Componentwise Sensitivity and Stability Analysis

Componentwise small perturbations

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Componentwise small perturbations: Let A be any $n \times m$ matrix. A perturbation $A + \delta A$ to A is componentwise small if there exists a small number $0 < \epsilon \ll 1$ such that $|\delta a_{ij}| \leq \epsilon |a_{ij}|$ for all $1 \leq i \leq n$, and $1 \leq j \leq m$.

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Exercise: Prove that if a perturbation to a matrix A is componentwise small, then it is also normwise small.

Componentwise sensitivity analysis

Componentwise sensitivity analysis of the solution of a system of $n \times n$ equations $Ax = b$ is a measure of the change in the solution with respect to perturbations in A and/or b where perturbations are considered in the componentwise manner.

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<u>Perturbing only b:</u>

Suppose

$$A(x + \delta x) = b + \delta b \tag{2}$$

where $|\delta b| \leq \epsilon |b|$ for some $\epsilon > 0$.

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Then $|\delta x| \leq \epsilon |A^{-1}||A||x|$. Exercise!

Taking the $\|\cdot\|_\infty$ norm on both sides, gives

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq \epsilon \||A^{-1}||A|\|_\infty. \tag{3}$$

Componentwise sensitivity analysis

Perturbing only A:

Suppose δA is a perturbation to A such that $|\delta A| \leq \epsilon |A|$ and

$$(A + \delta A)(x + \delta x) = b$$

If $\epsilon < 1/\||A^{-1}||A|\|_\infty$, then

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq \frac{\epsilon \||A^{-1}||A|\|_\infty}{1 - \epsilon \||A^{-1}||A|\|_\infty} \qquad \text{Exercise!} \tag{4}$$

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The quantity $\||A^{-1}||A|\|_\infty$ plays an important role in the analysis. It is called the *skeel condition number of A and is denoted by* $\mathrm{skeel}\, A = \||A^{-1}||A|\|_\infty$.

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Clearly, $\operatorname{skeel} A \leq \kappa_\infty(A)$.

Therefore bounds (3) and (4) can be tighter than those obtained via normwise sensitivity analysis.

Error bounds via componentwise sensitivity and stability analysis

An algorithm for solving a system of equations $Ax = b$ is said to *componentwise backward stable* if the computed solution x_c is the exact solution of a system of equations

$$(A + \delta A)z = b + \delta b$$

where there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ of the order of u such that

$$|\delta A| \leq \epsilon_1 |A| \text{ and } |\delta b| \leq \epsilon_2 |b|.$$

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Theorem Let G be any nonsingular lower (upper) triangular $n \times n$ matrix and b be any nonzero column vector of length n. If y_c be the computed solution of the system $Gw = b$ using any variant of forward (backward) substitution in floating point arithmetic, then y_c satisfies

$$(G + \delta G)y_c = b$$

where δG is an $n \times n$ matrix such that $|\delta G| \leq Cu|G|$ for some modest constant C. If y be the exact solution and $\operatorname{skeel} G < 1/Cu$, then

$$\frac{\|y_c - y\|_\infty}{\|y\|_\infty} \leq \frac{Cu \operatorname{skeel} G}{1 - Cu \operatorname{skeel} G}. \tag{5}$$

Properties of the Skeel condition number

Exercise: Given any nonsingular matrix A and a diagonal nonsingular matrix D, show that $\operatorname{skeel} DA = \operatorname{skeel} A$.

Using the above information, construct an example to show that given any $\epsilon > 0$, there exists a nonsingular matrix A such that $\frac{\kappa_\infty(A)}{\operatorname{skeel} A} \geq 1/\epsilon$.

Further show that if $|\delta A| \leq \epsilon |A|$ where $\epsilon < 1/\operatorname{skeel} A$, then $A + \delta A$ is also nonsingular.