

RCDS Statistics I - Imperial College London

Introduction to random sampling and hypothesis testing

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Random sampling and hypothesis testing

Chapter 1. Random events and probability theory

- i) What is probability
- ii) Discrete distributions
- iii) Continuous distributions

Chapter 2. Confidence intervals and central limit theorem

- i) Confidence intervals
- ii) Z and t distribution
- iii) The Central Limit Theorem

Chapter 3. Hypothesis testing

- i) Formulate hypotheses H_0, H_1
- ii) Quantify significance. p-value
- iii) Comparing means. t-test.

1.1 Random events and probability

* Random events ("stochastic")

Something whose output we don't know

* Probability: number $\in [0, 1]$ quantifying certainty / "surprise".

Example: tossing coins (H, T)

$P(H) = 0 \rightarrow$ certain I will never get H

$P(H) = 1 \rightarrow$ always get H

$0 < P(H) < 1 \rightarrow$ level of uncertainty / "surprise"

* Unitarity: The sum of probabilities for all possible outcomes x_i must add up to 1.

$$\sum_{\forall x_i} P(x_i) = 1$$

Example: tossing coins

$$P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1 \quad \checkmark$$

Example: rolling dice

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = \frac{1}{6} + \frac{1}{6} + \dots + \frac{1}{6} = 1 \quad \checkmark$$

1.2 Discrete probability distributions

* Discrete: number of possible outcomes is a finite number. } dice

i) Binomial distribution

"How many times X I get a specific result in n trials,

if the probability of each success is $p"$

$$\left\{ B(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x} \right. \quad \left. \begin{array}{l} x: \text{number of successes} \\ n: \text{number of trials} \\ p: \text{probability each success} \end{array} \right.$$

Example: Probability of 5 times H tossing 10 times a coin

$$B(5; 10, \frac{1}{2}) = \binom{10}{5} \left(\frac{1}{2}\right)^5 \left(1-\frac{1}{2}\right)^{(10-5)} = 0.246$$

~~Exercice~~

Example: Probability of 3 times a 6 in 10 dice

$$B(3; 10, \frac{1}{6}) = \binom{10}{3} \left(\frac{1}{6}\right)^3 \left(1-\frac{1}{6}\right)^{(10-3)} = 0.155$$

~~Exercice~~

Example: Probability of passing exam (A, B, C) of 10 quest.

$$B(5; 10, \frac{1}{3}) = \binom{10}{5} \left(\frac{1}{3}\right)^5 \left(1-\frac{1}{3}\right)^{(10-5)} = 0.137$$

~~Exercice~~

* What if $x=n$?

Example : probability of tossing 10 Heads out of 10

$$\begin{aligned} \mathcal{B}(10; 10, \frac{1}{2}) &= \binom{10}{10} \left(\frac{1}{2}\right)^{10} \left(1-\frac{1}{2}\right)^{0} \\ &= 1 \cdot \left(\frac{1}{2}\right)^{10} \cdot 1 = \left(\frac{1}{2}\right)^{10} \quad \checkmark \end{aligned}$$

* General : recover the individual probability, raised to the number of attempts.

$$\mathcal{B}(n; n, p) = \underbrace{\binom{n}{n}}_1 p^n \underbrace{(1-p)}_1^0 = \underline{\underline{p^n}}$$

iii) Poisson distribution

"How many times I observe an event in an interval, know λ "

$$\left\{ \begin{array}{l} P(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \\ \quad \quad \quad \left\{ \begin{array}{l} x: \text{number of times I observe} \\ \lambda: \text{observed average.} \end{array} \right. \end{array} \right.$$

Example: Probability of observing 3 new patients, with $\lambda = 5$.

$$P(3; 5) = \frac{e^{-5} 5^3}{3!} = 0.140$$

~~Excel~~

Example: Probability of observing 5 or less patients in same λ .

