M1F Foundations of Analysis, Problem Sheet 2, solutions.

1. Recall that one way to check that two sets A and B are equal is: first prove $A \subseteq B$ and then prove $B \subseteq A$.

(a) $\bigcup_{n=0}^{\infty}[n, n+1)$ equals $[0, \infty)$. Why is this? Because $n \geq 0$ in the union, we have $0 \leq n < n+1 < \infty$, so certainly the union is contained within $[0, \infty)$. Conversely if $r \in (0, \infty)$ then there is some integer n such that $n \leq r < n+1$ (we'll prove this in the course; alternatively you might want to argue that it's "obvious" and whether it is or not depends on your viewpoint of what mathematics is). This integer n must be at least zero, as if $n < 0$ then $n \leq -1$, so $n+1 \leq 0$, which implies $r < n+1 \leq 0$, a contradiction. Hence $r \in [n, n+1)$ and $n \geq 0$, so this is in the union on the left hand side.

(b) This union is $(0, 1]$. For if $n \geq 1$ then $1/n > 0$ and hence $[1/n, 1] \subseteq (0, 1]$, so we can deduce that the union is contained within $(0, 1]$. Conversely, if $r > 0$ then we showed in lectures that there's some positive integer n with $0 < 1/n < r$ (or maybe this is "obvious"), and hence $r \in [1/n, 1]$.

(c) This union is all of $\mathbf{R}$. It's clearly contained in $\mathbf{R}$, and conversely if r is any real number and we choose an integer $N > 0$ with $N > r$, and an integer $M > 0$ with $M > -r$, and let n be the maximum of N and M, we have $r < N \leq n$ and $-r < M \leq n$ so $-n < r$, and we conclude $r \in (-n, n)$.

(d) The intersection is just $(-1, 1)$. For if $n \geq 1$ then $-n \leq -1 < 1 \leq n$ and hence $(-1, 1) \subseteq (-n, n)$, meaning that $(-1, 1)$ is contained in the intersection; conversely the intersection is contained in each of the sets in the intersection and in particular within $(-1, 1)$.

2) Informally, we are going to argue that there can be no largest element, because if s is in $(0, 1)$ then the average of s and 1 will be a bit larger. Let me write this down more formally though.

We prove the result by contradiction. Let's assume for a contradiction that s is a largest element of $(0, 1)$. Then let's consider $t := \frac{s+1}{2}$. Because $s \in (0, 1)$ we have $s < 1$, and hence $s + 1 < 2$ so $t = \frac{s+1}{2} < 1$. Because $s > 0$ we have $s + 1 > 0$ and hence $t = \frac{s+1}{2} > 0$. We deduce that $t \in (0, 1)$. Now if s were a largest element of $(0, 1)$ then we must have $t \leq s$, but I claim that in fact $t > s$. For $t - s = \frac{s+1}{2} - \frac{2s}{2} = \frac{1-s}{2} > 0$ because $s < 1$ hence $1 - s > 0$.

3) By contradiction. Let's say 3 divides n^2 but it doesn't divide n. Then the remainder when we divide n by 3 must be 1 or 2, in other words $n = 3m + 1$ or $3m + 2$.

In the first case $n^2 = (3m + 1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1$ is not a multiple of 3.

In the second case $n^2 = (3m + 2)^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$ is also not a multiple of 3.

So in either case we have our contradiction, meaning that if 3 divides n^2 then 3 must divide n.

4) (a) This is false. For example if $a = \sqrt{2}$ and $b = -\sqrt{2}$ then both a and b are irrational, but their sum is zero, which is rational.

(b) This is also false. For example if $a = \sqrt{2}$ and $b = 0$ then $ab = 0$ is rational.

5) (a) This is true. We need to show that if $x \in \mathbf{R}$ is arbitrary, then there exists some $y \in \mathbf{R}$ such that $x + y = 2$, and this is easy: we can just let $y = 2 - x$.

(b) This is not true. The claim is that there is some magical number $y \in \mathbf{R}$ which has the property that whatever real number x we choose, we will have $x + y = 2$. But this cannot be true. Let's prove it by contradiction. Let's assume for a contradiction that such a number y really did exist, and now let's try some values of x. For example let's choose $x = 0$; then we have $y + 0 = 2$ and hence $y = 2$. But now let's choose $x = 1$; then we must have $2 + 1 = y + 1 = 2$, and hence $3 = 2$, a contradiction. So no such y can exist.

(6) Note that $\sqrt{2}$, $\sqrt{6}$ and $\sqrt{15}$ are all positive. Let's prove $\sqrt{2} + \sqrt{6} < \sqrt{15}$ by contradiction. So let's assume

$$\sqrt{2} + \sqrt{6} \geq \sqrt{15}$$

(NB lose a mark for $>$; the opposite of $<$ is $\geq$).

Both sides are positive so we can square both sides and deduce

$$(\sqrt{2}+\sqrt{6}) \geq 15.$$

Now expand out the bracket and tidy up, to get

$$2\sqrt{12} \geq 15-8=7.$$

Again both sides are positive so we can square both sides and conclude

$$48 \geq 49$$

and this is a contradiction.

Hence $\sqrt{2}+\sqrt{6}<\sqrt{15}$.

IMPORTANT NOTE. If you wrote something like this:

$$\begin{aligned} &\sqrt{2}+\sqrt{6}<\sqrt{15} \\ &\Rightarrow 2+6+2\sqrt{12}<15 \\ &\Rightarrow 2\sqrt{12}<7 \\ &\Rightarrow 48<49 \end{aligned}$$

then you get no marks at all. This is because if P is the statement that $\sqrt{2}+\sqrt{6}<\sqrt{15}$ then the argument just above shows that P implies $48<49$, so P implies something true. What can we deduce about P from this? Nothing! Because true implies true, and false implies true.

If however you wrote

$$\begin{aligned} &\sqrt{2}+\sqrt{6}<\sqrt{15} \\ &\Leftarrow 2+6+2\sqrt{12}<15 \\ &\Leftarrow 2\sqrt{12}<7 \\ &\Leftarrow 48<49 \end{aligned}$$

then this would logically be fine, although arguably it would also be upside-down, and also strictly speaking it doesn't follow from what we proved in the course about inequalities because we only proved $0<a<b$ implies $0<a^2<b^2$ rather than the other way around. Can you see how to prove that $0<a^2<b^2$ and $a,b>0$ implies $a<b$?

7) (a) Proof by contradiction. If $\sqrt{2}+\sqrt{3/2}$ were rational, then its square would be too. But its square is $2+3/2+2\sqrt{3}$, and if this were rational then $2\sqrt{3}$ and hence $\sqrt{3}$ would be too, contradicting Q3.

(b) This must be irrational because if it were rational then adding -1 would leave it rational, but adding 1 gives part (a) which is irrational.

(c) This is rational because it's zero :P