Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

European Options:

We will now establish some upper and lower bounds on the prices of European call and put options,

1. $C^E < S(0)$.

   Proof:

   Suppose the reverse inequality is satisfied, that is, $C^E \geq S(0)$. Then at time $t = 0$, we can sell a call for C^E, buy a stock for $S(0)$ and invest the remaining amount $C^E - S(0)$ at riskfree rate r. At time $t = T$, we can sell the stock for $\min(S(T), X)$. The final net amount would then be $(C^E - S(0))e^{rT} + \min(S(T), X) > 0$ leading to arbitrage. Thus $C^E < S(0)$.

2. $S(0) - Xe^{-rT} \leq C^E$.

   Proof:

   $P^E = C^E - S(0) + Xe^{-rT} \geq 0 \Rightarrow S(0) - Xe^{-rT} \leq C^E$.

3. $P^E < Xe^{-rT}$.

   Proof:

   $P^E - Xe^{-rT} = C^E - S(0) < 0 \Rightarrow P^E < Xe^{-rT}$.

4. $-S(0) + Xe^{-rT} \leq P^E$.

   Proof:

   $C^E = P^E + S(0) - Xe^{-rT} \geq 0 \Rightarrow -S(0) + Xe^{-rT} \leq P^E$.

These results can be summarized as follows:

Result:

The prices of European call and put options on a stock paying no dividends satisfy the inequalities:

$$\max\left(0, S(0) - Xe^{-rT}\right) \leq C^E < S(0)$$
$$\max\left(0, -S(0) + Xe^{-rT}\right) \leq P^E < Xe^{-rT}.$$

Result:

The prices of European call and put options on a stock paying no dividends satisfy the inequalities:

$$\max\left(0, S(0) - div_0 - Xe^{-rT}\right) \leq C^E < S(0) - div_0$$
$$\max\left(0, -S(0) + div_0 + Xe^{-rT}\right) \leq P^E < Xe^{-rT}.$$

Theorem:

The prices of European and American call options on a non-dividend paying stock are equal, that is, $C^A = C^E$, for the same strike price X and expiration T.

Proof:

We already know that $C^A \geq C^E$. Suppose that $C^A > C^E$. Then at time $t = 0$, we sell an American call for C^A and buy an European call for C^E and invest the balance $C^A - C^E$ at riskfree rate r.

1. If the American call is exercised at time $t \leq T$, then we short sell a stock for X to settle the short call option position and invest X at riskfree rate r. Then, at time T we use the European call to buy a share for X and return the stock to the owner of the short sold stock. The arbitrage profit will be $(C^A - C^E)e^{rT} + Xe^{rT} - X > 0$.

2. If the American option is not exercised at all, then we will end up with an arbitrage profit of $(C^A - C^E)e^{rT}$.

Thus proves that $C^A = C^E$.

American Options:

We first consider American options on a non-dividend paying stock. As already seen, in this case, the price of an American call is equal to that of an European call, $C^A = C^E$. So it must satisfy the same bounds as for an European call option. For an American put option we have the following:

1. $-S(0) + X \leq P^A$

   Proof:

   This is true since the price P^A of an American option cannot be less than the payoff of the option at time 0. Another way of looking at this is the following: Suppose $-S(0) + X > P^A$. Then we buy a put option for P^A, buy a stock for $S(0)$ and immediately exercise the option for X (all at time $t = 0$), thereby making an arbitrage profit of $-S(0) + X - P^A > 0$

2. $P^A < X$

   Proof:

   Suppose $P^A \geq X$. Then we can sell an American put for P^A and invest this amount at riskfree rate r.

   (a) If the put is exercised at time $t \leq T$, then a share of the underlying stock will have to be bought for X and which can then be sold for $S(t)$. The net balance will be $P^A e^{rt} - X + S(t) > 0$.

   (b) If the option is not exercised at all, the net balance will be $P^A e^{rT} > 0$ at expiration T.

   Thus $P^A < X$

Result:

The prices of American call and put options on a stock paying no dividends satisfy the inequalities

$$\max\left(0, S(0) - Xe^{-rT}\right) \leq C^A < S(0)$$
$$\max\left(0, -S(0) + X\right) \leq P^A < X.$$

Result:

The prices of American call and put options on a dividend-paying stock satisfy the following inequalities

$$\max\left(0, S(0) - div_0 - Xe^{-rT}, S(0) - X\right) \leq C^A < S(0),$$
$$\max\left(0, -S(0) + div_0 + Xe^{-rT}, -S(0) + X\right) \leq P^A < X.$$