LECTURE-4

Optimization Theory:

Minimize $f(x)$
subject to $x \in F$

Goal: To find optimal solutions and study properties of optimal solutions

Basics:
(a) Existence of optimal solutions
(b) Characterization of optimal solutions
- necessary conditions
- sufficient conditions
- Uniqueness
- parametric variation
- etc...

(c) Computation algorithms for finding (approximate) optimal solutions

Existence of Solutions

Q: Under what conditions on the objective function f and the constraint set F are we guaranteed that solutions will always exist?

Recall:

- Neighbourhood of $x \in \mathbb{R}^n$: $\{ y \in \mathbb{R}^n \mid \|y - x\| < \varepsilon \}$ (open ball with radius ε and centre x) where $\varepsilon > 0$ and $\|x\| = \sqrt{x^T x}$.
- $x \in S$ is an interior point of the set S if the set S contains some neighbourhood of x. The set of all interior points of S is called the interior of S.

- x is a boundary point of the set S if every neighbourhood of x contains a point in S and a point not in S.
  The set of all boundary points of S is called the boundary of S.
- A set is said to be open if it contains a neighbourhood of each of its points.
- A set is said to be closed if it contains its boundary.
- A set that is contained in a ball of finite radius is said to be bounded.
- A set is compact if it is both closed and bounded.

The Weierstrass Theorem:

Let $F \subseteq \mathbb{R}^n$ be compact and let $f : F \to \mathbb{R}$ be a continuous function on F. Then f attains a maximum and a minimum on F, ie, $\exists\, z_1, z_2 \in F \ni$

$$f(z_1) \geq f(x) \geq f(z_2) \quad , \forall x \in F.$$

Proof Sketch: 1. F compact, f cts $\Rightarrow f(F)$ is compact

2. $f(F)$ compact $\Rightarrow \sup f(F) \in f(F)$ and $\inf f(F) \in f(F)$.

Eg-1 : Let $F = \mathbb{R}$, $f(x) = x^3$ $\forall x \in \mathbb{R}$.

└ not bounded

$f(F) = \mathbb{R}$. No maximum ~~or~~ minimum for f.

Eg-2 : Let $F = (0,1)$, $f(x) = x$, $\forall x \in (0,1)$.

└ not closed.

$f(F) = (0,1)$. No max. or min. for f on F.

Eg-3: Let $F = [-1,1]$, $f(x) = \begin{cases} 0 & , x = -1 \text{ or } x = 1 \\ x & , -1 < x < 1 \end{cases}$

└ not continuous

$f(F) = (-1,1)$. No max or min for f on F.

Eg-4: Let $F = (0,\infty)$, $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{otherwise} \end{cases}$

└ neither closed nor bounded └ not continuous

$f(F) = \{0,1\}$ Max and min are attained.

Utility Maximization Problem (Recall)

Maximize $u(x)$ subject to $p^T x \le a$, $x \geqslant 0$.

$p > 0$. $a > 0$

$F = \{ x \in \mathbb{R}^n \mid p^T x \le a, x \geqslant 0 \}$

– compact