Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

By the properties of the logarithm, for each $t = nh$,

$$k_N(0,t) = k_N(h) + k_N(2h) + k_N(3h) + \cdots + k_N(nh).$$

Since the terms in this sum are independent and identically distributed random variables, therefore,

$$\begin{aligned} E(k_N(0,t)) &= nE(k_N(h)), \\ Var(k_N(0,t)) &= nVar(k_N(h)). \end{aligned}$$

Hence,

$$\mu = \frac{1}{h}E(k_N(h)),$$

is the expected logarithmic return per unit time and

$$\sigma^2 = \frac{1}{h}Var(k_N(h)),$$

is the variance of the logarithmic return per unit time, with σ being sometimes referred to as the volatility of the stock.

We can now express the single-step returns in terms of μ and σ as follows:

$$1 + U_N = e^{\mu h + \sigma\sqrt{h}},\ 1 + D_N = e^{\mu h - \sigma\sqrt{h}}.$$

This result can be proved as follows:

Proof:

If a random variable X takes two values $a < b$, with probabilities p and q, respectively, then $E(X) = pa+qb$ and $Var(X) = (b-a)^2pq$. In the case under consideration we have, $p = q = \frac{1}{2}$ and the random variable $X = k_N(h)$. Thus we get,

$$\frac{a+b}{2} = \mu h, \quad \frac{b-a}{2} = \sigma\sqrt{h}.$$

Solving for a and b we get,

$$a = \mu h + \sigma\sqrt{h}, \quad b = \mu h - \sigma\sqrt{h}.$$

In the context of our discussion $a = 1 + D_N$ and $b = 1 + U_N$.

Hence,

$$S_N(h) = \begin{cases} S(0)e^{\mu h + \sigma\sqrt{h}}, \\ S(0)e^{\mu h - \sigma\sqrt{h}}. \end{cases}$$

This can now be rewritten as,

$$S_N(h) = S(0)e^{\mu h + Y_1 \sigma \sqrt{h}},$$

where,

$$Y_1 = \begin{cases} +1, & \text{with probability } \frac{1}{2}, \\ -1, & \text{with probability } \frac{1}{2}. \end{cases}$$

Extending this repeatedly, we obtain,

$$S_N(t) = S(0)e^{\mu h + \sigma W_N(t)},$$

for each $t = nh$, $n = 0, 1, \ldots, N$, where $h = \dfrac{T}{N}$ and

$$W_N(t) = \sqrt{h}(Y_1 + Y_2 + \cdots + Y_n),$$

for an infinite sequence of independent and identically distributed random variables $Y_1, Y_2, \ldots$ such that

$$Y_n = \begin{cases} +1, & \text{with probability } \frac{1}{2}, \\ -1, & \text{with probability } \frac{1}{2}. \end{cases}$$

In particular, $W_N(0) = 0$. We call $W_N(t)$ a *scaled random walk* (with time steps of h and jumps $\pm\sqrt{h}$).

Wiener Process:

We now consider the scaled random walk $W_N(t)$ as $N \to \infty$ (*i.e.*, $h = \dfrac{T}{N} \to 0$). Now, $W_N(T)$ can be written as,

$$W_N(T) = \sqrt{h}(Y_1 + Y_2 + \cdots + Y_N).$$

It can be easily seen that the expectation and variance of each of these independent and identically distributed random variables Y_i are 0 and 1 respectively. Therefore the expectation of the sum $Y_1 + Y_2 + \cdots + Y_N$ is 0 and it's variance is N. Using $h = \dfrac{T}{N}$ in the above relation, we obtain,

$$\frac{W_N(T)}{\sqrt{T}} = \frac{Y_1 + \cdots + Y_N}{\sqrt{N}}$$

whose distribution, according to the Central Limit Theorem (as $N \to \infty$) tends to the standard normal distribution with mean 0 and variance 1. In other words,

$$P\left(a \leq \frac{W_N(T)}{\sqrt{T}} \leq b\right) \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx, \quad \text{as } N \to \infty.$$

Consequently, the limit distribution of $W_N(T)$ is normal with mean 0 and variance T.

A similar line of argument shows that the limit distribution of

$$W_N(t) - W_N(s) = \sqrt{h} \sum_{s \leq nh < t} Y_n$$

is normal with mean 0 and variance $(t - s)$.

Definition:

Wiener process or *Brownian motion* is a family of random variables $W(t)$ defined for $t \in [0, \infty)$ such that $W(0) = 0$, the increments $W(t) - W(s)$ have normal distribution with mean 0 and variance $(t - s)$ for $0 \leq s < t$, and $W(t_n) - W(t_{n-1}), \ldots, W(t_2) - W(t_1)$ are independent for all sequences $0 \leq t_1 < t_2 < \cdots < t_n$.

Black-Scholes Model:

Recall that the stock price for a N-step binomial model is given by,

$$S_N(t) = S(0)e^{\mu t + \sigma W_N(t)}$$

where $W_N(t)$ is the scaled random walk. Replacing $W_N(t)$ by the Wiener process $W(t)$ (corresponding to the limit as $N \to \infty$), a new model for stock prices, denoted by $S(t)$, defined for all $t \geq 0$ is given by,

$$S(t) = S(0)e^{\mu t + \sigma W(t)}.$$

This is what is sometimes referred to as the *Black-Scholes* model for stock prices. It follows immediately that,

$$\ln S(t) = \ln S(0) + \mu t + \sigma W(t).$$

Since $W(t)$ has a normal distribution $\mathcal{N}(0, t)$, it follows that $\ln S(t)$ has a normal distribution $\mathcal{N}(\ln S(0) + \mu t, \sigma^2 t)$. It is for this reason that the stock price $S(t)$ in the Black-Scholes model is said to have the *lognormal distribution.*

Result:

$$E(S(t)) = S(0)e^{(\mu + \frac{1}{2}\sigma^2)t}.$$

Proof:

$$\begin{aligned}
E[S(t)] &= E\left[S(0)e^{\mu t + \sigma W(t)}\right] \\
&= S(0)e^{\mu t} E\left[e^{\sigma W(t)}\right] \\
&= S(0)e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}\sigma} e^{x} e^{-\frac{x^2}{2\sigma^2 t}} dx \\
&= S(0)e^{\mu t} e^{\frac{1}{2}\sigma^2 t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{1}{2\sigma^2 t}\left(x - \frac{\sigma}{\sqrt{2t}}\right)^2} dx \\
&= S(0)e^{(\mu t + \frac{1}{2}\sigma^2)t}.
\end{aligned}$$

Remark:

From the above relation it can be seen that

$$\nu = \mu + \frac{1}{2}\sigma^2$$

is the "growth rate" of the expected stock price. It is often used as a parameter instead of the expected logarithmic return μ, in the Black-Scholes model. In this case the model becomes

$$S(t) = S(0)e^{(\nu - \frac{1}{2}\sigma^2)t + \sigma W(t)}.$$