# MA 322: Scientific Computing

# Lecture - 12

# Approximation of weighted definite integrals

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This shows that $w_1 = -w_0 = 4/\pi$.

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Gaussian quadrature rules are based on polynomial interpolation, but nodes as well as weights are chosen to maximize the degree of exactness.

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Thus the degree of exactness of 1-point Gaussian quadrature rule is 1.

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It is difficult to know whether this system has a solution, or how many, if any, additional equations can be added.

We now show that an $(n+1)$-point Gaussian quadrature rule $G_n(f)$ is exact for polynomials of degree $\leq 2n + 1$.

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Weighted inner product: If $\mu(x)$ is a positive function in $C[a, b]$ then $\langle f,\, g\rangle_\mu := \int_a^b f(x)g(x)\mu(x)dx$ is called a weighted inner product.

# Nodes for Gaussian quadrature rules

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Suppose that $G_n(f) = w_0 f(x_0) + \cdots + w_n f(x_n)$ is interpolatory. Then

$$G_n(f) = \int_a^b p_n(x)dx \text{ and } w_j = \int_a^b \ell_j(x)dx, \ j = 0 : n,$$

where $p_n(x)$ is the interpolating polynomial of degree $n$ that interpolates the data $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$.

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$$\int_a^b f(x)dx - \int_a^b p_n(x)dx = \int_a^b \frac{f^{(n+1)}(\theta_x)}{(n+1)!} w(x)dx = 0$$

when $f \in \mathcal{P}_{2n+1} \iff w(x) \perp \mathcal{P}_n$.

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Proof: Let $f \in \mathcal{P}_{2n+1}$. Then $f = qp + r$ for some $q, r \in \mathcal{P}_n$.

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Proof: Let $f \in \mathcal{P}_{2n+1}$. Then $f = qp + r$ for some $q, r \in \mathcal{P}_n$. Hence $f(x_j) = r(x_j),\ j = 0 : n$. Now

$$\int_a^b f(x)dx = \int_a^b r(x)dx = G_n(r) = \sum_{j=0}^{n} w_j r(x_j) = \sum_{j=0}^{n} w_j f(x_j) = G_n(f).$$

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Proof: Let $f \in \mathcal{P}_{2n+1}$. Then $f = qp + r$ for some $q, r \in \mathcal{P}_n$. Hence $f(x_j) = r(x_j),\ j = 0 : n$. Now

$$\int_a^b f(x)dx = \int_a^b r(x)dx = G_n(r) = \sum_{j=0}^{n} w_j r(x_j) = \sum_{j=0}^{n} w_j f(x_j) = G_n(f).$$

Note: Having determined the nodes $x_j$, the weights $w_j$ can be determined by method of undetermined coefficients.

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# Properties of Gaussian quadrature

Consider the Gaussian quadrature
$G_n(f) = w_0 f(x_0) + \cdots + w_n f(x_n) \approx \int_a^b f(x)dx$. Then the following hold:

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$$\left| \int_a^b f(x)dx - G_n(f) \right| \leq 2(b - a)E_n(f) \longrightarrow 0 \text{ as } n \to \infty.$$

# Properties of Gaussian quadrature

Proof:

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By Weierstrass theorem, $E_n(f) \to 0$ as $n \to \infty$. Hence the desired result follows. ■

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Given an inner product on $\mathcal{P}_n$, there exist orthogonal polynomials $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ of degree at most $n$ such that

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# Example: Legendre polynomial

Consider the inner product $\langle p, q \rangle := \int_{-1}^{1} p(x)q(x)dx$. Then

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$$\begin{aligned} \phi_0(x) &:= 1,\ \phi_1(x) := x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 = x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} 1 \cdot dx} \cdot 1 = x, \\ \phi_2(x) &= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 - \frac{\langle x^2, x \rangle}{\langle x, x \rangle} \cdot x = x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} \cdot x \\ &= x^2 - \frac{1}{3} - 0 \cdot x = x^2 - \frac{1}{3}. \end{aligned}$$

The polynomial $\phi_j(x)$ is called Legendre polynomial. The roots of $\phi_2(x) = x^2 - \frac{1}{3}$ are $\pm 1/\sqrt{3}$. This yields the two-point Gaussian quadrature rule

$$\int_{-1}^{1} f(x)dx \approx w_0 f\left(-\frac{1}{\sqrt{3}}\right) + w_1 f\left(\frac{1}{\sqrt{3}}\right).$$

Exactness at 1 and $x$ yield $w_0 = w_1 = 1$.