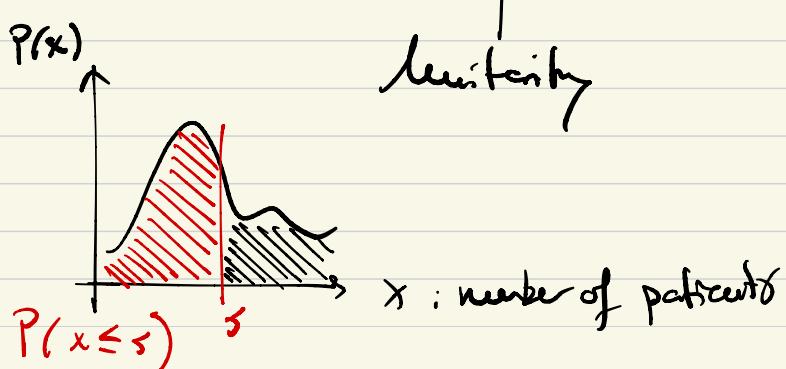
$$\begin{aligned} P(x \leq 5; 5) &= P(1) + P(2) + \dots + P(5) \\ &= \sum_{x=1}^5 P(x; \lambda=5) = 0.616 \end{aligned}$$

~~Excel~~

Accumulative distribution function cdf

Example: Probability of more than 5

$$P(x > 5, \lambda=5) = 1 - cdf(5) = 0.394$$



1.3 Continuous distributions

* Continuous: amount of possible outcomes is infinite / uncountable

Case 5: 2 outcomes (H, T) $\rightarrow P = \frac{1}{2}$ Discrete cases "Mass dist."

Dice: 6 outcomes ($1, 2, 3, 4, 5, 6$) $\rightarrow P = \frac{1}{6}$ Frequentist definition /

Continuous variable ($T, h, \text{conc.}$): ∞ outcomes $\rightarrow P = \frac{1}{\infty} = 0$ WTF

↳ Frequentist approach does not work

Need a new mathematical object "Density distribution"

i) Discrete case

Probability $P \in [0, 1]$

Unitarity $\sum_{x_i} P(x_i) = 1$

ii) Continuous case

Density $f(x)$

Unitarity $\int_{-\infty}^{\infty} dx f(x) = 1$

* Gaussian

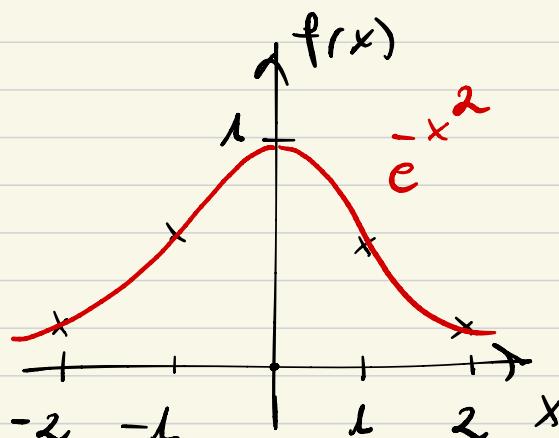
i) Gaussian distribution

"Random variable x centered at some μ and spreaded $\sigma"$

$$\left\{ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \right. \left. \begin{array}{l} \mu: \text{mean value} \\ \sigma: \text{standard deviation} \end{array} \right\}$$

