MATH50003 Numerical Analysis (2022–23)

Problem Sheet 6

This problem sheet concerns Cholesky factorisations and matrix norms.

Problem 1 Use the Cholesky factorisation to determine which of the following matrices are symmetric positive definite:

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

Problem 2.1 An inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ on $\mathbb{R}^n$ satisfies, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}$ and $a, b \in \mathbb{R}$:

1. Symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
2. Linearity: $\langle \mathbf{x}, a\mathbf{y} + b\mathbf{z} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle + b\langle \mathbf{x}, \mathbf{z} \rangle$
3. Posive-definite: $\langle \mathbf{x}, \mathbf{x} \rangle > 0, x \neq 0$

Prove that $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product if and only if

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top K \mathbf{y}$$

where K is a symmetric positive definite matrix.

Problem 2.2 Show that a symmetric positive definite matrix has strictly positive eigenvalues. Hint: you can use the fact that symmetric matrices have real eigenvalues and eigenvectors.

Problem 2.3 Show that a matrix is symmetric positive definite if and only if it has a *reverse* Cholesky factorisation of the form

$$A = UU^\top$$

where U is upper triangular with positive entries on the diagonal.

Problem 3.1 Use the Cholesky decomposition to prove that the following $n \times n$ matrix is symmetric positive definite for any n:

$$\Delta_n := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Hint: replace $\Delta_n[1,1]$ with $\alpha > 1$ and use a proof by induction.

Problem 3.2 Deduce its Cholesky and reverse Cholesky factorisations: $\Delta_n = L_n L_n^\top = U_n U_n^\top$ where L_n is lower triangular and U_n is upper triangular.

Problem 4.1 Prove the following:

$$\|A\|_\infty = \max_k \|A[k,:]\|_1$$
$$\|A\|_{1\to\infty} = \|\mathrm{vec}(A)\|_\infty = \max_{kj} |a_{kj}|$$

Problem 4.2 For a rank-1 matrix $A = \mathbf{x}\mathbf{y}^\top$ prove that

$$\|A\|_2 = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

Hint: use the Cauchy–Schwartz inequality which states $|\mathbf{y}^\top \mathbf{z}| \leq \|\mathbf{y}\|_2 \|\mathbf{z}\|_2$.

Problem 5.1 Show for any orthogonal matrix $Q \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m\times n}$ that

$$\|QA\|_F = \|A\|_F$$

by first showing that $\|A\|_F = \sqrt{\mathrm{tr}(A^\top A)}$ using the *trace* of an $m \times m$ matrix:

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{mm}.$$

Problem 5.2 Show that $\|A\|_2 \leq \|A\|_F \leq \sqrt{r}\|A\|_2$ where r is the rank of A.