

## Numerical Analysis MATH50003 (2023–24) Problem Sheet 8

**Problem 1** Give explicit formulae for  $\hat{f}_k$  and  $\hat{f}_k^n$  for the following functions:

$$\cos \theta, \cos 4\theta, \sin^4 \theta, \frac{3}{3 - e^{i\theta}}, \frac{1}{1 - 2e^{i\theta}}$$

**Problem 2** Prove that if the first  $\lambda - 1$  derivatives  $f(\theta), f'(\theta), \dots, f^{(\lambda-1)}(\theta)$  are  $2\pi$ -periodic and  $f^{(\lambda)}$  is uniformly bounded that

$$|\hat{f}_k| = O(|k|^{-\lambda}) \quad \text{as } |k| \rightarrow \infty$$

Use this to show for the Taylor case ( $0 = \hat{f}_{-1} = \hat{f}_{-2} = \dots$ ) that

$$|f(\theta) - \sum_{k=0}^{n-1} \hat{f}_k e^{ik\theta}| = O(n^{1-\lambda}) \quad \text{as } n \rightarrow \infty$$

**Problem 3(a)** If  $f$  is a trigonometric polynomial ( $\hat{f}_k = 0$  for  $|k| > m$ ) show for  $n \geq 2m + 1$  that we can exactly recover  $f$ :

$$f(\theta) = \sum_{k=-m}^m \hat{f}_k^n e^{ik\theta}$$

**Problem 3(b)** For the general (non-Taylor) case and  $n = 2m + 1$ , prove convergence for

$$f_{-m:m}(\theta) := \sum_{k=-m}^m \hat{f}_k^n e^{ik\theta}$$

to  $f(\theta)$  as  $n \rightarrow \infty$ . What is the rate of convergence if the first  $\lambda - 1$  derivatives  $f(\theta), f'(\theta), \dots, f^{(\lambda-1)}(\theta)$  are  $2\pi$ -periodic and  $f^{(\lambda)}$  is uniformly bounded?

**Problem 4(a)** Show for  $0 \leq k, \ell \leq n - 1$

$$\frac{1}{n} \sum_{j=1}^n \cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for  $\theta_j = \pi(j - 1/2)/n$ . Hint: Be careful as the  $\theta_j$  differ from before, and only cover half the period,  $[0, \pi]$ . Using symmetry may help. You may also consider replacing  $\cos$  with complex exponentials:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

**Problem 4(b)** Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos \theta_1 & \cdots & \cos \theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for  $\theta_j = \pi(j - 1/2)/n$ . Prove that  $C_n$  is orthogonal:  $C_n^\top C_n = C_n C_n^\top = I$ .

**Problem 5** What polynomial interpolates  $\cos z$  at  $1, \exp(2\pi i/3)$  and  $\exp(-2\pi i/3)$ ?