MA 322: Scientific Computing

Lecture - 10

Outline

- Quadrature formula

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- Newton-Cotes formula

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- Gaussian quadrature formula

Quadrature

Integrals such as

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Note that $Q_n : C[a,b] \longrightarrow \mathbb{R}$ is a linear functional, that is, $Q_n(\alpha f + g) = \alpha Q_n(f) + Q_n(g)$ for $f, g \in C[a,b]$ and $\alpha \in \mathbb{R}$.

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The functional $Q_n(f)$ is called a quadrature formula or a quadrature rule.

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Consider the nodes $[x_0, \ldots, x_n]$ in $[a, b]$. Consider the Lagrange interpolating polynomial $p_n(x)$ given by
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Note that the quadrature weights do not depend on the function $f(x)$ and hence can be computed independently of $f(x)$ and stored.

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$$|E_n(f)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \int_a^b |(x-x_0)(x-x_1)\cdots(x-x_n)| dx$$

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- Integrate piecewise polynomial interpolant.

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Set $E_1(f) := I(f) - T(f)$. If $f \in C^2[a, b]$ then there exists $\theta \in [a, b]$ such that

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Proof: $E_1(f) = \int_a^b \frac{f''(\xi_x)}{2}(x-a)(x-b)dx = \frac{f''(\theta)}{2} \int_a^b (x-a)(x-b)dx = -\frac{f''(\theta)}{12}(b-a)^3$.

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$$\begin{aligned} T_n(f) &:= \sum_{j=1}^{n} \frac{h}{2}[f(x_{j-1}) + f(x_j)] \\ &= h\left[\frac{f(x_0)}{2} + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right] \end{aligned}$$

we obtain composite trapezoid rule.

Error in composite trapezoid rule

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Consequently, there exists $\theta \in [a, b]$ such that

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Hence we have $|E_{1,h}(f)| \leq \dfrac{h^2}{12}(b-a)\|f''\|_\infty$.

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$$S(f) = \left[w_0 f(a) + w_1 f\left(\frac{a+b}{2}\right) + w_2 f(b) \right] \approx \int_a^b f(x)dx,$$

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A messy calculation shows that $w_0 = w_2 = (b-a)/6$ and $w_1 = 2(b-a)/3$. Hence

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Observe that $S(f) = \frac{1}{3}T(f) + \frac{2}{3}M(f),$ where $T(f)$ is the trapezoid rule and $M(f) = (b-a)f\left(\frac{a+b}{2}\right)$ is the midpoint rule.

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$$\ell_0(x) = 2\left(x - \frac{1}{2}\right)(x-1), \;\; \ell_1(x) = -4x(x-1), \;\; \ell_2(x) = 2x\left(x - \frac{1}{2}\right).$$

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Observe that for the present example, $S(f)$ is exact for polynomial of degree ≤ 3. As per the error analysis of quadrature rule, $S(f)$ is expected to be exact for polynomials of degree ≤ 2.