* Consider simplest case $\mu=0$; $\sigma=L$; ignore normalization factor

$$f(x) = e^{-x^2} ; \text{ plot same values}$$



$$x = 0; f(0) = e^{-0} = 1$$

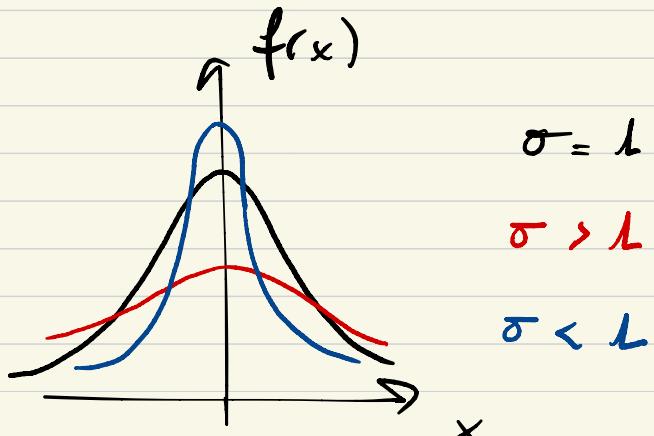
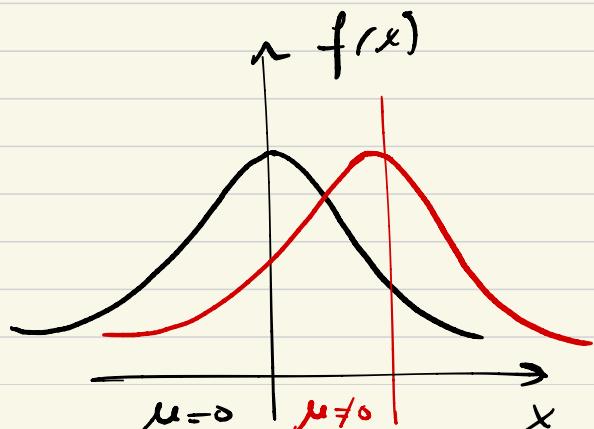
$$x = L; f(L) = e^{-L^2} = \frac{1}{e} = 0.37$$

$$x = -L; f(-L) = e^{-L^2} = \frac{1}{e} = 0.37$$

$$x = \pm 2; f(\pm 2) = e^{-4} = \frac{1}{e^4} = 0.018$$

* Switch back on μ and σ $\left. \begin{array}{l} \mu \text{ will displace the central value} \\ \sigma \text{ will drive sharper/wider shape} \end{array} \right\}$

$$f(x) = e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$



* Add normalization factor

i) Remember discrete probability : Normality $\sum_{x_i} P(x_i) = 1$

ii) Continuous case. Density distributions $\int_{-\infty}^{+\infty} dx f(x) = 1$

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = \sigma \sqrt{2\pi}$$

*
Exercise

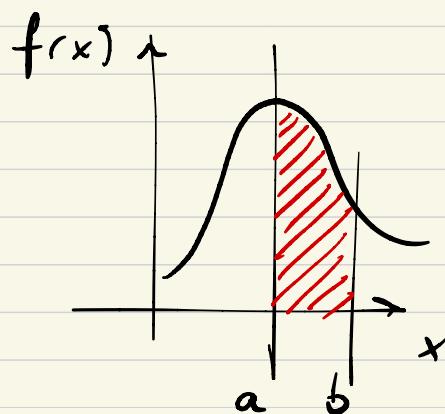
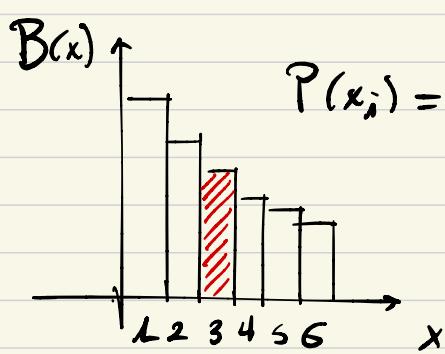
* for $f(x)$ to behave as a probability, we add a normalization factor

$$\left\{ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \right\} \left\{ \begin{array}{l} \text{Gaussian distribution ; } \int_{-\infty}^{\infty} dx f(x) = 1 \\ (\text{Normal distribution}) \end{array} \right.$$

* Particular case $\mu=0; \sigma=1$

$$\left\{ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \right\} \left\{ \begin{array}{l} \text{Standard Normal distribution } N(0,1) \\ (\text{Z distribution ; } Z(0,1)) \end{array} \right.$$

* In continuous distributions, we can only compute probability in a given range. Never a single value.

