
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Definition:

Given two partitions $\mathcal{P}$ and $\mathcal{P}'$, we say that $\mathcal{P}'$ is finer than $\mathcal{P}$ if each element of $\mathcal{P}$ can be represented as a union of some elements of $\mathcal{P}'$.

Note that the random variables $S(1)$ and $S(2)$ are constant on the components of the corresponding partition $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.

Definition:

In general, for any integer $n \leq N$ and any $v_1, \dots, v_n \in \{u, d\}$, we define $B_{v_1,\dots,v_n}$ to be the set of all $\omega \in \Omega$ such that the first n elements in the sequence $\omega = \omega_1\omega_2 \dots \omega_n$ are $\omega_1 = v_1, \dots, \omega_n = v_n$. The partition $\mathcal{P}_n$ is defined as the family of all such subsets $B_{v_1,\dots,v_n}$ of Ω.

Filtration:

Given a finite partition $\mathcal{P}$, we consider the family of sets $\mathcal{F}$, consisting of the empty set ϕ and all possible union of components in $\mathcal{P}$.

Definition:

We denote by $\mathcal{F}_n$, the field extending the partition $\mathcal{P}_n$. For example, when $N = 3$,

$$\begin{aligned}
\mathcal{F}_0 &= \{\phi, \Omega\} \\
\mathcal{F}_1 &= \{\phi, \Omega\, B_u, B_d\} \\
\mathcal{F}_2 &= \{\phi, \Omega, B_{uu}, B_{ud}, B_{du}, B_{dd}, B_{uu} \cup B_{ud}, B_{uu} \cup B_{du}, \\
&\quad B_{uu} \cup B_{dd}, B_{ud} \cup B_{du}, B_{ud} \cup B_{dd}, B_{du} \cup B_{dd}, \\
&\quad \Omega \backslash B_{uu}, \Omega \backslash B_{ud}, \Omega \backslash B_{du}, \Omega \backslash B_{dd}\} \\
\mathcal{F}_3 &= \{\text{All subsets of } \Omega\}.
\end{aligned}$$

We can see that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$. This is an example of filtration, which in general is a sequence of fields such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

Remark:

The family $\mathcal{F}_n$ consists of all sets A such that for every scenario $\omega \in \Omega$, it is possible to tell whether or not $\omega \in A$, by knowing the stock prices up to and including time n only.

Option Pricing:

Investment Strategies:

We assume that we trade in a stock following the binomial model and buy a risk-free investment. Let the price of the stock and the bond be $S(0) = 100$ and $A(0) = 100$, respectively. Further, suppose that $U = 10\%$, $D = -10\%$ and $R = 5\%$.

[For the risk-free investment, the prices are determined by the interest rate R as $A(n) = A(0)(1 + R)^n$.]

Suppose the goal is to invest an amount of $V(0) = 300$, which is the initial portfolio value.

We can buy $x(1) = 2$ stocks and invest the rest in the bond, *i.e.*, buy $y(1) = \dfrac{V(0) - x(1)S(0)}{A(0)} = 1$ bonds. At time $t = 1$, the portfolio takes one of the two values,

$$\begin{aligned} V^u(1) &= x(1)S^u(1) + y(1)A(1) = 325 \text{ on } B_u \\ V^d(1) &= x(1)S^d(1) + y(1)A(1) = 285 \text{ on } B_d. \end{aligned}$$

These are the amounts available for the next time-step. We call such a strategy as self-financing, assuming that there is no withdrawal or injection of funds.

1. If at time $t = 1$, the state is u, we may decide that the position of the stock will now be $x^u(2) = 1$, which implies that
$$y^u(2) = \frac{V^u(1) - x^u(2)S^u(1)}{A(1)} = 2.0476.$$
At time $t = 2$, the portfolio takes one of the two values,
$$\begin{aligned} V^{uu}(2) &= x^u(2)S^{uu}(2) + y^u(2)A(2) = 346.75 \text{ on } B_{uu} \\ V^{ud}(2) &= x^u(2)S^{ud}(2) + y^u(2)A(2) = 324.75 \text{ on } B_{ud}. \end{aligned}$$

2. If at time $t = 1$, the state is d, we may decide that the position of the stock will now be $x^d(2) = 1$, which implies that
$$y^d(2) = \frac{V^d(1) - x^d(2)S^d(1)}{A(1)} = 0.1429.$$
At time $t = 2$, the portfolio takes one of the two values,
$$\begin{aligned} V^{du}(2) &= x^d(2)S^{du}(2) + y^d(2)A(2) = 312.75 \text{ on } B_{du} \\ V^{dd}(2) &= x^d(2)S^{dd}(2) + y^d(2)A(2) = 258.75 \text{ on } B_{dd}. \end{aligned}$$

In general, we assume that we build a strategy, that is, a sequence of portfolios $(x(n), y(n))$ according to the following principles:

1. The strategy is self-financing, that is,
$$x(n)S(n) + y(n)A(n) = x(n+1)S(n) + y(n+1)A(n).$$
This means that the portfolio restructuring at time n involves the choice of $x(n+1)$ and consequently $y(n+1)$.

2. The decision about the choice of $(x(n+1), y(n+1))$ is taken on the basis of information available at time n.

The value of a strategy is given by,

$$\begin{aligned} V(0) &= x(1)S(0) + y(1)A(0), \\ V(n) &= x(n)S(n) + y(n)A(n) \ , \ \text{for } n > 0. \end{aligned}$$

In a market with options, it is possible to create a portfolio (x, y, z) consisting of x shares of stock, y bonds and z options. The time 0 value of such a portfolio will be

$$V(0) = xS(0) + yA(0) + zC(0).$$

At time T it will be worth

$$V(T) = xS(T) + yA(T) + zC(T).$$

Here $C(t), t = 0, 1$ will denote the price of the option at time t, while $S(0), S(T), A(0)$ and $A(T)$ have already been defined earlier.

<u>No-Arbitrage Principle:</u>

There is no portfolio (x, y, z) that includes a position z in call options and has initial value $V(0) = 0$ such that $V(T) \geq 0$ with probability 1 and $V(T) > 0$ with non-zero probability.

The price $C(0)$ of the option at time $t = 0$ can be found in two steps, as outlined below:

1. We construct an investment in x stocks and y bonds such that the value of the investment at time T is the same as the option, *i.e.*,

   $$xS(T) + yA(T) = C(T).$$

   This is known as replicating the option.

2. The price $C(0)$ of the option may be calculated as follows:

   $$C(0) = xS(0) + yA(0).$$

<u>Example:</u>

Consider bond with $A(0) = 100$ and $A(T) = 110$.

Also, consider a long call option, with strike price $X = 100$, on a stock with $S(0) = 100$ and

$$S(T) = \begin{cases} 120 \text{ with probability } p, \\ 80 \text{ with probability } (1-p), \end{cases}$$

where $0 < p < 1$. The relation

$$xS(T) + yA(T) = C(T),$$

yields the two following equations:

$$\begin{aligned} x \times 120 + y \times 110 &= \max(120 - 100, 0) = 20, \\ x \times 80 + y \times 110 &= \max(80 - 100, 0) = 0. \end{aligned}$$

Solving the two equations, we get, $x = 1/2$ and $y = -4/11$.

Accordingly, the price of the option then is,

$$C(0) = xS(0) + yA(0) = \frac{1}{2} \times 100 - \frac{4}{11} \times 100 \approx 13.6364.$$