Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Risk and Expected Return on a Portfolio:

The expected return on a portfolio consisting of two securities can be expressed in terms of the weights and the expected returns on the components as,

$$E(K_V) = w_1 E(K_1) + w_2 E(K_2).$$

This follows at once (from the previous proof) by the additivity property of mathematical expectation.

Example:

Consider three scenarios with the probabilities given below (a trinomial model). Let the returns on two different stocks in these scenarios be as follows:

Scenario	Probability	Return (K_1)	Return (K_2)
ω_1 (Recession)	0.2	-10%	-30%
ω_2 (Stagnation)	0.5	0%	20%
ω_2 (Boom)	0.3	10%	50%

The expected returns are

$$\begin{aligned} E(K_1) &= -0.2 \times 10\% + 0.5 \times 0\% + 0.3 \times 10\% = 1\%, \\ E(K_2) &= -0.2 \times 30\% + 0.5 \times 20\% + 0.3 \times 50\% = 19\%. \end{aligned}$$

Suppose that 60% of available funds is invested in stock 1 and 40% in stock 2. The expected return on such a portfolio will be

$$\begin{aligned} E(K_V) &= w_1 E(K_1) + w_2 E(K_2) \\ &= 0.6 \times 1\% + 0.4 \times 19\% = 8.2\% \end{aligned}$$

Theorem:

The variance of the return on a portfolio is given by

$$Var(K_V) = w_1^2 Var(K_1) + w_2^2 Var(K_2) + 2 w_1 w_2 Cov(K_1, K_2).$$

Proof:

Substituting $K_V = w_1 K_1 + w_2 K_2$ and collecting the terms with w_1^2, w_2^2 and $w_1 w_2$, we get

$$\begin{aligned} Var(K_V) &= E(K_V^2) - E(K_V)^2 \\ &= w_1^2[E(K_1^2) - E(K_1)^2] + w_2^2[E(K_2^2) - E(K_2)^2] \\ &\quad +2 w_1 w_2 [E(K_1 K_2) - E(K_1)E(K_2)] \\ &= w_1^2 Var(K_1) + w_2^2 Var(K_2) + 2 w_1 w_2 Cov(K_1, K_2). \end{aligned}$$

To avoid clutter, we introduce the following notation:

$$\mu_V = E(K_V), \qquad \sigma_V = \sqrt{Var(K_V)},$$
$$\mu_1 = E(K_1), \qquad \sigma_1 = \sqrt{Var(K_1)},$$
$$\mu_2 = E(K_2), \qquad \sigma_2 = \sqrt{Var(K_2)},$$

$$c_{12} = Cov(K_1, K_2).$$

The above formulae can be written as

$$\begin{aligned} \mu_V &= w_1\mu_1 + w_2\mu_2 \\ \sigma_V^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2c_{12}. \end{aligned}$$

We shall also use the correlation coefficient

$$\rho_{12} = \frac{c_{12}}{\sigma_1\sigma_2}.$$

Note that the correlation coefficient is undefined when $\sigma_1\sigma_2 = 0$. Note that this condition means that at least one of the assets is risk free.

Remark:

For risky securities, the returns K_1 and K_2 are non-constant random variables. Because of this $\sigma_1\sigma_2 > 0$ and ρ_{12} is well defined.

Example:

We use the following data:

Scenario	Probability	Return (K_1)	Return (K_2)
ω_1	0.2	-10%	5%
ω_2	0.4	0%	30%
ω_2	0.4	20%	-5%

We want to compare the risk of a portfolio such that $w_1 = 40\%$ and $w_2 = 60\%$ with the risk of each of its components. Direct computations give

$$\sigma_1^2 \simeq 0.0144, \sigma_2^2 \simeq 0.0254, \rho_{12} \simeq -0.6065.$$

Then

$$\sigma_V^2 \simeq (0.4)^2 \times 0.0144 + (0.6)^2 \times 0.0254 + 2 \times 0.4 \times 0.6 \times (-0.6065) \times \sqrt{0.0144} \times \sqrt{0.0254} \simeq 0.00588.$$

Observe that the variance σ_V^2 is smaller than both σ_1^2 and σ_2^2.

Example:

Consider another portfolio with weights $w_1 = 10\%$ and $w_2 = 90\%$, all other things being the same as in the previous example. Then

$$\sigma_V^2 \simeq (0.1)^2 \times 0.0144 + (0.9)^2 \times 0.0254 + 2 \times 0.1 \times 0.9 \times (-0.6065) \times \sqrt{0.0144} \times \sqrt{0.0254} \simeq 0.01863$$

which is between σ_1^2 and σ_2^2.

Proposition:

The variance σ_V^2 of a portfolio cannot exceed the greater of the variances σ_1^2 and σ_2^2 of the components,

$$\sigma_V^2 \leq \max\left\{\sigma_1^2, \sigma_2^2\right\}$$

if short sales are not allowed.

Proof:

Let us assume that $\sigma_1^2 \leq \sigma_2^2$. If short sales are not allowed, then $w_1, w_2 \geq 0$ and

$$w_1\sigma_1 + w_2\sigma_2 \leq (w_1 + w_2)\sigma_2 = \sigma_2.$$

Since the correlation coefficient satisfies $-1 \leq \rho_{12} \leq 1$, it follows that

$$\begin{aligned}
\sigma_V^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2. \\
&\leq w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2. \\
&= (w_1\sigma_1 + w_2\sigma_2)^2 \leq \sigma_2^2
\end{aligned}$$

If $\sigma_1^2 \geq \sigma_2^2$, the proof is analogous.

Example:

Now consider a portfolio with weights $w_1 = -50\%$ and $w_2 = 150\%$ (allowing short sales of security 1), all the other data being the same as in the previous example. The variance of this portfolio is

$$\sigma_V^2 \simeq (-0.5)^2 \times 0.0144 + (1.5)^2 \times 0.0254 + 2 \times (-0.5) \times 1.5 \times (-0.6065) \times \sqrt{0.0144} \times \sqrt{0.0254} \simeq 0.2795$$

which is greater than both σ_1^2 and σ_2^2.

Feasible Set:

The collection of all portfolios that can be manufactured by investing in two given assets is called the *feasible* (or *attainable*) *set*. Each portfolio can be represented by a point with coordinates σ_V and μ_V in the σ, μ plane. The set consists of all points with coordinates

$$\begin{aligned}
\mu_V &= w_1\mu_1 + w_2\mu_2. \\
\sigma_V^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2c_{12},
\end{aligned}$$

where $w_1, w_2 \in \mathbb{R}$ and

$$w_1 + w_2 = 1.$$

Portfolios in the feasible set can be parameterized by one of the weights. Here we shall use $s = w_1$ as a parameter. Then $1 - s = w_2$, and the above expressions for μ_V and σ_V^2 can be written as

$$\begin{aligned}
\mu_V &= s\mu_1 + (1-s)\mu_2. \\
\sigma_V^2 &= s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)c_{12},
\end{aligned}$$

where $s \in \mathbb{R}$.