Lecture 14

Numerical Approximations for IVP

MA 322: Scientific Computing

by

BHUPEN DEKA

Department of Mathematics

IIT Guwahati, ASSAM

1 Nonlinear First Order Differential Equations

Now it is time to turn our attention to the numerical approximations for the IVP

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{1}$$

Theorem 1.1 (Picard's Theorem) Let $D = \{(x, y) \ : \ |x - x_0| < a, \quad |y - y_0| < b\}$ be a rectangle. Let $f(\cdot, \cdot)$ be continuous and bounded on D, i.e., there exists a number K such that

$$|f(x, y)| \leq K \ \ \forall (x, y) \in D.$$

Further, let f satisfy the *Lipschitz condition* with respect to y in D, i.e., there exists a number M such that

$$|f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1| \quad \forall (x, y_1), \, (x, y_2) \in D. \tag{2}$$

Then, the IVP (1) has a unique solution $y(x)$. This solution is defined for all x in the interval

$$|x - x_0| < \alpha, \quad \text{where} \quad \alpha = \min\left\{a, \ \frac{b}{K}\right\}$$

and characterized as a uniform limit of the sequence

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_n(s))ds, \quad n \geq 0. \tag{3}$$

Example 1.1 Consider the IVP

$$y' = y^2 + \cos^2 x, \quad y(0) = 0.$$

Determine the largest interval where solution is valid.

*S*olution. Here

$$f(x, y) = y^2 + \cos^2 x \ \ \& \ \ \frac{\partial f}{\partial y} = 2y$$

are continuous everywhere. Therefore, the IVP has an unique solution in an interval containing initial point $(0, 0)$. The equation is neither separable nor exact. Therefore it is not possible to determine the interval of definition by solving the problem and so we have to rely on Picard method.

Let a and b are parameter, which need to determine, such that the solution is valid in the rectangle

$$R = \{(x, y) : \ |x - 0| \leq a, \ |y - 0| \leq b\}.$$

Then

$$\begin{aligned} M &= \sup\{|f(x, y)| : \ (x, y) \in R\} \\ &= \sup\{y^2 + \cos^2 x : \ (x, y) \in R\} \\ &= b^2 + 1. \end{aligned}$$

Thus the unique solution, by Picard result, is valid in the interval $|x| < h$ with

$$h = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{b}{b^2 + 1}\right\} = \min\left\{a, g(b)\right\}, \ g(b) = \frac{b}{b^2 + 1}.$$

For the largest interval, we set $g'(b) = 0$. This gives $b = \pm 1$ and hence the maximum value of g is given by $g(2) = 1/2$. Thus

$$h = \min\left\{a, \frac{1}{2}\right\} = \begin{cases} a, & \text{if } a < 1/2; \\ 1/2, & \text{if } a > 1/2; \\ 1/2, & \text{if } a = 1/2. \end{cases}$$

For all three cases the largest interval is valid given by $|x| < 1/2$. In this example even if we do not find the solution by traditional method, but still one can able to find the interval where the solution is valid.

In some cases the existence of a solution of the initial value problem (1) can be established directly by actually solving the problem and exhibiting a formula for the solution. However, in general, this approach is not feasible because there is no method of solving the differential equation that applies in all cases (see, Example 1.1). Therefore, for the general case it is necessary to adopt some numerical technique that provide a practical means of finding the solution for the initial value problem (1).

