Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

The material for Lecture 31 to Lecture 40 will be from Shreve-I.

We will revisit the binomial asset pricing model. We begin with the one-period binomial model, where the beginning and the end of the one-period are designated as time 0 and time 1, respectively. Let the price of the stock at time 0 be denoted by S_0. At time 1, the price of the stock will take two values, namely $S_1(H)$ and $S_1(T)$, with H and T standing for head and tail, respectively. The motivation is that we imagine a coin is tossed and the outcome of the coin toss determines the price of the stock at time 1. Further, the coin is not assumed to be unbiased, that is, probability of either of the outcomes need not be $\frac{1}{2}$. Accordingly, the assumption is that the probability of the head is p and that of the tail is $q = 1 - p$, both of which are positive. We introduce two positive numbers, $u = \frac{S_1(H)}{S_0}$ and $d = \frac{S_1(T)}{S_0}$. Without loss of generality, we assume that $d < u$. We refer to u as the *up-factor* and d as the *down-factor*. *Intuitively* $u > 1$ and $d < 1$. We also introduce the interest-rate r, so that an amount of 1 invested at time 0 in a money market account will grow to $(1 + r)$ at time 1. While in practice $r \geq 0$, mathematically we only require the condition that $r > -1$. Further, in order to rule out arbitrage, we need the assumption that $0 < d < 1 + r < u$.

While not necessary, it is commonly assumed that $d = \frac{1}{u}$.

Let us now consider an European call option with the strike price of K *i.e.,* with the payoff of $(S_1 - K)^+$. *The key question in option pricing is the determination of the option price at time* 0. *Accordingly, the "arbitrage pricing theory" approach to the option pricing problem is to replicate the option through investments in stocks and money market accounts.*

Let is begin with the following example:

Example:

Consider a one-period binomial model with $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, and $r = \frac{1}{4}$. Then $S_1(H) = 8$ and $S_1(T) = 2$. Now consider an European call option with strike $K = 5$. Further, suppose that we begin with an initial wealth level of $X_0 = 1.20$ with which we buy $\Delta_0 = \frac{1}{2}$ stocks at time 0 at a cost of 2, for which we need to borrow an amount of 0.80. The net cash position at time 0, then is $X_0 - \Delta_0 S_0 = -0.80$ (debt of 0.80). At time 1, our cash position will be $(1 + r)(X_0 - \Delta_0 S_0) = -1.00$ (debt of 1.00). One the other hand, at time 1, the value of the holding in the stock will be either $\frac{1}{2}S_1(H) = 4$ or $\frac{1}{2}S_1(T) = 1$, in which case the value of our portfolio will be $X_1(H) = \frac{1}{2}S_1(H) + (1 + r)(X_0 - \Delta_0 S_0) = 3$ or $X_1(T) = \frac{1}{2}S_1(T) + (1 + r)(X_0 - \Delta_0 S_0) = 0$. These correspond to the payoffs $(S_1(H) - 5)^+ = 3$ or $(S_1(T) - 5)^+ = 0$, respectively.

We conclude that the initial wealth level of 1.20 needed in order to set up the replicating portfolio is the "*no-arbitrage price of the option at time* 0".

Some assumptions on which the Example is dependent are:

1. The purchase or sale of fraction of stocks is possible.

2. The interest rate is identical for both lending and borrowing.

3. The purchase and sale price of the stock price is identical.

4. At any point of time, the stock can take only two values in the next one period.

We will now move onto generalizing the Example, in the paradigm of a single period model.
Let the "*derivative security*" pay an amount $V_1(H)$ at time 1, if there is a head and an amount $V_1(T)$ at time 1, if there is a tail.
In case of an European call option we have $V_1(H) = (S_1(H) - K)^+$ *and* $V_1(T) = (S_1(T) - K)^+$.
The question that we seek to answer is, what is the price V_0 *at time* 0 *for a derivative security, using replication.*
Following the Example, we begin with wealth X_0 and buy Δ_0 stocks, resulting in the cash position $X_0 - \Delta_0 S_0$. Therefore the value of the portfolio of stock and money market account at time 1 is

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = (1+r)X_0 + \Delta_0 (S_1 - (1+r) S_0).$$

In order to replicate, we must choose X_0 and Δ_0 in a manner such that $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$. Thus we have,

$$(1+r)X_0 + \Delta_0 (S_1(H) - (1+r) S_0) = V_1(H) \Rightarrow X_0 + \Delta_0 \left(\frac{1}{1+r} S_1(H) - S_0 \right) = \frac{1}{1+r} V_1(H).$$

Similarly,

$$X_0 + \Delta_0 \left(\frac{1}{1+r} S_1(T) - S_0 \right) = \frac{1}{1+r} V_1(T).$$

In order to solve these equations, we multiply the first equation by $\widetilde{p}$ and the second equation by $\widetilde{q} = 1 - \widetilde{p}$ (both $\widetilde{p}$ and $\widetilde{q}$ are yet to be determined) and then add them to obtain:

$$X_0 + \Delta_0 \left(\frac{1}{1+r} [\widetilde{p} S_1(H) + \widetilde{q} S_1(T)] - S_0 \right) = \frac{1}{1+r} [\widetilde{p} V_1(H) + \widetilde{q} V_1(T)].$$

If we choose $\widetilde{p}$ such that

$$S_0 = \frac{1}{1+r} [\widetilde{p} S_1(H) + \widetilde{q} S_1(T)],$$

then the term with Δ_0 becomes zero and we have the following formula for X_0:

$$X_0 = \frac{1}{1+r} [\widetilde{p} V_1(H) + \widetilde{q} V_1(T)].$$

Now, we need to determine $\widetilde{p}$. Accordingly,

$$S_0 = \frac{1}{1+r} [\widetilde{p} S_1(H) + \widetilde{q} S_1(T)] = \frac{1}{1+r} [\widetilde{p} u S_0 + (1-\widetilde{p}) d S_0] = \frac{S_0}{1+r} [(u-d)\widetilde{p} + d]$$

implies that

$$\widetilde{p} = \frac{(1+r) - d}{u - d} \text{ and } \widetilde{q} = \frac{u - (1+r)}{u - d}.$$

Finally, we obtain Δ_0 to get the Delta hedging formula,

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}.$$

In conclusion, if we begin with an amount X_0 and invest in a replicating portfolio, then we have either $V_1(H)$ or $V_1(T)$, and we say that "the short position in the derivative has been hedged". Hence a derivative security that pays V_1 at time 1 should at time 0, be priced as

$$V_0 = \frac{1}{1+r} \left[\widetilde{p} V_1(H) + \widetilde{q} V_1(T) \right].$$

Note:

1. $\widetilde{p}$ and $\widetilde{q}$ are called the risk-neutral probabilities.

2. For the real world probabilities p and q, we have, $pS_1(H) + qS_1(T) > S_0(1+r)$.