# Example: Chebyshev polynomial

Consider the inner product $\langle p, q\rangle := \int_{-1}^{1} p(x)q(x)(1-x^2)^{-1/2}dx$. Then $\phi_0(x) = 1$ and

$$\phi_1(x) \quad = \quad x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} \cdot 1 = x - \frac{\int_{-1}^{1} x(1-x^2)^{-1/2}dx}{\int_{-1}^{1}(1-x^2)^{-1/2}dx} \cdot 1 = x,$$

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Recall that Chebyshev polynomials satisfy the recurrence relation

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Proof: Note that $\phi_{n+1} - x\phi_n \in \mathcal{P}_n$, hence

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# Zeros of orthogonal polynomials

Theorem: Let $\phi_0(x), \ldots, \phi_{n+1}(x)$ be orthogonal polynomials such that

$$\phi_{n+1}(x) = (x - \alpha_n)\phi_n(x) - \beta_n\phi_{n-1}(x).$$

Define the Jacobi matrix

$$A = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ & & & \sqrt{\beta_{n-1}} & \alpha_n \end{bmatrix}.$$

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Proof: Define $\psi_j(x) := \phi_j(x)/\|\phi_j\|$. Dividing the recurrence relation by $\|\phi_n\|$ and noting that $\sqrt{\beta_n} = \|\phi_n\|//\|\phi_{n-1}\|$, we have

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which gives

$$x\psi_n(x) = \sqrt{\beta_{n+1}}\psi_{n+1}(x) + \alpha_n\psi_n + \sqrt{\beta_n}\psi_{n-1}.$$

Set $\Psi(x) := [\psi_0(x), \ldots, \psi_n(x)]^\top$. Then

$$x\Psi(x) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ & & & \sqrt{\beta_{n-1}} & \alpha_n \end{bmatrix} \begin{bmatrix} \psi_0(x) \\ \psi_1(x) \\ \vdots \\ \vdots \\ \psi_n(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{\beta_{n+1}}\psi_{n+1}(x) \end{bmatrix}.$$

# Zeros of orthogonal polynomials

Proof: Define $\psi_j(x) := \phi_j(x)/\|\phi_j\|$. Dividing the recurrence relation by $\|\phi_n\|$ and noting that $\sqrt{\beta_n} = \|\phi_n\|//\|\phi_{n-1}\|$, we have

$$\sqrt{\beta_{n+1}}\psi_{n+1}(x) = (x-\alpha_n)\psi_n(x) - \sqrt{\beta_n}\psi_{n-1}$$

which gives

$$x\psi_n(x) = \sqrt{\beta_{n+1}}\psi_{n+1}(x) + \alpha_n\psi_n + \sqrt{\beta_n}\psi_{n-1}.$$

Set $\Psi(x) := [\psi_0(x), \ldots, \psi_n(x)]^\top$. Then

$$x\Psi(x) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ & & & \sqrt{\beta_{n-1}} & \alpha_n \end{bmatrix} \begin{bmatrix} \psi_0(x) \\ \psi_1(x) \\ \vdots \\ \vdots \\ \psi_n(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{\beta_{n+1}}\psi_{n+1}(x) \end{bmatrix}.$$

This shows that

$$\phi_{n+1}(x_j) = 0 \iff \psi_{n+1}(x_j) = 0 \iff \det(A - x_j I) = 0.$$