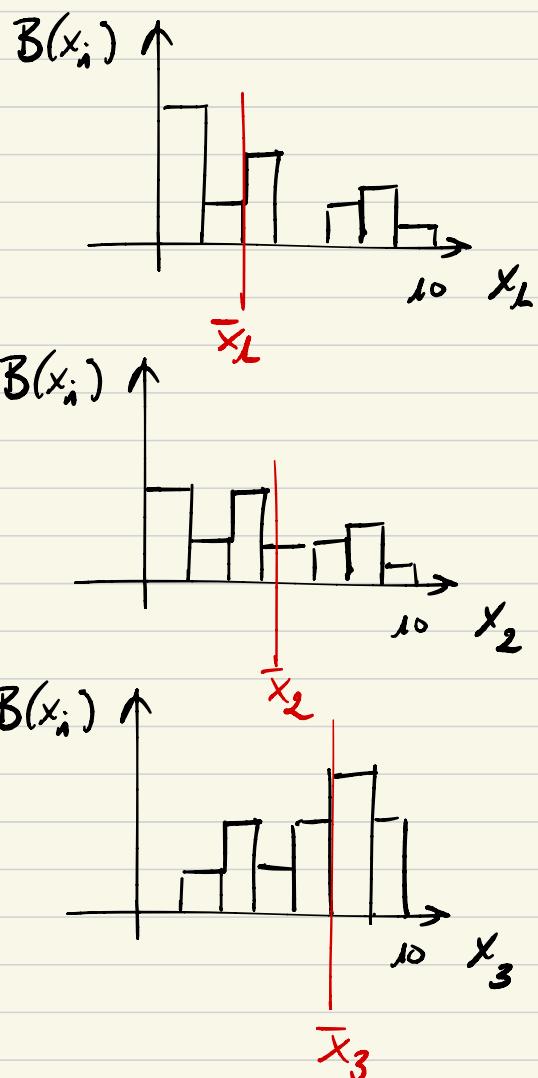


$$P(a < x < b) = \text{cdf}(b) - \text{cdf}(a)$$

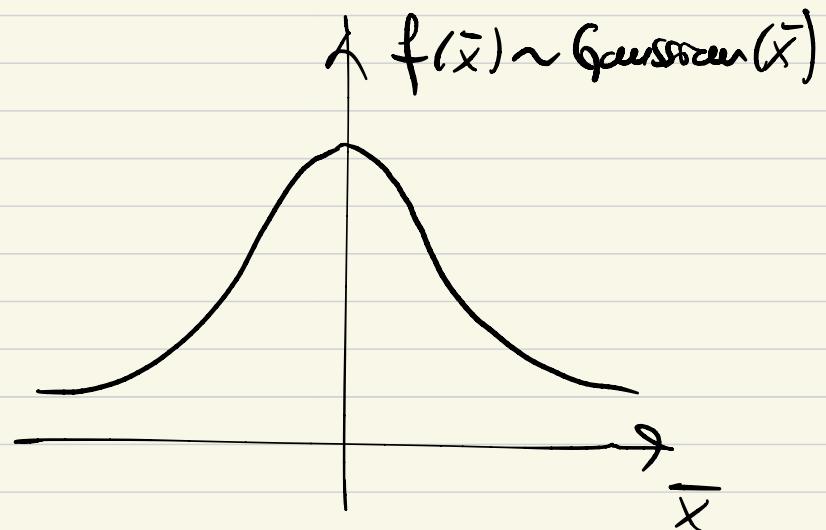
2.1 The Central Limit Theorem

"For a set of n independent and identically distributed (iid) random variables x_i , the sample means tends to a gaussian distribution for large values of n , regardless of how x_i are themselves distributed"

- Consider 10 rolls of a dice. How many times I observe a 6. Binomial distributed $B(x; 10, 1/6)$. Repeat N times.



* As the sample size n increases, the distribution of means \bar{x}_i tends to a gaussian distribution



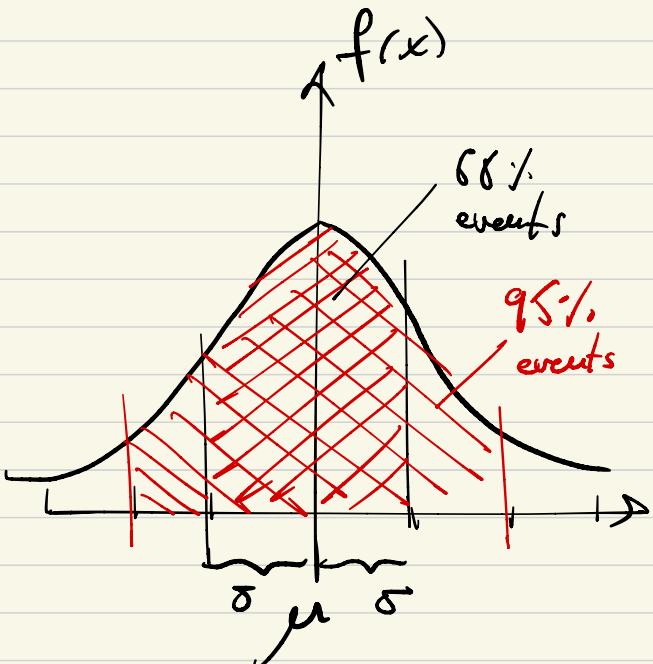
- Build new random variable $\bar{x} = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N\}$

