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MA 322: Scientific Computing Lab

Lab 06

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Using the Euler's Method:

1.2 Euler Method

Here, we introduce the oldest and simplest numerical method originated by Euler about 1768 to find an approximate solution of (1). It is called the tangent line method or the Euler method.

The Fundamental Theorem of Calculus, when coupled with the differential equation itself, suggests one simple scheme for computing the value of the solution of a differential equation numerically. The underlying idea is for this computational method is that the graph of the solution for (1) can be obtained by simply plotting points. To motivate towards this method, recall that the Fundamental Theorem of Calculus states that for any function $y(x)$ and for any numbers $x_0,\ x_1 (x_0 < x_1)$, we have

$$y(x_1) = y_0 + \int_{x_0}^{x_1} y'(s)ds.$$

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We assume that x_1 is not much larger than x_0, the value of the integral can be approximated as

$$y'(x_0)(x_1 - x_0).$$

Making this substitution shows that for x_1 not much larger than x_0, $y(x_1)$ (value of the actual solution at x_1) can be approximated by the formula

$$y_0 + y'(x_0)(x_1 - x_0) = y_1 \ \text{(say)}.$$

Since $y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$, thus

$$y_1 = y_0 + f(x_0, y_0)(x_0)(x_1 - x_0).$$

To proceed further, we can try to repeat the process in $[x_1, x_2]$ so that the approximate value for $y(x_2)$ can be obtained as

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1).$$

Here, we have used the approximate value y_1 for $y(x_1)$ since we do not know the value of the actual solution $y(x)$ at $x = x_1$. In general, we consider the following points

$$x_0,\ x_1 = x_0 + h,\ x_2 = x_1 + h, \ldots x_{n+1} = x_n + h,$$

h is called the step size. Then the approximate value y_{n+1} for $y(x_{n+1})$ is given by

$$y_{n+1} = y_n + f(x_n, y_n)(x_{n+1} - x_n),\ \ n = 0,\ 1,\ 2, \ldots.$$

For the convenience of presentation we introduce the notation $f_n = f(x_n, y_n)$, then we can rewrite equation (5) as

$$y_{n+1} = y_n + f_n h,\ \ n = 0,\ 1,\ 2, \ldots.$$

Question 1.

In this question we will use Euler's method to approximate the solution for each IVP.

In each case we first write obtain $f(t, y)$, and then use the recursion $y_{n+1} = y_n + f_n h$. We have been given h and y_0 in the question.

```
(a)
The approx solution in this case is given as:
y(0.0) = 1.0
y(0.5) = 1.1839397205857212
y(1.0) = 1.436252215343503
(b)
The approx solution in this case is given as:
y(1.0) = 2.0
y(1.5) = 2.3333333333333335
y(2.0) = 2.7083333333333335
(c)
The approx solution in this case is given as:
y(2.0) = 2.0
y(2.25) = 2.2071067811865475
y(2.5) = 2.4909989078432044
y(2.75) = 2.854680348463929
y(3.0) = 3.3025964649736848
(d)
The approx solution in this case is given as:
y(1.0) = 2.0
y(1.25) = 1.2273243567064205
y(1.5) = 0.8321501570804852
y(1.75) = 0.5704467722825309
y(2.0) = 0.3788261467612455
```

Question 2.

