Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Cox-Ross-Rubinstein Formula:

The payoff for a call option with strike price X satisfies $f(x) = 0$ for $x \leq X$. Accordingly, the summation starts with the smallest value "m" of k such that,

$$S(0)(1+U)^m(1+D)^{N-m} > X.$$

Hence we have the price of a call option as,

$$\begin{aligned} C_E(0) &= (1+R)^{-N} \sum_{k=m}^{N} \binom{N}{k} p_*^k (1-p_*)^{N-k} \left(S(0)(1+U)^k (1+D)^{N-k} - X \right), \\ &= S(0) \sum_{k=m}^{N} \binom{N}{k} q^k (1-q)^{N-k} - (1+R)^{-N} X \sum_{k=m}^{N} \binom{N}{k} p_*^k (1-p_*)^{N-k}, \end{aligned}$$

where,

$$q = p_* \frac{1+U}{1+R} \text{ and } 1-q = (1-p_*) \frac{1+D}{1+R}.$$

A similar formula can be derived for the put option, either directly, or by using the put-call parity.

Let us now denote by $\Phi(m, N, p)$, the cumulative binomial distribution with N trials, with the probability of success in each trial being p, *i.e.,*

$$\Phi(m, N, p) = \sum_{k=0}^{m} \binom{N}{k} p^k (1-p)^{N-k}.$$

We now summarize the results as follows:

Cox-Ross-Rubinstein Formula:

In the binomial model, the price of an European call and an European put option, with strike X, to be exercised after N time steps is given by,

$$\begin{aligned} C_E(0) &= S(0) \left[1 - \Phi(m-1, N, q) \right] - (1+R)^{-N} X \left[1 - \Phi(m-1, N, p_*) \right], \\ P_E(0) &= -S(0)\Phi(m-1, N, q) + (1+R)^{-N} X \Phi(m-1, N, p_*). \end{aligned}$$

American Claims:

An American option can be exercised at any time n such that $0 \leq n \leq N$, with the payoff being $f(S(n))$. The function f remains unchanged for all n. The price of the American claim at time n will be denoted by $H_A(n)$.

For simplicity, we first analyze a claim expiring after two time periods. If it is not exercised till expiration, then its worth at that time is

$$H_A(2) = f(S(2)),$$

with three possible values, due to three possible values of $S(2)$. At time $t = 1$, the holder of the option has two choices: exercise immediately, with payoff $f(S(1))$, or wait until time $t = 2$, when the value of the claim would be $f(S(2))$. The value of an option which is not exercised at time $t = 1$ is given by

$$\frac{1}{1+R}\left[p_* f(S(1)(1+U)) + (1-p_*) f(S(1)(1+D))\right].$$

Thus, the holder has the choice between this or the immediate payoff $f(S(1))$. The value of the American claim at time $t = 1$ would be the higher of these two, *i.e.,*

$$H_A(1) = \max\left\{f(S(1)), \frac{1}{1+R}\left[p_* f(S(1)(1+U)) + (1-p_*) f(S(1)(1+D))\right]\right\} = f_1(S(1)),$$

where

$$f_1(x) = \max\left\{f(x), \frac{1}{1+R}\left[p_* f(x(1+U)) + (1-p_*) f(x(1+D))\right]\right\}.$$

A similar argument results in the following value at time $t = 0$,

$$H_A(0) = \max\left\{f(S(0)), \frac{1}{1+R}\left[p_* f_1(S(0)(1+U)) + (1-p_*) f_1(S(0)(1+D))\right]\right\}.$$

<u>Example:</u>

Consider an American put option with $S(0) = 80$, $X = 80$, $T = 2$, $U = 0.1$, $D = -0.05$ and $R = 0.05$. The stock values, then, are

$$S(0) = 80.00; S^u(1) = 88.00, S^d(1) = 76.00; S^{uu}(2) = 96.80, S^{ud}(2) = S^{du}(2) = 83.60, S^{dd}(2) = 72.20.$$

The price of the American put at time n is denoted by $P_A(n)$.

Therefore, at the expiration time $t = 2$,

$$P_A^{uu}(2) = 0.00, P_A^{ud}(2) = P_A^{du}(2) = 0.00 \text{ and } P_A^{dd} = 7.80.$$

The values at time $t = 1$ are

$$P_A^u(1) = \max(0.00, 0.00) = 0.00 \text{ and } P_A^d(1) = \max(4.0, 2.8) = 4.0.$$

Finally, at time $t = 0$,

$$P_A(0) = \max(0.00, 1.27) = 1.27.$$

Thus the price of the American put is 1.27. It can be shown that the price of the corresponding European put is 0.79.