Wed : 5 to 7pm ⇒ $\boxed{1.30\text{ h}, \ 30\text{ min}}$

Fri : 6 to 7pm ⇒ Quiz Theory

A : $1 + r\,4\cos^2\theta/2$ ⇒ Eigen values of A

$\Downarrow$

≥ 1 ⇒ A^{-1} exists.

$\therefore V^{j+1} = A^{-1} L V^{j}$, $L = \begin{bmatrix} 1-2r & r & 0 & \cdots & 0 \\ r & 1-2r & r & \cdots & 0 \\ \vdots & & & & r \\ 0 & & & r & 1-2r \end{bmatrix}$

For stability, $\|A^{-1}L\|_2 \leq 1$

Eigen-values of $A^{-1}L$ (ρ) $= \dfrac{\text{eigen values of } L}{\text{eigen values of } A}$

$= \dfrac{1 - 4r\cos^2\theta/2}{1 + 4r\cos^2\theta/2}$

$\therefore \rho(A^{-1}L) \leq 1$

VIVA

$\therefore$ C N scheme is unconditionally stable scheme

T.E $= O(k^2) + O(h^2)$

→ Explicit / Implicit ?? Implicit since first find $A^{-1} \Rightarrow A^{-1}L \Rightarrow V^{j+1}$

Lecture - 6 (8/4)

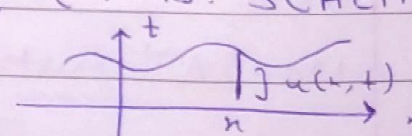
WAVE EQN (F.D. SCHEMES)

→ Vibration of string :

$u_{tt} - c^2 u_{xx} = 0$ — ①

$\Downarrow$

parameter (physical)



Initial condⁿ : $u(x,0) = f(x)$ ⇒ Initial shape of the string

$u_t(x,0) = g(x)$ ⇒ Initial vel. of the string

If only i.c. given ⇒ Infinite string problem

$\downarrow$

D' Alembert formula

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + g(x+ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} g(u)\,du$$

⇒ D'Alembert Formula

info at whole computation domain is not required.

Interval $[x-ct, x+ct]$ ⇒ Domain of dependence (Theoretical)

First order Eqn: $u_t + a u_x = 0$ ⇒ represents a wave eqn.

If u is sufficiently smooth, then

$$(u_{tt}) = -a(u_x)_t = -a(u_t)_x \quad \{u_{xt} = u_{tx} \text{ for sufficient smooth}\}$$

$$= -a(-a u_x)_x$$

$\Rightarrow u_{tt} = a^2 u_{xx}$ ⇒ so indirectly represents a 2nd order eqn.

Infinite string Problem:

$$u_t + a u_x = 0, \quad u(x,0) = f(x)$$

$$x \in (-\infty, \infty), \quad t \in (0, T], \quad T < \infty$$

→ Suppose we need sol^n at (x,t). Based on (x,t), we select a computational domain (can't use $(-\infty,\infty)$) $[a,b] \times (0,T]$ s.t. $(x,t) \in [a,b] \times [0,T]$

→ Discretize the computational domain :-

(x_i, t_j) ⇒ Grid points $\quad x_0 = a, \; x_{i+1} = x_i + h, \; x_{N+1} = b$

$t_0 = 0, \; t_{j+1} = t_j + k, \; t_n = T$

→ We select h & k s.t. $\exists\, i, j$ s.t $x_i = x, \; t_j = t$

→ $(x_i, t_j) = (x, t)$

→ $u_t(x_i, t_j) + a u_x(x_i, t_j) = 0$ — (2)

Different schemes for approximating (2)

1. FTBS (Forward time, Backward space)
2. FTFS
3. CTCS

1. FTBS (Forward time & ~~Forward~~ Backward space)

→ $u_t(x_i, t_j) + a u_x(x_i, t_j) = 0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} + \frac{u(x_i, t_j) - u(x_{i-1}, t_j)}{h} = 0$$

