Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Convexity:

Convexity is the measure of the error we are exposed to, by using the first derivative to predict the impact of a change in interest rates on the price of a portfolio of bonds. If we consider the second derivative of the bond price with respect to the yield, then,

$$\frac{\partial^2 P}{\partial y^2} = \sum_{i=1}^{T} i(i+1)\frac{C_i}{(1+y)^{i+2}}$$

We see that it is positive, and therefore, the relationship between the price of a bond and its yield is convex. The stronger the convexity (the larger the second derivative), the larger the error we are exposed to by using the duration for immunization purposes. The convexity is defined as,

$$C = \frac{1}{P}\frac{\partial^2 P}{\partial y^2}.$$

Using a Taylor's Series expansion up to the second-order term we see,

$$\Delta P = \frac{\partial P}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(\Delta y)^2 = -\frac{D}{1+y}P\Delta y + \frac{1}{2}CP(\Delta y)^2.$$

Risk and Return:

An investment in a risky security always carries the burden of possible losses or poor performance. In the next few lectures we will analyze the advantages of spreading the investment among several securities to keep the inevitable risk under control.

We will assume a simple single time step setup but consider many risky securities. The length of the time step can be arbitrary, and our typical example will be one year. We will use the notation 0 for the beginning and T for the end of the time period.

Expected Return:

Suppose we make a single period investment in some stock with known current price $S(0)$. The future price $S(T)$ is unknown, hence assumed to be a random variable

$$S(T) : \Omega \to [0, +\infty)$$

on a probability space Ω. The return

$$K = \frac{S(T) - S(0)}{S(0)}$$

is a random variable with expected value $E(K) = \mu_K = \mu$. We introduce the convention of using the Greek letter μ for expectations of various random returns, with subscripts indicating the context, if necessary. By the linearity of mathematical expectation

$$\mu = \frac{E(S(T)) - S(0)}{S(0)}.$$

The relationships between the prices and returns can, thus, be written as

$$S(T) = S(0)(1+K),$$
$$E(S(T)) = S(0)(1+\mu),$$

which indicates the possibility of reversing the approach, that is, given the returns, we can find the prices.

Standard Deviation as Risk Measure:

First of all, we need to identify a suitable quantity to measure risk. The uncertainty is understood as the scatter of returns around some reference point. A natural candidate for the reference value is the expected return. The extent of scatter can then be conveniently measured by standard deviation. This notion takes care of two aspects of risk:

1. The distances between possible values and the reference point.

2. The probabilities of attaining these values.

Definition:

By (the measure of) *risk* we mean the standard deviation

$$\sigma_K = \sqrt{Var(K)}$$

of the return K,

$$Var(K) = E(K-\mu)^2 = E(K^2) - \mu^2$$

being the variance of the return.

Example:

Let the return on an investment be $K = 3\%$ or -1%, both with probability 0.5. Then the risk is $\sigma_K = 0.02$. Now suppose that the return on another investment is double that on the first investment, being equal to $2K = 6\%$ or -2%, also with probability 0.5 each. Then the risk of the second investment will be $\sigma_{2K} = 0.04$. The risk as measured by the standard deviation is doubled.

This illustrates the following general rule:

$$\sigma_{aK} = |a|\sigma_K$$
$$Var(aK) = a^2 Var(K)$$

for any real number a.

Two Securities:

We begin a detailed discussion of the relationship between risk and expected return in the simple situation of a portfolio with just two risky securities.

Example:

Suppose that the prices of two stocks behave as follows:

If we split our money equally between these two stocks, then we shall earn 2.5%. Even though an investment in either stock separately involves risk, we have reduced the overall risk to nil by splitting the investment

Scenario	Probability	Return (K_1)	Return (K_2)
ω_1	0.5	10%	−5%
ω_2	0.5	−5%	10%

between the two stocks. This is a simple example of diversification, which is particularly effective here because the returns are negatively correlated.

