Chapter 5

Alternative Risk Measures and Capital Allocation

The definition of economic capital as introduced in Chapter 1 appears fully satisfactory at first glance. Starting with the path-breaking paper by Artzner et al. [5], several studies revealed a number of methodological weaknesses in the VaR concept by drawing up a catalogue of mathematical and material attributes that a risk measure should fulfill, and proving that the VaR concept only partly fulfills them. Risk measures that satisfy these axioms are called *coherent*. Before we describe their basic features in the next section we briefly reiterate the main notations (cf. Chapters 1 and 2):

The portfolio loss variable (compare Equation (2. 51)) is given by

$$L = \sum_{i=1}^{m} w_i \eta_i L_i ,$$

For a portfolio of m obligors, the portfolio loss relative to the portfolio's total exposure is given by

where

$$w_i = \frac{\mathrm{EAD}_i}{\sum_{i=1}^{m} \mathrm{EAD}_i}$$

are the exposure weights, so all portfolio quantities are calculated in percent of the portfolio's total exposure. The severity is abbreviated to $\eta_i = \mathrm{LGD}_i$; $L_i = \mathbf{1}_{D_i}$ are Bernoulli random variables with default events D_i. Accordingly, the default probability (e.g., for a one-year horizon) for obligor i is given by

$$\mathrm{DP}_i = \mathbb{P}(D_i) .$$

Default correlations are denoted as

$$\rho_{ij} = \mathrm{Corr}[L_i, L_j] = \mathrm{Corr}[\mathbf{1}_{D_i}, \mathbf{1}_{D_j}] .$$

In this chapter we assume a deterministic severity, η_i, but, if severities are assumed to be independent of defaults, then it is straightforward to extend the following to random severities. For notational convenience we denote by

$$\hat{L}_i := \eta_i L_i$$

the loss variable of some obligor i.

5.1 Coherent Risk Measures and Conditional Shortfall

Denote by $L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the space of bounded real random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The definition of a *coherent risk measure* as suggested by Artzner et al. [4,5] can then be stated in the following way:

5.1.1 Definition *A mapping $\gamma : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called a coherent risk measure if the following properties hold:*

(i) **Subadditivity** $\forall X, Y \in L^\infty$: $\gamma(X + Y) \le \gamma(X) + \gamma(Y)$

(ii) **Monotonicity** $\forall X, Y \in L^\infty$ *with* $X \le Y$ *a.s.:* $\gamma(X) \le \gamma(Y)$

(iii) **Positive homogeneity** $\forall \lambda > 0$, $\forall X \in L^\infty$: $\gamma(\lambda X) = \lambda \gamma(X)$

(iv) **Translation invariance** $\forall x \in \mathbb{R}$, $\forall X \in L^\infty$: $\gamma(X + x) = \gamma(X) + x$.

Note that the definition here slightly differs from the original set of axioms as they were introduced by Artzner et al. [4,5]. Our definition equals the one given by Frey and McNeil in [46], because we want to think about X in terms of a portfolio loss and about $\gamma(X)$ as the amount of capital required as a cushion against the loss X, according to the credit management policy of the bank. In the original approach by Artzner et al., X was interpreted as the future value of the considered portfolio. Let us now briefly explain the four axioms in an intuitive manner:

Subadditivity: Axiom (i) reflects the fact that due to diversification effects the risk inherent in the union of two portfolios should be less than the sum of risks of the two portfolios considered separately. We will later see that quantiles are not subadditive in general, such that the economic capital (EC) as introduced in Chapter 1 turns out to be not coherent.

Monotonicity: Let us say we are considering portfolios A and B with losses X_A and X_B. If almost surely the losses of portfolio A are lower than the losses of portfolio B, i.e., $X_A \leq X_B$ a.s., then the required risk capital $\gamma(X_A)$ for portfolio A should be less than the required risk capital $\gamma(X_B)$ of portfolio B. Seen from this perspective, monotonicity is a very natural property.

Homogeneity: Axiom (iii) can best be illustrated by means of the following example. Consider a credit portfolio with loss X and scale all exposures by a factor λ. Then, of course, the loss X changes to a scaled loss λX. Accordingly, the originally required risk capital $\gamma(X)$ will also change to $\lambda\gamma(X)$.

