MA 322: Scientific Computing

Lecture - 9

Convergence of Lagrange's Interpolating polynomial

Result

- Suppose $x_0,\ x_1, \dots, x_n$ are distinct real numbers, and let f be a given real valued function with $n+1$ continuous derivatives on the interval $I_t = \mathcal{H}\{t, x_0,\ x_1, \dots, x_n\}$, with t some given real number. Then there exits a number $\xi_t \in I_t$ such that

$$f(t) - p_n(t) = \frac{f^{(n+1)}(\xi_t)}{(n+1)!}(t - x_0)(t - x_1)\dots(t - x_n), \tag{1}$$

where $p_n(t) = \sum_{j=0}^{n} f(x_j)\ell_j(t)$ is the Lagrange interpolating polynomial of f with degree n.

Proof. We define the error function E by

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and a user defined function G by

$$G(x) = E(x) - \frac{\Psi(x)}{\Psi(t)}E(t) \tag{3}$$

with $\Psi(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$.

Convergence Analysis Contd...

Then for the user defined function

$$G(x) = E(x) - \frac{\Psi(x)}{\Psi(t)}E(t), \quad \Psi(x) = (x - x_0)(x - x_1)\dots(x - x_n), \tag{4}$$

observe that

$$G(x_i) = E(x_i) - \frac{\Psi(x_i)}{\Psi(t)}E(t) = 0, \quad i = 0,\ 1,\ 2, \dots, n \tag{5}$$

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Remark:

- Now, we wish to calculate the distance between f and p_n, which is done by introducing norm over function spaces.

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  - Therefore, in order to define the distance between two functions in $C[a, b]$, we first try to associate vector space $C[a, b]$ a norm.

Concept of Distance Between two Functions

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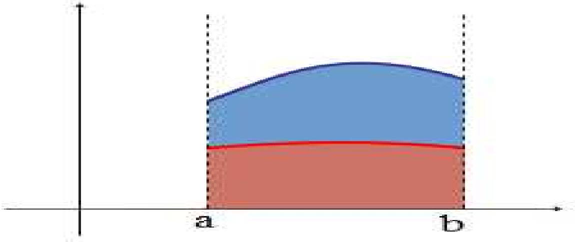

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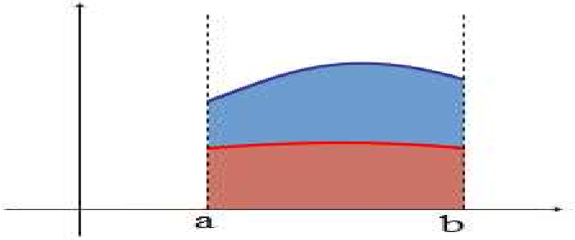

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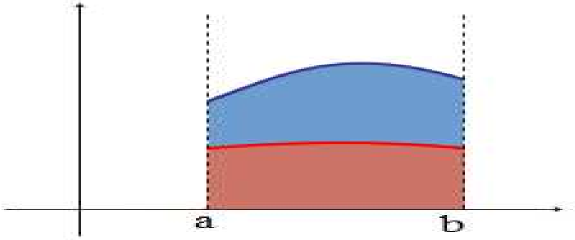

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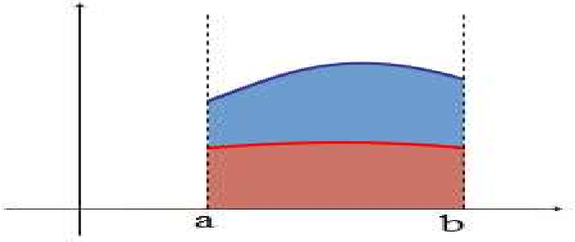

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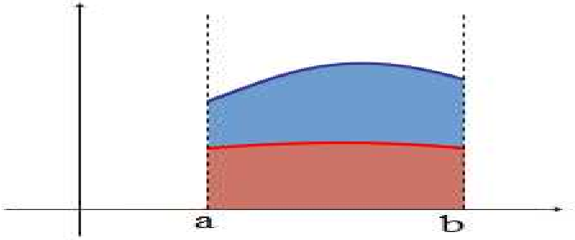

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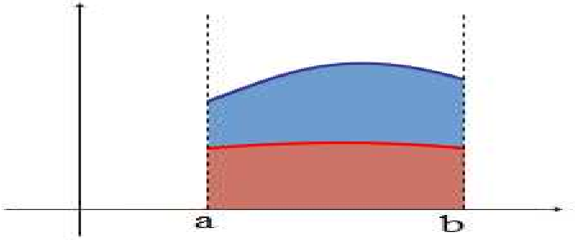

