Example

1	2	3
x	y	y
y	z	x
z	x	z

Suppose individual 1 is the almost decisive individual over x, y. We have yP_2x and yP_3x. So social preference over x, y is xPy. We have $yP_iz, i = 1, 2, 3$. So from Pareto principle, we have yPz. Now using transitivity we have $xPy \& yPz \rightarrow xPz$.

1	2	3
x	y	z
y	z	y
z	x	x

Here in this example individual 1 is almost decisive over x, y. So we have xPy. For other pair (x, z) and (y, z), we cannot say anything by simply using the properties U, P, I. Thus, in this case we do not have a social preference relation.
Here, the if part in the second part of the statement of lemma is false. So lemma is trivially true when there is an almost decisive individual.

Theorem: There is no social welfare function satisfying conditions U, P, I and D.

Proof: For any pair of alternatives, there is atleast one decisive set, the set of all individuals. It is due to Pareto principle. Thus, for every pair of alternatives there is also at least one almost decisive set, since a decisive set is also almost decisive. Lets compare all the sets of individuals that are almost decisive for some pair, it may not be same pair of alternatives. From these set of almost decisive sets chose the smallest decisive set. Let this set be called V and let it be almost decisive for x against y.

If V contains only one individual, then the lemma we get that individual is decisive over all pairs of alternative, thus dictator.
Suppose V contains two or more individuals. We divide V into two parts, V_1 containing a single individual, and V_2 contains the rest of V. V_3 contains all individuals not contain in V.
Due to U we can take any logically possible combination of individual orderings. Suppose we have
For all $i \in V_1$, $xP_iy\&yP_iz$.
For all $j \in V_2$, $zP_jx\&xP_jy$.
For all $k \in V_3$, $yP_kx\&zP_kx$.
Since V is almost decisive for x against y and since X is preferred to y for all individuals in V and y is preferred to x for all in V_3, so xPy.

Note that zP_iy for $jinV_2$ and for the rest we have y is preferred to z. It implies that V_2 is almost decisive for y, z , if we have zPy. Then we get a contradiction because V_2 is smaller than V. We have assume that V is the smallest almost decisive set. Therefore, we cannot have zPy. It implies that for y and z pair we have yRz. Now $xPy\&yRz \rightarrow xPz$. Now only individual in V_1 prefers x to z and all others prefer z to x. Thus, we have V_1 as almost decisive set for the pair x and z. Thus, we get a contraction since we assume V is the smallest almost decisive set. Thus, our assumption that V contains two or more is not true. Thus, V contains only one individual. Using the lemma we get that this individual is decisive for all the pairs of alternatives. Thus, there is a dictator whenever a social welfare function satisfies U, P and I.

Example:

$$\begin{array}{ccc} 1 & 2 & 3 \\ x & z & x \\ z & x & y \\ y & y & z \end{array}$$

What can we say about the social preference?

From Pareto principle we can say xPy. We cannot say anything further based on the four conditions of Arrow's Theorem.

Suppose We have one more additional information that yPz. Then using transitivity we have xPz. Thus the social preference is

$$f\begin{pmatrix} x & z & x \\ z & x & y \\ y & y & z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Note here that individual 3 has become the dictator.
The additional information is giving that individual 3 is almost decisive for the pair y, z.

Instead if we had the additional information zPy. In this we have xPy and zPy. We cannot say anything about z and x.
We need further information, this can be either zPx or xPz . If zPx then we individual 2 as dictator. If xPz the we have individual 1 as dictator.
If we have xIz. We dont have any dictator. In this case also we need additional restriction. Here it is not in the form of almost decisive set.

Example:

1	2	3
x	z	x
y	x	z
z	y	y
t	t	t

From the Pareto principle we get
xPy, yPt, zPt, and xPt.
We need additional information on y and z pair. If yPz, then by transitivity we have xPz. Thus, the social preference ordering is

$$f\begin{pmatrix}1 & 2 & 3\\ x & z & x\\ y & x & z\\ z & y & y\\ t & t & t\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\\ t\end{pmatrix}.$$

Note Individual 1 is decisive, so the dictator.

Condorcet Paradox:
There exists profiles of individual preferences such that method of majority decision does not constitute a social welfare function.

Proof: Suppose there are three individual, The preference profile is

1	2	3
x	y	z
z	x	y
y	z	x

The method of majority decision gives social preference relation in the following way

$xRy \leftrightarrow N(i \in N, xP_iy) \geq N(i \in N, yP_ix)$.

$xRy \leftrightarrow N(i \in N, xR_iy) \geq N(i \in N, yR_ix)$.

Here, $N(yP_ix) = 2 > N(xP_iy) = 1$ it implies yPx.
$N(zP_iy) = 2 > N(yP_iz) = 1$, it implies zPy.
$N(xP_iz) = 2 > N(zP_ix) = 1$, it implies xPz.
We have $zPy \& yPx \ \rightarrow zPx$, from transitivity. But we have xPz. It violates transitivity. Therefore, it is not a social welfare function.