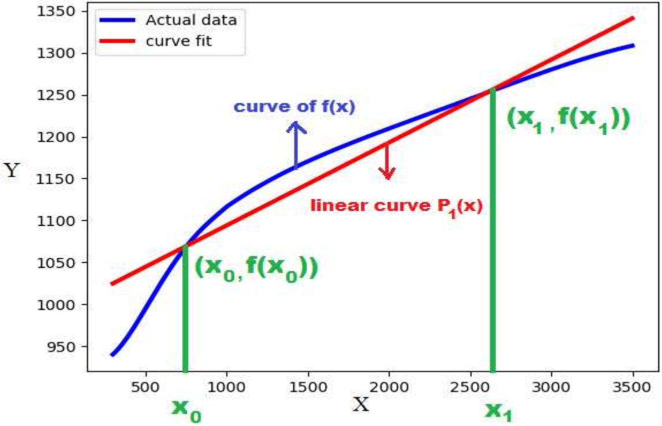
MA 322: Scientific Computing

Lecture - 7

Linear Curve Fitting

Geometrically, linear curve fitting can be described by following figure



- Observe that the fitted curve (red line) is a straightline passing through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, with slope $\frac{f(x_1)-f(x_0)}{x_1-x_0}$.

Linear Curve Fitting Contd.

- Hence, it is given by the equation

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

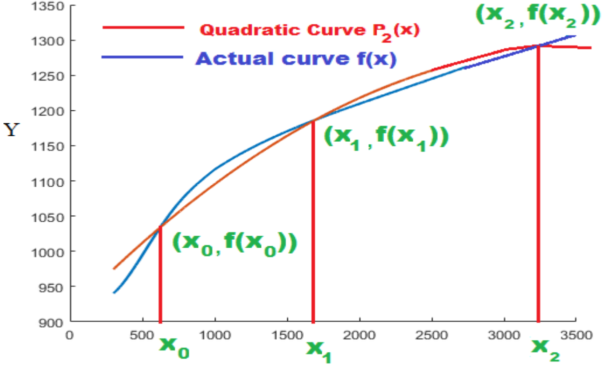
Which is a linear polynomial and we write as

$$\begin{aligned} P_1(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \\ &= f(x_0) + \frac{(x - x_0)}{h}\Delta f(x_0), \end{aligned}$$

with $h = x_1 - x_0$.

- Similarly, in quadratic curve fitting, we find a polynomial of degree two which intersect given curve at at least three points.

Quadratic Curve Fitting



- Observe that the fitted curve (red line) passing through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is a polynomial $P_2(x)$ of degree two given by (Homework)

$$\begin{aligned} P_2(x) &= f(x_0) + \frac{(x-x_0)}{h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f(x_0) \\ &= P_1(x) + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f(x_0), \quad h = x_1 - x_0 = x_2 - x_1. \end{aligned}$$

Newton's Forward Interpolation

- General Result on Polynomial Fitting

  - From a given set of $n+1$ values

$$\{(x_0, f(x_0)),\ (x_1, f(x_1)),\ \ldots,\ (x_n, f(x_n))\},\ \ x_i > x_{i-1},$$

we find a polynomial P_n of degree n such that

$$P_n(x_i) = f(x_i),\ \ 0 \le i \le n$$

and $P_n(x)$, for $x \in [x_0, x_n]$, is given by following formula

$$\begin{aligned} P_n(x) &= f(x_0) + \frac{(x - x_0)}{h}\Delta f(x_0) + \frac{(x - x_0)(x - x_1)}{2!h^2}\Delta^2 f(x_0) \\ &+ \ldots + \frac{(x - x_0)(x - x_1)\ldots(x - x_n)}{n!h^n}\Delta^n f(x_0) \end{aligned}$$

with $h = x_{i+1} - x_i$.

- Note that $P_n(x) = f(x),\ \ x \in [x_0, x_n]$ when $f(x)$ is a polynomial of degree n.

Newton's Forward Interpolation Contd.

- Newton's forward formula given by

$$
\begin{aligned}
P_n(x) &= f(x_0) + \frac{(x-x_0)}{h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f(x_0) \\
&\quad + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{n!h^n}\Delta^n f(x_0),
\end{aligned}
$$

can be simplified by introducing a variable $u = \frac{x-x_0}{h}$ and observing the fact that

$$
\begin{aligned}
\frac{(x-x_0)(x-x_1)}{2!h^2} &= u\left(\frac{x-x_0-h}{h}\right)\frac{1}{2!} = u(u-1)\frac{1}{2!}, \\
\frac{(x-x_0)(x-x_1)(x-x_2)}{3!h^3} &= u\left(\frac{x-x_0-h}{h}\right)\left(\frac{x-x_0-2h}{h}\right)\frac{1}{3!} \\
&= u(u-1)(u-2)\frac{1}{3!} \text{ and so on,}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
P_n(x) &= f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) \\
&\quad + \ldots + \frac{u(u-1)(u-2)\ldots(u-(n-1))}{n!}\Delta^n f(x_0),
\end{aligned}
$$

