

Numerical Analysis MATH50003 (2023–24) Problem Sheet 4

Problem 1 Suppose $x = 1.25$ and consider 16-bit floating point arithmetic (F_{16}). What is the error in approximating x by the nearest float point number $\text{fl}(x)$? What is the error in approximating $2x$, $x/2$, $x + 2$ and $x - 2$ by $2 \otimes x$, $x \oslash 2$, $x \oplus 2$ and $x \ominus 2$?

SOLUTION None of these computations have errors since they are all exactly representable as floating point numbers. **END**

Problem 2 Show that $1/5 = 2^{-3}(1.1001100110011\dots)_2$. What are the exact bits for $1 \oslash 5$, $1 \oslash 5 \oplus 1$ computed using half-precision arithmetic ($F_{16} := F_{15,5,10}$) (using default rounding)?

SOLUTION

For the first part we use Geometric series:

$$\begin{aligned} 2^{-3}(1.1001100110011\dots)_2 &= 2^{-3} \left(\sum_{k=0}^{\infty} \frac{1}{2^{4k}} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^{4k}} \right) \\ &= \frac{3}{2^4} \frac{1}{1 - 1/2^4} = \frac{3}{2^4 - 1} = \frac{1}{5} \end{aligned}$$

Write $-3 = 12 - 15$ hence we have $q = 12 = (01100)_2$. Since $1/5$ is below the midpoint (the midpoint would have been the first magenta bit was 1 and all other bits are 0) we round down and hence have the bits:

0 01100 1001100110

Adding 1 we get:

$$1 + 2^{-3} * (1.1001100110)_2 = (1.001100110011)_2 \approx (1.0011001101)_2$$

Here we write the exponent as $0 = 15 - 15$ where $q = 15 = (01111)_2$. Thus we have the bits:

0 01111 0011001101

END

Problem 3 Prove the following bounds on the *absolute error* of a floating point calculation in idealised floating-point arithmetic $F_{\infty,S}$ (i.e., you may assume all operations involve normal floating point numbers):

$$\begin{aligned} (\text{fl}(1.1) \otimes \text{fl}(1.2)) \oplus \text{fl}(1.3) &= 2.62 + \varepsilon_1 \\ (\text{fl}(1.1) \ominus 1) \oslash \text{fl}(0.1) &= 1 + \varepsilon_2 \end{aligned}$$

such that $|\varepsilon_1| \leq 11\epsilon_m$ and $|\varepsilon_2| \leq 40\epsilon_m$, where ϵ_m is machine epsilon.

SOLUTION

The first problem is very similar to what we saw in lecture. Write

$$(\text{fl}(1.1) \otimes \text{fl}(1.2)) \oplus \text{fl}(1.3) = (1.1(1 + \delta_1)1.2(1 + \delta_2)(1 + \delta_3) + 1.3(1 + \delta_4))(1 + \delta_5)$$

where we have $|\delta_1|, \dots, |\delta_5| \leq \epsilon_m/2$. We first write

$$1.1(1 + \delta_1)1.2(1 + \delta_2)(1 + \delta_3) = 1.32(1 + \varepsilon_1)$$

where, using the bounds:

$$|\delta_1\delta_2|, |\delta_1\delta_3|, |\delta_2\delta_3| \leq \epsilon_m/4, |\delta_1\delta_2\delta_3| \leq \epsilon_m/8$$

we find that

$$|\varepsilon_1| \leq |\delta_1| + |\delta_2| + |\delta_3| + |\delta_1\delta_2| + |\delta_1\delta_3| + |\delta_2\delta_3| + |\delta_1\delta_2\delta_3| \leq (3/2 + 3/4 + 1/8) \leq 5/2\epsilon_m$$

Then we have

$$1.32(1 + \varepsilon_1) + 1.3(1 + \delta_4) = 2.62 + \underbrace{1.32\varepsilon_1 + 1.3\delta_4}_{\varepsilon_2}$$

where

$$|\varepsilon_2| \leq (15/4 + 3/4)\epsilon_m \leq 5\epsilon_m.$$

Finally,

$$(2.62 + \varepsilon_2)(1 + \delta_5) = 2.62 + \underbrace{\varepsilon_2 + 2.62\delta_5 + \varepsilon_2\delta_5}_{\varepsilon_3}$$

