LhARA linear optics documentation

A.N. Other

1 Introduction

Introduction to the documentation! Mention:

- Goal to define consistent notation and document code;
- Presently treats proton only.

2 Coordinate systems

2.1 Laboratory coordinate system

The origin of the LhARA coordinate system, the "laboratory coordinate system" or "laboratory reference frame", is at the position of the laser focus at the position of the laser-target interaction. The z axis is horizontal and parallel to the nominal capture axis, pointing in the downstream direction. The y axis points vertically upwards, and the x axis completes a right-handed orthogonal coordinate system.

Unit vectors along the x, y and z axes are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively. The position of the reference particle, its momentum and energy are described as functions of the distance it has travelled from the origin of coordinates to its current position. The distance travelled is defined to be s, making the position, $\boldsymbol{r}_0$, momentum, $\boldsymbol{p}_0$, and energy, E_0, of the reference particle at position s:

$$\begin{aligned} \boldsymbol{r}_0 &= \boldsymbol{r}_0(s)\,; \\ \boldsymbol{p}_0 &= \boldsymbol{p}_0(s)\,;\text{and} \\ E_0 &= E_0(s)\,. \end{aligned} \tag{1}$$

The magnitude of the reference particle velocity is v_0 and the relativistic parameters that determine the reference particle energy and momentum are:

$$\begin{aligned} \beta_0 &= \frac{v_0}{c}\,;\text{and} \\ \gamma_0 &= \frac{1}{\sqrt{1-\beta_0^2}}\,; \end{aligned}$$

where c is the speed of light. The time, t, at which the reference particle is at s is also a function of s:

$$t = t(s) = \frac{s}{v_0} = \frac{s}{c}\frac{E_0}{cp_0}\,; \tag{2}$$

where $p_0 = |\boldsymbol{p}_0|$.

2.2 Reference particle local coordinate system

A coordinate system defined relative to the position of the reference particle, the "reference particle local coordinate" (RPLC) system, may be defined using the direction in which the particle is travelling. The position of the particle defines the origin of the RPLC system, see figure 1. The tangent to the reference particle trajectory at s defines the $z_{\rm RPLC}$ axis with unit vector $\boldsymbol{k}_{\rm RPLC}$. In the laboratory frame, the presence of local

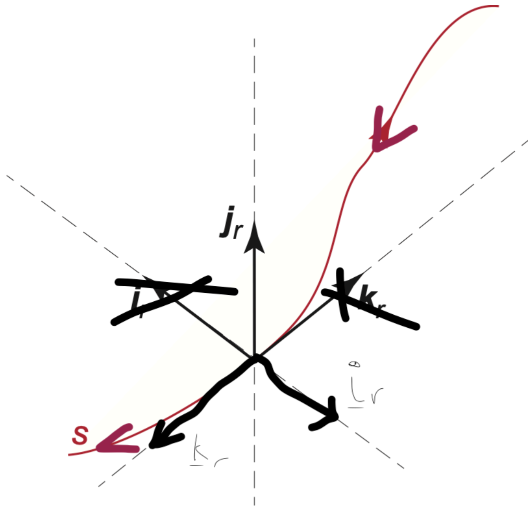

Figure 1: Reference particle local coordinate system. The trajectory of the reference particle is shown as the red line. The distance the reference particle has travelled, measured from the origin of coordinates in the laboratory frame, is labelled s. The origin of the RPLCs is coincident with the position of the reference particle. The directions of unit vectors along each of three righthanded, orthogonal coordinate axes are shown as black arrows labelled $\boldsymbol{i}_0$, $\boldsymbol{j}_0$, and $\boldsymbol{j}_0$.

electric or magnetic fields may cause the reference particle's trajectory to change. In the neighbourhood of the particle, the curved trajectory may be described in terms of an arc of a circle. The $x_{\rm RPLC}$ axis (with unit vector $\boldsymbol{i}_{\rm RPLC}$) is then taken to be in the direction pointing away from the centre of the circle. The third coordinate axis, $y_{\rm RPLC}$, is defined to complete the right-handed orthogonal coordinate system; the unit vector along the $y_{\rm RPLC}$ axis being given by $\boldsymbol{j}_{\rm RPLC} = \boldsymbol{k}_{\rm RPLC} \times \boldsymbol{i}_{\rm RPLC}$.