This question is fairly straightforward. We merely substitute the values of t (obtained in Question 1), in the solutions given and then compute the error in estimation by comparing exact and approximate values.

```
(a)
Approx Sol: [1.         1.18393972 1.43625222]
Exact Sol: [1.         1.21402306 1.48988013]
Error for y(0.0) = 0.0
Error for y(0.5) = 0.030083340043987716
Error for y(1.0) = 0.05362791030124714
(b)
Approx Sol: [2.         2.33333333 2.70833333]
Exact Sol: [2.         2.35410197 2.74165739]
Error for y(1.0) = 0.0
Error for y(1.5) = 0.020768632916351226
Error for y(2.0) = 0.03332405344060785
(c)
Approx Sol: [2.         2.20710678 2.49099891 2.85468035 3.30259646]
Exact Sol: [2.         2.24412111 2.56445195 2.96519383 3.45128665]
Error for y(2.0) = 0.0
Error for y(2.25) = 0.03701432914995362
Error for y(2.5) = 0.07345304138842979
Error for y(2.75) = 0.11051348602782429
Error for y(3.0) = 0.14869018729061345
(d)
Approx Sol: [2.         1.22732436 0.83215016 0.57044677 0.37882661]
Exact Sol: [2.         1.40319897 1.01641015 0.73800977 0.5296871 ]
Error for y(1.0) = 0.0
Error for y(1.25) = 0.1758746125735129
Error for y(1.5) = 0.18425998959802636
Error for y(1.75) = 0.16756299926745333
Error for y(2.0) = 0.15086048336343416
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Question 3.

(a) We first approximate the solution using Euler's method and h = 0.05. We also compare it with the actual values of $y(t) = -\frac{1}{t}$, and obtain the absolute error in each case.

	Evaluate	Approx	Actual	Error
0	y(1.0)	-1.000000	-1.000000	0.000000
1	y(1.05)	-0.950000	-0.952381	0.002381
2	y(1.1)	-0.904535	-0.909091	0.004555
3	y(1.15)	-0.863007	-0.869565	0.006558
4	y(1.2)	-0.824917	-0.833333	0.008416
5	y(1.25)	-0.789848	-0.800000	0.010152
6	y(1.3)	-0.757447	-0.769231	0.011784
7	y(1.35)	-0.727415	-0.740741	0.013326
8	y(1.4)	-0.699495	-0.714286	0.014791
9	y(1.45)	-0.673467	-0.689655	0.016188
10	y(1.5)	-0.649141	-0.666667	0.017525
11	y(1.55)	-0.626350	-0.645161	0.018811
12	y(1.6)	-0.604949	-0.625000	0.020051
13	y(1.65)	-0.584812	-0.606061	0.021249
14	y(1.7)	-0.565825	-0.588235	0.022410
15	y(1.75)	-0.547890	-0.571429	0.023539
16	y(1.8)	-0.530918	-0.555556	0.024637
17	y(1.85)	-0.514832	-0.540541	0.025708
18	y(1.9)	-0.499561	-0.526316	0.026754
19	y(1.95)	-0.485043	-0.512821	0.027778
20	y(2.0)	-0.471220	-0.500000	0.028780

(b) In this part we make use of Lagrange interpolation method to obtain the approx interpolated values of y at the 3 points, namely y(1.052), y(1.555) and y(1.978). We also then make comparisons with actual values and print the error.

```
(I)
Estimated value of y(1.052) from interpolation
=-0.9480987022541826
Actual value of y = -0.950570342205323
The error between them = 0.00247163995114053
(II)
Estimated value of y(1.555) from interpolation
=-0.6241497188955423
Actual value of y = -0.643086816720257
The error between them = 0.0189370978247150
(III)
Estimated value of y(1.978) from interpolation
=-0.4772194469532174
Actual value of y = -0.505561172901921
The error between them = 0.0283417259487038
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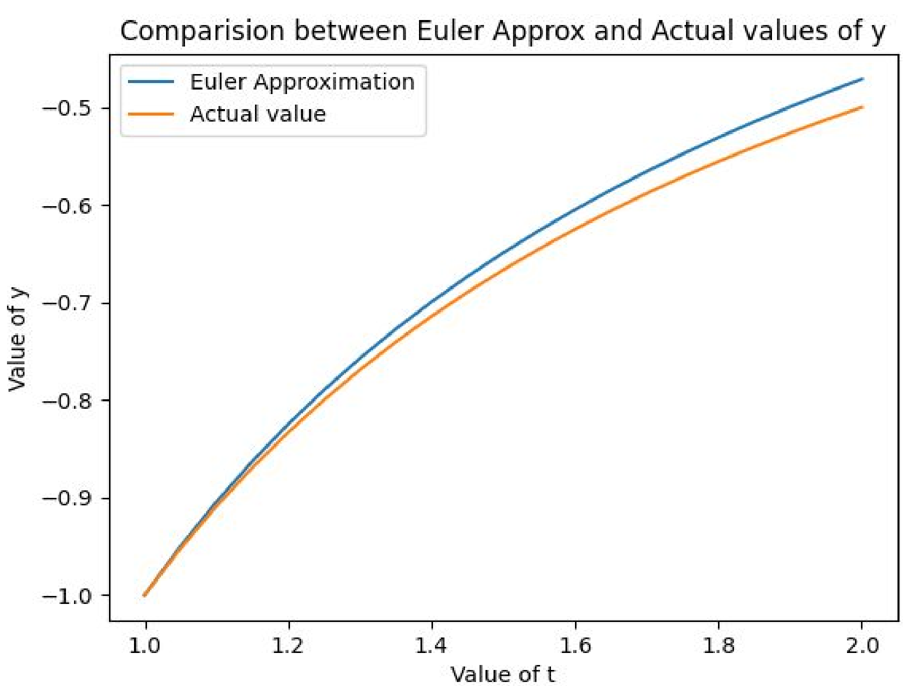

Question 4.

