

## II.6 Singular Value Decomposition

In this chapter we discuss the *Singular Value Decomposition (SVD)*: a matrix factorisation that encodes how much a matrix "stretches" a random vector. This includes *singular values*, the largest of which dictates the 2-norm of the matrix.

**Definition 1 (singular value decomposition)** For  $A \in \mathbb{C}^{m \times n}$  with rank  $r > 0$ , the (reduced) singular value decomposition (SVD) is

$$A = U\Sigma V^*$$

where  $U \in \mathbb{C}^{m \times r}$  and  $V \in \mathbb{C}^{r \times n}$  have orthonormal columns and  $\Sigma \in \mathbb{R}^{r \times r}$  is diagonal whose diagonal entries, which we call *singular values*, are all positive and non-increasing:  $\sigma_1 \geq \dots \geq \sigma_r > 0$ . The *full singular value decomposition (SVD)* is

$$A = U\Sigma V^*$$

where  $U \in U(m)$  and  $V \in U(n)$  are unitary matrices and  $\Sigma \in \mathbb{R}^{m \times n}$  has only diagonal non-zero entries, i.e., if  $m > n$ ,

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_n & \\ & & 0 & \\ & & \vdots & \\ & & 0 & \end{bmatrix}$$

and if  $m < n$ ,

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_m & 0 & \dots & 0 \end{bmatrix}$$

where  $\sigma_k = 0$  if  $k > r$ .

In particular, we discuss:

1. Existence of the SVD: we show that an SVD exists by relating it to the eigenvalue Decomposition of  $A^*A$  and  $AA^*$ .
2. 2-norm and SVD: the 2-norm of a matrix is defined in terms of the largest singular value.
3. Best rank- $k$  approximation and compression: the best approximation of a matrix by a smaller rank matrix can be constructed

using the SVD, which gives an effective way to compress matrices.

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In [1]: using LinearAlgebra, Plots
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[ Info: Precompiling Plots [91a5bcdd-55d7-5caf-9e0b-520d859cae80]

## 1. Existence

To show the SVD exists we first establish some properties of a *Gram matrix* ( $A^*A$ ):

**Proposition 1 (Gram matrix kernel)** The kernel of  $A$  is the also the kernel of  $A^*A$ .

**Proof** If  $A^*Ax = 0$  then we have

$$0 = \mathbf{x}^* A^* A \mathbf{x} = \|A\mathbf{x}\|^2$$

which means  $A\mathbf{x} = 0$  and  $\mathbf{x} \in \ker(A)$ . ■

**Proposition 2 (Gram matrix diagonalisation)** The Gram-matrix satisfies

$$A^*A = Q\Lambda Q^* \in \mathbb{C}^{n \times n}$$

is a Hermitian matrix where  $Q \in U(n)$  and the eigenvalues  $\lambda_k$  are real and non-negative. If  $A \in \mathbb{R}^{m \times n}$  then  $Q \in O(n)$ .

**Proof**  $A^*A$  is Hermitian so we appeal to the spectral theorem for the existence of the decomposition, and the fact that the eigenvalues are real. For the corresponding (orthonormal) eigenvector  $\mathbf{q}_k$ ,

$$\lambda_k = \lambda_k \mathbf{q}_k^* \mathbf{q}_k = \mathbf{q}_k^* A^* A \mathbf{q}_k = \|A\mathbf{q}_k\|^2 \geq 0.$$

■

This connection allows us to prove existence:

**Theorem 1 (SVD existence)** Every  $A \in \mathbb{C}^{m \times n}$  has an SVD.

**Proof** Consider

$$A^*A = Q\Lambda Q^*.$$

Assume (as usual) that the eigenvalues are sorted in decreasing modulus, and so  $\lambda_1, \dots, \lambda_r$  are an enumeration of the non-zero eigenvalues and

$$V := [\mathbf{q}_1 | \dots | \mathbf{q}_r]$$

the corresponding (orthonormal) eigenvectors, with

$$K = [\mathbf{q}_{r+1} | \dots | \mathbf{q}_n]$$

the corresponding kernel. Define

$$\Sigma := \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_r} \end{bmatrix}$$

Now define

$$U := AV\Sigma^{-1}$$

which is orthogonal since  $A^*AV = V\Sigma^2$ :

$$U^*U = \Sigma^{-1}V^*A^*AV\Sigma^{-1} = I.$$

Thus we have

$$U\Sigma V^* = AVV^* = A \underbrace{[V|K]}_Q \underbrace{\begin{bmatrix} V^* \\ K^* \end{bmatrix}}_{Q^*}$$

where we use the fact that  $AK = 0$  so that concatenating  $K$  does not change the value.

