Theory Assignment for Module 1 & 2

MA423 : Matrix Computations S. Bora

Total Marks: 20

Important instructions:

1. **Only pdf files neatly typed in LaTex will be accepted. File name should be M12GyT.pdf where y is group number. For example for Group 11 it will be M12G11T.pdf**

2. **Write the Group number and names of all group members on the top of the file.**

1. Prove that running Gaussian elimination with complete pivoting on a square matrix A produces permutation matrices P and Q, a unit lower triangular matrix L and an upper triangular matrix U such that $PAQ = LU$. **(8 marks)**

2. From the backward error analysis it is well known that given floating point numbers u_i, w_i, $i = 1, \ldots, n$, there exist $\gamma_i, i = 1, \ldots, n$, satisfying $|\gamma_i| \leq nu + O(u^2)$, such that in the presence of rounding errors,
$$\mathrm{fl}\left(\sum_{i=1}^{n} u_i w_i\right) = \sum_{i=1}^{n} u_i w_i (1 + \gamma_i). \tag{1}$$
<u>*irrespective of the order of summation.*</u> Answer the following question using (1). You may assume basic inequalities like $|AB| \leq |A||B|$, $|Ax| \leq |A||x|$ for matrices A and B and vectors x such that the products are defined.

   Let V be any $n \times n$ invertible matrix. Prove that given any $n \times n$ matrix A, there exists an $n \times n$ matrix δA, such that $\mathrm{fl}(VA) = V(A + \delta A)$ with $|\delta A_j| \leq \gamma |V^{-1}||V||A_j|$, for a scalar γ (which does not depend on j) such that $|\gamma| \leq nu + O(u^2)$ for all $j = 1, \ldots, n$, A_j and δA_j being the j^{th} columns of A and δA respectively. **(6 marks)**

3. Given $A = [a_{ij}]_{n \times n}$ let $A^{(k)} = [a_{ij}^{(k)}]_{n \times n}$ be the matrix obtained at the end of step k of Gaussian elimination in theory. The pivotal growth factor of the process is defined as
$$\rho(A) = \left(\max_{\substack{1 \leq i,j \leq n \\ 1 \leq k \leq n-1}} |a_{ij}^{(k)}| \right) \Big/ \left(\max_{1 \leq i,j \leq n} |a_{ij}| \right).$$
Prove that for GEPP, $\frac{\|U\|_\infty}{\|A\|_\infty} \leq n\rho(A) \leq n2^{n-1}$. **(6 marks)**