MA 322: Scientific Computing

Lecture - 2

The Newton's Method

The Newton's Method

Idea:

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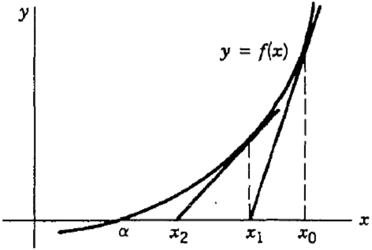

Figure: Newton's method.

- Assume that an initial estimate x_0 is known for the desired root α of equation $f(x) = 0$.

The Newton's Method

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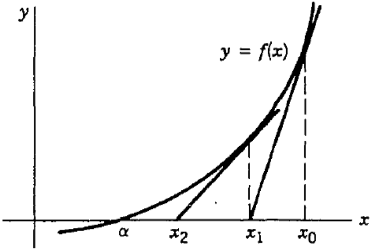

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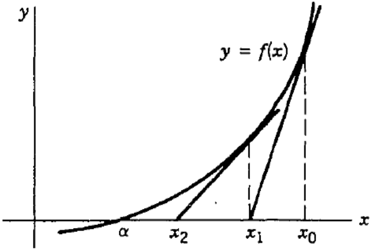

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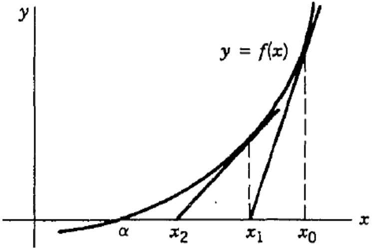

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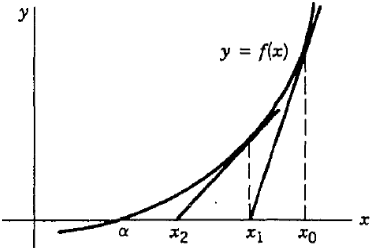

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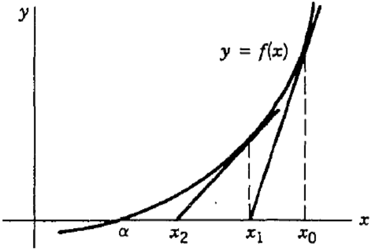

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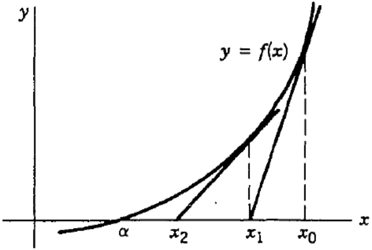

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Since this tangent line expected to cut x-axis at $(x_1, 0)$, we obtain

$$-f(x_0) = f'(x_0)(x_1 - x_0) \ \text{ Or } \ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{2}$$

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Example: Find an approximation for the largest root $\alpha \approx 1.1347241$ of the equation

$$x^6 - x - 1 = 0.$$

Newton's method

n	x_n
0	2.0
1	1.680628273
2	1.430738989
3	1.254970957
4	1.161538433
5	1.136353274
6	1.134730528
7	1.134724139

Bisection method

n	c_n	n	c_n
	$2=b$, $1=a$	8	1.13672
1	1.5	9	1.13477
2	1.25	10	1.13379
3	1.125	11	1.13428
4	1.1875	12	1.13452
5	1.15625	13	1.13464
6	1.14063	14	1.13470
7	1.13281	15	1.13474

Newton's Method Contd.....

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Example: Find an approximation for the root $\alpha \approx 1.414213562373095$ of the equation

