Chapter 6

Term Structure of Default Probability

So far, default has mostly been modeled as a binary event (except the intensity model), suited for single-period considerations within the regulatory framework of a fixed planning horizon. However, the choice of a specific period like one year is more or less arbitrary. Even more, default is an inherently time-dependent event. This chapter serves to introduce the idea of a term structure of default probability. This credit curve represents a necessary prerequisite for a time-dependent modeling as in Chapters 7 and 8. In principle, there are three different methods to obtain a credit curve: from historical default information, as implied probabilities from market spreads of defaultable bonds, and through Merton's option theoretic approach. The latter has already been treated in a previous chapter, but before introducing the other two in more detail we first lay out some terminology used in survival analysis (see [15,16] for a more elaborated presentation).

6.1 Survival Function and Hazard Rate

For any model of default timing, let $S(t)$ denote the probability of surviving until t. With help of the "time-until-default" τ (or briefly "default time"), a continuous random variable, the *survival function* $S(t)$ can be written as

$$S(t) = \mathbb{P}[\tau > t], \quad t \geq 0.$$

That is, starting at time $t = 0$ and presuming no information is available about the future prospects for survival of a firm, $S(t)$ measures the likelihood that it will survive until time t. The probability of default between time s and $t \geq s$ is simply $S(s) - S(t)$. In particular, if $s = 0$, and because $S(0) = 1$, then the probability of default $F(t)$ is

$$F(t) = 1 - S(t) = \mathbb{P}[\tau \leq t], \quad t \geq 0. \tag{6. 1}$$

$F(t)$ is the distribution function of the random default time τ. The corresponding probability density function is defined by

$$f(t) = F'(t) = -S'(t) = \lim_{\Delta \to 0^+} \frac{\mathbb{P}[t \leq \tau < t + \Delta]}{\Delta},$$

if the limit exists. Furthermore, we introduce the *conditional* or *forward* default probability

$$p(t|s) = \mathbb{P}[\tau \leq t | \tau > s], \quad t \geq s \geq 0\,,$$

i.e., the probability of default of a certain obligor between t and s conditional on its survival up to time s, and

$$q(t|s) = 1 - p(t|s) = \mathbb{P}[\tau > t | \tau > s] = S(t)/S(s), \quad t \geq s \geq 0,$$

the forward survival probability. An alternative way of characterizing the distribution of the default time τ is the *hazard function*, which gives the instantaneous probability of default at time t conditional on the survival up to t. The hazard function is defined via

$$\mathbb{P}[t < \tau \leq t + \Delta t | \tau > t] = \frac{F(t + \delta t) - F(t)}{1 - F(t)} \approx \frac{f(t)\Delta t}{1 - F(t)}$$

as

$$h(t) = \frac{f(t)}{1 - F(t)}\,.$$

Equation(6. 1) yields

$$h(t) = \frac{f(t)}{1 - F(t)} = -\frac{S'(t)}{S(t)},$$

and solving this differential equation in $S(t)$ results in

$$S(t) = e^{-\int_0^t h(s)ds}\,. \tag{6. 2}$$

This allows us to express $q(t|s)$ and $p(t|s)$ as

$$q(t|s) = e^{-\int_s^t h(u)du}, \tag{6. 3}$$

$$p(t|s) = 1 - e^{-\int_s^t h(u)du}\,. \tag{6. 4}$$

Additionally, we obtain

$$F(t) = 1 - S(t) = 1 - e^{-\int_0^t h(s)ds},$$

and

$$f(t) = S(t)h(t).$$

One could assume the hazard rate to be piecewise constant, i.e., $h(t) = h_i$ for $t_i \leq t < t_{i+1}$. In this case, it follows that the density function of τ is

$$f(t) = h_i e^{-h_i t} \mathbf{1}_{[t_i, t_{i+1}[}(t),$$

showing that the survival time is exponentially distributed with parameter h_i. Furthermore, this assumption entails over the time interval $[t_i, t_{i+1}[$ for $0 < t_i \leq t < t_{i+1}$

$$q(t|t_i) = e^{-\int_{t_i}^t h(u)du} = e^{-h_i(t-t_i)}.$$

Remark The "forward default rate" $h(t)$ as a basis of a default risk term structure is in close analogy to a forward interest rate, with zero-coupon bond prices corresponding to survival probabilities. The hazard rate function used to characterize the distribution of survival time can also be called a "credit curve" due to its similarity to a yield curve. If h is continuous then $h(t)\Delta t$ is approximately equal to the probability of default between t and $t+\Delta t$, conditional on survival to t. Understanding the first arrival time τ as associated with a Poisson arrival process, the constant mean arrival rate h is then called intensity and often denoted by λ[1]. Changing from a deterministically varying intensity to random variation, and thus closing the link to the stochastic intensity models [32], turns Equation (6. 3) into

$$q(t|s) = \mathbb{E}_s \left[e^{-\int_s^t h(u)du} \right],$$

where $\mathbb{E}_s$ denotes expectation given all information available at time s.

