Rational powers.

Raising things to powers a^b is a slightly problematic issue. For example consider equation (‡) below:

$$-1 = (-1)^1 = (-1)^{\frac{1}{2}+\frac{1}{2}} = (-1)^{\frac{1}{2}} \times (-1)^{\frac{1}{2}} = ((-1)\times(-1))^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1. \qquad (‡)$$

This argument is a "proof" that $-1 = 1$.

Here's the problem. Very early on in life you are told what x^a means *when a is a positive integer*. It means the obvious thing: $x^1 = x$, $x^2 = x \times x$, $x^3 = x \times x \times x$ and so on. One can then go on to prove the following things:

1) $x^a \times x^b = x^{a+b}$;
2) $(x^a)^b = x^{ab}$;
3) $x^a \times y^a = (xy)^a$.

If x is furthermore assumed to be non-zero, then we can even define x^a for $a = 0$ or a a negative integer: we set $x^0 = 1$, $x^{-1} = 1/x$, $x^{-2} = 1/(x^2)$ and so on. We can then check that facts (1), (2) and (3) still hold for $x, y \neq 0$ and $a, b \in \mathbf{Z}$.

On the other hand, if we use facts (1), (2) and (3) for general numbers x, y, a and b then we run into trouble, as the example (‡) at the top shows. This shows that we have to be more careful!

In this project let us assume that we believe the standard definition of x^a and that (1), (2), (3) are true *for x a positive real number and for a an integer*. Let us also assume the following fact from M1F:

Fact: If $x > 0$ is a positive real number and $n \in \mathbf{Z}_{\geq 1}$ is a positive integer, then there is a unique positive real number y such that $y^n = x$.

Proposed definition.

If $x \in \mathbf{R}_{>0}$ and $n \in \mathbf{Z}_{\geq 1}$ then let's *define* $x^{1/n}$ to be the unique positive real number y such that $y^n = x$. For example, $2^{1/2}$ is, by definition, the unique positive real number y such that $y^2 = 2$, so $2^{1/2} = \sqrt{2}$. Similarly $10^{1/3}$ is the (positive real) cube root of 10 and so on.

Now if $x \in \mathbf{R}_{>0}$ and $q = a/b$ is a rational number with $a, b \in \mathbf{Z}$ and $b > 0$, let's *define* x^q to mean $(x^{1/b})^a$.

Questions to mull over.

Here x, y are always positive reals, and k, ℓ, m, n are integers, and a, b are rational numbers.

Q1) If m/n is not in lowest terms, then we need to be careful – does our definition make sense? Is $x^{2/4}$ definitely equal to $x^{1/2}$? Can you *prove* that x^a is well-defined (in the sense that if $a = m/n = k/\ell$ with $n, \ell > 0$ then $x^{m/n} = x^{k/\ell}$?

Q2) Can you prove (1), (2), (3) above if $a, b \in \mathbf{Q}$?

Q3) Is (‡) now a valid proof that $-1 = 1$? Why not?

Q4) Can you now define x^r if x is a positive real number and r is any real number? How might you go about trying to do this? What are the problems you might face here? Are there better ways to define x^r? Are (1), (2), (3) true if a, b are real numbers? How might one prove them?