Imperial College London

Finite Elements

Colin Cotter

February 22, 2019

In the previous sections, we introduced the concept of finite element spaces, which contain certain functions defined on a domain Ω. Finite element spaces are examples of vector spaces (hence the use of the word "space").

Definition 1

Vector spaces A vector space over the real numbers $\mathbb{R}$ is a set V, with an addition operator $+ : V \times V \to V$, plus a scalar multiplication operator $* : \mathbb{R} \times V \to V$, such that:

1. There exists a unique zero element $e \in V$ such that:
   1.1 $ke = e$ for all $k \in \mathbb{R}$,
   1.2 $0v = e$ for all $v \in V$,
   1.3 $e + v = v$ for all $v \in V$.
2. V is closed under addition and multiplication, i.e., $au + v \in V$ for all $u, v \in V$, $a \in \mathbb{R}$.

Proposition 2

Let V be a finite element space. Then V is a vector space.

Proof.

First, we note that the zero function $u(x) := 0$ is in V, and satisfies the above properties. Further, let $u, v \in V$, and $a \in \mathbb{R}$. Then, when restricted to each triangle K_i, $u + av \in P_i$. Also, for each shared mesh entity, the shared nodal variables agree, i.e. $N_{i,j}[u + av] = N_{i,j}[u + av]$, by linearity of nodal variables. Therefore, $u + av \in V$. $\square$

We now develop/recall some definitions.

Definition 3 (Bilinear form)

A bilinear form $b(\cdot,\cdot)$ on a vector space V is a mapping $b : V \times V \to \mathbb{R}$, such that

1. $v \to b(v,w)$ is a linear map in v for all w.
2. $v \to b(w,v)$ is a linear map in v for all w.

It is symmetric if in addition, $b(v,w) = b(w,v)$, for all $v,w \in V$.

Example 4

Let V_h be a finite element space. The following are bilinear forms on V_h,

$$b(u, v) = \int_\Omega uv \, \mathrm{d}x, \tag{1}$$

$$b(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x. \tag{2}$$

Definition 5 (Inner product)

A real inner product, denoted by $(\cdot,\cdot)$, is a symmetric bilinear form on a vector space V with

1. $(v,v) \geq 0\ \forall v \in V$,
2. $(v,v) = 0 \iff v = 0$.

Definition 6 (Inner product space)

We call a vector space $(V,(\cdot,\cdot))$ equipped with an inner product an inner product space.

Example 7 (L^2 inner product)

Let f, g be two functions in $L^2(\Omega)$. The L^2 inner product between f and g is defined as

$$(f, g)_{L^2} = \int_\Omega fg \, \mathrm{d}x. \tag{3}$$

Remark 8

The L^2 inner product satisfies condition 2 provided that we understand functions in L^2 as being equivalence classes of functions under the relation $f \equiv g \iff \int_\Omega (f-g)^2 \, \mathrm{d}x = 0$.

Example 9 (H^1 inner product)

Let f, g be two C^0 finite element functions. The H^1 inner product between f and g is defined as

$$(f,g)_{H^1} = \int_\Omega fg + \nabla f \cdot \nabla g \, \mathrm{d}x. \tag{4}$$

Remark 10

The H^1 inner product satisfies condition 2 since

$$(f,f)_{L^2} \leq (f,f)_{H^1}. \tag{5}$$

We have already used the Schwarz inequality for L^2 inner products in our approximation results.

Theorem 11 (Schwarz inequality)

If $(V, (\cdot,\cdot))$ is an inner product space, then

$$|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}. \tag{6}$$

Equality holds if and only if $u = \alpha v$ for some $\alpha \in \mathbb{R}$.

Proof.

See a course on vector spaces. $\square$

We have already mentioned some norms.

Definition 12 (Norm)

Given a vector space V, a norm $\|\cdot\|$ is a function from V to $\mathbb{R}$, with

1. $\|v\| \geq 0, \forall v \in V$,
2. $\|v\| = 0 \iff v = 0$,
3. $\|cv\| = |c|\|v\| \forall c \in \mathbb{R}, v \in V$,
4. $\|v + w\| \leq \|v\| + \|w\|$.

