Notes on uniform structures

Let X be a set (or a type). There's something called a *uniform structure* on X, but different authors appear to use the phrase differently. One thing that everyone is agreed on is that a *uniform space* is a set or type X, equipped with a uniform structure.

Three definitions of a uniform space can be found on Wikipedia.

The first one (the only one I knew about, two weeks ago) defines a uniform structure on X to be a collection of *entourages*, which are subsets of $X \times X$ obeying a bunch of axioms. This definition is very much reminiscent of the definition of a topological space as a collection of *open sets*, which are subsets of X obeying the axioms for a topology. Some of the axioms for the sets in a uniform structure are that these sets form a *filter* on $X \times X$, and this can be used to simplify the definition a bit. One of the axioms stands out – if U is an entourage, and $U \subset V$, then V must also be an entourage.

Say we are given a uniform structure on X, i.e. a bunch of entourages. A set B of these entourges is called a *basis* for the uniform structure if for every entourage in the uniform structure, there's an element of B which is a subset of it.

Key example. Let X be given the structure of a metric space (i.e. say we have $d : X^2 \to \mathbb{R}$ satisfying the usual axioms). Then let's say that $U \subset X \times X$ is an *entourage* if there exists some $\epsilon > 0$ such that the set $d^{-1}([0, \epsilon]$ is a subset of U. More concretely, this says that U is an entourage if there exists $\epsilon > 0$ such that if $x, y \in X$ and $d(x, y) < \epsilon$ then $(x, y) \in U$.

Exercises

Note: you might want to do (3) first.

1) Check this this construction of entourages does give us a collection of entourages which satisfy the axioms of a uniform strucure as in Wikipedia.

2) Check that the sets $d^{-1}([0, \epsilon])$ form a basis for this uniform structure.

3) Check that everything still works for pseudometrics too (i.e. we never used the axiom $d(x, y) = 0 \iff x = y$ in the key example or (1) and (2)).