
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Example on Speculating on Forward Rates:

The one year pure discount bond with nominal 100 trades at 95, and the two year pure discount bond with nominal 100 trades at 89. The one year spot rate r_{1y} is given by $95(1 + r_{1y}) = 100$ with the solution $r_{1y} = 5.2632\%$, and the two year spot rate r_{2y} is given by $89(1+r_{2y})^2 = 100$ with the solution $r_{2y} = 5.9998\%$. The forward rate $f_{1y,2y}$ is given by,

$$(1.052632) \times (1 + f_{1y,2y}) = (1.059998)^2$$

with the solution $f_{1y,2y} = 6.7416\%$. If the speculator thinks that the rates will not change very much over the next year, she/he considers the forward rate to be relatively high. She/he wants to lock in the forward rate for her/his investment in the period between year one and year two. In order to do so (without a forward contract), she/he buys the two year pure discount bond at 89 and sells short 89/95 units of the one year pure discount bond at 95. The cost of this portfolio today is zero. One year from now, the one year pure discount bond matures and the speculator has to pay $(89/95) \times 100 = 93.6842$. Another year later, the two year pure discount bond matures, and she/he receives 100. The interest rate the investor gets in the second year with that trading strategy is $\frac{100-93.6842}{93.6842} = 0.067416$, that is, the forward rate. If one year from now the spot rate is less than 6.7416% as the investor had predicted, she/he can sell the one year bond short, get more than 93.6842, cover her/his short position in the 89/95 units of the previous one-year bond by paying 93.6842 and making a profit. At the end of the second year she/he gets 100 from the two-year bond, which is exactly enough to cover is short position in the one-year bond.

Prices of bonds are the result of three factors: the likelihood of default, the term structure of interest rates and the time left until maturity or coupon payments.

1. Time change gradually and in a totally predictable way, so there is basically no risk associated with the last factor.
2. We will ignore the risk of default and will focus on “risk-free” bonds.

Thus the price risk of this type of bonds depends exclusively on interest rate. The approach we will use is to consider bonds as derivatives, whose underlying variable is the interest rate. A peculiarity of this underlying variable is that it is not directly traded in the markets (unlike the underlying of plain vanilla stock options, for example). We consider the main risk measure for bonds, called duration. We explore the notion of Macaulay duration for measuring interest-rate risk, introduced in 1938. Although it is based on several restrictive assumptions, its simplicity and the fact that it captures the main factor that affects the bond price volatility makes it the fundamental tool for bond portfolio management.

Definition and Interpretation:

We first focus on the simplest case. Consider a risk-free pure discount bond that will pay a nominal amount (say, 100) in exactly one period (for example, one year) from today. The one-year spot rate is r. Therefore, the price P of this bond is,

$$P = \frac{100}{1+r}.$$

An investor who holds the bond only has to worry about one type of risk, a possible change (more explicitly, an increase) in the interest rate. Therefore, the natural measure of the risk of holding the bond is the derivative of the bond price with respect to the interest rate,

$$\frac{\partial P}{\partial r} = -\frac{100}{(1+r)^2} = -\frac{P}{1+r}.$$

We see that the effect on the price of the bond, of an increase in the level of interest rate will be larger (in absolute value), the larger the price of the bond. We can easily extend the same type of analysis to a pure discount bond that has several periods, say, T, to maturity. Suppose that the T-period spot rate is r. The price of the bond is,

$$P = \frac{100}{(1+r)^T},$$

and its sensitivity to a change in the interest rate is,

$$\frac{\partial P}{\partial r} = -T\frac{100}{(1+r)^{T+1}} = -T\left(\frac{P}{1+r}\right),$$

which is similar to the previous equation, with the additional factor of time left to maturity. That is, the price of a bond is more sensitive to a change in interest rates, the longer its maturity.

