
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Best Linear Predictor:

Suppose we wish to approximate y using some linear function $\beta x+\alpha$. The error is $\epsilon = y-\beta x-\alpha$ (Residual Random Variable). The best fit is generally considered as the one that minimizes the mean square error

$$MSE = E(\epsilon^2) = E\left[(y-\beta x-\alpha)^2\right] = E(y^2) - 2\beta E(xy) - 2\alpha E(y) + \beta^2 E(x^2) + 2\alpha\beta E(x) + \alpha^2.$$

The minimum value is obtained by setting the partial derivatives w.r.t α and β to be equal to zero, which gives,

$$\begin{aligned} \beta E(x^2) + \alpha E(x) &= E(xy) \\ \beta E(x) + \alpha &= E(y). \end{aligned}$$

Solving we get $\beta = \dfrac{\sigma_{xy}}{\sigma_x^2}$ and $\alpha = E(y) - \beta E(x)$. The best linear predictor (BLP) of y w.r.t x is the linear function $\beta x + \alpha$ that minimizes the mean square error $E(\epsilon^2)$. β is called the beta of y w.r.t x. The line $y = \beta x + \alpha$ is called the regression line.

The Risk-Return of an Asset Compared with the Market Portfolio:

Let us consider any particular asset a_k in the market portfolio. We want to use the best linear predictor to approximate the return R_k of asset a_k by a linear function of the return R_M of the entire market portfolio. We can write $R_k = \beta_k R_M + \alpha_k + \epsilon_k$, where $\beta_k = \dfrac{Cov(R_k, R_M)}{\sigma_M^2}$ and $\alpha_k = E(R_k) - \beta_k E(R_M)$. Here ϵ_k is the residual random variable and β_k is the beta of the assets return w.r.t the market portfolios return and is the slope of the linear regression line. Therefore,

$$\sigma_k^2 = Var(\beta_k R_M + \alpha_k + \epsilon_k) = \underbrace{\beta_k^2\sigma_M^2}_{\text{Systematic Risk}} + \underbrace{\sigma_{\epsilon_k}^2}_{\text{Unsystematic Risk}}.$$

Note that the systematic risk $\beta_k^2\sigma_M^2$ of asset a_k is proportional to the market risk with a proportionality factor of β_k^2. According to economic theory, when adding an asset to a diversified portfolio, the unique risk of that asset cancel out by other assets in that portfolio.

Theorem:

The expected return and risk of an asset a_k in the market portfolio is related to the assets beta w.r.t the market portfolio as follows: $\mu_k = \beta_k(\mu_M - \mu_{rf}) + \mu_{rf}$ and $\sigma_k^2 = \beta_k^2\sigma_M^2 + \sigma_{\epsilon_k}^2$.

The graph of the line $\mu_k = \beta_k(\mu_M - \mu_{rf}) + \mu_{rf}$ is called the Security Market Line. This equation shows that the expected return of an asset is equal to the return of the risk free asset + risk premium $\beta_k(\mu_M - \mu_{rf})$ of the asset.

Proof:

Consider a portfolio where weight s is invested in asset a_k and $(1-s)$ invested in the market portfolio. Then the expected return and risk on the portfolio are:

$$\begin{aligned} \mu_P &= s\mu_k + (1-s)\mu_M \\ \sigma_P &= \left(s^2\sigma_k^2 + 2s(1-s)\sigma_{kM} + (1-s)^2\sigma_M^2\right)^{1/2} \end{aligned}$$

In particular, $s = 0$ corresponds to the market portfolio. The curve cannot cross the CML, since it would violate the definition of CML as the efficient boundary of the feasible set. The tangent condition can be translated into the condition that the slope of the curve is equal to the slope of the CML.

$$\frac{d\mu_P}{d\sigma_P} = \frac{d\mu_P/ds}{d\sigma_P/ds} = \frac{(\mu_k - \mu_M)\sigma_P}{s(\sigma_k^2 + \sigma_M^2 - 2\sigma_{kM}) + (\sigma_{kM} - \sigma_M^2)}$$

