1 Integration

One possible definition for an integral is the limit of a Riemann sum, for example:

$$\int_0^1 f(x)\mathrm{d}x = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n f(k/n).$$

This suggests an algorithm known as the *(right-sided) rectangular rule* for approximating an integral: choose n large so that

$$\int_0^1 f(x)\mathrm{d}x \approx \frac{1}{n}\sum_{k=1}^n f(k/n).$$

In the lab we explore practical implementation of this approximation, and observe that the error in approximation is bounded by C/n for some constant C. This can be expressed using "Big-O" notation:

$$\int_0^1 f(x)\mathrm{d}x = \frac{1}{n}\sum_{k=1}^n f(k/n) + O(1/n).$$

In these notes we consider the "Analysis" part of "Numerical Analysis": we want to *prove* the convergence rate of the approximation, including finding an explicit expression for the constant C.

To tackle this question we consider the error incurred on a single "rectangle", then sum up the errors on rectangles.

Now for a secret. There are only so many tools available in analysis (especially at this stage of your career), and one can make a safe bet that the right tool in any analysis proof is either (1) integration-by-parts, (2) geometric series or (3) Taylor series. In this case we use (1):

Lemma (rect. rule on one panel) Assuming f is differentiable we have

$$\int_0^h f(x)\mathrm{d}x = hf(0) + \delta_h$$

where $|\delta_h| \le Mh^2$ for $M = \sup_{0\le x\le h} |f'(x)|$.

Proof We write

$$\int_0^h f(x)\mathrm{d}x = \int_0^h (x)' f(x)\mathrm{d}x = [xf(x)]_0^h - \int_0^h xf'(x)\mathrm{d}x = hf(h) + \underbrace{- \int_0^h xf'(x)\mathrm{d}x}_{\delta_h}.$$

Recall that we can bound the absolute value of an integral by the sepremum of the integrand times the width of the integration interval:

$$|\int_a^b g(x)\mathrm{d}x| \le (b-a) \sup_{0\le x\le h} |g(x)|.$$

The lemma thus follows since

$$|\int_0^h xf'(x)\mathrm{d}x| \le h \sup_{0\le x\le h} |xf'(x)| \le Mh^2.$$