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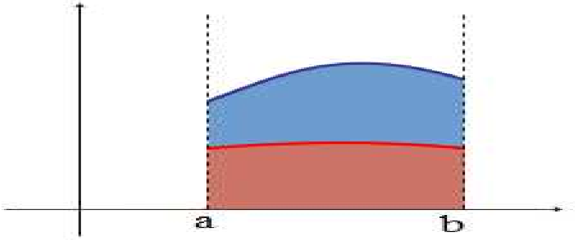

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- It is an easy exercise to verify that $\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$ defines a norm on $C[a, b]$, known as infinity norm. The distance associated with this norm is given by

$$d_\infty(f, g) = \max_{x\in[a,b]} |(f - g)(x)| = \max_{x\in[a,b]} |f(x) - g(x)|.$$

Notion of Convergence

- In $(C[a, b], d_\infty)$

  - A sequence $\langle f_n \rangle$ in $C[a, b]$ is said to be convergent w.r.t d_∞, if there exists a $f \in C[a, b]$ s. t.

$$d_\infty(f_n, f) \to 0 \text{ or equivalently } \|f_n - f\|_\infty \to 0 \text{ as } n \to \infty. \quad (12)$$

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  - Uniform limit of a sequence of continuous functions is also continuous.
  - If $f_n \to f$ converges in $(C[a,b], d_\infty)$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx. \tag{13}$$

About Uniform Convergence of Interpolating Polynomials

Consider again the error formula

$$f(x) - p_n(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_n)}{(n+1)!} f^{(n+1)}(\xi_x), \tag{14}$$

where $\xi_x \in \mathcal{H}\{x, x_0, x_1, \dots, x_n\}$.

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where $\xi_x \in \mathcal{H}\{x, x_0, x_1, \dots, x_n\}$. Then for any interval I that contains $\mathcal{H}\{x, x_0, x_1, \dots, x_n\}$ and for $f \in C^{(n+1)}(I)$, we obtain

$$\max_{x\in I} |f(x) - p_n(x)| \leq \frac{1}{(n+1)!} \max_{x\in I} |f^{(n+1)}(\xi_x)| \max_{x\in I} |\Psi_n(x)|, \qquad (15)$$

where $\Psi_n(x) = (x-x_0)(x-x_1)\dots(x-x_n)$.

Remark:

- Now, $p_n \to f$ uniformly provided quantities

$$M_{n+1} = \max_{x\in I} |f^{(n+1)}(\xi_x)| \ \text{ and } \ \max_{x\in I} |\Psi_n(x)|$$

are uniformly bounded in I.

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Consider again the error formula

$$f(x) - p_n(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_n)}{(n+1)!} f^{(n+1)}(\xi_x), \qquad (14)$$

where $\xi_x \in \mathcal{H}\{x, x_0, x_1, \dots, x_n\}$. Then for any interval I that contains $\mathcal{H}\{x, x_0, x_1, \dots, x_n\}$ and for $f \in C^{(n+1)}(I)$, we obtain

$$\max_{x \in I} |f(x) - p_n(x)| \leq \frac{1}{(n+1)!} \max_{x \in I} |f^{(n+1)}(\xi_x)| \max_{x \in I} |\Psi_n(x)|, \qquad (15)$$

where $\Psi_n(x) = (x - x_0)(x - x_1)\dots(x - x_n)$.

Remark:

- Now, $p_n \to f$ uniformly provided quantities

  $$M_{n+1} = \max_{x \in I} |f^{(n+1)}(\xi_x)| \;\text{ and }\; \max_{x \in I} |\Psi_n(x)|$$

  are uniformly bounded in I. More precisely, we may expect $\|f - p_n\|_\infty \to 0$ as $n \to \infty$ for C^∞ (infinitely many times differentiable functions) functions when

  $$\left(M_{n+1} \cdot \max_{x \in I} |\Psi_n(x)|\right) \text{ are uniformly bounded in } I.$$

About U.C. of Interpolating Polynomials Contd....

Unfortunately, this is not so, since the sequence

$$\left(M_{n+1} \cdot \max_{x \in I} |\Psi_n(x)| \right)$$

may tend to ∞, as $n \to \infty$, faster than the sequence $\frac{1}{(n+1)!}$ tends to 0.

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Example:

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Example: Consider the function $f(x) = \frac{1}{1+x^2}, \;\; x \in [-5, 5]$ and let us try to calculate $\|f - p_n\|$ for different values of n.

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Example: Consider the function $f(x) = \frac{1}{1+x^2}$, $x \in [-5, 5]$ and let us try to calculate $\|f - p_n\|$ for different values of n.

Degree n	Max error
2	0.65
4	0.44
6	0.61
8	1.04
10	1.92
12	3.66
14	7.15
16	14.25
18	28.74
20	58.59
22	121.02
24	252.78