Finite Elements: class test 1

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1. Let V_h be a C^0 finite element space defined on a triangulation $\mathcal{T}_h$. For $u \in V_h$, the finite element partial derivative $\frac{\partial^{FE}}{\partial x_i}u$ satisfies
$$\frac{\partial^{FE} u}{\partial x_i}|_{K_i} = \frac{\partial u}{\partial x_i}|_{K_i}.$$
Show that this uniquely defines the finite element partial derivative in L^2.
Solution: *Let us say that we have two functions $v, w \in V_h$ that both satisfy the condition. Then*
$$\|v-w\|_{L^2}^2 = \int_\Omega (v-w)^2 \,\mathrm{d}\,x = \sum_{K_i \in \mathcal{T}_h} \int_{K_i} (v-w)^2 \,\mathrm{d}\,x = \sum_{K_i \in \mathcal{T}_h} \int_{K_i} (u-u)^2 \,\mathrm{d}\,x = 0,$$
which means that the two functions are equivalent in L^2.

2. Let $u \in C^1(\Omega)$. Show that the finite element partial derivative and the usual derivative are equal in $L^2(\Omega)$.
Solution:
$$\left\|\frac{\partial^{FE} u}{\partial x} - \frac{\partial u}{\partial x}\right\|_{L^2}^2 = \int_\Omega \left(\frac{\partial^{FE} u}{\partial x} - \frac{\partial u}{\partial x})^2\right) \mathrm{d}\,x = \sum_{K_i \in \mathcal{T}_h} \int_{K_i}^2 \left(\frac{\partial^{FE} u}{\partial x} - \frac{\partial u}{\partial x})^2\right) \mathrm{d}\,x = \sum_{K_i \in \mathcal{T}_h} \int_{K_i}^2 \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x})^2\right) \mathrm{d}\,x = 0,$$
so the two functions are equivalent in L^2.

3. For a domain K and shape space P, is the following functional a nodal variable? Explain your answer.
$$N_0(p) = \int_K p^2 \,\mathrm{d}\,x.$$
Solution: *No, because it is not a linear map from P to $\mathbb{R}$.*

4. Consider the finite element defined by:
   (a) K is the unit interval $[0, 1]$
   (b) P is the space of quadratic polynomials on K,
   (c) The nodal variables are:
      i. $N_0[v] = v(0)$,
      ii. $N_1[v] = v(1)$,
      iii. $N_2[v] = \int_0^1 v(x)\,\mathrm{d}\,x$.

   Using the fact that
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & 1 & 1/3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -4 & 3 \\ 0 & -2 & 3 \\ 0 & 6 & -6 \end{pmatrix},$$
find the corresponding nodal basis for P in terms of the monomial basis $\{1, x, x^2\}$. Provide the C^0 geometric decomposition for the finite element (demonstrating that it is indeed C^0).
Solution: *The matrix on the left is the Vandermonde matrix V with $V_{ij} = N_j[x^{i-1}]$. Hence the nodal basis is given by*
$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 3 \\ 0 & -2 & 3 \\ 0 & 6 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = \begin{pmatrix} 1 - 4x + 3x^2 \\ -2x + 3x^2 \\ 6x - 6x^2 \end{pmatrix}$$

The C^0 geometric decomposition is to associate N_0 with the vertex 0, N_1 with the vertex 1, and N_2 with the interval itself. Clearly N_0 determines the value of the polynomial at $x = 0$ and N_1 determines the value of the polynomial at $x = 1$, so we have a C^0 geometric decomposition.

5. The nodal variables for the cubic Hermite element on triangles are: evaluations of the polynomial and both components of its gradient at each vertex, plus evaluation of the polynomial at the centre of the triangle. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.
   Solution: *Let P be a cubic polynomial on the triangle K such that $N_i(P) = 0$, $i = 1, 2, 3, 4, 5, 6$. Let L_1, L_2 and L_3 be the three edges of the triangle opposite vertices z_1, z_2 and z_3 respectively. Restricting P to L_1, we find that $P(z_0) = P'(z_0) = P(z_1) = P'(z_1)$ where $'$ indicates differentiation in the direction of L_1. Therefore $P|_{L_1}$ has double roots at z_0 and z_1, and so $P|_{L_1} = 0$. Similar argument for the other three edges means that $P = cL_1L_2L_3$ for constant c. But, P also vanishes at the midpoint z_4, so $c = 0$.*

6. Let $\mathcal{I}_K f$ be the interpolant for a finite element K. Show that $\mathcal{I}_K$ is a linear operator.
   Solution:

$$\begin{aligned}\mathcal{I}_K(f+\alpha g) &= \sum_{i=1}^{n} \phi_i(x)(f+\alpha g)(x_i), \\ &= \sum_{i=1}^{n} \phi_i(x) f(x_i) + \alpha \sum_{i=1}^{n} \phi_i(x) g(x_i), \\ &= \mathcal{I}_K f + \alpha \mathcal{I}_K g.\end{aligned}$$

7. Let K be a rectangle, P be the polynomial space spanned by $\{1, x, y, xy\}$, let $\mathcal{N}$ be the set of dual elements corresponding to each vertex of the rectangle. Show that $\mathcal{N}$ determines the finite element.
   Solution: *Let p be a polynomial that vanishes at each vertex. Let Π_1 be the line defined by $y = 0$, and Π_2 be the line defined by $x = 0$. Restricted to Π_1, p is a linear polynomial in x vanishing at $x = 0$ and $x = 1$, so p vanishes on Π_1. Hence, $p = yq_1(x, y)$ where q_1 is a linear polynomial (p is quadratic). Similarly, if we restrict to Π_2, we conclude that $p = xq_2(x, y)$, where q_2 is another linear polynomial. Equating gives*

$$yq_1(x, y) = xq_2(x, y).$$

   When restricting to $x = 1$, we have

$$yq_1(0, y) = q_2(1, y).$$

   On the left we have a quadratic polynomial in y, whilst on the right we have a linear polynomial in y. This means that q_1 must be in fact independent of y, and linear in x only. Now restricting to Π_2, similar arguments mean that $p(x, y) = cxy$ where c is a constant. Finally, evaluation at the vertex $(x, y) = (1, 1)$ means that $c = 0$, and we conclude that $p \equiv 0$ as required.

8. **Extra question**. Let K be a triangle, and P be the space of quadratic polynomials. Let N be the set of nodal variables given by point evaluation at each edge midpoint, plus integral of the function along each edge. Show that N does *not* determine P.