Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Example 2:

We consider the lookback option again. For time n, the price of the option can be represented as a function of the stock price S_n and the maximum stock price $M_n = \max_{0 \le k \le n} S_k$ to date. The set (S_3, M_3) has six possible values, namely, $(32, 32)$, $(8, 16)$, $(8, 8)$, $(2, 8)$, $(2, 4)$ and $(0.50, 4)$.

We define $v_3(s, m)$ to be the payoff of the option at time $t = 3$, if $S_3 = s$ and $M_3 = m$. We have $v_3(32, 32) = 0$, $v_3(8, 16) = 8$, $v_3(8, 8) = 0$, $v_3(2, 8) = 6$, $v_3(2, 4) = 2$ and $v_3(0.50, 4) = 3.50$.

In general, let $v_n(s, m)$ denote the value of the option at time n if $S_n = s$ and $M_n = m$. Then the algorithm in the Theorem can be rewritten in terms of v_n as,

$$v_n(s, m) = \frac{2}{5}\left[v_{n+1}\left(2s, m \vee (2s)\right) + v_{n+1}\left(\frac{1}{2}s, m\right)\right],$$

where $m \vee (2s)$ denotes the maximum of m and $2s$. Accordingly we compute,

$$\begin{aligned}
v_2(16, 16) &= \frac{2}{5}\left[v_3(32, 32) + v_3(8, 16)\right] = 3.20, \\
v_2(4, 8) &= \frac{2}{5}\left[v_3(8, 8) + v_3(2, 8)\right] = 2.40, \\
v_2(4, 4) &= \frac{2}{5}\left[v_3(8, 8) + v_3(2, 4)\right] = 0.80, \\
v_2(1, 4) &= \frac{2}{5}\left[v_3(2, 4) + v_3(0.50, 4)\right] = 2.20.
\end{aligned}$$

Then we compute

$$\begin{aligned}
v_1(8, 8) &= \frac{2}{5}\left[v_2(16, 16) + v_2(4, 8)\right] = 2.24, \\
v_1(2, 4) &= \frac{2}{5}\left[v_2(4, 4) + v_2(1, 4)\right] = 1.20.
\end{aligned}$$

Then we obtain,

$$v_0(4, 4) = \frac{2}{5}\left[v_1(8, 8) + v_1(2, 4)\right] = 1.376.$$

Finally we obtain the number of stocks held in the replicating portfolio as,

$$\delta_n\left(s, m\right) = \frac{v_{n+1}\left(2s, m \vee (2s)\right) - v_{n+1}\left(\frac{1}{2}s, m\right)}{2s - \frac{1}{2}s}.$$

Probability Theory on Coin Toss Space:

Finite Probability Space:

A finite probability space is used to model a situation in which a random experiment with finitely many possible outcomes is conducted.

Definition:

A finite probability space consists of a sample space Ω and a probability measure $\mathbb{P}$. The sample space Ω is a nonempty finite set and the probability measure $\mathbb{P}$ is a function that assigns to each element ω of Ω a number in $[0,1]$ so that

$$\sum_{\omega\in\Omega}\mathbb{P}(\omega)=1.$$

An event is a subset of Ω, and we define the probability of an event A to be:

$$\mathbb{P}(A)=\sum_{\omega\in\Omega}\mathbb{P}(\omega).$$

Random Variables, Distributions, and Expectations:

A random experiment generally generates numerical data, which gives rise to the concept of a random variable.

Definition:

Let $(\Omega,\mathbb{P})$ be a finite probability space. A random variable is a real-valued function defined on Ω. (We sometimes also permit a random variable to take the values $+\infty$ and $-\infty$).

Example:

Recall the Example of the sample space Ω of three independent coin-tosses. We define the stock prices as:

$$\begin{aligned}
S_0(\omega_1\omega_2\omega_3) &= 4 \text{ for all } \omega_1\omega_2\omega_3\in\Omega,\\
S_1(\omega_1\omega_2\omega_3) &= \begin{cases} 8, \text{ if } \omega_1=H,\\ 2, \text{ if } \omega_1=T,\end{cases}\\
S_2(\omega_1\omega_2\omega_3) &= \begin{cases} 16, \text{ if } \omega_1=\omega_2=H,\\ 4, \text{ if } \omega_1\neq\omega_2,\\ 1, \text{ if } \omega_1=\omega_2=T,\end{cases}\\
S_3(\omega_1\omega_2\omega_3) &= \begin{cases} 32, \text{ if } \omega_1=\omega_2=\omega_3=H,\\ 8, \text{ if there are two heads and one tail },\\ 2, \text{ if there is one head and two tails},\\ 0.5, \text{ if } \omega_1=\omega_2=\omega_3=T.\end{cases}
\end{aligned}$$

Here S_0 is not actually random and is sometimes called a *degenerate random variable.* It is customary to write the argument of the random variable as ω, even when ω is a sequence such as $\omega=\omega_1\omega_2\omega_3$. Note that these two notations will be used interchangeably, depending on the context of usage.