Question 04

(a) Given IVP, $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

and $\frac{\partial f}{\partial y} \leq 0$ $\forall x \in [x_0, x_n]$ and for all y.

By Euler's formula,

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{--- (1)}$$

Using Taylor's expansion,

$$Y(x_{n+1}) = Y(x_n + h)$$
$$= Y(x_n) + hy'(x_n) + \frac{h^2}{2} y''(\xi_k) \quad \text{--- (2)}$$

for some $\xi_k \in (x_{n-1}, x_n)$

$$\therefore e_{n+1} = y(x_{n+1}) - y_{n+1} \quad \{\text{using 1 and 2}\}$$
$$= y(x_n) + hy'(x_n) + \frac{h^2}{2} y''(\xi_k) - y_n - hf(x_n, y_n)$$
$$= (y(x_n) - y_n) + h(y'(x_n) - f(x_n, y_n)) + \frac{h^2}{2} y''(\xi_k) \quad \text{--- (3)}$$

Using mean-value theorem:-

$$f(x_n, y(x_n)) = f(x_n, y_n) + (y(x_n) - y_n) \frac{\partial f}{\partial y}(x_n, \eta_k)$$

where $\eta_k \in (y(x_n), y_n)$.

Using this in (3),

$$e_{n+1} = e_n + e_n \cdot \frac{\partial f}{\partial y}(x_n, \eta_k) + \frac{h^2}{2} y''(\xi_k)$$

Since $\frac{\partial f}{\partial y} \leq 0$, applying triangle inequality,

$$|e_{n+1}| \leq |e_n| + \frac{h^2}{2} y''(\xi_k)$$

or, $\boxed{|e_n| \leq |e_{n-1}| + \frac{h^2}{2} y''(\xi_k)}$ Thus proved.

(b) From the conclusion in (a),

$$|e_n| \leq |e_{n-1}| + \frac{h^2}{2} f''(\xi_{n-1}), \qquad \xi_{n-1} \in (x_{n-1}, x_n)$$

$$|e_{n-1}| \leq |e_{n-2}| + \frac{h^2}{2} f''(\xi_{n-2}) \qquad \xi_{n-2} \in (x_{n-2}, x_{n-1})$$

$$\vdots$$

$$|e_1| \leq |e_0| + \frac{h^2}{2} f''(\xi_0) \qquad \xi_0 \in (x_0, x_1)$$

Adding these up,

$$\boxed{|e_n| \leq |e_0| + nh^2 Y.} \quad \text{where } Y = \frac{1}{2} \max |y''(x)| \text{ where } x_0 \leq x \leq x_n$$

Question 5.

Using Euler's method with h = 0.5 and $\lambda = -20$, we compute y(3). On computing the actual value of y(3), using $y(x) = \sin(x)$, we see that there is a huge error.

We then make use of error bound as obtained in Q4(ii), i.e. $nh^2 Y \, where \, Y = \frac{1}{2} \max_{x_0 \leq x \leq x_n} |y''(x)|$ and compute it's value.

```
Approximate value of y(3) = -785.2886498351327
The actual value of y(3) = 0.1411200080598672
The error in this case = 785.4297698431925
The error bound in this case = 0.748121239953041
Clearly, absolute error with h = 0.5 greatly exceeds the
error bound computed using (I) in Q4

If we reduce h by 10 times, i.e make it 0.05 we observe
that:
Approximate value of y(3) = 0.14133753721110437
The actual value of y(3) = 0.1411200080598672
The error in this case = 0.00021752915123715577
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Clearly the actual error greatly exceeds the error bound.

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According to the theorem there exists an h, for which the
above bound will hold. In the case above we have taken
h=0.5, which makes the error go beyond the bounds, but that
does not mean that it's the only h. We can always find an h
which will ensure that error is within bounds. In our case
we make use of h=0.05 and the error is 0.0002175 which is
well within the max-error bounds.
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