[1]Note that some authors explicitly distinguish between the intensity $\lambda(t)$ as the arrival rate of default at t conditional on *all* information available at t, and the forward default rate $h(t)$ as arrival rate of default at t, conditional *only* on survival until t.

6.2 Risk-neutral vs. Actual Default Probabilities

When estimating the risk and the value of credit-related securities we are faced with the question of the appropriate probability measure, *risk-neutral* or *objective probabilities*. But in fact the answer depends on the objective we have. If one is interested in estimating the economic capital and risk charges, one adopts an actuarial-like approach by choosing historical probabilities as underlying probability measure. In this case we assume that actual default rates from historical information allow us to estimate a capital quota protecting us against losses in worst case default scenarios. The objective is different when it comes to pricing and hedging of credit-related securities. Here we have to model under the risk-neutral probability measure. In a risk-neutral world all individuals are indifferent to risk. They require no compensation for risk, and the expected return on all securities is the risk-free interest rate. This general principle in option pricing theory is known as risk-neutral valuation and states that it is valid to assume the world is risk-neutral when pricing options. The resulting option prices are correct not only in the risk-neutral world, but in the real world as well. In the credit risk context, risk-neutrality is achieved by calibrating the default probabilities of individual credits with the market-implied probabilities drawn from bond or credit default swap spreads. The difference between actual and risk-neutral probabilities reflects risk-premiums required by market participants to take risks. To illustrate this difference suppose we are pricing a one-year par bond that promises its face value 100 and a 7% coupon at maturity. The one-year risk-free interest rate is 5%. The actual survival probability for one year is $1 - \text{DP} = 0.99$; so, if the issuer survives, the investor receives 107. On the other hand, if the issuer defaults, with actual probability $\text{DP} = 0.01$, the investor recovers 50% of the par value. Simply discounting the expected payoff computed with the actual default probability leads to

$$\frac{(107 \times 0.99 + 50\% \times 100 \times 0.01)}{1 + 5\%} = 101.36\,,$$

which clearly overstates the price of this security. In the above example we have implicitly adopted an actuarial approach by assuming that the price the investor is to pay should exactly off set the expected loss due to a possible default. Instead, it is natural to assume that investors are concerned about default risk and have an aversion to bearing more risk. Hence, they demand an additional risk premium and the pricing should somehow account for this risk aversion. We therefore turn the above pricing formula around and ask which probability results in the quoted price, given the coupons, the risk-free rate, and the recovery value. According to the risk-neutral valuation paradigm, the fact that the security is priced at par implies that

$$100 = \frac{(107 \times (1 - \mathrm{DP}^*) + 50\% \times 100 \times \mathrm{DP}^*)}{1 + 5\%}.$$

Solving for the market-implied risk-neutral default probability yields $\mathrm{DP}^* = 0.0351$. Note that the actual default probability $\mathrm{DP} = 0.01$ is less than DP^*. Equivalently, we can say that the bond is priced as though it were a break-even trade for a "stand-in" investor who is not risk adverse but assumes a default probability of 0.0351. The difference between DP and DP^* reflects the risk premium for default timing risk. Most credit market participants think in terms of spreads rather than in terms of default probabilities, and analyze the shape and movements of the spread curve rather than the change in default probabilities. And, indeed, the link between credit spread and probability of default is a fundamental one, and is analogous to the link between interest rates and discount factors in fixed income markets. If s represents a multiplicative spread over the risk-free rate one gets

$$\mathrm{DP}^* = \frac{1 - \frac{1}{1+s}}{1 - REC} \approx \frac{s}{1 - REC},$$

where the approximation is also valid for additive spreads.

