Lecture #30

① $ds = \mu s\,dt + \sigma s\,dW(t)$

Since $V = V(S, t)$, therefore

$$dV = \underbrace{\frac{\partial v}{\partial t}dt}_{T1} + \underbrace{\frac{\partial v}{\partial s}ds}_{T2} + \overbrace{\frac{1}{2}\frac{\partial^2 v}{\partial t^2}(dt)^2}^{T3} + \overbrace{\frac{1}{2}\frac{\partial^2 v}{\partial s^2}(ds)^2}^{T4}$$

$$+ \underbrace{\frac{\partial^2 v}{\partial t \partial s}(ds)(dt)}_{T5} + \underbrace{\text{Higher Order Terms}}_{T6}$$

T1 : $\frac{\partial v}{\partial t}dt$

T2 : $\frac{\partial v}{\partial s}ds = \frac{\partial v}{\partial s}(\mu s\,dt + \sigma s\,dW(t))$

T3 : $\frac{1}{2}\frac{\partial^2 v}{\partial t^2} \cdot (dt)^2 = O(dt)^2$

T4 : $\frac{1}{2}\frac{\partial^2 v}{\partial s^2}(ds)^2 = \frac{1}{2}\frac{\partial^2 v}{\partial s^2}(\mu s\,dt + \sigma s\,dW(t))^2$

$$= \frac{1}{2}\frac{\partial^2 v}{\partial s^2}\left(\mu^2 s^2 dt^2 + \sigma^2 s^2\,dW(t)^2 + 2\mu s\sigma^2\,dt\,dW(t)\right)$$

$$= \frac{1}{2}\frac{\partial^2 v}{\partial s^2}\sigma^2 s^2 dt + O((dt)^{3/2})$$

(Since $dW(t) \cong \sqrt{dt}$).

T5 : $\frac{\partial^2 v}{\partial t \partial s}(ds)(dt) = \frac{\partial^2 v}{\partial t \partial s}(\mu s\,dt + \sigma s\,dW(t))(dt)$

$$= O((dt)^{3/2}) \quad (\text{Since } dW(t) \cong \sqrt{dt})$$

T6 : Higher Order Terms

Adding and neglecting higher order terms

$$dV = \frac{\partial v}{\partial t}dt + \mu s\frac{\partial v}{\partial s}dt + \sigma s\frac{\partial v}{\partial s}dW(t) + \frac{1}{2}\frac{\partial^2 v}{\partial s^2}\sigma^2 s^2 dt$$

$$= \sigma s\frac{\partial v}{\partial s}dW(t) + \left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 v}{\partial s^2} + \mu s\frac{\partial v}{\partial s}\right)dt$$