The trajectory of the reference particle will be a straight line as it traverses a drift space or when the particle's energy is increased (or decreased) through an electric field applied parallel to its direction of motion. In such cases the RPLC axes are taken to coincide with either the laboratory coordinate system or the RPLC system defined at the exit of the beam-line element that preceded the drift space or accelerating element.

2.3 Transforming to and from reference particle local coordinates to laboratory coordinates

In the RPLC system, the trajectory of the reference particle, $\boldsymbol{R}_0$, is:

$$\boldsymbol{R}_0(s) = \boldsymbol{0}\,. \tag{3}$$

The position of a test particle in the RPLC frame, $\boldsymbol{R}$, is described with reference to the position of the reference particle. In the laboratory frame, the position of the test particle is:

$$\boldsymbol{r}(s) = \boldsymbol{r}_0(s) + \boldsymbol{\delta r}(s)\,; \tag{4}$$

where:

$$\boldsymbol{\delta r}(s) = \underline{\underline{R}}(s)\boldsymbol{R}(s)\,;\,\text{and} \tag{5}$$

$\underline{\underline{R}}(s)$ is a rotation matrix that takes the RPLCs at s to the laboratory frame coordinates.

In the laboratory frame, the unit vectors $\boldsymbol{i}_{\rm RPLC}$, $\boldsymbol{j}_{\rm RPLC}$ and $\boldsymbol{k}_{\rm RPLC}$ are given by:

$$
\begin{aligned}
\boldsymbol{i}_{\rm RPLC} &= \begin{pmatrix} i_{rx} \\ i_{ry} \\ i_{rz} \end{pmatrix} ; \\
\boldsymbol{j}_{\rm RPLC} &= \begin{pmatrix} j_{rx} \\ j_{ry} \\ j_{rz} \end{pmatrix} ; \text{and} \\
\boldsymbol{k}_{\rm RPLC} &= \begin{pmatrix} k_{rx} \\ k_{ry} \\ k_{rz} \end{pmatrix} .
\end{aligned}
\tag{6}
$$

The rotation matrix, $\underline{\underline{R}}$, may now be written:

$$
\underline{\underline{R}}(s) = \begin{bmatrix} i_{{\rm RPLC}x} & j_{{\rm RPLC}x} & k_{{\rm RPLC}x} \\ i_{{\rm RPLC}y} & j_{{\rm RPLC}y} & k_{{\rm RPLC}y} \\ i_{{\rm RPLC}z} & j_{{\rm RPLC}z} & k_{{\rm RPLC}z} \end{bmatrix} .
\tag{7}
$$

3 Phase space and trace space

Para 1:

- Quantum effects required to describe, e.g., development of polarisation in electron storage rings;
- Description of beam dynamics largely done using classical, Hamiltonian, mechanics;
- In classical mechanics, the equations of motion are solved to give the evolution of the position, momentum, and energy of a particle as a function of time;
- Check Goldstein, time as parameter.

Para 2:

- Relativistic mechanics exploits four-vector position ($\underline{R} = (\boldsymbol{r}, ct)$) and four-vector momentum ($\underline{P} = (\boldsymbol{p}, E)$);
- Hamiltonian mechanics uses time as a paramter, i.e. $\underline{R} = \underline{R}(t)$ and $\underline{P} = \underline{P}(t)$;
- Time, or a quantity directly related to time such as the path length, s, is therefore a paramter. In addition, E and $\boldsymbol{p}$ are related by the invariant mass of particle.

Para 3:

- Phase space is the position of the particle in coordinate and and momentun space;
- Six phase-space coordinates are required and are usually to taken to be $\boldsymbol{r}$ and $\boldsymbol{p}$;
- This section defines the 6-dimensional spaces used to describe particle phase space in the linear optics code.

