
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

We now consider an example to apply the Theorem to a "so-called" path-dependent option.

Example:

Let us consider $S_0 = 4, u = 2, d = \frac{1}{2}$. Let $r = \frac{1}{4}$. Then we obtain $\widetilde{p} = \widetilde{q} = \frac{1}{2}$. Now, we consider a "lookback option" (for a three period binomial model) with the payoff $V_3 = \max_{0 \le n \le 3} S_n - S_3$, at time $t = 3$.

Then

$$
\begin{aligned}
V_3(HHH) &= S_3(HHH) - S_3(HHH) = 32 - 32 = 0, \\
V_3(HHT) &= S_2(HH) - S_3(HHT) = 16 - 8 = 8, \\
V_3(HTH) &= S_1(H) - S_3(HTH) = 8 - 8 = 0, \\
V_3(HTT) &= S_1(H) - S_3(HTT) = 8 - 2 = 6, \\
V_3(THH) &= S_3(THH) - S_3(THH) = 8 - 8 = 0, \\
V_3(THT) &= S_2(TH) - S_3(THT) = 4 - 2 = 2, \\
V_3(TTH) &= S_0 - S_3(TTH) = 4 - 2 = 2, \\
V_3(TTT) &= S_0 - S_3(TTT) = 4 - 0.50 = 3.50.
\end{aligned}
$$

Then the price of the option at time $t = 2$ are:

$$
\begin{aligned}
V_2(HH) &= \frac{4}{5}\left[\frac{1}{2}V_3(HHH) + \frac{1}{2}V_3(HHT)\right] = 3.20, \\
V_2(HT) &= \frac{4}{5}\left[\frac{1}{2}V_3(HTH) + \frac{1}{2}V_3(HTT)\right] = 2.40, \\
V_2(TH) &= \frac{4}{5}\left[\frac{1}{2}V_3(THH) + \frac{1}{2}V_3(THT)\right] = 0.80, \\
V_2(TT) &= \frac{4}{5}\left[\frac{1}{2}V_3(TTH) + \frac{1}{2}V_3(TTT)\right] = 2.20.
\end{aligned}
$$

Accordingly, the price of the option at time $t = 1$ are:

$$
\begin{aligned}
V_1(H) &= \frac{4}{5}\left[\frac{1}{2}V_2(HH) + \frac{1}{2}V_2(HT)\right] = 2.24, \\
V_1(T) &= \frac{4}{5}\left[\frac{1}{2}V_2(TH) + \frac{1}{2}V_2(TT)\right] = 1.20.
\end{aligned}
$$

Finally, we obtain,

$$
V_0 = \frac{4}{5}\left[\frac{1}{2}V_1(H) + \frac{1}{2}V_1(T)\right] = 1.376.
$$

If an investor sells the lookback option for 1.376 (at time $t = 0$), then this short position in the option can be hedged by buying

$$
\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{2.24 - 1.20}{8 - 2} = 0.1733.
$$

Computational Considerations:

If we consider a large number of periods in the binomial model, say $n = 100$, then there are $2^{100} \approx 10^{30}$ possible outcomes. Calculating 2^{100} values for V_{100} is not computationally feasible. However, it is possible to organize the Theorem in a computationally efficient manner, which will be illustrated through two examples, one for path-independent option and the other for path-dependent option.

Example 1:

Let us consider $S_0 = 4, u = 2, d = \dfrac{1}{2}$ and $r = \frac{1}{4}$. Consider the problem of pricing an European put option with strike price $K = 5$ and expiration at time $t = 3$. As before, the risk-neutral probabilities are $\widetilde{p} = \widetilde{q} = \dfrac{1}{2}$. The values of the payoff $(5-S_3)^+$ are: $V_3(HHH) = 0$, $V_3(HHT) = V_3(HTH) = V_3(THH) = 0$, $V_3(HTT) = V_3(THT) = V_3(TTH) = 3$ and $V_3(TTT) = 4.50$. In order to simplify things with $2^3 = 8$ entries, we denote by $v_3(s)$, the payoff for the option at time $t = 3$, when the stock price at time $t = 3$ is s. While the argument for V_3 is a sequence of three coin tosses, the argument for v_3 is the stock price. Note that at time $t = 3$, there are only four possible values of the stock price. Accordingly, $v_3(32) = 0$, $v_3(8) = 0$, $v_3(2) = 3$ and $v_3(0.5) = 4.50$.

In case of $n = 100$, while the argument of V_{100} would range over 2^{100} possible outcomes, the argument of v_{100} would range over 101 possible stock price at time 100. This is a huge reduction in the computational complexity. Accordingly, we compute V_2 as follows:

$$V_2(\omega_1\omega_2) = \frac{2}{5}\left[V_3(\omega_1\omega_2 H) + V_3(\omega_1\omega_2 T)\right],$$

which represents four equations, resulting from four possible values of $\omega_1\omega_2$. Let $v_2(s)$ denote the price of the put option at time $t = 2$, if the stock price at time $t = 2$ is s. Accordingly,

$$v_2(s) = \frac{2}{5}\left[v_3(2s) + v_3\left(\frac{1}{2}s\right)\right].$$

This results in

$$\begin{aligned} v_2(16) &= \frac{2}{5}\left[v_3(32) + v_3(8)\right] = 0, \\ v_2(4) &= \frac{2}{5}\left[v_3(8) + v_3(2)\right] = 1.20, \\ v_2(1) &= \frac{2}{5}\left[v_3(2) + v_3(0.50)\right] = 3. \end{aligned}$$

Similarly,

$$\begin{aligned} v_1(8) &= \frac{2}{5}\left[v_2(16) + v_2(4)\right] = 0.48, \\ v_1(2) &= \frac{2}{5}\left[v_2(4) + v_2(1)\right] = 1.68, \end{aligned}$$

where $v_1(s)$ denotes the price of the put option at time $t = 1$, if the stock price at time $t = 1$ is s. The price of the put option at time $t = 0$ is,

$$v_0(4) = \frac{2}{5}\left[v_1(8) + v_1(2)\right] = 0.864.$$

At each time $n = 0, 1, 2$, if the stock price is s, then the number of stocks in the replicating portfolio is,

$$\delta_n(s) = \frac{v_{n+1}(2s) - v_{n+1}\left(\frac{1}{2}s\right)}{2s - \frac{1}{2}s}.$$