Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Factors Determining Option Prices:

The option price depends on a number of factors, such as the strike price X, the asset price S, the expiration T etc. We shall now analyze option prices as functions of one variable keeping the other variables constant.

European Option:

Dependence on the Strike Price:

We shall consider options on the same underlying asset and with the same exercise time T, but with different values of the strike price X. The call and the put option prices will be denoted by $C^E(X)$ and $P^E(X)$, respectively. All remaining variables such as the exercise time T, running time t and the underlying asset price S are also kept fixed.

Result:

If $X' < X''$, then

$$C^E(X') \geq C^E(X'')\ ,\ P^E(X') \leq P^E(X'').$$

This means that $C^E(X)$ is a non-increasing and $P^E(X)$ a non-decreasing function of X.

Proof:

Suppose that $X' < X''$, but $C^E(X') < C^E(X'')$. We can sell a call with strike price X'' and buy a call with strike price X', investing the difference $C^E(X'') - C^E(X')$ at risk-free rate r. If the option with strike price X'' is exercised at T, we pay $(S(T) - X'')^+$. We will then simultaneously exercise the option with strike price $X'$and receive $(S(T) - X')^+$. Since $X' < X''$, we have $(S(T) - X')^+ > (S(T) - X'')^+$. Thus there will be an arbitrage profit of $\left(C^E(X'') - C^E(X')\right) e^{rT} + (S(T) - X')^+ - (S(T) - X'')^+$. In case the call with strike price X'' is not exercised then there will be an arbitrage profit of $\left(C^E(X'') - C^E(X')\right) e^{rT}$. The proof for put option is similar.

Result:

If $X' < X''$, then

$$\begin{aligned} C^E(X') - C^E(X'') &\leq e^{-rT}(X'' - X'), \\ P^E(X'') - P^E(X') &\leq e^{-rT}(X'' - X'). \end{aligned}$$

Proof:

By put-call parity

$$\begin{aligned} C^E(X') - P^E(X') &= S(0) - X'e^{-rT} \\ C^E(X'') - P^E(X'') &= S(0) - X''e^{-rT}. \end{aligned}$$

Subtracting, we get

$$(C^E(X') - C^E(X'')) + (P^E(X'') - P^E(X')) = (X'' - X')e^{-rT}$$

Since both terms on the LHS are non-negative, so each of the terms on the LHS cannot exceed the RHS term. This proves the result.

Remark:

The above inequalities mean that the call and the put prices as functions of the strike price satisfy the Lipschitz condition with constant $e^{-rT} < 1$,

$$\begin{aligned}|C^E(X'') - C^E(X')| &\leq e^{-rT}|X'' - X'| \\ |P^E(X'') - P^E(X')| &\leq e^{-rT}|X'' - X'|.\end{aligned}$$

Result:

Let $X' < X''$ and let $\alpha \in (0, 1)$. Then

$$\begin{aligned}C^E(\alpha X' + (1-\alpha)X'') &\leq \alpha C^E(X') + (1-\alpha)C^E(X''), \\ P^E(\alpha X' + (1-\alpha)X'') &\leq \alpha P^E(X') + (1-\alpha)P^E(X'').\end{aligned}$$

In other words, $C^E(X)$ and $P^E(X)$ are convex functions of X.

Proof:

For convenience, let $X = \alpha X' + (1-\alpha)X''$. Suppose that

$$C^E(X) > \alpha C^E(X') + (1-\alpha)C^E(X'').$$

We sell a call option with strike price X, and purchase α call options with strike price X' and $1-\alpha$ call options with strike price X'' and invest the balance $C^E(X) - \alpha C^E(X') - (1-\alpha)C^E(X'') > 0$ at risk-free rate r. If the option with strike price X is exercised at expiry, then we have to pay $(S(T) - X)^+$. This can be covered by the amount $\alpha(S(T) - X')^+ + (1-\alpha)(S(T) - X'')^+$ by exercising the α and $1-\alpha$ calls with strike X' and X'' respectively. There is an arbitrage profit since

$$\alpha(S(T) - X')^+ + (1-\alpha)(S(T) - X'')^+ \geq (S(T) - X)^+$$

holds, the proof of which is as follows:

We consider four possible cases:

1. If $S(T) \leq X' < X < X''$, then the relation reduces to $0 \leq 0$.

2. If $X' < S(T) \leq X < X''$, then the relation becomes $0 \leq \alpha(S(T) - X')$, which is true when $S(T) > X'$.

3. If $X' < X < S(T) \leq X''$, then the relation is $(S(T) - X) \leq \alpha(S(T) - X')$.
   This is equivalent to $(1-\alpha)S(T) \leq X - \alpha X' = (\alpha X' + (1-\alpha)X'') - \alpha X' \iff (1-\alpha)S(T) \leq (1-\alpha)X'' \iff S(T) \leq X''$ which is true.

4. Finally if $X' < X < X'' \leq S(T)$, then the above relation becomes an equality since $X = \alpha X' + (1-\alpha)X''$

If the option with strike price X is not exercised at expiry, then there is an arbitrage profit of $\left(C^E(X) - \alpha C^E(X') - (1-\alpha)C^E(X'')\right) e^{rT}$.

The proof for put option is similar.