
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Dependence on the Underlying Asset Price:

The current price $S(0)$ of the underlying asset is fixed. However, in order to examine the dependence on the underlying asset price, we consider a portfolio consisting of x shares, so that the worth is $S = xS(0)$. In this case, the payoff of an European call with strike price X and expiration T will be $(xS(T) - X)^+$. Similarly, for a put option the payoff is $(X - xS(T))^+$. We will examine the dependence of option prices on S. For this purpose, we will denote the call and the put prices by $C^E(S)$ and $P^E(S)$, respectively, assuming that the other variables are fixed.

Result:

If $S' < S''$, then

$$\begin{aligned} C^E(S') &\leq C^E(S'') \\ P^E(S') &\geq P^E(S'') \end{aligned}$$

In other words, $C^E(S)$ and $P^E(S)$ are non-decreasing and non-increasing functions of S, respectively.
Suppose that $C^E(S') > C^E(S'')$ for some $S' < S''$, where $S' = x'S(0)$ and $S'' = x''S(0)$. We sell a call on x' shares and buy a call on x'' shares with same X and T. The balance will be $C^E(S') - C^E(S'')$ which is invested at the risk-free rate r. If the sold option is exercised, then we can exercise the option we hold and make an arbitrage profit since, $(x''S(T) - X)^+ \geq (x'S(T) - X)^+$.
The proof for put option is similar.

Result:

If $S' < S''$, then

$$\begin{aligned} C^E(S'') - C^E(S') &< (S'' - S'), \\ P^E(S') - P^E(S'') &< (S'' - S'). \end{aligned}$$

Proof:

By the put-call parity

$$\begin{aligned} C^E(S'') - P^E(S'') &= S'' - Xe^{-rT}, \\ C^E(S') - P^E(S') &= S' - Xe^{-rT}. \end{aligned}$$

Subtracting, we get

$$(C^E(S'') - C^E(S')) + (P^E(S') - P^E(S'')) = (S'' - S').$$

Since both terms on the LHS are non-negative, so each of the terms on the LHS cannot exceed the RHS term. This proves the result.

Remark:

The above inequalities mean that the call and the put prices as functions of the asset price satisfy the Lipschitz condition with constant 1,

$$\begin{aligned} |C^E(S'') - C^E(S')| &< |S'' - S'|, \\ |P^E(S'') - P^E(S')| &< |S'' - S'|. \end{aligned}$$

Result:
Let $S' < S''$ and let $\alpha \in (0,1)$. Then

$$\begin{aligned} C^E(\alpha S' + (1-\alpha)S'') &\leq \alpha C^E(S') + (1-\alpha)C^E(S''), \\ P^E(\alpha S' + (1-\alpha)S'') &\leq \alpha P^E(S') + (1-\alpha)P^E(S''). \end{aligned}$$

In other words, $C^E(S)$ and $P^E(S)$ are convex functions of S.
Proof:
For convenience, let $S = \alpha S' + (1-\alpha)S''$. Also $S' = x'S(0)$ and $S'' = x''S(0)$, so that $x = \alpha x' + (1-\alpha)x''$. Suppose that

$$C^E(S) > \alpha C^E(S') + (1-\alpha)C^E(S'').$$

We sell a call option on a portfolio with x shares, and purchase α and $(1-\alpha)$ options on a portfolio of x' and x'' shares, respectively. We invest the balance $C^E(S) - \alpha C^E(S') - (1-\alpha)C^E(S'')$ at risk-free rate r. If the option sold is exercised at expiry, then we have to pay $(xS(T) - X)^+$. This can be covered by the amount $\alpha(x'S(T) - X)^+ + (1-\alpha)(x''S(T) - X)^+$ by exercising the α and $(1-\alpha)$ options, respectively. There is an arbitrage profit since,

