Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Capital Asset Pricing Model:

Now that we have discussed the *Markowitz portfolio theory*, we are ready to take a look at the *Capital Asset Pricing Model* or CAPM. The major factor that turns Markowitz portfolio into capital market theory is the inclusion of a riskfree asset in the model.

Imagine a portfolio that consists of a riskfree asset a_{rf} with weight w_{rf} and risky assets $a_1, a_2, \ldots, a_n$ with weights $w_1, w_2, \ldots, w_n$, respectively. Note that now the sum of the risky assets will be at most 1.

Thus,

$$w_{rf} + \sum_{i=1}^{n} w_i = 1 \text{ and } w_{risky} = \sum_{i=1}^{n} w_i \leq 1.$$

The expected return on the complete portfolio is

$$\mu = w_{rf}\mu_{rf} + \sum_{i=1}^{n} w_i\mu_i = w_{rf}\mu_{rf} + \mu_{risky}.$$

Also, the variance of the riskfree asset is 0. Since the return of the riskfree asset is a constant, hence it's covariance with any other return is 0 and so,

$$\sigma^2 = Var\left(w_{rf}K_{rf} + \sum_{i=1}^{n} w_iK_i\right) = Var\left(\sum_{i=1}^{n} w_iK_i\right) = \sigma^2_{risky}.$$

Therefore $\sigma = \sigma_{risky}$. We also want to consider the portfolio formed by removing the riskfree asset and beefing up the weights of the risky assets by the same factor to make the sum of weights equal to 1. Now we make a connection between the <u>original</u> portfolio and the newly <u>derived</u> portfolio.

$$\begin{aligned}
\mu &= w_{rf}\mu_{rf} + \sum_{i=1}^{n} w_i\mu_i = w_{rf}\mu_{rf} + w_{risky}\sum_{i=1}^{n}\frac{w_i}{w_{risky}}\mu_i = w_{rf}\mu_{rf} + w_{risky}\mu_{der},\\
\sigma^2 &= Var\left(\sum_{i=1}^{n} w_iK_i\right) = w^2_{risky}Var\left(\sum_{i=1}^{n}\frac{w_i}{w_{risky}}K_i\right) = w^2_{risky}\sigma^2_{der}.
\end{aligned}$$

Now using $w_{rf} + w_{risky} = 1$ we get $\mu = \mu_{rf} + w_{risky}\left(\mu_{der} - \mu_{rf}\right)$ and $\sigma = w_{risky} \cdot \sigma_{der}$. As w_{risky} ranges over all real numbers, the previous equation traces out a straight line. Solving for w_{risky} we get,

$$\mu = \frac{\mu_{der} - \mu_{rf}}{\sigma_{der}}\sigma + \mu_{rf}.$$

Two points on this line are given by,

1. $w_{risky} = 0 \Rightarrow (\sigma, \mu) = (0, \mu_{rf})$.
2. $w_{risky} = 1 \Rightarrow (\sigma, \mu) = (\sigma_{der}, \mu_{der})$.

So where do we stand ? An investor who invests in a riskfree asset along with some risky assets will have the risk-expected return point lying somewhere on the line joining the points $(0, \mu_{rf})$ and $(\sigma_{def}, \mu_{def})$. But it is clear from the geometry that amongst all the lines joining the point $(0, \mu_{rf})$ with various points $(\sigma_{der}, \mu_{der})$ in the Markowitz bullet, the line that produces the points with the highest expected return for a given risk is the tangent line to the upper potion of the Markowitz bullet. This tangent line is called the *Capital Market Line (CML)* and the point of tangency on the Markowitz efficient frontier is called the *Capital Market Portfolio*.

The Equation of the Capital Market Line (CML):

If the market portfolio has the risk-expected return point σ_M, μ_M then the equation of the capital market line is:

$$\mu = \frac{\mu_M - \mu_{rf}}{\sigma_M}\sigma + \mu_{rf}.$$

For any point (σ, μ) on this line, the value $\mu - \mu_{rf} = \frac{\mu_M - \mu_{rf}}{\sigma_M}\sigma$, (which is the additional expected return above the expected return on the riskfree asset) is called the risk-premium. It is the additional return that one may expect for assuming the risk.

