Finite Elements: mock question paper

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Answer all questions from Section A, and choose two questions from three in Section B.

Section A

1. Provide a weak formulation for the following equation for u.

$$u'' = -f, \quad u(0) = 7,\, u'(1) = 3.$$

2. Which of the following dual bases define a one-dimensional C^0 finite element with quadratic polynomials P_2, and for $K = [x_n, x_{n+1}]$?

   (a) $N_1(f) = f(x_n)$, $N_2(f) = f(0.5(x_n + x_{n+1}))$, $N_3(f) = f(x_{n+1})$.

   (b) $N_1(f) = f(x_n)$, $N_2(f) = f(x_{n+1})$, $N_3(f) = \int_{x_n}^{x_{n+1}} f \,\mathrm{d}\, x$.

   (c) $N_1(f) = f'(x_n)$, $N_2(f) = f'(x_{n+1})$, $N_3(f) = \int_{x_n}^{x_{n+1}} f \,\mathrm{d}\, x$.

3. Obtain the nodal basis function $\phi_1(x)$ for the last dual basis introduced in the previous question.

4. For a linear variational problem with bilinear form $a(\cdot, \cdot)$ and a Ritz-Galerkin approximation of it, derive a Galerkin Orthogonality result.

5. What is the maximum possible approximation order of a triangular finite element with $\mathcal{P}$ being the polynomial space spanned by polynomials of degree ≤ 2 and the cubic function that vanishes on all edges of the triangle?

6. Suggest a nodal basis for the polynomial space introduced in the previous question.

7. A quadrature rule on a reference element K has degree m if it produces the exact answer for all polynomials of degree m or less. What is the minimum degree of quadrature rule required to exactly assemble the matrix for the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\, x,$$

   using cubic Lagrange elements?

8. Describe a dual basis for cubic Lagrange elements.

Section B

1. The cubic Hermite element $(K, \mathcal{P}, \mathcal{N})$ is defined by

   - K is a non-degenerate triangle,
   - $\mathcal{P}$ is the set of polynomials of degree 3 or less,
   - $\mathcal{N}$ is the dual basis comprising function evaluation at the vertices, gradient evaluation at the vertices, and function evaluation at the triangle midpoint.

   (a) Show that the dual basis determines the element.

   (b) Show that the resulting finite element space defined on a triangulation of a polygonal domain Ω is C^0.

   (c) Is the resulting finite element space defined above C^1?

   (d) For a right-angled triangle, determine the nodal basis function corresponding to the triangle midpoint node.

2. The following approximation theory result holds on the reference element K with vertices $(0,0)$, $(1,0)$ and $(1,1)$,

$$|e|_{W_2^1(K)} \leq c_0 |e|_{W_2^2(K)}, \quad \forall e \in W_{2,0}^2(K),$$

where $c_0 > 0$ is a constant that is independent of e, and $W_{2,0}^2(K)$ is the subspace of $W_2^2(K)$ functions that vanish at the vertices of K.

(a) For a mesh element K_h with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), with $x_i = x_0 + ih$, $y_j = y_0 + jh$, show that

$$|e|_{W_2^1(K_h)} \leq c_0 h |e|_{W_2^2(K_h)}, \quad \forall e \in W_{2,0}^2(K_h).$$

(b) Use this result to show that

$$|u - \mathcal{I}_h u|_{W_2^1(\Omega)} \leq C h |u|_{W_2^1(\Omega)},$$

where $\mathcal{I}_h$ is the global interpolation operator to linear Lagrange elements defined on a mesh of a square domain Ω constructed from $h \times h$ squares subdivided into right-angled triangles.

3. The inhomogeneous Helmholtz equation in two dimensions is given by

$$\alpha(x)u - \nabla^2 u = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $\partial\Omega$ is the boundary of the problem domain Ω, and $\alpha(x)$ is a $C^\infty(\Omega)$ function with bounds $1 \leq \alpha(x) \leq 2$ for all $x \in \Omega$.

(a) Derive a variational formulation for this problem, in the form

$$a(u, v) = F(v), \quad \forall v \in W_2^1(\Omega).$$

(b) Show that $a(\cdot, \cdot)$ is continuous and coercive.

(c) Hence, show that the linear Lagrange finite element approximation satisfies

$$\|u - u_h\|_{W_2^1(\Omega)} \leq Ch|u|_{W_2^2(\Omega)},$$

for $C > 0$, independent of u. (You may make use of the approximation theory estimate

$$\|u - \mathcal{I}_h u\|_{w_2^1(\Omega)} \leq \hat{C}|u|_{W_2^2(\Omega)},$$

for $\hat{C} > 0$, independent of u.)

(d) Show that

$$\|u - u_h\|_{L^2(\Omega)} \leq \gamma h^2 |u|_{W_2^2(\Omega)},$$

for $\gamma > 0$.