$$\alpha(x'S(T) - X)^+ + (1-\alpha)(x''S(T) - X)^+ > (xS(T) - X)^+.$$

holds.
The proof for put option is similar.
American Option:
In general, American options have similar properties to their European counterparts. Difficulties, however, arise because of the absence of put-call parity and the possibility of early exercise.
Dependence on the Strike Price:
In this case, the call and put prices will be denoted by $C^A(X)$ and $P^A(X)$, respectively. All remaining variables are kept fixed.
Result:
If $X' < X''$, then

$$C^A(X') \geq C^A(X'')\ ,\ P^A(X') \leq P^A(X'').$$

This means that $C^A(X)$ is a non-increasing and $P^A(X)$ a non-decreasing function of X.
Proof:
Suppose that $X' < X''$, but $C^A(X') < C^A(X'')$. We can sell a call with strike price X'' and buy a call with strike price X', investing the difference $C^A(X'') - C^A(X')$ at risk-free rate r. If the option with strike price X'' is exercised at time $t \leq T$, we pay $(S(t) - X'')^+$. We will then immediately exercise the option with strike price $X'$and receive $(S(t) - X')^+$. Since $X' < X''$, we have $(S(t) - X')^+ > (S(t) - X'')^+$. Thus there will be an arbitrage profit.
The proof for put option is similar.
Result:
If $X' < X''$, then

$$\begin{aligned} C^A(X') - C^A(X'') &\leq (X'' - X'), \\ P^A(X'') - P^A(X') &\leq (X'' - X'). \end{aligned}$$

Proof:
Suppose that $X' < X''$, but $C^A(X') - C^A(X'') > (X'' - X')$ We sell a call with strike price X' and buy a

call with strike X'', investing the balance $C^A(X') - C^A(X'')$ at risk-free rate r. If the written option for X' is exercised at time $t \leq T$, the we have to pay $(S(t) - X')^+$. An immediate exercise of the other option results in receipt of an amount $(S(t) - X'')^+$. The balance from this is $-(X'' - X')$. The net arbitrage profit thus is $(C^A(X') - C^A(X''))e^{rt} - (X'' - X')$ which is positive.
The proof for put option is similar.

Result:

Suppose $X' < X''$ and let $\alpha \in (0, 1)$. Then

$$
\begin{aligned}
C^A(\alpha X' + (1 - \alpha)X'') &\leq \alpha C^A(X') + (1 - \alpha)C^A(X''), \\
P^A(\alpha X' + (1 - \alpha)X'') &\leq \alpha P^A(X') + (1 - \alpha)P^A(X'').
\end{aligned}
$$

Proof:

For convenience, let $X = \alpha X' + (1 - \alpha)X''$. Suppose that

$$
C^A(X) > \alpha C^A(X') + (1 - \alpha)C^A(X'').
$$

We sell an option with strike price X, and purchase α options with strike price X' and $(1 - \alpha)$ options with strike price X'' and invest the balance $C^E(X) - \alpha C^E(X') - (1 - \alpha)C^E(X'') > 0$ at risk-free rate r. If the option is not exercised at all, then we make an arbitrage profit from the risk-free investment. If the option with strike price X is exercised at $t \leq T$, then we have to pay $(S(t) - X)^+$. A simultaneous exercise of the two options being held by us will yield $\alpha(S(t) - X')^+ + (1 - \alpha)(S(t) - X'')^+$ resulting in arbitrage profit since $\alpha(S(t) - X')^+ + (1 - \alpha)(S(t) - X'')^+ \geq (S(t) - X)^+$.
The proof for put option is similar.

Dependence on the Underlying Asset Price:

As in case of European options we consider a portfolio consisting of x shares. We will denote the call and the put prices on this portfolio by $C^A(S)$ and $P^A(S)$, respectively, with $S = xS(0)$ and all the other variables being fixed. The payoffs at time $t \leq T$ are $(xS(t) - X)^+$ for calls and $(X - xS(t))^+$ for puts.