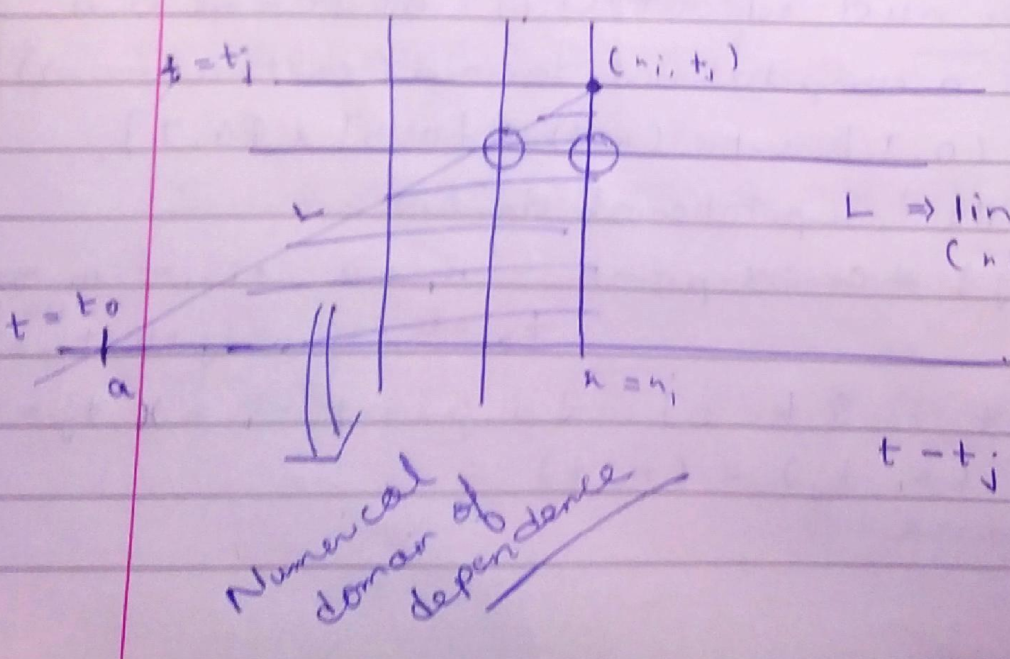
→ Now, $u(x_i, t_j) \approx u_{i,j}$

$$\therefore \frac{u_{i,j+1} - u_{i,j}}{k} + a\frac{u_{i,j} - u_{i-1,j}}{h} = 0$$

$\Rightarrow u_{i,j+1} = (1 - ar)u_{i,j} + ar\, u_{i-1,j}$, $r = \frac{k}{h}$

↓ need at (x_i, t_j)

$u_{i,j} = (1 - ar)u_{i,j-1} + ar\, u_{i-1,j-1}$



L ⇒ line passing through (x_i, t_j) & (x_{i-1}, t_{j-1})

$$t - t_j = (x - x_i)\frac{t_j - t_{j-1}}{x_i - x_{i-1}}$$

$\Rightarrow t = t_j + \frac{k}{h}(x - x_i)$

$\therefore (a - x_i)\frac{k}{h} + t_j = 0 \quad \{t_0 = 0\}$

$\Rightarrow a = x_i - \frac{h}{k} t_j$

& $b = x_i$

$\Rightarrow$ But, it is not allowed due to CFL cond^n

> CFL cond^n: For wave eqn, numerical domain of dependence should contain theoretical domain of dependence.

For (x_i, t_j), theoretical domain of dependence is $[x_i - ct_j, x_i + ct_j]$

& numerical domain is $[a, x_i]$

Now, by CFL, $[x_i - ct_j, x_i + ct_j] \not\subseteq [a, x_i]$
$\times$

→ Select h & k s.t. $h/k \geq c$, so that CFL cond^n is satisfied.

→ $a = x_i - \frac{h}{k} t_j$, ~~$h$ & k s.t. $\frac{h}{k} \leq c$~~

~~$b \geq x_i + ct_j$~~ → $b = x_i + \frac{h}{k} t_j$

Ex: $u_t + u_x = 0$, $x \in (-\infty, \infty)$, $t \in (0, 1]$

$u(x, 0) = 0$

F.O sol^n at $\left(\frac{1}{4}, \frac{1}{4}\right)$ → Numerical domain of dependence

