Lecture 19

Numerical Approximations for IVP

MA 322: Scientific Computing

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1 Linear Multistep Methods

While RungeKutta methods give an improvement over Eulers method in terms of accuracy, this is achieved by investing additional computational effort; in fact, RungeKutta methods require more evaluations of $f(\cdot,\cdot)$ than would seem necessary. For example, the fourth-order method involves four function evaluations per step. For comparison, by considering three consecutive points $x_{n-1},\ x_n = x_{n-1}+h,\ x_{n+1} = x_{n-1}+2h$, integrating the differential equation $y' = f(x,y)$ between x_{n-1} and x_{n+1}, yields

$$y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(x, y(x))dx, \tag{1}$$

and applying Simpsons rule to approximate the integral on the righthand side then leads to the method

$$y_{n+1} = y_{n-1} + \frac{h}{3}\Big[f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})\Big], \tag{2}$$

requiring only three function evaluations per step. In contrast with the one-step methods considered in the previous section where only a single value y_n was required to compute the next approximation y_{n+1}, here we need two preceding values, y_n and y_{n-1}, to be able to calculate y_{n+1}, and therefore (2) is not a one-step method. In this lecture we consider a class of methods of the type (2) for the numerical solution of the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0, \tag{3}$$

called linear multistep methods. Further, due to implicit dependence on y_{n+1} the method is then called implicit. The method (2) is called linear because it involves only linear combinations of the y_{n+j} and the $f(x_{n+j}, y_{n+j})$. For the sake of notational simplicity, henceforth we shall often write f_n instead of $f(x_n, y_n)$. Remark:

- The method (2) is known as classical Milne's corrector method. Like Improved Euler method, we predict y_{n+1} and then correct/improved it using Milne's corrector method (2). Therefore, we need Milne's predictor method.

1.1 Milne's Predictor Method

Integrating the differential equation $y' = f(x,y)$ between x_{n-3} and x_{n+1}, yields

$$y(x_{n+1}) = y(x_{n-3}) + \int_{x_{n-3}}^{x_{n+1}} f(x, y(x))dx, \tag{4}$$

Then use Lagrange polynomial approximation for $g(x) = f(x, y(x))$ based on four mesh points

$$(x_{n-3}, f_{n-3}),\ (x_{n-2}, f_{n-2}),\ (x_{n-1}, f_{n-1}),\ (x_n, f_n)$$

to have

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\big(2f_{n-2} - f_{n-1} + 2f_n\big), \tag{5}$$

which is an explicit multistep method.

Example 1.1 Consider $y' = 1 + y^2,\ \ y(0) = 0$. Find approximations at $0.2,\ 0.4$ and 0.6 using RK-4 method. Then using Milne's method, evaluate the approximation at 0.8 and 1.0.

Solution: Recall classical fourth-order method:

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned}
k_1 &= f(x_n, y_n), \\
k_2 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1), \\
k_3 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2), \\
k_4 &= f(x_n + h, y_n + hk_3).
\end{aligned}$$

We take $h = 0.2$, so that grid points are

$$x_0 = 0,\ x_1 = x_0 + h = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,\ x_5 = 1.0,\ \ldots.$$

Then calculate $y_1,\ y_2$ and y_3. In fact

$$y_1 = 0.2027,\ y_2 = 0.4228,\ y_3 = 0.6841.$$

Using Milne's predictor method (5), we obtain

$$\begin{aligned}
y_4 &= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \\
&= 0 + \frac{4 \times 0.2}{3}\Big(2f(x_1, y_1) - f(x_2, y_2) + 2f(x_3, y_3)\Big) = 1.0239.
\end{aligned}$$

Then, to correct the value y_4, apply Milne's corrector method (2) so that

$$\begin{aligned}
y_4 &= y_3 + \frac{h}{3}\Big[f(x_2, y_2) + 4f(x_3, y_3) + f(x_4, y_4)\Big] && (6) \\
&= 1.0294 && (7)
\end{aligned}$$

1.2 Adams Methods

First observe that

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))dx = \int_{x_n}^{x_{n+1}} g(x)dx. \tag{8}$$

Adams methods are based on the idea of approximating the integrand with a polynomial within the interval (x_n, x_{n+1}). Using a pth order polynomial results in a $p+1$th order method. There are two types of Adams methods, the explicit and the implicit types. The explicit type is called the Adams-Bashforth (AB) methods and the implicit type is called the Adams-Moulton (AM) methods. AB method is used as predictor method and AM method is used as corrector method.
Case-1: Fourth order Adams-Bashforth Method:

- In this case we wish to approximate $g(x)$ by a polynomial of degree 3 so that (8) yields a fourth order method. Since we are expecting an explicit scheme, we approximate $g(x)$ taking $x_{n-3},\ x_{n-2},\ x_{n-1}$ and x_n as grid point. Using Newton's backward interpolation

$$\begin{aligned}
g(x) \approx\ & g(x_n) + \frac{(x - x_n)}{h}\nabla g(x_n) + \frac{(x - x_n)(x - x_{n-1})}{2!h^2}\nabla^2 g(x_n) \\
& + \frac{(x - x_n)(x - x_{n-1})(x - x_{n-2})}{3!h^3}\nabla^3 g(x_n) + \\
& + \frac{(x - x_n)(x - x_{n-1})(x - x_{n-2})(x - x_{n-3})}{4!h^4}\nabla^4 g(x_n)
\end{aligned}$$

  Substitute above approximation in (8) to obtain following approximation

$$y_{n+1} = y_n + \frac{h}{24}\Big(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\Big) \tag{9}$$

Case-2: Fourth order Adams-Moulton Method:

- Since we are expecting an implicit scheme, we approximate $g(x)$ taking $x_{n-2},\ x_{n-1},\ x_n$ and x_{n+1} as grid point. The derivation is exactly the same for the Adams-Bashforth method. In this case, the scheme is given by

$$y_{n+1} = y_n + \frac{h}{24}\Big(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\Big) \tag{10}$$