Pareto Principle (Condition P): For any $x, y \in X$, for all i
$xP_iy \rightarrow xPy$.

If everyone prefers x to y then society must also prefer x to y. The social welfare function violates Pareto principle when everybody prefer x to y and the social relation gives y is preferred to x.

Example

1	2	3
x	y	x
y	z	y
z	x	z

$$f\begin{pmatrix} x & y & y \\ y & z & y \\ z & x & z \end{pmatrix} = \begin{pmatrix} x \\ yz \end{pmatrix}$$

Independence of irrelevant alternatives (Condition I): let R and $R^{'}$ be the social binary relations determined by f corresponding respectively to two sets of indidividual preferences, $(R_1, R_2....R_n)$ and $R_1^{'}, R_2^{'}, ..., R_n^{'})$. If for all pairs of alternatives x, y in a subset of S of X, $xR_iy \leftrightarrow xR_i^{'}y$, for all i then $C(S, R)$ and $C(S, R^{'})$ are the same.
If for any two alternatives x and y, the two binary relations R_i and $R_i^{'}$ have same relations then social preference must be same over x and y.

Borda Rule:

The preferences of 3 individuals over $\{x, y, z\}$ are;

1	2	3
x	y	x
y	z	z
z	x	y

In Borda rule rule assign number integers to each alternatives of individuals. The indiividuals have report their full preference ordering. Since there are three alternatives, so 2 points is given to the best alternatives, 1 is given to the second position and 0 is given to the last position. We aggregate the scores of each alternatives based on the individual preference ordering. The social preference relation is based on the aggregate score.

1	2	3
x 2	y 2	x 2
y 1	z 1	z 1
z 0	x 0	y 0

The aggregate score of x is $2+0+2=4$

$y: \; 1+2+0=3$

$z: 0+1+1=2$

The social preference relation based on these scores is $xPy, \; yPz$ and xPz. This is Borda outcome.

$$f\begin{pmatrix} x & y & x \\ y & z & z \\ z & x & y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Take another preference profile of three individuals, name is B and previous one is A

1	2	3
x	y	x
y	z	zy
z	x	

If we consider preference profile of A and B over the alternative x, y, we see that they are same.

$xP_1y,\ yP_2x,\ xP_3y$ in A.

$xP_1^{'}y,\ yP_2^{'}x,\ xP_3^{'}y$ in B.

Only difference between A and B is between z, and y for individual 3.

According to independence of irrelevant alternatives, the social preference relation between x and y should not be different in case of A and B.

Borda count of B is

1	2	3
x 2	y 2	x 2
y 1	z 1	zy 1
z 0	x 0	

So $x; 2+0+2=4$
$y: 1+2+1=4$
$z: 0+1+1=2$.
The social preference relation based on these scores is xIy, xPz and yPz.
We see that the social preference relation between x, y is not same in A and B. Therefore Borda rule violates independence of irrelevant alternative condition.

Majority rule: it satisfies independence of irrelevant alternatives.
Individual 1: xyz, Individual 2: zxy, Individual 3: yxz
$n(xpy) = 2$ and $n(yPx) = 1$.
Now change the position of z.
Like In Individual 1 $x(yz)$, zxy, $(zx)y$.
Individual 2 $x(yz)$, $(zx)y$, xzy.
Individual 3: yzx, zyx, $(yz)x$ and $y(xz)$. With any of these changes in the preferences of z, $n(xpy) = 2$ and $n(yPx) = 1$ remain same.

Non-dictatorship (Condition D); There is no individual i such that for every element in the domain of rule f, for all $x, y \in X$, $xP_iy \ \rightarrow xPy$.

There should not any individual such that the social preference is based on the preference of this person.
For example if xP_1y and yP_ix for all i except 1 and social preference is xPy. It violates non dictatorship.
Further we assume that there are atleast two individuals and atleast three alternatives.

A set of individuals V is almost decisive for x against y if xPy whenever xP_iy for every $i \in V$. and yP_ix for every $i \notin V$.
For example

$$\begin{array}{ccc} 1 & 2 & 3 \\ x & y & x \\ y & z & z \\ z & x & y \end{array} \qquad f\begin{pmatrix} x & y & x \\ y & z & z \\ z & x & y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For x, y individual 1 and 3 forms a almost decisive set.

A set of individuals V is decisive for x against y if xPy when xP_iy for every $i \in V$.

1	2	3
x	z	x
y	yx	z
z		y

$$f\begin{pmatrix} x & z & x \\ y & yx & z \\ z & & y \end{pmatrix} = \begin{pmatrix} x \\ yz \\ \ \end{pmatrix}$$

Individual 1 and 3 forms a decisive set for x, y. It is almost decisive for x, z.

Suppose there is an individual J who is almost decisive over x and y and denote it by $D(x, y)$. $D(\bar{x}, y)$ denotes that J is decisive over x, y.