
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Single Step Pricing:

We begin with the case $N = 1$ and extend the procedure done in the previous lecture, to European contingent claim whose payoff is of the general form $f(S(1))$ for some function f. The value of such an option at time t will be denoted by $H(t)$, for $t = 0, 1$. At exercise time $t = 1$, we have,

$$H(1) = f(S(1)).$$

As already seen, the option can be priced by replication of the two possible values, $H^u(1)$ and $H^d(1)$, of $H(1)$, and then computing the time $t = 0$ value of the replicating portfolio. For this, we need to solve the following system of equations:

$$\begin{aligned} x(1)S^u(1) + y(1)A(1) &= H^u(1), \\ x(1)S^d(1) + y(1)A(1) &= H^d(1). \end{aligned}$$

The solution of the above system of equations is given by,

$$\begin{aligned} x(1) &= \frac{H^u(1) - H^d(1)}{S^u(1) - S^d(1)} = \frac{H^u(1) - H^d(1)}{S(0)(U - D)}, \\ y(1) &= \frac{1}{A(1)}[H^u(1) - x(1)S^u(1)] = \frac{1}{A(0)(1+R)} \frac{H^d(1)(1+U) - H^u(1)(1+D)}{U - D}. \end{aligned}$$

Here $x(1)$ is the replicating position in the stock, called the *Delta* of the option. Using the no arbitrage principle, we have the price of the claim as,

$$H(0) = x(1)S(0) + y(1)A(0),$$

which implies that

$$H(0) = \frac{1}{1+R}\left(H^u(1)\frac{R - D}{U - D} + H^d(1)\frac{U - R}{U - D}\right).$$

The coefficients $\dfrac{(R-D)}{(U-D)}$ and $\dfrac{(U-R)}{(U-D)}$ add up to 1. Also, both are greater than 0 and less than 1, since the condition $D < R < U$ holds in order to satisfy the no-arbitrage condition. These terms can now be regarded as probabilities and, accordingly, we use the notation:

$$p_* = \frac{R - D}{U - D}.$$

Therefore,

$$\begin{aligned} H(0) &= \frac{1}{1+R}\Big(H^u(1)p_* + H^d(1)(1 - p_*)\Big) \\ &= \frac{1}{1+R}E_*[H(1)] = \frac{1}{1+R}E_*[f(S(1))]. \end{aligned}$$

Result:

$$E_*[K(1)] = R.$$

Proof:

$$E_*[K(1)] = Up_* + D(1-p_*) = U\frac{R-D}{U-D} + D\frac{U-R}{U-D} = R.$$

Definition:

A probability $(p_*, 1-p_*)$ such that $p_*, 1-p_* \in (0,1)$ under which the expected return $E_*[K(1)]$ is equal to the risk-free rate R is called *risk-neutral*.

Now, the following notations are introduced for convenience

$$\widetilde{S}(1) = \frac{S(1)}{1+R},\ \widetilde{V}(1) = \frac{V(1)}{1+R},\ \widetilde{H}(1) = \frac{H(1)}{1+R}$$

and $\widetilde{S}(1),\ \widetilde{V}(1),\ \widetilde{H}(1)$ are called discounted values of $S(1), V(1), H(1)$, respectively.

Result:

Under risk-neutral probability

$$E_*[\widetilde{S}(1)] = S(0),\ E_*[\widetilde{V}(1)] = V(0),\ E_*[\widetilde{H}(1)] = H(0)$$

Proof:

1. $E_*[\widetilde{S}(1)] = \dfrac{1}{1+R}E_*[S(1)] = \dfrac{S(0)}{1+R}E_*[1+K(1)] = \dfrac{S(0)}{1+R}(1+E_*[K(1)]) = S(0).$
2. $E_*[\widetilde{V}(1)] = E_*[x(1)\widetilde{S}(1) + y(1)A(0)] = x(1)S(0) + y(1)A(0) = V(0).$
3. $E_*[\widetilde{H}(1)] = \dfrac{1}{1+R}E_*[H(1)] = H(0).$

Exotic Option:

Consider an interesting example with $N = 2$ and with the payoff,

$$C(2) = \max\left(\frac{S(1)+S(2)}{2} - X, 0\right).$$

This is an European call option with the average stock price as the underlying asset and is an example of what is known as *path-dependent* or *exotic option*. When $U = 10\%$, $D = -10\%$, $S(0) = 100$ and $X = 90$, then,

$$C(2) = \begin{cases} C^{uu}(2) = 25.5 \\ C^{ud}(2) = 14.5 \\ C^{du}(2) = 4.5 \\ C^{dd}(2) = 0. \end{cases}$$

With $R = 5\%$, we get $p_* = 0.75$. Then,

$$\begin{aligned} C^u(1) &= \frac{1}{1+R}\left(p_*C^{uu}(2) + (1-p_*)C^{ud}(2)\right) = 21.67 \\ C^d(1) &= \frac{1}{1+R}\left(p_*C^{du}(2) + (1-p_*)C^{dd}(2)\right) = 3.21. \end{aligned}$$

This gives,

$$C(0) = \frac{1}{1+R}\left(p_*C^u(1) + (1-p_*)C^d(1)\right) = 6.24.$$

Note that, the single-step method is recursively applied here, starting from the final values and going backwards in time.