Translation invariance: If x is some capital which will be lost/gained on a portfolio with certainty at the considered horizon, then the risk capital required for covering losses in this portfolio can be increased/reduced accordingly. Translation invariance implies the natural property $\gamma(X - \gamma(X)) = 0$ for every loss $X \in L^\infty$.

Typical risk measures discussed by Artzner et al. are the *value-at-risk* and *expected shortfall capital*, which will be briefly discussed in the sequel.

Value-at-Risk Value-at-risk (VaR) has already been mentioned in Section 1.2.1 as a synonymous name for EC. Here, VaR will be defined for a probability measure $\mathbb{P}$ and some confidence level α as the α-quantile of a loss random variable X,

$$\mathrm{VaR}_\alpha(X) = \inf\{x \geq 0 \mid \mathbb{P}[X \leq x] \geq \alpha\} .$$

VaR as a risk measure defined on L^∞ is

- translation invariant, because shifting a loss distribution by a fixed amount will shift the quantile accordingly,
- positively homogeneous, because scaling a loss variable will scale the quantile accordingly,
- monotone, because quantiles preserve monotonicity, but
- not subadditive, as the following example will show.

Because VaR is not subadditive, it is *not coherent*. Now let us give a simple example showing that VaR is not subadditive.

Consider two independent loans, represented by two loss indicator variables $\mathbf{1}_{D_A}, \mathbf{1}_{D_B} \sim B(1;p)$ with, e.g., $0.006 \leq p < 0.01$. Assume LGDs equal to 100% and exposures equal to 1. Define two portfolios A and B, each consisting of one single of the above introduced loans. Then, for the portfolio losses $X_A = \mathbf{1}_{D_A}$ and $X_B = \mathbf{1}_{D_B}$ we have

$$\mathrm{VaR}_{99\%}(X_A) \;=\; \mathrm{VaR}_{99\%}(X_B) \;=\; 0\,,$$

Now consider a portfolio C defined as the union of portfolios A and B, and denote by $X_C = X_A + X_B$ the corresponding portfolio loss. Then

$$\mathbb{P}[X_C = 0] \;=\; (1-p)^2 \;<\; 99\%\,.$$

Therefore, $\mathrm{VaR}_{99\%}(X_C) > 0$, so that

$$\mathrm{VaR}_{99\%}(X_A + X_B) \;>\; \mathrm{VaR}_{99\%}(X_A) + \mathrm{VaR}_{99\%}(X_B)\,.$$

This shows that in general VaR is not a coherent risk measure.

Expected Shortfall The *tail conditional expectation*, or *expected shortfall*, w.r.t. a confidence level α is defined as

$$\mathrm{TCE}_\alpha(X) = \mathbb{E}[X \mid X \geq \mathrm{VaR}_\alpha(X)]\,.$$

In the literature one can find several slightly different versions of TCE definitions. Tasche [122] showed that expected shortfall to a great extent enjoys the coherence properties; for further reading see also [5,66]. For example, expected shortfall is coherent when restricted to loss variables X with continuous distribution function[1]; see [122]. Therefore, TCE provides a reasonable alternative to the classical VaR measure. Figure 5.1 illustrates the definition of expected shortfall capital. From an

[1] In [1], Acerbi and Tasche introduce a so-called *generalized expected shortfall* measure, defined by

$$\mathrm{GES}_\alpha \;=\; \frac{1}{1-\alpha}\Big(\mathbb{E}[X \mid X \geq \mathrm{VaR}_\alpha(X)] + \mathrm{VaR}_\alpha(X)(1-\alpha-\mathbb{P}[X \geq \mathrm{VaR}_\alpha(X)])\Big).$$

If the distribution of X is continuous, the second term in the definition of GES_α vanishes, and the generalized expected shortfall coincides with (normalized) expected shortfall in this case. It is shown in [1] that generalized expected shortfall is a coherent risk measure.

