Numerical Analysis MATH50003 (2023–24) Problem Sheet 10

Problem 2(a) What are the upper 3×3 sub-block of the Jacobi matrix for the monic and orthonormal polynomials with respect to the following weights on $[-1, 1]$:

$$1 - x, \sqrt{1 - x^2}, 1 - x^2$$

Problem 2(b) Compute the roots of the Legendre polynomial $P_3(x)$, orthogonal with respect to $w(x) = 1$ on $[-1, 1]$, by computing the eigenvalues of a 3×3 truncation of the Jacobi matrix.

Problem 3(a) Compute the interpolatory quadrature rule for $w(x) = \sqrt{1 - x^2}$ with the points $[-1, 1/2, 1]$.

Problem 3(b) Compute the 2-point interpolatory quadrature rule associated with roots of orthogonal polynomials for the weights $\sqrt{1 - x^2}$, 1, and $1 - x$ on $[-1, 1]$ by integrating the Lagrange bases.

Problem 3(b) Compute the 2-point and 3-point Gaussian quadrature rules associated with $w(x) = 1$ on $[-1, 1]$.

Problem 4(a) Give an explicit diagonalisation of

$$X_n = \begin{bmatrix} 0 & 1/2 & & \\ 1/2 & 0 & \ddots & \\ & \ddots & \ddots & 1/2 \\ & & 1/2 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

for all n by relating it to the Jacobi matrix for $U_n(x)$.

Problem 4(b) Show for $w(x) = 1/\sqrt{1 - x^2}$ that the Gaussian quadrature rule is

$$Q_n^w[f] = \frac{\pi}{n} \sum_{j=1}^{n} f(x_j)$$

where $x_j = \cos(\theta_j)$ for $\theta_j = (j - 1/2)\pi/n$.

Problem 4(c) Solve Problem 4.2 from PS8 using **Lemma III.6.3 (discrete orthogonality)** with $w(x) = 1/\sqrt{1 - x^2}$ on $[-1, 1]$. That is, use the connection of $T_n(x)$ with $\cos n\theta$ to show that the Discrete Cosine Transform

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for $\theta_j = \pi(j - 1/2)/n$ is an orthogonal matrix.