Finite Elements: examples 3

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March 12, 2019

1. Let V be a discontinuous Lagrange finite element space of degree k defined on a triangulation $\mathcal{T}$ of a domain Ω. Show that functions in V do not have weak derivatives in general.

2. Show that
$$D^\beta Q_B^k f = Q_B^{k-|\beta|} D^\beta f,$$
where Q_B^l is the degree l averaged Taylor polynomial of f, and D^β is the β-th derivative where β is a multi-index.

3. Consider the variational problem of finding $u \in H^1([0,1])$ such that
$$\int_0^1 vu + v'u' \,\mathrm{d}\,x = \int_0^1 vx \,\mathrm{d}\,x + v(1) - v(0), \quad \forall v \in H^1([0,1]).$$
After dividing the interval $[0,1]$ into N equispaced cells and forming a $P1$ C^0 finite element space V_N, the error $\|u - u_h\|_{H^1} = 0$ for any $N > 0$.

   Explain why this is expected.

4. Let $\mathring{H}^1([0,1])$ be the subspace of $H^1([0,1])$ such that $u(0) = 0$. Consider the variational problem of finding $u \in \mathring{H}^1([0,1])$ with
$$\int_0^1 v'u' \,\mathrm{d}\,x = \int_0^{1/2} v \,\mathrm{d}\,x, \quad \forall v \in \mathring{H}([0,1]).$$
The interval $[0,1]$ is divided into $3N$ equispaced cells (where N is a positive integer). After forming a $P1$ C^0 finite element space V_N, the error $\|u - u_h\|_{H^1}$ is found not to converge to zero. Explain why this is expected?

5. (a) Let Ω be a convex polygonal 2D domain.Consider the following two problems.
   i. Find $u \in H^2$ such that
$$\|\nabla^2 u + f\|_{L^2(\Omega)} = 0, \quad \|u\|_{L^2(\partial\Omega)} = 0,$$
   which we write in a shorthand as
$$-\nabla^2 u = f, \quad u|_{\partial\Omega} = 0.$$
   ii. Find $u \in \mathring{H}^1(\Omega)$ such that
$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\,x = \int_\Omega fv \,\mathrm{d}\,x, \quad \forall v \in \mathring{H}^1(\Omega),$$
   where $\mathring{H}^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of functions whose trace vanishes on the boundary.

   Under assumptions on u which you should state, show that a solution to problem ii is a solution to problem i.

   (b) Let h be the maximum triangle diameter of a triangulation T_h of Ω, with V_h the corresponding linear Lagrange finite element space. Construct a finite element approximation to Problem ii above. Briefly give the main arguments as to why the $H^1(\Omega)$ norm of the error converges to zero linearly in h as $h \to 0$, giving your assumptions.