* Mean value μ of the \bar{x}_i variable
* Standard error (SE)

$$SE = \frac{\sigma_i}{\sqrt{n}} ; \text{ as } n \text{ increase, } \sqrt{n} \downarrow SE$$

2.2 Confidence Intervals.

i) Consider a Gaussian distribution $f(x; \mu, \sigma)$



* 68% of events are contained in the $(\mu - \sigma, \mu + \sigma)$ interval. 1σ conf. int.

* 95% of events are contained in the $(\mu - 2\sigma, \mu + 2\sigma)$ interval. 2σ conf. int.

* 99% of events are contained in the $(\mu - 3\sigma, \mu + 3\sigma)$ interval. 3σ conf. int.

ii) Any random variable can be normalized / "standardized" just by subtracting the mean and dividing by the standard deviation.

x : random distributed; with mean μ and σ

standardized x ; } $z = \frac{x-\mu}{\sigma}$ { follows a $N(0,1)$ distribution,
also referred to as z score or z distribution.

iii) We want to estimate the mean μ

The best estimator is the sample mean \bar{x} , or "average"
Normally, we want to build

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} ; \text{ such that } P(\bar{x} - z < \mu < \bar{x} + z) = P_{\text{critical value}}$$

③ Hypothesis testing

i) Check if an observation is compatible with a given hypothesis H_0

* statistic test : a function / number computed out of our data
(t-test, χ^2 test, Fisher test, Wald test, ...)

* p-value : probability of obtaining a result at least as extreme as the one actually observed, under the assumption that our null hypothesis H_0 was correct.

(how likely / unlikely was to observe this result)

} i) \uparrow p-value : it was likely to obtain this result. **Accept H_0 .**
ii) \downarrow p-value : // unlikely // . **Reject H_0 .**

ii) General approach

1. Formulate null hypothesis H_0 and significance level α .
2. Make measurement / observations.
3. Compute statistic test
4. Compute p-value
5. If $p\text{-value} < \alpha$, reject H_0 . Otherwise, accept H_0 .

iii) A bit of history

Karl Pearson (1857-1936). Pearson's "p" values for χ^2 test.

Mathematical statistics.

Ronald Fisher (1890-1962) "Statistical methods for research" (1925)

3.1 One-sided t-test.

* Compare mean of observed die rolls to the expected mean of a fair die.

* For a series of observations $X = \{x_1, x_2, \dots, x_n\}$

$$\left\{ t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \right\}$$

\bar{x} : observed average ("sample mean")

μ : expected average under H_0 ("population mean")

s : observed standard deviation ("sample standard dev.")

n : length of collected data ("sample size")

Example: Check if a die is biased. (1)

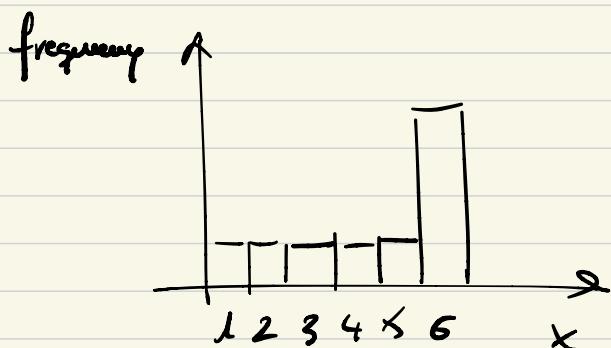
* You suspect your opponent is using a loaded die

i) Formulate null hypothesis $H_0: P = \frac{1}{6}$

ii) alternative // $H_1: P > \frac{1}{6}$

Choose significance level $\alpha = 0.01$

iii) Collect data. Roll die 15 times.



