From the definition of maximal element and best element, it is clear that if an element is a best element of a set, then it is also maximal element. So, the choice set is a subset of maximal set. $C(S, R) \subset M(S, R)$.

A maximal element may not be best element. Suppose $S = \{x, y, z\}$ and $R = \{(x, y), (z, y)\}$. In this example we have xPy and zPy. We do not have xRz and zRx. So The set of best element is empty. We have maximal elements $x, z \in M(S, R)$.

This implies that $M(S, R)$ is not a subset of $C(S, R)$.
We can have situations where both choice set and maximal set is empty.
Suppose $S = \{x, y, z\}$ and $R = \{(x, y), (y, z), (z, x)\}$. We have xPy, yPz and zPx. We dont have any $x \in S$ such that xRy for all $y \in S$. There is always an x such that xPz for all y. So maximal set is empty.

Indifference relation
for all $x, y \in S$, xRy and yRx then xIy

Preference relation
for all $x, y \in S$, xRy and $\sim yRx$ then xPy

Some results on strict preference and indifference

If R is an ordering, then for all $x, y, z \in S$
1) xIy & $yIz \rightarrow xIz$.
2) xPy & $yIz \rightarrow xPz$.
3) xIy & $yPz \rightarrow xPz$.
4) xPy & $yPz \rightarrow xPz$.

Proof of 1). xIy & $yIz \rightarrow$ $(xRy$ & $yRx)$&$(yRz$ &$zRy)$.
$\rightarrow (xRy$ & $yRz)$&$(zRy$ &$yRz)$
$\rightarrow xRz$ & zRx from transitivity
$\rightarrow xIz$.

Proof of 2. xPy & $yIz \to (xRy$ & $\sim yRx)$ & $(yRz$ & $zRy)$.
We have xRy & yRz implying xRz from transitivity.
Suppose zRx it implies xIz. We have yIz. Therefore using the first result we get xIy. We are given xPz. A contradiction. Therefore zRx is not true. We have xRz and $\sim zRx$, it implies xPz.

The proof of 3 is similar to the proof of 2.

Proof of 4. xPy & $yPz \rightarrow (xRy$ & $\sim yRx)$ & $(yRz$ & $\sim zRy)$. It implies $(xRy$ & $yRz)$. It implies xRz from transitivity. Suppose zRx. We get zIx. Take yPz and zIx. Using the second result we have yPx. It is a contradiction because we have xpy. Thus, we cannot have zRx. Therefore, we have xRz and $\sim zRx$. It implies xPz.

Result:
Any finite quasi ordered set has at least one maximal element.

Proof. Suppose the set of maximal set is empty. It implies that that for every $y \in S$ there exist one x such that xPy. Take for example $x_1, x_2, x_3 \in S$. From above we have $x_1 P x_2$, $x_2 P x_3$ and $x_3 P x_1$. This implies that it violates transitivity. Therefore, R is not quasi ordered. This implies that set of maximal element is not empty.

We define choice function.
A choice function $C(S, R)$ defined over S is a functional relation such that the choice set $C(S, R)$ is non-empty for every non empty subset s of S.
It means that there is a non-empty choice set or there are best elements for every non-empty subset of S.
We have seen that whenever completeness is violated, choice set is empty. It implies that choice function does not exist.
If reflexivity is violated then also choice set is empty . For single element set, reflexivity is required for non-empty choice set.

If transitivity is violated, choice set can be empty. Example xPy, yPz and zPx. In this situation choice set is empty for $\{x, y, z\}$. We cannot have a choice function for this subset of S.

If R is an ordering defined over a finite set S, then a choice function $C(S, R)$ is defined over S.

Proof: Suppose the choice set over S is empty. This means that choice function is not defined. If choice set is empty it implies that it violates any one of the following property;
completeness, reflexivity, or transitivity.
This implies that R is not an ordering if anyone of the above property is violated. This implies that choice set is not empty since R is an ordering. Thus, choice function defined fo S.