3.1 Phase space

The 6D phase-space vector is defined in terms of the three-vector position and three vector momentum as:

$$
\begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \end{bmatrix}
\tag{8}
$$

The trajectory of the particle may be evaluated as a function of time or s.

3.2 Trace space

Trace space is defined to simplify the calculation of the trajectory of particles through the accelerator lattice and is derived from the phase space expressed in the RPLC frame. Consider a particle with position $(x_{\rm RPLC}, y_{\rm RPLC}, z_{\rm RPLC})$ and momentum $p_{\rm RPLC}$ with components $(p_{\rm x\,RPLC}, p_{\rm y\,RPLC}, p_{\rm z\,RPLC})$. Taking the momentum of the reference particle in the laboratory frame to be p_0, the trace-space coordinates are given by:

$$\boldsymbol{\phi} = \begin{pmatrix} x_{\rm RPLC} \\ x'_{\rm RPLC} \\ y_{\rm RPLC} \\ y'_{\rm RPLC} \\ z_{\rm RPLC} \\ \delta_{\rm RPLC} \end{pmatrix} ; \tag{9}$$

where:

$$x'_{\rm RPLC} = \frac{\partial x}{\partial s} = \frac{c p_{x\,\rm RPLC}}{c p_0} ; \tag{10}$$

$$y'_{\rm RPLC} = \frac{\partial y}{\partial s} = \frac{c p_{y\,\rm RPLC}}{c p_0} ; \tag{11}$$

$$z_{\rm RPLC} = \frac{s}{\beta_0} - ct = \frac{\Delta s}{\beta_0} ; \tag{12}$$

$$\delta_{\rm RPLC} = \frac{E}{c p_0} - \frac{1}{\beta_0} = \frac{\Delta E}{c p_0} ; \text{and} \tag{13}$$

Δs and ΔE are the differences between the reference particle position, s_0, and its energy, E_0, and the position and energy of the test particle, s and E respectively. Δs and ΔE are given by $\Delta s = s - s_0$ and $\Delta E = E - E_0$.

4 Transfer matrices

Description of beam transport often carried out by:

- Breaking lattice down into a series of "elements", e.g. drift, quadrupole, dipole, solenoid, etc.;
- Transport of particle between the start and end of a particular element of the lattice linearised such that the trace space at the end of the element, $\boldsymbol{\phi}_f$, is written in terms of the trace space at the start of the element, $\boldsymbol{\phi}_i$:

$$\boldsymbol{\phi}_f = \underline{\underline{T}}\, \boldsymbol{\phi}_i \,; \text{and} \tag{14}$$

- The distance along the reference particle trajectory is increased by δs, where:

$$s_{\rm end} = s_{\rm start} + \delta s \,; \tag{15}$$

  and $s_{\rm start}$ and $s_{\rm end}$ are the length of the reference particle trajectory at the start and end of the beam-line element respectively.

Many excellent descriptions of derivation of transfer matrices, $\underline{\underline{T}}$, so only quote results here.

4.1 Drift

A "drift" space refers to a region in which the beam propagates in the absence of any electromagnetic fields. In a drift, particles propagate in straight lines, therefore:

$$\underline{\underline{T}}_{\rm drift} = \begin{pmatrix} 1 & l & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & l & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{l}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; \tag{16}$$

where l is the length of the drift. The increment in the reference particle trajectory is:

$$\delta s = l \, . \tag{17}$$

4.2 Quadrupole

The passage of a beam particle through a quadrupole magnet may be described by specifying the field gradient, g, within the magnet and the length, l_q, of the quadrupole measured along its axis. The impact of a quadrupole on the trajectory of a particle in the xy plane is independent of the impact of the magnet on the particle's trajectory in the yz plane. In this sense quadrupole focusing in the xz and yz planes is said to be "uncoupled".