10 times out of 100
we get a 6. Suspicious!

iv) Perform 1-sample t-test.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{5 - 3.5}{1.6 / \sqrt{15}} \approx 3.6$$

μ : expected average is a fair die
(all faces with same probability)

$$\mu = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

$$\bar{x} = 5 ; s = 1.6 ; n = 15$$

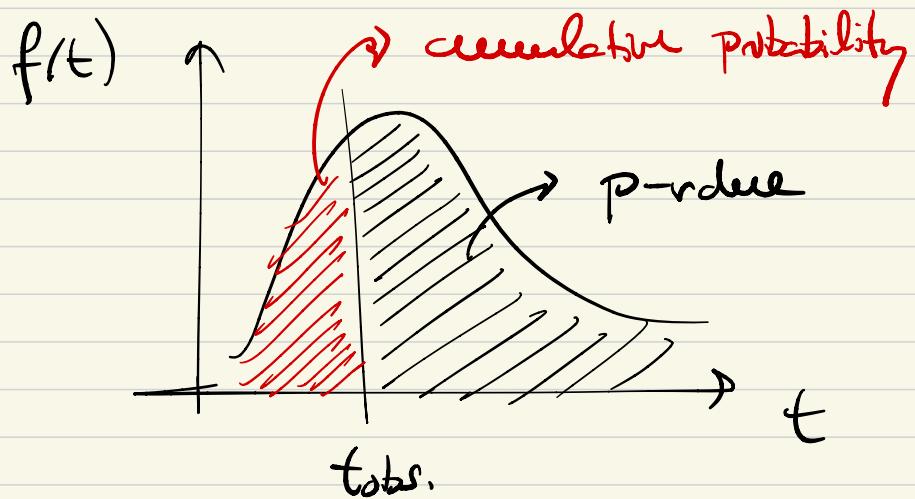
(*)

(*)

iv) P-value. Probability of obtaining a value at least
as extreme as the one we obtained for our t-variable,
given H_0 was true.

If H_0 is true, then t variable follows

a "Student's t distribution"



$$p\text{-value} = P(t \geq t_{\text{obs}}) = 1 - \text{cdf}(t_{\text{obs}}) \quad (*)$$

$\left. \begin{array}{c} \text{Student's t tables} \\ \text{Computer simulations} \end{array} \right\}$

Example: Check if a die is biased. (11)

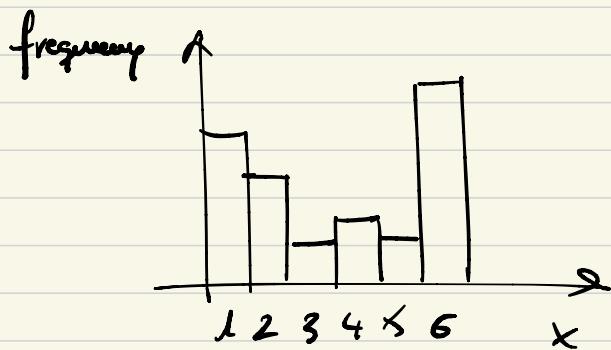
* You suspect your opponent is using a loaded die

i) Formulate null hypothesis $H_0: p = \frac{1}{6}$

ii) alternative $H_1: p > \frac{1}{6}$

Choose significance level $\alpha = 0.01$

iii) Collect data. Roll die 100 times



43 times out of 100
we get a 6.

iv) Compute probability of 43 or less 6's, given H_0 .

Given by Binomial distribution with $p = 1/6$

$$B(43; 100, \frac{1}{6}) \sim L_{\text{calf}}$$

Exercice

v) Compute p-value. Probability of obtaining at least a value as extreme as 43.

$$\text{p-value} = 1 - B_{\text{calf}}(43; 100, \frac{1}{6}) = 1^{-10}$$

↪ p-value $< \alpha$. → evidence to reject H_0 .

