C. Adjoints and Normal Matrices

Here we review

1. Complex inner-products
2. Adjoints
3. Normal Matrices and the spectral theorem

1. Complex-inner products

We will use bars to denote the complex-conjugate: if $z = x + \mathrm{i}y$ then $\bar{z} = x - \mathrm{i}y$.

Definition 1 (inner-product) An *inner product* $\langle \cdot, \cdot \rangle$ over $\mathbb{C}$ satisfies, for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n$ and $a, b \in \mathbb{C}$,

1. Conjugate symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
2. Linearity: $\langle \mathbf{x}, a\mathbf{y} + b\mathbf{z} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle + b\langle \mathbf{x}, \mathbf{z} \rangle$. (Some authors use linearity in the first argument.)
3. Positive-definiteness: for $\mathbf{x} \neq 0$ we have $\langle \mathbf{x}, \mathbf{x} \rangle > 0$.

We will usually use the standard inner product defined on $\mathbb{C}^n$:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \bar{\mathbf{x}}^\top \mathbf{y} = \sum_{k=1}^n \bar{x}_k y_k$$

Note that $\overline{zw} = \bar{z}\bar{w}$ and $\overline{z + w} = \bar{z} + \bar{w}$ together imply that:

$$\overline{A\mathbf{x}} = \bar{A}\bar{\mathbf{x}}.$$

2. Adjoints

Definition 2 (adjoint) Given an inner product, an adjoint of a matrix $A \in \mathbb{C}^{m \times n}$ is the unique matrix $A^\star$ satisfying

$$\langle \mathbf{x}, A\mathbf{y} \rangle = \langle \bar{A}\mathbf{x}, \mathbf{y} \rangle$$

for all $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{y} \in \mathbb{C}^n$. (Note this definition can be extended to other inner products.)

Proposition 1 (adjoints are conjugate-transposes) For the standard inner product, $A^\star = \bar{A}^\top$. If $A \in \mathbb{R}^{m \times n}$ then it reduces to the transpose $A^\star = A^\top$.

Proof

$$A_{k,j} = \langle \mathbf{e}_k, A\mathbf{e}_j \rangle = \langle A^\star \mathbf{e}_k, \mathbf{e}_j \rangle = \mathbf{e}_k^\top \overline{(A^\star)}^\top \mathbf{e}_j = \overline{A^\star_{j,k}}$$

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In this module we will assume the standard inner product (unless otherwise stated), that is, we will only use the standard adjoint $A^\star = \bar{A}^\top$. Note if $A = A^\star$ a matrix is called *Hermitian*. If it is real it is also called *Symmetric*.

3. Normal matrices

Definition 3 (normal) A (square) *normal matrix* commutes with its adjoint:

$$AA^\star = A^\star A$$

Examples of normal matrices include: 2. Symmetric and Hermitian: ($A^\star A = A^2 = AA^\star$) 3. Orthogonal and Unitary: ($Q^\star Q = I = QQ^\star$)

An important property of normal matrices is that they are diagonalisable using unitary eigenvectors:

Theorem 1 (spectral theorem for normal matrices) If $A \in \mathbb{C}^{n\times n}$ is normal then it is diagonalisable with unitary eigenvectors:

$$A = Q\Lambda Q^\star$$

where $Q \in U(n)$ and Λ is diagonal.

We will prove this later in the module. There is an important corollary for symmetric matrices that you may have seen before:

Corollary 1 (spectral theorem for symmetric matrices) If $A \in \mathbb{R}^{n\times n}$ is symmetric then it is diagonalisable with orthogonal eigenvectors:

$$A = Q\Lambda Q^\top$$

where $Q \in O(n)$ and Λ is real and diagonal.

Proof

$A = Q\Lambda Q^\star$ since its normal hence we find that:

$$\bar{\Lambda} = \Lambda^\star = (Q^\star AQ)^\star = Q^\star A^\top Q = Q^\star AQ = \Lambda$$

which shows that Λ is real. The fact that Q is real follows since the column $\mathbf{q}_k = Q\mathbf{e}_k$ is in the null space of the real matrix $A - \lambda_k I$.

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