Numerical Analysis MATH50003 (2023–24) Problem Sheet 8

Problem 1 Give explicit formulae for $\hat{f}_k$ and $\hat{f}_k^n$ for the following functions:

$$\cos\theta, \cos 4\theta, \sin^4\theta, \frac{3}{3-\mathrm{e}^{\mathrm{i}\theta}}, \frac{1}{1-2\mathrm{e}^{\mathrm{i}\theta}}$$

Hint: You may wish to try the change of variables $z = \mathrm{e}^{-\mathrm{i}\theta}$.

Problem 2 Prove that if the first $\lambda - 1$ derivatives $f(\theta), f'(\theta), \ldots, f^{(\lambda-1)}(\theta)$ are 2π-periodic and $f^{(\lambda)}$ is uniformly bounded that

$$|\hat{f}_k| = O(|k|^{-\lambda}) \qquad \text{as } |k| \to \infty$$

Use this to show for the Taylor case ($0 = \hat{f}_{-1} = \hat{f}_{-2} = \cdots$) that

$$|f(\theta) - \sum_{k=0}^{n-1} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}| = O(n^{1-\lambda}) \qquad \text{as } n \to \infty$$

Problem 3(a) If f is a trigonometric polynomial ($\hat{f}_k = 0$ for $|k| > m$) show for $n \geq 2m+1$ that we can exactly recover f:

$$f(\theta) = \sum_{k=-m}^{m} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

Problem 3(b) For the general (non-Taylor) case and $n = 2m+1$, prove convergence for

$$f_{-m:m}(\theta) := \sum_{k=-m}^{m} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

to $f(\theta)$ as $n \to \infty$. What is the rate of convergence if the first $\lambda-1$ derivatives $f(\theta), f'(\theta), \ldots, f^{(\lambda-1)}(\theta)$ are 2π-periodic and $f^{(\lambda)}$ is uniformly bounded?

Problem 4(a) Show for $0 \leq k, \ell \leq n-1$

$$\frac{1}{n}\sum_{j=1}^{n} \cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_j = \pi(j-1/2)/n$. Hint: Be careful as the θ_j differ from before, and only cover half the period, $[0, \pi]$. Using symmetry may help. You may also consider replacing cos with complex exponentials:

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}.$$

Problem 4(b) Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} \\ & \sqrt{2/n} \\ && \ddots \\ &&& \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for $\theta_j = \pi(j-1/2)/n$. Prove that C_n is orthogonal: $C_n^\top C_n = C_n C_n^\top = I$. Hint: $C_n C_n^\top = I$ might be easier to show than $C_n^\top C_n = I$ using the previous problem.