Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Theorem:

Consider the binomial model with N periods. Let $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ be an adapted portfolio process, let X_0 be a real number, and let the wealth process $X_1, X_2, \ldots, X_N$ be generated recursively by

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)\left(X_n - \Delta_n S_n\right),\ n = 0, 1, \ldots, N-1.$$

Then the discounted wealth process $\dfrac{X_n}{(1+r)^n}$, $n = 0, 1, \ldots, N$, is a martingale under the risk-neutral measure, *i.e.*,

$$\frac{X_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right],\ n = 0, 1, \ldots, N-1.$$

Proof:

We compute:

$$\begin{aligned}
\widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \widetilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \\
&= \widetilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + \widetilde{\mathbb{E}}_n\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \ \textit{(linearity)} \\
&= \Delta_n \widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \frac{X_n - \Delta_n S_n}{(1+r)^n} \ \textit{(taking out what is known)} \\
&= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n} \ \textit{(Martingale condition of discounted stock prices)} \\
&= \frac{X_n}{(1+r)^n}.
\end{aligned}$$

Corollary:

Under the conditions of the above Theorem, we have

$$\widetilde{\mathbb{E}}_n \frac{X_n}{(1+r)^n} = X_0,\ n = 0, 1, \ldots, N.$$

Proof:

The corollary follows from the fact that the expected value of a martingale cannot change with time and so must always be equal to the time-zero value of the martingale. Applying this fact to the $\widetilde{\mathbb{P}}$-martingale $\dfrac{X_n}{(1+r)^n}$, $n = 0, 1, \ldots, N$, we obtain the Corollary.

There are two consequences of the Theorem, one which leads to the *First Fundamental Theorem of Asset Pricing* (to be discussed in a subsequent course) and another is the following version of the *risk-neutral pricing formula*:

Theorem:

Consider a N-period binomial asset-pricing with $0 < d < 1 + r < u$ and with the risk-neutral probability measure $\widetilde{\mathbb{P}}$. Let V_N be a random variable (a derivative security paying off at time N) depending on the coin tosses. Then, for n between 0 and N, the price of the derivative security at time n is given by the risk-neutral pricing formula

$$V_n = \widetilde{\mathbb{E}}_n \left[\frac{V_N}{(1+r)^{N-n}} \right].$$

Furthermore, the discounted price of the derivative security is a martingale under $\widetilde{\mathbb{P}}$, *i.e.*,

$$\frac{V_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n \left[\frac{V_{n+1}}{(1+r)^{n+1}} \right], \quad n = 0, 1, \ldots, N-1.$$

Theorem (Cash Flow Valuation):

Consider an N-period binomial asset pricing-model with $0 < d < 1 + r < u$, and with risk-neutral probability measure $\widetilde{\mathbb{P}}$. Let $C_0, C_1, \ldots, C_N$ be a sequence of random variables such that each C_n depends only on $\omega_1\omega_2\ldots\omega_n$. The price at time n of the derivative security that makes payments $C_n, C_{n+1}, \ldots, C_N$ at times $n, n+1, \ldots, N$, respectively, is

$$V_n = \widetilde{\mathbb{E}}_n \left[\sum_{k=n}^{N} \frac{C_k}{(1+r)^{k-n}} \right], \quad n = 0, 1, \ldots, N.$$

The price process $V_n,\ n = 0, 1, \ldots, N$ satisfies

$$C_n(\omega_1\omega_2\ldots\omega_n) = V_n(\omega_1\omega_2\ldots\omega_n) - \frac{1}{1+r}\left[\widetilde{p}V_{n+1}(\omega_1\omega_2\ldots\omega_n H) + \widetilde{q}V_{n+1}(\omega_1\omega_2\ldots\omega_n T)\right].$$

We define

$$\Delta_n(\omega_1\omega_2\ldots\omega_n) = \frac{V_{n+1}(\omega_1\omega_2\ldots\omega_n H) - V_{n+1}(\omega_1\omega_2\ldots\omega_n T)}{S_{n+1}(\omega_1\omega_2\ldots\omega_n H) - S_{n+1}(\omega_1\omega_2\ldots\omega_n T)},$$

where n ranges between 0 and $N-1$. If we set $X_0 = V_0$ and define recursively forward in time, the portfolio values $X_1, X_2, \ldots, X_N$ by

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n),$$

then we have

$$X_n(\omega_1\omega_2\ldots\omega_n) = V_n(\omega_1\omega_2\ldots\omega_n),$$

for all n and all $\omega_1\omega_2\ldots\omega_n$.

Markov Process:

Definition:

Consider the binomial asset-pricing model. Let $X_0, X_1, \ldots, X_N$ be an adapted process. If, for every n between 0 and $N-1$ and for every function $f(x)$, there is another function $g(x)$ (depending on n and f) such that

$$\mathbb{E}_n\left[f(X_{n+1})\right] = g(X_n),$$

we say that $X_0, X_1, \ldots, X_N$ is a Markov process.

By definition, $\mathbb{E}_n\left[f(X_{n+1})\right]$ is random and depends on the first n coin tosses. The *Markov property* says that this dependence on the coin tosses occurs through X_n. In other words, the information about the coin tosses that one needs in order to evaluate $\mathbb{E}_n\left[f(X_{n+1})\right]$ is summarized by X_n. We are not so concerned with determining a formula for the function g right now as we are with asserting its existence because its mere existence tells us that if the payoff of a derivative security is random only through its dependence on X_N, then there is a version of the derivative security pricing algorithm in which we do not need to store path information.