
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Definition:

An American *derivative security* or *contingent claim* with payoff function f, expiring at time N, is a sequence of random variables defined by the following backward inductive relations:

$$\begin{aligned}
H_A(N) &= f(S(N)),\\
H_A(N-1) &= \max\left\{f(S(N-1)), \frac{1}{1+R}\left[p_* f(S(N-1)(1+U)) + (1-p_*) f(S(N-1)(1+D))\right]\right\}\\
&:= f_{N-1}(S(N-1)),\\
H_A(N-2) &= \max\left\{f(S(N-2)), \frac{1}{1+R}\left[p_* f_{N-1}(S(N-2)(1+U)) + (1-p_*) f_{N-1}(S(N-2)(1+D))\right]\right\}\\
&:= f_{N-2}(S(N-2)),\\
&\vdots\\
H_A(1) &= \max\left\{f(S(1)), \frac{1}{1+R}\left[p_* f_2(S(1)(1+U)) + (1-p_*) f_2(S(1)(1+D))\right]\right\}\\
&:= f_1(S(1)),\\
H_A(0) &= \max\left\{f(S(0)), \frac{1}{1+R}\left[p_* f_1(S(0)(1+U)) + (1-p_*) f_1(S(0)(1+D))\right]\right\}.
\end{aligned}$$

Continuous Time Model:

Shortcomings of Discrete Models:

1. Restricts the range of asset price movements.

2. Restricts the set of time instances at which these movements may occur.

The goal is to arrive at the classical Black-Scholes market model as a limit of a sequence of binomial models.

Let $T > O$ denote the time window (measured in years). In a binomial model with N steps, the length of each time step or interval will be $h = \dfrac{T}{N}$. The time instances, t between 0 and T will be given by $t = nh$, where $n = 0, 1, \ldots, N$. For these time instances, the stock price and the risk-free asset price in the N-step binomial model will be denoted by $S_N(t)$ and $A_N(t)$ respectively.

Choice of N-step Binomial Model:

Changes in the number of steps in the binomial model affects the length of the time steps, as well as the returns. For this reason, we introduce the following notation,

$$\begin{aligned}
A_N(t+h) &= (1+R_N)A_N(t),\\
S_N(t+h) &= (1+K_N(t))S_N(t).
\end{aligned}$$

This holds for $t = nh, n = 0, 1, \ldots,$ and the returns $K_N(t)$ are independently and identically distributed with,

$$K_N(t) = \begin{cases} U_N, & \text{if the stock price goes up in step } n, \\ D_N, & \text{if the stock price goes down in step } n, \end{cases}$$

with the no-arbitrage condition $D_N < R_N < U_N$ being satisfied. It is also assumed that

$$P(K_N(t) = U_N) = P(K_N(t) = D_N) = \frac{1}{2}.$$

Since these are real world probabilities and the option prices in the binomial model depend only on the risk-neutral probabilities, this assumption on the real world probabilities can be made.

For the sake of concreteness, we define the probability space $\Omega = [0, 1]$, with the probability P such that $P([a, b]) = b - a$, for any $0 \leq a \leq b \leq 1$.

Then $K_N(t) : \Omega \to \{U_N, D_N\}$ $\left(t = nh,\ h = \dfrac{T}{N}\right)$ are defined $\omega \in \Omega = [0, 1]$ by,

$$K_N^{\omega}(t) = \begin{cases} U_N, & \text{if } \omega \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \text{ for even } k, \\ D_N, & \text{otherwise.} \end{cases}$$

It can be shown that these random variables are independent and

$$P\left(K_N(t) = U_N\right) = P\left(K_N(t) = D_N\right) = \frac{1}{2}.$$

While moving to continuous time limit we have,

$$A(t) = A(0)e^{rt}$$

for any $t \geq 0$, where $e^{rh} = 1 + R_N$. For simplicity, we take $A(0) = 1$.

We define the single-step *logarithmic returns* by,

$$k_N(t) = \ln\left(1 + K_N(t)\right) = \ln \frac{S_N(t+h)}{S_N(t)}$$

for $t = nh$, $n = 0, 1, \ldots, N-1$. These returns are independent and identically distributed random variables such that

$$k_N(t) = \begin{cases} \ln(1 + U_N), & \text{if the stock price goes up in step } n \\ \ln(1 + D_N), & \text{if the stock price goes down in step } n. \end{cases}$$

In general, the *logarithmic return* on a stock between time instants $t < u$ is given by,

$$k_N(t, u) = \ln \frac{S_N(u)}{S_N(t)}.$$

We assume that the expectation and variance of the random variable $k_N(0, t)$ are of a special form:

$$\begin{aligned} E\left(k_N(0, t)\right) &= \mu t, \\ Var\left(k_N(0, t)\right) &= \sigma^2 t. \end{aligned}$$

for some $\mu \in \mathbb{R}$, $\sigma > 0$ independent of N.