
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Overview of Stochastic Calculus and Ito's Formula

The formula, $S_N(t) = S(0)e^{\mu t+\sigma W_N(t)}$ for the binomial model and $S(t) = S(0)e^{\mu t+\sigma W(t)}$ for the Black-Scholes model can be written as $S_N(t) = f(t, W_N(t))$ and $S(t) = f(t, W(t))$, respectively, where $f(t, x) = S(0)e^{\mu t+\sigma x}$.

For a function f that is sufficiently smooth, we can write (using the Taylor's formula),

$$\Delta f(t, W_N(t)) = f_t(t, W_N(t))h + f_x(t, W_N(t))\Delta W_N(t) + \frac{1}{2}f_{xx}(t, W_N(t))h + O\left(h^{\frac{3}{2}}\right),$$

for $t = nh$, $n = 0, 1, \ldots, N$. Here f_t, f_x and f_{xx} denote $\dfrac{\partial f}{\partial t}$, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial^2 f}{\partial x^2}$, respectively.
Further, $\Delta f(t, W_N(t)) = f(t + h, W_N(t + h)) - f(t, W_N(t))$ and $\Delta W_N(t) = W_N(t + h)) - W_N(t)$.
The term $O\left(h^{\frac{3}{2}}\right)$ means terms of order $h^{\frac{3}{2}}$ or higher. We have also used the fact that $\Delta W_N(t)$ has values $\pm\sqrt{h}$, so that $(\Delta W_N(t))^2 = h$ and $(\Delta W_N(t))^k = O\left(h^{\frac{3}{2}}\right)$ for all $k \geq 3$.
Summing up over the n time steps, we obtain, for $0 \leq nh < T$,

$$\begin{aligned} f(T, W_N(T)) &= f(0, W_N(0)) + \sum_{0\leq nh<T} f_t(nh, W_N(nh))h \\ &+ \sum_{0\leq nh<T} f_x(nh, W_N(nh))\Delta W_N(nh) \\ &+ \frac{1}{2}\sum_{0\leq nh<T} f_{xx}(nh, W_N(nh))h + O\left(h^{\frac{1}{2}}\right). \end{aligned}$$

Note that the sum of $N = \dfrac{T}{h}$, the terms of $O\left(h^{\frac{3}{2}}\right)$ gives $O\left(h^{\frac{1}{2}}\right)$.
The corresponding formula as $N \to \infty$ is given by,

$$f(T, W(T)) = f(0, W(0)) + \int_0^T f_t(t, W(t))dt + \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt.$$

While the first and the third integral are the usual Riemann integrals, the second one is a special integral called the stochastic or Ito integral.
The above Ito formula can be written in the form:

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dt.$$

We now use the Ito's formula with

$$f(t, x) = S(0)e^{\mu t+\sigma x},$$

to obtain,

$$dS(t) = \left(\mu + \frac{1}{2}\sigma^2\right) S(t)dt + \sigma S(t)dW(t).$$

This is the stochastic differential equation that is satisfied by the stock price $S(t)$ in the Black-Scholes model.

Recalling that $\nu = \mu + \frac{1}{2}\sigma^2$ (the growth rate of the expected stock price), we obtain

$$dS(t) = \nu S(t)dt + \sigma S(t)dW(t).$$

The Black-Scholes Equation

Suppose that we have an option whose value $V(S,t)$ depends only on S and t where,

$$dS = \nu S dt + \sigma S dW(t).$$

Then we have (using Ito's Lemma) :

$$dV = \sigma S \frac{\partial V}{\partial S} dW + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt.$$

Now, we construct a portfolio with one option and $-\Delta$ number of the underlying asset. Then the value of the portfolio is,

$$\Pi = V - \Delta S.$$

The change in the value of this portfolio in the time step dt is,

$$d\Pi = dV - \Delta dS.$$

Here Δ is held fixed during the time step dt. Then we get,

$$d\Pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta\right) dW + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S\right) dt.$$

To eliminate the random or risky component we need to choose,

$$\Delta = \frac{\partial V}{\partial S}.$$

This gives us,

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

Also an investment of Π at the risk free rate r grows by $r\Pi dt$ in time dt.

Thus we have (using the no-arbitrage principle),

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\Pi dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt.$$

This leads to the classical Black-Scholes equation,

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0}$$

The solution to the Black-Scholes equation using the appropriate initial/final and boundary conditions gives the value or price of the option at time t.

Boundary and Final Conditions for European Options

1. Call Option ($V(S,t) = C(S,t)$):

   (a) $C(S,T) = \max(S-K, 0)$.

   (b) $C(0,t) = 0$.

   (c) $C(S,t) \sim S$ as $S \longrightarrow \infty$.

2. Put Option ($V(S,t) = P(S,t)$):

   (a) $P(S,T) = \max(K-S, 0)$.

   (b) $P(0,t) = Ke^{-r(T-t)}$.

   (c) $P(S,t) \longrightarrow 0$ as $S \longrightarrow \infty$.

The Black-Scholes Formula for European Option :

1. Price of a call option is given by,

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2).$$

   The delta is,

$$\Delta_C = \frac{\partial C}{\partial S} = N(d_1).$$

2. Price of a put option is given by,

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1).$$

   The delta is,

$$\Delta_P = \frac{\partial P}{\partial S} = N(d_1) - 1.$$

Here, $d_1 = \dfrac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = \dfrac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$ and $N(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy$.