MA 322: Scientific Computing

Lecture - 13

Numerics for ODEs

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- In an ODE, we have only one independent variable (say, x) and one dependent variable (say, y). Mathematically, a ODE is a relation of the form

$$F(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x), \ldots) = 0, \quad x \in [a,b]. \tag{3}$$

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where $y'(x) = \frac{dy}{dx}$, $y''(x) = \frac{d^2y}{dx^2}, \cdots, y^{(n)}(x) = \frac{d^ny}{dx^n}$.

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- In general, we have following set relation

$$C[a,b] \supset C^1[a,b] \supset C^2[a,b] \supset C^3[a,b] \ldots$$

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  - In other sense, y and y' both are continuous functions in $[a, b]$.
- How do you define a solution for the equation (5)?
  - A continuously differentiable function $y : [a, b] \to \mathbb{R}$ satisfying equation (5) for all $x \in [a, b]$. More precisely, a solution to the first order ODE (5) is a function $y \in C^1[a, b]$ such that

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  - Therefore, we need a tool which further simplify the first order ODE.

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$$G(X, Y) = AX + BY + C.$$

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  - Hence, continuity of P and Q is sufficient for the existence of unique solution to the linear IVP (17).

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- Any solution to a first order non-linear IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0 \tag{22}$$

is given by

$$y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s))ds. \tag{23}$$

- Thus, in general, solution y can not be expressed explicitly in terms of known data.
- There is no general formula for solutions of nonlinear equations.
- The singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution.

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$$|f(x, y)| \leq K \ \ \forall (x, y) \in D.$$

Further, let f satisfy the Lipschitz condition with respect to y in D, i.e., there exists a number M such that

$$|f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1| \quad \forall (x, y_1), (x, y_2) \in D. \tag{25}$$

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- One can think to construct the sequence (27) from fixed point iteration for differential operators. Recall the IVP

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- In this case, we try to construct a sequence of functions which is expected to converse point-wise to the actual solution y.