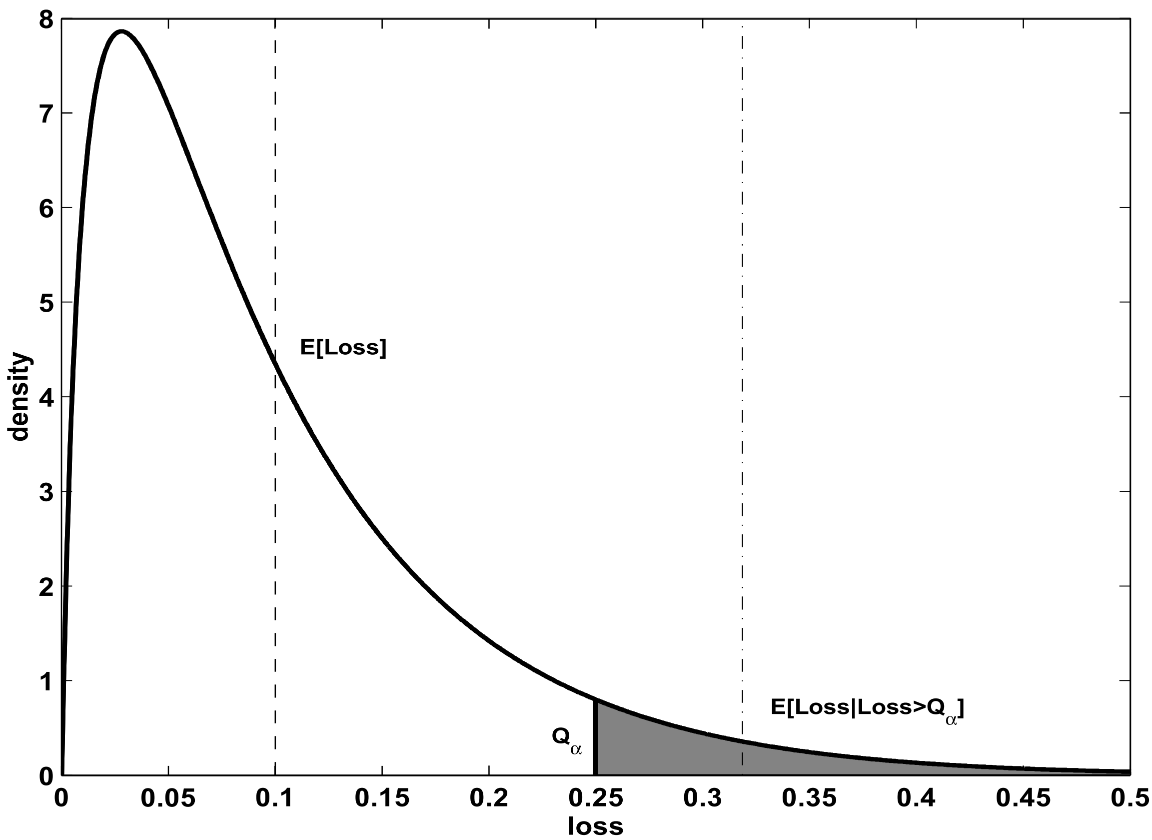

FIGURE 5.1
Expected Shortfall $\mathbb{E}[X \mid X \geq \mathbf{VaR}_\alpha(X)]$**.**

insurance point of view, expected shortfall is a very reasonable measure: Defining by $c = \mathrm{VaR}_\alpha(X)$ a *critical loss threshold* corresponding to some confidence level α, expected shortfall capital provides a cushion against the mean value of losses exceeding the critical threshold c. In other words, TCE focusses on the expected loss in the tail, starting at c, of the portfolio's loss distribution. The critical threshold c, driven by the confidence level α, has to be fixed by the senior management of the bank and is part of the bank's credit management policy.

The following proposition provides another interpretation of coherency and can be found in [5].

5.1.2 Proposition *A risk measure γ is coherent if and only if there exists a family $\mathcal{P}$ of probability measures such that*

$$\gamma \;=\; \gamma_{\mathcal{P}} \;,$$

where $\gamma_{\mathcal{P}}$ is defined by

$$\gamma_{\mathcal{P}}(X) \;=\; \sup_{Q\in\mathcal{P}} \mathbb{E}_Q[X] \qquad \text{for all } X \in L^\infty \;.$$

The probability measures in $\mathcal{P}$ are called generalized scenarios.

The challenge underlying Proposition 5.1.2 is to find a suitable set $\mathcal{P}$ of probability distributions matching a given coherent risk measure γ such that $\gamma = \gamma_{\mathcal{P}}$.

