Black-Scholes Equation

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- Let V denote the value of an option that depends on the value of the underlying asset S and time t, i.e., $V = V(S, t)$.
- It is not necessary at this stage to specify whether V is a call or a put; indeed, V can even be the value of a whole portfolio of various options.
- For simplicity, readers may think of a simple call or put.
- Assume that in a time step dt, the underlying asset pays out a dividend SD_0dt, where D_0 is a constant known as the dividend yield.

Suppose S satisfies the equation

$$\frac{dS}{S} = \mu(S,t)dt + \sigma(S,t)dX.$$

According to Itô's lemma, the random walk followed by V is given by

$$dV = \frac{\partial V}{\partial S}dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt. \tag{1.1}$$

Here we require V to have at least one t derivative and two S derivatives.

- Now construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified.
- The value of this portfolio is

$$\Pi = V - \Delta S. \tag{1.2}$$

- Because the portfolio contains one option and a number $-\Delta$ of the underlying asset, and the owner of the portfolio receives SD_0dt for every asset held, the earnings for the owner of the portfolio during the time step dt is

$$d\Pi = dV - \Delta(dS + SD_0dt).$$

- Using the relation (1.1), we find that Π follows the random walk

$$d\Pi = \left(\frac{\partial V}{\partial S} - \Delta\right) dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right) dt.$$

- The random component in this random walk can be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}. \tag{1.3}$$

- This result in a portfolio whose increment is wholly deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right) dt. \tag{1.4}$$

- Because the return for any risk-free portfolio should be r, we have

$$r\Pi dt = d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right) dt. \tag{1.5}$$

- Substituting the relations (1.2) and (1.3) into Equation (1.5) and dividing by dt, we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0. \tag{1.6}$$

- When we take different Π for different S and t, we can conclude that Equation (1.6) holds on a domain.
- Equation (1.6) is called the Black-Scholes partial differential equation, or the Black-Scholes equation, even though $D_0 = 0$ in the equation originally given by Black and Scholes. With its extensions and variants, it plays the major role.

About the derivation of this equation and the equation itself, we give more explanation here.

- The key idea of deriving this equation is to eliminate the uncertainty or the risk.
- $d\Pi$ is not a differential in the usual sense. It is the earning of the holder of the portfolio during the time step dt. Therefore, $\Delta S D_0 dt$ appear.
- In the derivation, in order to eliminate any small risk, Δ is chosen before an uncertainty appears and does not depend on the coming risk. Therefore, no differential of Δ is needed.

- The linear differential operator given by

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r$$

has a financial interpretation as a measure of the difference between the return on a hedged option portfolio

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + -D_0 S\frac{\partial}{\partial S}$$

and the return on a bank deposit

$$r\left(1 - S\frac{\partial}{\partial S}\right).$$

- Although the difference between the two returns is identically zero for European options, we will later see that the difference between the two returns may be nonzero for American options.

- From the Black-Scholes equation (1.6), we know that the parameter μ does not affect the option price, i.e., the option price determined by this equation is independent of the average return rate of an asset price per unit time.
- From the derivation procedure of the Black-Scholes equation we know that the Black-Scholes equation still holdsif $r$and D_0 are functions of S and t.
- If dividends are paid only on certain dates, then the money the owner of the portfolio will get during the time period $[t, t + dt]$ is

$$dV - \Delta dS - \Delta D(S, t)dt,$$

where $D(S, t)$ is a sum of several Dirac delta functions.

- Suppose that a stock pays dividend $D_1(S)$ at time t_1 and $D_2(S)$ at time t_2 for a share, where $D_1(S) \leq S$ and $D_2(S) \leq S$. Then

$$D(S,t) = D_1(S)\delta(t - t_1) + D_2(S)\delta(t - t_2),$$

where the Dirac delta function $\delta(t)$ is defined as follows:

$$\delta(t) = \begin{cases} 0, & if\ t \neq 0 \\ \infty, & if\ t = 0 \end{cases} \quad \text{and} \int_{-\infty}^{\infty} \delta(t) = 1.$$

- In this case, the modified Black-Scholes equation is in the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + [rS - D(S,t)]\frac{\partial V}{\partial S} - rV = 0. \tag{1.7}$$