We introduce weights to describe the allocation of funds between the securities as a convenient alternative to specifying portfolios in terms of the number of shares of each security. The *weights* are defined by

$$w_1 = \frac{x_1 S_1(0)}{V(0)}, w_2 = \frac{x_2 S_2(0)}{V(0)}$$

where x_k denotes the number of shares of kind $k = 1, 2$ in the portfolio. This means that w_k is the percentage of the initial value of the portfolio invested in security k. Observe that the weights always add up to 100%,

$$w_1 + w_2 = \frac{x_1 S_1(0) + x_2 S_2(0)}{V(0)} = \frac{V(0)}{V(0)} = 1.$$

If short selling is allowed, then one of the weights may be negative and the other one greater than 100%.

<u>Example:</u>

Suppose that the prices of two kinds of stock are $S_1(0) = 30$ and $S_2(0) = 40$. We prepare a portfolio worth $V(0) = 1,000$ by purchasing $x_1 = 20$ shares of stock 1 and $x_2 = 10$ shares of stock 2. The allocation of funds between the two securities is

$$w_1 = \frac{30 \times 20}{1,000} = 60\%, w_2 = \frac{10 \times 40}{1,000} = 40\%.$$

These are the weights in the portfolio. If the stock prices change to $S_1(T) = 35$ and $S_2(T) = 39$, then the portfolio will be worth $V(T) = 20 \times 35 + 10 \times 39 = 1,090$. Observe that this amount is no longer split between the two securities as 60% to 40%, but as follows:

$$\frac{20 \times 35}{1,090} \simeq 64.22\%, \frac{10 \times 39}{1,090} \simeq 35.78\%,$$

even though the actual number of shares of each stock in the portfolio remains unchanged.

<u>Remark:</u>

In reality the number of shares has to be an integer, which places a constraint on possible weights. To simplify matters, however, we shall assume divisibility of assets. This means that the weights can be any real numbers that add up to one. Since not all real markets allow short selling, sometimes we need to distinguish a special case when the weights are non-negative.

<u>Example:</u>

Suppose that a portfolio worth $V(0) = 1,000$ is constructed by taking a long position in stock 1 and a short position in stock 2 in the previous example, with weights $w_1 = 120\%$ and $w_2 = -20\%$. The portfolio will consist of

$$x_1 = w_1 \frac{V(0)}{S_1(0)} = 120\% \times \frac{1,000}{30} = 40,$$
$$x_2 = w_2 \frac{V(0)}{S_2(0)} = -20\% \times \frac{1,000}{40} = -5$$

shares of type 1 and 2. If the stock prices change as in the previous example, then this portfolio will be worth

$$\begin{aligned} V(T) &= x_1 S_1(T) + x_2 S_2(T) = V(0)\left(w_1 \frac{S_1(T)}{S_1(0)} + w_2 \frac{S_2(T)}{S_2(0)}\right) \\ &= 1,000\left(120\% \times \frac{35}{30} - 20\% \times \frac{39}{40}\right) = 1,205, \end{aligned}$$

benefiting from both the rise in the price of stock 1 and the fall in stock 2.

Proposition:

The return K_V on a portfolio consisting of two securities is the weighted average

$$K_V = w_1 K_1 + w_2 K_2,$$

where w_1 and w_2 are the weights and K_1 and K_2 the returns on the two components.

Proof:

Suppose that the portfolio consists of x_1 shares of security 1 and x_2 shares of security 2. Then the initial and final values of the portfolio are

$$\begin{aligned} V(0) &= x_1 S_1(0) + x_2 S_2(0), \\ V(T) &= x_1 S_1(0)(1 + K_1) + x_2 S_2(0)(1 + K_2), \\ &= V(0)(w_1(1 + K_1) + w_2(1 + K_2)). \\ &= V(0)(1 + w_1 K_1 + w_2 K_2). \end{aligned}$$

As a result, the return on the portfolio is

$$K_V = \frac{V(T) - V(0)}{V(0)} = w_1 K_1 + w_2 K_2.$$