Economic capital, based on shortfall risk, can be defined as the mean loss above a threshold c minus the expected loss:

$$\mathrm{EC}_{\mathrm{TCE}}(c) = \mathbb{E}[X \mid X \geq c] - \mathbb{E}[X].$$

This calculation method for risk capital also includes events above the critical loss threshold c, e.g., $c = \mathrm{VaR}_\alpha$, and answers the question "how bad is bad" from a mean value's perspective. If $c = \mathrm{VaR}_\alpha(X)$, we write

$$\mathrm{EC}_{\mathrm{TCE}_\alpha} \;=\; \mathrm{EC}_{\mathrm{TCE}}(\mathrm{VaR}_\alpha(X))$$

in the sequel in order to keep the notation simple.

Quantiles The following table shows a comparison of expected shortfall EC and VaR-EC over the 99%-quantile for different distributions. For example, one can see that if X had a normal distribution, the expected shortfall EC would not much differ from the VaR-EC. In contrast, for a t-distribution and the loss distribution $F_{p,\varrho}$ defined by a uniform (limit) portfolio as introduced right before Proposition 2.5.7, the difference between the ECs is quite significant.

	t(3)	N(0,1.73)	LN(0,1)	N(1.64,2.16)	Weil(1,1)	N(1,1)
std	1.73	1.73	2.16	2.16	1	1
$\mathrm{EC}_{\mathrm{VaR}}(0.99)$	4.54	4.02	8.56	5.02	3.6	2.32
$\mathrm{EC}_{\mathrm{TCE}}(0.99)$	6.99	4.61	13.57	5.76	4.6	2.66
rel.diff. (%)	54	15	58	15	27	14

	$F_{0.003,0.12}$
std	0.0039
$\mathrm{EC}_{\mathrm{VaR}}(0.99)$	0.0162
$\mathrm{EC}_{\mathrm{TCE}}(0.99)$	0.0237
rel.diff. (%)	46

This table highlights the sensitivity of the determination of economic capital w.r.t. its definition (VaR-based or shortfall-based) and the choice of the underlying probability measure. Here, the probability measures are given by the following distributions:

- the t-distribution $t(3)$ with 3 degrees of freedom,
- three normal distributions with different parameter pairs,
- the $(0, 1)$-log-normal distribution,
- the $(1, 1)$-Weibull distribution, and
- the uniform portfolio distribution $F_{p,\varrho}$ with uniform default probability $p = 30$bps and uniform asset correlation $\varrho = 12\%$.

In the table, the first row contains the respective standard deviations, the second row shows the EC based on VaR at the 99%-confidence level, and the third row shows the EC based on expected shortfall with threshold $c = \mathrm{VaR}_{99\%}$. The fourth row shows the relative difference between the two EC calculations.

5.2 Contributory Capital

If the economic capital EC of the bank is determined, one of the main tasks of risk management is to calculate *contributory economic capital* for each business division of the bank. Such a contributory EC could be interpreted as the *marginal capital* a single transaction respectively business unit adds or contributes to the overall required capital. Tasche [120] showed that there is only one definition for *risk contributions* that is suitable for *performance measurement*, namely the *derivative of the underlying risk measure in direction of the asset weight of the considered entity.* Additionally, Denault [24] showed by arguments borrowed from game theory that in the case of a so-called *1-homogeneous risk measure* its gradient is the only "allocation principle" that satisfies some "coherency" conditions.

So when talking about capital allocation, we are therefore faced with the problem of *differentiability* of some considered risk measure. To allow for differentiation, the risk measure must be "sufficiently smooth" for guaranteeing that the respective derivative exists. It turns out that the standard deviation as a risk measure has nice differentiability properties. This observation is the "heart" of Markovitz's classical portfolio theory (var/covar approach). Now, VaR-based EC is a *quantile-based* quantity, and only in case of normal distributions quantiles and standard deviations are reconcilable from a portfolio optimization point of view. Because we know from previous chapters that credit portfolio loss distributions are typically skewed with fat tails, the "classical" approach can not be applied to credit risk in a straightforward manner without "paying" for the convenience of the approach by allowing for inconsistencies.

Fortunately, the var/covar approach for *capital allocation*, adapted to credit risk portfolios, in many cases yields acceptable results and is, due to its simplicity, implemented in most standard software packages. The following section explains the details.

