Properties

- Resource monotonicity: Given a claims problem (C, M), for each $M' \in R_+$, if $\sum_{i=1}^{N} c_i \geq M" > M$, then $x_i(C, M') \geq x_i(C, M)$ for all $i \in N$.
  If the size of the cake or endowment increases, then all the claimants should get atleast what they were getting previously.
- Invariance under claims truncation: Given a claims problem (C, M), $x(C, M) = x((\min\{c_i, M\}_{i \in N}, M)$.
  Suppose the claims of some claimants are greater than M the size of cake, their claim is truncated and made M. This rule says that the allocation is such that each claimant should get same amount when their claim is truncated and when it is not truncated.
  Suppose claims are; $(10, 20, 100)$ and $M = 60$, if we follow constraint equal awards rule $x = (10, 20, 30)$.
  Instead, if we truncate the claim of claimant 3, the claim vector is $(10, 20, 60)$, constraint equal awards rule allocate,

If you use proportional rule, the allocation is $(\frac{60}{130}, \frac{120}{130}, \frac{600}{130})$. After truncation, the allocation is $x = (\frac{20}{3}, \frac{40}{3}, 40)$. They are not same. Constraint equal award rule satisfies this rule.

- Minimal rights first: Given a claims problem (C, M),
  $x(C, M) = m(C, M) + x(c - m(C, M), M - \sum_{i=1}^{N} m_i(C, M))$,
  where $m_i(c, M) = \max\{M - \sum_{j \in N-\{i\}} c_j, 0\}$ for each claimant i.
  Each claimant is given its minimal right. The minimal right of a player is the amount remaining after giving all the other claimants its claim. The sum of the minimal amount is less than the size of the cake. After giving the minimal rights amount, the remaining is allocated using a division rule, here the claim of each claimant is their original claim minus the minimal amount.

Example: Claim, $C = (10, 20, 60)$ and $M = 40$
The minimal amounts are:
$m_1 = \max\{40 - (20 + 60, 0\} = 0, \quad m_2 = \max\{40 - (10 + 60), 0\} = 0, \quad m_3 = \max\{40 - (10 + 20), 0\} = 10.$
Now we have to allocated $40 - 10 = 30$ among the three claimants. Their claims are $(10 - 0, 20 - 0, 60 - 10)$. We can use an allocation rule for this division.
A division rule satisfying minimal rights first must allocate in the same way, if the allocation is done based original claims problem and when first minimal rights amount is given to each claimants and then the remaining cake is allocated using the same rule taking the clams after subtracting the minimal amount of each player.
In the above example, if we use constraint equal loss rule, the allocation is $x = (0, 0, 40)$.
The minimal rights are; $m = (0, 0, 10)$. The claims problem is $(10, 20, 50)$ and $40 - 10 = 30$ size of the cake.

Apply constraint equal loss rule to $C - m = (10, 20, 50)$ and $M - \sum_{i=1}^{3} m_i = 30$, the allocation is $(0, 0, 30)$. So, the final allocation is $(0, 0, 40)$. It is same as constraint equal loss rule. Constraint equal awards rule does not satisfy minimal rights rule for all claims problem.

- Composition down: Given a claims problem (C, M), for each $M' < M$, we have $x(C, M') = x(x(c, M), M')$.
  Suppose allocation is promised for the problem with claims C and M as size of the cake using a particular division rule. If the size of the cake depreciates and its M', then the allocation is done based on taking promised allocations as the new claims using the same division rule. The division rule satisfying this property must give same allocation when the allocation is done using C as claims and M' as the size of the cake and when $x(C, M)$ as the claims and M' as the size of the cake.

Example: $C = (10, 30, 40)$ and $M = 40$. If we use proportional rule, the allocation is $x(C, M) = (5, 15, 20)$.
Now suppose the size of the cake $M' = 30$, we take the claims to be $x(C, M) = (5, 15, 20)$. If we use proportional use, the allocation is $x(x(C, M), M') = (\frac{15}{4}, \frac{45}{4}, 15)$.
If we apply proportional rule to the problem $(10, 30, 40)$ and 30, the allocation is $x(C, M') = (\frac{30}{8}, \frac{90}{8}, 15)$. It is same as the above.
Proportional rule satisfies composition down.

- Composition up: Given a claims problem (C, M) and each M', if $\sum\limits_{i=1}^{N} c_i \geq M' > M$, then
  $x(C, M') = x(C, M) + x(c - x(C, M), M' - M)$.
  Suppose given a claims problem, there is a promised allocation to the claimants using a particular division rule. Now, suppose the size of the cake appreciates and it is $M' > M$. The allocation is done first as promised earlier. Next, the claims of each claimant is reduced by the amount received and the remaining size of the cake is divided using the same rule.

Example:
$C = (10, 30, 40)$ and $M = 30$. Suppose we use proportional rule, the allocation is $x = (\frac{30}{8}, \frac{90}{8}, 15)$. Now suppose the size of the cake increases to 40.
We first allocation based on the claims C and cake 30, so $x = (\frac{30}{8}, \frac{90}{8}, 15)$. The remaining cake is $40 - 30 = 10$, the reduced claims are $c_1' = 10 - \frac{30}{8}, \quad c_2' = 30 - \frac{90}{8}, \quad c_3' = 40 - 15 = 25$.
Apply proportional rule to this problem, the allocation is $x(c', M' - M) = (\frac{5}{4}, \frac{15}{4}, 5)$. The final allocation is $(5, 15, 20)$.
When proportional rule is applied to the problem $(10, 30, 40)$ and 40, the allocation is $(5, 15, 20)$. Proportional rule satisfies it.

- No advantageous transfer: Given a claims problem (C, M), for each $S \subset N$ and each $(c_i^{'})_{i \in S} \in R_+^{|S|}$, if $\sum_{i \in S} c_i^{'} = \sum_{i \in S} c_i$ then $\sum_{i \in S} x_i(C, M) = \sum_{i \in S} x_i((c_i^{'})_{i \in S}, c_{N-S}, M)$.
  Suppose a subset of claimants form a group S, the claims in S transfer their claims among themselves but the sum of the claims remain same, these claimants should not gain any additional amount by this transfer of claims.
  Example:
  $C = (10, 30, 40)$ and $M = 30$. Suppose there is group of claimant 1 and 2. The sum of their claims is $10 + 30 = 40$. If their is transfer between them say $c_1^{'} = 10 + a, c_2^{'} = 30 - a$, the sum remains same. If we apply proportional rule , the allocation is $(\frac{(10+a)30}{80}, \frac{(30-a)30}{80}, 15)$.

The sum of allocation of claimant 1 and 2 is $\frac{((10+a)+(20-a))30}{80}$. It remains same, whatever may be the transfer. So claimant 1 and 2 have no advantage of forming a group (coalition).