C. Spectral theorem for symmetric and normal matrices

Here we review the proof of the spectral theorem for symmetric and normal matrices, as well as adjoints (complex-conjugation). Here we use the standard inner product defined on $\mathbb{C}^n$:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \bar{\mathbf{x}}^\top \mathbf{y} = \sum_{k=1}^n \bar{x}_k y_k$$

where the bars indicate complex conjugate: if $z = x + \mathrm{i}y$ then $\bar{z} = x - \mathrm{i}y$. Note that $\overline{zw} = \bar{z}\bar{w}$ and $\overline{z + w} = \bar{z} + \bar{w}$ together imply that:

$$\overline{A\mathbf{x}} = \bar{A}\bar{\mathbf{x}}.$$

1. Adjoints

Definition 1 (adjoint) An adjoint of a matrix $A \in \mathbb{C}^{m \times n}$ is its conjugate transpose: $A^\star := \bar{A}^\top$. If $A \in \mathbb{R}^{m \times n}$ then it reduces to the transpose $A^\star = A^\top$.

Note adjoints have the important product that for the standard inner product they satisfy:

$$\langle \mathbf{x}, A\mathbf{y} \rangle = \bar{\mathbf{x}}^\top (A\mathbf{y}) = (A^\top \bar{\mathbf{x}})^\top \mathbf{y} = (\overline{\bar{A}^\top \mathbf{x}\ })^\top \mathbf{y} = \langle \bar{A}\mathbf{x}, \mathbf{y} \rangle$$

2. Spectral theorem for symmetric matrices

Theorem (spectral theorem for symmetric matrices) If A is symmetric ($A = A^\top$) then

$$A = Q \Lambda Q^\top$$

where Q is orthogonal ($Q^\top Q = I$) and Λ is real.

Proof

Recall every eigenvalue λ has at least one eigenvector $\mathbf{q}$ which we can normalize, but this can be complex. Thus we have

$$\lambda = \lambda \mathbf{q}^\star \mathbf{q} = \mathbf{q}^\star A \mathbf{q} =$$

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