
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Cash Flows:

According to the broad interpretation, an investment is defined in terms of its resulting cash flow sequence- the amounts of money that will flow to and from an investor over time. Usually these cash flows (either positive or negative) occur at known specific dates, such as at the end of each quarter or at the end of each year. The stream can then be described by listing the flow at each of these times. This is simplest if the flows are known deterministically. The magnitude of future cash flows can be uncertain.

The Basic Theory of Interest:

Interest is frequently called the *time value of money*. We will outline the basic elements of interest rate theory. The relative return of risk free securities (bonds or bank accounts) is called interest rate.

Computation of Interest Rates:

Consider a pure discount bond that sells today at a price $P(0)$ and matures at time T with nominal payment of $P(T)$. We will say that the bond's interest rate is the value r that solves,

$$P(0)(1+r) = P(T) \Rightarrow r = \frac{P(T) - P(0)}{P(0)}.$$

This is the interest rate in a single period model. In a multiperiod model this would be the way to compute the interest rate per period. Traditionally interest rates have been classified as simple or compound, depending on whether interest is paid on the interest received, or not.

Simple Interest:

The general rule for simple interest is that if an amount $P(0)$ is left in an account at simple interest r, the value after T years is,

$$P(T) = P(0)(1 + rT).$$

If the proportional rule holds for fractional years, then after any time t (measured in years), the account value is,

$$P(t) = P(0)(1 + rt).$$

Note that the account grows linearly with time.

Compound Interest:

In case of compound interest, the interest is paid on accumulated interest as long as the accumulated interest is not withdrawn from the investment. Under yearly compounding, an amount of $P(0)$, left in an account, after T years will grow to,

$$P(T) = P(0)(1+r)^T.$$

When two interest rates refer to the payments of different frequencies, it is important to be able to compare them and see which one results in higher returns. In order to make this comparison, the common practice is to transform the rates into the corresponding rates for a common time interval, usually a year. That is, the rate will be converted into an “annual” or “annualized” interest rate.

Typically, in practice, a bank will quote a nominal rate (for example 6%). If the bank does the compounding on a quarterly basis, the nominal rate of 6% means that the quarterly rate r_Q is $6/4 = 1.5\%$. The actual annual rate of interest called the “effective annual interest rate” (denoted r_{3m}), that corresponds to the quarterly compound rate r_Q, is determined from,

$$(1 + r_{3m}) = (1 + r_Q)^4.$$

With $r_Q = 1.5\%$ we get $r_{3m} = 6.1364\%$. Thus the effective rate is higher than the nominal rate. In case of semiannual compounding, the semiannual rate r_S and annual rate r_{6m} we have,

$$(1 + r_{6m}) = (1 + r_S)^2.$$

In our example $r_S = 3\% \Rightarrow r_{6m} = 6.09\%$. Compounding can be carried out with any frequency. The general method is that a year is divided into equally spaced periods - say n periods. The interest rate for each of the n periods is thus r/n, where r is the nominal annual rate. After a full year of n periods, the account grows by a factor $(1 + r/n)^n$. The effective interest rate is the number r' that satisfies,

$$(1 + r') = \left(1 + \frac{r}{n}\right)^n.$$

<u>Continuous Compounding:</u>

The continuously compounded interest rate is the rate paid at infinite frequency. That is, we say that the bank pays a continuous rate r when the effective rate is r' and obtained from

$$(1 + r') = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r.$$

For example, if the quoted rate is 6%, then the effective annual rate is $r = e^{0.06} - 1 = 6.1837\%$.
If the continuous rate r is paid over a period T different from a year, the value of $P(0)$ invested today will grow to

$$P(T) = P(0)e^{rT},$$

where T represents the time period in years.

<u>Present Value:</u>

A topic of great importance is the derivation of criteria that would allow us to choose today between two different future payments or cash flow of several payments. A typical approach is to assign to each cash flow its today’s value, and compare those values. For the time being, we consider the simplest case of deterministic payments. We are interested in transforming the future payments into a value today. This value, the today’s worth of the future payments, is known as the <u>present value</u>.

<u>Definition:</u>

Present value of a future payment is that amount which, when invested today at a given interest rate, would result in the given value of the future payment. The present value $V(0)$ is obtained as,

$$V(0) = \frac{V}{1 + r},$$

where V is the value of the future payment and r is the interest rate per period in question. We say that the future payoff amount V is discounted, since the present value is less than the future payment V. The factor $d = \dfrac{1}{1 + r}$ by which we multiply the future payoff value is called the discount factor.
Consider now the case of a payment at a future date different from a year. The corresponding interest rate will be expressed in its equivalent annual rate. If the payment takes place in $1/n$ years and r is the corresponding equivalent annual rate, the discount factor is

$$d = \frac{1}{(1 + \frac{r}{n})}$$

in case of simple interest, and

$$d = \frac{1}{(1+r)^{\frac{1}{n}}}$$

in case of compound interest. In case of continuous compounding with a continuous rater r_C and a period of t years the discount factor is,

$$d = e^{-r_C t}.$$

<u>Present and Future Values of Cash Flow:</u>

The previous discussion is easily extended to a stream of several payments, a <u>cash flow</u>. Consider a time interval $[0, T]$ consisting of m equally spaced periods with interest rate r per period. Suppose that we receive payments of $P_0, P_1, \ldots, P_{m-1}$ at the end of period $1, 2, \ldots, m-1$, respectively, and invest them immediately upon receipt. Further, at moment $T(\equiv m)$ we receive the payment P_m. Note that $P_i < 0$ means borrowing.

Then, the future value of cash flow computed every period is given by,

$$P(T) = P_0(1+r)^m + P_1(1+r)^{m-1} + \cdots + P_m.$$

In case of continuous compounding we replace $(1+r)^i$ with $e^{r(T-t_i)}$, where t_i is the time of payment P_i.

Similarly, we can extend the notion of present value to cash flows of several payments. Given a cash flow of payments with values $V_0, V_1, \ldots, V_m$, where V_i is the payment at the end of the i-th period, the present value of the cash flow (compounded every period) is,

$$V(0) = V_0 + \frac{V_1}{1+r} + \cdots + \frac{V_m}{(1+r)^m}.$$

We replace $(1+r)^i$ with e^{rt_i} for continuous compounding, where t_i is the time of payment V_i.

It can be checked that the present value $V(0)$ and the future value $V(T)$ of cash flows $V_0, V_1, \ldots, V_m$ satisfy the following relation:

$$V(0) = \frac{V(T)}{(1+r)^m}.$$