Result:

If $S' < S''$, then

$$
\begin{aligned}
C^A(S') &\leq C^A(S''), \\
P^A(S') &\geq P^A(S'').
\end{aligned}
$$

Suppose that $C^A(S') > C^A(S'')$ for some $S' < S''$, where $S' = x'S(0)$ and $S'' = x''S(0)$. We sell a call on x' shares and buy a call on x'' shares with same X and T. The balance will be $C^A(S') - C^A(S'')$ which is invested at risk-free rate r. If the sold option is exercised at time $t \leq T$, then we can exercise the option we hold and make an arbitrage profit since $(x'S(t) - X)^+ \leq (x''S(t) - X)^+$.
The proof for put option is similar.

Result:

Suppose that $S' < S''$. Then

$$
\begin{aligned}
C^A(S'') - C^A(S') &\leq S'' - S', \\
P^A(S') - P^A(S'') &\leq S'' - S'.
\end{aligned}
$$

Proof:

The result for the call option follows in a straight forward way from a similar relation for European call options and using the fact that the price of an American call is equal to that of a European call for a non-dividend paying stock.

For the put option suppose that $P^A(S') - P^A(S'') > S'' - S'$, for some $S' < S''$, with $S' = x'S(0)$ and $S'' = x''S(0)$. We buy $x'' - x'$ shares and a put on x'' shares and sell a put on x' shares. The balance $-(S'' - S') - P^A(S'') + P^A(S')$ is invested at risk-free rate. In case the holder of the option decides to exercise at time $t \leq T$, we will have to pay $(X - x'S(t))^+$, which can be covered by selling $x'' - x'$ shares and exercising the option on x'' shares, since $(x'' - x')S(t) + (X - x''S(t))^+ \geq (X - x'S(t))^+$ for $x'' > x'$. In case the holder of the option does not exercise then the arbitrage profit is the return from the risk-free investment.

Result:

Let $S' < S''$ and let $\alpha \in (0, 1)$. Then

$$\begin{aligned} C^A(\alpha S' + (1-\alpha)S'') &\leq \alpha C^A(S') + (1-\alpha)C^A(S''), \\ P^A(\alpha S' + (1-\alpha)S'') &\leq \alpha P^A(S') + (1-\alpha)P^A(S''). \end{aligned}$$

Proof:

For convenience, let $S = \alpha S' + (1-\alpha)S''$. Also $S' = x'S(0)$ and $S'' = x''S(0)$, so that $x = \alpha x' + (1-\alpha)x''$. Suppose that

$$C^A(S) > \alpha C^A(S') + (1-\alpha)C^A(S'').$$

We sell a call option on a portfolio with x shares, and purchase α and $(1-\alpha)$ options on a portfolio of x' and x'' shares, respectively. All these options have the same X and T. We invest the balance $C^A(S) - \alpha C^A(S') - (1-\alpha)C^A(S'')$ at risk-free rate r. If the option sold is exercised at time $t \leq T$, then we have to pay $(xS(t) - X)^+$. This can be compensated by the amount $\alpha(x'S(t) - X)^+ + (1-\alpha)(x''S(t) - X)^+$ by exercising the α and $(1-\alpha)$ options, respectively. There is an arbitrage profit since

$$\alpha(x'S(t) - X)^+ + (1-\alpha)(x''S(t) - X)^+ \geq (xS(t) - X)^+.$$

holds.

The proof for put option is similar.

Dependence on the Expiration Time

For American options one can derive the dependence of the option price on the expiration time T. For this purpose we will use the notations, $C^A(T)$ and $P^A(T)$ for American call and put, respectively, assuming that all other variables are fixed.

Result:

If $T' < T''$, then

$$\begin{aligned} C^A(T') &\leq C^A(T'') \\ P^A(T') &\leq P^A(T''). \end{aligned}$$

Proof:

Suppose that $C^A(T') > C^A(T'')$. We then sell an option for expiration T' and buy an option for expiration T'', both with the same strike X. We invest the balance $C^A(T') - C^A(T'')$ at risk-free rate r. If the written option is exercised at time $t \leq T$, the option that we hold will be exercised to cover this position. Thus the arbitrage profit would be the return from the risk-free investment.

The proof for put option is similar.

Note that such a relationship in case of a European option is uncertain.