If the field gradient along the x and y axes is identical, then:

$$g_x = \frac{\partial B_{qx}}{\partial x} = g_y = \frac{\partial B_{qy}}{\partial y} = g \, ; \tag{18}$$

where the field in the quadrupole, $\boldsymbol{B_q}$, has components $(B_{qx}, B_{qy}, 0)$.

In the "hard-edge" approximation, where the field falls to zero at the start and end of the quadrupole, the transfer matrix for a quadrupole focusing in the xz plane (a "focusing quadrupole") may be written:

$$\underline{\underline{T}}_{\rm Fquad} = \begin{pmatrix} \cos(\sqrt{k_q}l_q) & \frac{\sin(\sqrt{k_q}l_q)}{\sqrt{k_q}} & 0 & 0 & 0 & 0 \\ -\sqrt{k_q}\sin(\sqrt{k_q}l_q) & \cos(\sqrt{k_q}l_q) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{k_q}l_q) & \frac{\sinh(\sqrt{k_q}l_q)}{\sqrt{k_q}} & 0 & 0 \\ 0 & 0 & \sqrt{k_q}\sinh(\sqrt{k_q}l_q) & \cosh(\sqrt{k_q}l_q) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{l_q}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; \tag{19}$$

where:

$$k_q = \frac{gc}{p} \times 10^{-3} \, , \tag{20}$$

and c is the speed of light in metres per second, p is the magnitude of the momentum of the particle in MeV/c, and the field gradient, g, is given in T/m. As before, β_0 is the relativistic velocity of the reference particle and $\gamma_0 = (1 - \beta_0^2)^{-\frac{1}{2}}$. The increment in the reference particle trajectory is:

$$\delta s = l_q \, . \tag{21}$$

It is important to include a description of the effect of dispersion on focusing in the LhARA beam line since the laser-driven proton and ion source provides a broad energy spectrum. Reference [1] provides two methods for the description of dispersion in a linear approximation. The first is to use the reference momentum to calculate the quadrupole focusing strength ($k_{0q} = \frac{gc}{p_0} \times 10^{-3}$) and to include terms in the expressions for x, x', y, and y' dependent on δ. The second is to use equation 20 to calculate the effective quadrupole focusing strength. The second approach has been adopted here.

In the same notation, the transfer matrix for a quadrupole focusing in the yz plane (a "defocusing quadrupole") may be written:

$$\underline{\underline{T}}_{\text{Dquad}} = \begin{pmatrix} \cosh(\sqrt{k_q} l_q) & \frac{\sinh(\sqrt{k_q} l_q)}{\sqrt{k_q}} & 0 & 0 & 0 & 0 \\ \sqrt{k_q}\sinh(\sqrt{k_q} l_q) & \cosh(\sqrt{k_q} l_q) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\sqrt{k_q} l_q) & \frac{\sin(\sqrt{k_q} l_q)}{\sqrt{k_q}} & 0 & 0 \\ 0 & 0 & -\sqrt{k_q}\sin(\sqrt{k_q} l_q) & \cos(\sqrt{k_q} l_q) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{l_q}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{22}$$

4.3 Solenoid

The passage of a beam particle through a solenoid is determined by the magnetic field strength, $\boldsymbol{B}_s$, within the solenoid and the length of the solenoid, l, measured along its axis. As the particle enters the solenoid, the fringe fields impart momentum transverse to the axis of the magnet. This results in the particle executing a helical trajectory, the axis of the helix being parallel to the solenoid axis. The sense of the rotation depends on the particle charge and the polarity of the field. The helical motion means that the evolution of the particle motion in the xz plane is coupled with the evolution of the particle motion in the yz plane.

In the "hard-edge" approximation, the magnetic field inside the magnet is given by $\boldsymbol{B}_s = (0, 0, B_{s0})$, where the solenoid axis lies along the z_{RPLC} axis. The solenoid field strength parameter is then given by:

$$k_s = \left[\frac{B_{s0} c}{2p} \times 10^{-3} \right]^2 ; \tag{23}$$

where B_{s0} is measured in T, p in MeV/c and c in m/s.

