Numerical Analysis MATH50003 (2023–24) Revision Sheet

Problem 1(a) State which integers are represented by an 8-bit signed (`Int8`) and unsigned (`UInt8`) integer, both with bits

`10000001`

Problem 1(b) State which real number is represented by an IEEE 16-bit floating point number (with $\sigma = 15, Q = 5$, and $S = 10$) with bits

`1 01000 0000000001`

Problem 2(a) Consider the truncated Taylor series for $1/(1-x)$:

$$g_n(x) := \sum_{k=0}^{n} x^k$$

Write a one-line mathematical function $g_n^t(x)$ approximating $g_n(x)$ using floating point operations involving $\oplus$, $\oslash$, and $\otimes$. You may find it convenient to use the notation:

$$\bigoplus_{k=1}^{n} x_k := x_1 \oplus \cdots \oplus x_n = (\cdots((x_1 \oplus x_2) \oplus x_3) \cdots \oplus x_{n-1}) \oplus x_n$$
$$\bigotimes_{k=1}^{n} x_k := x_1 \otimes \cdots \otimes x_n = (\cdots((x_1 \otimes x_2) \otimes x_3) \cdots \otimes x_{n-1}) \otimes x_n$$

Problem 2(b) Recall for $E_{n,\epsilon} := \frac{n\epsilon}{1-n\epsilon}$ and $x_1, \ldots, x_n \in F$ that

$$x_1 \otimes \cdots \otimes x_n = x_1 \cdots x_n (1 + \theta_{n-1}),$$
$$x_1 \oplus \cdots \oplus x_n = x_1 + \cdots + x_n + \sigma_n$$

where $|\theta_n| \leq E_{n,\epsilon_{\mathrm{m}}/2}$ and $|\sigma_n| \leq M E_{n-1,\epsilon_{\mathrm{m}}/2}$, for $M = \Sigma_{k=1}^{n} |x_k|$ and assuming $n\epsilon_{\mathrm{m}} < 2$. Show for $x \in F$ such that $|x| < 1$

$$g_n^t(x) = \sum_{k=0}^{n} x^k + \varepsilon_n$$

where

$$|\varepsilon_n| \leq \frac{2E_{n,\epsilon_{\mathrm{m}}/2} + E_{n,\epsilon_{\mathrm{m}}/2}^2}{1 - |x|}.$$

You may assume all operations are normal.

Problem 3 What is the dual extension of square-roots? I.e. what should $\sqrt{a + b\epsilon}$ equal assuming $a > 0$?

Problem 6 Use the Cholesky factorisation to determine whether the following matrix is symmetric positive definite:

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

Problem 5 Use reflections to determine the entries of an orthogonal matrix Q such that

$$Q \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 8 For the function $f(\theta) = \sin 3\theta$, state explicit formulae for its Fourier coefficients

$$\hat{f}_k := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta$$

and their discrete approximation:

$$\hat{f}_k^n := \frac{1}{n} \sum_{j=0}^{n-1} f(\theta_j) \mathrm{e}^{-\mathrm{i}k\theta_j}.$$

for *all* integers k, $n = 1, 2, \ldots$, where $\theta_j = 2\pi j/n$.

Problem 9 Consider orthogonal polynomials

$$H_n(x) = 2^n x^n + O(x^{n-1})$$

as $x \to \infty$ and $n = 0, 1, 2, \ldots$, orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) w(x) \mathrm{d}x, \qquad w(x) = \exp(-x^2)$$

Construct $H_0(x)$, $H_1(x)$, $H_2(x)$ and hence show that $H_3(x) = 8x^3 - 12x$. You may use without proof the formulae

$$\int_{-\infty}^{\infty} w(x) \mathrm{d}x = \sqrt{\pi}, \int_{-\infty}^{\infty} x^2 w(x) \mathrm{d}x = \sqrt{\pi}/2, \int_{-\infty}^{\infty} x^4 w(x) \mathrm{d}x = 3\sqrt{\pi}/4.$$

Problem 10 Solve Problem 4(b) from PS8 using **Lemma 12 (discrete orthogonality)** with $w(x) = 1/\sqrt{1 - x^2}$ on $[-1, 1]$. That is, use the connection of $T_n(x)$ with $\cos n\theta$ to show that the Discrete Cosine Transform

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for $\theta_j = \pi(j - 1/2)/n$ is an orthogonal matrix.

Problem 5 Compute the 2-point and 3-point Gaussian quadrature rules associated with $w(x) = 1$ on $[-1, 1]$.