According to the definition, a random variable is a function that maps a sample space Ω to the real numbers. The *distribution* of a random variable is a specification of the probabilities that the random variable takes various values. A random variable is not a distribution and a distribution is not a random variable. We consider an example to illustrate this.

Example:

Let us consider the outcome of a coin being tossed three times, with the set of possible outcomes being

$$\Omega=\{HHH,HHT,HTH,HTT,THH,THT,TTH,TTT\}.$$

Define the random variables X and Y to denote the total numbers of heads and tails, respectively. Accordingly, $X(HHH) = 3$, $X(HHT) = X(HTH) = X(THH) = 2$, $X(HTT) = X(THT) = X(TTH) = 1$, $X(TTT) = 0$ and $Y(TTT) = 3$, $Y(TTH) = Y(THT) = Y(HTT) = 2$, $Y(THH) = Y(HTH) = Y(HHT) = 1$, $Y(HHH) = 0$. At this point we do not need to have the information about the probabilities of various outcomes so as to specify these random variables. However, once the probability measure on Ω is specified, we can enumerate the distribution for X and Y.

For example, if we specify the probability measure $\widetilde{\mathbb{P}}$ under which the probability of head on each toss is $\frac{1}{2}$ and the probability of each element in Ω is $\frac{1}{8}$. Then

$$\begin{aligned}
\widetilde{\mathbb{P}}\{\omega \in \Omega;\ X(\omega) = 0\} &= \widetilde{\mathbb{P}}(TTT) = \frac{1}{8} \\
\widetilde{\mathbb{P}}\{\omega \in \Omega;\ X(\omega) = 1\} &= \widetilde{\mathbb{P}}(HTT, THT, TTH) = \frac{3}{8} \\
\widetilde{\mathbb{P}}\{\omega \in \Omega;\ X(\omega) = 2\} &= \widetilde{\mathbb{P}}(HHT, HTH, THH) = \frac{3}{8} \\
\widetilde{\mathbb{P}}\{\omega \in \Omega;\ X(\omega) = 3\} &= \widetilde{\mathbb{P}}(HHH) = \frac{1}{8}.
\end{aligned}$$

For brevity, the notation $\widetilde{\mathbb{P}}\{\omega \in \Omega;\ X(\omega) = j\}$ will be represented as $\widetilde{\mathbb{P}}\{X = j\}$, which refers to the probability of a subset of Ω, the set of elements ω for which $X(\omega) = j$. Under $\widetilde{\mathbb{P}}$, the probability that X takes the four values 0, 1, 2 and 3 are:

$$\widetilde{\mathbb{P}}\{X = 0\} = \frac{1}{8},\ \widetilde{\mathbb{P}}\{X = 1\} = \frac{3}{8},\ \widetilde{\mathbb{P}}\{X = 2\} = \frac{3}{8},\ \widetilde{\mathbb{P}}\{X = 3\} = \frac{1}{8}.$$

Now the random variable Y is different from X, since it counts tails rather than heads. However, under $\widetilde{\mathbb{P}}$, the distribution of Y is the same as the distribution of X:

$$\widetilde{\mathbb{P}}\{Y = 0\} = \frac{1}{8},\ \widetilde{\mathbb{P}}\{Y = 1\} = \frac{3}{8},\ \widetilde{\mathbb{P}}\{Y = 2\} = \frac{3}{8},\ \widetilde{\mathbb{P}}\{Y = 3\} = \frac{1}{8}.$$

On the other hand, suppose we choose a probability measure $\mathbb{P}$ for Ω wherein, the probability of a head on each toss is $\frac{2}{3}$ and that of a tail on each toss is $\frac{1}{3}$. Then

$$\mathbb{P}\{X = 0\} = \frac{1}{27},\ \mathbb{P}\{X = 1\} = \frac{6}{27},\ \mathbb{P}\{X = 2\} = \frac{12}{27},\ \mathbb{P}\{X = 3\} = \frac{8}{27}.$$

Note that the random variable X is the same as before, but has a different distribution under $\mathbb{P}$, than under $\widetilde{\mathbb{P}}$. Accordingly, we have

$$\mathbb{P}\{Y = 0\} = \frac{8}{27},\ \mathbb{P}\{Y = 1\} = \frac{12}{27},\ \mathbb{P}\{Y = 2\} = \frac{6}{27},\ \mathbb{P}\{Y = 3\} = \frac{1}{27}.$$

Again, note that the random variable Y is the same as before, but has a different distribution under $\mathbb{P}$, than under $\widetilde{\mathbb{P}}$.