
Note: This document is a part of the lectures given during the Jan-May 2020 Semester.

Value of a Forward Contract:

Every forward contract has value zero when initiated. As time passes, the price of the underlying asset may change. Consequently, the value of the forward contract will keep changing and will no longer be zero (as was the case initially). At the time of delivery, the value of a long forward contract will be $S(T) - F(0,T)$ which could either be positive, zero or negative.

Suppose that the price for a forward contract for time T initiated at time t is $F(t,T)$, where $0 < t < T$. Then the net gain/loss of the investor with the long position is $F(t,T) - F(0,T)$ as compared to an investor entering into a new long forward contract at time t with the same delivery date T. To find the value of the original forward position at time t one needs to discount this gain back to time t. This discounted amount would be received (or paid, if negative) by the investor with a long position should the forward contract initiated at time 0 be closed out at time t, which is earlier than the agreed delivery date T.

For any t such that $0 \leq t \leq T$ the time t value of a long forward contract with forward price $F(0,T)$ is given by:

$$V(t) = [F(t,T) - F(0,T)]e^{-r(T-t)}.$$

1. Suppose that:

$$V(t) < [F(t,T) - F(0,T)]e^{-r(T-t)}$$

   (a) At time t:

      i. Borrow the amount $V(t)$ to enter into a long forward contract with forward price $F(0,T)$ and delivery time T.

      ii. Initiate a short forward position with forward price $F(t,T)$ at no cost.

   (b) At time T:

      i. Close out the forward contracts collecting the amounts $S(T) - F(0,T)$ for the long position and $-S(T) + F(t,T)$ for the short position.

      ii. Pay back the loan with interest amounting to $V(t)e^{r(T-t)}$ in total.

   The final balance $F(t,T) - F(0,T) - V(t)e^{r(T-t)} > 0$ will be the arbitrage profit.

2. Suppose that:

$$V(t) > [F(t,T) - F(0,T)]e^{-r(T-t)}.$$

   (a) At time t:

      i. Receive the amount $V(t)$ to acquire a short forward contract with forward price $F(0,T)$ and delivery date T. (Borrow and pay $V(t)$, if negative).

ii. Initiate a new long forward contact with forward price $F(t, T)$ at no cost.

(b) Then at time T:

i. Close out both forward contracts receiving the amounts $F(0, T) - S(T)$ and $S(T) - F(t, T)$, respectively. (Pay the amounts, if negative).

ii. Collect $V(t)e^{r(T-t)}$ from the risk-free investment, with interest.

The final balance $-F(t, T) + F(0, T) + V(t)e^{r(T-t)} > 0$ will be the arbitrage profit.

Futures:

We assume that time is discrete with steps of length τ , typically a day. Just like a forward contract, a futures contract involves an underlying asset (say stock) with prices $S(n)$ for $n = 0, 1, \ldots$ and time T, say. In addition to the usual stock prices, the market dictates the so called futures prices $f(n, T)$ for each step $n = 0, 1, \ldots$ such that $n\tau \leq T$. These prices are unknown at time 0, except for $f(0, T)$, and we shall treat them as random variables. As in the case of a forward contract, it costs nothing to initiate a futures position. The difference lies in the cash flow during the lifetime of the contract. A long forward contract involves just a single payment $S(T) - F(0, T)$ at delivery. A futures contract involves a random cash flow, known as marking to market. This means that, at each time step $n = 1, 2, \ldots$ such that $n\tau \leq T$, the holder of a long futures position will receive the amount $f(n, T) - f(n - 1, T)$.

Example:

Suppose that the initial margin is set at 10% and the maintenance margin at 5% of the futures price. The table below shows a scenario with futures prices $f(n, T)$. The columns labeled "Margin 1" and "Margin 2" show the deposit at the beginning and at the end of each day, respectively. The "Payment" column contains the amounts paid to top up the deposit (negative numbers) or withdrawn (positive numbers).

n	$f(n,T)$	Cash Flow	Margin 1	Payment	Margin 2
0	140	Opening	0	-14	14
1	138	-2	12	0	12
2	130	-8	4	-9	13
3	140	$+10$	23	$+9$	14
4	150	$+10$	24	$+9$	15
		Closing	15	$+15$	0

On day 0 a futures position is opened and a 10% deposit paid. On day 1 the futures price drops by 2, which is subtracted from the deposit. On day 2 the futures price drops further by 8, triggering a margin call because the deposit falls below 5%. The investor has to pay 9 to restore the deposit to the 10% level. On day 3 the forward price increases and 9 is withdrawn, leaving a 10% margin. On day 4 the forward price goes up again, allowing the investor to withdraw another 9. At the end of the day the investor decides to close the position, collecting the balance of the deposit. The total of all payments is 10 $(-14 + 0 - 9 + 9 + 9 + 15)$, the increase in the futures price between day 0 and 4.