Numerical Analysis MATH50003 (2023–24) Problem Sheet 9

Problem 1 Construct $\pi_0(x), \pi_1(x), \pi_2(x), \pi_3(x)$, monic OPs for the weights $\sqrt{1-x^2}$ and $1-x$ on $[-1,1]$. Hint: for the first weight, first compute $\int_{-1}^1 x^k \sqrt{1-x^2}\mathrm{d}x$ for $0 \leq k \leq 2$ using a change-of-variables.

Problem 2(a) Prove Theorem III.3.1: a precisely degree n polynomial

$$p(x) = k_n x^n + O(x^{n-1})$$

satisfies

$$\langle p, f_m \rangle = 0$$

for all polynomials f_m of degree $m < n$ of degree less than n if and only if $p(x) = c\pi_n$ for some constant c, where π_n are monic orthogonal polynomials.

Problem 2(b) Show that if $\{p_n\}$ are OPs then there exist real constants $A_n \neq 0$, B_n, and C_n such that

$$\begin{aligned} p_1(x) &= (A_0 x + B_0) p_0(x) \\ p_{n+1}(x) &= (A_n x + B_n) p_n(x) - C_n p_{n-1}(x) \end{aligned}$$

Write this as a lower triangular linear system, given $p_0(x) = \mu \in \mathbb{R}$:

$$L_x \begin{bmatrix} p_0(x) \\ \vdots \\ p_{n+1}(x) \end{bmatrix} = \begin{bmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

What are the bandwidths of L_x?

Problem 2(c) If $w(-x) = w(x)$ for a weight supported on $[-b, b]$ show that $a_n = 0$. Hint: first show that the (monic) polynomials $p_{2n}(x)$ are even and $p_{2n+1}(x)$ are odd.

Problem 3(a) Recall the Chebyshev polynomials $T_0(x) = 1$, $T_n(x) = 2^{n-1}x^n + O(x^{n-1})$ which are orthogonal with respect to $1/\sqrt{1-x^2}$ and $U_n(x) = 2^n x^n + O(x^{n-1})$ which are orthogonal with respect to $\sqrt{1-x^2}$, both on $[-1,1]$. Use the fact that orthogonal polynomials are uniquely determined by their leading order coefficient and orthogonality to lower dimensional polynomials to show that:

$$T_n'(x) = nU_{n-1}(x).$$

Hint: use integration-by-parts.

Problem 3(b) Prove that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

Problem 3(c) Show that

$$\begin{aligned} xU_0(x) &= U_1(x)/2 \\ xU_n(x) &= \frac{U_{n-1}(x)}{2} + \frac{U_{n+1}(x)}{2}. \end{aligned}$$

Problem 4(a) Consider Hermite polynomials orthogonal with respect to the weight $\exp(-x^2)$ on $\mathbb{R}$ with the normalisation

$$H_n(x) = 2^n x^n + O(x^{n-1}).$$

Prove the Rodrigues formula

$$H_n(x) = (-1)^n \exp(x^2) \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp(-x^2).$$

Hint: use integration-by-parts.

Problem 4(b) What are $k_n^{(1)}$ and $k_n^{(2)}$ such that

$$H_n(x) = 2^n x^n + k_n^{(1)} x^{n-1} + k_n^{(2)} x^{n-2} + O(x^{n-3})$$

Problem 4(c) Deduce the 3-term recurrence relationship for $H_n(x)$.

Problem 4(d) Prove that $H_n'(x) = 2nH_{n-1}(x)$. Hint: show orthogonality of H_n' to all lower degree polynomials, and that the normalisation constants match.