
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Example on Spot Rate Arbitrage:

In the same setting as the previous example, suppose that a one year pure discount bond is also traded. It's price today (time zero) has to be,

$$P(0,1) = \frac{100}{1+r_{1y}} = \frac{100}{1.044996} = 95.6942,$$

or else there are arbitrage opportunities. In order to show this fact, suppose that the price of the one-year pure discount bond is 95.00, suggesting that the bond is underpriced. In order to take advantage of this arbitrage opportunity, we should buy the underpriced security and sell the equivalent portfolio.

In this case we buy one unit of the one-year pure discount bond and pay 95.00 and take a short position in 100/103 units of the portfolio described in the previous example (which pays 103 after a year). Recall that the portfolio buys the coupon bond and sells 0.03 units of the six-month pure discount bond. This strategy will initially generate $98.565 \times (100/103) = 95.6942$. After a year it pays $(100/103) \times 103 = 100$, the same as the one year pure discount bond. The combination of the long position in the one-year pure discount bond at the price of 95.00 and the short position in the portfolio behaving like a one-year discount bond at the price of 95.6942 generates a total profit of $95.6942 - 95.00 = 0.6942$. This result means that there is an arbitrage opportunity, that is, we make a positive initial profit and the future profit or loss is zero.

Deriving Points of the Term Structure:

Assume that today's date is 1st January, 2003. We want to determine as many points as possible in today's term structure of interest rates. We know that there are three risk-free bonds with the following characteristics.

1. A pure discount bond that matures on 30th June 2003 is selling today at 98.
2. A 6% bond (that is, it pays annually 6% of its nominal value as coupons) that matures on 30th June 2004 is selling today at 101. The coupon is paid out once a year on 30th June, so that the bond will pay coupons on 30th June 2003 and 30th June 2004.
3. An 8% bond that matures on 30th June 2005 is selling today at 105. The coupon is paid once a year on 30th June.

All the bonds have a face value of 100.

We first compute the six-month interest rate from,

$$98 = \frac{100}{1+r_1} \Rightarrow r_1 = 2.041\%.$$

We want to construct the term structure in annual terms. We can use the simple interest rate so that the annual rate r_{6m} of the six-month bond is $r_{6m} = 2 \times r_1 = 4.082\%$. Similarly we can compute the 18 month interest rate using the already computed rate. We then have,

$$101 = \frac{6}{1.02041} + \frac{106}{1+r_2} \Rightarrow r_2 = 11.438\%.$$

In annual terms, using the simple interest rate we have,

$$r_{18m} = (2/3) \times r_2 = 7.625\%.$$

We proceed in a similar way with the next maturity,

$$105 = \frac{8}{1.02041} + \frac{8}{1.11438} + \frac{108}{1+r_3} \Rightarrow r_3 = 20.025\%.$$

We find the annual rate as,

$$r_{30m} = (2/5) \times r_3 = 8.01\%.$$

Forward Rates:

An alternative way of analyzing the term structure of interest rates is by considering the so called forward rates. Forward rates are determined by the relationship between spot rates of different maturities. Consider, for example, the one-year and the two year spot rates r_{1y} and r_{2y}. The forward rate that connect these two rates, denoted by $f_{1y,2y}$ is given by,

$$(1 + r_{1y})(1 + f_{1y,2y}) = (1 + r_{2y})^2.$$

Effectively, the forward rate $f_{t,u}$ for the period $[t, u]$ is the interest rate for the money invested between the dates t and u in future, but agreed upon today. The preceding is a theoretical construction, and rates obtained this way are sometimes called the implied forward rates. In practice, the actual market forward rates might be different because of various reasons including transaction costs and only an approximate knowledge of the term structure of interest rates. Forward rates are similar to spot rates in their nature, and, therefore the rules about the computation of annual equivalents apply.

Example:

Using the six-month annualized spot rate r_{6m} and the one year spot rate r_{1y}, we get the annualized forward rate $f_{6m,1y}$ between the sixth month and the year as,

$$(1 + r_{6m})^{1/2}(1 + f_{6m,1y})^{1/2} = (1 + r_{1y}).$$

In general, the forward rate $f_{t,u}$ for the period $[t, u]$ with t and u expressed in years is determined from this annualized forward rate formula with annual compounding:

$$(1 + r_{t_y})^t(1 + f_{t,u})^{u-t} = (1 + r_{u_y})^u.$$

Forward Rate Formulas:

1. For yearly compounding with $j > i$,

$$(1 + r_j)^j = (1 + r_i)^i(1 + f_{i,j})^{j-i} \Rightarrow f_{ij} = \left[\frac{(1 + r_j)^j}{(1 + r_i)^i}\right]^{\frac{1}{j-i}} - 1.$$

2. For m-period per year compounding with $j > i$,

$$\left(1 + \frac{r_j}{m}\right)^j = \left(1 + \frac{r_i}{m}\right)^i \left(1 + \frac{f_{i,j}}{m}\right)^{j-i} \Rightarrow f_{ij} = m\left[\frac{(1 + \frac{r_j}{m})^j}{(1 + \frac{r_i}{m})^i}\right]^{\frac{1}{j-i}} - m.$$

3. For continuous compounding with $t_2 > t_1$,

$$e^{r_{t_2}t_2} = e^{r_{t_1}t_1}e^{f_{t_1,t_2}(t_2-t_1)} \Rightarrow f_{t_1,t_2} = \frac{r_{t_2}t_2 - r_{t_1}t_1}{t_2 - t_1}.$$