MA 322: Scientific Computing

Lecture - 3

Fixed Point Iteration

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$$\text{(i) } x = x^2 + x - a = g(x), \;\; \text{(ii) } x = \frac{a}{x} = g(x), \;\; \text{(iii) } x = \frac{1}{2}\left(x + \frac{a}{x}\right) = g(x)$$

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- It is natural to ask what is the choice for g such that it leads to a solution of the equation $f(x) = 0$. The objective of this lecture is to study the choice of g and its relationship with f.

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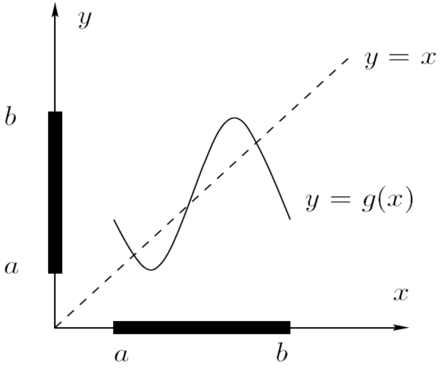
observe that $f(a) \cdot f(b) \leq 0$, and so $f(x)$ has at least one root α in the interval $[a, b]$. Therefore, we obtain

$$f(\alpha) = 0 \text{ Or } \alpha = f(\alpha).$$



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- In order to relate this example to previous theorem, let us rewrite the equation $f(x) = 0$ in the equivalent form

$$x = g(x), \quad \text{where} \quad g(x) = \log(2x + 1), \ x \in [1, 2]. \tag{1}$$

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Definition (Contraction):

- Suppose that $g : [a, b] \to \mathbb{R}$ is a real-valued and continuous function. Then, g is said to be a contraction on $[a, b]$ if there exists a constant L such that $0 < L < 1$ and

$$|g(x) - g(y)| \leq L|x - y| \ \ \forall x, y \in [a, b]. \tag{6}$$

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4	1.224563
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The results are shown for 11 iterates.

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Hence, $|\alpha - x_n| \leq \epsilon$ if we set $L^n|\alpha - x_0| \leq \epsilon$. Thus, finding n such that relation (15) holds depend on $x_0,\ \alpha$ and L. But, α might not be available before computation. What is next?

Observations:

- Note that

$$|\alpha - x_0| \leq |\alpha - x_1| + |x_1 - x_0| \leq L|\alpha - x_0| + |x_1 - x_0|, \tag{17}$$

which implies

$$|\alpha - x_0| \leq \frac{1}{1-L}|x_1 - x_0| \tag{18}$$

Now, use above estimate in (16) to have

$$|\alpha - x_n| \leq \frac{L^n}{1-L}|x_1 - x_0|. \tag{19}$$

Stopping Criterion Contd....

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- Alternative Criterion:
  - Since the approximating sequence $\langle x_n \rangle$ converges, so, we can apply following stoping criterion

$$|x_{n+1} - x_n| \leq \epsilon.$$

Stopping Criterion Contd....

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Example: Now we can return to previous example to answer the question posed there about the maximum number of iterations required, with starting value $x_0 = 1$, to ensure that the last iterate computed is correct to six decimal digits.

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- In fact, 33 is a somewhat pessimistic overestimate of the number of iterations required. Computing the iterates x_n successively shows that already x_{25} is correct to six decimal digits, giving $\alpha = 1.256431$.

Few Fixed Point Results Contd....

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  Hence, g' is monotonic decreasing on $[1, 2]$. Hence $g'(1) \geq g'(\eta) \geq g'(2)$, that is $g'(\eta) \in [2/5, 2/3]$ and $L = 2/3$.

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- Assume that g is continuously differentiable on $[a, b]$, that $g([a, b]) \subset [a, b]$, and that $L = \max_{x \in [a,b]} |g'(x)| < 1$.

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Remark: It is an easy exercise to verify that $L = \frac{1}{2}(1 + |g'(\alpha)|)$.

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