
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Definition:

Let X be a random variable defined on a finite probability space $(\Omega, \mathbb{P})$. The expectation (or expected value) of X is defined to be:

$$\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega).$$

When we compute the expectation using the risk-neutral probability measure $\widetilde{\mathbb{P}}$, we use the notation:

$$\widetilde{\mathbb{E}}X = \sum_{\omega \in \Omega} X(\omega)\widetilde{\mathbb{P}}(\omega).$$

The variance of X is:

$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}X)^2\right].$$

It is clear from its definition that expectation is linear: If X and Y are random variables and c_1 and c_2 are constants then:

$$\mathbb{E}\left(c_1 X + c_2 Y\right) = c_1 \mathbb{E}X + c_2 \mathbb{E}Y.$$

Theorem: Jensen's Inequality:

Let X be a random variable on a finite probability space, and let $\varphi(x)$ be a convex function of a dummy variable x. Then:

$$\mathbb{E}\left[\varphi(X)\right] \geq \varphi\left(\mathbb{E}X\right).$$

Conditional Expectations:

Recall the formulas for risk-neutral probabilities:

$$\widetilde{p} = \frac{1 + r - d}{u - d} \text{ and } \widetilde{q} = \frac{u - 1 - r}{u - d}.$$

It can be easily verified that

$$\frac{\widetilde{p}u + \widetilde{q}d}{1 + r} = 1.$$

Consequently, for every time n and for every sequence of coin tosses $\omega_1\omega_2 \dots \omega_n$, we have

$$S_n\left(\omega_1\omega_2 \dots \omega_n\right) = \frac{1}{1 + r}\left[\widetilde{p}S_{n+1}\left(\omega_1\omega_2 \dots \omega_n H\right) + \widetilde{q}S_{n+1}\left(\omega_1\omega_2 \dots \omega_n T\right)\right].$$

For the sake of brevity, we define the notation:

$$\widetilde{\mathbb{E}}_n\left[S_{n+1}\right]\left(\omega_1\omega_2 \dots \omega_n\right) = \widetilde{p}S_{n+1}\left(\omega_1\omega_2 \dots \omega_n H\right) + \widetilde{q}S_{n+1}\left(\omega_1\omega_2 \dots \omega_n T\right).$$

Accordingly, we have,

$$S_n = \frac{1}{1 + r}\widetilde{\mathbb{E}}_n\left[S_{n+1}\right].$$

Definition:

Let n satisfy $1 \leq n \leq N$, and let $\omega_1\omega_2\dots\omega_n$ be fixed (as of now). There are 2^{N-n} possible continuations $\omega_{n+1}\omega_{n+2}\dots\omega_N$ of the fixed sequence $\omega_1\omega_2\dots\omega_n$. Denote by $\#H(\omega_{n+1}\omega_{n+2}\dots\omega_N)$ and $\#T(\omega_{n+1}\omega_{n+2}\dots\omega_N)$, the number of heads and tails, respectively, of the continuation $\omega_{n+1}\omega_{n+2}\dots\omega_N$. We define

$$\widetilde{\mathbb{E}}_n[X](\omega_1\omega_2\dots\omega_n) = \sum_{\omega_{n+1}\omega_{n+2}\dots\omega_N} \widetilde{p}^{\#H(\omega_{n+1}\omega_{n+2}\dots\omega_N)}\widetilde{q}^{\#T(\omega_{n+1}\omega_{n+2}\dots\omega_N)} X(\omega_1\omega_2\dots\omega_n\omega_{n+1}\omega_{n+2}\dots\omega_N),$$

and call $\widetilde{\mathbb{E}}_n[X]$ the conditional expectation of X based on the information at time n.

Definition (Continued):

The two extreme cases of conditioning are $\widetilde{\mathbb{E}}_0[X]$, the conditional expectation of X based on no information, which we define by,

$$\widetilde{\mathbb{E}}_0[X] = \widetilde{E}X,$$

and $\widetilde{\mathbb{E}}_N[X]$, the conditional expectation of X based on knowledge of all N coin tosses, which we define by,

$$\widetilde{\mathbb{E}}_N[X] = X.$$

While the conditional expectation has been defined above for the risk-neutral probabilities, it can also be computed using the actual probabilities, wherein the notation will be $\mathbb{E}_n$.

Theorem: Fundamental Properties of Conditional Expectation:

Let N be a positive integer, and let X and Y be random variables depending on the first N coin tosses. Let $0 \leq n \leq N$ be given. Then the following properties hold:

1. *Linearity of conditional expectations:* For all constants c_1 and c_2, we have

$$\mathbb{E}_n[c_1X + c_2Y] = c_1\mathbb{E}_n[X] + c_2\mathbb{E}_n[Y].$$

2. *Taking out what is known:* If X actually depends only on the first n coin tosses, then

$$\mathbb{E}_n[XY] = X \cdot \mathbb{E}_n[Y].$$

3. *Iterated conditioning:* If $0 \leq n \leq m \leq N$, then

$$\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X].$$

In particular,

$$\mathbb{E}[\mathbb{E}_m[X]] = \mathbb{E}X.$$

4. *Independence:* If X depends only on tosses $n+1$ through N, then

$$\mathbb{E}_n[X] = \mathbb{E}X.$$

5. *Conditional Jensen's inequality:* If $\varphi(x)$ is a convex function of the dummy variable x, then

$$\mathbb{E}_n[\varphi(X)] \geq \varphi(\mathbb{E}_n[X]).$$