$$x^2 - 2 = 0.$$

A Matlab Code:

```
n=5;   x(1)=2;
for i=1:n
x(i+1)=x(i)-(x(i)^2-2)/(2*x(i));
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$$x_1 = 2, \quad x_2 = 1.5000000000$$

$$x_3 = 1.416666666666667,$$

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- Recall following Taylor series expansion

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + \frac{(x - x_n)^2}{2} f''(\xi_n), \tag{4}$$

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with ξ_n between x & x_n. Letting $x = \alpha$ and using $f(\alpha) = 0$, we have

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(\alpha - x_n)^2}{2} \frac{f''(\xi_n)}{f'(x_n)}, \tag{5}$$

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Convergence of Newton's Method: Assume f, f' and f'' are continuous for all x in some neighborhood of α, and assume that $f(\alpha) = 0,\ f'(\alpha) \neq 0$. Then if x_0 is chosen sufficiently close to α, the sequence $\langle x_n \rangle$ given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

will converge to α. Moreover,

$$\lim_{n\to\infty} \frac{(\alpha - x_{n+1})}{(\alpha - x_n)^2} = -\frac{f''(\alpha)}{2f'(\alpha)}$$

providing that the iterates have an order of convergence $p = 2$.

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- We can apply the same argument to $x_1,\ x_2\ \ldots,$ inductively, so that $|\alpha - x_n| \leq \epsilon$ and $M|\alpha - x_n| < 1$ for all $n \geq 1$.

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- Again error relation

$$\frac{(\alpha - x_{n+1})}{(\alpha - x_n)^2} = -\frac{f''(\xi_n)}{2f'(x_n)} \qquad (11)$$

yields

$$\lim_{n\to\infty} \frac{(\alpha - x_{n+1})}{(\alpha - x_n)^2} = -\lim_{n\to\infty} \frac{f''(\xi_n)}{2f'(x_n)} = -\frac{f''(\alpha)}{2f'(\alpha)} \quad \square \qquad (12)$$

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- These are very restrictive assumptions and we would like to remove them, if possible.

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Newton's Method: Algorithm

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Algorithm *Newton* $(f, df, x_0, \epsilon, \text{root}, \text{itmax}, \text{ier})$

1. Remark: df is the derivative function $f'(x)$, itmax is the maximum number of iterates to be computed, and ier is an error flag to the user.

2. $\text{itnum} := 1$

3. $\text{denom} := df(x_0)$.

4. If $\text{denom} = 0$, then $\text{ier} := 2$ and exit.

5. $x_1 := x_0 - f(x_0)/\text{denom}$

6. If $|x_1 - x_0| \le \epsilon$, then set $\text{ier} := 0$, $\text{root} := x_1$, and exit.

7. If $\text{itnum} = \text{itmax}$, set $\text{ier} := 1$ and exit.

8. Otherwise, $\text{itnum} := \text{itnum} + 1$, $x_0 := x_1$, and go to step 3.

(Source: An Introduction to Numerical Analysis by Atkinson)

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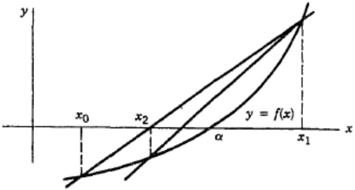

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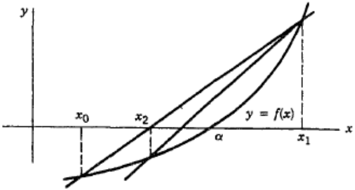

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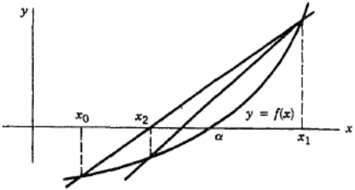

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is used in the Newton formula, we obtain the secant formula. But, above approximation is not quadratic. Hence, we can not expect quadratic convergence for secant method.

Convergence of Secant's Method: Assume f, f' and f'' are continuous for all x in some neighborhood of α, and assume that $f(\alpha) = 0$, $f'(\alpha) \neq 0$. Then if x_0 and x_1 are chosen sufficiently close to α, the sequence $\langle x_n \rangle$ given by

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

will converge to α.

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$$x^6 - x - 1 = 0.$$

Secant method

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Bisection method

n	c_n	n	c_n
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3	1.125	11	1.13428
4	1.1875	12	1.13452
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Remark:

- It is better than Bisection method but not good as Newton's method.

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Thus, secant method converges with order $p = \frac{1+\sqrt{5}}{2} = 1.618\ldots$.