Theorem:

For any expected riskfree return μ_{rf} the capital market portfolio has weights

$$\mathbf{w}_M = \frac{(\mathbf{m} - \mu_{rf}\mathbf{u})C^{-1}}{(\mathbf{m} - \mu_{rf}\mathbf{u})C^{-1}\mathbf{u}^\top}$$

Proof:

For any point (σ, μ) in the Markowitz bullet, the slope of the line from $(0, \mu_{rf})$ to (σ, μ) is

$$s = \frac{\mu - \mu_{rf}}{\sigma} = \frac{\sum_{i=1}^{n} w_i\mu_i - \mu_{rf}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_iw_j\right)^{1/2}}.$$

It is intuitively clear that the point of tangency is the point with the property that the slope is a maximum among all points (σ, μ) in the Markowitz bullet. So the problem is to maximize $s = \frac{\mu - \mu_{rf}}{\sigma}$ subject to $\sum_{i=1}^{n} w_i = 1$. Accordingly, define the Lagrangian as:

$$F = \frac{\sum_{i=1}^{n} w_i\mu_i - \mu_{rf}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_iw_j\right)^{1/2}} - \lambda\left(\sum_{i=1}^{n} w_i - 1\right).$$

Now differentiating F w.r.t w_k and setting equal to 0 we get,

$$\begin{aligned}
\frac{\partial F}{\partial w_k} &= \frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_iw_j\right)^{1/2}\mu_k - \left(\sum_{i=1}^{n} w_i\mu_i - \mu_{rf}\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_iw_j\right)^{-1/2}\left(\sum_{i=1}^{n} c_{ik}w_i\right)}{\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_iw_j} - \lambda \\
&= \left((\mathbf{w}C\mathbf{w})^\top\mu_k - (\mathbf{m}\mathbf{w}^\top - \mu_{rf})C_k\mathbf{w}^\top - \lambda(\mathbf{w}C\mathbf{w}^\top)^{3/2}\right)/(\mathbf{w}C\mathbf{w}^\top)^{3/2} \\
&= (\sigma^2\mu_k - (\mu - \mu_{rf})C_k\mathbf{w}^\top - \lambda\sigma^3)/\sigma^3 = 0.
\end{aligned}$$

Since this holds for all k therefore we have,

$$\begin{aligned}
&\sigma^2\mathbf{m}^\top - (\mu - \mu_{rf})C\mathbf{w}^\top = \lambda\sigma^3\mathbf{u}^\top \\
\Rightarrow\ &\sigma^2\mathbf{m} - (\mu - \mu_{rf})\mathbf{w}C = \lambda\sigma^3\mathbf{u} \\
\Rightarrow\ &\sigma^2\mathbf{m}\mathbf{w}^\top - (\mu - \mu_{rf})\mathbf{w}C\mathbf{w}^\top = \lambda\sigma^3\mathbf{u}\mathbf{w}^\top = \lambda\sigma^3 \\
\Rightarrow\ &\sigma^2\mu - (\mu - \mu_{rf})\sigma^2 = \lambda\sigma^3.
\end{aligned}$$

This gives $\lambda = \dfrac{\mu_{rf}}{\sigma}$. Substituting this value of λ we get,

$$\sigma^2\mathbf{m} - (\mu - \mu_{rf})\mathbf{w}C = \mu_{rf}\sigma^2\mathbf{u}.$$

Therefore

$$\mathbf{w} = \frac{(\mathbf{m} - \mu_{rf}\mathbf{u})C^{-1}}{(\mu - \mu_{rf})/\sigma^2}.$$

But

$$\frac{(\mu - \mu_{rf})}{\sigma^2} = \frac{(\mu - \mu_{rf})}{\sigma^2}\mathbf{w}\mathbf{u}^\top = (\mathbf{m} - \mu_{rf}\mathbf{u})C^{-1}\mathbf{u}^\top.$$

Finally we get,

$$\mathbf{w}_M = \frac{(\mathbf{m} - \mu_{rf}\mathbf{u})C^{-1}}{(\mathbf{m} - \mu_{rf}\mathbf{u})C^{-1}\mathbf{u}^\top}.$$