But the equilibrium market portfolio contains asset k since it contains all assets. Therefore $s = 0$ and $\sigma_P = \sigma_M$ which gives $\frac{d\mu_P}{d\sigma_P} = \frac{(\mu_k - \mu_M)\sigma_M}{(\sigma_{kM} - \sigma_M^2)}$. This must be equal to the slope of the market line. Thus,

$$\begin{aligned} & \frac{(\mu_k - \mu_M)\sigma_M}{(\sigma_{kM} - \sigma_M^2)} = \frac{\mu_M - \mu_{rf}}{\sigma_M}, \\ \Rightarrow\ & (\mu_k - \mu_M)\sigma_M^2 = (\mu_M - \mu_{rf})(\sigma_{kM} - \sigma_M^2), \\ \Rightarrow\ & \mu_k = \mu_{rf} + \frac{\sigma_{kM}(\mu_M - \mu_{rf})}{\sigma_M^2}, \\ \Rightarrow\ & \mu_k = \mu_{rf} + \beta_k(\mu_M - \mu_{rf}). \end{aligned}$$

Forward and Futures Contract:

Recall that a forward contract is an agreement to buy or sell an asset on a fixed date in the future called the delivery time for a price specified in advance, called the forward price. The party to the contract which agrees to sell the asset is said to be taking a short forward position. The other party which agrees to buy the asset at delivery is said to be taking a long forward position.

Let us denote the time of agreement on the forward contract as 0 and the delivery time by T. Let $F(0,T)$ be the corresponding forward price. The price of the underlying asset at any time t will be denoted by $S(t)$. No payment is made by the either party at time $t = 0$, when the forward contract is exchanged.

At delivery, the party with long forward position will benefit if $F(0,T) < S(T)$. They can buy the asset for $F(0,T)$ and sell for $S(T)$ to make a profit of $S(T) - F(0,T)$. The party holding the short forward position will suffer a loss of $S(T) - F(0,T)$. If $F(0,T) > S(T)$ then the situation will be reversed.

Accordingly, the payoff at delivery are $S(T) - F(0,T)$ for a long forward position and $F(0,T) - S(T)$ for the short forward position, respectively.

Note: If the contract is initiated at time $t < T$ rather than 0, then we shall write $F(t,T)$ for the forward price, the payoff at delivery being $S(T) - F(t,T)$ for a long forward position and $F(t,T) - S(T)$ for a short forward position.

Forward Price for Stock Paying no Dividends:

We begin with the simplest case of a stock paying no dividends. For a stock paying no dividends the forward price is:

$$F(0,T) = S(0)e^{rT},$$

where r is a constant risk-free interest rate under continuous compounding. If the contract is initiated at time $t \leq T$, then

$$F(t,T) = S(t)e^{r(T-t)}.$$

Proof:

1. Suppose that $F(0,T) > S(0)e^{rT}$. In this case:

   (a) At time $t = 0$:

      i. Borrow an amount $S(0)$ at rate r till time T.

      ii. Buy one share for $S(0)$.

      iii. Take a short forward contract with forward price $F(0,T)$ at time T.

   (b) At time $t = T$:

      i. Sell the stock for $F(0,T)$.

      ii. Pay $S(0)e^{rT}$ to clear the loan with interest.

   This will bring a profit of $F(0,T) - S(0)e^{rT} > 0$ in violation of the no-arbitrage principle.

2. Suppose that $F(0,T) < S(0)e^{rT}$. In this case:

   (a) At time $t = 0$:

      i. Short sell one share for $S(0)$.

      ii. Invest the proceeds at the risk free rate r for time T.

      iii. Enter into a long forward contract with forward price $F(0,T)$.

   (b) At time $t = T$:

      i. Cash the risk free investment with interest collecting $S(0)e^{rT}$.

      ii. Buy the stock for $F(0,T)$ using forward contract.

      iii. Close out the short position in stock by returning it to the owner.

   This will bring a profit of $S(0)e^{rT} - F(0,T) > 0$ in violation of the no-arbitrage principle.

Thus we can only have,

$$F(0,T) = S(0)e^{rT}$$

In a similar way one can derive,

$$F(t,T) = S(t)e^{r(T-t)}.$$

Finally, under periodic compounding the forward price is given by,

$$F(0,T) = S(0)\left(1 + \frac{r}{m}\right)^{mT}.$$