Sum of games:

A cooperative game is a pair $(N; v)$ where N is the set of players and v is the characteristic function.

Different characteristic functions give different games.

$(N; u)$ game is

$N = \{1, 2, 3\}$

$u(1) = u(2) = u(3) = 0,\ u(\emptyset) = 0.$

$u(1, 2) = 10,\ \ u(1, 3) = 20,\ \ u(2, 3) = 30$

$u(1, 2, 3) = 40.$

Shown in fig 1.

Another game $(N; w)$
$N = \{1, 2, 3\}$
$w(1) = 5,\ w(2) = 10,\ w(3) = 15, w(\emptyset) = 0$
$w(1, 2) = 20,\ w(1, 3) = 25,\ w(2, 3) = 30$
$w(1, 2, 3) = 35.$
Shown in figure 2.

Another game $(N; v)$ is
$N = \{1, 2, 3\}$
$w(1) = 5,\ w(2) = 10,\ w(3) = 15, w(\emptyset) = 0$
$w(1, 2) = 30,\ w(1, 3) = 45,\ w(2, 3) = 60$
$w(1, 2, 3) = 75.$
Shown in figure 3.

Note we can write
$v(S) = u(S) + w(S)$ for all $S \subset N$ in the above example.

The game $(N; v)$ is called the sum of two games $(N; u)$ and $(N; w)$ if for every colaition S from the set of players $N(S \subseteq N)$
$v(S) = u(S) + w(S)$

Example:
$N\{1, 2, 3\}$ and the characteristic function is
$v(1) = 10,\ v(2) = 5,\ v(3) = 15,\ v(\emptyset) = 0$
$v(1, 2) = 15,\ v(1, 3) = 30,\ v(2, 3) = 25$
$v(1, 2, 3) = 40.$
We can split the above game in the following way
$(N; u)$
$u(1) = 5,\ u(2) = 5,\ u(3) = 5,\ u(\emptyset) = 0$
$u(1, 2) = 10,\ u(1, 3) = 15,\ u(1, 3) = 15$
$u(1, 2, 3) = 20$

Another game is
$(N; w)$
$w(1) = 5,\ w(2) = 0,\ w(3) = 10$
$w(1,2) = 5,\ w(2,3) = 10,\ w(1,3) = 15$
$w(1,2,3) = 20$

We have
$v(S) = u(S) + w(S)$ for every $S \subset N$ in this example.
It is shown in figure 4.

Shapley Value:

Consider a game (N, v).

A value function ϕ assigns to each possible characteristics function of an n- person game v, an n- tuple $\phi(v) = (\phi_1, \phi_2, \phi_3, ...\phi_N)$ of real numbers. Each ϕ_i represents the worth or value of player i in the game with characteristic function v.

For example

In game

$(N; w)$

$w(1) = 5,\ w(2) = 0,\ w(3) = 10$

$w(1, 2) = 5,\ w(2, 3) = 10,\ w(1, 3) = 15$

$w(1, 2, 3) = 20$

$w(1, 2, 3)$ can be distributed among the players $(5 + \frac{5}{3}, \frac{5}{3}, 10 + \frac{5}{3})$.

Here, $\phi(v) = (5 + \frac{5}{3}, \frac{5}{3}, 10 + \frac{5}{3})$.

How to get this division?

Axiom 1: The total amount $v(N)$ is divided among all the players.
Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$.

Axiom 2: Symmetric players get equal payoffs.
If i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S not containing i and j then $\phi_i(v) = \phi_j(v)$.

It gives fair division, if players are equal in terms of its contribution to a coalition, they are treated equally.

Axiom 3: The payoff to a null player is zero.
If i is such that $v(S) = v(S \cup \{i\})$ for every colaition S not containing i, then $\phi_i(v) = 0$.

If a player contributes nothing to the coalition, it gets nothing.

Axiom 4: If we split the original game into a sum of individual game games, the division of payoffs among the players in the original game should be the sum of divisions obtained in the individual games.

If u and v are characteristic functions, then $\phi(u + v) = \phi(u) + \phi(v)$.

Using these four axioms, we derive the Shapley value of a game.
Consider the following game

$N = \{1, 2, 3\}$

$v(1) = 6,\ v(2) = 12,\ v(3) = 18$

$v(1, 2) = 30,\ v(1, 3) = 60,\ v(2, 3) = 90$

$v(1, 2, 3) = 120,\ v(\emptyset) = 0.$

In figure 5, we derive the Shapley value.

We split the game into two games in such a way that , one of the game has a special property, it contains a coalition S such that $v(T) = v(S)$ whenever T contains S and $V(T) = 0$ for every other coalition. Such a game is called a carrier game and coalition S is called its carrier.

A carrier game $(N; v)$ is a game in which there is a coalition S called the carrier of the game, such that
$v(T) = v(S)$, whenever $S \subseteq T$
$v(T) = 0$, otherwise.