where, using $|\varepsilon_2\delta_5| \leq 3\epsilon_m$ we get,

$$|\varepsilon_3| \leq (5 + 3/2 + 3)\epsilon_m \leq 10\epsilon_m.$$

For the second part, we do:

$$(\mathfrak{fl}(1.1) \ominus 1) \oslash \mathfrak{fl}(0.1) = \frac{(1.1(1 + \delta_1) - 1)(1 + \delta_2)}{0.1(1 + \delta_3)}(1 + \delta_4)$$

where we have $|\delta_1|, \dots, |\delta_4| \leq \epsilon_m/2$. Write

$$\frac{1}{1 + \delta_3} = 1 + \varepsilon_1$$

where, using that $|\delta_3| \leq \epsilon_m/2 \leq 1/2$, we have

$$|\varepsilon_1| \leq \left| \frac{\delta_3}{1 + \delta_3} \right| \leq \frac{\epsilon_m}{2} \frac{1}{1 - 1/2} \leq \epsilon_m.$$

Further write

$$(1 + \varepsilon_1)(1 + \delta_4) = 1 + \varepsilon_2$$

where

$$|\varepsilon_2| \leq |\varepsilon_1| + |\delta_4| + |\varepsilon_1||\delta_4| \leq (1 + 1/2 + 1/2)\epsilon_m = 2\epsilon_m.$$

We also write

$$\frac{(1.1(1 + \delta_1) - 1)(1 + \delta_2)}{0.1} = 1 + \underbrace{11\delta_1 + \delta_2 + 11\delta_1\delta_2}_{\varepsilon_3}$$

where

$$|\varepsilon_3| \leq (11/2 + 1/2 + 11/4) \leq 9\epsilon_m$$

Then we get

$$(\mathfrak{fl}(1.1) \ominus 1) \oslash \mathfrak{fl}(0.1) = (1 + \varepsilon_3)(1 + \varepsilon_2) = 1 + \underbrace{\varepsilon_3 + \varepsilon_2 + \varepsilon_2\varepsilon_3}_{\varepsilon_4}$$

and the error is bounded by:

$$|\varepsilon_4| \leq (9 + 2 + 18)\epsilon_m \leq 29\epsilon_m.$$

END

Problem 4 Let $x \in [0, 1] \cap F_{\infty, S}$. Assume that $f^{\text{FP}} : F_{\infty, S} \rightarrow F_{\infty, S}$ satisfies $f^{\text{FP}}(x) = f(x) + \delta_x$ where $|\delta_x| \leq c\epsilon_m$ for all $x \in [0, 1]$. Show that

$$\frac{f^{\text{FP}}(x+h) \ominus f^{\text{FP}}(x-h)}{2h} = f'(x) + \varepsilon$$

where absolute error is bounded by

$$|\varepsilon| \leq \frac{|f'(x)|}{2}\epsilon_m + \frac{M}{3}h^2 + \frac{2c\epsilon_m}{h},$$

where we assume that $h = 2^{-n}$ for $n \leq S$.

SOLUTION

In floating point we have

$$\begin{aligned} \frac{f^{\text{FP}}(x+2h) \ominus f^{\text{FP}}(x-2h)}{2h} &= \frac{f(x+h) + \delta_{x+h} - f(x-h) - \delta_{x-h}}{2h}(1 + \delta_1) \\ &= \frac{f(x+h) - f(x-h)}{2h}(1 + \delta_1) + \frac{\delta_{x+h} - \delta_{x-h}}{2h}(1 + \delta_1) \end{aligned}$$

Applying Taylor's theorem we get

$$(f^{\text{FP}}(x+h) \ominus f^{\text{FP}}(x-h))/(2h) = f'(x) + \underbrace{f'(x)\delta_1 + \delta_{x,h}^T(1 + \delta_1) + \frac{\delta_{x+h} - \delta_{x-h}}{2h}(1 + \delta_1)}_{\delta_{x,h}^{\text{CD}}}$$

where

$$|\delta_{x,h}^{\text{CD}}| \leq \frac{|f'(x)|}{2}\epsilon_m + \frac{M}{3}h^2 + \frac{2c\epsilon_m}{h}$$

END

Problem 5 For intervals $X = [a, b]$ and $Y = [c, d]$ satisfying $0 < a < b$ and $0 < c < d$, and $n > 0$ prove that

$$\begin{aligned} X/n &= [a/n, b/n] \\ XY &= [ac, bd] \end{aligned}$$

Generalise (without proof) these formulæ to the case $n < 0$ and to where there are no restrictions on positivity of a, b, c, d . You may use the min or max functions.

