Completeness can be violated. For example xP_1y and yP_2x then $x\bar{R}y$ and $y\bar{R}x$ are not defined.

Result:
A necessary and sufficient condition for $\bar{R}$ to be an ordering and for $R = \bar{R}$ to be decisive collective choice rule is that for all $x, y \in X$ if there exist i such that xP_iy then xR_jy for all j.

Proof: For any pair x, y if xI_iy for all i, then the condition is trivially true. If for i xP_iy then for all j we must have xR_jy. It implies $x\bar{R}y$.

If condition is violated than there exist i such that xP_iy and there exist j such that yP_jx. It implies that $x\bar{R}y$ is not possible and $y\bar{R}x$ is not possible. So $\bar{R}$ is not complete.

We define Pareto Optimality:
For any n tuple of individual preferences $(R_1, R_2, ..., R_n)$, a state $x \in X$ is Pareto-optimal in X if and only if $\sim [\exists y \in X : y\bar{P}x]$.

A Pareto-optimal state is also called economically efficient.
It also implies that if an alternative or social state is preferred over other alternatives by all the individuals then society should also prefer it over other alternatives.
If xP_iy for all i then society should prefer x over y.
If x is not Pareto optimal it means that there exist an alternative Y which is preferred to x by all individuals.

Result:
For every set of individual preferences $(R_1, R_2, ...R_n)$ over any finite set of social states X, there is atleast one Pareto optimal state.

Proof: The binary relation $\bar{R}$ already defined is Pareto relation. We have already shown that $\bar{R}$ is quasi ordering. It is due to R_i being orderings. And Pareto optimal set is the maximal set of X with respect to $\bar{R}$. So set of Pareto optimal set is $M(X, \bar{R})$. We have also proved that Maximal set is non-empty when X is finite and the binary relation is quasi ordering. Since $\bar{R}$ is quasi ordering and X is finite, so $M(X, \bar{R})$ is non-empty.

Example: $X = \{x, y, z\}$.
Preference of the individuals:

1	2	3
x	x	x
z	zy	y
y		z

Here in this example x is only Pareto optimal state. y is not Pareto optimal because xPy for all individuals. Similarly z is not Pareto optimal, xPz for all individuals.

Another example

$X = \{x, y, z\}$.

Individual preferences:

1	2	3
x	y	x
y	x	z
z	z	y

Here in this example x and y are Pareto optimal state. z is not Pareto optimal because xPz for all individuals.

Assumptions

A social welfare function (SWF) is a collective choice rule f,, the range of which is restricted to the set of ordering over X.
We impose certain conditions on this social welfare function f.

Unrestricted domain (Condition U); The domain of the rule f must include all logically possible combinations of individual orderings.
f must have an image for all possible individual orderings over social states.
Example: $X = \{x, yz\}$. Examples of possible orderings are

1	2	3
x	y	z
y	z	x
z	x	y

1	2	3
z	xyz	y
y		z
x		x

1	2	3
y	x	zx
z	z	y
x	y	

Take the example of Pareto principle, we have $\bar{R}$ based on R_i. This $\bar{R}$ is defined if it is valid for all individuals. If we have

$$\begin{array}{ccc} 1 & 2 & 3 \\ x & y & z \\ y & z & x \\ z & x & y \end{array}.$$

We dont have $x\bar{R}y$, $x\bar{R}y$, $z\bar{R}y$, $y\bar{R}z$, $x\bar{R}z$ $z\bar{R}x$.

$\bar{R}$ is not complete, in this case.

Majority decision rule:
It is a collective choice rule where x is socially at least s good as y if and only if at least as many people prefer x to y as prefer y to x that $N(xPy) \geq N(yPx)$.

It violates unrestricted domain.
Take example , 6 individuals, xP_iy for 4 individuals, yP_iz for 4 individuals and zp_ix for 3 individuals. The social welfare violates transitivity.

Pareto Principle (Condition P): For any $x, y \in X$, for all i
$xP_iy \to xPy$.

If everyone prefers x to y then society must also prefer x to y. The social welfare function violates Pareto principle when everybody prefer x to y and the social relation gives y is preferred to x.

Independence of irrelevant alternatives (Condition I): let R and $r^{'}$ be the social binary relations determined by f corresponding respectively to two sets of indidividual preferences, $(R_1, R_2....R_n)$ and $R_1^{'}, R_2^{'}, ..., R_n^{'})$. If for all pairs of alternatives x, y in a subset of S of X, $xR_iy \leftrightarrow xR_i^{'}y$, for all i then $C(S, R)$ and $C(S, R^{'})$ are the same.

If for any two alternative x and y, the two binary relations R_i and $R_i^{'}$ have same relations then social preference must be same over x and y.