cos\theta_2 & & -\sin\theta_2 \\ & 1 & \\ \sin\theta_2 & & \cos\theta_2 \end{bmatrix}}_{Q_2} \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & \\ \sin\theta_1 & \cos\theta_1 & \\ & & 1 \end{bmatrix}}_{Q_1}$$

so that

$$\|Q\mathbf{a}\| = \begin{bmatrix} \|\mathbf{a}\| \\ 0 \\ 0 \end{bmatrix}.$$

Hint: you do not need to determine θ_1, θ_2, instead you can write the entries of Q_1, Q_2 directly using just square-roots.

Problem 2.1 Show that every matrix $A \in \mathbb{R}^{m \times n}$ has a QR factorisation such that the diagonal of R is non-negative. Make sure to include the case of more columns than rows (i.e. $m < n$). You may use the fact from lectures that a QR factorisation exists (but not necessarily with positive diagonals) whenever $m \geq n$.

Problem 2.2 Show that the QR factorisation of a square invertible matrix $A \in \mathbb{R}^{n \times n}$ is unique, provided that the diagonal of R is positive. You may use Problem 3.5 from PS4, which states that an orthogonal matrix whose eigenvalues are all 1 must be equal to the identity. Hint: a product of two orthogonal matrices is also orthogonal.

Problem 3.1 Show that if U is triangular and normal ($U^\star U = UU^\star$) then it is diagonal.

Problem 3.2 Show that every matrix $A \in \mathbb{C}^{n \times n}$ has a *Schur decomposition*: it can be written as

$$A = QUQ^\star$$

where U is upper-triangular with the eigenvalues of A on the diagonal. Hint: you may use the Jordan canonical form

$$A = VJV^{-1}$$

where J is upper-triangular with eigenvalues of A on the diagonal.

Problem 3.3 Prove the spectral theorem for normal matrices: for every normal matrix $A \in \mathbb{C}^{n \times n}$ there exists a unitary matrix Q such that

$$A = Q\Lambda Q^\star$$

where Λ is diagonal.