Lecture 16

Numerical Approximations for IVP

MA 322: Scientific Computing

by

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1 Convergence of Euler Method

Our IVP

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{1}$$

1.1 Example

Like earlier, let us discuss Euler method for a numerical example.

Example: Consider the IVP

$$y' = \frac{1}{1+x^2} - 2y^2, \quad y(0) = 0. \tag{2}$$

The true solution is given by

$$y(x) = \frac{x}{1+x^2} \tag{3}$$

and the numerical solutions at different points for different h are presented in the following table (Source: An Introduction to Numerical Analysis, by Atkinson) using the Euler scheme

$$y_{n+1} = y_n + hf(x_n, y_n), \quad y_0 = 0 \;\; \& \;\; x_{n+1} = x_n + h, \quad x_0 = 0. \tag{4}$$

Euler's method

	x	$y_h(x)$	$y(x)$	$y(x) - y_h(x)$
$h = .2$	0.00	0.0	0.0	0.0
	.40	.37631	.34483	−.03148
	.80	.54228	.48780	−.05448
	1.20	.52709	.49180	−.03529
	1.60	.46632	.44944	−.01689
	2.00	.40682	.40000	−.00682
$h = .1$	.40	.36085	.34483	−.01603
	.80	.51371	.48780	−.02590
	1.20	.50961	.49180	−.01781
	1.60	.45872	.44944	−.00928
	2.00	.40419	.40000	−.00419
$h = .05$	.40	.35287	.34483	−.00804
	.80	.50049	.48780	−.01268
	1.20	.50073	.49180	−.00892
	1.60	.45425	.44944	−.00481
	2.00	.40227	.40000	−.00227

In order to assess the accuracy of the numerical method (4), we define the error, e_n, by

$$e_n = y(x_n) - y_n. \tag{5}$$

Now, we wish to show that, for Euler method, the error at each grid point decreases to zero as h decreases to 0. The associated theory is called convergence theory. In developing convergence theory, we will require some preliminary results.

Lemma 1.1 *If the numbers $|E_i|$, $i = 0,\ 1,\ 2,\ 3, \ldots,\ n$, satisfy*

$$|E_{i+1}| \leq A|E_i| + B, \quad i = 0,\ 1,\ 2,\ 3, \ldots,\ n-1 \tag{6}$$

where A and B are nonnegative constants and $A \neq 1$, then

$$|E_i| \leq A^i|E_0| + \frac{A^i - 1}{A - 1}B, \; i = 1,\ 2,\ 3, \ldots,\ n. \tag{7}$$

Proof. For $i = 0$, (6) yields

$$|E_1| \leq A|E_0| + B = A|E_0| + \frac{A-1}{A-1}B,$$

so that (7) is valid for $i = 1$. The proof is now completed by induction. Assume that for fixed i, (7) is valid, that is

$$|E_i| \leq A^i|E_0| + \frac{A^i - 1}{A-1}B. \tag{8}$$

Then we must prove that

$$|E_{i+1}| \leq A^{i+1}|E_0| + \frac{A^{i+1} - 1}{A-1}B. \tag{9}$$

Using (8) in (6), we obtain

$$|E_{i+1}| \leq A\left[A^i|E_0| + \frac{A^i - 1}{A-1}B\right] + B = A^{i+1}|E_0| + \frac{A^{i+1} - 1}{A-1}B$$

and the proof is completed.

Remark: The value of Lemma 1.1 is as follows:

- If each term of a sequence $\{|E_i| : \ i = 0,\ 1,\ 2,\ \ldots\}$ is related to the previous term by (6), then Lemma 1.1 enables one to relate each term directly to $|E_0|$ only, that is, to the very first term of the sequence.

Now, we are in a position to discuss the main result of this lecture.

Theorem 1.1 Suppose y is an unique solution for the IVP

$$y' = f(x, y), \ \ y(0) = \alpha \tag{10}$$

in $I = [0, L],\ L > 0$. Assume that y' and y'' are continuous and that there exist positive constants $M,\ N$ such that

$$|y''(x)| \leq N \ \ \& \ \ \left|\frac{\partial f}{\partial y}\right| \leq M, \quad 0 \leq x \leq L, \quad -\infty < y < \infty. \tag{11}$$

Next, let I be subdivided into n equal parts by the grid points $x_0 < x_1 < x_2 < \ldots < x_n$, where $x_0 = 0$ and $x_n = L$. The grid size is given by $h = \frac{L}{n}$. Let y_k be the numerical solution of (10) by Euler's method on the grid point x_k, so that

$$y_{k+1} = y_k + hf(x_k, y_k), \ \ y_0 = \alpha, \ \ k = 0, 1, 2, \ldots, n-1. \tag{12}$$

