Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Basis and Optimal Hedge Ratio:

The difference between the spot price and the futures price is called the *basis* and is given by,

$$b(t,T) = S(t) - f(t,T).$$

The basis converges to zero as $t \to T$, since $f(T,T) = S(T)$.

We consider the problem of designing a hedging strategy. Suppose that we wish to sell the asset at time $t < T$. In order to hedge ourselves against a potential fall in the price of the asset, we short a futures contact with futures price $f(0,T)$ at time $t = 0$. Accordingly, at time t we receive $S(t)$ from selling the asset in addition to a cash flow of $f(0,T) - f(t,T)$ resulting from marking to market (we neglect any intermediate cash flow). The total cash flow thus is,

$$f(0,T) + S(t) - f(t,T) = f(0,T) + b(t,T).$$

The price $f(0,T)$ is known at time 0, so the risk involved with the hedging position is reflected by the basis. The hedger seeks to minimize the risk associated with the basis. In order to determine an optimal hedge ratio, the hedger enters into N futures contract, where N need not necessarily be the number of units of the underlying asset. In order to determine the optimal N, we compute the risk as measured by the variance of the basis ($b_N(t,T) = S(t) - Nf(t,T)$), that is,

$$Var(b_N(t,T)) = \sigma^2_{S(t)} + N^2\sigma^2_{f(t,T)} - 2N\sigma_{S(t)}\sigma_{f(t,T)}\rho_{S(t),f(t,T)}.$$

The variance is a quadratic in N and attains a minimum at,

$$N = \rho_{S(t),f(t,T)}\frac{\sigma_{S(t)}}{\sigma_{f(t,T)}},$$

which is the optimal hedge ratio.

Futures on Stock Index:

A stock exchange index is a weighted average of a selection of stock prices, with weights proportional to the market capitalization of stocks. An index is approximately proportional to the value of the market portfolio. For the purposes of futures markets, the index can be treated as a security. The futures prices $f(n,T)$, expressed in index points, are assumed to satisfy the conditions outlined earlier. Marking to market is given by the difference $f(n,T) - f(n-1,T)$ multiplied by a fixed amount. Our purpose is to study applications of index futures for hedging, based on CAPM. Recall that the expected return on a portfolio over a time step of length τ is given by,

$$\mu_V = r_F + (\mu_M - r_F)\beta_V,$$

where r_F is the risk-free rate for a single period, μ_M is the expected return on the market portfolio and β_V is the beta coefficient of the portfolio, Let $V(n)$ denote the value of the portfolio at the n-th time step.

We assume for simplicity that the index is equal to value of the market portfolio, so that the futures prices are given by,

$$f(n,T) = M(n)(1+r_F)^{T-n},$$

where $M(n)$ is the value of the market portfolio at the n-th time step. (Note that here discrete compounding has been used to be consistent with portfolio theory).

We now form a new portfolio (with value $\widetilde{V}(n)$ at the n-th time step), by adding to the original portfolio, N short futures contracts on the index, with delivery time T. The value $\widetilde{V}(n)$ of the new portfolio is the same as the value $V(n)$ of the original portfolio, since there is no investment made to initiate N futures contracts. At the $n+1$-th step, the value of the new portfolio is given by,

$$\widetilde{V}(n+1) = V(n+1) - N(f(n+1,T) - f(n,T)),$$

includes the cash flow resulting from marking-to-market. The return on the new portfolio over the first step is given by,

$$K_{\widetilde{V}} = \frac{\widetilde{V}(n+1) - \widetilde{V}(n)}{\widetilde{V}(n)} = \frac{V(n+1) - N(f(n+1,T) - f(n,T)) - V(n)}{V(n)} = K_V - \frac{N(f(n+1,T) - f(n,T))}{V(n)}.$$

It can be shown that the beta of the new portfolio, $\beta_{\widetilde{V}}$ can be modified by an appropriate choice of N. Accordingly we have the result

<u>Result:</u>

If

$$N = (\beta_V - a)\,\frac{(1+r_F)V(n)}{f(n,T)},$$

then $\beta_{\widetilde{V}} = a$ for any given number a.

<u>Proof:</u>

We calculate the beta from the definition,

$$\begin{aligned}
\beta_{\widetilde{V}} &= Cov(K_{\widetilde{V}}, K_M)/\sigma_M^2 \\
&= Cov(K_V, K_M)/\sigma_M^2 - \frac{1}{V(n)} Cov(N(f(n+1,T) - f(n,T)), K_M)/\sigma_M^2 \\
&= \beta_V - \frac{1}{V(n)} Cov(N(f(n+1,T) - f(n,T)), K_M)/\sigma_M^2
\end{aligned}$$

where K_M is the return on the market portfolio and K_V is the return on the portfolio without futures. Since $Cov(f(n,T), K_M) = 0$ and covariance is linear with respect to each argument,

$$Cov(N(f(n+1,T) - f(n,T)), K_M) = NCov(f(n+1,T), K_M)$$

But $f(n+1,T) = M(n+1)(1+r_F)^{T-(n+1)}$. Thus we have,

$$Cov(f(n+1,T), K_M) = (1+r_F)^{T-(n+1)} Cov(M(n+1), K_M).$$

Again by linearity of covariance in each argument,

$$Cov(M(n+1), K_M) = M(n)Cov\left(\frac{M(n+1) - M(n)}{M(n)}, K_M\right) = M(n)\sigma_M^2.$$

Substituting, we get,

$$\beta_{\widetilde{V}} = \beta_V - \frac{(1+r_F)^{T-(n+1)} NM(n)}{V(n)} = \beta_V - N\frac{f(n,T)}{V(n)(1+r_F)}.$$

This implies,

$$N = \left(\beta_V - \beta_{\widetilde{V}}\right)\frac{(1+r_F)V(n)}{f(n,T)}.$$