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Finite Elements

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- For some elliptic problems we have seen how to show that

$$\|u-u_h\|^2_{H^1(\Omega)} \leq Ch^k\|u\|_{H^2(\Omega)}, \quad \|u-u_h\|^2_{L^2(\Omega)} \leq Ch^{k+1}\|u\|_{H^2(\Omega)}.$$

- This means that the error can be arbitrarily reduced by reducing h.

This might be very inefficient if the error is mostly concentrated in one area of the grid.

Model problem (from porous media):

$$-\nabla \cdot (\sigma(x)\nabla)u = f, \quad u = 0 \text{ on } \partial\Omega.$$

σ is C^0 but might have large variations in value, and there exist $0 < a < \sigma(x) < b < \infty$.
Variational formulation, find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega \sigma \nabla v \cdot \nabla u \, \mathrm{d}x = L(v) := \int_\Omega vf \, \mathrm{d}x, \quad \forall H_0^1(\Omega),$$

Galerkin discretisation: $V_h \in H_0^1(\Omega)$. Find $u_h \in V_h$ such that

$$a(u, v) := \int_\Omega \sigma \nabla v \cdot \nabla u \, \mathrm{d}x = L(v) := \int_\Omega vf \, \mathrm{d}x, \quad \forall v \in V_h.$$

Definition 1

Let $h(x)$ be the P_1-continuous finite element function such that

$$h(x_i) = \max_{K \in \mathcal{T}_h : x_i \in K} \mathrm{diam}(K).$$

We see immediately that $h(x) \leq \mathrm{diam}(K)$ for all $x \in K$.

Since we have coercivity and continuity of a (exercise), we have from Céa's lemma,

$$\|u - u_h\|^2_{H^1(\Omega)} \leq C\|u - \mathcal{I}_h u\|^2_{H^1(\Omega)}.$$

We also have

$$\begin{aligned}
\|u - \mathcal{I}_h u\|^2_{H^1(\Omega)} &= \sum_K \|u - \mathcal{I}_K u\|^2_{H^1(\Omega)} \\
&\leq \sum_K d_K^{2k-2}\|u\|^2_{H^k(\Omega)} \\
&= \sum_K \sum_{|\alpha|=k} \int_K h(x)^{2k-2}|D^\alpha u|^2 \, \mathrm{d}x.
\end{aligned}$$

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This suggests that we can adaptively choose $h(x)$ to reduce the local error. But how can we estimate it if we don't know u?

Lemma 2

Let $u_h \in V_h$ be the Galerkin approximation to u, and let $e_h = u - u_h$. Then

$$a(e_h, v) = R(v), \quad \forall v \in V,$$

where R is the residual,

$$R(v) = \sum_K \int_K (f + \nabla \cdot (\sigma \nabla u_h)) v \, dx + \int_\Gamma [[\sigma n \cdot \nabla u_h]] v \, dS,$$

where $[[\sigma n \cdot \nabla u]] = \sigma^+ n^+ \cdot \nabla u^+ + \sigma^- n^- \cdot \nabla u^-$.

Proof.

$$a(e_h, v) = a(u, v) - a(u_h, v) = \int_\Omega fv \, \mathrm{d}x - a(u_h, v),$$

$$\sum_K \int_K fv - \sigma \nabla v \cdot \nabla u \, \mathrm{d}x,$$

and integration by parts in each cell gives the result. □

Choosing $v = e_h$ in the above expression, we find that

$$\gamma |e_h|^2_{H^1(\Omega)} \leq |R(e_h)| \leq \|R\|_{H^{-1}(\Omega)} \|e_h\|_{H^1(\Omega)},$$

so

$$\|e_h\|^2_{H^1(\Omega)} \leq C|e_h|^2(H^1\Omega) \leq \frac{C}{\gamma} \|R\|_{H^{-1}(\Omega)} \|e_h\|_{H^1(\Omega)},$$

and so

$$\|e_h\|_{H^1(\Omega)} \leq C_1 \|R\|_{H^{-1}(\Omega)}.$$

Only problem is that R is as hard to compute as e_h. But we can try to estimate it.

We assume that we have an interpolant $\mathcal{I}_h$ such that

$$\|v - \mathcal{I}_h v\|_{L^2(K)} \leq \gamma_0 h_K |v|_{H^1(\hat{K})},$$
$$\|v - \mathcal{I}_h v\|_{L^2(e)} \leq \gamma_1 h_e^{1/2} |v|_{H^1(\hat{K}_e)},$$

where $\hat{K}$ is the patch of cells touching K, and $\hat{K}_e$ is the patch of cells touching K_e, and K_e is the pair of cells touching e.
(This is called a Clemént interpolant but we don't have time to define it or prove the estimates.)

Theorem 3

Let u, u_h, e_h, R be as above. Then

$$|e_h|_{H^1(\Omega)} \leq D\left(\sum_{e\in\Gamma}\mathcal{E}_e(u_h)^2\right)^{1/2},$$

where

$$\mathcal{E}_e(u_h)^2 = \sum_{K\in K_e} h_L^2\|f+\nabla\cdot(\sigma\nabla u_h)\|^2_{L^2(K)} + h_e\|[[\sigma n\cdot\nabla u]]\|^2_{L^2(e)}.$$

Proof

We start by noting that if $v \in V_h$, then $R(v) = 0$. Then by linearity of R we have $R(v) = R(v - \mathcal{I}_h v)$.

Proof (Cont.)

Then,

$$
\begin{aligned}
|R(v)| &= |R(v - \mathcal{I}_h v)|, \\
&= \left| \sum_K \int_K (f + \nabla \cdot (\sigma \nabla u_h))(v - \mathcal{I}_h v)\, \mathrm{d}x \right. \\
&\qquad \left. + \sum_e \int_e [[\sigma n \cdot \nabla u_h]](v - \mathcal{I}_h v)\, \mathrm{d}S \right|
\end{aligned}
$$

Proof.

$$
\begin{aligned}
|R(v)| \leq & \left| \sum_K \int_K (f + \nabla \cdot (\sigma \nabla u_h)) \gamma_0 \, \mathrm{d}x h_K |v|_{H^1(\hat{K})} \right. \\
& \left. + \sum_e \int_e [[\sigma n \cdot \nabla u_h]] \, \mathrm{d}S \gamma_1 h_e^{1/2} |v|_{H^1(\hat{K}_e)} \right|, \\
\leq & C \sum_e \left(\sum_K h_k^2 (f + \nabla \cdot (\sigma \nabla u_h))^2 \, \mathrm{d}x \right. \\
& \left. + h_e \int_e ([[\sigma n \cdot \nabla u_h]])^2 \, \mathrm{d}S \right)^{1/2} |v|_{H^1(\Omega)},
\end{aligned}
$$

and hence the result. □

This gives the starting point to an adaptive mesh algorithm.

Start with an initial mesh.

1. On the current mesh, obtain the numerical solution u_h.
2. Compute the local error measure for each edge.
3. If the local error measure is above a given tolerance, refine the mesh around the edge and return to 1.

Still need to check that local error measure being small $\implies$ local error is small. (Beyond the scope of this brief summary!).