# 1 Unseen questions

Recall that one can define the real numbers $\mathbb{R}$ as an ordered field that has the **least upper bound** property and contains $\mathbb{Q}$ (the rationals). The fact that such a field exists is not obvious.

In the following series of exercises we will construct $\mathbb{R}$ and show it has the least upper bound property.

### 1.0.1 Definitions

- An **totally ordered set** is a set $S$ with a binary relation $<$ that is transitive and $\forall x, y \in S$, exactly one of the following is true

$$x < y, \; x = y, \; y < x.$$

  For example, $\mathbb{Q}$ is ordered if we say $x < y$ whenever $y - x$ is a positive rational number.

- Let $(S, <)$ be an ordered set and $E \subseteq S$ a subset. We say $E$ is **bounded above** (in $S$) if $\exists \beta \in S$ such that $x \leq \beta \; \forall x \in E$. We call $\beta$ an **upper bound** of $E$. Lower bounds are defined similarly.

- Let $(S, <)$ be an ordered set and $E \subseteq S$ a subset. We say $\alpha \in S$ is a **least upper bound/supremum of** $E$, and write $\alpha = \sup(E)$, if $\alpha$ is an upper bound of $E$ *and* $\forall \gamma \in S$ such that $\gamma < \alpha$, $\gamma$ is not an upper bound of $E$. Intuitively this definition means exactly what it says; $\alpha$ is an upper bound and it is the smallest possible upper bound. We can define greatest lower bounds/infimums similarly.

- Let $(S, <)$ be an ordered set. We say $(S, <)$ has the **least upper bound property** if for any subset $E \subseteq S$ such that $E \neq \emptyset$ and $E$ is bounded above (i.e. $E$ has an upper bound), then $\exists \sup(E) \in S$, i.e. $E$ has a least upper bound.

The reason why we don't go into detail about greatest lower bounds is that a set has the least upper bound property *if and only if* it has the greatest lower bound property. You can prove this yourself if you want a warm-up exercise.

### 1.0.2 Problems

1. In this question we construct the reals using Dedekind cuts. A reference for this section is the first chapter of Principals of Mathematical Analysis by Walter Rudin ($3^{rd}$ edition). All the answers are in there so don't look unless you want spoilers.

   (a) A subset $\alpha \subseteq \mathbb{Q}$ is called a **cut** if (i) $\alpha \neq \emptyset, \mathbb{Q}$, (ii) if $p \in \alpha, q \in \mathbb{Q}$ and $q < p$ then $q \in \alpha$, and (iii) if $p \in \alpha$ then $p < r$ for some $r \in \alpha$. Give an example of a cut. Prove that if $p \in \alpha, q \notin \alpha$, then $p < q$. Prove if $r \notin \alpha$ and $r < s$ then $s \notin \alpha$.

   (b) Define $\alpha < \beta$ if $\alpha$ is a strict subset of $\beta$. Show this is an total ordering on cuts.

   (c) Define $\mathbb{R}$ (as a set) as the set of all cuts. Prove this set has the least upper bound property.

   (d) Define a binary relation $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and show it satisfies the desired properties of addition. What is the identity element $0_{\mathbb{R}}$? Once you've done this, show that if $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

   (e) Multiplication is a bit harder so we start by defining it on the positive reals $\mathbb{R}_{>0}$ (what does this mean? the set of cuts such that $0_{\mathbb{R}} < \alpha$). Define a binary relation $\tilde{\times} : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and show it satisfies the desired properties of multiplication. What is the identity element $1_{\mathbb{R}}$?

   (f) Extend your multiplication map to a map $\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and show it satisfies the desired properties. Prove the distributive law: $\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma$. Hint: split your definition into cases, same with the proofs.

(g) Conclude that at this point we have shown $\mathbb{R}$ is an totally ordered field (c.f. definition 1.17 in chapter 1 of Rudin) that has the least upper bound property.

(h) For each $r \in \mathbb{Q}$, define the cut $\alpha_r = \{x \in \mathbb{Q} : x < r\}$ (convince yourself this is a cut). Prove that $\alpha_p + \alpha_q = \alpha_{p+q}$, $\alpha_p \times \alpha_q = \alpha_{pq}$, and $\alpha_p < \alpha_q$ if and only if $p < q$. Conclude that $\tilde{\mathbb{Q}} = \{\alpha_r : r \in \mathbb{Q}\}$ is an ordered subfield of $\mathbb{R}$.

(i) Convince yourself that there is an *isomorphism of ordered fields* $\mathbb{Q} \xrightarrow{\sim} \tilde{\mathbb{Q}}$ of ordered fields. Conclude that $\mathbb{Q}$ embeds as an ordered subfield of $\mathbb{R}$.

2. Recall that a sequence $(a_n)$ is Cauchy if it satisfies the following condition:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall m, n \geq N, |a_n - a_m| < \epsilon$$

We can use this to construct the real numbers. The idea is that every Cauchy sequence of rational numbers converges to a real number, and any real number has at least one Cauchy sequence that converges to it. Lets define $S := \{(a_n) : \forall n \in \mathbb{N}, a_n \in \mathbb{Q}\}$

(a) First we need to define an equivalence relation on $S$, so that $(a_n) \sim (b_n)$ if and only if $(a_n)$ and $(a'_n)$ converge to the same limit $c$. But $c$ may be in $\mathbb{R}$, so we can't use it in the definition of $\sim$. How can we define $\sim$? Make sure this is an equivalence relation.

(b) Come up with a definition of $+$ on $S$, and make sure that if $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$, then $(a_n) + (b_n) \sim (a'_n) + (b'_n)$.

(c) Do the same for $\times$

(d) Now let us define $R$ to be the set of equivalence classes $\{[(a_n)]_\sim : (a_n) \in S\}$. We have shown that $+$ and $\times$ is compatible with $\sim$, and it follows easily that $R$ satisfies the axioms of $+$ and $\times$ over the reals. How can we define $<$ over $R$? Then prove that $R$ is totally ordered.

(e) Now show that $R$ satisfies the least upper bound property.
Hint: for any $q \in \mathbb{Q}$, the equivalence class of the sequence $(q, q, q, \cdots)$ is in $R$. We can abuse notation by writing $q \in R$. First show that for any $\alpha \in R$ there exist $l, u \in \mathbb{Q}$ such that $l < \alpha < u$. We can then construct a sequence of upper bounds $(u_n)$, and "not upper bounds" $(l_n)$ such that $(u_n) \sim (l_n)$. The equivalence class of this will be our least upper bound.

3. Using the completeness of the reals, show that for $d > 0$, $\sqrt{d}$ exists.