Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Time Value of Options:

We say that at time t a call option with strike price X is

1. *In the money* if $S(t) > X$,
2. *At the money* if $S(t) = X$,
3. *Out of the money* if $S(t) < X$.

Similarly, for a put option we say that it is

1. *In the money* if $S(t) < X$,
2. *At the money* if $S(t) = X$,
3. *Out of the money* if $S(t) > X$.

Also convenient, though less precise, are the terms *deep in the money* and *deep out of the money*, which mean that the difference between the two sides in the respective inequalities is considerable.

Definition:

At time $t \leq T$, the *intrinsic value* of a call option with strike price X is equal to $(S(t) - X)^+$. Similarly, the intrinsic value of a put option with the same strike price is $(X - S(t))^+$. Thus the intrinsic value is positive if the option is in the money and zero if it is either out of the money or at the money.

Definition:

The *time value* of an option is the difference between the price of the option and its intrinsic value. In particular the time value for various options are:

1. European call: $C^E(t) - (S(t) - X)^+$.
2. European put: $P^E(t) - (X - S(t))^+$.
3. American call: $C^A(t) - (S(t) - X)^+$.
4. American put: $P^A(t) - (X - S(t))^+$.

Binomial Model:

Definition of the Model:

Recall that the future stock price $S(t)$ at time t, $0 < t \leq T$ is a random variable, with the initial stock price $S(0) > 0$ given. We consider evenly spaced discrete time points $t = 0, h, 2h, \ldots, nh, \ldots$ with the final time being $T = Nh$. Here, $h > 0$ is the length of a single time-step. For convenience, we will use the notation $S(n)$ to mean $S(nh)$.

The goal is to build a concrete model for the random variable $S(n)$.

Single Step Model:

We consider a single step model for the random variable $S(1)$. In order to define $S(1)$, we will need a probability space. Since, in a binomial model we allow $S(1)$ to take only two values, it is enough to consider $\Omega = \{u, d\}$ along with a field $\mathcal{F}$ of all subsets of Ω and a probability P determined by a single number p such that $P(u) = p$, $P(d) = 1 - p$. It is possible to have other alternatives for Ω.

Let

$$S(1) : \Omega \to (0, +\infty)$$

be given by,

$$S(1) = S(0)(1 + K),$$

where the rate of return K is random and has the form

$$K^{\omega} = \begin{cases} U \text{ , if } \omega = u, \\ D \text{ , if } \omega = d, \end{cases}$$

where $-1 < D < U$. Then,

$$S^{\omega}(1) = \begin{cases} S(0)(1+U) \text{ , if } \omega = u, \\ S(0)(1+D) \text{ , if } \omega = d. \end{cases}$$

Two Step Model:

We assume that the second step is of the same form as the first step. Also the return for the second step is independent of the first step. For this purpose we need to extend the probability space and define $\Omega = \{uu, ud, du, dd\}$ with the field $\mathcal{F}$ of all subsets of Ω and probability determined by

$$P(uu) = p^2, P(ud) = P(du) = p(1-p), P(dd) = (1-p)^2.$$

Then

$$\begin{aligned} K^{\omega}(1) &= \begin{cases} U \text{ , if } \omega = uu \text{ or } \omega = ud, \\ D \text{ , if } \omega = du \text{ or } \omega = dd, \end{cases} \\ K^{\omega}(2) &= \begin{cases} U \text{ , if } \omega = uu \text{ or } \omega = du, \\ D \text{ , if } \omega = ud \text{ or } \omega = dd, \end{cases} \end{aligned}$$

are independent. We define,

$$\begin{aligned} S(1) &= S(0)(1 + K(1)), \\ S(2) &= S(1)(1 + K(2)) = S(0)(1 + K(1))(1 + K(2)). \end{aligned}$$

Thus, at time $t = 2$ we have three possible stock prices

$$S^{\omega}(2) = \begin{cases} S(0)(1+U)^2 \text{ , if } \omega = uu, \\ S(0)(1+U)(1+D) \text{ , if } \omega = ud \text{ or } \omega = du, \\ S(0)(1+D)^2 \text{ , if } \omega = dd. \end{cases}$$

Note that in this case, the ud and du scenarios yield the same result. This is a result of recombining and as such the binomial tree is sometimes called the recombining tree. Because of this there are fewer stock prices at time n than scenarios.

General Case:

For N steps we take $\Omega = \{u, d\}^N$ (sequence of N u's and d's) with the field $\mathcal{F}$ of all subsets of Ω and probability

$$P(\omega) = p^k(1-p)^{N-k}$$

wherever there are k no of u's and $N-k$ no of d's. The return $K(n)$ is defined as

$$K^\omega(n) = \begin{cases} U \text{ , if } \omega_n = u, \\ D \text{ , if } \omega_n = d. \end{cases}$$

In general

$$S(n+1) = S(n)(1 + K(n+1)).$$

For example,

$$S^\omega(3) = \begin{cases} S(0)(1+U)^3 \text{ , if } \omega = uuu, \\ S(0)(1+U)^2(1+D) \text{ , if } \omega = uud \text{ or } \omega = udu \text{ or } \omega = duu \\ S(0)(1+U)(1+D)^2 \text{ , if } \omega = ddu \text{ or } \omega = dud \text{ or } \omega = udd \\ S(0)(1+D)^3 \text{ , if } \omega = ddd. \end{cases}$$

Flow of Information:

With the passage of time, more information about the possible scenarios ω are realised. We will restrict our discussion to the case $N = 3$. At $t = 0$, each element of

$$\Omega = \{uuu, uud, udu, udd, duu, dud, ddu, ddd\}$$

can be realized in the future. After $t = 1$ with more information being available, the possibilities are restricted to either

$$B_u = \{uud, uud, udu, udd\}$$

if the stock moves up from $t = 0$ to $t = 1$ or the possible scenarios could be

$$B_d = \{duu, dud, ddu, ddd\}$$

if the stock moves down from $t = 0$ to $t = 1$. After two time steps the scenarios could be one of the following:

$$B_{uu} = \{uud, uud\}, B_{ud} = \{udu, udd\}, B_{du} = \{duu, dud\}, B_{dd} = \{ddu, ddd\}.$$

Mathematically, these are so-called partitions of Ω, meaning that $\Omega = B_u \cup B_d$ or $\Omega = B_{uu} \cup B_{ud} \cup B_{du} \cup B_{dd}$.

Definition:

A family $\mathcal{P} = \{B_i\}$ of events is partition of Ω if $B_i \neq \phi$, $B_i \cap B_j = \phi$ for $i \neq j$ and $\Omega = \cup_i B_i$.

Note that

$$B_u = B_{uu} \cup B_{ud}, B_d = B_{du} \cup B_{dd}.$$

We say that $\mathcal{P}_2 = \{B_{uu}, B_{ud}, B_{du}, B_{dd}\}$ is a finer partition than $\mathcal{P}_1 = \{B_u, B_d\}$.