Theorem: There exists a unique function ϕ satisfying the four axioms of Shapley. It is given by

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(N-|S|)!}{N!} \Big[v(S) - v(S - \{i\}) \Big],$$

$i = 1, 2, 3 \ldots N.$

Proof:
For a given coalition $S \subset N$, suppose w_S is a characteristic function such that

$$w_S(T) = \begin{cases} 1 \text{ if} S \subset T \\ 0 \quad otherwise \end{cases}$$

So, $(N; w_S)$ is a carrier game.
From above, it is clear that $w_S(S \cup \{i\}) = w_S(S) = 1$ when $i \notin S$.
This implies that $\phi_1(W_S) = 0$. From the null player axiom.
Now if $i, j \in S$, then $\phi_i(w_S) = \phi_j(w_s)$ using the axiom that symmetric players must get same payoffs. Take a coalition $S^{'} \subset S$ and $i, j \notin S^{'}$. $w_S(S^{'}) = 0$, then $w_S(S^{'} \cup \{i\}) = w_S(S^{'} \cup \{j\})$.

Now for a given S we get $\sum_{i \in N} \phi_i(w_S) = w_S(N) = 1$ from efficiency axiom. This implies $\phi_i(w_S) = \frac{1}{|S|}$ for all $i \in S$.

Suppose we take the characteristic function cW_S where c is real number, using the similar steps we get

$$\phi_i(cw_S) = \begin{cases} \frac{c}{|S|} & \text{for} i \in S \\ 0 & \text{for} i \in S \end{cases}$$

Now we show that any characteristic function v defining a coalition game, can be represented as a weighted sum of these characteristic function giving us a carrier game. It is $v = \sum_{S \subset N} c_S w_S$ where c_S is chosen appropriately.

Using the axiom that a characteristic function can be sum of characteristic function, we get that

$$\phi_i(v) = \sum_{S \subset N, i \in N} \frac{c_S}{|S|}, \ i \in N$$

where summation is taken over all coalitions in which i belongs. Here c_S can be negative also.

We have to show that any v can be represented as $v = \sum_{S \subset N} c_S w_S$.

To do this we need to find the c_S. Assume that $c_\emptyset = 0$. Note that c_S is indexed on coalitions.
For all $T \subset N$, we define
$c_T = v(T) - \sum_{S \subset T,\ S \neq T} c_S$.
Each c_T is defined in terms of c_S where S has less number of members than T. We are using induction, $c_i = v(i)$ for all $i \in N$.
We have $v(T) = c_T + \sum_{S \subset T,\ S \neq T} c_S$
This implies $c_T + \sum_{S \subset T,\ S \neq T} c_S = \sum_{S \subset T} c_S$.
This can be written as
$\sum_{S \subset T} c_S = \sum_{S \subset N} c_S w_S$. Thus, $v(T) = \sum_{S \subset N} c_S w_S$.

Now, we have to show that $\phi_i(v) = \sum_{S \subset N, i \in N} \frac{c_S}{|S|}$ satisfies all the four axioms. This part is obvious.

Weighted majority game

The game is defined in the following way

$[q, w_1, w_2, w_3..., w_N]$, where N players, weight of each player is w_i and q is the quota.

If $\sum_{i \in S} w_i \geq q$ then S is a winning coalition. The characteristic function is

$$v(S) = \begin{cases} 1 \text{ `if } S \textit{ is a winning coalition} \\ 0 \textit{ if } S \textit{ is a losing coalition} \end{cases}$$

In these types of game, we measure the power or strength of a party.