Examples

Problem-1: Find $f(14)$ from the following data

x	12	16	20	24
$f(x)$	1	5	10	17

Solution: We can apply Newton's forward interpolation formula because arguments are equidistant and $14 \in [12, 24]$. Since, four points are given we can fit a polynomial $P_3(x)$ of degree 3 and $P_3(x) = f(x)$ when f is a polynomial of degree 3. Since, it is asked to find $f(14)$, so we assume that f is polynomial of degree 3 so that $f(14) = P_3(14)$. Now, using Newton's forward interpolation, we obtain

$$P_3(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0),$$

with

$$u = \frac{x - x_0}{h} = \frac{x - 12}{4}.$$

- Clearly, for the terms $\Delta f(x_0)$, $\Delta^2 f(x_0) \ldots$, we need finite difference tables.

Tabular Representation of Forward Differences

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
$12 = x_0$	$1 = f(x_0)$			
	$\longrightarrow$	$\Delta f(x_0) = 4$		
$16 = x_1$	$5 = f(x_1)$	$\longrightarrow$	$\Delta^2 f(x_0) = 1$	
				$\Delta^3 f(x_0) = 1$
	$\longrightarrow$	$\Delta f(x_1) = 5$		
$20 = x_2$	$10 = f(x_2)$	$\longrightarrow$	$\Delta^2 f(x_1) = 2$	
	$\longrightarrow$	$\Delta f(x_2) = 7$		
$24 = x_3$	$17 = f(x_3)$			

Now, use Newton's forward formula and $u = \frac{14-12}{4} = 0.5$ to have

$$\begin{aligned} f(14) = P_3(14) &= 1 + 0.5 \times 4 + \frac{0.5(0.5-1)}{2} \times 1 \\ &\quad + \frac{0.5(0.5-1)(0.5-2)}{3} \times 1 = 2.94 \end{aligned}$$

Drawback of Newton's Forward Interpolation

- Newton's forward interpolation formula gives good estimate of the function if the interpolating value lies in the beginning of x series.
- Therefore, to overcome this drawback, we go for Newton's Backward Interpolation.
    - This is done by writing the data set

$$\{(x_0, f(x_0)),\ (x_1, f(x_1)),\ \ldots,\ (x_n, f(x_n))\},\ \ x_i > x_{i-1},$$

in reverse order as

$$\{(x_n, f(x_n)),\ (x_{n-1}, f(x_{n-1})),\ \ldots,\ (x_0, f(x_0))\},\ \ x_i > x_{i-1}$$

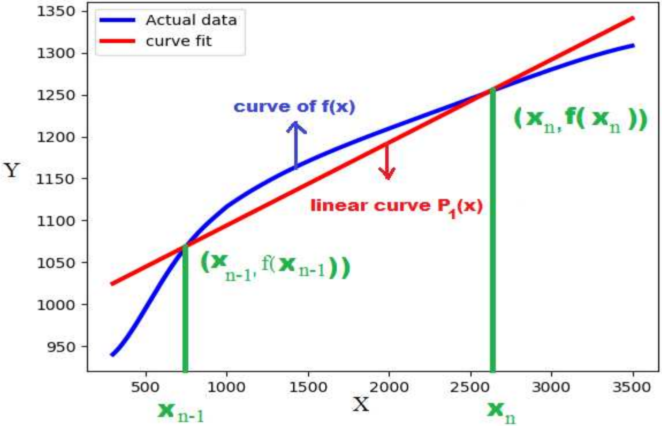
and then, we try to fit polynomial of different degrees as it was done for Newton's Forward case.

- Let us try to fit a linear curve for the data set

$$\{(x_n, f(x_n)),\ (x_{n-1}, f(x_{n-1})).$$

Linear Curve Fitting

Geometrically, linear curve fitting can be described by following figure



- Observe that the fitted curve (red line) is a straightline passing through the points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$, with slope $\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$.

General Curve Fitting

- Hence, it is given by the equation

$$y - f(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

Which is a linear polynomial and we write as

$$\begin{aligned} P_1(x) &= f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_n) \\ &= f(x_n) + \frac{(x - x_n)}{h}\nabla f(x_n), \quad \nabla f(x_n) = f(x_n) - f(x_{n-1}), \end{aligned}$$

with $h = x_n - x_{n-1}$ and ∇ is called backward operator.

- Similarly, other higher degree polynomials could be fitted. In general, we find a polynomial P_n of degree n such that $P_n(x_i) = f(x_i), \; 0 \le i \le n$, and $P_n(x)$, for $x \in [x_0, x_n]$, is given by following formula

$$\begin{aligned} P_n(x) &= f(x_n) + \frac{(x - x_n)}{h}\nabla f(x_n) + \frac{(x - x_n)(x - x_{n-1})}{2!h^2}\nabla^2 f(x_n) \\ &\quad + \ldots + \frac{(x - x_n)(x - x_{n-1})\ldots(x - x_1)}{n!h^n}\nabla^n f(x_n) \end{aligned}$$

with $h = x_{i+1} - x_i$.

