
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Example:

Consider two investment opportunities: Project A with cash flow of $(-1.0, 1.5, 2.0)$ in millions and Project B with cash flow of $(-1.2, 1.2, 2.6)$ in millions. Assuming that the interest rate is likely to be at 6% over the next two years, the present value of Project A is $-1.0 + \frac{1.5}{1.06} + \frac{2.0}{(1.06)^2} = 2.1951$ and the present value of Project B is $-1.2 + \frac{1.2}{1.06} + \frac{2.6}{(1.06)^2} = 2.2461$.

In order to compute the present values in various examples, it is convenient to recall the following formula for a geometric sequence,

$$\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^m} = \frac{1}{r}\left[1 - \frac{1}{(1+r)^m}\right]$$

This expression means that the cash flow of equal payments of P after each of m periods has the present value,

$$V(0) = \frac{P}{r}\left[1 - \frac{1}{(1+r)^m}\right],$$

where r is the interest rate per period. The payment of P are sometimes are sometimes called annuities. Equivalently, inverting the above relation, a loan of $V(0)$ in equal installments of P at the end of each period for m periods, the amount P is set equal to,

$$P = \frac{r(1+r)^m V(0)}{(1+r)^m - 1}.$$

We then say that the loan is amortized over m periods.

Example of Amortization on a Loan:

Consider a 30 year loan on $400,000$, at the fixed interest rate of 8%. We want to compute the monthly payments on this loan. Since there are 12 months in a year, the number of periods is $m = 30 \times 12 = 360$. The interest rate per period is $0.08/12 = 0.0067$. The value of the loan is $V(0) = 400,000$. Using the formula we get $P = 2,946$.

Now the balance is computed as follows:

After the first month, the interest of $0.0067 \times 400,000 = 2,680$ is added to the initial balance of $400,000$, resulting in a balance of $402,680$. After the first installment of $2,946$ has been paid, the new balance will be $402,680 - 2,946 = 399,734$. After the second month, before the payment the balance will be $399,734(1 + 0.0067)$, and so on.

Example of Loan Fees:

When a bank advertises its mortgage products there are usually two rates listed. One is the Mortgage Interest Rate and the other is called the APR or the Annual Percentage Rate. The latter rate includes the fees for providing the loan. These fees are added to the loan amount and also paid through monthly installments. As an example, consider a bank that offers a 30 year mortgage loan of $400,000$ at the rate

7.8% compounded monthly, while the APR is 8.00%. As already computed, the monthly payment at this APR is $2,946$. Now we used this monthly payment of $2,946$ and the rate of $7.8/12 = 0.65\%$ in,

$$V(0) = \frac{P}{r}\left(1 - \frac{1}{(1+r)^m}\right),$$

to find that the total balance actually being paid is $409,240.27$. The total fees are therefore,

$$409,240.27 - 400,000 = 9,240.27.$$

Bond Yield:

Closely related to the interest-rate/present-value analysis is the analysis of a bond price and a bond yield, also called yield to maturity or YTM. The yield of a bond is defined as that interest rate value for which the present value of the bond's payments (the coupons and the face value) equals the current bond price. For general investments, this concept is called the internal rate of return or IRR. The IRR of a bond is its yield. Suppose a bond pays a face value equal to V and n coupon payments per year, each of size C/n, at regular intervals (so that the annual coupon payment is C and the size of the intervals between payments, expressed in years, is $1/n$). Suppose also that there are T years left until maturity, for a total of $T \cdot n = m$ periods until maturity, and the current bond price is $P(t)$. Then the current yield $y(t)$ of the bond is determined from the equation,

$$P(t) = \frac{V}{[1+y(t)]^T} + \sum_{i=1}^{m} \frac{C/n}{[1+y(t)]^{i/n}}.$$

This equation gives the annualized yield using the compound interest rate. Very often, the yield is annualized using the simple interest rate. The above expression then becomes,

$$P(t) = \frac{V}{[1+y(t)/n]^m} + \sum_{i=1}^{m} \frac{C/n}{[1+y(t)/n]^i}.$$

We indeed see that the yield $y(t)$ is the interest rate implied by the bond. We also observe that the price of the bond and the yield move in opposite directions – higher yield implies lower price and vice versa – because paying less for a bond today means getting a higher return later.