
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Example 1 (Linearity of conditional expectations):

We again consider the example with $S_0 = 4$, $u = 2$, and $d = \frac{1}{2}$. We have $S_1(H) = 8$, $S_1(T) = 2$, $S_2(HH) = 16$, $S_2(HT) = S_2(TH) = 4$, $S_2(TT) = 1$, $S_3(HHH) = 32$, $S_3(HHT) = S_3(HTH) = S_3(THH) = 8$, $S_3(HTT) = S_3(THT) = S_3(TTH) = 2$ and $S_3(TTT) = 0.50$. With $p = \frac{2}{3}$ and $q = \frac{1}{3}$, we compute:

$$\begin{aligned} \mathbb{E}_1[S_2](H) &= \frac{2}{3} \cdot 16 + \frac{1}{3} \cdot 4 = 12, \\ \mathbb{E}_1[S_3](H) &= \frac{4}{9} \cdot 32 + \frac{2}{9} \cdot 8 + \frac{2}{9} \cdot 8 + \frac{1}{9} \cdot 2 = 18. \end{aligned}$$

Consequently,

$$\mathbb{E}_1[S_2](H) + \mathbb{E}_1[S_3](H) = 12 + 18 = 30.$$

On the other hand

$$\mathbb{E}_1[S_2 + S_3](H) = \frac{4}{9} \cdot (16 + 32) + \frac{2}{9} \cdot (16 + 8) + \frac{2}{9} \cdot (4 + 8) + \frac{1}{9} \cdot (4 + 2) = 30.$$

Similarly we can obtain that,

$$\mathbb{E}_1[S_2 + S_3](T) = 7.50 = \mathbb{E}_1[S_2](T) + \mathbb{E}_1[S_3](T).$$

In conclusion,

$$\mathbb{E}_1[S_2 + S_3] = \mathbb{E}_1[S_2] + \mathbb{E}_1[S_3].$$

Example 2 (Taking out what is known):

Continuing with the setup of Example 1, we compute

$$\mathbb{E}_1[S_2](H) = \frac{2}{3} \cdot 16 + \frac{1}{3} \cdot 4 = 12.$$

Now we have the following computation:

$$\mathbb{E}_1[S_1 S_2](H) = \frac{2}{3} \cdot 128 + \frac{1}{3} \cdot 32 = 96 = 8 \cdot 12 = S_1(H)\mathbb{E}_1[S_2](H).$$

A similar computation shows that,

$$\mathbb{E}_1[S_1 S_2](T) = 6 = S_1(T)\mathbb{E}_1[S_2](T).$$

In conclusion,

$$\mathbb{E}_1[S_1 S_2] = S_1 \mathbb{E}_1[S_2].$$

Example 3 (Iterated conditioning):

We first calculate the estimated values of S_3, based on the information at time 2:

$$\begin{aligned}
\mathbb{E}_2[S_3](HH) &= \frac{2}{3}\cdot 32 + \frac{1}{3}\cdot 8 = 24,\\
\mathbb{E}_2[S_3](HT) &= \frac{2}{3}\cdot 8 + \frac{1}{3}\cdot 2 = 6,\\
\mathbb{E}_2[S_3](TH) &= \frac{2}{3}\cdot 8 + \frac{1}{3}\cdot 2 = 6,\\
\mathbb{E}_2[S_3](TT) &= \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot \frac{1}{2} = 1.50.
\end{aligned}$$

We now estimate the estimate, based on the information at time 1:

$$\begin{aligned}
\mathbb{E}_1\left[\mathbb{E}_2[S_3]\right](H) &= \frac{2}{3}\cdot \mathbb{E}_2[S_3](HH) + \frac{1}{3}\cdot \mathbb{E}_2[S_3](HT) = \frac{2}{3}\cdot 24 + \frac{1}{3}\cdot 6 = 18,\\
\mathbb{E}_1\left[\mathbb{E}_2[S_3]\right](T) &= \frac{2}{3}\cdot \mathbb{E}_2[S_3](TH) + \frac{1}{3}\cdot \mathbb{E}_2[S_3](TT) = \frac{2}{3}\cdot 6 + \frac{1}{3}\cdot 1.50 = 4.50.
\end{aligned}$$

We now estimate S_3 directly based on the information at time 1:

$$\begin{aligned}
\mathbb{E}_1[S_3](H) &= \frac{4}{9}\cdot 32 + \frac{2}{9}\cdot 8 + \frac{2}{9}\cdot 8 + \frac{1}{9}\cdot 2 = 18,\\
\mathbb{E}_1[S_3](T) &= \frac{4}{9}\cdot 8 + \frac{2}{9}\cdot 2 + \frac{2}{9}\cdot 2 + \frac{1}{9}\cdot \frac{1}{2} = 4.50.
\end{aligned}$$

In conclusion,

$$\mathbb{E}_1\left[\mathbb{E}_2[S_3]\right] = \mathbb{E}_1[S_3].$$

Example 4 (Independence):

The quotient $\frac{S_2}{S_1}$ takes the values 2 or $\frac{1}{2}$. In particular, $\frac{S_2}{S_1}$ does not depend on the first coin toss. We compute:

$$\begin{aligned}
\mathbb{E}_1\left[\frac{S_2}{S_1}\right](H) &= \frac{2}{3}\cdot\frac{S_2(HH)}{S_1(H)} + \frac{1}{3}\cdot\frac{S_2(HT)}{S_1(H)} = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot\frac{1}{2} = \frac{3}{2},\\
\mathbb{E}_1\left[\frac{S_2}{S_1}\right](T) &= \frac{2}{3}\cdot\frac{S_2(TH)}{S_1(T)} + \frac{1}{3}\cdot\frac{S_2(TT)}{S_1(T)} = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot\frac{1}{2} = \frac{3}{2}.
\end{aligned}$$

We see that $\mathbb{E}_1\left[\frac{S_2}{S_1}\right]$ does not depend on the outcome of the first coin toss. In fact it is equal to

$$\mathbb{E}_1\frac{S_2}{S_1} = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot\frac{1}{2} = \frac{3}{2}.$$

Martingales:

Recall the relation:

$$S_n = \frac{1}{1+r}\widetilde{\mathbb{E}}_n\left[S_{n+1}\right].$$

If we divide both the sides by $(1+r)^n$, we get the equation

$$\frac{S_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right].$$

This equation expresses an important fact that, for a non-dividend paying stock, the best estimate based on the information at time n, of the value of the discounted stock price at time $(n+1)$ is the discounted stock price at time n.