
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Definition:

Consider the binomial asset pricing model. Let $\left\{\left(X_n^1, X_n^2, \dots, X_n^K\right),\ n = 0, 1, \dots, N\right\}$ be a K-dimensional adapted process, *i.e.,* K one-dimensional adapted processes. If, for every n between 0 and $N-1$ and for every function $f\left(x^1, x^2, \dots, x^K\right)$, there is another function $g\left(x^1, x^2, \dots, x^K\right)$ (depending on n and f) such that,

$$\mathbb{E}_n\left[f\left(X_{n+1}^1, X_{n+1}^2, \dots, X_{n+1}^K\right)\right] = g\left(X_n^1, X_n^2, \dots, X_n^K\right),$$

we say that $\left\{\left(X_n^1, X_n^2, \dots, X_n^K\right),\ n = 0, 1, \dots, N\right\}$ is a K-dimensional Markov process.

Example:

In a N-period binomial model, consider the two-dimensional adapted process $\{(S_n, M_n),\ n = 0, 1, \dots, N\}$, where S_n is the stock price at time n and $M_n = \max\limits_{0 \le k \le n} S_k$ is the stock price maximum-to-date. We show that this two-dimensional process is Markov. In order to do that, we define $Y = \dfrac{S_{n+1}}{S_n}$, which depends only on the $(n+1)$-th coin toss. Then

$$S_{n+1} = S_n Y$$

and

$$M_{n+1} = M_n \vee S_{n+1} = M_n \vee (S_n Y),$$

where $x \vee y = \max\{x, y\}$. We wish to compute

$$\mathbb{E}_n\left[f\left(S_{n+1}, M_{n+1}\right)\right] = \mathbb{E}_n\left[f\left(S_n Y, M_n \vee (S_n Y)\right)\right].$$

Now, we replace S_n by a dummy variable s and replace M_n by a dummy variable m, to compute,

$$g(s, m) = \mathbb{E} f\left(sY, m \vee (sY)\right) = pf\left(us, m \vee (us)\right) + qf\left(ds, m \vee (ds)\right).$$

Then

$$\mathbb{E}_n\left[f\left(S_{n+1}, M_{n+1}\right)\right] = g\left(S_n, M_n\right).$$

Since we have obtained a formula for $\mathbb{E}_n\left[f\left(S_{n+1}, M_{n+1}\right)\right]$ in which the only randomness enters through the random variables S_n and M_n, we conclude that the two-dimensional process is Markov.

Note that in this example, we have used the actual probability measure, but the same argument shows that $\{(S_n, M_n),\ n = 0, 1, \dots, N\}$ is Markov under the risk-neutral probability measure $\widetilde{\mathbb{P}}$.

We conclude with one last theorem.

Theorem:

Let $X_0, X_1, \ldots, X_N$ be a Markov process under the risk-neutral probability measure $\widetilde{\mathbb{P}}$ in the binomial model. Let $v_N(x)$ be a function of the dummy variable x, and consider a derivative security whose payoff at time N is $v_N(x_N)$. Then, for each n between 0 and N, the price V_n of this derivative security is some function v_n of X_n, *i.e.*,

$$V_n = v_n\left(X_n\right), \; n = 0, 1, \ldots, N.$$

There is a recursive algorithm for computing v_n whose exact formula depends on the underlying Markov process $X_0, X_1, \ldots, X_N$. Analogous results hold if the underlying Markov process is multidimensional.