The transfer matrix for passage of a positive particle through a solenoid with field pointing in the positive z_{RPLC} direction may may be written:

$$\underline{\underline{T}}_{\text{Sol}} = \begin{pmatrix} \cos^2(\sqrt{k_s} l) & \frac{1}{2\sqrt{k_s}}\sin(\sqrt{k_s} l) & \frac{1}{2}\sin(2\sqrt{k_s} l) & \frac{1}{\sqrt{k_s}}\sin^2(\sqrt{k_s} l) & 0 & 0 \\ -\frac{\sqrt{k_s}}{2}\sin(2\sqrt{k_s} l) & \cos^2(\sqrt{k_s} l) & -\sqrt{k_s}\sin^2(\sqrt{k_s} l) & \frac{1}{2}\sin(2\sqrt{k_s} l) & 0 & 0 \\ -\frac{1}{2}\sin(2\sqrt{k_s} l) & -\frac{1}{\sqrt{k_s}}\sin^2(\sqrt{k_s} l) & \cos^2(\sqrt{k_s} l) & \frac{1}{2\sqrt{k_s}}\sin(2\sqrt{k_s} l) & 0 & 0 \\ \sqrt{k_s}\sin^2(\sqrt{k_s} l) & -\frac{1}{2}\sin(2\sqrt{k_s} l) & -\frac{\sqrt{k_s}}{2}\sin(2\sqrt{k_s} l) & \cos^2(\sqrt{k_s} l) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{l}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{24}$$

As in the case of the quadrupoles, dispersion is accounted for using equation 23.

4.4 Non-neutral (electron) plasma (Gabor) lens

A dense gas of electrons confined in a Penning-Malmberg trap provides an electric field that can be used to focus a positive ion beam. The electron gas is confined axially in the lens by an electostatic potential created using a central anode of length l. The gas is confined radially using the uniform field of a solenoid. Assuming a uniform electron density, n_e, the focusing parameter, k_G, may be written:

$$k_G = \frac{e}{2\epsilon_0} \frac{m_p \gamma}{p^2} n_e ; \tag{25}$$

where e is the charge on the electron, ϵ_0 is the permittivity of free space, and m_p is the proton mass. As in the case of the quadrupoles and solenoid, dispersion is accounted for using equation 25. The force on a particle passing through the electron gas is towards the axis of the lens and is proportional to the radia distance of the particle from the axis. Focusing is therefore cylindrically symmetric and does not couple motion in the the xz and yz planes.

In the "hard-edge" approximation, the electric field inside the lens falls to zero at the end of the electron gas and the contribution of the magnetic field used to confine the electron gas in the transverse direction has a negligible effect on particles passing through hthe lens. The transfer matrix for passage of a positive particle through the lens may be written:

$$\underline{\underline{T}}_G = \begin{pmatrix} \cos(\sqrt{k_G}l) & \frac{\sin(\sqrt{k_G}l)}{\sqrt{k_G}} & 0 & 0 & 0 & 0 \\ -\sqrt{k_G}\sin(\sqrt{k_G}l) & \cos(\sqrt{k_G}l) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\sqrt{k_G}l) & \frac{\sin(\sqrt{k_G}l)}{\sqrt{k_G}} & 0 & 0 \\ 0 & 0 & -\sqrt{k_G}\sin(\sqrt{k_G}l) & \cos(\sqrt{k_G}l) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{l}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{26}$$

5 Source

Para 1:

- A variety of options for generating the particle distribution at source are included in the package;
- The principle (default) option is the model of the TNSA mechanism presented in [2].