Example: check if coin is fair

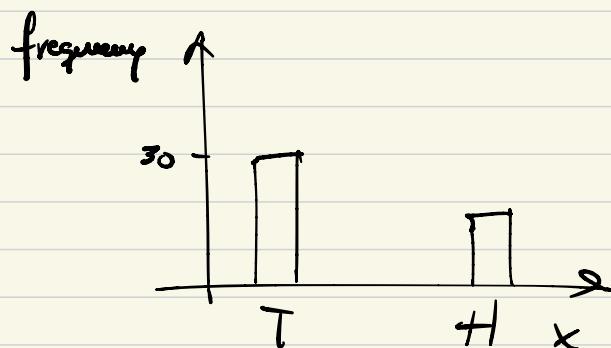
* You suspect your opponent is using a biased coin

i) Formulate null hypothesis $H_0: p = \frac{1}{2}$

ii) alternative " $H_1: p \neq \frac{1}{2}$

Choose significance level $\alpha = 0.05$

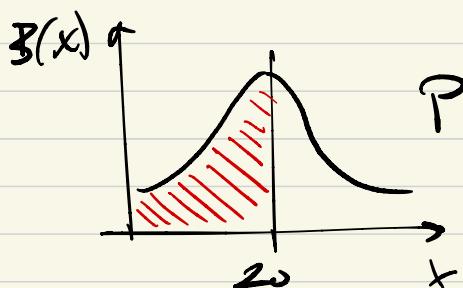
iii) Collect data. Roll die 100 times



20 heads in 50
flips of a coin

iv) Compute probability of 20 or less H, given H_0 .

Given by Binomial distribution with $p = 1/2$



$$P(X=20) = B_{\text{cdf}}(20; 50, \frac{1}{2}) = 0.1$$



v) Compute p-value.

Now H_1 is just $p \neq \frac{1}{2}$, either larger or smaller.

p-value = $2 \cdot \text{cdf}(20) = 0.2 < \alpha$ \nexists evidence
to reject H_0 .

3.3 Comparing means. t-test

* Compare if two samples have significantly different means

Given two sets of observations x_1 and x_2 , of sample size n_1 and n_2 , and sample means \bar{x}_1 and \bar{x}_2 .

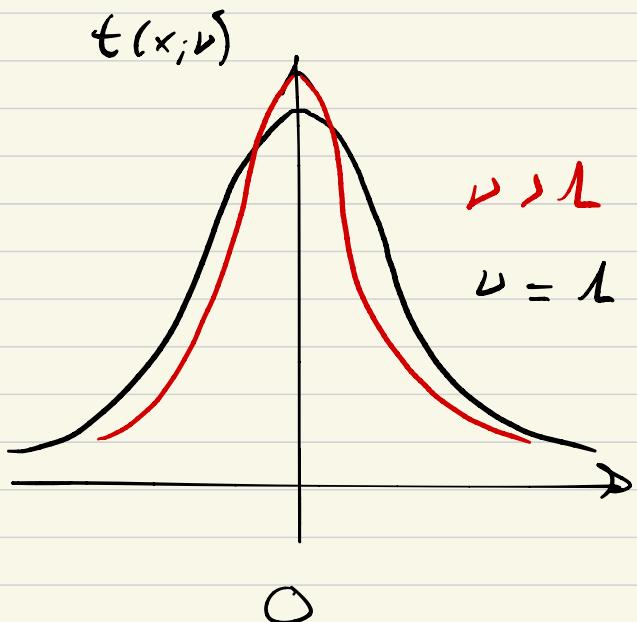
* Compute t variable (t "statistic")

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad \left\{ \begin{array}{l} \\ \end{array} \right.$$

being s_p the pooled / mixed standard deviation

$$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}}$$

* Under null hypothesis of equal means, the t statistic follows a Student's t distribution with (n_1+n_2-2) dof.



i) For $v=1$; $t(x; v)$ is referred

to as the Cauchy distribution

$$\left\{ f(x, x_0, \gamma) = \frac{1}{\pi} \frac{\gamma}{(x-x_0)^2 + \gamma^2} \right\}$$

ii) For v large; $t(x; v)$

\times tends to a normal distribution

* A bit of history

1876. t-distribution derived first by Helmert and Bünnell

1895. General form from Pearson, as Pearson type IV distribution

1908. William Sealy Gosset publishes under name "Student"

1925. Popularized by Fisher as "Student's distribution"

Example : Comparing means of 2 populations

* Check if the mean of birth weights are significantly different between smokers and no smokers.

