Lecture 17

Numerical Approximations for IVP

MA 322: Scientific Computing

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1 Consistency of Euler Method

Consistency of a numerical method is related to the truncation error. If the truncation error tends to zero when the mesh parameter tends to zero, then the numerical scheme is called consistent. Recall IVP

$$y'(x) = f(x, y(x)), \;\; y(x_0) = y_0. \tag{1}$$

In operator notation, above equation can be written as

$$(T(y))(x) = f(x, y(x)), \;\; T(y) = y'. \tag{2}$$

In particular at $x = x_n$, we obtain

$$y'(x_n) = f(x_n, y(x_n)), \;\; y(x_0) = y_0. \tag{3}$$

Now, expanding $y(x_{n+1})$ about x_n, we have

$$\begin{aligned} y(x_{n+1}) = y(x_n + h) &= y(x_n) + hy'(x_n) + \frac{h^2}{2} y''(\xi_n), \;\; x_n \leq \xi_n \leq x_{n+1}, \\ &= y(x_n) + hy'(x_n) + \mathrm{O}(h^2). \end{aligned} \tag{4}$$

Above equation (4) yields

$$y'(x_n) = \frac{y(x_{n+1}) - y(x_n)}{h} + \mathrm{O}(h) \approx \frac{y(x_{n+1}) - y(x_n)}{h}. \tag{5}$$

Therefore

$$y'(x_n) \approx \frac{y(x_{n+1}) - y(x_n)}{h}. \tag{6}$$

Hence, at $x = x_n$, the equation in (1) can be approximated by following equation

$$\frac{y(x_{n+1}) - y(x_n)}{h} = f(x_n, y(x_n)). \tag{7}$$

It is important to note that above approximation leads to Euler scheme. In operator notation, above equation (7) can be written as

$$(T_h(y))(x_n) = f(x_n, y(x_n)), \;\; (T_h(y))(x_n) = \frac{y(x_{n+1}) - y(x_n)}{h}. \tag{8}$$

Finally, at each grid point we have following error

$$((T - T_h)(y))(x_n) = (T(y))(x_n) - (T_h(y))(x_n) = y'(x_n) - \frac{y(x_{n+1}) - y(x_n)}{h}. \tag{9}$$

Above error is known as truncation error. Therefore, using (6), the truncation error for the Euler method is given by

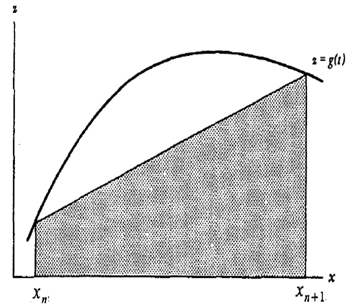
$$((T - T_h)(y))(x_n) = \mathrm{O}(h), \tag{10}$$

which tends to 0 as $h \to 0$. Hence, Euler method is consistent.

Remark:

- Above technique will be frequently used while discussing truncation errors for the approximations differential equations (ODE/ PDE). Thus, consistency means the convergence of the scheme that is $T - T_h$, which can be measured with respect to
  - the operator norm: $\|T - T_h\|$,
  - the norm in suitable function space (eg. $C[a, b]$, $C^1[a, b]$ etc.): $\|(T - T_h)(y)\|$,
  - the norm in $\mathbb{R}$: $\|((T - T_h)(y))(x)\| = |((T - T_h)(y))(x)|$,
- Another issue which is related with the round of error. During computation, the input data is always influenced by round of error. So, a numerical scheme is called stable if the round of error does not grow exponentially. In laymen sense, you should have controlled over round of error. This will be discussed in a separate lecture.

