
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

We now move on to the *Multiperiod Binomial Model.*

Accordingly, we first consider the general two period model with the goal of determining the no-arbitrage price for an option at time 0.

As before, we assume that at time 0, an amount of V_0 (to be determined) is received by selling an option. Then Δ_0 stocks are purchased, with the remaining amount of $V_0 - \Delta_0 S_0$ being invested (borrowed, if negative) at rate r.

At time 1, the portfolio is valued at

$$X_1 = \Delta_0 S_1 + (1+r)(V_0 - \Delta_0 S_0) = V_1,$$

which results in two equations as follows:

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) = V_1(H), \tag{1}$$

$$X_1(T) = \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) = V_1(T). \tag{2}$$

At time 1, the hedging position is allowed to be readjusted. Accordingly, at time 1, the number of stocks being held is Δ_1, which depends on the outcome of the first toss. The remaining wealth $X_1 - \Delta_1 S_1$ is invested (borrowed, if negative) at rate r. At time 2, the portfolio is valued at,

$$X_2 = \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1) = V_2,$$

which results in four equations as follows:

$$X_2(HH) = \Delta_1(H) S_2(HH) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) = V_2(HH), \tag{3}$$

$$X_2(HT) = \Delta_1(H) S_2(HT) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) = V_2(HT), \tag{4}$$

$$X_2(TH) = \Delta_1(T) S_2(TH) + (1+r)(X_1(T) - \Delta_1(T) S_1(T)) = V_2(TH), \tag{5}$$

$$X_2(TT) = \Delta_1(T) S_2(TT) + (1+r)(X_1(T) - \Delta_1(T) S_1(T)) = V_2(TT). \tag{6}$$

We now have six equations, in six unknowns, namely, V_0, Δ_0, $X_1(H)$, $X_1(T)$, $\Delta_1(H)$ and $\Delta_1(T)$. Solving equations (1)-(6), we obtain,

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)},$$

$$\Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)},$$

$$X_1(H) = V_1(H) = \frac{1}{1+r}\left[\widetilde{p} V_2(HH) + \widetilde{q} V_2(HT)\right],$$

$$X_1(T) = V_1(T) = \frac{1}{1+r}\left[\widetilde{p}V_2(TH) + \widetilde{q}V_2(TT)\right],$$

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)},$$

$$X_0 = V_0 = \frac{1}{1+r}\left[\widetilde{p}V_1(H) + \widetilde{q}V_1(T)\right].$$

In summary, we have three *stochastic processes*, (Δ_0, Δ_1), (X_0, X_1, X_2) and (V_0, V_1, V_2). By *stochastic process*, we mean a sequence of random variables indexed by time.

We now move on to a N-period model. Accordingly, the value of the portfolio, beginning with the wealth level of X_0 is defined recursively using the following *wealth equation*,

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)\left(X_n - \Delta_n S_n\right). \tag{7}$$

Theorem (Replication in the Multiperiod Binomial Model)

Consider N- period binomial asset pricing model, with $0 < d < 1+r < u$, and with

$$\widetilde{p} = \frac{1+r-d}{u-d} \text{ and } \widetilde{q} = \frac{u-1-r}{u-d}. \tag{8}$$

Let V_N be a random variable (a derivative security paying off at time N) depending on the first N coin tosses $\omega_1\omega_2\ldots\omega_N$. We define recursively (backward in time), the sequence of random variables $V_{N-1}, V_{N-2}, \ldots, V_0$ by

$$V_n\left(\omega_1\omega_2\ldots\omega_n\right) = \frac{1}{1+r}\left[\widetilde{p}\; V_{n+1}\left(\omega_1\omega_2\ldots\omega_n H\right) + \widetilde{q}\; V_{n+1}\left(\omega_1\omega_2\ldots\omega_n T\right)\right] \tag{9}$$

so that each V_n depends on the first n coin tosses $\omega_1\omega_2\ldots\omega_n$, where $n = N-1, N-2, \ldots, 0$. We next define

$$\Delta_n\left(\omega_1\omega_2\ldots\omega_n\right) = \frac{V_{n+1}\left(\omega_1\omega_2\ldots\omega_n H\right) - V_{n+1}\left(\omega_1\omega_2\ldots\omega_n T\right)}{S_{n+1}\left(\omega_1\omega_2\ldots\omega_n H\right) - S_{n+1}\left(\omega_1\omega_2\ldots\omega_n T\right)}, \tag{10}$$

where $n = 0, 1, \ldots, N-1$. If we set $X_0 = V_0$ define $X_1, X_2, \ldots, X_N$ using relation (7), then we will have,

$$X_N\left(\omega_1\omega_2\ldots\omega_N\right) = V_N\left(\omega_1\omega_2\ldots\omega_N\right),\ \forall\ \omega_1\omega_2\ldots\omega_N. \tag{11}$$

