Chapter 3

Credit Spreads and Bond Price-Based Pricing

Default-free and defaultable bonds

- Let $B(t,T)$ and $\bar{B}(t,T)$ be the prices of default-free and defautable zero-coupon bonds.
- We assume that at time t we know the prices of all $B's$ and $\bar{B}'s$.
- Let $I(t) = 1_{\tau > t}$ be the default indicator.
- Assumption: $\{B(t,T) | T \geq t\}$ and τ are independent.
- We know that $B(t,T) = E(e^{-\int_t^T r(s)ds})$.
- Similarly, $\bar{B}(t,T) = E(e^{-\int_t^T r(s)ds} I(T))$.
- So, by independence:

$$\bar{B}(t,T) = E(e^{-\int_t^T r(s)ds} I(T)) =$$

$$E(e^{-\int_t^T r(s)ds})E(I(T)) = B(t,T)P(t,T)$$

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Implied Survival Probability

- $P(t, T)$ is the implied survival probability in $[t, T]$.
- Then the implied default probability over $[t, T]$ is $P^{def}(t, T) = 1 - P(t, T)$.
- The conditional survival probability over $[T_1, T_2]$ is given by:

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)}$$

Forward Rates

- Simply compounded fwd rate:

$$F(t, T_1, T_2) = \frac{B(t, T_1)/B(t, T_2) - 1}{T_2 - T_1}$$

- Defaultable simple compounded fwd rate:

$$\bar{F}(t, T_1, T_2) = \frac{\bar{B}(t, T_1)/\bar{B}(t, T_2) - 1}{T_2 - T_1}$$

- The continuously compounded versions are:

$$-\frac{\partial}{\partial T} lnB(t, T) \text{ and } -\frac{\partial}{\partial T} ln\bar{B}(t, T)$$

Implied Hazard rates

- Conditional prob of def per time unit Δt between T and $T + \Delta t$ as seen from time $t < T$ is

$$\frac{1}{\Delta t} = P^{def}(t, T, T + \Delta t) = \frac{1}{\Delta t}(1 - P(t, T, T + \Delta t)$$

- Discrete implied hazard rate is defined as:

$$H(t, T, T + \Delta t) = \frac{1}{\Delta t} \frac{P(t, T) - P(t, T + \Delta t)}{P(t, T + \Delta t)}$$

- Continuous implied hazard rate is the limit:

$$h(t, T) = -\frac{1}{P(t, T)} \frac{\partial}{\partial T} P(t, T)$$

Hazard/Forward rates

The hazard rate wrt the probability of default is defined analogously to the forward rates wrt to the bond prices.

Relation between hazard rates and forward rates

- Using that $P(t, T) = \frac{\bar{B}(t,T)}{B(t,T)}$ we get:
- $H(t, T_1, T_2) = \frac{B(0,T_2)}{B(0,T_1)}(\bar{F}(t, T_1, T_2) - F(t, T_1, T_2)$
- $h(t, T) = \bar{f}(t, T) - f(t, T)$
- The prob of default in a short time interval is roportional to the length of the interval with proportionality factor: $\bar{f}(t, T) - f(t, T)$.
- Or: in a short period of time (as $f(t, t) = r)t$) the credit spread is the proportionality factor of the default probability.

Recovery Modeling

- Assuming independence between the term structure of interest rates and the probability of default we can compute the value of \$1 payable at time $T + \Delta t$ if a default happens between time T and time $T + \Delta t$.
- $e(t, T, T + \Delta t) = E^Q(\beta(t, T + \Delta t)(I(T) - I(T + \Delta t))|\mathcal{F}_t)$
- It ends up being:

$$e(t, T, T + \Delta t) = \Delta t \bar{B}(t, T + \Delta t) H(t, T, T + \Delta t)$$

.

- or $e(t, T) = \bar{B}(t, T) h(t, T)$.

Building blocks for pricing

$$B(0, T_k) = \prod_{i=1}^{k} \frac{1}{1+\delta_{i-1}F(0,T_{i-1},T_i)}$$

$$\bar{B}(0, T_k) = B(0, T_k)P(0, T_k) = B(0, T_K) \prod_{i=1}^{k} \frac{1}{1+\delta_{i-1}H(0,T_{i-1},T_i)}$$

$$e(0, T_k, T_{k+1}) = \delta_k H(0, T_k, T_{k+1})\bar{B}(0, T_k)$$

or,

$$B(0, T_k) = e^{-\int_0^{T_k} f(0,s)ds}$$

$$\bar{B}(0, T_k) = e^{-\int_0^{T_k} h(0,s)+f(0,s)ds}$$

$$e(0, T_k) = h(0, T_k)\bar{B}(0, T_k)$$

Credit Default Swaps

- In a CDS a fixed payment is matched up with a payoff should default occur.
- The spread ends up being:

$$\bar{s} = (1-\pi)\frac{\sum_{k=1}^{k_N} \delta_{k-1} H(0, T_{k-1}, T_k)\bar{B}(0, T_k)}{\sum_{n=1}^{N} \delta'_{n-1} \bar{B}(0, T_{k_n})}$$

- If tenor dates and coupon dates coincide:

$$\bar{s} = (1-\pi)\sum_{n=1}^{N} w_n H(0, T_{k-1}, T_k)$$

  where the weights add up to one.
- That formula is analogous to the one linking the swap rate and forward rates.