

# 1 Unseen questions

Recall that one can define the real numbers  $\mathbb{R}$  as an ordered field that has the **least upper bound** property and contains  $\mathbb{Q}$  (the rationals). The fact that such a field exists is not obvious.

In the following series of exercises we will construct  $\mathbb{R}$  and show it has the least upper bound property.

## 1.0.1 Definitions

- An **totally ordered set** is a set  $S$  with a binary relation  $<$  that is transitive and  $\forall x, y \in S$ , exactly one of the following is true

$$x < y, \quad x = y, \quad y < x.$$

For example,  $\mathbb{Q}$  is ordered if we say  $x < y$  whenever  $y - x$  is a positive rational number.

- Let  $(S, <)$  be an ordered set and  $E \subseteq S$  a subset. We say  $E$  is **bounded above** (in  $S$ ) if  $\exists \beta \in S$  such that  $x \leq \beta \quad \forall x \in E$ . We call  $\beta$  an **upper bound** of  $E$ . Lower bounds are defined similarly.
- Let  $(S, <)$  be an ordered set and  $E \subseteq S$  a subset. We say  $\alpha \in S$  is a **least upper bound/supremum of  $E$** , and write  $\alpha = \sup(E)$ , if  $\alpha$  is an upper bound of  $E$  and  $\forall \gamma \in S$  such that  $\gamma < \alpha$ ,  $\gamma$  is not an upper bound of  $E$ . Intuitively this definition means exactly what it says;  $\alpha$  is an upper bound and it is the smallest possible upper bound. We can define greatest lower bounds/infimums similarly.
- Let  $(S, <)$  be an ordered set. We say  $(S, <)$  has the **least upper bound property** if for any subset  $E \subseteq S$  such that  $E \neq \emptyset$  and  $E$  is bounded above (i.e.  $E$  has an upper bound), then  $\exists \sup(E) \in S$ , i.e.  $E$  has a least upper bound.

The reason why we don't go into detail about greatest lower bounds is that a set has the least upper bound property *if and only if* it has the greatest lower bound property. You can prove this yourself if you want a warm-up exercise.

## 1.0.2 Problems

1. In this question we construct the reals using Dedekind cuts. A reference for this section is the first chapter of *Principals of Mathematical Analysis* by Walter Rudin (3<sup>rd</sup> edition). All the answers are in there so don't look unless you want spoilers.
  - (a) A subset  $\alpha \subseteq \mathbb{Q}$  is called a **cut** if (i)  $\alpha \neq \emptyset, \mathbb{Q}$ , (ii) if  $p \in \alpha, q \in \mathbb{Q}$  and  $q < p$  then  $q \in \alpha$ , and (iii) if  $p \in \alpha$  then  $p < r$  for some  $r \in \alpha$ . Give an example of a cut. Prove that if  $p \in \alpha, q \notin \alpha$ , then  $p < q$ . Prove if  $r \notin \alpha$  and  $r < s$  then  $s \notin \alpha$ .
  - (b) Define  $\alpha < \beta$  if  $\alpha$  is a strict subset of  $\beta$ . Show this is a total ordering on cuts.
  - (c) Define  $\mathbb{R}$  (as a set) as the set of all cuts. Prove this set has the least upper bound property.
  - (d) Define a binary relation  $+$  :  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and show it satisfies the desired properties of addition. What is the identity element  $0_{\mathbb{R}}$ ? Once you've done this, show that if  $\alpha, \beta, \gamma \in \mathbb{R}$  and  $\beta < \gamma$ , then  $\alpha + \beta < \alpha + \gamma$ .
  - (e) Multiplication is a bit harder so we start by defining it on the positive reals  $\mathbb{R}_{>0}$  (what does this mean? the set of cuts such that  $0_{\mathbb{R}} < \alpha$ ). Define a binary relation  $\tilde{\times}$  :  $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$  and show it satisfies the desired properties of multiplication. What is the identity element  $1_{\mathbb{R}}$ ?
  - (f) Extend your multiplication map to a map  $\times$  :  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and show it satisfies the desired properties. Prove the distributive law:  $\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma$ . Hint: split your definition into cases, same with the proofs.

- (g) Conclude that at this point we have shown  $\mathbb{R}$  is an totally ordered field (c.f. definition 1.17 in chapter 1 of Rudin) that has the least upper bound property.
- (h) For each  $r \in \mathbb{Q}$ , define the cut  $\alpha_r = \{x \in \mathbb{Q} : x < r\}$  (convince yourself this is a cut). Prove that  $\alpha_p + \alpha_q = \alpha_{p+q}$ ,  $\alpha_p \times \alpha_q = \alpha_{pq}$ , and  $\alpha_p < \alpha_q$  if and only if  $p < q$ . Conclude that  $\tilde{\mathbb{Q}} = \{\alpha_r : r \in \mathbb{Q}\}$  is an ordered subfield of  $\mathbb{R}$ .
- (i) Convince yourself that there is an *isomorphism of ordered fields*  $\mathbb{Q} \xrightarrow{\sim} \tilde{\mathbb{Q}}$  of ordered fields. Conclude that  $\mathbb{Q}$  embeds as an ordered subfield of  $\mathbb{R}$ .

2. Recall that a sequence  $(a_n)$  is Cauchy if it satisfies the following condition:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall m, n \geq N, |a_n - a_m| < \epsilon$$

We can use this to construct the real numbers. The idea is that every Cauchy sequence of rational numbers converges to a real number, and any real number has at least one Cauchy sequence that converges to it. Lets define  $S := \{(a_n) : (a_n) \text{ is Cauchy and } \forall n \in \mathbb{N}, a_n \in \mathbb{Q}\}$

- (a) First we need to define an equivalence relation on  $S$ , so that  $(a_n) \sim (a'_n)$  if and only if  $(a_n)$  and  $(a'_n)$  converge to the same limit  $c$ . But  $c$  may be in  $\mathbb{R}$ , so we can't use it in the definition of  $\sim$ . How can we define  $\sim$ ? Make sure this is an equivalence relation.
- (b) Come up with a definition of  $+$  on  $S$ , and make sure that if  $(a_n) \sim (a'_n)$  and  $(b_n) \sim (b'_n)$ , then  $(a_n) + (b_n) \sim (a'_n) + (b'_n)$ .
- (c) Do the same for  $\times$
- (d) Now let us define  $R$  to be the set of equivalence classes  $\{[(a_n)]_\sim : (a_n) \in S\}$ . We have shown that  $+$  and  $\times$  is compatible with  $\sim$ , and it follows easily that  $R$  satisfies the axioms of  $+$  and  $\times$  over the reals. How can we define  $<$  over  $R$ ? Then prove that  $R$  is totally ordered.
- (e) Now show that  $R$  satisfies the least upper bound property.  
Hint: for any  $q \in \mathbb{Q}$ , the equivalence class of the sequence  $(q, q, q, \dots)$  is in  $R$ . We can abuse notation by writing  $q \in R$ . First show that for any  $\alpha \in R$  there exist  $l, u \in \mathbb{Q}$  such that  $l < \alpha < u$ . We can then construct a sequence of upper bounds  $(u_n)$ , and "not upper bounds"  $(l_n)$  such that  $(u_n) \sim (l_n)$ . The equivalence class of this will be our least upper bound.

3. Using the completeness of the reals, show that for  $d > 0$ ,  $\sqrt{d}$  exists.