Consider a bond that pays the coupons of size C_i at the end of each period, for T periods (we assume that the final coupon C_T includes the principal). Suppose further that the corresponding rates are denoted by $r_i, i = 1, 2, \ldots, T$ so that the price of the bond is,

$$P = \sum_{i=1}^{T} \frac{C_i}{(1+r_i)^i}.$$

We see that we cannot extend in a straightforward manner the analysis we performed for pure discount bonds, since the price of the bond is determined by a collection of interest rates that change with time. However, in practice, interest rate for different maturities are highly correlated and they frequently (though not always) move in parallel, that is, if the one-year spot rate goes up, so does the five year spot rate, for example. The problem is, then, to find a single number that will be a good representation of the whole term structure and whose changes will be a good summary of changes in the term structure. The obvious candidate is the yield or internal rate of return of the bond. Recall that the yield of the bond is the number that y that satisfies,

$$P = \sum_{i=1}^{T} \frac{C_i}{(1+y)^i}.$$

Comparing the equations, it is clear that the yield depends only on the rates r_i that affect the price of the bond. For example, if all the rates r_i go up, the yield y also goes up. Then, similar to our analysis of risk for pure discount bonds, we consider the derivative of the coupon bond with respect to its yield,

$$\frac{\partial P}{\partial y} = -\sum_{i=1}^{T} i\frac{C_i}{(1+y)^{i+1}} = -\frac{P}{1+y}\sum_{i=1}^{T} i\frac{\frac{C_i}{(1+y)^i}}{P}$$

where the second equality is simply the result of multiplying and dividing by P. The second factor of the equation is called <u>duration</u> or <u>Macaulay duration</u>.

<u>Definition:</u>

The duration of a bond is a measure of the sensitivity of the bond to interest rate movements. More explicitly, the duration D of a bond that pays coupons C_i and has yield y and maturity T is given by,

$$D := \sum_{i=1}^{T} i \frac{\frac{C_i}{(1+y)^i}}{P}.$$

It is clear that the sensitivity of a bond to interest rate changes (measured through the change is y) is larger, the larger the duration of the bond. The holder of a bond with high duration has more interest rate exposure than the holder of a bond with low duration.

The duration is the weighted average of all the time points i at which the bond makes a payment. Each of these numbers is assigned a weight $\frac{1}{P}\frac{C_i}{(1+y)^i}$. This actually adds up to one, since $\sum_{i=1}^{T} \frac{C_i}{(1+y)^i} = P$. The weight corresponding to time point i is the proportion of the value of the bond that corresponds to the payment received at i. This average places more weight on the time points that are more important for the value of the bond. For example, a bond that pays low coupons initially and large coupons later on will have a higher duration than a bond that pays high coupons early and low coupons later. The duration of a pure discount bond is equal to its maturity.

Examples:

1. Consider a bond with a nominal value of 100 that pays a coupon of 8 at the end of each year and has five years left until maturity. Suppose that the yield of this bond is 5%. The price of the bond is,

$$P = \sum_{i=1}^{5} \frac{8}{(1.05)^i} + \frac{100}{(1.05)^5} = 112.99.$$

   Duration of the bond is,

$$D = \sum_{i=1}^{5} i \frac{\frac{8}{(1.05)^i}}{112.99} + 5 \frac{\frac{100}{(1.05)^5}}{112.99} = 4.36.$$

2. Now consider the above example with 30 years left until maturity. Then,

$$P = \sum_{i=1}^{30} \frac{8}{(1.05)^i} + \frac{100}{(1.05)^{30}} = 146.12$$

   and

$$D = \sum_{i=1}^{30} i \frac{\frac{8}{(1.05)^i}}{146.12} + 30 \frac{\frac{100}{(1.05)^{30}}}{146.12} = 14.82.$$

3. Finally consider the previous example, but with an yield of 10% instead of 5%. We get,

$$P = \sum_{i=1}^{30} \frac{8}{(1.1)^i} + \frac{100}{(1.1)^{30}} = 81.15$$

   and

$$D = \sum_{i=1}^{30} i \frac{\frac{8}{(1.1)^i}}{81.15} + 30 \frac{\frac{100}{(1.1)^{30}}}{81.15} = 10.65.$$