
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Forward Price for Stock Paying Dividend:

The forward price of a stock paying dividend div at time t, where $0 < t < T$ is:

$$F(0,T) = [S(0) - e^{-rt}div]e^{rT}.$$

1. Suppose that
$$F(0,T) > [S(0) - e^{-rt}div]e^{rT}.$$
   (a) At time $t = 0$:
      i. Enter into a short forward contract with forward price $F(0,T)$ and delivery time T.
      ii. Borrow $S(0)$ dollars and buy one share.
   (b) At time $t = t$:
      i. Cash the dividend div and invest it at risk free rate for the remaining time $T - t$.
   (c) At time $t = T$:
      i. Sell the share for $F(0,T)$.
      ii. Pay $S(0)e^{rT}$ to clear the loan with interest.
      iii. Collect $e^{r(T-t)}div$.

   The final balance is $F(0,T) - S(0)e^{rT} + e^{r(T-t)}div > 0$ in violation of the no-arbitrage principle.

2. Suppose that
$$F(0,T) < [S(0) - e^{-rt}div]e^{rT}.$$
   (a) At time $t = 0$:
      i. Enter into a long forward contract with forward price $F(0,T)$ and delivery at time T.
      ii. Sell short one share and invest the proceeds $S(0)$ at the risk free rate r.
   (b) At time $t = t$:
      i. Borrow div dollars and pay a dividend to the stock owner.
   (c) At time $t = T$:
      i. Buy one share for $F(0,T)$ and close out the short position in the stock.
      ii. Cash the risk free investment with interest collecting the amount $S(0)e^{rT}$.
      iii. Pay $e^{r(T-t)}div$ to clear the loan with interest.

   The final balance is $-F(0,T) + S(0)e^{rT} - e^{r(T-t)}div > 0$ in violation of the no-arbitrage principle.

Thus,

$$F(0,T) = [S(0) - e^{-rt}div]e^{rT}.$$

Forward Price for Stock Paying Dividends Continuously:

The forward price of a stock paying dividends continuously at rate r_{div} is:

$$F(0,T) = S(0)e^{(r-r_{div})T}.$$

1. Suppose that

$$F(0,T) > S(0)e^{(r-r_{div})T}.$$

   (a) At time $t = 0$:

      i. Enter into a short forward contract.

      ii. Borrow the amount $S(0)e^{-r_{div}T}$ to buy $e^{-r_{div}T}$ shares.

   (b) Between time 0 and T collect the dividends paid continuously, reinvesting them in the stock. At time T you will have 1 share.

   (c) At time $t = T$:

      i. Sell the share for $F(0,T)$, closing out the short forward position.

      ii. Pay $S(0)e^{(r-r_{div})T}$ to clear the loan with interest.

   The final balance is $F(0,T) - S(0)e^{(r-r_{div})T} > 0$ in violation of the no-arbitrage principle.

2. Suppose that

$$F(0,T) < S(0)e^{(r-r_{div})T}.$$

   (a) At time $t = 0$:

      i. Take a long forward position.

      ii. Short sell a fraction $e^{-r_{div}T}$ of a share investing the proceeds $S(0)e^{-r_{div}T}$ risk free.

   (b) Between time 0 and T you will need to pay dividends to the stock owner, raising cash by shorting the stock. Your short position in the stock will thus increase to 1 share at time T.

   (c) At time $t = T$:

      i. Buy one share for $F(0,T)$ and return it to the owner, closing out the long forward position and the short position in the stock.

      ii. Receive $S(0)e^{(r-r_{div})T}$ from the risk free investment.

   The final balance is $S(0)e^{(r-r_{div})T} - F(0,T) > 0$ in violation of the no-arbitrage principle.

Thus,

$$F(0,T) = S(0)e^{(r-r_{div})T}.$$