
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

<u>Definition:</u>

Consider the binomial asset-pricing model. Let $M_0, M_1, \ldots, M_N$ be a sequence of random variables, with each M_n depending only on the first n coin tosses (and M_0 constant). Such a sequence of random variables is called an adapted stochastic process

1. If $M_n = \mathbb{E}_n\left[M_{n+1}\right],\ n = 0, 1, 2, \ldots, N-1$, we say this process is a martingale.

2. If $M_n \leq \mathbb{E}_n\left[M_{n+1}\right],\ n = 0, 1, 2, \ldots, N-1$, we say the process is a sub-martingale (even though it may have a tendency to increase).

3. If $M_n \geq \mathbb{E}_n\left[M_{n+1}\right],\ n = 0, 1, 2, \ldots, N-1$, we say the process is a super-martingale (even though it may have a tendency to decrease).

While the martingale property is a "one-step" condition, it implies a similar condition for any number of steps. Indeed, if $M_0, M_1, \ldots, M_N$ is a martingale and $n \leq N-2$, then the martingale property implies

$$M_{n+1} = \mathbb{E}_{n+1}\left[M_{n+2}\right].$$

Now, taking conditional expectations on both sides based on the information at time n, and using the *iterated conditioning property*, we obtain,

$$\mathbb{E}_n\left[M_{n+1}\right] = \mathbb{E}_n\left[\mathbb{E}_{n+1}\left[M_{n+2}\right]\right] = \mathbb{E}_n\left[M_{n+2}\right].$$

Now using the martingale property, we obtain the "two-step-ahead" property

$$M_n = \mathbb{E}_n\left[M_{n+2}\right].$$

Iterating this argument, we can obtain the "multi-step-ahead" version of the martingale property, for $0 \leq n \leq m \leq N$,

$$M_n = \mathbb{E}_n\left[M_m\right].$$

<u>Remark:</u>

The expectation of a martingale is constant over time *i.e.,* if $M_0, M_1, \ldots, M_N$ is a martingale, then

$$M_0 = \mathbb{E}M_n,\ n = 0, 1, \ldots, N.$$

To see this, we take expectation on both sides of the martingale condition to obtain

$$\mathbb{E}M_n = \mathbb{E}\mathbb{E}_n\left[M_{n+1}\right],$$

which making use of the *iterated conditioning* results in

$$\mathbb{E}M_n = \mathbb{E}\left[M_{n+1}\right].$$

Hence it follows that

$$M_0 = \mathbb{E}M_0 = \mathbb{E}M_1 = \mathbb{E}M_2 = \cdots = \mathbb{E}M_{N-1} = \mathbb{E}M_N.$$

<u>Theorem:</u>

Consider the general binomial model with $0 < d < 1 + r < u$. Let the risk-neutral probabilities be given by

$$\widetilde{p} = \frac{1 + r - d}{u - d}, \ \widetilde{q} = \frac{u - 1 - r}{u - d}.$$

Then, under the risk-neutral measure, the discounted stock price is a martingale, that is, the condition for martingale holds at every time n and for every sequence of coin tosses.

<u>Proof 1:</u>

Let n and $\omega_1\omega_2 \ldots \omega_n$ be given. Then

$$\begin{aligned}
& \widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right](\omega_1\omega_2 \ldots \omega_n), \\
= \ & \frac{1}{(1+r)^n} \cdot \frac{1}{1+r}\left[\widetilde{p}S_{n+1}(\omega_1\omega_2 \ldots \omega_n H) + \widetilde{q}S_{n+1}(\omega_1\omega_2 \ldots \omega_n T)\right], \\
= \ & \frac{1}{(1+r)^n} \cdot \frac{1}{1+r}\left[\widetilde{p}uS_n(\omega_1\omega_2 \ldots \omega_n) + \widetilde{q}dS_n(\omega_1\omega_2 \ldots \omega_n)\right], \\
= \ & \frac{S_n(\omega_1\omega_2 \ldots \omega_n)}{(1+r)^n} \cdot \frac{\widetilde{p}u + \widetilde{q}d}{1+r}, \\
= \ & \frac{S_n(\omega_1\omega_2 \ldots \omega_n)}{(1+r)^n}.
\end{aligned}$$

<u>Proof 2:</u>

We note that $\dfrac{S_{n+1}}{S_n}$ depends only on the $(n+1)$-st coin toss. Accordingly, we compute:

$$\begin{aligned}
& \widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right], \\
= \ & \widetilde{\mathbb{E}}_n\left[\frac{S_n}{(1+r)^{n+1}} \cdot \frac{S_{n+1}}{S_n}\right], \\
= \ & \frac{S_n}{(1+r)^n}\widetilde{\mathbb{E}}_n\left[\frac{1}{1+r} \cdot \frac{S_{n+1}}{S_n}\right], \ \textit{(taking out what is known)} \\
= \ & \frac{S_n}{(1+r)^n} \cdot \frac{1}{1+r}\widetilde{\mathbb{E}}\frac{S_{n+1}}{S_n}, \ \textit{(independence)} \\
= \ & \frac{S_n}{(1+r)^n}\frac{\widetilde{p}u + \widetilde{q}d}{1+r}, \\
= \ & \frac{S_n}{(1+r)^n}.
\end{aligned}$$

In a binomial model with N coin tosses, recall that an investor takes a position of Δ_n at time n and holds this position until time $(n+1)$, at which point, the investor takes the position Δ_{n+1}. The "portfolio variable" Δ_n may depend on the first n coin tosses, and Δ_{n+1} may depend on the first $n+1$ coin tosses. In other words, the *portfolio process* $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ is *adapted*.