Proposition 13

Let $(V, (\cdot,\cdot))$ be an inner product space. Then $\|v\| = \sqrt{(v,v)}$ defines a norm on V.

Proof

From bilinearity we have

$$\|\alpha v\| = \sqrt{(\alpha v, \alpha v)} = \sqrt{\alpha^2 (v,v)} = |\alpha|\|v\|,$$

from bilinearity, hence property 3.
$\|v\| = (v,v)^{1/2} \geq 0$, hence property 1.
If $0 = \|v\| = (v,v)^{1/2} \implies (v,v) = 0 \implies v = 0$, hence property 2.

Proof (Cont.)

We finally check the triangle inequality (property 4).

$$
\begin{aligned}
\|u+v\|^2 &= (u+v, u+v) && (7)\\
&= (u,u) + 2(u,v) + (v,v) && (8)\\
&= \|u\|^2 + 2(u,v) + \|v\|^2 && (9)\\
&\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{[Schwarz]}, && (10)\\
&= (\|u\| + \|v\|)^2, && (11)
\end{aligned}
$$

hence $\|u+v\| \leq \|u\| + \|v\|$. □

Definition 14 (Normed space)

A vector space V with a norm $\|\cdot\|$ is called a normed vector space, written $(V, \|\cdot\|)$.

Definition 15 (Cauchy sequence)

A Cauchy sequence on a normed vector space $(V, \|\cdot\|)$ is a sequence $\{v_i\}_{i=1}^{\infty}$ satisfying $\|v_j - v_k\| \to 0$ as $j, k \to \infty$.

Definition 16 (Complete normed vector space)

A normed vector space $(V, \|\cdot\|)$ is complete if all Cauchy sequences have a limit $v \in V$ such that $\|v - v_j\| \to 0$ as $j \to \infty$.

Definition 17 (Hilbert space)

An inner product space $(V, (\cdot,\cdot))$ is a Hilbert space if the corresponding normed space $(V, \|\cdot\|)$ is complete.

Remark 18

All finite dimensional normed vector spaces are complete. Hence, C^0 finite element spaces equipped with L^2 or H^1 inner products are Hilbert spaces. Later we shall understand our finite element spaces as subspaces of infinite dimensional Hilbert spaces.

We will now quote some important results from the theory of Hilbert spaces.

Definition 19 (Continuous linear functional)

Let H be a Hilbert space.

1. A functional L is a map from H to $\mathbb{R}$.
2. A functional $L : H \to \mathbb{R}$ is linear if $u, v \in H$, $\alpha \in \mathbb{R}$ $\implies L(u + \alpha v) = L(u) + \alpha L(v)$.
3. A functional $L : H \to \mathbb{R}$ is continuous if there exists $C > 0$ such that

$$|L(u) - L(v)| \leq C\|u - v\|_H \quad \forall u, v \in H.$$

Definition 20 (Bounded functional)

A functional $L : H \to \mathbb{R}$ is bounded if there exists $C > 0$ such that

$$|L(u)| \leq C\|u\|_H, \quad \forall u \in H.$$

Lemma 21

Let $L : H \to \mathbb{R}$ be a linear functional. Then L is continuous if and only if it is bounded.

Proof

L bounded $\implies L(u) \leq C\|u\|_H \implies |L(u) - L(v)| =$
$|L(u - v)| \leq C\|u - v\|_H \quad \forall u, v \in H$, i.e. L is continuous.

Proof (Cont.)

L continuous $\implies |L(u-v)| \leq C\|u-v\|_H \quad \forall u, v \in H$.
Pick $v=0$, then $|L(u)| = |L(u-0)| \leq C|u-0|_H = C|u|_H$, i.e. L is bounded. $\square$

Definition 22 (Dual space)

Let H be a Hilbert space. The dual space H' is the space of continuous (or bounded) linear functionals $L : H \to \mathbb{R}$.

