Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

We begin with finding the portfolio with the smallest variance (that is, smallest risk) among all the feasible portfolios.

Proposition:

If $\rho_{12} < 1$ or $\sigma_1 \neq \sigma_2$, then σ_V^2 as a function of s attains its minimum value at

$$s_0 = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$$

The corresponding values of the expected return μ_V and variance σ_V^2 are given by:

$$\begin{aligned} \mu_0 &= \frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2 - (\mu_1 + \mu_2)c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}} \\ \sigma_0^2 &= \frac{\sigma_1^2\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}} \end{aligned}$$

If $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$, then all feasible portfolios have the same variance equal to $\sigma_1^2 = \sigma_2^2$.

Proof:

To find the value of s for which σ_V^2 attains a minimum, we differentiate σ_V^2 with respect to s and equate the derivative to zero. This gives an equation for s,

$$\frac{d\left(\sigma_V^2\right)}{ds} = 2s(\sigma_1^2 + \sigma_2^2 - 2c_{12}) - 2(\sigma_2^2 - c_{12}) = 0,$$

which has a solution s_0 as given in the above expression, as long as the denominator in the expression is non-zero. This is guaranteed by the condition that $\rho_{12} < 1$ or $\sigma_1 \neq \sigma_2$. Indeed if $\rho_{12} < 1$, then

$$\sigma_1^2 + \sigma_2^2 - 2c_{12} > \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 = (\sigma_1 - \sigma_2)^2 \geq 0,$$

and if $\sigma_1 \neq \sigma_2$, then

$$\sigma_1^2 + \sigma_2^2 - 2c_{12} \geq \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 > (\sigma_1 - \sigma_2)^2 \geq 0.$$

For the second derivative, we observe that

$$\frac{d^2\left(\sigma_V^2\right)}{ds^2} = 2(\sigma_1^2 + \sigma_2^2 - 2c_{12}) > 0.$$

Hence, we conclude that σ_V^2 attains its minimum at s_0. The expressions for μ_0 and σ_0^2 follow by substituting s_0 for s in the expressions for μ_V and σ_V^2, respectively.

If $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$, then $c_{12} = \sigma_1\sigma_2$, and for any $s \in \mathbb{R}$

$$\sigma_V^2 = s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)c_{12} = (s\sigma_1 + (1-s)\sigma_2)^2 = \sigma_1^2 = \sigma_2^2.$$

Proposition:

Suppose that $\rho_{12} = 1$ and $\sigma_1 \neq \sigma_2$. Then $\sigma_V = 0$ if and only if

$$w_1 = -\frac{\sigma_2}{\sigma_1 - \sigma_2}, \ w_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2}.$$

This involves short selling, since either w_1 or w_2 is negative.

Suppose that $\rho_{12} = -1$. Then $\sigma_V = 0$ if and only if

$$w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \quad w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

No short selling is necessary, since both w_1 and w_2 are positive.

Proof:

Let $\rho_{12} = 1$ and $\sigma_1 \neq \sigma_2$. Then

$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2 = (w_1\sigma_1 + w_2\sigma_2)^2$$

and $\sigma_V^2 = 0$ if and only if $w_1\sigma_1 + w_2\sigma_2 = 0$. Using this with $w_1 + w_2 = 1$ we obtain the first result.

Now let $\rho_{12} = -1$. Then

$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 - 2w_1w_2\sigma_1\sigma_2 = (w_1\sigma_1 - w_2\sigma_2)^2$$

and $\sigma_V^2 = 0$ if and only if $w_1\sigma_1 - w_2\sigma_2 = 0$. Using this with $w_1 + w_2 = 1$ we obtain the second result.

Corollary:

Suppose that $\sigma_1 \leq \sigma_2$. The following five cases are possible:

1. If $\rho_{12} = 1$, then there is a feasible portfolio V with short selling such that $\sigma_V = 0$, whenever $\sigma_1 < \sigma_2$. Each portfolio V in the feasible set has the same σ_V whenever $\sigma_1 = \sigma_2$.
2. If $\frac{\sigma_1}{\sigma_2} < \rho_{12} < 1$, then there is a feasible portfolio V with short selling such that $\sigma_V < \sigma_1$, but for each portfolio without short selling $\sigma_V \geq \sigma_1$.
3. If $\rho_{12} = \frac{\sigma_1}{\sigma_2}$, then $\sigma_V \geq \sigma_1$ for each feasible portfolio V.
4. If $-1 < \rho_{12} < \frac{\sigma_1}{\sigma_2}$, then there is a feasible portfolio V without short selling such that $\sigma_V < \sigma_1$.
5. If $\rho_{12} = -1$, then there is a feasible portfolio V without short selling such that $\sigma_V = 0$.

