Largrange Interpolation

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Given points $\{(z_i, f_i)\}_{i=0}^n$, where $\forall i, z_i, f_i \in \mathbb{C}$, and z_i are distinct. We would like to find a polynomial $p_n \in \mathbb{P}_n$ such that $\forall i, p_n(z_i) = f_i$.

Natural questions -

- Does p_n exists at all?
- Is p_n unique?

Answers -

- Yes! The following polynomials does our job -

$$p_n(z) = \sum_{i=0}^{n} f_i l_i(z)$$

where $l_i(z)$ is the Largrange's basis function

$$l_i(z) = \prod_{\substack{0 \le k \le n \\ k \ne i}} \frac{z - z_k}{z_i - z_k}$$

- Yes! It is unique.

Why are $l_i(z)$ special, and why do we call them basis function? Certainly $l_i(z)$ is a polynomial of degree n, so $\{l_i(z)\}_{i=0}^n$ spans $\mathbb{P}_n$. But can we prove that $\{_i(z)\}_{i=0}^n$ is actually a *basis* of $\mathbb{P}_n$?

We may substitute some numbers in $l_i(z)$ to see what we get. In fact, if we substitute $z_k (k \ne i)$, we get 0, and if we substitute z_i, we get 1! (It's not that hard to see yourself). This is important, since we now have $\forall i, p_n(z_i) = f_i$. Moreover, if $\sum_{i=0}^n c_i \phi(z_i) \equiv 0$ then by substiuting z_i we see that $\forall i, c_i = 0$, indicating that $\{l_i(z)\}_{i=0}^n$ is a linearly independent set. Therefore $\{l_i(z)\}_{i=0}^n$ is indeed a *basis* of $\mathbb{P}_n$!

The polynomial is unique as well. Let's say we have another polynomial $p'_n(x)$ which satisfies $\forall i, p'_n(z_i) = f_i$ as well. Then $\forall z_i, p_n(z) - p'_n(z) = 0$. So $p_n(z) - p'_n(z)$ has $n + 1 > n$ zero points, indicating that $p_n(z) - p'_n(z) \equiv 0$. Therefore $p_n(z)$ is unique.

This is a better way of fitting a polynomial then to solve linear equation.

Example
Obtain the cubic interpolating polynomial which passes through points P_0, P_1, P_2, P_3, where

$$P_0 = (-1,-3), P_1 = (0,-1), P_2 = (2,4), P_3 = (5,1)$$

Step 1 - Obtaining Largrange Basis

$$l_0(z) = \frac{(z-0)(z-2)(z-5)}{(-1-0)(-1-2)(-1-5)} = -\frac{1}{18}(z^3 - 7z^2 + 10z)$$

$$l_1(z) = \frac{(z+1)(z-2)(z-5)}{(0+1)(0-2)(0-5)} = \frac{1}{10}(z^3 - 6z^2 + 3z + 10)$$

$$l_2(z) = \frac{(z+1)(z-0)(z-5)}{(2+1)(2-0)(2-5)} = -\frac{1}{18}(z^3 - 4z^2 - 5z)$$

$$l_3(z) = \frac{(z+1)(z-0)(z-2)}{(5+1)(5-0)(5-2)} = \frac{1}{90}(z^3 - z^2 - 2z)$$

Step 2 - Find the polynomial

$$\begin{aligned}
p_n &= (-3)l_0 + (-1)l_1 + (4)l_2 + (1)l_3 \\
&= \frac{1}{6}(z^3 - 7z^2 + 10z) - \frac{1}{10}(z^3 - 6z^2 + 3z + 10) - \frac{2}{9}(z^3 - 4z^2 - 5z) + \frac{1}{90}(z^3 - z^2 - 2z) \\
&= \frac{-13}{90}z^3 + \frac{14}{45}z^2 + \frac{221}{90}z - 1
\end{aligned}$$