Another way to compute Shapley value
Consider the following game
$N = \{1, 2, 3\}$
$v(1) = 6,\ v(2) = 12,\ v(3) = 18$
$v(1, 2) = 30,\ v(1, 3) = 60,\ v(2, 3) = 90$
$v(1, 2, 3) = 120,\ v(\emptyset) = 0.$

In the last class using carrier coalitions, we compute the Shapley value as $(22, 40, 58)$.

Allocation based on marginal contribution of each player to the coalition.
$v(1,2) - v(2) =$ marginal contribution of player 1.
$v(1,2) - v(1) =$ marginal contribution of player 2.
$v(1,2,3) - v(1,2) =$ marginal contribution of player 3.
We can take the following sequence
$v(1) = 6$, marginal contribution of player 1
$v(1,2) - v(1) = 30 - 6 = 24$ marginal contribution of player 2
$v(1,2,3) - v(1,2) = 120 - 30 = 90$ marginal contribution of player 3.
The sequence is player 1, player 2, player 3.

Suppose the sequence is player 2, player 3, player 1.
$v(2) = 12$, marginal contribution of player 2.
$v(2,3) - v(2) = 90 - 12 = 78$ marginal contribution of player 3.
$v(1,2,3) - v(2,3) = 120 - 90 = 30$ marginal contribution of player 1.
As we change the position, the marginal contribution changes.

Shapley value is the average of marginal contribution of each player taking into account all the possible order.

In case of the above game, the computation is shown below.

Order of entry	Player 1	Player 2	Player 3
1 2 3	6	30-6=24	120-30=90
1 3 2	6	120-60=60	60-6=54
2 3 1	120-90=30	12	90-12=78
2 1 3	30-12=18	12	120-30 90
3 1 2	60-18=42	120-60= 60	18
3 2 1	120-60=30	90-18=72	18
average	22	40	58

$$\phi_1 = \frac{6+6+30+18+42+30}{6},$$
$$\phi_2 = \frac{24+60+12+12+60+72}{6}$$
$$\phi_3 = \frac{90+54+78+90+18+18}{6}$$

For any coalition game $(N; v)$, the Shapley value is

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(N-|S|)!}{N!} \Big[v(S) - v(S - \{i\}) \Big],$$

$i = 1, 2, 3...N$ and $S \subset N$

In a game with N players, total number of order of players is $N!$. Suppose player i enter the coalition. It will join a coalition which will have $S - \{i\}$ players. The contribution of player i is $v(S) - v(S - \{i\})$. $S - \{i\}$ player can come first in $(|S| - 1)!$ ways. The remaining $N - |S|$ players can be ordered in $(N - |S|)!$ ways. So, in $(|S| - 1)!(N - |S|)!$ ways out of $N!$ ways, player i is going to join the coalition after $S - \{i\}$ players have joined.

In the above example, Probability of Player 1 in first position; $\frac{2!0!}{3!}$

Probability of Player 1 in second position; $\frac{1!1!}{3!}$ and $\frac{1!1!}{3!}$

Probability of player 1 in third position ; is $\frac{2!1!}{3!}$

Similarly we can find for player 2 and player 3.

Compare core and Shapley value