

Numerical Analysis MATH50003 (2023–24) Problem Sheet 7

Problem 1(a) Show for a unitary matrix $Q \in U(n)$ and a vector $\mathbf{x} \in \mathbb{C}^n$ that multiplication by Q preserve the 2-norm: $\|Q\mathbf{x}\| = \|\mathbf{x}\|$.

SOLUTION

$$\|Q\mathbf{x}\|^2 = (Q\mathbf{x})^* Q\mathbf{x} = \mathbf{x}^* Q^* Q\mathbf{x} = \mathbf{x}^* \mathbf{x} = \|\mathbf{x}\|^2$$

END

Problem 1(b) Show that the eigenvalues λ of a unitary matrix Q are on the unit circle: $|\lambda| = 1$. Hint: recall for any eigenvalue λ that there exists a unit eigenvector $\mathbf{v} \in \mathbb{C}^n$ (satisfying $\|\mathbf{v}\| = 1$).

SOLUTION Let \mathbf{v} be a unit eigenvector corresponding to λ : $Q\mathbf{v} = \lambda\mathbf{v}$ with $\|\mathbf{v}\| = 1$. Then

$$1 = \|\mathbf{v}\| = \|Q\mathbf{v}\| = \|\lambda\mathbf{v}\| = |\lambda|.$$

END

Problem 1(c) Show for an orthogonal matrix $Q \in O(n)$ that $\det Q = \pm 1$. Give an example of $Q \in U(n)$ such that $\det Q \neq \pm 1$. Hint: recall for any real matrices A and B that $\det A = \det A^\top$ and $\det(AB) = \det A \det B$.

SOLUTION

$$(\det Q)^2 = (\det Q^\top)(\det Q) = \det Q^\top Q = \det I = 1.$$

An example would be a 1×1 complex-valued matrix $\exp(i)$.

END

Problem 1(d) A normal matrix commutes with its adjoint. Show that $Q \in U(n)$ is normal.

SOLUTION

$$QQ^* = I = Q^*Q$$

END

Problem 1(e) The spectral theorem states that any normal matrix is unitarily diagonalisable: if A is normal then $A = V\Lambda V^*$ where $V \in U(n)$ and Λ is diagonal. Use this to show that $Q \in U(n)$ is equal to I if and only if all its eigenvalues are 1.

SOLUTION

Note that Q is normal and therefore by the spectral theorem for normal matrices we have

$$Q = V\Lambda V^* = VV^* = I$$

since V is unitary.

END

Problem 2 Consider the vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2i \\ 2 \end{bmatrix}.$$

Use reflections to determine the entries of orthogonal/unitary matrices Q_1, Q_2, Q_3 such that

$$Q_1\mathbf{a} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, Q_2\mathbf{a} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, Q_3\mathbf{b} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

SOLUTION

For Q_1 : we have

$$\begin{aligned}\mathbf{y} &= \mathbf{a} - \|\mathbf{a}\|\mathbf{e}_1 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} \\ \mathbf{w} &= \frac{\mathbf{y}}{\|\mathbf{y}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\ Q_1 = Q\mathbf{w} &= I - \frac{2}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} = I - \frac{2}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\end{aligned}$$

For Q_2 : we have

$$\begin{aligned}\mathbf{y} &= \mathbf{a} + \|\mathbf{a}\|\mathbf{e}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \\ \mathbf{w} &= \frac{\mathbf{y}}{\|\mathbf{y}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\ Q_1 = Q\mathbf{w} &= I - \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = I - \frac{1}{3} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -1 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}\end{aligned}$$

For Q_3 we just need to be careful to conjugate:

$$\begin{aligned}\mathbf{y} &= \mathbf{b} - \|\mathbf{b}\|\mathbf{e}_1 = \begin{bmatrix} -2 \\ 2i \\ 2 \end{bmatrix} \\ \mathbf{w} &= \frac{\mathbf{y}}{\|\mathbf{y}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} \\ Q_1 = Q\mathbf{w} &= I - \frac{2}{3} \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -i & 1 \end{bmatrix} = I - \frac{2}{3} \begin{bmatrix} 1 & i & -1 \\ -i & 1 & i \\ -1 & -i & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & -2i & 2 \\ 2i & 1 & -2i \\ 2 & 2i & 1 \end{bmatrix}\end{aligned}$$

