We have proved in the last class

If R is an ordering, then for all $x, y, z \in S$

xPy & $yPz \rightarrow xPz$

This is all the definition of quasi-transitivity.

If for all $x, y, z \in S$, xPy and $yPz \ \rightarrow \ xPz$, then R is quasi transitive.

We have shown that reflexivity, completeness and transitivity of R over S implies that there is a choice function $C(S, R)$.

We take a weaker condition quasi transitivity of R.

Result:
If R is reflexive, complete and quasi-transitive over a finite set X, then a choice function $C(S, R)$ is defined over X.

Proof: Suppose there are n elements in S and $S \subset X$. The elements are $x_1, x_2, x_3..., x_n$. Take the first pair (x_1, x_2), by reflexivity and completeness of R, there is a best element in this pair (x_1, x_2).
Consider a set of j elements $(x_1, x_2, ...x_j)$. Suppose a_j is a best element of for the set $(x_1, x_2, ...x_j)$. This implies that $a_j R x_k$ for $k = 1, 2, 3...j$.
Now consider the set $(x_1, x_2..., x_j, x_{j+1})$, in this set we can have $a_j R x_{j+1}$ or $x_{j+1} P a_j$. If we have $a_j R x_{j+1}$ then a_j is a best element for the set $(x_1, x_2..., x_j, x_{j+1})$. So, there exist a best element for this set. It implies that choice function exist.

Suppose $x_{j+1}Pa_j$. In this x_{j+1} is not a best element only when x_kPx_{j+1} for some $k = 1, 2, 3...j$. Let this be x_k. We have x_kPx_{j+1} and $x_{j+1}Pa_j$. By quasi transitivity we have x_kPa_j. But we have a_jRx_k since a_j is the best element of the set $(x_1, x_2, ...x_j)$. A contradiction. So we cannot have x_kPx_{j+1} for some $k = 1, 2, 3...j$. Thus, x_{j+1} is a best element when $x_{j+1}Pa_j$.

Thus, we get that reflexivity, completeness and quasi transitivity implies that choice function can be defined.

We define Acyclicity
R is acyclical over S if and only the following holds:
for all $x_1, x_2..., x_j \in S$, if $x_1 P x_2$ & $x_2 P x_3$ & ....&$x_{j-1} P x_j$ then $x_1 R x_j$.
It is much weaker than quasi transitivity condition.

Result:
If R is reflexive and complete, then a necessary and sufficient condition for $C(S, R)$ to be defined over finite S is that R be acyclical over S.

Proof: We first proof there necessary part, that is if $C(S, R)$ is defined over S then R must be acyclical over S given reflexivity and completeness are satisfied by R.
Suppose R is not acyclical. This implies that there is a subset of of j alternatives in S such that $x_1 P x_2, ... x_{j-1} P x_j$ and $x_j P x_1$. This implies that there is no best element in this subset, so choice function is not defined over S.

Sufficiency part. Suppose all the alternatives are indifferent to each other. It implies all the elements are in the best element. Suppose there is only one pair satisfying the strict preference that is $x_2 P x_1$. We have x_2 is the best element. But x_2 cannot be best element of S if there is some element say x_3 in S such that $x_3 P x_2$. If $x_1 P x_3$ then by acyclicity we have $x_1 R x_2$. It contradicts $x_2 P x_1$. So x_3 is the best element of the set x_1, x_2, x_3. If we continue in this way, we can exhaust all the elements and have best choice set being non-empty. We get the acyclicaity is a sufficient condition for the choice function to be defined when reflexivity and completeness are given.

The existence of a choice function explains rational choice. By specifying certain properties on choice function we define rational choice.
For example suppose we choose x from the set x, y, z and choose y from the set x, y. We cannot rationalize this choice outcome.
We define certain properties. They are consistency properties.

Property α: $x \in S_1 \subset S_2 \rightarrow [x \in C(S_2) \rightarrow x \in C(S_1)]$ for all x.
If some element of a subset S_1 of S_2 is best in the set S_2 then it must be best in S_1.
It means if x is best in $\{x, y, z\}$ then x must be best in $\{x, z\}$ subset of $\{x, y, z\}$.

Property β :
$[x, y \in C(S_1) \& S_1 \subset S_2] \rightarrow [x \in C(S_2)$ if and only if $y \in C(S_2)]$.

If x, Y are best element of a set S_1 which is a subset of S_2, then if x is a best element of S_2 implies y is also a best element of S_2 and if y is a best element of S_2 it implies x is also a best element of S_2.

Result:
Every choice function $C(S, R)$ generated by a binary relation R satisfies property α but not necessarily property β.

Proof: If x belongs to $C(S, R)$ then xRy for all $y \in S$. Therefore xRy for all y in all the subsets of S. This is property α.
Suppose we have xIy, xPz and zPy for a triple x, y, z. Note that acyclicity is satisfied. $C([x, y]) = [x, y]$ and $C([x, y, z]) = [x]$. It violates property β.