Definition 23 (Dual norm)

Let L be a continuous linear functional on H, then

$$\|L\|_{H'} = \sup_{0\neq v\in H} \frac{L(v)}{\|v\|_H}.$$

Lemma 24

Let $u \in H$. Then the functional $L_u : H \to \mathbb{R}$ defined by

$$L_u(v) = (u, v), \quad \forall v \in H,$$

is linear and continuous.

Proof.

For $v, w \in H$, $\alpha \in \mathbb{R}$ we have

$$L_u(v + \alpha w) = (u, v + \alpha w) = (u, v) + \alpha(u, v) = L_u(v) + \alpha L_u(w).$$

Hence L_u is continuous. L_u is bounded by Schwarz inequality,

$$|L_u(v)| = |(u, v)| \leq C\|v\|_H \text{ with } C = \|u\|_H.$$

$\square$

The following theorem states that the converse is also true.

Theorem 25 (Riesz representation theorem)

For any continuous linear functional L on H there exists $u \in H$ such that

$$L(v) = (u, v) \quad \forall v \in H.$$

Further,

$$\|u\|_H = \|L\|_{H'}.$$

Proof.

See a course or textbook on Hilbert spaces. □

We will consider finite element methods that can be formulated in the following way.

Definition 26 (Linear variational problem)

Let $b(u,v)$ be a bilinear form on a Hilbert space V, and F be a linear form on V. This defines a linear variational problem: find $u \in V$ such that

$$b(u,v) = F(v), \quad \forall v \in V.$$

Example 27 (Pk discretisation of (modified) Helmholtz problem with Neumann bcs)

For some known function f,

$$b(u, v) = \int_\Omega uv + \nabla u \cdot \nabla v \,\mathrm{d}x, \quad F(v) = \int_\Omega vf \,\mathrm{d}x,$$

and V is the Pk continuous finite element space on a triangulation of Ω.

Example 28 (Pk discretisation of Poisson equation with partial Dirichlet bcs)

For some known function f,

$$b(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad F(v) = \int_\Omega vf \, \mathrm{d}x,$$

and V is the subspace of the Pk continuous finite element space on a triangulation of Ω such that functions vanishes on $\Gamma_0 \subseteq \partial\Omega$.

Example 29 (Pk discretisation of Poisson equation with pure Neumann bcs)

For some known function f,

$$b(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad F(v) = \int_\Omega vf \, \mathrm{d}x,$$

and V is the subspace of the Pk continuous finite element space on a triangulation of Ω such that functions satisfy

$$\int_\Omega u \, \mathrm{d}x = 0.$$

Definition 30 (Continuous bilinear form)

A bilinear form is continuous on a Hilbert space V if there exists a constant $0 < M < \infty$

$$|b(u, v)| \leq M\|u\|_V\|v\|_V.$$

Definition 31 (Coercive bilinear form)

A bilinear form is coercive on a Hilbert space V if there exists a constant $0 < \gamma < \infty$

$$|b(u,u)| \geq \gamma \|u\|_V \|u\|_V.$$

Theorem 32 (Lax-Milgram theorem)

Let b be a bilinear form, F be a linear form, and $(V, \|\cdot\|)$ be a Hilbert space. If b is continuous and coercive, and F is continuous, then a unique solution $u \in V$ to the linear variational problem exists, with

$$\|u\|_V \leq \frac{1}{\gamma}\|F\|_{V'}.$$

Proof.

See a course or textbook on Hilbert spaces. □

We are going to use this result to show solveability for FE discretisations.

Definition 33 (Stability)

Consider a sequence of triangulations $\mathcal{T}_h$ with corresponding finite element spaces V_h labelled by a maximum triangle diameter h, applied to a variational problem with bilinear form $b(u, v)$ and linear form L. For each V_h we have a corresponding coercivity constant γ_h.

$$\text{If } \gamma_h \to \gamma > 0, \quad \|F\|_{V_h'} \to c < \infty,$$

then we say that the finite element discretisation is stable.

With this in mind it is useful to consider h-independent definitions of $\|\cdot\|_V$ (such as L^2 norm and H^1 norm).

