Optimization Problems : Some Examples

Eg-1: (Farmer's Problem)

'a' acres avaible, To cultivate sugarcane & Rice

'r' investment amount

'm' man-days of laborer

Return: Sugarcane: Rs. 1700/acre   Rice: 800/acre

Cost of seeding & cultivating one acre of land:

Rs. 300 for sugarcane & Rs. 120 for rice.

Labour needed: Sugarcane: 6 man-days/acre

Rice: 3 man-days/acre

Objective: How many acres each of sugarcane and rice he should grow so that his return is maximum.

Decision variables:

x_1 = acres for plantation of sugarcane

x_2 = acres for plantation of rice.

Maximize $\vartheta = 1700x_1 + 800x_2$

subject to

$$x_1 + x_2 \le a$$

$$6x_1 + 3x_2 \le m$$

$$300x_1 + 120x_2 \le r$$

$$x_1 \ge 0,\ x_2 \ge 0$$

LPP

Eg.2 : (Approximation)

Let $g : [a,b] \to \mathbb{R}$ be a continuous function whose values $g(x_k)$ at x_k $(k = 0,1,2,\ldots,m)$ are known.

To find a polynomial of degree $n < m$ which approximates $g(\cdot)$ in a certain sense.

Let a general polynomial of degree n be of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Define

$$\begin{aligned} \varepsilon_k(a_0, a_1, \ldots, a_n) &= g(x_k) - p(x_k) \\ &= g(x_k) - \sum_{k=0}^{n} a_i x_k^i, \quad k = 0,1,\ldots,m. \end{aligned}$$

Objective: To choose $a_0, a_1, \ldots, a_n$ such that

$$f(a_0, a_1, \ldots, a_n) = \sum_{k=0}^{m} \varepsilon_k^2 (a_0, a_1, \ldots, a_n)$$ is minimized

$$\text{Minimize } f(a_0, a_1, \ldots, a_n)$$

$$\text{s.to. } \quad \underset{\sim}{a} \in \mathbb{R}^{n+1}.$$

Modifications: (eg) $\underset{\sim}{a} \geq 0$

Eg-3: (Utility Maximization)

Let a consumer consume 'n' commodities with the amount of the i^{th} commodity being $x_i \geq 0$.

The satisfaction level of consuming x_i units of the i^{th} commodity can be described by the value $u(x) = u(x_1, \ldots, x_n)$ of a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$.

Suppose the price for the i^{th} commodity is $p_i > 0$ and the consumer has a total budget of $a > 0$.

$$\begin{aligned} \text{Maximize} \quad & z = u(\underset{\sim}{x}) \\ \text{subject to} \quad & \underset{\sim}{p}^T \underset{\sim}{x} \leq a \\ & \underset{\sim}{x} \geq \underset{\sim}{0} \end{aligned}$$

$$\underset{\sim}{p}^T \underset{\sim}{x} = \sum_{i=1}^{n} p_i x_i$$

A concrete example for $u(x)$:

Cobb - Douglas function

$$u(x_1, x_2, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad a_i > 0 \ \forall i, \quad \sum_{i=1}^{n} a_i = 1$$

Eg-4: (Expenditure Minimization)

Flip side of utility maximization

Given $p \in \mathbb{R}^n_+$, what is the minimum amount of income needed to give a utility-maximizing consumer a utility level of at least $\bar{u}$ ($\bar{u}$ fixed).

$$\text{Minimize } \underline{p}^T \underline{x}$$

$$\text{subject to } \quad u(\underline{x}) \geq \bar{u}$$

$$\underline{x} \geq 0$$

Eg-5: (Portfolio Optimization)

Given n securities (eg. stocks) and their characteristics in terms of their expected returns (μ_i) and the variance of their returns (σ_i^2), along with the covariances (σ_{ij}).

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \qquad C = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & & & \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \qquad \sum_{i=1}^{n} w_i = 1$$

$$\mu_V = w^T m \quad , \quad \sigma_V^2 = w^T C w$$

portfolio return $\qquad\qquad$ portfolio risk

MVP: $\quad$ Minimize σ_V^2

Subject to $\sum_{i}^{n} w_i = 1$.

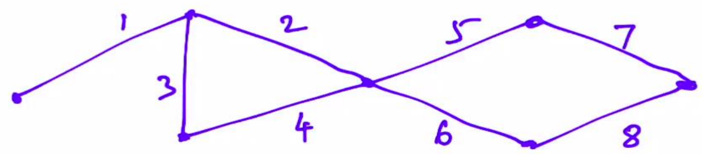
Variation: (i) add the constraint

$$\mu_V = w^T m = \mu^0 \quad (\mu^0 \text{ fixed})$$

(ii) $\quad a_i \le w_i \le b_i \qquad 0 \le a_i \le b_i \le 1$

(iii) $\quad w_i \geq 0$

Eg-6: (Matching and Covering Problems)



Maximum Cardinality Matching Problem (MCMP)

Given a graph G with n edges and m nodes, determine a matching containing the largest number of edges

$M = \{1, 2, \ldots, m\}$ → set of nodes

$N = \{1, 2, \ldots, n\}$ → set of edges

Define, for $j \in N$

$$x_j = \begin{cases} 1 & \text{if the edge } j \text{ is in the matching} \\ 0 & \text{if the edge } j \text{ is not in the matching} \end{cases}$$

MCMP: Maximize $\sum_{j \in N} x_j$

subject to $x_i + x_j \leq 1$, for i and j sharing a common endpoint and $i, j \in N$

$x_j = 0$ or 1, $j \in N$.

Minimum Cardinality Covering Problem (MCCP)

→ determine a covering containing the fewest number of edges.

Define, for each $j \in N$

$$y_j = \begin{cases} 1 & \text{if the edge } j \text{ is in the covering} \\ 0 & \text{if the edge } j \text{ is not in the covering} \end{cases}$$

Also, let S_i be the set of edges that intersect at node $i \in M$.

MCCP: Minimize $\sum_{j \in N} y_j$

subject to $\sum_{j \in S_i} y_j \geq 1$, $i \in M$

$y_j = 0$ or 1, $j \in N$.

Integer linear programming problems

More examples can be found in almost all areas of science, engineering, economics, etc.

$\rightarrow$ Prevalent almost everywhere.