Lecture 15

Numerical Approximations for IVP

MA 322: Scientific Computing

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1 Some View Points on Euler Method

1.1 A Geometric View Point

Let y be the exact solution for the IVP

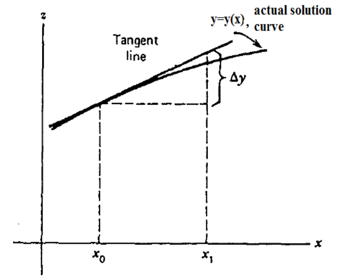
$$y' = f(x, y(x)), \quad y(x_0) = y_0. \tag{1}$$

Consider the graph of the solution $y = y(x)$, as shown in the figure. Form the tangent line to the graph of $y = y(x)$ at x_0, and use this line as an approximation to the curve for $x_0 \leq x \leq x_1$. Then

$$y(x_1) \approx y(x_0) + \Delta y \;\; \& \;\; \frac{\Delta y}{h} = \tan\theta = y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0).$$

Combining above equations, we obtain

$$y(x_1) \approx y(x_0) + hf(x_0, y_0) = y_1 \;\; \text{(say)}.$$



Geometric interpretation of Euler's method

Remark: By repeating this argument on $[x_1, x_2], [x_2, x_3], \ldots$, we obtain the general formula

$$y(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n, y_n). \tag{2}$$

Note that $y(x_n)$ denotes the actual value of the true solution at x_n and y_n is the approximation of $y(x_n)$ at x_n. For this, we have used the notation $y(x_n) \approx y_n$. Here, the points $x_0,\ x_1,\ x_2,\ \ldots$, with $x_{i+1} = x_i + h$, are known as grid points and the collection $\{x_0,\ x_1,\ x_2,\ \ldots\}$ are known as mesh and $h > 0$ is called the mesh parameter. Since the grid points are equally spaced, so the mesh $\{x_0,\ x_1,\ x_2,\ \ldots\}$ is called uniform mesh. Otherwise it is known as non-uniform mesh.

1.2 Taylor Series

Expanding $y(x_{n+1})$ about x_n, we have

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2} y''(\xi_n), \quad x_n \leq \xi_n \leq x_{n+1}. \tag{3}$$

Note, that for twice continuously differential solution y, we have

$$\frac{h^2}{2} y''(\xi_n) \leq \frac{h^2}{2} \|y''\|_\infty = Ch^2,$$

where $C = \|y''\|_\infty > 0$. For the simplicity of the presentation, we call the remainder term $\frac{h^2}{2} y''(\xi_n)$ of order h^2 and it is denoted by

$$\frac{h^2}{2} y''(\xi_n) = \mathrm{O}(h^2). \tag{4}$$

Remark: In general, a term T is called of order $h^k,\ k \in \mathbb{N}$, if there exists a positive constant C, independent of mesh parameter h, such that

$$T \leq Ch^k.$$

In that case, we write $T = \mathrm{O}(h^k)$. This definition will be frequently used when we discuss the finite difference methods for partial differential equations.

Using (4) in (3), we obtain

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hf(x_n, y(x_n)) + \mathrm{O}(h^2). \tag{5}$$

By dropping the last term and using the notation $y(x_n) \approx y_n$, we obtain

$$y(x_{n+1}) \approx y_n + hf(x_n, y_n)) = y_{n+1}. \tag{6}$$

1.3 Numerical Differentiation

Consider the slope of the tangent at (x_n, y_n), that is $y'(x_n)$, and which is given by

$$\lim_{h\to 0} \frac{y(x_n + h) - y(x_n)}{h} = y'(x_n) \ \text{ Or}$$
$$\lim_{h\to 0} \frac{y(x_{n+1}) - y(x_n)}{h} = y'(x_n),$$

which implies

$$y'(x_n) = \frac{y(x_{n+1}) - y(x_n)}{h} + r(h), \tag{7}$$

where $r(h) \to 0$ as $h \to 0$. Therefore, $y'(x_n)$ can be approximated by

$$\frac{y(x_{n+1}) - y(x_n)}{h}.$$

More precisely,

$$\frac{y(x_{n+1}) - y(x_n)}{h} \approx y'(x_n) = f(x_n, y(x_n)) \approx f(x_n, y_n). \tag{8}$$