Definition

For $n = 1, 2, \ldots, N$, the price of the derivative security at time n if the outcomes of the first n tosses are $\omega_1\omega_2\ldots\omega_n$ is defined to be the random variable $V_n\left(\omega_1\omega_2\ldots\omega_n\right)$ of the Theorem. The price of the derivative security at time 0 is defined to be V_0.

Proof of Theorem

We prove the following by induction:

$$X_n\left(\omega_1\omega_2\ldots\omega_n\right) = V_n\left(\omega_1\omega_2\ldots\omega_n\right),\ \forall\ \omega_1\omega_2\ldots\omega_n, \tag{12}$$

where $n = 0, 1, \ldots, N$. Recall that for $n = 0$, $X_0 = V_0$. We will prove the case of $n = N$.

We assume that equation (12) holds for some $n < N$. We will prove the result for $n+1$. Accordingly, let $\omega_1\omega_2\ldots\omega_n\omega_{n+1}$ be fixed but arbitrary. We assume that the induction holds for the particular fixed

$\omega_1\omega_2\ldots\omega_n$. We will consider both the cases for ω_{n+1}, *i.e.*, $\omega_{n+1} = H$ or $\omega_{n+1} = T$. We first consider the former. From (7) we have

$$\begin{aligned} X_{n+1}\left(\omega_1\omega_2\ldots\omega_n H\right) &= \Delta_n\left(\omega_1\omega_2\ldots\omega_n\right) uS_n\left(\omega_1\omega_2\ldots\omega_n\right) \\ &+ (1+r)\left(X_n\left(\omega_1\omega_2\ldots\omega_n\right) - \Delta_n\left(\omega_1\omega_2\ldots\omega_n\right) S_n\left(\omega_1\omega_2\ldots\omega_n\right)\right). \end{aligned}$$

For the sake of brevity, we suppress $\omega_1\omega_2\ldots\omega_n$ to obtain,

$$X_{n+1}(H) = \Delta_n uS_n + (1+r)\left(X_n - \Delta_n S_n\right). \tag{13}$$

Similarly from (10) we have

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}.$$

Substituting this into (13) and using the induction hypothesis in (12) and the definition of V_n in (9) we obtain,

$$\begin{aligned} X_{n+1}(H) &= (1+r)\,X_n + \Delta_n S_n\left(u - (1+r)\right) \\ &= (1+r)\,V_n + \frac{\left(V_{n+1}(H) - V_{n+1}(T)\right)\left(u - (1+r)\right)}{u-d} \\ &= (1+r)\,V_n + \widetilde{q}V_{n+1}(H) - \widetilde{q}V_{n+1}(T) \\ &= \widetilde{p}V_{n+1}(H) + \widetilde{q}V_{n+1}(T) + \widetilde{q}V_{n+1}(H) - \widetilde{q}V_{n+1}(T) \\ &= V_{n+1}(H). \end{aligned}$$

Reinstating $\omega_1\omega_2\ldots\omega_n$, we have,

$$X_{n+1}\left(\omega_1\omega_2\ldots\omega_n H\right) = V_{n+1}\left(\omega_1\omega_2\ldots\omega_n H\right).$$

A similar argument shows that,

$$X_{n+1}\left(\omega_1\omega_2\ldots\omega_n T\right) = V_{n+1}\left(\omega_1\omega_2\ldots\omega_n T\right).$$

Consequently, we have,

$$X_{n+1}\left(\omega_1\omega_2\ldots\omega_n\omega_{n+1}\right) = V_{n+1}\left(\omega_1\omega_2\ldots\omega_n\omega_{n+1}\right).$$

Since $\omega_1\omega_2\ldots\omega_n\omega_{n+1}$ is arbitrary, the proof is complete.