MA 373 : Financial Engineering II

January - May 2021

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Exercises 1

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1. Use Ito's-formula to write the following stochastic process $Y(t)$ on the standard form
$$dY(t) = b(t, Y(t))dt + \sigma(t, Y(t))dW(t)$$
a) $Y(t) = W(t) + 4t$
b) $Y(t) = W^2(t)$
c) $Y(t) = t^2W(t) - 2\int_0^t sW(s)ds$
d) $Y(t) = e^{W(t)} + t^2 + 1$
e) $Y(t) = (\frac{1}{3}W(t) + a)^3$
f) $Y(t) = e^{ct+\alpha W(t)}$
g) $Y(t) = e^{\int_0^t h(s)dW(s) - \frac{1}{2}\int_0^t h(s)^2 ds}$

2. Solve the following stochastic differential equations:
a) $dX(t) = X(t)dt + dW(t),\ X(0) = x_0$
b) $dX(t) = -X(t)dt + e^{-t}dW(t),\ X(0) = x_0$
c) $dX(t) = rdt + \alpha X(t)dW(t),\ X(0) = x_0$
d) $dX(t) = \frac{1}{2}X(t)dt + X(t)dW(t),\ X(0) = 1$
e) $dX(t) = -\frac{1}{1+t}X(t)dt + \frac{1}{1+t}dW(t),\ X(0) = 0$

3. Compute the stochastic differential for $Z(t)$ when $Z(t) = 1/X(t)$ and $dX(t) = \mu X(t)dt + \sigma X(t)dW(t),\ X(0) = x_0$.

4. The mean-reverting Ornstein-Uhlenbeck process is the solution $X(t)$ of the stochastic differential equation
$$dX(t) = (m - X(t))dt + \sigma dW(t),\ X(0) = x_0,$$
where m, σ are real constants.
a) Solve this equation
b) Find $\mathbb{E}[X(t)]$ and $Var[X(t)] =: \mathbb{E}[(X(t) - \mathbb{E}[X(t)])^2]$.

5. For fixed $a, b \in \mathbb{R}$ consider the following SDE
$$dX(t) = \frac{b - X(t)}{1 - t}dt + dW(t),\ 0 \le t \le 1,\ X(0) = a$$
Verify that
$$X(t) = a(1-t) + bt + (1-t)\int_0^t \frac{dW(s)}{1-s},\ 0 \le t < 1$$
solve the equation. This process is called the Brownian bridge from a to b.

6. Let $f(x)$ be a bounded and continuous. Suppose that $u(t,x)$ is a bounded function that satisfies the partial differential equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) - 2u(t,x),$$

and the initial condition

$$u(0,x) = f(x).$$

Then prove that

$$u(t,x) = \mathbb{E}[e^{-2t} f(W(t))],$$

where the process $\{W(t)\}_{t\geq 0}$ is Brownian motion started at x.

7. Let X, Y satisfy the following system of SDE's

$$\begin{aligned} dX(t) &= \alpha X(t)dt + Y(t)dW(t),\ X(0) = x_0 \\ dY(t) &= \alpha Y(t)dt - X(t)dW(t),\ Y(0) = y_0. \end{aligned}$$

where x_0, y_0 are real constants.
(i) Compute $\mathbb{E}[X(t)]$, $\mathbb{E}[Y(t)]$ and $\mathrm{Cov}(X(t), Y(t)) =: \mathbb{E}[(X(t)Y(t)] - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]$.
(ii) Show that $R(t) = X^2(t) + Y^2(t)$ is deterministic (non-random).

8. Let $f(t), g(t)$, and $h(t)$ be continuous function on $[0,T]$. Show that the solution of the stochastic differential equation

$$dX(t) = (h(t) + g(t)X(t))dt + f(t)dW(t),\ X(0) = x_0,$$

is a Gaussian process. Find the mean function and covariance function.

9. Suppose that the process X has a stochastic differential

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),\ X(0) = x_0,$$

and that $\mu(t)$ is continuous and $\mu(t) > 0$. Show that this implies that X is a submartingale.

10. A function $h(x_1, x_2)$ is said to be harmonic if it satisfies the condition

$$\sum_{i=1}^{2} \frac{\partial^2 h}{\partial x_i^2} = 0.$$

It is subharmonic if it satisfies the condition

$$\sum_{i=1}^{2} \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Let W_1, W_2 be independent standard Brownian motions, and define the process X by $X(t) = h(W_1(t), W_2(t))$. Show that X is a martingale (submartingale) if h is harmonic (subharmonic).