KMB, 20/11/18

M1F Foundations of Analysis, Problem Sheet 7.

1. Here is a cool decimal expansion. Recall that the Fibonacci sequence starts with the terms 1,1 and then each term after that is the sum of the two previous terms, so it goes $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. Now get out your calculator/computer/phone and work out the decimal expansion of $x = 100/9899$ (if you have an Android phone and you install paridroid you'll be able to work it out to thousands of decimal places; probably there are iPhone apps which will do the same). You'll see that it's $0.0101020305081321\cdots$. Note how the Fibonacci sequence lives in this decimal expansion! Can you explain why? Do you think the pattern continues forever? Hint: check $100+x+100x = 10000x$ and then do an informal (i.e. the way you did things at school) decimal expansion calculation.

2. Grab your calculator and work out the highest common factor of 7261 and 10001. If you like, you could try the method that many of you learnt at school (factor everything) – good luck with this (even with a calculator). Alternatively you might want to try Euclid's algorithm (and then consider writing a letter to your old maths teacher if they told you to compute hcf's by factoring both numbers, telling them that they're teaching a syllabus which is 2000 years out of date...).

3.

(a) Set $a = 46$ and $b = 18$; now find the highest common factor d of a and b. You could use the "factor everything" method that many of you learnt at school but you'll then be in trouble in part (b).

(b) Now find integers λ and μ such that $46\lambda + 18\mu = d$.

(c) Now find another solution in integers to $46\lambda + 18\mu = d$, this time with $\lambda > 10^6$.

(d) Now find a solution in integers ρ, σ to $\rho a + \sigma b = 4000000$. Note: ρ is called "rho" and pronounced "row"; it's nothing to do with p.

4*. True or false?

(i) If a and b are positive integers, and there exist integers λ and μ such that $\lambda a + \mu b = 1$, then $\gcd(a, b) = 1$.

(ii) If a and b are positive integers, and there exist integers λ and μ such that $\lambda a + \mu b = 7$, then $\gcd(a, b) = 7$.

5. In the lectures I proved that every positive integer is uniquely the product of prime numbers. Here is a number system where that isn't true.

Let S be the set of positive integers which end in 1, and let's pretend that these are the only positive integers which exist. First convince yourself that if a and b are in S, then so is $a \times b$, so this is a perfectly good system for multiplying, primes, and factoring.

Let's call an element of S a "prim number" if it is bigger than 1 and can't be factored into smaller elements of S. So, for example, 11 and 21 and 31 and 41 and 51 are all prim numbers, because the smallest number greater than 1 in our system is 11, so the smallest non-prim number is $11^2 = 121$.

Find an element of S which is the product of prims in more than one way (even if you count changing the order of the factors as being the same factorization). Hint: think about primes as well as prims. We see 21 is prim because 3 and 7 are not in our system – can we do a similar trick with 13×17, or 13×7...? Can you see where this is going?

Bonus question: can you find an element of S which is equal to the product of two prims, and also the product of three prims?

6. (i) Let a and b be coprime positive integers (recall that *coprime* here means $\gcd(a, b) = 1$). I open a fast food restaurant which sells chicken nuggets in two sizes – you can either buy a box with a nuggets in, or a box with b nuggets in. Prove that there is some integer N with the property that for all integers $m \geq N$, it is possible to buy exactly m nuggets.

(ii) (harder) If $a > 1$ and $b > 1$ then you certainly can't buy 1 nugget. Can you come up with a formula for the *largest* number of nuggets you can't buy?