Finally, define the error e_k at each grid point x_k by

$$e_k = y(x_k) - y_k \ \ k = 0, 1, 2, \ldots, n. \tag{13}$$

Then

$$|e_k| \leq \frac{\Big[(1 + Mh)^k - 1\Big]Nh}{2M}, \quad k = 0, 1, 2, \ldots, n. \tag{14}$$

Proof. First observe that

$$|e_{k+1}| = |y(x_{k+1}) - y_{k+1}| = |y(x_k + h) - y_{k+1}|.$$

Introducing Taylor expansion for $y(x_k+h)$ implies

$$\begin{aligned}
|e_{k+1}| &= |y(x_k+h)-y_{k+1}| = |y(x_k)+hy'(x_k)+\frac{h^2}{2}y''(\xi_k)-(y_k+hf(x_k,y_k))| \\
&= |(y(x_k)-y_k)+h(y'(x_k)-f(x_k,y_k))+\frac{h^2}{2}y''(\xi_k)| \\
&= |e_k+h(f(x_k,y(x_k))-f(x_k,y_k))+\frac{h^2}{2}y''(\xi_k)| \\
&= |e_k+h(f(x_k,Y_k)-f(x_k,y_k))+\frac{h^2}{2}y''(\xi_k)|, \quad Y_k=y(x_k). \qquad (15)
\end{aligned}$$

Now, by the MVT for a function of two variables, we obtain

$$f(x_k,Y_k)-f(x_k,y_k)=(Y_k-y_k)\frac{\partial f}{\partial y}(x_k,\eta_k)=e_k\frac{\partial f}{\partial y}(x_k,\eta_k), \qquad (16)$$

where η_k is areal number between $Y_k=y(x_k)$ and y_k. Using above identity in (15), we arrive at

$$\begin{aligned}
|e_{k+1}| &= \left|e_k+he_k\frac{\partial f}{\partial y}(x_k,\eta_k)+\frac{h^2}{2}y''(\xi_k)\right| \\
&\le |e_k|+h|e_k|M+\frac{h^2}{2}N. \qquad (17)
\end{aligned}$$

Here, we have used (11). Therefore, we arrive at

$$|e_{k+1}|\le|e_k|(1+Mh)+\frac{h^2}{2}N \qquad (18)$$

which together with Lemma 1.1, for $A=(1+Mh)$ and $B=\frac{h^2}{2}N$, we obtain

$$|e_k|\le(1+Mh)^k|e_0|+\frac{(1+Mh)^k-1}{(1+Mh)-1}\left(\frac{h^2}{2}N\right). \qquad (19)$$

However, since $e_0=y(x_0)-y_0=\alpha-\alpha=0$, estimate (20) simplifies to

$$\begin{aligned}
|e_k| &\le \frac{(1+Mh)^k-1}{Mh}\left(\frac{h^2}{2}N\right) \\
&= \frac{\Big[(1+Mh)^k-1\Big]Nh}{2M}. \qquad (20)
\end{aligned}$$

This completes the rest of the proof.

Corollary 1.1 *Now, for $x\ge-1$, use the fact that*

$$0\le(1+x)^m\le e^{mx}$$

to have

$$\begin{aligned}
|e_k| &\le \frac{\big(e^{Mhk}-1\big)Nh}{2M}=\frac{\big(e^{Mx_k}-1\big)Nh}{2M}, \quad x_k=x_0+kh=kh \\
&\le \frac{\big(e^{Mx_n}-1\big)Nh}{2M}=\frac{\big(e^{ML}-1\big)Nh}{2M}, \quad L=x_n. \qquad (21)
\end{aligned}$$

Thus, there exists a positive constant C independent of the mesh parameter h such that

$$|e_k|\le Ch, \quad C=\frac{\big(e^{ML}-1\big)N}{2M}, \; k=0,1,2,\ldots,n. \qquad (22)$$

In particular, we obtain

$$|e_n|\le Ch\to0 \;\; as \;\; h\to0. \qquad (23)$$

Remark 1.2 At each grid point, roughly, we obtain

$$|\text{error}| \approx Ch, \quad \text{where } C \text{ is independent of } h.$$

Therefore, when h is halved, we obtain

$$|\text{new error}| \approx C\frac{h}{2},$$

so that

$$\frac{|\text{new error}|}{|\text{error}|} \approx \frac{1}{2}.$$

Hence, the error at each point grid point decreases by about half when h is halved.