
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Definition:

A *path-dependent* European option with exercise time N, and underlying asset S, is a contingent claim with payoff of the form $H(N) = f(S(1), S(2), \ldots, S(N))$ available to the holder only at time N, where f is a function of N variables. A *path-independent* European contingent claim has the payoff of the form $H(N) = f(S(N))$.

Two Step Pricing:

We consider an European claim with path-independent payoff. For $N = 2$, we have,

$$H(2) = f(S(2)),$$

which has three possible values.

The claim price $H(1)$ had two values

$$H(1) = \begin{cases} \frac{1}{1+R}\left[p_* f(S^{uu}(2)) + (1-p_*) f(S^{ud}(2))\right] & \text{on } B_u \\ \frac{1}{1+R}\left[p_* f(S^{du}(2)) + (1-p_*) f(S^{dd}(2))\right] & \text{on } B_d. \end{cases}$$

This gives

$$H(1) = \frac{1}{1+R}\left[p_* f(S(1)(1+U)) + (1-p_*) f(S(1)(1+D))\right] = g(S(1)),$$

where

$$g(x) = \frac{1}{1+R}\left[p_* f(x(1+U)) + (1-p_*) f(x(1+D))\right].$$

As a result, $H(1)$ can be regarded as a derivative security with expiration time $t = 1$, with the payoff defined by the function g, even though it cannot be exercised at time $t = 1$. It can however be sold for $H(1) = g(S(1))$. Hence the one-step approach applied between time $t = 2$ to time $t = 1$, can now be applied again from $t = 1$ to $t = 0$.

We therefore have,

$$H(0) = \frac{1}{1+R}\left[p_* g(S(0)(1+U)) + (1-p_*) g(S(0)(1+D))\right].$$

It follows that,

$$\begin{aligned} H(0) &= \frac{1}{1+R}\left[p_* g(S^u(1)) + (1-p_*) g(S^d(1))\right] \\ &= \frac{1}{(1+R)^2}\left[p_*^2 f(S^{uu}(2)) + 2p_*(1-p_*) f(S^{ud}(2)) + (1-p_*)^2 f(S^{dd}(2))\right]. \end{aligned}$$

The last expression in square brackets is the expectation of $f(S(2))$ under the risk-neutral probability P_* defined as

$$P_*(uu) = p_*^2,\ P_*(ud) = P_*(du) = p_*(1-p_*),\ P_*(dd) = (1-p_*)^2.$$

Hence we have the following result.

Theorem:

The expectation of the discounted payoff computed with respect to the risk-neutral probability is equal to the present value of the European claim,

$$H(0) = E_* \left(\frac{1}{(1+R)^2} f(S(2) \right).$$

Several Step Pricing:

We now extend the preceding results to a multi-step model. Beginning with the payoff at expiration, we proceed backwards by applying the one-step procedure repeatedly. For the case of $N = 3$, we have,

$$\begin{aligned}
H(3) &= f(S(3)), \\
H(2) &= \frac{1}{1+R} \left[p_* f(S(2)(1+U)) + (1-p_*) f(S(2)(1+D)) \right] = g(S(2)) \\
H(1) &= \frac{1}{1+R} \left[p_* g(S(1)(1+U)) + (1-p_*) g(S(1)(1+D)) \right] = h(S(1)) \\
H(0) &= \frac{1}{1+R} \left[p_* h(S(0)(1+U)) + (1-p_*) h(S(0)(1+D)) \right].
\end{aligned}$$

where

$$\begin{aligned}
g(x) &= \frac{1}{1+R} \left[p_* f(x(1+U)) + (1-p_*) f(x(1+D)) \right]. \\
h(x) &= \frac{1}{1+R} \left[p_* g(x(1+U)) + (1-p_*) g(x(1+D)) \right].
\end{aligned}$$

Hence,

$$\begin{aligned}
H(0) &= \frac{1}{1+R} \left[p_* h(S^u(1)) + (1-p_*) h(S^d(1)) \right] \\
&= \frac{1}{(1+R)^2} \left[p_*^2 g(S^{uu}(2)) + 2p_*(1-p_*) g(S^{ud}(2)) + (1-p_*)^2 g(S^{dd}(2)) \right]. \\
&= \frac{1}{(1+R)^3} \left[p_*^3 f(S^{uuu}(3)) + 3p_*^2(1-p_*) f(S^{uud}(3)) + 3p_*(1-p_*)^2 f(S^{udd}(3)) + (1-p_*)^3 f(S^{ddd}(3)) \right].
\end{aligned}$$

Observing the above pattern we arrive at the option price for an N-step model as follows,

$$H(0) = \frac{1}{(1+R)^N} \sum_{k=0}^{N} \binom{N}{k} p_*^k (1-p_*)^{N-k} f\left(S(0)(1+U)^k (1+D)^{N-k} \right).$$

Note that the expectation of $f(S(N))$ under the risk-neutral probability

$$P_*(\omega) = p_*^k (1-p_*)^{N-k},$$

where k and $N-k$ are the number of upward and downward stock movements, respectively, is evident in the above formula.

Theorem:

The value of an European path-independent contingent claim with payoff $f(S(N))$ in the N-step binomial model is the expectation of the discounted payoff under the risk-neutral probability:

$$H(0) = E_* \left(\frac{1}{(1+R)^N} f(S(N) \right).$$