2 Modified Euler Method



Integrate $y' = f(x, y(x))$ over $[x_n, x_{n+1}]$ to have

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))dt = y(x_n) + \int_{x_n}^{x_{n+1}} g(t)dt. \quad (11)$$

Then consider the Trapezoidal rule

$$\int_{x_n}^{x_{n+1}} g(t)dt \approx \frac{x_{n+1} - x_n}{2}\big(g(x_n) + g(x_{n+1})\big). \quad (12)$$

Applying (12) in the equation (11), we obtain

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))dt = y(x_n) + \int_{x_n}^{x_{n+1}} g(t)dt \\ &\approx y(x_n) + \frac{x_{n+1} - x_n}{2}\big(g(x_n) + g(x_{n+1})\big) \\ &= y(x_n) + \frac{h}{2}\big(f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\big). \end{aligned} \quad (13)$$

Again from the Euler method, the quantity $y(x_{n+1})$ in the right hand side can be approximated as

$$y(x_{n+1}) \approx y_n + hf(x_n, y_n),$$

which together with (13), we arrive at

$$y(x_{n+1}) \approx y_n + \frac{h}{2}\big(f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\big) = y_{n+1}. \quad (14)$$

Above approximation is known as Modified Euler method.

Remark: Modified Euler method can be interpreted as follows:

- First, we predict the approximation of $y(x_{n+1})$ by Euler method, then the resulting approximation is corrected by modified Euler method given by (14). Such method is known as **predictor and corrector method**.
- It is natural to expect that modified Euler method should perform better than classical Euler method in terms of order of convergence. This has been proved in Section 3.
- From (14), it is clear that evaluation of y_{n+1} depends on the y_n, so modified Euler method is a single step method. Some text book refer it as improved Euler method. But, we will not distinguish between modified and improved.

3 General One Step Method

Now, we turn our discussion to the convergence of general one step method. A general explicit one-step method may be written as in the form

$$y_{n+1} = y_n + h\Phi(x_n, y_n; h). \quad (15)$$

For example,

- in the case Euler method
$$\Phi(x_n, y_n, h) = f(x_n, y_n),$$

- in the case of improved Euler method

$$\Phi(x_n, y_n; h) = \frac{1}{2}\big(f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n)\big).$$

In order to asses the accuracy of the numerical scheme (15), we define error e_n at each grid point by

$$e_n = y(x_n) - y_n.$$

Further, for our convenience, we define

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n); h) \text{ Or } y(x_{n+1}) = y(x_n) + h\Phi(x_n, y(x_n); h) + hT_n. \tag{16}$$

Then next theorem provides a bound on the error e_n.

Theorem 3.1 Consider the general one step method (15). We assume that Φ is continuous in a region $R \subset \mathbb{R}^2$ containing the initial point (x_0, y_0) and there exists a positive constant L_Φ such that

$$|\Phi(x, y; h) - \Phi(x, z; h)| \leq L_\Phi |y - z| \;\; \forall\, (x, y),\ (x, z) \in R. \tag{17}$$

Then we have following error bound

$$|e_k| \leq \exp^{L_\Phi(x_k - x_0)} |e_0| + \Big[\frac{\exp^{L_\Phi(x_k - x_0)} - 1}{L_\Phi}\Big] T, n = 0,\ 1,\ 2, \ldots,\ n, \tag{18}$$

where $T = \max_{0 \leq k \leq n-1} |T_k|$.

Proof. Subtracting (15) from (16) we arrive at

$$e_{k+1} = e_k + h\big[\Phi(x_k, y(x_k); h) - \Phi(x_k, y_k; h)\big] + hT_k. \tag{19}$$

Then the condition (17) yields

$$\begin{aligned} |e_{k+1}| \leq |e_k| + hL_\Phi |e_k| + hT_k &= (1 + hL_\Phi)|e_k| + h|T_k| \\ &\leq (1 + hL_\Phi)|e_k| + h|T|. \end{aligned} \tag{20}$$

Recursively, we obtain

$$\begin{aligned} |e_{k+1}| &\leq (1 + hL_\Phi)|e_k| + h|T| \\ &\leq (1 + hL_\Phi)^k |e_0| + h\big[1 + (1 + hL_\Phi) + (1 + hL_\Phi)^2 + \ldots + (1 + hL_\Phi)^{k-1}\big]|T| \\ &= (1 + hL_\Phi)^k |e_0| + \big[(1 + hL_\Phi)^k - 1\big] T / L_\Phi. \end{aligned} \tag{21}$$

Finally, use the fact that $1 + hL_\Phi \leq \exp^{hL_\Phi}$ to obtain the desire result.