Proposition 34 (Solving the (modified) Helmholtz problem)

Let b, L be the forms from the Helmholtz problem, with $\|f\|_{L^2} < \infty$. Let V_h be a Pk continuous finite element space defined on a triangulation $\mathcal{T}$. Then the finite element approximation u_h exists and the discretisation is stable in the H^1 norm.

Proof

First we show continuity of F. We have

$$F(v) = \int_\Omega fv \,\mathrm{d}x \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H^1},$$

since $\|v\|_{L^2} \leq \|v\|_{H^1}$.

Proof (Cont.)

Continuity of b:

$$|b(u, v)| = |(u, v)_{H^1}| \leq \|u\|_{H^1}\|v\|_{H^1}.$$

Coercivity of b:

$$b(u, u) = \|u\|_{H^1}^2 \geq \|u\|_{H^1}^2.$$

The continuity and coercivity constants are both 1, independent of h, so the discretisation is stable. □

For the Helmholtz problem, we have

$$b(u, v) = \int_\Omega uv + \nabla u \cdot \nabla v \, \mathrm{d}x = (u, v)_{H^1}.$$

For the Poisson problem, we have

$$b(u, v) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x = |u|^2_{H^1} \neq \|u\|^2_{H^1}.$$

Some additional results are required.

Lemma 35 (Mean estimate for finite element spaces)

Let u be a member of a C^0 finite element space, and define

$$\bar{u} = \frac{\int_\Omega u \, \mathrm{d}x}{\int_\Omega \mathrm{d}x}.$$

Then there exists a positive constant C, independent of the triangulation but dependent on (convex) Ω, such that

$$\|u - \bar{u}\|_{L^2} \leq C|u|_{H^1}.$$

Proof

Let x and y be two points in Ω. We note that $f(s) = u(y + s(x - y))$ is a C^0, piecewise polynomial function of s, with $f'(s) = (x - y) \cdot \nabla u(y + s(x - y))$, where ∇u is the finite element derivative of u, and thus

$$u(x) = u(y) + \int_0^1 f'(s)\,\mathrm{d}s = u(y) + \int_0^1 (x - y) \cdot \nabla u(y + s(x - y))\,\mathrm{d}s,$$

Then

$$\begin{aligned} u(x) - \bar{u} &= \frac{1}{|\Omega|} \int_\Omega u(x) - u(y)\,\mathrm{d}y \\ &= \frac{1}{|\Omega|} \int_\Omega (x - y) \cdot \int_{s=0}^1 \nabla u(y + s(x - y))\,\mathrm{d}s\,\mathrm{d}y, \end{aligned}$$

Proof (Cont.)

Therefore

$$
\begin{aligned}
&\|u-\bar{u}\|^2_{L^2(\Omega)} \\
=&\frac{1}{|\Omega|^2}\int_\Omega\left(\int_\Omega (x-y)\cdot\int_{s=0}^1 \nabla u(y+s(x-y))\,\mathrm{d}s\,\mathrm{d}y\right)^2\mathrm{d}x, \\
\leq&\frac{1}{|\Omega|^2}\int_\Omega\int_\Omega |x-y|^2\,\mathrm{d}y\int_\Omega\int_{s=0}^1 |\nabla u(y+s(x-y))|^2\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x, \\
\leq& C\int_\Omega\int_\Omega\int_{s=0}^1 |\nabla u(y+s(x-y))|^2\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Proof (Cont.)

Split into two parts (to avoid singularities):

$$\|u - \bar{u}\|^2_{L^2(\Omega)} \leq C(I + II),$$

where

$$I = \int_\Omega \int_{s=0}^{1/2} \int_\Omega |\nabla u(y + s(x-y))|^2 \, \mathrm{d}y \, \mathrm{d}s \, \mathrm{d}x,$$
$$II = \int_\Omega \int_{s=1/2}^{2} \int_\Omega |\nabla u(y + s(x-y))|^2 \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}y,$$

which we will now estimate separately.

Proof (Cont.)