$a = 0$, $b = 1/2$

$a = \left(x_i - \frac{h}{k} t_j\right)$

$(x_i, t_j) = \left(\frac{1}{4}, \frac{1}{4}\right)$

$= \frac{1}{4} - \frac{h}{k} \cdot \frac{1}{4} = 0$

$b = x_i + \frac{h}{k} t_j = 1/2$

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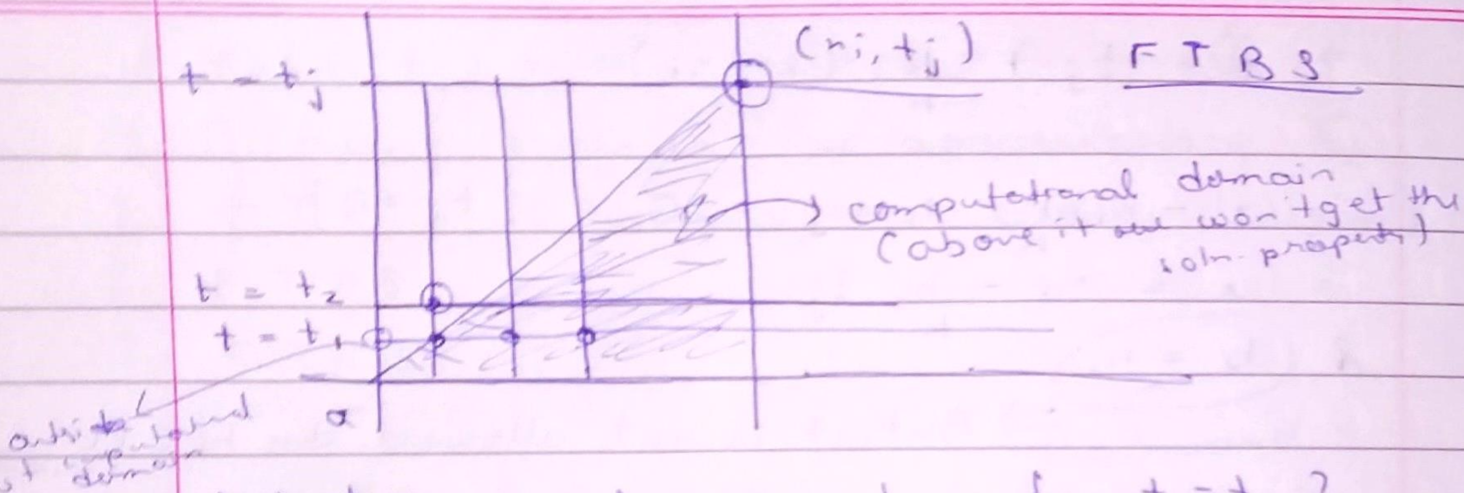


FTBS

What are unknown soln. for $t = t_1$?

2. FTFS (Forward Time & Forward Space)

→ Fix the computational domain.

$$\therefore \frac{u_{i,j+1} - u_{i,j}}{k} + a\frac{u_{i+1,j} - u_{i,j}}{h} = 0$$

(VIVA)

ALGORITHM

→ Fix computational domain $[a,b] \times [0,T]$ with:-

(i) $(x_i, t_j) \in [a,b] \times [0,T]$

(ii) select h, k s.t. $x_i = x$, $t_j = t$, i.e. (x,t) becomes a grid point.

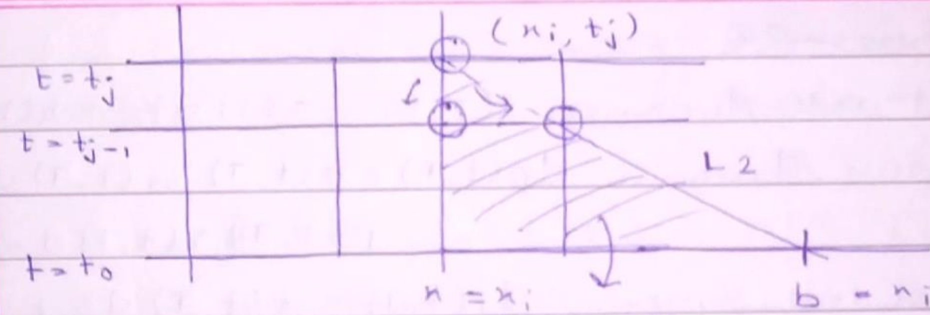
→ $u_t(x_i, t_j) + a\, u_x(x_i, t_j) = 0$

$$\Downarrow \qquad \Downarrow$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} + a\frac{u_{i+1,j} - u_{i,j}}{h} = 0$$

$$\Rightarrow u_{i,j+1} = (1 + ar)u_{i,j} + ar\,u_{i+1,j}$$

$$\Rightarrow u_{i,j} = (1 + ar)u_{i,j-1} + ar\,u_{i+1,j-1}$$



Numerical region of dependency
$[x_i, b]$

→ But due to CFL condn, we have to modify it to satisfy; $[a, b] \supset [x_i - c t_j, x_i + c t_j]$

condn $\left\{ \begin{array}{l} \therefore h/k \geq c, \quad b = x_i + h/k \, t_j \\ \qquad\qquad\quad a = x_i - h/k \, t_j \end{array} \right.$