M5MA47 Finite elements mastery exercise

Due by 1600 on Friday 1 May 2015

Provide written answers to the following questions. A neatly handwritten or a typed document are equally acceptable. You should submit your answers to Anna Lisowska in Huxley 652.

All of the questions concern the following differential equation:

$$\phi - \nabla \cdot (1 - x_0/2)\nabla\phi = f \quad \text{on } \Omega \tag{1}$$

$$\nabla\phi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \tag{2}$$

where Ω is the unit square $(0 \leqslant x_0 \leqslant 1, 0 \leqslant x_1 \leqslant 1)$, Γ is the boundary of Ω, and $\mathbf{n}$ is the outward pointing normal to the domain.

1. Find $f(x_0, x_1)$ such that:

   $$\phi = \cos(2\pi x_0)\cos(2\pi x_1) \tag{3}$$

   is a solution to (1). You may assume in all subsequent questions that f takes this value.

   (4 marks)

2. Let $V_h \subset H^1(\Omega)$ be a finite element space. Derive a symmetric variational problem from (1) whose answer lies in V_h.

   (4 marks)

3. Prove that this variational problem has a unique solution.

   (4 marks)

4. Use the finite element code you developed during this module to implement a solver for this variational problem. When you incorporate the $(1 - x_0/2)$ term, ensure that you use the global coordinate x and not the local coordinate X.

   For each of piecewise linear and piecewise quadratic polynomials, solve the problem on successively finer meshes and calculate the rate of convergence in the L^2 norm numerically. Demonstrate numerically that your solution converges at a good approximation to the theoretically optimal rate.

   For this question you must submit:

   (a) The numerical results.

   (b) The bitbucket commit code (the git sha eg 16455ad) of the version of your software you used to produce those results. The repository must include a file `mastery.py` such that `python mastery.py` will run the convergence tests and print out the results.

   (8 marks)