Finite Elements: class test 2

1. Let $\mathcal{T}_h$ be a triangulation on the 1×1 unit square domain Ω, and let V be a C^0 Lagrange finite element space of degree k defined on $\mathcal{T}_h$. A finite element discretisation for the Poisson equation with Neumann boundary conditions is given by:
   find $u \in V$ such that
   $$\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}\, x = \int_\Omega v f \,\mathrm{d}\, x, \quad \forall v \in V,$$
   for some known function f. Explain why the bilinear form for this problem is not coercive in V.
   Hint: find a function in V that breaks the coercivity definition.

2. Let V be a C^0 finite element space on $[0,1]$, defined on a one-dimensional mesh with vertices $0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 1$. Show that $u \in V$ satisfies the fundamental theorem of calculus, *i.e.*
   $$\int_0^1 u' \,\mathrm{d}\, x = u[1] - u[0],$$
   where u' is the usual finite element derivative defined in $L^2([0,1])$ by taking the usual derivative when restricting u to any subinterval $[x_k, x_{k+1}]$.

3. Let
   $$a(u,v) = \int_0^1 (u'v' + u'v + uv) \,\mathrm{d}\, x.$$
   Let V be a C^0 finite element space on $[0,1]$ and let $\mathring{V}$ be the subspace of functions that vanish at $x = 0$ and $x = 1$. Using the finite element version of the fundamental theorem of calculus above, prove that
   $$a(v,v) = \int_0^1 \left((v')^2 + v^2\right) \mathrm{d}\, x := \|v\|_{H^1}^2, \quad \forall v \in \mathring{V}.$$
   Hence conclude that the bilinear form is coercive on $\mathring{V}$.

4. Consider the variational problem with bilinear form
   $$a(u,v) = \int_0^1 (u'v' + u'v + uv) \,\mathrm{d}\, x,$$
   corresponding to the differential equation
   $$-u'' + u' + u = f.$$
   Prove that $a(\cdot,\cdot)$ is continuous and coercive on a C^0 finite element space V defined on $[0,1]$, with respect to the H^1 inner product.
   Hints: for continuity, just use the triangle inequality and the relationship between L^2 and H^1 norms. For coercivity, try completing the square for the integrand in a.

5. (a) For $f \in L^2(\Omega)$, $\sigma \in C^1(\Omega)$, find a finite element formulation of the problem
   $$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sigma(x)\frac{\partial u}{\partial x_i}\right) = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$
   (b) If there exist $0 < a < b$ such that $a < \sigma(x) < b$ for all $x \in \Omega$, show continuity and coercive for your formulation with respect to the H^1 norm.

6. Find a C^0 finite element formulation for the Poisson equation
   $$-\nabla^2 u = f, \quad u = g \text{ on } \partial\Omega,$$
   for a function g which is C^2 and whose restriction to $\partial\Omega$ is in $L^2(\partial\Omega)$. Derive conditions under the discretisation has a unique solution.