To evaluate I, change variables $y \to y' = y + s(x-y)$, defining $\Omega_s' \subset \Omega$ as the image of Ω under this transformation. Then,

$$\begin{aligned}
I &= \int_\Omega \int_{s=0}^{1/2} \frac{1}{(1-s)^2} \int_{\Omega_s'} |\nabla u(y')|^2 \,\mathrm{d}y' \,\mathrm{d}s \,\mathrm{d}x, \\
&\leq \int_\Omega \int_{s=0}^{1/2} \frac{1}{(1-s)^2} \int_{\Omega} |\nabla u(y')|^2 \,\mathrm{d}y' \,\mathrm{d}s \,\mathrm{d}x, \\
&= \frac{|\Omega|}{2} |\nabla u|^2_{H^1(\Omega)}.
\end{aligned}$$

Proof (Cont.)

To evaluate II, change variables $x \to x' = y + s(x-y)$, defining $\Omega'_s \subset \Omega$ as the image of Ω under this transformation. Then,

$$\begin{aligned} II &= \int_\Omega \int_{s=1/2}^{2} \frac{1}{s^2} \int_{\Omega'_s} |\nabla u(x')|^2 \,\mathrm{d}x' \,\mathrm{d}s \,\mathrm{d}y, \\ &\leq \int_\Omega \int_{s=0}^{1/2} \frac{1}{s^2} \int_{\Omega} |\nabla u(x')|^2 \,\mathrm{d}x' \,\mathrm{d}s \,\mathrm{d}y, \\ &= |\Omega| |\nabla u|^2_{H^1(\Omega)}. \end{aligned}$$

$$\text{Combining, } \|u - \bar{u}\|^2_{L^2(\Omega)} \leq C(I + II) = \frac{3C|\Omega|}{2}|u|^2_{H^1(\Omega)},$$

which has the required form. □

Proposition 36 (Solving the Poisson problem with pure Neumann conditions)

Let b, L, V, be the forms for the pure Neumann Poisson problem, with $\|f\|_{L^2} < \infty$. Let V_h be a Pk continuous finite element space defined on a triangulation $\mathcal{T}$, and define

$$\bar{V}_h = \{u \in V_h : \bar{u} = 0\}.$$

Then for $\bar{V}_h$, the finite element approximation u_h exists and the discretisation is stable in the H^1 norm.

Proof.

Using the mean estimate, for $u \in \bar{V}_h$, we have

$$\|u\|_{L^2}^2 = \|u - \underbrace{\bar{u}}_{=0}\|_{L^2}^2 \leq C^2|u|_{H^1}^2.$$

Hence we obtain the coercivity result,

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + |u|_{H^1}^2 \leq (1 + C^2)|u|_{H^1}^2 = (1 + C^2)b(u,u).$$

Continuity follows from Schwarz inequality,

$$|b(u,v)| \leq |u|_{H^1}|v|_{H^1} \leq \|u\|_{H^1}\|v\|_{H^1}.$$

$\square$

Proving the coercivity for the case with Dirichlet or partial Dirichlet boundary conditions requires some additional results.

Lemma 37 (Finite element divergence theorem)

Let ϕ be a C^1 vector-valued function. and $u \in V$ be a member of a C^0 finite element space. Then

$$\int_\Omega \nabla \cdot (\phi u)\,\mathrm{d}x = \int_{\partial\Omega} \phi \cdot n u\,\mathrm{d}S,$$

where n is the outward pointing normal to $\partial\Omega$.

Proof.

$$\begin{aligned}\int_\Omega \nabla\cdot(\phi u)\,\mathrm{d}x &= \sum_{K\in\mathcal{T}}\int_K \nabla\cdot(\phi u)\,\mathrm{d}x,\\ &= \sum_{K\in\mathcal{T}}\int_{\partial K}\phi\cdot n_K u\,\mathrm{d}S,\\ &= \int_{\partial\Omega}\phi\cdot n u\,\mathrm{d}S + \underbrace{\int_\Gamma \phi\cdot(n^+ + n^-)u\,\mathrm{d}S}_{=0}.\end{aligned}$$

$\square$

Proposition 38 (Trace theorem for continuous finite elements)

Let V_h be a continuous finite element space, defined on a triangulation $\mathcal{T}$, on a polygonal domain Ω. Then

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}, \tag{12}$$

where C is a constant that depends only on the geometry of Ω.