SOLUTION

For X/n : if $x \in X$ then $a/n \leq x/n \leq b/n$ means $x \in [a/n, b/n]$. Similarly, if $z \in [a/n, b/n]$ then $a \leq nz \leq b$ hence $nz \in X$ and therefore $z \in X/n$.

For XY : if $x \in X$ and $y \in Y$ then $ac \leq xy \leq bd$ means $xy \in [ac, bd]$. Note $ac, bd \in XY$. To employ convexity we take logarithms. In particular if $z \in [ac, bd]$ then $\log a + \log c \leq \log z \leq \log b + \log d$. Hence write

$$\log z = (1-t)(\log a + \log c) + t(\log b + \log d) = \underbrace{(1-t)\log a + t\log b}_{\log x} + \underbrace{(1-t)\log c + t\log d}_{\log y}$$

i.e. we have $z = xy$ where

$$\begin{aligned} x &= \exp((1-t)\log a + t\log b) = a^{1-t}b^t \in X \\ y &= \exp((1-t)\log c + t\log d) = c^{1-t}d^t \in Y. \end{aligned}$$

The generalisation to negative cases proceeds by being a bit careful with the signs. Eg if $n < 0$ we need to swap the order hence we get:

$$A/n = \begin{cases} [a/n, b/n] & n > 0 \\ [b/n, a/n] & n < 0 \end{cases}$$

For multiplication we just use min and max in a naive fashion:

$$AB = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)].$$

END

Problem 6(a) Compute the following floating point interval arithmetic expression assuming half-precision F_{16} arithmetic:

$$[1, 1] \ominus ([1, 1] \oslash 6)$$

Hint: it might help to write $1 = (0.1111\dots)_2$ when doing subtraction.

SOLUTION Note that

$$\frac{1}{6} = \frac{1}{2} \frac{1}{3} = 2^{-3}(1.010101\dots)_2$$

Thus

$$[1, 1] \oslash 6 = 2^{-3}[(1.0101010101)_2, (1.0101010110)_2]$$

And hence

$$\begin{aligned} [1, 1] \ominus ([1, 1] \oslash 6) &= [1, 1] \ominus [(0.0010101010101)_2, (0.0010101010110)_2] \\ &= [\text{fl}^{\text{down}}(0.1101010101010111\dots)_2, \text{fl}^{\text{up}}(0.1101010101011111\dots)_2] \\ &= 2^{-1}[(1.1010101010)_2, (1.1010101011)_2] = [0.8330078125, 0.83349609375] \end{aligned}$$

END

Problem 6(b) Writing

$$\sin x = \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \delta_{x,2n+1}$$

Prove the bound $|\delta_{x,2n+1}| \leq 1/(2n+3)!$, assuming $x \in [0, 1]$.

SOLUTION

We have from Taylor's theorem up to order x^{2n+2} :

$$\sin x = \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \underbrace{\frac{\sin^{2n+3}(t)x^{2n+3}}{(2n+3)!}}_{\delta_{x,2n+1}}.$$

The bound follows since all derivatives of \sin are bounded by 1 and we have assumed $|x| \leq 1$.

END

Problem 6(c) Combine the previous parts to prove that:

$$\sin 1 \in [(0.11010011000)_2, (0.11010111101)_2] = [0.82421875, 0.84228515625]$$

You may use without proof that $1/120 = 2^{-7}(1.000100010001\dots)_2$.

SOLUTION Using $n = 1$ we have

$$\sum_{k=0}^1 \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x}{3!} \in x \ominus (x \oslash 6).$$

Thus we compute

$$\begin{aligned} \sin 1 &\in [1, 1] \ominus [1, 1] \oslash 6 \oplus [\mathfrak{fl}^{\text{down}}(-1/120), \mathfrak{fl}^{\text{up}}(1/120)] \\ &= [(0.11010101010)_2, (0.11010101011)_2] \oplus [-(0.0000001000100010)_2, (0.00000010001000101)_2] \\ &= [\mathfrak{fl}^{\text{down}}(0.11010011000\textcolor{violet}{11101111} \dots)_2, \mathfrak{fl}^{\text{up}}(0.110101111000\textcolor{violet}{00101})_2] \\ &= [(0.11010011000)_2, (0.11010111101)_2] = [0.82421875, 0.84228515625] \end{aligned}$$

END