Observations:

- The single step scheme (15) is consistent if and only if $\Phi(x, y; 0) = f(x, y)$.

  Proof. Observe that the truncation error is given by

$$\begin{aligned} ((T - T_h)(y))(x_n) &= y'(x_n) - f(x_n, y(x_n)) - \frac{y(x_{n+1}) - y(x_n)}{h} + \Phi(x_n, y(x_n); h) \\ &= y'(x_n) - \frac{y(x_{n+1}) - y(x_n)}{h} + \big[\Phi(x_n, y(x_n); h) - f(x_n, y(x_n))\big] \\ &= \mathrm{O}(h) + \big[\Phi(x_n, y(x_n); h) - f(x_n, y(x_n))\big]. \end{aligned}$$

  Clearly, the truncation error tends to zero provided

$$\lim_{h \to 0} \Phi(x, y; h) = \Phi(x, y; 0) = f(x, y).$$

- As an immediate consequence, we observe that the modified Euler scheme is consistent.

Remark:

- For the modified Euler method

$$\Phi(x, y(x); h) = \frac{1}{2}\big(f(x, y(x)) + f(x+h, y(x) + hf(x, y(x))\big)$$

and so

$$\begin{aligned}
T_n &= \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n); h) \\
&= \frac{y(x_{n+1}) - y(x_n)}{h} - \frac{1}{2}\big(f(x_n, y(x_n)) + f(x_{n+1}, y(x_n) + hf(x_n, y(x_n))\big) \\
&= \frac{y(x_{n+1}) - y(x_n)}{h} - \frac{1}{2}\big(g(x_n) + g(x_{n+1}) - g(x_{n+1}) + f(x_{n+1}, y(x_n) + hf(x_n, y(x_n)))\big) \\
&= \frac{y(x_{n+1}) - y(x_n)}{h} - \frac{g(x_n) + g(x_{n+1})}{2} \\
&\quad + \frac{1}{2}\big(g(x_{n+1}) - f(x_{n+1}, y(x_n) + hf(x_n, y(x_n))\big), \quad g(x) = f(x, y(x)) \\
&= \frac{y(x_{n+1}) - y(x_n)}{h} - \frac{g(x_n) + g(x_{n+1})}{2} \\
&\quad + \frac{1}{2}\big(f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y(x_n) + hf(x_n, y(x_n))\big)
\end{aligned}$$

Then try to establish following estimates

$$\begin{aligned}
\frac{y(x_{n+1}) - y(x_n)}{h} &= y'(x_{n+\frac{1}{2}}) + \mathrm{O}(h^2) = f(x_{n+\frac{1}{2}}, y(x_{n+\frac{1}{2}})) + \mathrm{O}(h^2) = g(x_{n+\frac{1}{2}})) + \mathrm{O}(h^2) \\
&= \frac{g(x_n) + g(x_{n+1})}{2} + \mathrm{O}(h^2) + \mathrm{O}(h^2).
\end{aligned}$$

Then use MVT to have

$$\begin{aligned}
f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y(x_n) + hf(x_n, y(x_n)) &= \frac{\partial f}{\partial y}(y(x_{n+1}) - y(x_n) - hf(x_n, y(x_n))) \\
&= \frac{\partial f}{\partial y} \times \mathrm{O}(h^2).
\end{aligned}$$

Finally, we have $T_n = \mathrm{O}(h^2)$.

- For the grid points

$$x_0, x_1, x_2, \ldots, x_n = b,$$

along with Theorem 3.1, we obtain

$$\begin{aligned}
|e_n| &\leq \Big[\frac{\exp^{L_\Phi(x_n - x_0)} - 1}{L_\Phi}\Big]\mathrm{O}(h^2), \text{ since } e_0 = 0 \\
&= \Big[\frac{\exp^{L_\Phi(b - x_0)} - 1}{L_\Phi}\Big]\mathrm{O}(h^2)
\end{aligned}$$

Therefore, modified Euler method has second order convergence.