Test 2 solution slides

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Q1

The constant function $v = 1$ is in V, since it is a polynomial and hence can be represented exactly by a degree k polynomial. We have

$$\int_\Omega |\nabla v|^2 \, \mathrm{d}x = 0,$$

which breaks coercivity.

Q2

$$
\begin{aligned}
\int_0^1 u' \, \mathrm{d}x &= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} u' \, \mathrm{d}x, \\
&= \sum_{k=1}^{n} \left(u(x_k) - u(x_{k-1}) \right) \\
&= u(1) - u(0).
\end{aligned}
$$

Q3

First we note that the fundamental theorem of calculus holds for C^0 finite element functions, by subdividing the integral into cells or partial cells as usual. Second we note that $v'v = (v^2)'/2$. So,

$$\int_0^1 v'v \,\mathrm{d}x = \int_0^1 \frac{1}{2}(v^2)' \,\mathrm{d}x = \left[\frac{1}{2}v^2\right]_0^1 = 0,$$

by the boundary conditions. Then,

$$a(v,v) = \int_0^1 \left((v')^2 + v'v + v^2\right) \mathrm{d}x = \int_0^1 \left((v')^2 + v^2\right) \mathrm{d}x.$$

Thus we have coercivity of a since $a(v,v) = \|v\|_{H^1}^2$ which is a special case of the coercivity condition $a(v,v) \leq C\|v\|_{H^1}^2$ with $C = 1$.

Q4

For continuity,

$$
\begin{aligned}
|a(u,v)| &\leq |(u,v)_{H^1}| + \left|\int_0^1 u'v\,\mathrm{d}x\right| \\
&\leq \|u\|_{H^1}\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L_2},
\end{aligned}
$$

$$\leq 2\|u\|_{H^1}\|v\|_{H^1}.$$

For coercivity,

$$
\begin{aligned}
a(v,v) &= \int_0^1 (v')^2 + v'v + v^2\,\mathrm{d}x, \\
&= \frac{1}{2}\int_0^1 (v'+v)^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 (v')^2 + v^2\,\mathrm{d}x \geq \frac{1}{2}\|v\|_{H^1}.
\end{aligned}
$$

Q5

Let V be a C^0 finite element space defined on Ω. Multiply by test function $v \in V$ and integrate by parts, to obtain

$$\int_\Omega \nabla v \cdot (\sigma \nabla u)\,\mathrm{d}x = \int_\Omega vf\,\mathrm{d}x + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial n}}_{=0} \sigma v\,\mathrm{d}x = \int_\Omega vf\,\mathrm{d}x.$$

Hence, the finite element discretisation requires to find $u_h \in V$, such that

$$a(u_h, v) = (f, v), \quad \forall v \in V,$$

where

$$a(u, v) = \int_\Omega \nabla v \cdot (\sigma \nabla u)\,\mathrm{d}x.$$

Q5 [ctd]

For continuity,

$$
\begin{aligned}
|a(u,v)| &= \left|\sum_i \int_\Omega \frac{\partial u}{\partial x_i}\sigma\frac{\partial v}{\partial x_i}\,\mathrm{d}x\right| \leq b\left|\sum_i \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x\right| \\
&\leq b\sum_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^2} \leq b\sum_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}\sum_j \left\|\frac{\partial v}{\partial x_j}\right\|_{L^2} \\
&\leq b\left(\|u\|_{L^2}^2 + \sum_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2\right)^{1/2}\left(\|v\|_{L^2}^2 + \sum_j \left\|\frac{\partial v}{\partial x_j}\right\|_{L^2}^2\right)^{1/2}, \\
&= b\|u\|_{H^1}\|v\|_{H^1}.
\end{aligned}
$$

Q5 [cdt]

For coercivity,

$$
\begin{aligned}
\|v\|^2_{H^1\Omega} &\leq \left(1 + C_\Omega^2\right) |v|^2_{H^1(\Omega)}, \quad \text{(From Lectures)} \\
&= \left(1 + C_\Omega^2\right) \int_\Omega \nabla v \cdot \nabla v \, \mathrm{d}x, \\
&= \left(1 + C_\Omega^2\right) \int_\Omega \frac{1}{\sigma} \sigma \nabla v \cdot \nabla v \, \mathrm{d}x, \\
&\leq \left(1 + C_\Omega^2\right) \frac{1}{a} \int_\Omega \sigma \nabla v \cdot \nabla v \, \mathrm{d}x = \frac{1}{a} \left(1 + C_\Omega^2\right) a(v, v).
\end{aligned}
$$

Hence, $a(\cdot, \cdot)$ is a symmetric bilinear form that is continous and coercive on $H^1(\Omega)$, and hence a unique solution exists.

Q7

Choosing a C^0 finite element space V, we define $\mathring{V}$ as the subspace of functions vanishing on $\partial\Omega$ as above. We write $u = u^H + u^g$, where $u^H \in \mathring{V}$, and $u^g = g$ on $\partial\Omega$. Then,

$$a(u^H, v) = (f, v) - a(g, v), \quad \forall v \in \mathring{V},$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x.$$

Q8

We have already checked coercivity and continuity of a in lectures, so we just need to check continuity of $L(v)$ given by

$$L(v) = (f, v) - a(g, v).$$

We have

$$|L(v)| \leq \|f\|_{L^2}\|v\|_{L^2} + \|g\|_{H^1}\|v\|_{H^1} = \left(\|f\|_{L^2} + \|g\|_{H^1}\right)\|v\|_{H^1},$$

so it is continuous as required.