■

## 2. 2-norm and SVD

Singular values tell us the 2-norm:

**Corollary 1 (singular values and norm)**

$$\|A\|_2 = \sigma_1$$

and if  $A \in \mathbb{C}^{n \times n}$  is invertible, then

$$\|A^{-1}\|_2 = \sigma_n^{-1}$$

**Proof**

First we establish the upper-bound:

$$\|A\|_2 \leq \|U\|_2 \|\Sigma\|_2 \|V^*\|_2 = \|\Sigma\|_2 = \sigma_1$$

This is attained using the first right singular vector:

$$\|A\mathbf{v}_1\|_2 = \|\Sigma V^*\mathbf{v}_1\|_2 = \|\Sigma \mathbf{e}_1\|_2 = \sigma_1$$

The inverse result follows since the inverse has SVD

$$A^{-1} = V\Sigma^{-1}U^* = (VW)(W\Sigma^{-1}W)(WU)^*$$

is the SVD of  $A^{-1}$ , i.e.  $VW \in U(n)$  are the left singular vectors and  $WU$  are the right singular vectors, where

$$W := P_\sigma = \begin{bmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{bmatrix}$$

is the permutation that reverses the entries, that is,  $\sigma$  has Cauchy notation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

■

We will not discuss in this module computation of singular value decompositions or eigenvalues: they involve iterative algorithms (actually built on a sequence of QR decompositions).

### 3. Best rank- $k$ approximation and compression

One of the main usages for SVDs is low-rank approximation:

**Theorem 2 (best low rank approximation)** The matrix

$$A_k := [\mathbf{u}_1 | \cdots | \mathbf{u}_k] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix} [\mathbf{v}_1 | \cdots | \mathbf{v}_k]^*$$

is the best 2-norm approximation of  $A$  by a rank  $k$  matrix, that is, for all rank- $k$  matrices  $B$ , we have

$$\|A - A_k\|_2 \leq \|A - B\|_2.$$

**Proof** We have

$$A - A_k = U \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \sigma_{k+1} & \\ & & & & \ddots \\ & & & & & \sigma_r \end{bmatrix} V^*.$$

Suppose a rank- $k$  matrix  $B$  has

$$\|A - B\|_2 < \|A - A_k\|_2 = \sigma_{k+1}.$$

For all  $\mathbf{w} \in \ker(B)$  we have

$$\|A\mathbf{w}\|_2 = \|(A - B)\mathbf{w}\|_2 \leq \|A - B\| \|\mathbf{w}\|_2 < \sigma_{k+1} \|\mathbf{w}\|_2$$

But for all  $\mathbf{u} \in \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{k+1})$ , that is,  $\mathbf{u} = V[:, 1 : k + 1]\mathbf{c}$  for some  $\mathbf{c} \in \mathbb{R}^{k+1}$  we have

$$\|A\mathbf{u}\|_2^2 = \|U\Sigma_k\mathbf{c}\|_2^2 = \|\Sigma_k\mathbf{c}\|_2^2 = \sum_{j=1}^{k+1} (\sigma_j c_j)^2 \geq \sigma_{k+1}^2 \|\mathbf{c}\|^2,$$

i.e.,  $\|A\mathbf{u}\|_2 \geq \sigma_{k+1} \|\mathbf{c}\|$ . Thus  $\mathbf{w}$  cannot be in this span.

