LaplaceTransformsExam3

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Part 1: Practice Problems

Question 1 (Inverse Laplace - Partial Fractions)

The inverse Laplace transform of the function

$$F(s) = \frac{5s+4}{(s-1)(s+2)}$$

is the function $f(t)$. Find the value of $f(0)$.

Question 2 (Inverse Laplace - Quadratic Denominators)

The inverse Laplace transform of the function

$$F(s) = \frac{3s^2}{(s^2+1)(s^2+4)}$$

is the function $f(t)$. Find the value of $f(\pi/2)$.

Question 3 (IVP - Non-Homogeneous)

Let $y(t)$ be the solution to the initial value problem:

$$y'' - 3y' + 2y = 6e^{3t}$$

with initial conditions $y(0) = 1$ and $y'(0) = 3$. Find the value of $y(\ln 2)$.

Question 4 (IVP - Coefficient Sum)

Let $y(t)$ be the solution to the initial value problem:

$$y'' - 4y' + 3y = e^{2t}$$

with initial conditions $y(0) = 2$ and $y'(0) = 3$. The solution can be written in the form $y(t) = Ae^{t} + Be^{3t} + Ce^{2t}$. Find the value of the sum of the coefficients: $A + B + C$.

Question 5 (Shifted Laplace)

Let $f(t)$ be the inverse Laplace transform of:

$$F(s) = \frac{5e^{-2s}}{(s-4)^2}$$

Find the value of $\frac{f(3)}{e^4}$.

Part 2: Solutions & Answer Key

Solution to Question 1

We must find the partial fraction decomposition to determine $f(t)$, then evaluate at $t = 0$.

$$\frac{5s+4}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

Find A (cover-up method at $s = 1$):

$$A = \frac{5(1)+4}{1+2} = \frac{9}{3} = 3$$

Find B (cover-up method at $s = -2$):

$$B = \frac{5(-2)+4}{-2-1} = \frac{-6}{-3} = 2$$

So, $f(t) = 3e^{t} + 2e^{-2t}$. Evaluate at $t = 0$:

$$f(0) = 3e^{0} + 2e^{0} = 3 + 2 = 5$$

Answer: 5

Solution to Question 2

We assume the form:

$$\frac{3s^2}{(s^2+1)(s^2+4)} = \frac{A}{s^2+1} + \frac{B}{s^2+4}$$

Multiply by common denominator and substitute $x = s^2$:

$$3x = A(x+4) + B(x+1)$$

At $x = -1$: $-3 = A(3) \implies A = -1$. At $x = -4$: $-12 = B(-3) \implies B = 4$.

$$F(s) = \frac{-1}{s^2+1} + \frac{4}{s^2+4}$$

Inverse Laplace (remember $\mathcal{L}^{-1}[\frac{\omega}{s^2+\omega^2}] = \sin(\omega t)$):

$$f(t) = -\sin(t) + \frac{4}{2}\sin(2t) = -\sin(t) + 2\sin(2t)$$

Evaluate at $t = \pi/2$:

$$f(\pi/2) = -\sin(\pi/2) + 2\sin(\pi)$$

$$f(\pi/2) = -1 + 2(0) = -1$$

Answer: -1

Solution to Question 3

1. **Homogeneous:** $r^2 - 3r + 2 = (r-1)(r-2) = 0 \implies y_h = c_1 e^t + c_2 e^{2t}$.
2. **Particular:** Guess $y_p = Ce^{3t}$.

$$(9C - 3(3C) + 2C)e^{3t} = 6e^{3t} \implies 2C = 6 \implies C = 3$$

   General Solution: $y = c_1 e^t + c_2 e^{2t} + 3e^{3t}$. Derivative: $y' = c_1 e^t + 2c_2 e^{2t} + 9e^{3t}$.

3. **Initial Conditions:** $y(0) = c_1 + c_2 + 3 = 1 \implies c_1 + c_2 = -2$ $y'(0) = c_1 + 2c_2 + 9 = 3 \implies c_1 + 2c_2 = -6$ Subtract equations: $c_2 = -4$. Solve for c_1: $c_1 - 4 = -2 \implies c_1 = 2$. Full Solution: $y(t) = 2e^t - 4e^{2t} + 3e^{3t}$.
4. **Evaluate:** $y(\ln 2) = 2e^{\ln 2} - 4e^{2\ln 2} + 3e^{3\ln 2}$ Using $e^{k \ln x} = x^k$: $y(\ln 2) = 2(2) - 4(4) + 3(8)$ $y(\ln 2) = 4 - 16 + 24 = 12$.

Answer: 12

Solution to Question 4

1. **Homogeneous:** $r^2 - 4r + 3 = (r-1)(r-3) \implies y_h = Ae^t + Be^{3t}$.
2. **Particular:** Guess $y_p = Ce^{2t}$.

$$(4C - 8C + 3C)e^{2t} = e^{2t} \implies -C = 1 \implies C = -1$$

   $y = Ae^t + Be^{3t} - e^{2t}$. $y' = Ae^t + 3Be^{3t} - 2e^{2t}$.

3. **Initial Conditions:** $y(0) = A + B - 1 = 2 \implies A + B = 3$. $y'(0) = A + 3B - 2 = 3 \implies A + 3B = 5$. Subtract: $2B = 2 \implies B = 1$. Solve for A: $A + 1 = 3 \implies A = 2$. Coefficients are $A = 2, B = 1, C = -1$.
4. **Sum:** $A + B + C = 2 + 1 + (-1) = 2$.

Answer: 2

Solution to Question 5

Base function inverse: $\mathcal{L}^{-1}\left\{\frac{5}{(s-4)^2}\right\} = 5te^{4t}$. Apply time shift $t \to (t-2)$ due to e^{-2s}: $f(t) = 5(t-2)e^{4(t-2)}u(t-2)$. Evaluate at $t = 3$ (note: $u(3-2) = u(1) = 1$): $f(3) = 5(3-2)e^{4(3-2)} = 5(1)e^4 = 5e^4$. Compute ratio: $\frac{5e^4}{e^4} = 5$. **Answer: 5**