5.2.1 Variance/Covariance Approach

At the core of var/covar is the question of what an individual credit or business unit contributes to the portfolio's standard deviation UL_{PF}. To answer this question, the classical var/covar approach splits the portfolio risk UL_{PF} into risk contributions RC_i in a way such that

$$\sum_{i=1}^{m} w_i \times \mathrm{RC}_i \;=\; \mathrm{UL}_{PF}\;.$$

In this way, the weighted risk contributions sum-up to the total risk of the portfolio, where "risk" is identified with volatility. It follows from Proposition 1.2.3 that

$$\begin{aligned}\mathrm{UL}_{PF} &= \frac{1}{\mathrm{UL}_{PF}} \sum_{i=1}^{m} w_i \cdot \sum_{j=1}^{m} w_j \eta_i \eta_j \sigma_{D_i} \sigma_{D_j} \rho_{ij} \\ &= \frac{1}{\mathrm{UL}_{PF}} \sum_{i=1}^{m} w_i \cdot \sum_{j=1}^{m} w_j \mathrm{UL}_i \mathrm{UL}_j \rho_{ij},\end{aligned}$$

where $\sigma_{D_i} = \sqrt{\mathrm{DP}_i(1-\mathrm{DP}_i)}$ denotes the standard deviation of the default indicator and $\mathrm{UL}_i = \eta_i \sigma_{D_i}$. Thus,

$$\mathrm{RC}_i = \frac{\mathrm{UL}_i}{\mathrm{UL}_{PF}} \sum_{j=1}^{m} w_j \mathrm{UL}_j \rho_{ij}$$

is a plausible quantity measuring the "risk portion" of credit i in a way such that all weighted risks sum-up to the portfolio's UL. It is straightforward to show that the quantity RC_i corresponds to the covariance of credit (business unit) i and the total portfolio loss, divided by the portfolio's volatility respectively UL. The definition of RC_i obviously is in analogy to *beta-factor models* used in market risk. Furthermore, RC_i is equal to the *partial derivative* of UL_{PF} w.r.t. w_i, the weight of the i-th credit in the portfolio, i.e.,

$$\mathrm{RC}_i = \frac{\partial \mathrm{UL}_{PF}}{\partial w_i}.$$

In other words, an increase in the weight of the considered credit by a small amount h in the portfolio, implies a growth of UL_{PF} by $h \times \mathrm{RC}_i$. Coming from this side, it can in turn be easily shown that the total sum of weighted partial derivatives again equals UL_{PF}.

Regarding the ratio between risk contributions and the standard deviation of the individual exposure, it is true in most cases that

$$\frac{\mathrm{RC}_i}{\mathrm{UL}_i} \;\leq\; 1.$$

This quantity is known as the *retained risk* of unit i. It is the portion of risk of the i-th entity that has not been absorbed by diversification in the portfolio. In contrast, the quantity

$$1 - \frac{\mathrm{RC}_i}{\mathrm{UL}_i}$$

is often called the corresponding *diversification effect.*

Capital multiplier Since the total risk capital is typically determined via quantiles, i.e.,

$$\mathrm{EC}_{\mathrm{VaR}_\alpha} = \mathrm{VaR}_\alpha(L) - \mathbb{E}[L] \ ,$$

the individual risk contributions have to be rescaled with the so-called *capital multiplier*

$$\mathrm{CM}_\alpha = \frac{\mathrm{EC}_{\mathrm{VaR}_\alpha}}{\mathrm{UL}_{PF}}$$

in order to imitate the classical approach from market risk. The contributory capital for credit i then equals

$$\delta_i = \mathrm{CM}_\alpha \times \mathrm{RC}_i, \quad \text{with} \quad \sum_{i=1}^{m} w_i \delta_i = \mathrm{EC}_{\mathrm{VaR}_\alpha} \ .$$

The quantity δ_i is called the *analytic capital contribution* of transaction i to the portfolio capital. For a business unit in charge for credits 1 to l, where $l < m$, the capital requirement is

$$\mathrm{CM}_\alpha \sum_{j=1}^{l} \mathrm{RC}_j .$$

Note that the capital multiplier is an auxiliary quantity depending on the particular portfolio, due to the fact that, in contrast to the normal distribution, the quantiles of credit portfolio loss distributions not only depend on the standard deviation, but also on other influences like correlations, default probabilities, and exposure weights. Therefore it is unrealistic, after changing the portfolio, to obtain the same capital multiplier CM_α as originally calculated.

In the next two sections we discuss capital allocation w.r.t. EC based on VaR and expected shortfall respectively.