The dimension of the span of  $\ker(B)$  is at least  $n - k$ , but the dimension of  $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{k+1})$  is at least  $k + 1$ . Since these two spaces cannot intersect (apart from at 0) we have a contradiction, since  $(n - r) + (r + 1) = n + 1 > n$ . ■

**Example 1 (Hilbert matrix)** Here we show an example of a simple low-rank approximation using the SVD. Consider the Hilbert matrix:

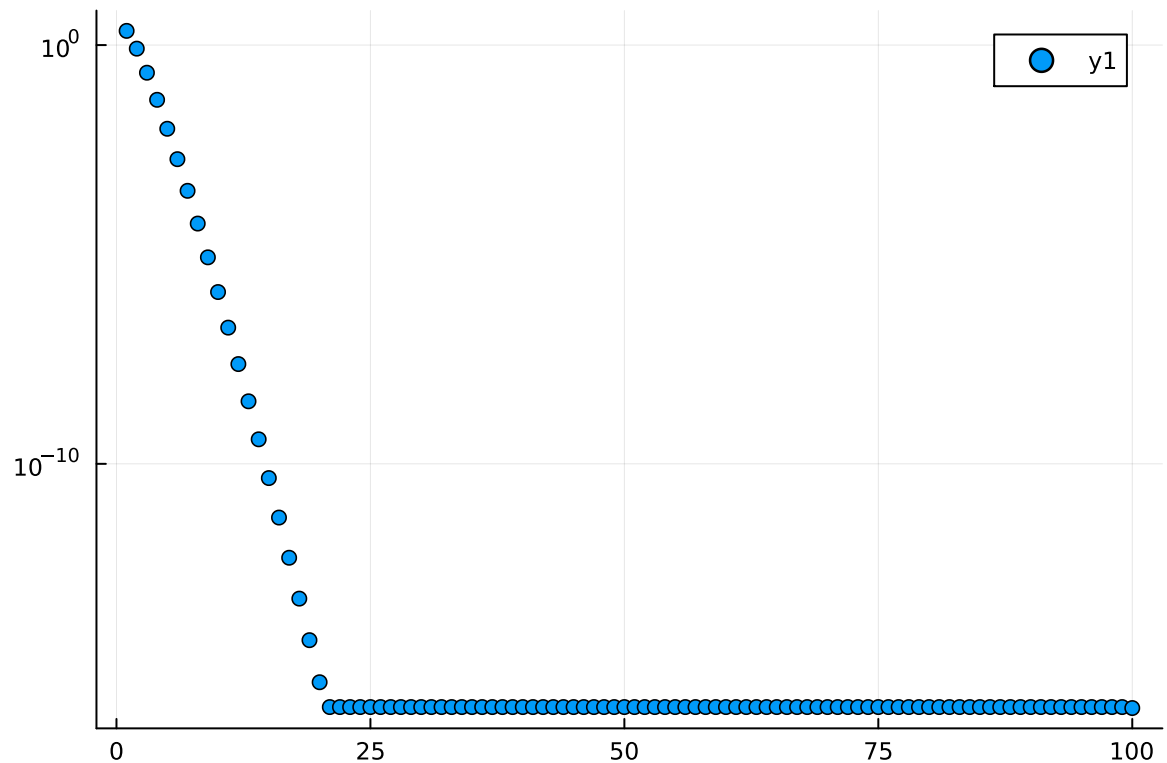
```
In [2]: hilbertmatrix(n) = [1/(k+j-1) for j = 1:n, k=1:n]
        hilbertmatrix(5)
```

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Out[2]: 5×5 Matrix{Float64}:
 1.0      0.5      0.333333  0.25      0.2
 0.5      0.333333  0.25      0.2      0.166667
 0.333333  0.25      0.2      0.166667  0.142857
 0.25      0.2      0.166667  0.142857  0.125
 0.2      0.166667  0.142857  0.125     0.111111
```

That is, the  $H[k, j] = 1/(k + j - 1)$ . This is a famous example of matrix with rapidly decreasing singular values:

```
In [3]: H = hilbertmatrix(100)
        U,σ,V = svd(H)
        scatter(σ; ylabel=:log10)
```

Out [3]:



Note numerically we typically do not get a exactly zero singular values so the rank is always treated as  $\min(m, n)$ . Because the singular values decay rapidly we can approximate the matrix very well with a rank 20 matrix:

```
In [4]: k = 20 # rank
        Σ_k = Diagonal(σ[1:k])
        U_k = U[:,1:k]
        V_k = V[:,1:k]
        opnorm(U_k * Σ_k * V_k' - H)
```

Out [4]: 8.20222266307798e-16

Note that this can be viewed as a *compression* algorithm: we have replaced a matrix with  $100^2 = 10,000$  entries by two matrices and a vector with 4,000 entries without losing any information. In the problem sheet we explore the usage of low rank approximation to smooth functions and to compress images.