## M1F Foundations of Analysis, Problem Sheet 6.

**1.** We all know that Most Hard Chair is an anagram of Richard Thomas. But how many permutations of RICHARDTHOMAS are there in total? Of course you can include permutations which are not words, and we don't care about spaces here. Either use a calculator or leave your answer as a multinomial coefficient.

**2.** (i) Find the coefficient of $x^{19}$ in $(1+x)^{21}$.
(ii) Find the coefficient of $x$ in $(x^3+1/x)^7$.
(iii) Find the coefficient of $x^6$ in $(1+3x+x^2)^5$.

**3.** Say $S \subseteq \mathbf{R}$ is a set of real numbers, with the property that $\forall s \in S, \exists t \in S, s < t$. Can $S$ be bounded above?

**4*.** For each of the following non-empty sets $S$, figure out whether or not they are bounded above. For those that are bounded above, figure out what the least upper bound is. Full proofs required!
a) $S = (-\infty, 0)$
b) $S = \mathbf{Q}$
c) $S = \{x \in \mathbf{R} : (x+1)^2 < x^2\}$
d) $S = \{x \in \mathbf{Q} : 1 < x < 2\}$

**5.** Say $S \subset \mathbf{R}$, and $S$ has an upper bound $x \in \mathbf{R}$ with the property that $x \in S$. Prove that $x$ is the least upper bound for $S$.

**6.** If $S$ is a set of real numbers, we say $S$ is *bounded below* if there exists some $x \in \mathbf{R}$ with $x \leq s$ for all $s \in S$; such an $x$ is called a *lower bound* for $S$; we say $z \in \mathbf{R}$ is a *greatest lower bound* (GLB) for $S$ if $z$ is a lower bound for $S$ and furthermore that if $y \in \mathbf{R}$ is any lower bound then $z \geq y$.

a) Prove that $S$ is bounded below if and only if $-S := \{-s : s \in S\}$ is bounded above. Then prove that $x$ is a greatest lower bound for $S$ if and only if $-x$ is a least upper bound for $-S$.

b) Prove that if $x_1$ and $x_2$ are both greatest lower bounds for $S$, then $x_1 = x_2$.

c) Assuming that any non-empty bounded-above set of reals has a LUB, prove that any non-empty bounded-below set of reals has a GLB.

**7.** Say we have a sequence of real numbers $a_1$, $a_2$, $a_3, \ldots$, which is bounded above in the sense that there exists some real number $B$ such that $a_i \leq B$ for all $i$.

Now let's define some sets $S_1$, $S_2$, $S_3, \ldots$ by

$$S_n = \{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$

For example $S_3 = \{a_3, a_4, a_5, \ldots\}$.

a) Prove that for all $n \geq 1$, $S_n$ is a non-empty set which is bounded above, and hence has a least upper bound $b_n$.

b) Prove that $b_{n+1} \leq b_n$ and hence $b_1, b_2, b_3$ is a decreasing sequence.

If the set $\{b_1, b_2, b_3, \ldots\}$ is bounded *below*, then its greatest lower bound $\ell$ is called the *limsup* of the sequence $(a_1, a_2, a_3, \ldots)$ (this is an abbreviation for Limit Superior).

c) Find the limsup of the following sequences (they do exist).
i) $1, 1, 1, 1, 1, \ldots$
ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
iii) $0, 1, 0, 1, 0, 1, 0, 1, \ldots$

d) If you like, then guess the definition of *liminf* (Limit Inferior) and compute it for examples (i) to (iii) of (c) above. Which of these sequences converges? (we will see a rigorous definition of this notion next term, but I think you probably know what it means). Can you tell just from looking at the limsup and liminf?