MATH50003 Numerical Analysis (2022–23)

Problem Sheet 3

This problem sheet explores the error in using divided differences and using dual numbers.

Please complete the problems using pen-and-paper, though some can be verified using Julia.

Problem 1 Suppose our floating point approximation $f^{\rm FP} : F \to F$ has *relative accuracy*:

$$f^{\rm FP}(x) = f(x)(1 + \delta_x^{\rm r})$$

where

$$|\delta_x^{\rm r}| \leq c\epsilon_{\rm m}.$$

Suppose further that $f(0) = f^{\rm FP}(0) = 0$ and assume that $f'(0) \neq 0$. Show that divided differences achieves relative accuracy:

$$\frac{f^{\rm FP}(h)}{h} = f'(0)(1 + \varepsilon_h)$$

where

$$|\varepsilon_h| \leq \frac{M}{2f'(0)} h(1 + c\epsilon_{\rm m}) + c\epsilon_{\rm m}$$

for $M = \sup_{0 \leq t \leq h} |f''(t)|$.

Problem 2.1 For

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \delta_{x,h}^{\rm T},$$

bound the absolute error $|\delta_{x,h}^{\rm T}|$ in terms of

$$M = \max_{y \in [x-h,x+h]} \left|f'''(y)\right|.$$

Problem 2.2 Assume that

$$f^{\rm FP}(x) = f(x) + \delta_x^f$$

where $|\delta_x^f| \leq c\epsilon_{\mathrm{m}}$. For the *absolute error* $\delta_{x,h}^{\mathrm{CD}}$ satisfying

$$\frac{f^{\mathrm{FP}}(x+h) \ominus f^{\mathrm{FP}}(x-h)}{2h} = f'(x) + \delta_{x,h}^{\mathrm{CD}}$$

find a bound on $|\delta_{x,h}^{\mathrm{CD}}|$ in terms of M. You may assume all operations result in numbers in the normalised range, $h = 2^{-n}$, $x \oplus h = x + h$ and $x \ominus h = x - h$.

Problem 3.1 For the second-order derivative approximation

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \delta_{x,h}^{\mathrm{T}}$$

bound the absolute error $|\delta_{x,h}^{\mathrm{T}}|$ in terms of

$$M = \max_{y \in [x-h, x+h]} |f'''(y)| .$$

Problem 3.2 Assume that

$$f^{\mathrm{FP}}(x) = f(x) + \delta_x^f$$

where $|\delta_x^f| \leq c\epsilon_{\mathrm{m}}$. For the *absolute error* $\delta_{x,h}^{\mathrm{2D}}$ satisfying

$$(f^{\mathrm{FP}}(x+h) \ominus 2f^{\mathrm{FP}}(x) \oplus f^{\mathrm{FP}}(x-h))/h = f''(x) + \delta_{x,h}^{\mathrm{2D}}$$

find a bound on $|\delta_{x,h}^{\mathrm{2D}}|$ in terms of M and $F = \sup_{x-h \leq t \leq x+h} |f(t)|$. You may assume all operations result in numbers in the normalised range, $h = 2^{-n}$, $x \oplus h = x + h$ and $x \ominus h = x - h$.

Problem 4 Show that dual numbers $\mathbb{D}$ are a *commutative ring*, that is, for all $a, b, c \in \mathbb{D}$ the following are satisfied:

1. *additive associativity*: $(a + b) + c = a + (b + c)$
2. *additive commutativity*: $a + b = b + a$
3. *additive identity*: There exists $0 \in \mathbb{D}$ such that $a + 0 = a$.
4. *additive inverse*: There exists $-a$ such that $(-a) + a = 0$.
5. *multiplicative associativity*: $(ab)c = a(bc)$
6. *multiplictive commutativity*: $ab = ba$
7. *multiplictive identity*: There exists $1 \in \mathbb{D}$ such that $1a = a$.
8. *distributive*: $a(b + c) = ab + ac$

Problem 5.1 What is the correct definition of division on dual numbers, i.e.,

$$(a + b\epsilon)/(c + d\epsilon) = s + t\epsilon$$

for what choice of s and t?

Problem 5.2 A *field* is a commutative ring such that $0 \neq 1$ and all nonzero elements have a multiplicative inverse, i.e., there exists a^{-1} such that $aa^{-1} = 1$. Can we use Problem 5.1 to define $a^{-1} := 1/a$ to make $\mathbb{D}$ a field? Why or why not?

Problem 6 Use dual numbers to compute the derivative of the following functions at $x = 0.1$:

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^{3} \left(\frac{x}{k} - 1\right), \text{ and } f_2^{\mathrm{s}}(x) = 1 + \frac{x - 1}{2 + \frac{x-1}{2}}$$