Newton's Backward Interpolation

- Like forward formula, the backward formula can be simplified by introducing a variable $v = \frac{x-x_n}{h}$ and observing the fact that

$$\frac{(x-x_n)(x-x_{n-1})}{2!h^2} = v\left(\frac{x-x_n+h}{h}\right)\frac{1}{2!} = v(v+1)\frac{1}{2!},$$
$$\frac{(x-x_n)(x-x_{n-1})(x-x_{n-2})}{3!h^3} = v\left(\frac{x-x_n+h}{h}\right)\left(\frac{x-x_n+2h}{h}\right)\frac{1}{3!}$$
$$= v(v+1)(v+2)\frac{1}{3!} \text{ and so on,}$$

Therefore

$$\begin{aligned} P_n(x) &= f(x_n) + v\nabla f(x_n) + \frac{v(v+1)}{2!}\nabla^2 f(x_n) \\ &\quad + \ldots + \frac{v(u+1)(v+2)\ldots(v+(n-1))}{n!}\nabla^n f(x_n), \end{aligned}$$

- Newton's backward interpolation formula gives good estimate of the function if the interpolating value lies at the end of x series.

Alternative Expression for Newton's Backward Formula

- Recall the backward formula

$$\begin{aligned} P_n(x) &= f(x_n) + v\nabla f(x_n) + \frac{v(v+1)}{2!}\nabla^2 f(x_n) \\ &\quad + \ldots + \frac{v(u+1)(v+2)\ldots(v+(n-1))}{n!}\nabla^n f(x_n), \end{aligned}$$

- Note that

$$\begin{aligned} \nabla f(x_n) &= f(x_n) - f(x_{n-1}) = \Delta f(x_{n-1}) \\ \nabla^2 f(x_n) &= \nabla(f(x_n) - f(x_{n-1})) = \Delta f(x_{n-1}) - \Delta f(x_{n-2}) = \Delta(\Delta f(x_{n-2})) \\ &= \Delta^2 f(x_{n-2}) \quad \text{and so on} \end{aligned}$$

- Therefore, an alternative formula is given by

$$\begin{aligned} P_n(x) &= f(x_n) + v\Delta f(x_{n-1}) + \frac{v(v+1)}{2!}\Delta^2 f(x_{n-2}) \\ &\quad + \ldots + \frac{v(u+1)(v+2)\ldots(v+(n-1))}{n!}\Delta^n f(x_0). \end{aligned}$$

Examples

From the following data, estimate $f(3)$ and $f(9)$

x:	2	4	6	8	10
$f(x)$:	4	13	25	43	64

Solution: Since the values of x are equidistant, we can apply forward formula to estimate $f(3)$ and backward formula to estimate $f(9)$.

- To estimate $f(3)$, we calculate

$$\begin{aligned} u &= \frac{x - x_0}{h} \\ &= \frac{3-2}{2} \\ &= \frac{1}{2} \\ &= 0.5 \end{aligned}$$

- To estimate $f(9)$, we need

$$\begin{aligned} v &= \frac{x - x_n}{h} \\ &= \frac{9-10}{2} \\ &= \frac{-1}{2} \\ &= -0.5 \end{aligned}$$

Example Contd..

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$2=x_0$	$4=y_0$				
		$9=\Delta f(x_0)$			
$4=x_1$	$13=y_1$		$3=\Delta^2 f(x_0)$		
		$12=\Delta f(x_1)$		$3=\Delta^3 f(x_0)$	
$6=x_2$	$25=y_2$		$6=\Delta^2 f(x_1)$		$-6=\Delta^4 f(x_0)$
		$18=\Delta f(x_2)$		$-3=\Delta^3 f(x_1)$	
$8=x_3$	$43=y_3$		$3=\Delta^2 f(x_2)$		
		$21=\Delta f(x_3)$			
$10=x_4$	$64=y_4$				

Now, use Newton's forward formula with $u = 0.5$ to have

$$\begin{aligned} P_4(3) &= 4 + 0.5 \times 9 + \frac{0.5(0.5-1)}{2} \times 3 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times 3 \\ &\quad + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \times (-6) = 8.0782. \end{aligned}$$

Now, use Newton's backward formula with $v = -0.5$ to have

$$\begin{aligned} P_4(9) &= 64 + (-.5) \times 21 + \frac{(-.5)(-.5+1)}{2} \times 3 + \frac{(-.5)(-.5+1)(-.5+2)}{3!}(-3) \\ &\quad + \frac{(-.5)(-.5+1)(-.5+2)(-.5+3)}{4!} \times (-6) = 53.92. \end{aligned}$$