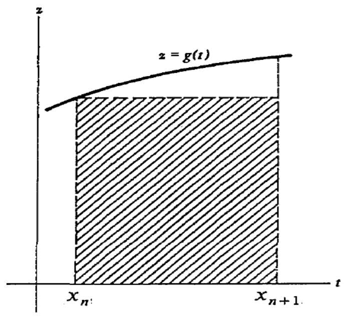
Here, we have used the fact that $y(x_n) \approx y_n$. Thus, finally, we arrive at

$$y(x_{n+1}) \approx y_n + hf(x_n, y_n) = y_{n+1}. \tag{9}$$

1.4 Numerical Integration

Integrate $y' = f(x, y(x))$ over $[x_n, x_{n+1}]$ to have

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))dt = y(x_n) + \int_{x_n}^{x_{n+1}} g(t)dt. \tag{10}$$



Then consider the simple numerical quadrature rule

$$\int_{x_n}^{x_{n+1}} g(t)dt \approx hg(x_n), \tag{11}$$

which is known as *left-hand rectangular rule*. Applying (11) in the equation (10), we obtain

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))dt = y(x_n) + \int_{x_n}^{x_{n+1}} g(t)dt \\ &= y(x_n) + hf(x_n, y(x_n)) \\ &\approx y_n + hf(x_n, y_n). \end{aligned} \tag{12}$$

Remark:

- Consider the Euler method
$$y_{n+1} = y_n + hf(x_n, y_n), \tag{13}$$
where $y_n \approx y(x_n)$. Therefore, approximation at $n+1$th step depends only on the information of nth step. Such methods are known as single-step method. Otherwise it is called multi-step method.
- Of the three analytical derivations method in Sections 1.2-1.4, both methods in Section 1.2 and Section 1.4 are the simplest cases of a set of increasingly accurate methods. Approach in Section 1.2 leads to the single-step methods, particularly the Runge-Kutta formulas. Approach describe in Section 1.4 leads to multistep methods, especially the predictor-corrector methods. Perhaps surprisingly, method describe in Section 1.3 often does not lead to other successful methods.

2 Convergence of Euler Method

2.1 Example

Before analyzing the convergence of Euler method, we give some numerical examples. They also serve as illustrations for the convergence and demonstrates the efficiency of the Euler method.

Example: Consider the equation $y' = y,\ y(0) = 1$. Its true solution is $y(x) = e^x$. Numerical results are given in the following table for several values of h (Source: An Introduction to Numerical Analysis, by Atkinson). For given mesh parameter h, let $y_n^h(x)$ be the approximation of $y(x)$. Then the error at x is given by $y(x) - y_n^h(x)$.

For the calculation of $y_n^h(x)$, for given h, we first fix n such that $x = x_{n+1} = x_n + h$. For the present example, suppose we wish to evaluate the approximate solution at $x = .40$ for $h = 0.2$. Now, calculate the grid points with $h = .20$ and $x_0 = 0$. The grid points are

$$x_0 = 0,\ \ x_1 = x_0 + h = 0.2,\ \ x_2 = x_1 + h = 0.4, \ldots. \tag{14}$$

Thus, we need to calculate y_2, which is given by

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + hy_1,\ \ y_1 = y_0 + hf(x_0, y_0) = 1 + .2 \times y_0 = 1.2,$$

which gives

$$y_2 = 1.2 + .2 \times 1.2 = 1.44.$$

Similarly, for a given h, we can calculate the approximation at any x.

Euler's method, example (1)

	x	$y_h(x)$	$Y(x)$	$Y(x) - y_h(x)$
$h = .2$	.40	1.44000	1.49182	.05182
	.80	2.07360	2.22554	.15194
	1.20	2.98598	3.32012	.33413
	1.60	4.29982	4.95303	.65321
	2.00	6.19174	7.38906	1.19732
$h = .1$	.40	1.46410	1.49182	.02772
	.80	2.14359	2.22554	.08195
	1.20	3.13843	3.32012	.18169
	1.60	4.59497	4.95303	.35806
	2.00	6.72750	7.38906	.66156
$h = .05$	.40	1.47746	1.49182	.01437
	.80	2.18287	2.22554	.04267
	1.20	3.22510	3.32012	.09502
	1.60	4.76494	4.95303	.18809
	2.00	7.03999	7.38906	.34907

Remark:

- Note that the error at each point x decreases by about half when h is halved. For example, see the errors at $x = .40$ for different h.

Error at x=.40

	x	$y_h(x)$	$Y(x)$	$Y(x) - y_h(x)$
$h = .2$	.40	1.44000	1.49182	.05182
$h = .1$	.40	1.46410	1.49182	.02772
$h = .05$	.40	1.47746	1.49182	.01437

- Our next objective is to justify why the error at each point x decreases by about half when h is halved. This is done by convergence analysis.