Voting Methods:

- Based on scoring method:
  - Plurality Method
  - Borda Count
- Pairwise comparison
  - Simple Majority
  - Absolute Majority
  - Two-third Majority
- Voting procedure
  - Instant run-off

Scoring Method
Based on preferences of the individual scores are given to the alternative. Based on aggregate score we derive the social preference.

Suppose individual preferences are

1	2	3
x	y	y
y	z	x
z	x	z

Plurality Method: scoring rule is $(1, 0, 0)$. Give one point to the most preferred alternative and zero to all other alternatives. So, the scores in above preference profile is

1	2	3
x 1	y 1	y 1
y 0	z 0	x 0
z 0	x 0	z 0

Aggregate scores are $x : 1$, $y : 2$, $z : 0$.
So the social preference is

$$\begin{pmatrix} y \\ x \\ z \end{pmatrix}$$

The winner in the election using plurality voting method is y.

Borda Count: scoring rule is $(2, 1, 0)$. If there are four alternatives, the scoring rule is $(3, 2, 1, 0)$. The scores in the given individual preference profile is

1	2	3
x 2	y 2	y 2
y 1	z 1	x 1
z 0	x 0	z 0

The aggregate scores are
$x : 3$, $y : 5$, $z : 1$.
The social preference ordering is

$$\begin{pmatrix} y \\ x \\ z \end{pmatrix}.$$

The winner is y, if this method is used.

The social ranking may not same in the case of Plurality and Borda count.

1	2	3
x	y	y
z	x	xz
y	z	

Plurality rule gives social preference as $\begin{pmatrix} y \\ x \\ z \end{pmatrix}$.

Borda Count method gives social preference as $\begin{pmatrix} xy \\ z \end{pmatrix}$.

Pairwise comparison

Simple majority : $xRy \leftrightarrow N(xP_iy) \geq N(yP_ix)$.

1	2	3
x	y	y
y	z	x
z	x	z

Simple majority rule: $N(xP_iy) = 1 < N(yP_ix) = 2$, $N(yP_iz) = 3 > N(zP_iy) = 0$, $N(xP_iz) = 2 < N(zP_ix) = 1$.

Social preference: yPx, yPz, xPz.

Absolute majority: $xRy \leftrightarrow N(xP_iy) \geq \frac{N}{2} + 1$. If we have xRy based on absolute majority , we will also have xPy since yRx is not possible.

1	2	3
x	y	y
y	z	x
z	x	z

Absolute majority: $N(xP_iy) = 1 < N(yP_ix) = 2$, $N(yP_iz) = 2 > N(zP_iy) = 1$, $N(xP_iz) = 2 < N(zP_ix) = 1$.
Social preference : yPx, yPz, xPz.

Two-third majority: $xRy \leftrightarrow N(xP_iy) \geq \frac{2N}{3}$.

1	2	3
x	y	y
y	z	x
z	x	z

Two-third majority: $N(xP_iy) = 1 < N(yP_ix) = 2$,
$N(yP_iz) = 3 > N(zP_iy) = 0$, $N(xP_iz) = 2 < N(zP_ix) = 1$.
Social preference : yPx, yPz, xPz.

Example:

$$\begin{array}{ccc} 1 & 2 & 3 \\ x & x & yz \\ y & z & x \\ z & y & \end{array}.$$

Simple majority: xPy, yIz and xPz.

Absolute majority: xPy and xPz, not possible to compare y and z. It is not complete.

Two-third majority: xPy and xPz, not possible to compare y and z. It is not complete.

Possibility result:
Simple majority violates transitivity. If we move away from social welfare function which requires transitivity, can we generate a reflexive, complete and acyclic social ordering. So that we have social choice function .

A social decision function is a collective choice rule f, the range of which is restricted to those preference relations$R$, each of which generates a choice function $C(S, R)$ over the whole set of alternatives X.

Result: There is an Social decision function satisfying condition U, P, I and D.

Proof: We provide one example. Since we have to show existence, it proofs the statement.

Suppose $xRy \leftrightarrow \sim [(\forall i : yR_ix) \& (\exists i : yP_ix)]$.

This R is reflexive and complete. It is easy to see that P is satisfied. For pair x, y, the social relation is based on individual preferences over x and y, there is no role any other alternative , say z. So it satisfies I.

Suppose xP_1y and yP_2x and xP_3y, here we have both xRy and yRx, so social preference is xIy. The condition D is satisfied.

We have to show that quasi-ordering is satisfied.

$[xPy \& yPz] \rightarrow [\{\forall i : xR_iy \& \exists i : xP_iy\} \& \forall i : yR_iz]$

$\rightarrow [\forall i : xR_iz \& \exists i : xP_iz]$.

$\rightarrow xPz$.

Thus, R is quasi-transitive. We know that if R is quasi transitive, complete and reflexive, there is going to be best element for each non-empty subset S.Thus, there exist a social decision function.

A collective choice rule gives x is preferred to y if it is Pareto-superior to y. xRy if y is not Pareto superior to x.
1: xyz , 2: yzx, 3: zxy.
We have y is not Pareto superior to x, x is not Pareto superior to y, y is not Pareto superior to z, z is not Pareto superior to y. So We have xIy and yIz. For x, z also, we have xIz.