

Lecture #07

Topic : Generation of geometric Brownian motion and jump-diffusion sample paths

We begin our discussion on methods for simulating paths for a variety of stochastic processes relevant to financial engineering.

The emphasis is on methods for exact simulation of continuous time processes at a discrete set of dates.

The word "exact" here implies that the distribution of the simulated values coincides with the distribution of the continuous time process on the simulation time grid.

§ Brownian Motion

One Dimension

By a "standard" one dimensional Brownian motion on $[0, T]$, we mean a "stochastic process" $\{W(t), 0 \leq t \leq T\}$

with the following properties:

(i) $W(0) = 0$

(ii) The mapping $t \mapsto W(t)$ is, with probability 1, a continuous function on $[0, T]$

(iii) The increments $\{W(t_1) - W(t_0), W(t_2) - W(t_1), \dots, W(t_k) - W(t_{k-1})\}$ are independent for any k and any $0 \leq t_0 < t_1 < \dots < t_k \leq T$.

(iv) $W(t) - W(s) \sim N(0, t-s)$, for any $0 \leq s < t \leq T$.

Note: $W(t) \sim N(0, t)$ for $0 < t \leq T$.

For constants μ and $\sigma > 0$, we call a process $X(t)$ a Brownian motion with drift μ and diffusion coefficient σ^2 , represented as $X \sim BM(\mu, \sigma^2)$ if

$$\frac{X(t) - \mu t}{\sigma}$$

is a standard Brownian motion.

Accordingly, a Brownian motion may be constructed from a standard Brownian motion by setting

$$X(t) = \mu t + \sigma W(t).$$

It follows that $X(t) \sim N(\mu t, \sigma^2 t)$.

Furthermore, $X(t)$ solves the SDE:

$$dX(t) = \mu dt + \sigma dW(t)$$

The assumption of $X(0) = 0$ is a normalization.

However, one may construct a Brownian motion with parameters μ and σ^2 with the initial value x by just adding x to each $X(t)$.

For time dependent (but deterministic) $\mu(t)$ and $\sigma(t) > 0$, a Brownian motion, with drift μ and diffusion coefficient $\sigma^2 > 0$, may be defined via the SDE:

$$dX(t) = \mu(t) dt + \sigma(t) dW(t).$$

Equivalently,

$$X(t) = X(0) + \int_0^t \mu(s) ds + \int_0^t \sigma(s) dW(s)$$

where $X(0)$ is a constant.

The process $X(t)$ has continuous sample paths and independent increments. Each increment $X(t) - X(s)$ is normally distributed with mean

$$E [X(t) - X(s)] = \int_s^t \mu(u) du$$



and variance

$$\text{Var} [X(t) - X(s)] = \text{Var} \left[\int_s^t \sigma(u) dW(u) \right]$$

$$= \int_s^t \sigma^2(u) du.$$

Random Walk Construction :

In the simulation of Brownian motion , we focus on
the simulation of $(W(t_1), W(t_2), \dots, W(t_n))$ or
 $(X(t_1), X(t_2), \dots, X(t_n))$ at a fixed set of points

$$0 < t_1 < t_2 < \dots < t_n$$

Let Z_1, Z_2, \dots, Z_n be independent $N(0, 1)$ variables , generated
using methods already discussed in the course.

For a standard Brownian motion, we set $t_0 = 0$ and $W(t_0) = 0$,
and the subsequent values are generated as follows:

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} Z_{i+1}, i = 0, 1, 2, \dots, n-1$$

For $X \sim BM(\mu, \sigma^2)$ with constant μ and σ , we set $t_0 = 0$
and $W(t_0) = 0$, and the subsequent values are generated as

follows: $X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma \sqrt{t_{i+1} - t_i} Z_{i+1}$

$$j = 0, 1, 2, \dots, n-1.$$

When the coefficients are time dependent, the recursion becomes:

$$X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} \mu(s) ds + \sqrt{\int_{t_i}^{t_{i+1}} \sigma^2(u) du} Z_{i+1},$$
$$i = 0, 1, 2, \dots, n-1.$$

Using the Euler approximation for the integrals, we obtain

$$X(t_{i+1}) = X(t_i) + \mu(t_i)(t_{i+1} - t_i) + \sigma(t_i) \sqrt{t_{i+1} - t_i} Z_{i+1}$$

$i = 0, 1, 2, \dots, n-1$.

ϕ Geometric Brownian Motion

One Dimension

A stochastic process $S(t)$ is a geometric Brownian motion if $\log S(t)$ is a Brownian motion with initial value $\log S(0)$.

In other words, a geometric Brownian motion is simply an exponential Brownian motion

Whereas ordinary Brownian motion can take negative values, which is undesirable in a stock price modelling, geometric Brownian motion is always positive because the exponential function only takes positive values.

In the context of asset pricing the percentage changes

$$\frac{S(t_2) - S(t_1)}{S(t_1)}, \frac{S(t_3) - S(t_2)}{S(t_2)}, \dots, \frac{S(t_n) - S(t_{n-1})}{S(t_{n-1})}$$

are independent for $t_1 < t_2 < t_3 < \dots < t_n$ (rather than absolute changes $S(t_{i+1}) - S(t_i)$).

A geometric Brownian motion process is specified through an SDE of the form :

$$\frac{ds(t)}{s(t)} = \mu dt + \sigma dw(t)$$

Using Ito's Lemma, we obtain,

$$d \log s(t) = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dw(t)$$

Here μ : Drift parameter and σ : Volatility parameter

Solving, with the initial value of $S(0)$, we obtain

$$S(t) = S(0) \exp \left((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \right)$$

More generally, if $u < t$, then

$$S(t) = S(u) \exp \left((\mu - \frac{1}{2}\sigma^2)(t-u) + \sigma (W(t) - W(u)) \right).$$

Since the increments of W are independent and normally

distributed, we therefore have the following recursive procedure for simulating values of S at $0 = t_0 < t_1 < t_2 < \dots < t_n$:

$$S(t_{i+1}) = S(t_i) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} Z_{i+1}\right),$$

$$i = 0, 1, 2, \dots, n-1$$

With Z_1, Z_2, \dots, Z_n being independent $N(0, 1)$ variables.