END

Problem 3(a) What simple rotation matrices $Q_1, Q_2 \in SO(2)$ have the property that:

$$Q_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}, Q_2 \begin{bmatrix} \sqrt{5} \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

SOLUTION

The rotation that takes $[x, y]$ to the x-axis is

$$\frac{1}{\sqrt{x^2 + y^2}} \begin{bmatrix} x & y \\ -y & x \end{bmatrix}.$$

Hence we get

$$Q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$
$$Q_2 = \frac{1}{3} \begin{bmatrix} \sqrt{5} & 2 \\ -2 & \sqrt{5} \end{bmatrix}$$

END

Problem 3(b) Find an orthogonal matrix that is a product of two simple rotations but acting on two different subspaces:

$$Q = \underbrace{\begin{bmatrix} \cos \theta_2 & & -\sin \theta_2 \\ & 1 & \\ \sin \theta_2 & & \cos \theta_2 \end{bmatrix}}_{Q_2} \underbrace{\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & \\ \sin \theta_1 & \cos \theta_1 & \\ & & 1 \end{bmatrix}}_{Q_1}$$

so that, for \mathbf{a} defined above,

$$Q\mathbf{a} = \begin{bmatrix} \|\mathbf{a}\| \\ 0 \\ 0 \end{bmatrix}.$$

Hint: you do not need to determine θ_1, θ_2 , instead you can write the entries of Q_1, Q_2 directly using just square-roots.

SOLUTION

We use Q_1 to introduce a 0 in the second entry by rotating the vector $[1, 2]$:

$$Q_1 = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & \\ -2/\sqrt{5} & 1/\sqrt{5} & \\ & & 1 \end{bmatrix}$$

so that

$$Q_1\mathbf{a} = \begin{bmatrix} \sqrt{5} \\ 0 \\ 2 \end{bmatrix}.$$

Now we use the matrix that rotates the vector $[\sqrt{5}, 2]$ whose norm is 3 to deduce the entries

$$Q_2 = \begin{bmatrix} \sqrt{5}/3 & & 2/3 \\ & 1 & \\ -2/3 & & \sqrt{5}/3 \end{bmatrix}$$

so that

$$Q_2Q_1 = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & \\ -2/(3\sqrt{5}) & -4/(3\sqrt{5}) & \sqrt{5}/3 \end{bmatrix}$$

END

Problem 4(a) Show that every matrix $A \in \mathbb{R}^{m \times n}$ has a QR factorisation such that the diagonal of R is non-negative. Make sure to include the case of more columns than rows (i.e. $m < n$).

SOLUTION

We first show for $m < n$ that a QR decomposition exists. Writing

$$A = [\mathbf{a}_1 | \cdots | \mathbf{a}_n]$$

and taking the first m columns (so that it is square) we can write $[\mathbf{a}_1 | \cdots | \mathbf{a}_m] = QR_m$. It follows that $R := Q^*A$ is right-triangular.

We can write:

$$D = \begin{bmatrix} \text{sign}(r_{11}) & & \\ & \ddots & \\ & & \text{sign}(r_{pp}) \end{bmatrix}$$

where $p = \min(m, n)$ and we define $\text{sign}(0) = 1$. Note that $D^\top D = I$. Thus we can write: $A = QR = QDDR$ where (QD) is orthogonal and DR is upper-triangular with positive entries.

END

Problem 4(b) Show that the QR factorisation of a square invertible matrix $A \in \mathbb{R}^{n \times n}$ is unique, provided that the diagonal of R is positive.

SOLUTION

Assume there is a second factorisation also with positive diagonal

$$A = QR = \tilde{Q}\tilde{R}$$

Then we know

$$Q^\top \tilde{Q} = R\tilde{R}^{-1}$$

Note $Q^\top \tilde{Q}$ is orthogonal, and $R\tilde{R}^{-1}$ has positive eigenvalues (the diagonal), hence all m eigenvalues of $Q^\top \tilde{Q}$ are 1. This means that $Q^\top \tilde{Q} = I$ and hence $\tilde{Q} = Q$ and $\tilde{R} = R$. **END**