MA 322: Scientific Computing

Lecture - 8

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- Our objective is to discuss few algorithm which are capable of constructing polynomial function $P_n(x)$ with minimum computational cost. Such few algorithms are Lagrange's Interpolation and Newton's Interpolation etc.

- Recall the advantages and disadvantages of Lagrange's and Newton's backward/forward interpolation. Now, our objective is to develop an interpolation method which can accommodate new data and does not require equidistant arguments.

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Observe that $C(x)$ is a polynomial of degree n such that

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Thus, $C(x) = a_n(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})$.

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Thus, $C(x) = a_n(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})$. Then unknown quantity a_n is calculated from (1) using the condition $P_n(x_n) = f(x_n)$, so that

$$a_n = \frac{f(x_n) - P_{n-1}(x_n)}{(x_n - x_0)\ldots(x_n - x_{n-1})}. \tag{3}$$

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First order Newton's divided difference:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \; f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \ldots .$$

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Second order Newton's divided difference:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \; f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}, \ldots$$

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n-th order Newton's divided difference:

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}, \ldots.$$

Divided Differences Contd....

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Let us recall the coefficients a_n, which is given by

$$a_n = \frac{f(x_n) - P_{n-1}(x_n)}{(x_n - x_0)\dots(x_n - x_{n-1})}. \tag{4}$$

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&= \left(\frac{f(x_2) - f(x_1)}{x_2 - x_0} + \frac{f(x_1) - f(x_0)}{x_2 - x_0} - f[x_0, x_1]\right) / (x_2 - x_1) \\
&= \frac{f[x_1, x_2]}{x_2 - x_0} + \frac{f[x_0, x_1]}{x_2 - x_1}\left[\frac{x_1 - x_0}{x_2 - x_0} - 1\right] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} =: f[x_0, x_1, x_2]
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Therefore, for the coefficients a_n, we write

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Hence, the correction term $C(x)$ can be written as

$$C(x) = (x - x_0)(x - x_1)(x - x_2)\dots(x - x_{n-1})f[x_0, x_1, x_2, \dots, x_n], \tag{6}$$

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Recursively, we now construct Newton's divided difference formula for the interpolating polynomial

$$\begin{aligned} P_0(x) &= f(x_0), \quad P_1(x) = f(x_0) + (x - x_0)f[x_0, x_1], \\ P_2(x) &= f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ &\quad \dots\dots\dots\dots\dots\dots\dots\dots \\ P_n(x) &= f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ &\quad + \dots (x - x_0)(x - x_1)(x - x_2)\dots(x - x_{n-1})f[x_0, x_1, x_2, \dots, x_n]. \end{aligned}$$

Divided Difference Table

Divided Difference Table

Format for constructing divided differences of $f(x)$

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}] \cdots$
x_0	f_0		
		$f[x_0, x_1]$	
x_1	f_1		$f[x_0, x_1, x_2] \cdots$
		$f[x_1, x_2]$	
x_2	f_2		$f[x_1, x_2, x_3] \cdots$
		$f[x_2, x_3]$	
x_3	f_3		$f[x_2, x_3, x_4] \cdots$
		$f[x_3, x_4]$	
x_4	f_4		$f[x_3, x_4, x_5] \cdots$
		$f[x_4, x_5]$	
x_5	f_5		
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Divided differences table

Consider the nodes $\mathbf{x} := [x_0, \dots, x_n]$ and the values $\mathbf{f} := [f_0, \dots, f_n]$.

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Then, we generate following table of divided differences:

x	**f**			
x_0	f_0			
x_1	f_1	$f[x_0, x_1]$		
x_2	f_2	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
.	.	$\ldots$	$\ldots$	$\ldots$

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x_2	f_2	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
.	.	$\dots$	$\dots$	$\dots$

The $n+1$ diagonal entries in the table of divided differences are the coefficients of the Newton interpolating polynomial $p(x)$.

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x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
.	.	...	...	...

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x	**f**			
0	3			
1	4	$(4-3)/(1-0) = 1$		
2	7	$(7-4)/(2-1) = 3$	$(3-1)/(2-0) = 1$	
4	19	$(19-7)/(4-2) = 6$	$(6-3)/(4-1) = 1$	$(1-1)/(4-0) = 0$

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x_2	f_2	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
.	.	$\ldots$	$\ldots$	$\ldots$

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$$p(x) = 3 + 1\cdot(x-0) + 1\cdot(x-0)(x-1) + 0\cdot(x-0)(x-1)(x-2) = 3 + x^2.$$

Divided differences at repeated nodes

Theorem: If $f \in C^n[a, b]$ and $[x_0, \ldots, x_n]$ are distinct nodes then

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\theta)}{n!}$$

for some θ between the smallest and the largest nodes $[x_0, \ldots, x_n]$.

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This defines the n-th order divided difference of f at $n+1$ times repeated node x_0.