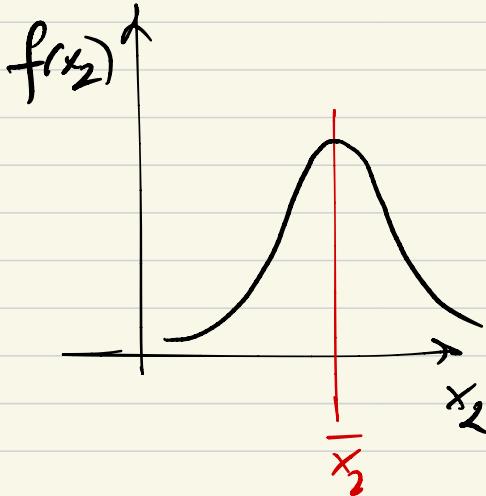
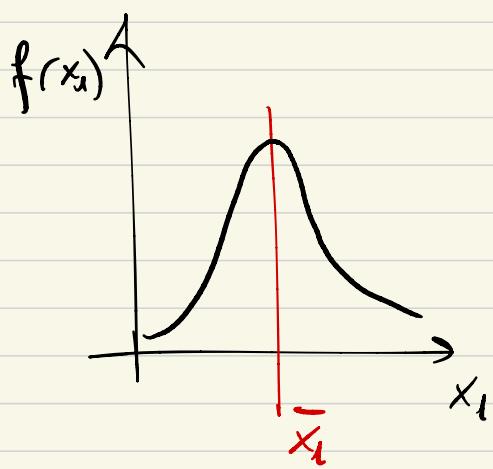
i) Formulate null hypothesis

$$H_0: \text{No difference b/w the means. } d = \bar{x}_1 - \bar{x}_2 = 0$$

$$H_1: \exists \text{ } d \neq 0$$

Choose significance level α

ii) Collect data . Smokers and non smokers .



n_1 non smokers

Sample mean \bar{x}_1

n_2 heavy smokers

Sample mean \bar{x}_2

standard deviation s_1

standard deviation s_2

iii) Compute t statistic iv) Compute d.o.f

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$D = n_1 + n_2 - 2$$

v) Check if t falls inside the
95% confidence interval.

Example: Comparing means of two populations.

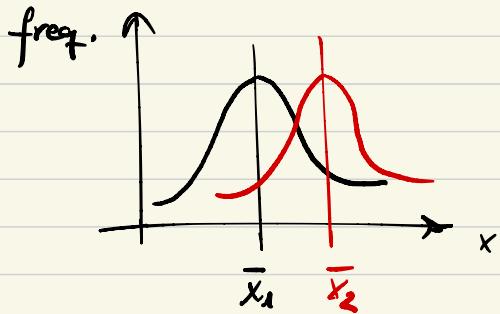
i) Formulate H_0, H_1

H_0 : both samples have the same mean $\bar{x}_1 = \bar{x}_2 = \mu$

H_1 : the samples have different means $\bar{x}_1 \neq \bar{x}_2$

ii) Collect data

Example: luminosity of 2 different types of stars.



$\{x_1\}$: Type IV stars ("Supergiant")

$\{x_2\}$: type V stars ("hypergiant")

iii) Compute t "statistic" (t variable)

$$\left\{ t = \frac{\bar{x}_1 - \bar{x}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right\} \quad \text{if } \bar{x}_1 = \bar{x}_2 \Rightarrow t=0 \quad \checkmark$$

$$S_p = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2} \quad \text{"pooled" standard deviation.}$$

iv) Compute p-value.

"What was the probability of obtaining a value at least as extreme as the one we got for t."

* If $\bar{x}_1 = \bar{x}_2$, the t variable follows a Student's t distribution with n_1+n_2-2 degrees of freedom. (*)