Proof

The first step is to construct a C^1 function ξ satisfying $\xi \cdot n = 1$ on Ω. We do this by finding a triangulation $\mathcal{T}_0$ (unrelated to $\mathcal{T}$), and defining an C^1 Argyris finite element space V_0 on it. We then choose ξ so that both Cartesian components are in V_0, satisfying the boundary condition.

Proof (Cont.)

Then,

$$
\begin{aligned}
\|u\|^2_{L^2(\partial\Omega)} = \int_{\partial\Omega} u^2 \,\mathrm{d}S &= \int_{\partial\Omega} \xi \cdot n u^2 \,\mathrm{d}S, \\
&= \int_{\Omega} \nabla \cdot (\xi u^2) \,\mathrm{d}x, \\
&= \int_{\Omega} u^2 \nabla \cdot \xi + 2u\xi \cdot \nabla u \,\mathrm{d}x, \\
&\leq \|u\|^2_{L^2} \|\nabla \cdot \xi\|_\infty + 2|\xi|_\infty \|u\|_{L^2} |u|_{H^1},
\end{aligned}
$$

Proof (Cont.)

So,

$$\begin{aligned}\|u\|^2_{L^2(\partial\Omega)} &\leq \|u\|^2_{L^2}\|\nabla\cdot\xi\|_\infty + |\xi|_\infty\left(\|u\|^2_{L^2} + |u|^2_{H^1}\right), \\ &\leq C\|u\|^2_{H^1},\end{aligned}$$

where we have used the geometric-arithmetic mean inequality $2ab \leq a^2 + b^2$. $\square$

Proposition 39 (Solving the Poisson problem with partial Dirichlet conditions)

Let b, L, V, be the forms for the (partial) Dirichlet Poisson problem, with $\|f\|_{L^2} < \infty$. Let V_h be a Pk continuous finite element space defined on a triangulation $\mathcal{T}$, and define

$$\mathring{V}_h = \{u \in V_h : u|_{\Gamma_0}\}.$$

Then for $\mathring{V}_h$, the finite element approximation u_h exists and the discretisation is stable in the H^1 norm.

Proof

Proof taken from Brenner and Scott. We have

$$\begin{aligned}
\|v\|_{L^2(\Omega)} &\leq \|v-\bar{v}\|_{L^2(\Omega)} + \|\bar{v}\|_{L^2(\Omega)}, \\
&\leq C|v|_{H^1(\Omega)} + \frac{|\Omega|^{1/2}}{|\Gamma_0|}\left|\int_{\Gamma_0} \bar{v}\,\mathrm{d}S\right|, \\
&\leq C|v|_{H^1(\Omega)} + \frac{|\Omega|^{1/2}}{|\Gamma_0|}\left(\left|\int_{\Gamma_0} v\,\mathrm{d}S + \int_{\Gamma_0} \bar{v} - v\,\mathrm{d}S\right|\right).
\end{aligned}$$

Imperial College London

Proof (Cont.)

We have

$$\left|\int_{\Gamma_0}(v-\bar{v})\,\mathrm{d}s\right| \leq |\Gamma_0|^{1/2}\|v-\bar{v}\|_{L^2(\partial\Omega)},$$
$$\leq |\Gamma_0|^{1/2}C|v|_{H^1(\Omega)}.$$

Combining, we get

$$\|v\|_{L^2(\Omega)} \leq C_1|v|_{H^1(\Omega)},$$

and hence coercivity,

$$\|v\|^2_{H^1(\Omega)} \leq (1+C_1^2)b(v,v).$$

□

We have developed some techniques for showing that variational problems arising from finite element discretisations for Helmholtz and Poisson problems have unique solutions, that are stable in the H^1-norm. This means that we can be confident that we can solve the problems on a computer and the solution won't become singular as the mesh is refined.

Now we would like to go further and ask what is happening to the numerical solutions as the mesh is refined. What are they converging to?