
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Proposition:

Assume that $\det \mathbf{C} \neq 0$. Then the minimum variance portfolio has weights

$$\mathbf{w}_{MVP} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top}.$$

Proof:

We need to find the minimum of $\mathbf{w}\mathbf{C}\mathbf{w}^\top$ subject to the constraint $\mathbf{w}\mathbf{u}^\top = 1$. According to the method of Lagrange multipliers, we define the Lagrangian

$$F(\mathbf{w}, \lambda) = \mathbf{w}\mathbf{C}\mathbf{w}^\top - \lambda(\mathbf{w}\mathbf{u}^\top - 1).$$

The first order necessary condition gives

$$2\mathbf{w}\mathbf{C} - \lambda\mathbf{u} = 0.$$

Hence we get,

$$\mathbf{w} = \frac{\lambda}{2}\mathbf{u}\mathbf{C}^{-1}.$$

Substituting this in the constraint we get

$$\frac{\lambda}{2}\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top = 1.$$

The properties of the covariance matrix guarantee that $\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top \neq 0$. Hence we get,

$$\frac{\lambda}{2} = \frac{\mathbf{1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top}.$$

Therefore,

$$\mathbf{w}_{MVP} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top}.$$

We have therefore verified the necessary condition for a minimum. Observe that $\mathbf{w}\mathbf{C}\mathbf{w}^\top$ is a quadratic function of the weights, bounded below by 0. It must therefore have a minimum, which completes the proof.

Efficient Frontier:

Given the choice between two securities a rational investor will, if possible, choose the one with higher expected return and lower standard deviation (that is, lower risk). This motivates the following definition.

Definition:

We say that a security with expected return μ_1 and standard deviation σ_1, dominates another security with expected return μ_2 and standard deviation σ_2, whenever

$$\mu_1 \geq \mu_2 \text{ and } \sigma_1 \leq \sigma_2.$$

This definition readily extends to portfolios, which can be considered as securities in their own right.

Definition:

A portfolio is called *efficient* if there is no other portfolio, except itself, that dominates it. The set of efficient portfolios among all feasible portfolios is called the *efficient frontier*.

Definition:

The family of portfolios V, parameterized by $\mu \in \mathbb{R}$, such that $\mu_V = \mu$ and $\sigma_V \leq \sigma^2_{V'}$ for each portfolio V' with $\mu_{V'} = \mu$ is called the *minimum variance line* (MVL).

To compute the weights of the portfolio on the minimum variance line, for any $\mu \in \mathbb{R}$, we shall solve the constrained minimum problem

$$\min \mathbf{w}\mathbf{C}\mathbf{w}^\top,$$

where the minimum is taken over all vectors $\mathbf{w} \in \mathbb{R}^n$ such that

$$\mathbf{w}\mathbf{m}^\top = \mu \text{ and } \mathbf{w}\mathbf{u}^\top = 1.$$

We introduce Lagrange multipliers λ_1, λ_2 and minimize the Lagrangian

$$G(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}\mathbf{C}\mathbf{w}^\top - \lambda_1(\mathbf{w}\mathbf{m}^\top - \mu) - \lambda_2(\mathbf{w}\mathbf{u}^\top - 1).$$

The gradient of G with respect to $\mathbf{w}$ and the partial derivatives with respect to λ_1, λ_2 give necessary conditions for a minimum

$$\begin{aligned} 2\mathbf{w}\mathbf{C} - \lambda_1\mathbf{m} - \lambda_2\mathbf{u} &= \mathbf{0} \\ \mathbf{w}\mathbf{m}^\top - \mu &= 0 \\ \mathbf{w}\mathbf{u}^\top - 1 &= 0. \end{aligned}$$

The first of these is a system of n equations, $\mathbf{0}$ representing a vector of zeros. Assuming invertibility of $\mathbf{C}$, we solve the system for $\mathbf{w}$, to obtain

$$2\mathbf{w} = \lambda_1\mathbf{m}\mathbf{C}^{-1} + \lambda_2\mathbf{u}\mathbf{C}^{-1}.$$

Substituting $\mathbf{w}$ in the constraints, we obtain

$$\begin{aligned} \lambda_1\mathbf{m}\mathbf{C}^{-1}\mathbf{m}^\top + \lambda_2\mathbf{u}\mathbf{C}^{-1}\mathbf{m}^\top &= 2\mu \\ \lambda_1\mathbf{m}\mathbf{C}^{-1}\mathbf{u}^\top + \lambda_2\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top &= 2. \end{aligned}$$

The matrix of the coefficients of the system will be denoted by

$$M = \begin{bmatrix} \mathbf{m}\mathbf{C}^{-1}\mathbf{m}^\top & \mathbf{u}\mathbf{C}^{-1}\mathbf{m}^\top \\ \mathbf{m}\mathbf{C}^{-1}\mathbf{u}^\top & \mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top \end{bmatrix}.$$

We can then solve the system of linear equations for λ_1 and λ_2 thereby obtaining a linear expression in μ:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 2M^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix}.$$

Thus we get,

$$\mathbf{w} = \mu\mathbf{a} + \mathbf{b}.$$

for some vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. The important point here is that the vectors $\mathbf{a}$ and $\mathbf{b}$ are the same for each portfolio on the minimum variance line.

<u>Two Fund Theorem:</u>

Let $\mathbf{w_1}, \mathbf{w_2}$ be the weights of any two portfolios V_1, V_2 on the minimum variance line with different expected returns $\mu_{V_1} \neq \mu_{V_2}$. Then each portfolio V on the minimum variance line can be obtained as linear combination of these two, that is,

$$\mathbf{w} = \alpha\mathbf{w_1} + (1-\alpha)\mathbf{w_2}$$

for some $\alpha \in \mathbb{R}$.

<u>Proof:</u>

We find α so that

$$\mu_V = \alpha\mu_{V_1} + (1-\alpha)\mu_{V_2}.$$

This is possible since the returns are different:

$$\alpha = \frac{\mu_V - \mu_{V_2}}{\mu_{V_1} - \mu_{V_2}}.$$

Since the two portfolios belong to the minimum variance line, they satisfy

$$\mathbf{w_1} = \mu_{V_1}\mathbf{a} + \mathbf{b}, \mathbf{w_2} = \mu_{V_2}\mathbf{a} + \mathbf{b}.$$

Consequently,

$$\alpha\mathbf{w_1} + (1-\alpha)\mathbf{w_2} = (\alpha\mu_{V_1} + (1-\alpha)\mu_{V_2})\mathbf{a} + \mathbf{b} = \mu_V\mathbf{a} + \mathbf{b} = \mathbf{w}.$$