Numerical Analysis MATH50003 (2023–24) Problem Sheet 2

Problem 1 Using dual number arithmetic, compute the following polynomials evaluated at the dual number $2+\epsilon$ and use this to deduce their derivative at 2:

$$2x^2 + 3x + 4, (x+1)(x+2)(x+3), (2x+1)x^3$$

Problem 2 Show that dual numbers $\mathbb{D}$ are a *commutative ring*, that is, for all $a, b, c \in \mathbb{D}$ the following are satisfied:

1. *additive associativity*: $(a+b)+c = a+(b+c)$
2. *additive commutativity*: $a+b = b+a$
3. *additive identity*: There exists $0 \in \mathbb{D}$ such that $a+0 = a$.
4. *additive inverse*: There exists $-a$ such that $(-a)+a = 0$.
5. *multiplicative associativity*: $(ab)c = a(bc)$
6. *multiplictive commutativity*: $ab = ba$
7. *multiplictive identity*: There exists $1 \in \mathbb{D}$ such that $1a = a$.
8. *distributive*: $a(b+c) = ab+ac$

Problem 3 What should the following functions applied to dual numbers return for $x = a+b\epsilon$:

$$f(x) = x^{100} + 1, g(x) = 1/x, h(x) = \tan x$$

State the domain where these definitions are valid.

Problem 4(a) What is the correct definition of division on dual numbers, i.e.,

$$(a+b\epsilon)/(c+d\epsilon) = s + t\epsilon$$

for what choice of s and t?

Problem 4(b) A *field* is a commutative ring such that $0 \neq 1$ and all nonzero elements have a multiplicative inverse, i.e., there exists a^{-1} such that $aa^{-1} = 1$. Can we use the previous part to define $a^{-1} := 1/a$ to make $\mathbb{D}$ a field? Why or why not?

Problem 5 Use dual numbers to compute the derivative of the following functions at $x = 0.1$:

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^{3} \left(\frac{x}{k} - 1\right), \text{ and } f_2^{\mathrm{s}}(x) = 1 + \frac{x-1}{2 + \frac{x-1}{2}}$$