Example:

Suppose there are two sellers and two buyers of a product. Each seller has one unit of the good and the worth of the object to the sellers is 150. Each buyer wants one unit and values it 200. What is the characteristic function of this game?

Set of players is $\{A_1, A_2, A_3, A_4\}$, $\{A_1, A_2\}$ is the set of sellers and $\{A_3, A_4\}$ is the set of buyers.

We have $v(\{A_1\}) = 150$, $v(\{A_2\}) = 150$, $v(\{A_3\}) = 0$, $v(\{A_4\}) = 0$.

$v(\{A_1, A_2\}) = 300$,

$v(\{A_3, A_4\}) = 0$, coalition of only buyers cannot generate any value.

$$v(\{A_1, A_3\}) = v(\{A_1, A_4\}) = v(\{A_2, A_3\}) = v(\{A_2, A_4\}) = 200.$$

Suppose seller 1 sells at p_1 to buyer 3. This $p_1 \geq 150$ and $p_1 \leq 200$, the valuation of the seller 1 is 150 and the valuation of the buyer is 200. If this coalition forms then seller 1 gets p_1 and buyer 3 gets $200 - p_1$. So $v(\{A_1, A_3\}) = p_1 + (200 - p_1) = 200$.

$$v(\{A_1, A_2, A_3\}) = v(\{A_1, A_2, A_4\}) = 350.$$

Suppose seller 1 and 2 want to sell to buyer 3. Seller 1 can give p_2 to seller 2 for not selling to buyer 3 and sell at p_3 to buyer 3. Gain to seller 1 is $p_3 - p_2$, so we want $p_3 > p_2$. The gain to seller 2 is $150 + p_2$ and to buyer 3 is $200 - p_3$. The value of the coalition is $v(\{A_1, A_2, A_3\}) = (p_3 - p_2) + (150 + p_2) + (200 - p_3) = 350$.

$v(\{A_1, A_3, A_4\}) = v(\{A_2, A_3, A_4\}) = 200$

Suppose buyer 3 and 4 want to buy from seller 1. Buyer 3 can bribe Buyer 4 by paying p_4 so that buyer 4 does not make any offer to seller 1. Buyer 3 will offer p_5 to seller 1 to get the object. The gain to seller 1 is p_5, gain to buyer 4 is p_4 and gain to buyer 3 is $200 - p_4 - p_5$. The value of the coalition is $v(\{A_1, A_3, A_4\}) = p_5 + p_4 + (200 - p_4 - p_5) = 200$.

$v(\{A_1, A_2, A_3, A_4\} = 400$.

Seller 1 sells to buyer 3 and seller 2 sells to buyer 4. Seller 1 sells at p_6 to buyer 3 and seller 2 sells at p_7 to buyer 4. The gain are, seller 1 gains p_6, seller 2 gains p_7, buyer 3 gains $200 - p_6$, buyer 4 gains $200 - p_7$. The value of the coalition is $v(\{A_1, A_2, A_3, A_4\} = p_6 + p_7 + (200 - p_6) + (200 - p_7) = 400$.

The outcome of a coalition game is payoffs to each player, an allocation of the value of the coalition.

Given a game $v \in G^N$, an outcome of the game or allocation (a payoff vector) is an $n-$ coordinated vector $x = (x_{A_1}, x_{A_2}, x_{A_3}, ...x_{A_n})$.

x_{A_i} is the $i-$ th coordinate of allocation vector $x = (x_{A_1}, x_{A_2}, x_{A_3}, ...x_{A_n})$ denotes the amount received by player i.

$x(S) = \sum_{A_i \in S} x_{A_i}$ for any subset S of N. It denotes the sum of the allocations or pay-offs received by players in coalition S.

While choosing which coalition to be part of each player is going to compare the payoffs received that coalition and what it receives if it remains on his own or stay alone.

Given a game $v \in G^N$, a payoff vector or allocation x is called individually rational if $x_{A_i} \geq v(\{A_i\})$ for all $A_i \in N$.
Each player should get in a coalition whatever he gets by staying alone.

Given a game $v \in G^N$, a pay-off vector or allocation x is called totally rational or Pareto efficient if $x(N) = v(N)$.
The aggregate amount generated under a payoff vector or allocation must be equal to the amount earned by the grand coalition.

$x(N) = v(N)$ is taken as two inequalities

$x(N) \leq v(N)$, sum of payoff vector or allocation must be feasible.
$x(N) \geq v(N)$, the grand coalition cannot value or earn more than $x(N)$.

Given a game $v \in G^N$, an imputation in v is a payoff vector or allocation which is individually rational and Pareto efficient (totally rational).
The set of all imputations associated with $v \in G^N$ is denoted by $I(v)$.
Imputation is payoff vector or allocation that assign each players at least as much as he can earn on his own and assign all the players together the maximum value they can create when the grand coalition is formed.

Core:
Given a game $v \in G^N$, the core of v is the set of all imputations x in $I(c)$ such that $x(S) \geq v(s)$ for all non-empty coalitions $S \subset N$. The core of a game $v \in G^N$ is denoted by $C(v)$.

If $x(S) < v(S)$ then by dividing $\frac{v(s) - x(S)}{|S|}$ among the members of S, each player in S can be made better off. So, an allocation x such that $x(S) < v(S)$ is not in core.
Further, in an allocation in core each player should receive what it gets when it stays alone. If an allocation is core allocation then to make one player better - off, another player has o be made worst-off.

Example: A bargaining problem between two person. They want to divide a cake of size 1 between them. If they fail to divide each get zero.

$x_{A_1} \geq 0, x_{A_2} \geq 0$ and $x_{A_1} + x_{A_2} = 1$. These are the core allocation.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 1$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

See figure 2