"Actuarial credit spreads" are those implied by assuming that investors are neutral to risk, and use historical data to estimate default probabilities and expected recoveries. Data from Fons [43] suggest that corporate yield spreads are much larger than the spreads suggested by actuarial default losses alone. For example, actuarially implied credit spreads on a A-rated 5-year US corporate debt were estimated by Fons to be six basis points. The corresponding market spreads have been in the order of 100 basis points. Clearly, there is more than default risk behind the difference between "actuarial credit spreads" and actual yield spreads, like liquidity risk, tax-related issues, etc. But even after measuring spreads relative to AAA yields (thereby stripping out treasury effects), actuarial credit spreads are smaller than actual market spreads, especially for high-quality bonds.

6.3 Term Structure Based on Historical Default Information

Multi-year default probabilities can be extracted from historical data on corporate defaults similarly to the one-year default probabilities. But before going into details we first show a "quick and dirty" way to produce a whole term structure if only one-year default probabilities are at hand.

6.3.1 Exponential Term Structure

The derivation of exponential default probability term structure is based on the idea that credit dynamics can be viewed as a two-state time-homogeneous Markov-chain, the two states being survival and default, and the unit time between two time steps being Δ. Suppose a default probability DP_T for a time interval T (e.g., one year) has been calibrated from data; then the survival probability for the time unit Δ (e.g., one day) is given by

$$\mathbb{P}[\tau > t + \Delta | \tau \geq t] = (1 - \mathrm{DP}_T)^{\Delta/T} \ , \tag{6. 5}$$

and the default probability for the time t, in units of Δ, is then

$$\mathrm{DP}_t = 1 - (1 - \mathrm{DP}_T)^{t/T} \ . \tag{6. 6}$$

In the language of survival analysis we can write for the probability of survival until T

$$1 - \mathrm{DP}_T = q(T|0) = e^{-\int_0^T h(u)du} = e^{-\bar{h}T},$$

where the last equation defines the average default rate $\bar{h}$,

$$\bar{h} = -\log\left(1 - \mathrm{DP}_T\right)/T.$$

Assuming a constant default rate over the whole life time of the debt, Equation (6. 6) reads

$$F(t) = q(t|0) = 1 - p(t|0) = 1 - e^{-\bar{h}t}.$$

have much more impact than years further back in history and the result does not reflect a typical year as averaged over economic cycles.

Having now extracted cumulative default probabilities at discrete points in time, DP_i, we might be interested in a continuous version of this term structure. The simplest answer is an adroit linear interpolation of the multi-year default probability table (the interpolation *between* rating classes ought to be done on a logarithmic scale).

A slightly more sophisticated method can be formulated with the help of the forward default rate h. The forward default probability between t_i and t_{i+1} is given by

$$p(t_{i+1}|t_i) = \frac{\mathrm{DP}_{i+1} - \mathrm{DP}_i}{1 - \mathrm{DP}_i} = 1 - \exp\left(-\int_{t_i}^{t_{i+1}} h(u)du\right).$$

Note that $\mathrm{DP}_i = F(t_i)$. Define for the time interval $[t_i, t_{i+1}]$ an average forward default rate by

$$\bar{h}_i = \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} h(u)du, \quad \text{for} \quad i = 0, \ldots, n.$$

In terms of the multi-year default probabilities the forward default rate for period i is

$$\bar{h}_i = -\frac{1}{t_{i+1} - t_i} \log\left(\frac{1 - \mathrm{DP}_{i+1}}{1 - \mathrm{DP}_i}\right).$$

Two hazard rate functions obtained from the multi-year default probabilities in Table 6.2 are depicted in Figure 6.2 and show a typical feature: investment grade securities tend to have an upward sloping hazard rate term structure, whereas speculative grades tend to have a downward sloping term structure.

The cumulative default probability until time t, $t_i \leq t < t_{i+1}$ boils down to

$$\begin{aligned} \mathrm{DP}_t = F(t) &= 1 - q(t_i|0)q(t|t_i) \\ &= 1 - (1 - \mathrm{DP}_i)\left(\frac{1 - \mathrm{DP}_{i+1}}{1 - \mathrm{DP}_i}\right)^{(t-t_i)/(t_{i+1}-t_i)}. \end{aligned}$$

For $0 < t < 1$ we obtain again the exponential term structure. For $t > t_n$ the term structure can be extrapolated by assuming a constant forward default rate $\bar{h}_{n-1}$ beyond t_{n-1},

$$\mathrm{DP}_t = 1 - (1 - \mathrm{DP}_{n-1})\left(\frac{1 - \mathrm{DP}_n}{1 - \mathrm{DP}_{n-1}}\right)^{(t-t_n)/(t_{i+1}-t_i)}.$$