KMB, 29/10/18

M1F Foundations of Analysis, Problem Sheet 4.

1. I will remind you, before we start on this question, that you're supposed to be discussing the starred parts of the sheet with your personal tutor each week.

a) Let $p = a + ib$ be a complex number. Draw a picture to show that $\overline{p}$ equals the reflection of p in the real axis.

b) By thinking of complex conjugation as reflection, and by thinking of addition of complex numbers as addition of vectors, draw a little picture to convince yourself that for p and q complex numbers, the statement

$$\overline{p+q} = \overline{p} + \overline{q}$$

is obvious.

c) By using the picture you drew in part (a), convince yourself that $\overline{re^{i\theta}} = re^{-i\theta}$.

d) We know from de Moivre that to multiply by $re^{i\theta}$ we first scale by r and then rotate by θ. Convince yourself, thinking geometrically, that

$$\overline{pq} = \overline{p}\,\overline{q}$$

is obvious.

e*) Do you think that your answers to (b) and (d) are rigorous mathematical proofs? Do you think that I think that they are? But do you think that your answers are "mathematics"? What is mathematics?

2. Recall that we showed in lectures that $\cos(3\theta) = 4c^3 - 3c$, where $c = \cos(\theta)$. Here's how we can use this fact to solve a cubic equation! Rather than getting bogged down with $Ax^3+Bx^2+Cx+D = 0$, let me just use numbers; the technique will work in general (kind of...).

a) Pull out your calculator – we're going to find the roots of $3x^3 - 18x^2 + 27x - 4 = 0$ using it. Make sure the cosine button is working (and the inverse cosine button).

b) First we do a linear change of variables to kill the x^2 term. So set $y = x - 2$ (the point being that then $3y^3 = 3x^3 - 18x^2 + \cdots$) and rewrite the cubic equation as a cubic equation in y instead (don't forget the $= 0$ bit, that's an important part of the equation).

c) Now we want to scale y to make that cubic equation look like $4c^3 - 3c + \cdots = 0$, and a bit of playing around should convince you that one way of doing this is by setting $c = y/2$ and then dividing the entire equation by 6.

Spoiler: if you've got it all right so far, you should have

$$4c^3 - 3c + 1/3 = 0.$$

d) Now substitute $c = \cos(\theta)$ and deduce that we want to solve $\cos(3\theta) = -1/3$. Solve this for θ using your calculator and hence work out c, y and then x.

e) Did it work? It did for me, I got $x = 3.6079128829148322904316053206617018144\ldots$ which does seem to be a root.

f) Aren't cubics supposed to have three roots? Can you get all three using this method?

g) Can you solve all quartics this way?

3. What is $\binom{100}{0} - \binom{100}{2} + \binom{100}{4} - \binom{100}{6} + \cdots + \binom{100}{100}$? Hint: do you remember me running into a question like this in lectures? [Whenever I was doing problem sheets like this, I would sometimes just mindlessly page through the notes I'd been taking for the course, looking for inspiration...]

4.

(a) By considering $(1+i)(\sqrt{3}-i)$ or otherwise, prove that $\cos(\pi/12) = \frac{\sqrt{6}+\sqrt{2}}{4}$.

(b) Deduce that $\cos(\pi/12)$ is irrational.

NB for those of you who never got the hang of radians, $\pi/12$ is 15 degrees and I'm trying to get you to use the amazing insight that $45 - 30 = 15$. But honestly, don't be like me, get the hang of radians.

5. (a) Draw a picture of the ten 10th roots of i in the complex plane. Which one is closest to i?

(b) Let z be a non-zero complex number. Prove that the three cube roots of z in the complex plane are at the vertices of an equilateral triangle.