1.1 Picard Approximation

The theorem on existence and uniqueness states that if both $f(x, y)$ and $\partial f/\partial y$ are continuous in some region around the point (x_0, y_0) then there is a unique solution to the initial value problem (1) valid in some interval around x_0. In other words, if the slope field is sufficiently smooth at each point, then there is a unique integral curve passing through any given point. The uniqueness of the solution has been proved in the previous section by transferring the initial value problem to an equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds = \int_{x_0}^{x} f(s, y(s))ds. \tag{4}$$

For simplicity of the exposition, we have assumed $x_0 = 0$ & $y_0 = 0$. Even though this looks like it is solved, it really is not because the function y is buried inside the integrand. To solve this, we attempt to use the following algorithm, known as Picard Iteration:

Step 1. Choose an initial function $y_0(x)$ either arbitrarily or to approximate in some way the solution of the initial value problem. The simplest choice is

$$y_0(x) = 0.$$

Step 2. For $n = 1,\ 2,\ 3, \ldots,$ set

$$y_{n+1}(x) = \int_0^x f(s, y_n(s))ds.$$

In this manner we generate the sequence of functions $\langle y_n \rangle = \{y_1, y_2, \ldots, y_n, \ldots\}$. Each member of the sequence satisfies the initial condition, but in general none satisfies the differential equation. However, if we can prove that there exists a function $\phi(x)$ such that $y_n \to \phi$ uniformly. Then using the continuity of f and the consequence of uniform convergence under the sign of integration, we have

$$y_{n+1}(x) \to \phi(x) \quad \& \quad \int_0^x f(s, y_{n-1}(s))ds \to \int_0^x f(s, \phi(s))ds.$$

In such case, clearly, ϕ satisfies the following integral equation

$$\phi(x) = \int_0^x f(s, \phi(s))ds,$$

and hence a solution to the differential equation. So the limit of the Picard iteration becomes the solution of the initial value problem.

To establish the limit of the picard iteration two principal questions must be answered:

1. Do all members of the sequence $\langle y_n \rangle$ exists?

   - It follows from the definition of y_n that

     $$y_n(0) = 0 \ \ \& \ \ y'_{n+1}(x) = f(x, y_n(x))$$

     for each n. Further recall that

     $$|f(x, y)| \le M, \ \ (x, y) \in R,$$

     R is a closed bounded region containing $(0, 0)$. Since $y_0(x) = 0$ for all $x \in R_x$, R_x is the restriction of x axis in R, therefore

     $$f(x, y_0(x)) \le M \quad \forall x \in R_x$$

     and hence

     $$y_1(x) = \int_0^x f(s, y_0(s))ds \le Mx \quad \forall x \in R_x.$$

     So y_1 will be in $R = \{(x, y) : \ |x - 0| \le a,\ |y - 0| \le b\}$ provided $Mx \le b$ that is for $|x| \le b/M$. In other sense y_1 is valid in a rectangle $D : \{(x, y) : \ |x| \le h,\ |y| \le b\}$, where h is either equal to a or to b/M, whichever is smaller. With this restriction, all members of the sequence $y_n(x)$ exists.

2. Do the sequence converge?

   - It is easy to see that

$$|y_{k+1}(x) - y_k(x)| \leq ML^k \frac{|x-x_0|^{k+1}}{(k+1)!},$$
$$L = \sup\Big\{\Big|\frac{\partial f}{\partial y}\Big| : (x,y) \in D\Big\}.$$

   Which ensure the convergence of the sequence $\langle y_n \rangle$. Let $y_n to \phi$ and in such case it has proved that ϕ is the solution of the problem.

Example 1.2 Solve

$$y' = 2x(1+y), \quad y(0) = 0$$

by Picard iteration method.

*S*olution. Here $f(x,y) = 2x(1+y)$. Setting $y_0(x) = 0$, we have the following first few iterations

$$\begin{aligned}
y_1(x) &= \int_0^x f(s, y_0(s))ds = \int_0^x 2s(1+y_0)ds = \int_0^x 2sds = x^2. \\
y_2(x) &= \int_0^x f(s, y_1(s))ds = \int_0^x 2s(1+y_1)ds \\
&= \int_0^x 2s(1+s^2)ds = x^2 + \frac{x^4}{2}. \\
y_3(x) &= \int_0^x f(s, y_2(s))ds = \int_0^x 2s(1+y_2)ds \\
&= \int_0^x 2s(1+s^2+\frac{s^4}{2})ds \\
&= x^2 + \frac{x^4}{2} + \frac{x^6}{2.3}
\end{aligned}$$

