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Test 2 solution slides

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Question 1

Choose a triangle $K_0 \in \mathcal{T}$, and define $u \in V$ as

$$u(x) = \begin{cases} 1 & \text{if } x \in K_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then if $D_w^x u$ exists,

$$\begin{aligned} \int_\Omega D_w^x u \phi \, \mathrm{d}x &= -\int_\Omega \phi_x u \, \mathrm{d}x, \\ &= -\int_{K_0} \phi_x \, \mathrm{d}x, \\ &= -\int_{\partial K_0} \phi n_1 \, \mathrm{d}S, \end{aligned}$$

where n_1 is the x-component of outward pointing normal to ∂K_0.

Question 1 [ctd]

Now we choose a sequence $\phi_i \in C_0^\infty(\Omega)$ such that

$$\phi_i|_{\partial K_0} \to 1 \text{ in } L^2(\partial K_0), \quad \phi_i|_{\Omega} \to 0 \text{ in } L^2(\Omega).$$

Then,

$$\int_\Omega \phi_i D_w^x \,\mathrm{d}x \to 0,$$

but we have just shown that

$$\int_\Omega D_w^x \phi_i \,\mathrm{d}x \to \int_{\partial K_0} n_1 \,\mathrm{d}S \neq 0,$$

so the weak derivative does not exist.

Question 2

We assume that $f \in C^\infty(B)$ and then pass to the limit and integrate over $y \in B$.

$$
\begin{aligned}
D_x^\beta (T_y^k f)(x) &= D_x^\beta \sum_{|\alpha|\leq k} D_y^\alpha f(y) \frac{(x-y)^\alpha}{\alpha!}, && (1)\\
&= \sum_{|\alpha|\leq k} D_y^\alpha f(y) \frac{\alpha!}{(\alpha-\beta)!} \frac{(x-y)^{\alpha-\beta}}{\alpha!}, && (2)\\
&= \sum_{|\alpha'|\leq k-|\beta|} D_y^{\alpha'} D_y^\beta f(y) \frac{(x-y)^{\alpha'}}{\alpha'!}, && (3)\\
&= (T_y^{k-|\beta|} D^\beta f)(x), && (4)
\end{aligned}
$$

after setting $\alpha' = \alpha - \beta$.

Question 3

The problem has the solution $u(x) = x$. Hence, $u \in V_N$ for any N, and $\min_v \|v - u\|_{H^1([0,1])} = 0$, hence $\|u - u_h\|_{H^1([0,1])} = 0$ from the previous result.

Question 4

The solution is

$$u(x) = \begin{cases} \frac{x-x^2}{2} & x < 1/2, \\ \frac{1}{8} & \text{otherwise.} \end{cases}$$

This solution is not in H^2, and so the usual interpolation estimate is not expected.

Question 5a

We assume that $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$. By assumption, the boundary condition is satisfied. Then, u is smooth enough for integration by parts, and we get

$$\int_\Omega (f - \nabla^2 u) v \, \mathrm{d}x = 0, \quad \forall v \in \mathring{H}^1(\Omega).$$

Since $C_0^\infty(\Omega)$ functions are dense in $L^2(\Omega)$ we can approximate $f - \nabla^2 u$ arbitrarily closely by v in $\mathring{H}^1(\Omega)$. Taking the limit gives

$$\int_\Omega (f - \nabla^2 u)^2 \, \mathrm{d}x = 0,$$

as required.

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Question 5b

The finite element approximation is to find $u_h \in \mathring{V}_h$ (which is the subspace of V_h of functions that vanish on the boundary) such that

$$\int_\Omega \nabla u_h \cdot \nabla v \, \mathrm{d}x = \int_\Omega vf \, \mathrm{d}x, \quad \forall v \in \mathring{V}_h.$$

Since $\mathring{V}_h \subset \mathring{H}^1(\Omega)$, Céa's lemma gives

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{M}{\gamma} \sup_{v \in V_h} \|v - u\|_{H^1(\Omega)}.$$

In particular,

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{M}{\gamma} \|I_h u - u\|_{H^1(\Omega)},$$

where I_h is the nodal interpolant into V_h.

Question 5b [ctd]

Then, if $u \in H^2(\Omega) \cap \mathring{H}^2(\Omega)$, we have

$$\|I_h u - u\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)},$$

hence

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{MCh}{\gamma}|u|_{H^2(\Omega)},$$

which is linear in h.