5.2.2 Capital Allocation w.r.t. Value-at-Risk

Calculating risk contributions associated with the VaR risk measure is a natural but difficult attempt, since in general the quantile function will not be differentiable with respect to the asset weights. Under certain continuity assumptions on the joint density function of the random variables X_i, differentiation of $\mathrm{VaR}_\alpha(X)$, where $X = \sum_i w_i X_i$, is guaranteed. One has (see [122])

$$\frac{\partial \mathrm{VaR}_\alpha}{\partial w_i}(X) \;=\; \mathbb{E}[X_i \mid X = \mathrm{VaR}_\alpha(X)]. \tag{5. 1}$$

Unfortunately, the distribution of the portfolio loss $L = \sum w_i \hat{L}_i$, as specified at the beginning of this chapter, is purely discontinuous. Therefore the derivatives of VaR_α in the above sense will either not exist or vanish to zero. In this case we could still define risk contributions via the right-hand-side of Equation (5. 1) by writing

$$\gamma_i \;=\; \mathbb{E}[\hat{L}_i \mid L = \mathrm{VaR}_\alpha(L)] - \mathbb{E}[\hat{L}_i] \; . \tag{5. 2}$$

For a clearer understanding, note that

$$\frac{\partial \mathbb{E}[L]}{\partial w_i} \;=\; \mathbb{E}[\hat{L}_i] \qquad \text{and} \qquad \sum_{i=1}^{m} w_i \gamma_i \;=\; \mathrm{EC}_{\mathrm{VaR}_\alpha} .$$

Additionally observe, that for a large portfolio and on an appropriate scale, the distribution of L will appear to be "close to continuous". Unfortunately, even in such "approximately good" cases, the loss distribution often is not given in an analytical form in order to allow for differentiations.

Remark For the CreditRisk$^+$ model, an analytical form of the loss distribution can be found; see Section 2.4.2 and Chapter 4 for a discussion of CreditRisk$^+$. Tasche [121] showed that in the CreditRisk$^+$ framework the VaR contributions can be determined by calculating the corresponding loss distributions several times with different parameters. Martin et al. [82] suggested an approximation to the partial derivatives of VaR via the so-called *saddle point method*.

Capital allocation based on VaR is not really satisfying, because in general, although $(\mathrm{RC}_i)_{i=1,\ldots,m}$ might be a reasonable partition of the portfolio's standard deviation, it does not really say much about the tail risks captured by the quantile on which VaR-EC is relying. Even if in general one views capital allocation by means of partial derivatives as useful, the problem remains that the var/covar approach completely neglects the dependence of the quantile on correlations. For example, var/covar implicitely assumes

$$\frac{\partial \mathrm{VaR}_\alpha(X)}{\partial \mathrm{UL}_{PF}} \;=\; \mathrm{const} \;=\; \mathrm{CM}_\alpha,$$

for the specified confidence level α. This is true for (multivariate) normal distributions, but generally not the case for loss distributions of credit portfolios. As a consequence it can happen that transactions require a contributory EC exceeding the original exposure of the considered transaction. This effect is very unpleasant. Therefore we now turn to expected shortfall-based EC instead of VaR-based EC.

5.2.3 Capital Allocations w.r.t. Expected Shortfall

At the beginning we must admit that shortfall-based risk contributions bear the same "technical" difficulty as VaR-based measures, namely the quantile function is not differentiable in general. But, we find in Tasche [122] that if the underlying loss distribution is "sufficiently smooth", then TCE_α is partially differentiable with respect to the exposure weights. One finds that

$$\frac{\partial \mathrm{TCE}_\alpha}{\partial w_i}(X) \;=\; \mathbb{E}[X_i \mid X \geq \mathrm{VaR}_\alpha(X)].$$

In case the partial derivatives do not exist, one again can rely on the right-hand side of the above equation by defining shortfall contributions for, e.g., discontinuous portfolio loss variables $L = \sum w_i \hat{L}_i$ by

$$\zeta_i \;=\; \mathbb{E}[\hat{L}_i \mid L \geq \mathrm{VaR}_\alpha(L)] - \mathbb{E}[\hat{L}_i]\;, \qquad (5.\ 3)$$

which is consistent with expected shortfall as an "almost coherent" risk measure. Analogous to what we saw in case of VaR-EC, we can write

$$\sum_{i=1}^{m} w_i \zeta_i \;=\; \mathrm{EC}_{\mathrm{TCE}_\alpha}\;,$$

such that shortfall-based EC can be obtained as a weighted sum of the corresponding contributions.