Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Options (General Properties):

Put-Call-Parity for European Options on a Stock That Pays No Dividends:

For a stock that pays no dividends the following relation holds between the prices of European call option (C^E) and European put option (P^E), both with exercise price X and exercise time T:

$$C^E - P^E = S(0) - Xe^{-rT}.$$

1. Suppose that $C^E - P^E > S(0) - Xe^{-rT}$.

   (a) At time $t = 0$:

      i. Sell one call option for C^E.

      ii. Buy one put option for P^E.

      iii. Buy one share for $S(0)$.

      iv. Invest the amount $C^E - P^E - S(0)$ at the rate r for time T.

   (b) At time $t = T$:

      i. Receive an amount $(C^E - P^E - S(0))e^{rT}$.

      ii. Sell the share for X either by exercising the put (if $S(T) \leq X$) or settling the short position in call (if $S(T) > X$).

   The balance will be $(C^E - P^E - S(0))e^{rT} + X > 0$, which violates the no-arbitrage principle.

2. Suppose that $C^E - P^E < S(0) - Xe^{-rT}$.

   (a) At time $t = 0$:

      i. Short sell one share for $S(0)$.

      ii. Buy one call option for C^E.

      iii. Sell one put option for P^E.

      iv. Invest the amount $S(0) - C^E + P^E$ at the rate r for time T.

   (b) At time $t = T$:

      i. Receive an amount $(S(0) - C^E + P^E)e^{rT}$.

      ii. Buy one share for X either by exercising the call (if $S(T) > X$) or settling the short position in put (if $S(T) \leq X$), and close the short position in stock.

   The balance will be $(S(0) - C^E + P^E)e^{rT} - X > 0$, which violates the no-arbitrage principle.

Put-Call-Parity for European Options on a Stock That Pays Dividends:

1. For a stock paying dividend between time 0 and expiry time T, with div_0 being the value of the dividend discounted to time 0,

$$C^E - P^E = S(0) - div_0 - Xe^{-rT}.$$

2. For a stock paying dividend continuously at a rate r_{div},

$$C^E - P^E = S(0)e^{-r_{div}T} - Xe^{-rT}.$$

<u>Put-Call-Parity Estimates for American Options on a Stock That Pays No Dividends:</u>
The prices of American call option (C^A) and American put option (P^A) with the same strike price X and expiry time T on a stock that pays no dividends satisfy:

$$S(0) - Xe^{-rT} \geq C^A - P^A \geq S(0) - X.$$

1. Suppose that $C^A - P^A - S(0) + Xe^{-rT} > 0$.

   (a) At time $t = 0$:

      i. Sell one call option for C^A.
      ii. Buy one put option for P^A.
      iii. Buy one share for $S(0)$.
      iv. Invest the amount $C^A - P^A - S(0)$ at the rate r.

   (b) At time $t \leq T$: If the holder of the American call chooses to exercise it at time $t \leq T$ then,

      i. Receive X for the share.
      ii. Receive an amount $(C^A - P^A - S(0))e^{rt}$.

      The balance will be

      $$(C^A - P^A - S(0))e^{rt} + X = (C^A - P^A - S(0) + Xe^{-rt})e^{rt} \geq (C^A - P^A - S(0) + Xe^{-rT})e^{rt} > 0,$$

      which violates the no-arbitrage principle.

   (c) At time $t = T$: If the call option is not exercised then,

      i. Receive an amount $(C^A - P^A - S(0))e^{rT}$.
      ii. Sell the share for X by exercising the put.

      The balance will be $(C^A - P^A - S(0))e^{rT} + X > 0$, which violates the no-arbitrage principle.

2. Suppose that $-C^A + P^A + S(0) - X > 0$.

   (a) At time $t = 0$:

      i. Sell one call option for C^A.
      ii. Buy one put option for P^A.
      iii. Short sell share for $S(0)$.
      iv. Invest the amount $-C^A + P^A + S(0)$ at the rate r.

(b) At time $t \leq T$: If the holder of the American put chooses to exercise it at time $t \leq T$ then,

i. Receive an amount $(-C^A + P^A + S(0))e^{rt}$.

ii. Buy a share for X and close the short sale position.

The balance will be $(-C^A + P^A + S(0))e^{rt} - X > Xe^{rt} - X > 0$, which violates the no-arbitrage principle.

(c) At time $t = T$: If the call option is not exercised then,

i. Receive an amount $(-C^A + P^A + S(0))e^{rT}$.

ii. Buy a share for X by exercising the call option and close the short sale position.

The balance will be $(-C^A + P^A + S(0))e^{rT} - X > Xe^{rT} - X > 0$, which violates the no-arbitrage principle.

Put-Call-Parity Estimates for American Options on a Stock That Pays Dividends:

1. For a stock paying dividend between time 0 and expiry time T, with div_0 being the value of the dividend discounted to time 0,

$$S(0) - Xe^{-rT} \geq C^A - P^A \geq S(0) - div_0 - X.$$

2. For a stock paying dividend continuously at a rate r_{div},

$$S(0) - Xe^{-rT} \geq C^A - P^A \geq S(0)e^{-r_{div}T} - X.$$

Bounds on Option Prices:

We first note the obvious inequalities

$$C^E \leq C^A, P^E \leq P^A,$$

for European and American options with the same strike price X and expiry time T. They hold because an American option gives at least the same rights as the corresponding European option.

It is also obvious that the price of a call or put option has to be non-negative because an option of this kind offers the possibility of a future gain with no liability. Therefore,

$$C^E \geq 0, P^E \geq 0.$$

Similar inequalities are of course valid for the more valuable American options, that is,

$$C^A \geq 0, P^A \geq 0.$$