
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Example:

Recall that in the binomial model, the price of the stock at time $n+1$ is given in terms of the price of the stock at time n by the formula:

$$S_{n+1}(\omega_1\omega_2\ldots\omega_n\omega_{n+1}) = \begin{cases} uS_n(\omega_1\omega_2\ldots\omega_n), \text{ if } \omega_{n+1} = H, \\ dS_n(\omega_1\omega_2\ldots\omega_n), \text{ if } \omega_{n+1} = T. \end{cases}$$

Therefore,

$$\mathbb{E}_n[f(S_{n+1})(\omega_1\omega_2\ldots\omega_n)] = pf(uS_n(\omega_1\omega_2\ldots\omega_n)) + qf(dS_n(\omega_1\omega_2\ldots\omega_n)),$$

and the right hand side depends on $\omega_1\omega_2\ldots\omega_n$ only through the value of $S_n(\omega_1\omega_2\ldots\omega_n)$. Omitting the coin tosses $\omega_1\omega_2\ldots\omega_n$, we can rewrite the equation as

$$\mathbb{E}_n[f(S_{n+1})] = g(S_n),$$

where the function $g(x)$ of the dummy variable x is defined by $g(x) = pf(ux) + qf(dx)$. This shows that the stock price process is Markov.

Indeed, the stock price process is Markov under either the actual (as seen above) or the risk-neutral probability measure. To determine the price V_n at time n of a derivative security whose payoff at time N is a function v_N of the stock price S_N, that is, $V_N = v_n(S_N)$, we use the risk-neutral pricing formula which reduces to

$$V_n = \frac{1}{1+r}\widetilde{\mathbb{E}}_n[V_{n+1}], \; n = 0, 1, 2, \ldots, N-1.$$

But $V_N = v_n(S_N)$ and the stock price process is Markov, so

$$V_{N-1} = \frac{1}{1+r}\widetilde{\mathbb{E}}_{N-1}[v_N(s_N)] = v_{N-1}(S_{N-1}),$$

for some function v_{N-1}. Similarly,

$$V_{N-2} = \frac{1}{1+r}\widetilde{\mathbb{E}}_{N-2}[v_{N-1}(S_{N-1})] = v_{N-2}(S_{N-2}),$$

for some function v_{N-2}. In general,

$$V_n = v_n(S_n),$$

for some function v_n. Moreover, we can compute these functions recursively by the algorithm

$$v_n(s) = \frac{1}{1+r}[\widetilde{p}v_{n+1}(us) + \widetilde{q}v_{n+1}(ds)], \; n = N-1, N-2, \ldots, 0.$$

This algorithm works in the binomial model for any derivative security whose payoff at time N is a function only of the stock price at time N. In particular, this can be used for calls and put options, with

$v_N(s) = (s-K)^+$ and $v_N(s) = (K-s)^+$.

Lemma (Independence):

In the N-period binomial asset pricing model, let n be an integer between 0 and N. Suppose the random variables, $X^1, X^2, \ldots, X^K$ depend only on coin tosses 1 through n and the random variables $Y^1, Y^2, \ldots, Y^L$ depend only on coin tosses $n+1$ through N. (Note that the superscripts $1, 2, \ldots, K$ on X and $1, 2, \ldots, L$ on Y are superscripts, and not exponents). Let $f\left(x^1, x^2, \ldots, x^K, y^1, y^2, \ldots, y^L\right)$ be a function of dummy variables $x^1, x^2, \ldots, x^K$ and $y^1, y^2, \ldots, y^L$, and define

$$g\left(x^1, x^2, \ldots, x^K\right) = \mathbb{E} f\left(x^1, x^2, \ldots, x^K, Y^1, Y^2, \ldots, Y^L\right).$$

Then

$$\mathbb{E}_n\left[f\left(X^1, X^2, \ldots, X^K, Y^1, Y^2, \ldots, Y^L\right)\right] = g\left(X^1, X^2, \ldots, X^K\right).$$

Proof:

Let $\omega_1\omega_2\ldots\omega_n$ be fixed but arbitrary. From the definition of conditional expectation

$$\begin{aligned} &\mathbb{E}_n\left[f\left(X, Y\right)\right]\left(\omega_1\omega_2\ldots\omega_n\right) \\ = &\sum_{\omega_{n+1}\omega_{n+2}\ldots\omega_N} f\left(X\left(\omega_1\omega_2\ldots\omega_n\right), Y\left(\omega_{n+1}\omega_{n+2}\ldots\omega_N\right)\right) p^{\#H(\omega_{n+1}\omega_{n+2}\ldots\omega_N)} q^{\#T(\omega_{n+1}\omega_{n+2}\ldots\omega_N)}. \end{aligned}$$

On the other hand,

$$g(x) = \mathbb{E} f(x, Y) = \sum_{\omega_{n+1}\omega_{n+2}\ldots\omega_N} f\left(x, Y\left(\omega_{n+1}\omega_{n+2}\ldots\omega_N\right)\right) p^{\#H(\omega_{n+1}\omega_{n+2}\ldots\omega_N)} q^{\#T(\omega_{n+1}\omega_{n+2}\ldots\omega_N)},$$

which implies that

$$\begin{aligned} &g\left(X\left(\omega_1\omega_2\ldots\omega_n\right)\right) \\ = &\sum_{\omega_{n+1}\omega_{n+2}\ldots\omega_N} f\left(X\left(\omega_1\omega_2\ldots\omega_n\right), Y\left(\omega_{n+1}\omega_{n+2}\ldots\omega_N\right)\right) p^{\#H(\omega_{n+1}\omega_{n+2}\ldots\omega_N)} q^{\#T(\omega_{n+1}\omega_{n+2}\ldots\omega_N)}. \end{aligned}$$

Hence we obtain

$$\mathbb{E}_n\left[f\left(X, Y\right)\right]\left(\omega_1\omega_2\ldots\omega_N\right) = g\left(X\left(\omega_1\omega_2\ldots\omega_N\right)\right).$$

Example (Non-Markov Process):

In the binomial model with $S_0 = 4$, $u = 2$ and $d = \frac{1}{2}$, consider the maximum-to-date process $M_n = \max\limits_{0\le k\le n} S_k$. With $p = \frac{2}{3}$ and $q = \frac{1}{3}$, we obtain,

$$\mathbb{E}_2[M_3](TH) = \frac{2}{3}M_3(THH) + \frac{1}{3}M_3(THT) = \frac{16}{3} + \frac{4}{3} = 6\frac{2}{3}.$$

But

$$\mathbb{E}_2[M_3](TT) = \frac{2}{3}M_3(TTH) + \frac{1}{3}M_3(TTT) = \frac{8}{3} + \frac{4}{3} = 4.$$

Since $M_2(TH) = M_2(TT) = 4$, there cannot be a function g such that $\mathbb{E}_2[M_3](TH) = g(M_2(TH))$ and $\mathbb{E}_2[M_3](TT) = g(M_2(TT)$. Thus the maximum-to-date process is not Markov.