Above expressions for $y_1,\ y_2,\ y_3$ suggest that

$$y_n(x) = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots + \frac{x^{2n}}{n!}.$$

Clearly the sequence convergence to

$$x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots + \frac{x^{2n}}{n!} + \ldots = e^{x^2} - 1$$

and hence the solution is given by

$$y(x) = e^{x^2} - 1.$$

1.2 Euler Method

Here, we introduce the oldest and simplest numerical method originated by Euler about 1768 to find an approximate solution of (1). It is called the tangent line method or the Euler method.

The Fundamental Theorem of Calculus, when coupled with the differential equation itself, suggests one simple scheme for computing the value of the solution of a differential equation numerically. The underlying idea is for this computational method is that the graph of the solution for (1) can be obtained by simply plotting points. To motivate towards this method, recall that the Fundamental Theorem of Calculus states that for any function $y(x)$ and for any numbers $x_0,\ x_1 (x_0 < x_1)$, we have

$$y(x_1) = y_0 + \int_{x_0}^{x_1} y'(s)ds.$$

We assume that x_1 is not much larger than x_0, the value of the integral can be approximated as

$$y'(x_0)(x_1 - x_0).$$

Making this substitution shows that for x_1 not much larger than x_0, $y(x_1)$ (value of the actual solution at x_1) can be approximated by the formula

$$y_0 + y'(x_0)(x_1 - x_0) = y_1 \ \text{(say)}.$$

Since $y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$, thus

$$y_1 = y_0 + f(x_0, y_0)(x_0)(x_1 - x_0).$$

To proceed further, we can try to repeat the process in $[x_1, x_2]$ so that the approximate value for $y(x_2)$ can be obtained as

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1).$$

Here, we have used the approximate value y_1 for $y(x_1)$ since we do not know the value of the actual solution $y(x)$ at $x = x_1$. In general, we consider the following points

$$x_0, \ x_1 = x_0 + h, \ x_2 = x_1 + h, \ldots x_{n+1} = x_n + h,$$

h is called the step size. Then the approximate value y_{n+1} for $y(x_{n+1})$ is given by

$$y_{n+1} = y_n + f(x_n, y_n)(x_{n+1} - x_n), \ \ n = 0, \ 1, \ 2, \ldots.$$

For the convenience of presentation we introduce the notation $f_n = f(x_n, y_n)$, then we can rewrite equation (5) as

$$y_{n+1} = y_n + f_n h, \ \ n = 0, \ 1, \ 2, \ldots.$$

Example 1.3 Consider the initial value problem

$$y' = x^2 + 5, \ y(0) = 0, \ x \in [0, 1].$$

Find first four approximation using Euler method for $h = .25$.

*S*olution. Here, $y_0 = 0, \ f_n = x_n^2 + 5$. The points of subdivisions are $x_0 = 0, \ x_1 = .25, \ x_2 = .5, \ x_3 = .75$ & $x_4 = 1$. Then use formula (5) for $n = 0, \ 1, \ 2, \ 3, \ 4$ to have

$$\begin{aligned} y_1 &= y_0 + f_0 h = 0 + .25(5) = 1.25 \\ y_3 &= y_2 + h f_2 \\ &= 2.5156 + .25(x_2^2 + 5) \\ &= 2.5156 + .25((.5)^2 + 5) \\ &= 3.8281 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h f_1 \\ &= 1.25 + .25(x_1^2 + 5) \\ &= 1.25 + .25((.25)^2 + 5) \\ &= 2.5156 \\ y_4 &= y_3 + h f_3 \\ &= 3.8281 + .25(x_3^2 + 5) \\ &= 3.8281 + .25((.75)^2 + 5) \\ &= 5.218725. \end{aligned}$$