Lecture 24

Finite Difference Methods for BVP

MA 322: Scientific Computing

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1 Higher Order Scheme: Crank-Nicolson

We consider following simple initial boundary value problem (IBVP)

$$u_t(x,t) - u_{xx}(x,t) = 0, \ \ (x,t) \in (a,b) \times (0,T], \ T < \infty, \tag{1}$$

with boundary conditions

$$u(a,t) = 0 \ \text{ and } \ u(b,t) = 0 \ \forall t > 0 \tag{2}$$

and initial condition

$$u(x,0) = g(x) \ \ x \in [a,b]. \tag{3}$$

As a first step towards numerical approximation, we now divide the computational domain $[a,b] \times [0,T]$ by the following points

$$(x_i, t_j), \ \ x_0 = a, \ x_{i+1} = x_i + h, \ \ldots, x_N = x_{N-1} + h = b \ \ \& \ \ t_0 = 0, \ t_{j+1} = t_j + k, \ \ t_M = T,$$

with mesh parameters

$$h = \frac{b-a}{n} \ \ \& \ \ k = \frac{T}{M}.$$

Already we have seen that backward scheme or forward scheme is of first order in time and second order in space. Now, our objective is to improve the order of convergence along time direction. This is motivated by the following result. For a smooth solution function $G : [a,b] \times [0,T] \to \mathbb{R}$ following result holds true.

Lemma 1.1 *Suppose $t_{j+\frac{1}{2}} = t_j + \frac{k}{2}$ is the mid point of the line joining of t_j and t_{j+1}, then*

$$\begin{aligned} &(a) \qquad G(x_i, t_{j+\frac{1}{2}}) = \frac{G(x_i,t_j) + G(x_i,t_{j+1})}{2} + O(k^2), \\ &(b) \qquad G_t(x_i, t_{j+\frac{1}{2}}) = \frac{G(x_i,t_{j+1}) - G(x_i,t_j)}{k} + O(k^2). \end{aligned}$$

Proof. Using Taylor series expansion, we obtain

$$\begin{aligned} G(x_i, t_{j+1}) &= G(x_i, t_{j+\frac{1}{2}} + \frac{k}{2}) \\ &= G(x_i, t_{j+\frac{1}{2}}) + \frac{k}{2} G_t(x_i, t_{j+\frac{1}{2}}) + \ldots \end{aligned} \tag{4}$$

Similarly, we arrive at

$$\begin{aligned} G(x_i, t_j) &= G(x_i, t_{j+\frac{1}{2}} - \frac{k}{2}) \\ &= G(x_i, t_{j+\frac{1}{2}}) - \frac{k}{2} G_t(x_i, t_{j+\frac{1}{2}}) + \ldots \end{aligned} \tag{5}$$

Adding above relations, we obtain

$$G(x_i, t_{j+1}) + G(x_i, t_j) = 2G(x_i, t_{j+\frac{1}{2}}) + \mathrm{O}(k^2). \tag{6}$$

This completes the proof of part (a). For part (b), we subtract (**??**) from (**??**) to obtain

$$G(x_i, t_{j+1}) - G(x_i, t_j) = kG_t(x_i, t_{j+\frac{1}{2}}) + \mathrm{O}(k^3).$$

This completes the rest of the proof.

Remark: Now, we can think to derive a second order scheme for the parabolic equation. This is known as Crank-Nicolson scheme.

2 Crank-Nicolson Scheme

We consider equation (??) at $(x_i, t_{j+\frac{1}{2}})$ to have

$$u_t(x_i, t_{j+\frac{1}{2}}) - u_{xx}(x_i, t_{j+\frac{1}{2}}) = 0.$$

Thus, we obtain following approximation

$$\frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} - \frac{1}{2}(u_{xx}(x_i, t_j) + u_{xx}(x_i, t_{j+1})) = 0$$

with T.E. of order k^2. For the space derivative, we use symmetric scheme to have

$$\begin{aligned}\frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} - \frac{1}{2}\Bigg(&\frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} \\ &+ \frac{u(x_{i+1}, t_{j+1}) - 2u(x_i, t_{j+1}) + u(x_{i-1}, t_{j+1})}{h^2}\Bigg) = 0,\end{aligned}$$

with T.E. of order k^2 and h^2. Now, using the notation $u(x_i, t_j) \approx u_{i,j}$, we finally have following scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \frac{1}{2}\Big(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\Big) = 0.$$

Now, separating both time levels, we obtain

$$-ru_{i-1,j+1} + (1 + 2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i+1,j} + (1 - 2r)u_{i,j} + ru_{i-1,j}. \tag{7}$$

Setting $i = 1, 2, \ldots, N - 1$ and using the boundary conditions, we have following system of equations

$$\begin{pmatrix} 1+2r & -r & 0 & \ldots & 0 & 0 \\ -r & 1+2r & -r & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & -r & 1+2r & -r \\ \ldots & \ldots & \ldots & \ldots & -r & 1+2r \end{pmatrix} \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{N-1,j+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1-2r & r & 0 & \ldots & 0 & 0 \\ r & 1-2r & r & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & r & 1-2r & r \\ \ldots & \ldots & \ldots & \ldots & r & 1-2r \end{pmatrix} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{N-1,j} \end{pmatrix}.$$

In matrix notation, we have

$$AV^{j+1} = LV^j,$$

where

$$\begin{aligned} A &= \begin{pmatrix} 1+2r & -r & 0 & \ldots & 0 & 0 \\ -r & 1+2r & -r & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & -r & 1+2r & -r \\ \ldots & \ldots & \ldots & \ldots & -r & 1+2r \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix} + r \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & -1 & 2 & -1 \\ \ldots & \ldots & \ldots & \ldots & -1 & 2 \end{pmatrix} \\ &= I + rP = f(P), \end{aligned}$$

where

$$P = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -1 & 2 & -1 \\ \dots & \dots & \dots & \dots & -1 & 2 \end{pmatrix}.$$

Similarly

$$L = I - rP = g(P).$$

Suppose λ_n $(1 \le n \le N-1)$ is an eigenvalue of matrix P, then $f(\lambda_n)$ is an eigenvalue of $f(P) = A$. Similarly $g(\lambda_n)$ is an eigenvalue of $g(P) = L$. Now, eigenvalues of matrix P are

$$\begin{aligned} \lambda_n &= 2 + 2\cos\frac{n\pi}{N}, \quad 1 \le n \le N-1 \\ &= 2(1+\cos 2\theta), \quad \theta = \frac{n\pi}{2N} \\ &= 4\cos^2\theta. \end{aligned}$$

Therefore, eigenvalues for the matrix A are given by $f(\lambda_n) = 1 + r\lambda_n \ge 1$. Thus, A^{-1} exists and the approximate solution at $j+1$ level is given by

$$V^{j+1} = A^{-1}LV^j = MV^j,$$

with eigenvalues of matrix $M = A^{-1}L$ are given by

$$\frac{1 - r\lambda_n}{1 + \lambda_n}$$

so that

$$\|M\|_2 \le 1.$$

Hence, Crank-Nicolson scheme is unconditionally stable.