Finite Elements: class test 1

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1. Let V_h be a C^0 finite element space defined on a triangulation $\mathcal{T}_h$. For $u \in V_h$, the finite element partial derivative $\frac{\partial^{FE}}{\partial x_i}u$ satisfies
$$\frac{\partial^{FE} u}{\partial x_i}|_{K_i} = \frac{\partial u}{\partial x_i}|_{K_i}.$$
Show that this uniquely defines the finite element partial derivative in L^2.

2. Let $u \in C^1(\Omega)$. Show that the finite element partial derivative and the usual derivative are equal in $L^2(\Omega)$.

3. For a domain K and shape space P, is the following functional a nodal variable? Explain your answer.
$$N_0(p) = \int_K p^2 \,\mathrm{d}\, x.$$

4. Consider the finite element defined by:
   (a) K is the unit interval $[0, 1]$
   (b) P is the space of quadratic polynomials on K,
   (c) The nodal variables are:
      i. $N_0[v] = v(0)$,
      ii. $N_1[v] = v(1)$,
      iii. $N_2[v] = \int_0^1 v(x) \,\mathrm{d}\, x$.

   Using the fact that
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & 1 & 1/3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -4 & 3 \\ 0 & -2 & 3 \\ 0 & 6 & -6 \end{pmatrix},$$
find the corresponding nodal basis for P in terms of the monomial basis $\{1, x, x^2\}$. Provide the C^0 geometric decomposition for the finite element (demonstrating that it is indeed C^0).

5. The nodal variables for the cubic Hermite element on triangles are: evaluations of the polynomial and both components of its gradient at each vertex, plus evaluation of the polynomial at the centre of the triangle. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.

6. Let $\mathcal{I}_K f$ be the interpolant for a finite element K. Show that $\mathcal{I}_K$ is a linear operator.

7. Let K be a rectangle, P be the polynomial space spanned by $\{1, x, y, xy\}$, let $\mathcal{N}$ be the set of dual elements corresponding to each vertex of the rectangle. Show that $\mathcal{N}$ determines the finite element.

8. **Extra question**. Let K be a triangle, and P be the space of quadratic polynomials. Let N be the set of nodal variables given by point evaluation at each edge midpoint, plus integral of the function along each edge. Show that N does *not* determine P.