Para 2:

- The energy spectrum of the protons per unit energy and unit solid angle is given by:

$$\frac{dN}{dE} = \frac{n_{i0}c_s t}{(2EE_0)^{\frac{1}{2}}} \exp\left[-\left(\frac{2E}{E_0} \right)^{\frac{1}{2}} \right] ; \tag{27}$$

  where n_{i0} is the ion number density at $t = 0$, the instant the laser pulse strikes the target; c_s is given by:

$$c_s = \left[\frac{Zk_{\rm B}T_e}{m_i} \right]^{\frac{1}{2}} ; \tag{28}$$

  where Z is the ion charge number, $k_{\rm B}$ is the Boltzmann constant; T_e is the electron temperature; m_i is the ion mass; t is the instant in time at which the spectrum is evaluated; and

$$E_0 = \left[\frac{n_{e0}k_{\rm B}T_e}{\epsilon_0} \right]^{\frac{1}{2}} \tag{29}$$

Para 3:

- To generate the proton spectrum at the source, a practical approach is taken. The leading behaviour of equation 27 is taken to be:

$$\frac{dN}{dE} = \Gamma \frac{\exp\left[-E^{\frac{1}{2}} \right]}{E^{\frac{1}{2}}} ; \tag{30}$$

- Normalisation of equation 30 between a minimum energy ($E_{\rm min}$) and a maximum energy ($E_{\rm max}$) is used to determine the constant Γ. Both $E_{\rm min}$ and $E_{\rm max}$ are user defined and set at the initialisation stage.

References

[1] A. Wolski, *Beam dynamics in high energy particle accelerators*. Imperial College Press, 57 Shelton Street, Covent Garden, London WC2H 9HE, 2014.

[2] P. Mora, "Plasma Expansion into a Vacuum," *Phys. Rev. Lett.* **90** (May, 2003) 185002. https://link.aps.org/doi/10.1103/PhysRevLett.90.185002.

A Set-up and run

Introduction

This section summarises the steps needed to set-up and run the linear optics simulation of the LhARA beam line. A summary of the tasks that the software suite performs will be documented in due course. The code has been developed in python; python 3 is assumed.

Getting the code

The linear optics package is maintained using the GitHub version-control system. The latest release can be downloaded from:

```
\centerline{
  \href{https://github.com/ImperialCollegeLondon/LhARAlinearOptics.git}{https:///c
}
```

Dependencies and required packages

The linear optics code requires the following packages:

- Python modules: `scipy` and `matplotlib`.

It may be convenient to run the package in a "virtual environment". To set this up, after updating your python installation to python 3.9, execute the following commands:

1. `python3 -m venv --system-site-packages venv`
   - This creates the director `venv` that contains files related to the virtual environment.
2. `source venv/bin/activate`
3. `python -m pip install pandas scipy matplotlib`

To exit from the virtual environment, execute the command `deactivate`. The command `source venv/bin/activat` places you back into the virtual environment.

The Imperial HEP linux cluster provides python 3.9.18 by default.

Unpacking the code, directories, and running the tests

After downloading the package from GitHub, or cloning the repository, you will find a "`README.md`" file which provides some orientation and instructions to run the code. In particular, a `bash` script "`startup.bash`" is provided which:

- Sets the "`LhARAOpticsPATH`" environment variable so that the files that hold constants etc. required by the code can be located; and
- Adds "`01-Code`" (see below) to the PYTHONPATH. The scripts in "`02-Tests`" (see below) may then be run with the command "`python 02-Tests/<filename>.py`".

Below the top directory, the directory structure in which the code is presented is:

`01-Code`: contains the python implementation as a series of modules. Each module contains a single class or a related set of methods.

`02-Tests`: contains self-contained test scripts that run the various methods and simulation packages defined in the code directory.

`11-Parameters`: contains the parameter set used to specifiy the various beam lines presently implemented.

The instructions in the `README.md` file should be followed to set up and run the code.

Running the code

Execute ”startup.bash” from the top directory (i.e. run the bash command ”source startup.bash”). This will:

- Set up ”LhARAOpticsPATH”; and
- Add ”01-Code” to the PYTHONPATH. The scripts in ”02-Tests” may then be run with the command ”python 02-Tests/<filename>.py”;
- Example scripts are provided in ”03-Scripts”, these can be used first to “Run” the simulation and then to “Read” the data file produced. Example scripts are provided for the DRACO, LION, and LhARA Stage 1 beam lines.