Numerical Analysis MATH50003 (2023–24) Problem Sheet 6

Problem 1 By computing the Cholesky factorisation, determine which of the following matrices are symmetric positive definite:

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

Problem 2 Show that a matrix is symmetric positive definite if and only if it has a *reverse* Cholesky factorisation of the form

$$A = UU^\top$$

where U is upper triangular with positive entries on the diagonal.

Problem 3(a) Use the Cholesky factorisation to prove that the following $n \times n$ matrix is symmetric positive definite for any n:

$$\Delta_n := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Hint: consider a matrix K_n^α that equals Δ_n apart from the top left entry which is $\alpha > 1$ and use a proof by induction.

Problem 3(b) Deduce its Cholesky factorisations: $\Delta_n = L_n L_n^\top$ where L_n is lower triangular and U_n is upper triangular.

Problem 4 Use Lagrange interpolation to interpolate the function $\cos x$ by a polynomial at the points $[0, 2, 3, 4]$ and evaluate at $x = 1$.

Problem 5(a) Compute the LU factorisation of the following transposed Vandermonde matrices:

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{bmatrix}$$

Can you spot a pattern? Test your conjecture with a 5×5 Vandermonde matrix.

Problem 5(b) Show that the determinant of a Vandermonde matrix is

$$\det V = \prod_{k=1}^{n} \prod_{j=k+1}^{n} (x_j - x_j).$$