Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Theorem:
If the interest rate r is constant, then $f(0,T) = F(0,T)$.
Proof:
For simplicity, suppose that the marking to market (for a futures contract) is done at only two intermediate time points t_1 and t_2, such that $0 < t_1 < t_2 < T$. The argument given below can easily be extended to more frequent marking to market.
Strategy Using Forward Contract:

1. At time $t = 0$:
   (a) We take a long forward position (at no cost) with the forward price $F(0,T)$.
   (b) Invest an amount of $e^{-rT}F(0,T)$ in a risk-free account.
2. At time $t = T$:
   (a) We receive an amount $F(0,T)$.
   (b) Close the forward position by buying a share for $F(0,T)$.
   (c) Sell the share for the market price $S(T)$.

Thus the final wealth is $S(T)$.
The idea is to replicate this payoff ($S(T)$) by using futures contracts
Strategy Using Futures Contract:

1. At time $t = 0$:
   (a) We take a fraction $e^{-r(T-t_1)}$ of a long futures position (at no cost).
   (b) We invest an amount $e^{-rT}f(0,T)$ in a risk-free account (this will grow to $v_0 = f(0,T)$ at time T).
2. At time $t = t_1$:
   (a) We receive (or pay if negative) the amount $e^{-r(T-t_1)}[f(t_1,T) - f(0,T)]$ as a result of marking to market.
   (b) We invest (or borrow if negative) $e^{-r(T-t_1)}[f(t_1,T) - f(0,T)]$ (this will grow to $v_1 = f(t_1,T) - f(0,T)$ at time T).
   (c) We increase our long futures position to $e^{-r(T-t_2)}$ of a contract (at no cost).
3. At time $t = t_2$:
   (a) We receive (or pay if negative) an amount $e^{-r(T-t_2)}[f(t_2,T) - f(t_1,T)]$ as a result of marking to market.
   (b) We invest (or borrow if negative) $e^{-r(T-t_2)}[f(t_2,T) - f(t_1,T)]$ (this will grow to $v_2 = f(t_2,T) - f(t_1,T)$ at time T).
   (c) We increase our long futures position to 1 of a contract (at no cost).

4. At time $t = T$:

    (a) We receive an amount $v_0 + v_1 + v_2 = f(t_2, T)$ from the risk-free investments.

    (b) We close the futures position by paying $f(t_2, T)$.

    (c) We sell the stock for $S(T)$.

The final wealth level will be $S(T)$, as before.
Thus, in order to avoid arbitrage we have,

$$e^{-rT} f(0,T) = e^{-rT} F(0,T) \Rightarrow f(0,T) = F(0,T).$$

Hedging with Futures:

One can hedge an exposure to stock price variations by entering into a forward contract. However, an appropriate forward contract might not be easily available, not to speak of the default risk. Instead one could hedge using the futures market.

Example:

Let $S(0) = 100$ and let the constant risk-free rate be $r = 8\%$. Assume that the marking to market takes place once a month, the time step being $1/12$. Suppose that we intend to sell the stock after three months. To hedge the exposure to stock prices we enter into a futures contract with delivery in three months. The payments from marking to market attract risk-free interest. The results for two such stock price scenarios are given below,

1.

n	$S(n)$	$f(n/12, 3/12)$	m2m	Interest
0	100	102.02		
1	102	103.37	-1.35	-0.02
2	101	101.68	$+1.69$	$+0.01$
3	105	105.00	-3.32	0.00
		Total	-2.98	-0.01

In this case, one can sell the stock for 105.00, but marking to market causes loss bringing the sum to $105.00 - 2.98 - 0.01 = 102.01$. If the marking to market did not attract interest than the realized sum would be $105.00 - 2.98 = 102.02$ which is exactly equal to the futures price $f(0, 3/12)$.

2.

n	$S(n)$	$f(n/12, 3/12)$	m2m	Interest
0	100	102.02		
1	98	99.32	$+2.70$	$+0.04$
2	97	97.65	$+1.67$	$+0.01$
3	92	92.00	$+5.65$	0.00
		Total	10.02	$+0.05$

In this case, one can sell the stock for 92.00 and along with the amount for marking to market and interest accrued, receive a final amount of $92.00 + 10.02 + 0.05 = 102.07$. Without the interest the final amount would be $92.00 + 10.02 = 102.02$ which is exactly equal to the futures price $f(0, 3/12)$.

Some limitations arise because of standardized nature of futures contract. As a result there are difficulties in matching the terms of the contract to the specific needs of the individual. The exercise dates for futures are typically certain fixed days in a year, for example, third Friday of March, June, September and December. If we want to close out our investment at the end of April, then we need to hedge with futures contracts with a delivery date after April, such as June.