Several Securities:

Risk and Expected Return on a Portfolio:

A portfolio constructed from n different securities can be described in terms of their weights

$$w_i = \frac{x_i S_i(0)}{V(0)}, i = 1, \ldots, n,$$

where x_i is the number of shares of security i in the portfolio, $S_i(0)$ is the initial price of security i, and $V(0)$ is the amount initially invested in the portfolio. It will prove convenient to arrange the weights into a one-row matrix

$$\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \ldots & w_n \end{bmatrix}.$$

Just like for two securities, the weights add up to one, which can be written in the matrix form as

$$\mathbf{w}\mathbf{u}^\top = 1,$$

where

$$\mathbf{u} = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}$$

is a one-row matrix with all n entries being equal to 1. Therefore $\mathbf{u}^\top$ (the transpose of $\mathbf{u}$) is a one-column matrix, and the usual matrix multiplication rules apply. The *feasible* (or *attainable*) *set* consists of all portfolios with weights $\mathbf{w}$ satisfying the above relation, called *feasible* (or *attainable*) *portfolios*.

Suppose that the returns on the securities are $K_1, \ldots, K_n$. The expected returns $\mu_i = E(K_i)$ for $i = 1, \ldots, n$ will also be arranged into a one-row matrix

$$\mathbf{m} = \begin{bmatrix} \mu_1 & \mu_2 & \ldots & \mu_n \end{bmatrix}.$$

The covariances between returns will be denoted by $c_{ij} = Cov(K_i, K_j)$. They are the entries of the $n \times n$ *covariance matrix*

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{bmatrix}.$$

The diagonal elements of $\mathbf{C}$ are simply the variances of returns, $c_{ii} = \sigma_i^2 = Var(K_i)$.

The covariance matrix is symmetric and non-negative definite. In what follows, we shall assume that $\det \mathbf{C} \neq 0$, which implies that $\mathbf{C}$ has an inverse $\mathbf{C}^{-1}$.

Proposition:

The expected return $\mu_V = E(K_V)$ and variance $\sigma_V^2 = Var(K_V)$ of the return $K_V = w_1 K_1 + \cdots + w_n K_n$ on a portfolio V with weights $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \ldots & w_n \end{bmatrix}$ are given by $\mu_V = \mathbf{w}\mathbf{m}^\top$ and $\sigma_V^2 = \mathbf{w}\mathbf{C}\mathbf{w}^\top$, respectively.

Proof:

The formula for μ_V follows by the linearity of expectation,

$$\mu_V = E(K_V) = E\left(\sum_{i=1}^n w_i K_i\right) = \sum_{i=1}^n w_i \mu_i = \mathbf{w}\mathbf{m}^\top.$$

For σ_V^2 we use the linearity of covariance with respect to each of its arguments,

$$\begin{aligned} \sigma_V^2 &= Var(K_V) = Var\left(\sum_{i=1}^n w_i K_i\right) \\ &= Cov\left(\sum_{i=1}^n w_i K_i, \sum_{j=1}^n w_j K_j\right) = \sum_{i,j=1}^n w_i w_j c_{ij} \\ &= \mathbf{w}\mathbf{C}\mathbf{w}^\top. \end{aligned}$$

Minimum Variance Portfolio:

The portfolio with the smallest variance among all the feasible portfolios will be called the *minimum variance portfolio* (MVP). To find this portfolio we need to minimize the variance $\sigma_V^2 = \mathbf{w}\mathbf{C}\mathbf{w}^\top$ over all weights $\mathbf{w}$. Because the weights must add up to 1, this leads to the constrained minimum problem

$$\min \mathbf{w}\mathbf{C}\mathbf{w}^\top,$$

where the minimum is taken over all vectors $\mathbf{w}$ that satisfy the condition

$$\mathbf{w}\mathbf{u}^\top = 1.$$

To compute this constrained minimum we can use the method of Lagrange multipliers.