LECTURE-5

Mathematical Concepts

Linear Algebra Review

$\mathbb{R}^n$ $\quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

↳ Linear space or Vector space

Linear subspace V

Affine subspace $u+V = \{u+v \mid v \in V\}$

Linear independence / dependence of $\{a_1, a_2 \dots a_k\}$

Linear combination of $a_1, a_2, \dots, a_k$

$$a = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_k a_k$$

Inner or dot product, norm of a vector, orthogonal vectors.

Span

Basis, dimension

Natural basis of $\mathbb{R}^n$: $\{e_1, e_2 \dots, e_n\}$

Matrices

Elementary matrix operations

Special matrices: Identity, Triangular, Symmetric, Orthogonal etc.

Determinant, rank, inverse, singular, nonsingular

minor

→ $Ax = b$ has a soln iff $\text{rank}(A) = \text{rank}[A\ b]$

Linear transformation (or linear map)

$L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map if

a) $L(ax) = a\,L(x) \quad \forall x \in \mathbb{R}^n,\ a \in \mathbb{R}$

b) $L(x+y) = L(x) + L(y) \quad \forall x, y \in \mathbb{R}^n$

$L(x) = Ax$

Eigenvalues and Eigenvectors

Characteristic Polynomial

Let $A \in \mathbb{R}^{m\times n}$ Range or image of A

$\mathcal{R}(A) = \{ Ax : x \in \mathbb{R}^n \}$

Nullspace or Kernel of A

$\mathcal{N}(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$

Geometry: $\mathbb{R}^n$

Line segment joining $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$ is the set of points (vectors) given by

$\{ x \mid x = \lambda x_1 + (1-\lambda) x_2,\ 0 \le \lambda \le 1 \}$.

Line joining x_1 and x_2 is the set

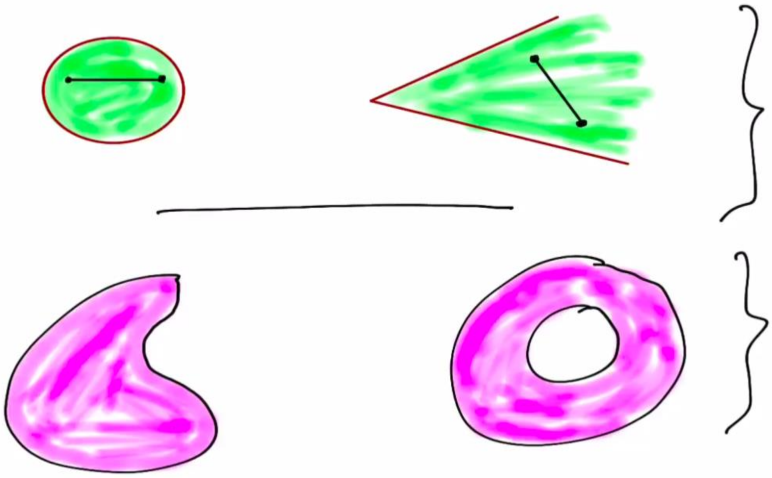
$\{ x \mid x = \lambda x_1 + (1-\lambda) x_2,\ \lambda \in \mathbb{R} \}$.

$x \in \mathbb{R}^n$ is a linear combination of $x_1, x_2, \dots x_m \in \mathbb{R}^n$ if $\exists\ \lambda_i \in \mathbb{R}$ such that $x = \sum_{i=1}^{m} \lambda_i x_i$

- Affine combination if $\sum_{i=1}^{m} \lambda_i = 1$.
- Convex combination if $\lambda_i \geq 0\ \forall i$ and $\sum_1^m \lambda_i = 1$.

A set $S \subseteq \mathbb{R}^n$ is called a convex set if

$$x_1, x_2 \in S \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S \quad \forall\ 0 \le \lambda \le 1 .$$



convex sets

non convex sets

Given a point $x_0 \in \mathbb{R}^n$ and a nonzero vector $d \in \mathbb{R}^n$, the set $\{x_0 + \lambda d \mid \lambda \geq 0\}$ is called a ray in $\mathbb{R}^n$.

Here, x_0 is the vertex of the ray, and d is the direction of the ray

Let $c \in \mathbb{R}$ and $a \in \mathbb{R}^n$, $a \neq 0$. Then the set

$$H = \{x \mid a^T x = c\}$$

is said to be a hyperplane in $\mathbb{R}^n$.

The sets $H_+ = \{x \mid a^T x \geq c\}$ ← positive half-space

$H_- = \{x \mid a^T x \leq c\}$ ← negative half-space

are called closed half-spaces generated by H.

$|||^{ly}$ open half-spaces

Let $b \in H$. Then $a^T b - c = 0$.

We can write

$$a^T x - c = a^T x - c - (a^T b - c) = a^T(x-b) = \langle a, x-b \rangle$$

ie, H consists of points x for which a and $x-b$ are orthogonal

We call a the normal to the hyperplane H.

A linear variety is a set of the form

$$\{x \in \mathbb{R}^n \mid Ax = b\} = \{b + Ax \mid Ax = 0 \ x \in \mathbb{R}^n\}$$

for some matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

If $\dim \mathcal{N}(A) = r$, we say that the linear variety has dimension r.

Let $C \subseteq \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

Then $\lambda C = \{x \mid x = \lambda c, \ c \in C\}$

$$C + D = \{x \mid x = c + d, \ c \in C, \ d \in D\}$$

Thm: Convex subsets of $\mathbb{R}^n$ have the following properties:

(i) The intersection of any collection of convex sets is convex

(ii) If C is a convex set and $\lambda \in \mathbb{R}$, then λC is a convex set.

(iii) If C and D are convex sets, then $C + D$ is a convex set.

Pf: Simple and exercise.

Thm: A set $S \subseteq \mathbb{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S.

Let $S \subseteq \mathbb{R}^n$ be a convex set.

A point $x \in S$ is called an extreme point or vertex of S if there exist no two distinct points x_1 and x_2 in S s.t.

$$x = \lambda x_1 + (1-\lambda) x_2 \quad \text{for } 0 < \lambda < 1.$$