KMB, 21/10/18

M1F Foundations of Analysis, Problem Sheet 3.

1. Using only the 4 standard $<$ assumptions (A1) to (A4) from lectures (and any equality facts about $+, -, \times$ etc that you need), write down a proof that if $0 < x$ and $0 < y$ then $0 < x + y$. Hint: you'll only need to use two of them.

2†. This question is a little harder and longer than the others, and so I've put a dagger by it. This question continues the development of the basic facts about inequalities for real numbers.

We say that a real number x is *positive* if $x > 0$ and *negative* if $x < 0$. Recall that (A3) says that every real number is exactly one of: positive, negative, or zero, and (A4) says that the product of two positive numbers is positive.

a) We proved in lectures that if $x > y$ and $c > 0$ then $cx > cy$. Deduce from this that the product of a positive number and a negative number is negative.

b) We showed in lectures that if $x < 0$ then $-x > 0$. Using this (and standard facts such as $(-1)^2 = 1$ and so on), prove that the product of two negative real numbers is positive.

c) Deduce if x and y are real numbers, and $xy = 0$, then either $x = 0$ or $y = 0$. Hint: rule out all other possibilities using the previous parts.

d) For this part you may assume that if $x > 0$ is a real number, then there is a unique positive real number $y > 0$ such that $y^2 = x$. Prove that there are exactly two real numbers z such that $z^2 = x$, and figure out what they are. Hint: you can use (c), but it's possible to do this part without it.

3. In this question you may *assume* that if $x > 0$ is real and $n > 0$ is an integer, then there's a unique positive real y, called *the nth root of x*, such that $y^n = x$. You can also assume all standard results about powers such as $\left(a^b\right)^c = a^{bc}$ for $b, c \in \mathbf{Z}_{\geq 1}$ and so on.

a) Which is bigger, the three trillionth root of 3 or the two trillionth root of 2? Remark: a trillion is 10^{12}. Hint: if $0 < x < y$ then $0 < x^n < y^n$ for $n \geq 1$ (you can assume this, or you can prove it by induction if you know about induction).

b) Which is bigger, 100^{10000} or 10000^{100}?

c) What's the square root of 2^{22}? What's the square root of $2^{2^{22}}$?

4*. Find the set of non-zero real numbers x such that $3x + \frac{1}{x} < 4$. Hint: *be careful.* It is *not* true that if $a < b$ then $xa < xb$: this is only true for $x > 0$. Note that it's easy to deduce from Q1 the standard facts such as the product of three negative numbers is negative etc – just assume all this too.

5. Let's define the *absolute value* $|x|$ of a real number x by $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

a) Prove that if t is a positive real number then $|x| < t$ if and only if $-t < x < t$.

b) Find all real numbers x such that $|x + 1| < 3$.

c) Find all real numbers x such that $|x - 2| < |x - 4|$.

6. Say $p = a + ib$ and $q = c + id$ and $r = e + if$ are complex numbers.

a) Prove from first principles that $(p+q)+r = p+(q+r)$. You may assume that the analogous fact holds for real numbers – what I'm asking is how to deduce this fact for complex numbers.

This fact ("it doesn't matter which + you do first when you add three things up"), has a proper fancy name, which is "associativity of addition".

b) Prove from first principles that $pq = qp$ (again you may assume that the analogous fact ("it doesn't matter which order you multiply numbers together") is is true for real numbers, and again it has a fancy name, namely "commutativity of multiplication").

c) Recall that the *complex conjugate* of $z = x + iy$ is the complex number $\overline{z} = x - iy$. Prove from first principles that $\overline{p}\,\overline{q} = \overline{pq}$.

7. Prove from first principles that if $z = x + iy$ and $z^2 = -1$ then $z = i$ or $z = -i$. Hint: Q2(c) and Q2(d) will be helpful.