Numerical Analysis MATH50003 (2022–23) Problem Sheet 1

Problem 1 Assuming f is differentiable, prove the left-point Rectangular rule error formula

$$\int_a^b f(x)\mathrm{d}x = h\sum_{k=0}^{n-1} f(x_k) + \delta$$

where $|\delta| \leq M(b-a)h$ for $M = \sup_{a\leq x\leq b}|f'(x)|$, $h = (b-a)/n$ and $x_k = a + kh$.

Problem 2(a) Assuming f is twice-differentiable, prove a one-panel Trapezium rule error bound:

$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2} + \delta$$

where $|\delta| \leq M(b-a)^3$ for $M = \sup_{a\leq x\leq b}|f''(x)|$.

Hint: Recall from the notes

$$\int_a^b \frac{(b-x)f(a)+(x-a)f(b)}{b-a}\mathrm{d}x = (b-a)\frac{f(a)+f(b)}{2}$$

and you may need to use Taylor's theorem. Note that the bound is not sharp and so you may arrive at something sharper like $|\delta| \leq 3(b-a)^3M/4$. The sharpest bound is $|\delta| \leq (b-a)^3M/12$ but that would be a significantly harder challenge to show!

Problem 2(b) Assuming f is twice-differentiable, prove a bound for the Trapezium rule error:

$$\int_a^b f(x)\mathrm{d}x = h\left[\frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_k) + \frac{f(b)}{2}\right] + \delta$$

where $|\delta| \leq M(b-a)h^2$ for $M = \sup_{a\leq x\leq b}|f''(x)|$.

Problem 3 Assuming f is twice-differentiable, for the left difference approximation

$$f'(x) = \frac{f(x)-f(x-h)}{h} + \delta,$$

show that $|\delta| \leq Mh/2$ for $M = \sup_{x\leq t\leq x+h}|f''(t)|$.

Problem 4 Assuming f is thrice-differentiable, for the central differences approximation

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} + \delta,$$

show that $|\delta| \leq Mh^2/6$ for $M = \sup_{x-h\leq t\leq x+h}|f'''(t)|$.

Problem 5 Assuming f is thrice-differentiable, for the second-order derivative approximation

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x) + \delta$$

show that $|\delta| \leq Mh/3$ for $M = \sup_{x-h\leq t\leq x+h}|f'''(t)|$.