We have seen in the last class that property α is always satisfied by choice function when it is defined. It means when that the binary relation satisfies reflexivity, completeness, and acyclicity.

Consider an example $xIy,\ xPz,\ zPy$, then the choice sets are $C([x,y],R) = [x,y]$ and $C([x,y,z],R) = [x]$. It violates property β.

We have done that if R is an ordering then, for all $x, y, z \in S$, xPy & $yIz \ \rightarrow xPz$.
When the above is satisfied by a binary relation R we call R to satisfy PI transitivity.

Relation R is PI transitive over S if and only if for all $x, y, z \in S$, xPy & $yIz \ \rightarrow \ xPz$.

Result:
A choice function $C(S, R)$ generated by a binary relation R satisfies property β if and only if R is PI transitive.

Proof: R must satisfy reflexivity and completeness then only choice function can be defined. We have seen it. We need one more property that is acyclicity to be able to define choice function. We have seen that reflexivity, completeness and acyclicity does not satisfy property β.
Suppose PI transitivity is not satisfied. It implies that here is a triple $x, y, z \in S$ such that xpy, yIz and zRx. From this we $C([y, z], R) = [y, z]$. And $C([x, y, z], R) = [z]$ and $y \in C([x, y, z], R)$ is not true. It violated property β. If R is not PI transitive then property β is violated. We get that PI transitivity is a necessary condition.

We proof the sufficiency part. Suppose property β is violated. Then we have a pair $x, y \in S_1$ such that $x, y \in C(S_1, R)$ and $x \in C(S_2, R)$, $y \notin C(S_2, R)$ when $S_1 \subset S_2$. This implies that there exist a $z \in S_2$ such that zPy and xRz. Since $x, y \in C(S_1, R)$. It implies that xIy. Suppose PI transitivity holds, so we have zPy and yIx implying zPx. We already have xRz . A contradiction. Therefore, PI transitivity cannot hold. Thus, we get PI implies that property β is satisfied.

We get that PI is a stronger condition then acyclicity.

We have when R satisfies reflexivity, completeness and PI transitivity, the choice function defined gives rational outcomes. It means that consistent outcomes.

If R_i is an ordering and it satisfies continuity and monotonicity then we can define a real valued function called utility function $U(x)$.

We move to social choice . We assume that X is the set of social sates. There are n individuals. The preference relation of the ith individual is $R_i,\ i = 1, 2, 3, ...n$. R denotes the social preference relation.

We assume that each individual i has an ordering over the elements in X. Each R_i satisfies reflexivity, completeness and transitivity. We dont assume that R social preference relation has to satisfy these three properties. We will find when R is going to be an ordering.

A collective choice rule is a functional relation F such that for any set of n individual orderings $R_1, R_2, ...R_n$ one and only social preference relation R is determined,
$R = f(R_1, R_2, ...R_n)$.
Suppose x, y, z. There are three individual. The preference relation are:

1	2	3
x	x	x
y	z	yz
z	y	

The collective choice rule is $f\begin{pmatrix} x & x & x \\ y & z & yz \\ z & y & \end{pmatrix}$

A collective choice rule is decisive if and only if its range is restricted to complete preference relations R.
For all $x, y \in X$
$x\bar{R}y$ if and only if for all $i : xR_iy$.
$x\bar{P}y$ if and only if $x\bar{R}y$ & $\sim (y\bar{R}x)$. Pareto preference.
$x\bar{I}y$ if and only if $x\bar{R}y$ & $y\bar{R}x$. Indifference.

From these Pareto relations we define collective choice rule under certain condition.

Result:
Relation $\bar{R}$ is a quasi -ordering for every logically possible combination of individual preferences.

Proof: $\bar{R}$ is reflexive as for all $x \in X : wehavexR_ix$. Since R_i are reflexive.
For all $x, y, z \in X$, $x\bar{R}y$ and $y\bar{R}z \rightarrow$ for all i xR_iy and yR_iz.
Since R_i are transitive.
It implies for all $i : xR_iz$. This implies $x\bar{R}z$. Thus, $\bar{R}$ is transitive.

Completeness can be violated. For example xP_1y and yP_2x then $x\bar{R}y$ and $y\bar{R}x$ are not defined.

Result:
A necessary and sufficient condition for $\bar{R}$ to be an ordering and for $R = \bar{R}$ to be decisive collective choice rule is that for all x,$x, y \in X$ if there exist i such that xP_iy then xR_jy for all j.

Proof: For any pair x, y if xI_iy for all i, then the condition is trivially true. If for i xP_iy then for all j we must have xR_jy. It implies $x\bar{R}y$.

If condition is violated than there exist i such that xP_iy and there exist j such that yP_jx. It implies that $x\bar{R}y$ is not possible and $y\bar{R}x$ is not possible. So $\bar{R}$ is not complete.