

## Daily Assignment 2

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1. Let  $F$  be any field. A *Fibonacci sequence* in  $F$  is a function  $f : \mathbb{N} \rightarrow F$  defined recursively by letting  $f(0), f(1)$  be elements of  $F$  and then setting  $f(n+2) = f(n+1) + f(n)$  for all  $n \in \mathbb{N}$ . Show that the set  $\mathcal{F}$  of all Fibonacci sequences in  $F$  is a subspace of  $F^{\mathbb{N}}$ . Then compute the dimension of this space.

**Solution.** The zero element of  $F^{\mathbb{N}}$ ,  $\mathbf{0}_{F^{\mathbb{N}}}$ , defined to be  $g : \mathbb{N} \rightarrow F$  where  $g(n) = 0$ , is in  $\mathcal{F}$  by letting  $f(0) = f(1) = 0$  and noticing that  $f(n+2) = f(n+1) + f(n)$  means that  $f(n) = 0$  for all  $n \in \mathbb{N}$ . We also have that for any two elements  $f, g \in \mathcal{F}$ , and for any  $\alpha, \beta \in F$ ,  $\alpha f + \beta g \in \mathcal{F}$  since  $\alpha f(n) + \beta g(n) = (\alpha f + \beta g)(n)$  for all  $n \in \mathbb{N}$  under the operations of addition and scalar multiplication. Therefore  $\mathcal{F}$  is a subspace of  $F^{\mathbb{N}}$ , and the dimension of this subspace is  $\dim_F(\mathcal{F}) = 2$  since any Fibonacci sequence can be defined by any scalar multiple of the first two elements  $f(0) = 1, f(1) = 1$  with basis  $S = \{1, 1\}$ .  $\square$

2. For each of the following, determine whether the set of vectors  $S$  is a spanning set for the vector space  $V$  over the field  $F$ . If  $S$  is a spanning set, determine whether it's a basis, and justify your answers.

- (a)  $F = \mathbb{Q}, V = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}, S = \{1 - \sqrt{2}, 4\}$ .

**Solution.** Yes,  $S$  is a spanning set for  $V$  since  $\text{Span}(S) = \{\alpha(1 - \sqrt{2}) + \beta(4) : \alpha, \beta \in \mathbb{Q}\} = V$  where  $\alpha(1 - \sqrt{2}) + \beta(4) = (\alpha + \beta) - \alpha\sqrt{2} = c + d\sqrt{2}$ , and  $c = \alpha + \beta \in \mathbb{Q}$  by closure and  $d = -\alpha \in \mathbb{Q}$ . Then  $S$  is also a basis for  $V$  since  $(\alpha + \beta) - \alpha\sqrt{2} = 0 = 0 + 0\sqrt{2}$  implies  $\alpha = \beta = 0$ , making the elements in  $S$  linearly independent.  $\square$

- (b)  $F = \mathbb{Z}_2, V = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in F^{2 \times 2} : a_{11} + a_{22} = 0 \right\}, S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$

**Solution.** Yes,  $S$  is a spanning set for  $V$  since

$$\text{Span}(S) = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$$

where

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+b+c & b \\ c & a+b+c \end{pmatrix}$$

and  $a_{11} + a_{22} = 2(a+b+c) \equiv 0 \pmod{2}$  for all  $a, b, c \in \mathbb{Z}_2$ , so  $\text{Span}(S) = V$ . Also,  $S$  is a basis for  $V$  since

$$\begin{pmatrix} a+b+c & b \\ c & a+b+c \end{pmatrix} = \mathbf{0}^{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

implies that  $a+b+c = 0$  since  $b = c = 0$ , so the elements of  $S$  are linearly independent as well.  $\square$

- (c)  $F = \mathbb{Z}_2, V = F_2[x] = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in F\}, S = \{1 + x, 1 + x^2\}$

**Solution.** Yes,  $\text{Span}(S) = \{a(1 + x) + b(1 + x^2) : a, b \in \mathbb{Z}_2\} = V$  since

$$a(1 + x) + b(1 + x^2) = (a + b) + ax + bx^2 = c + ax + bx^2$$

where  $c = a + b \in \mathbb{Z}_2$  by closure. Also,  $S$  is a basis since  $(a + b) + ax + bx^2 = 0 = 0 + 0x + 0x^2$  implies  $a = b = 0$ , making the  $S$  linearly independent.  $\square$

3. Let  $M := \text{Mag}_3(\mathbb{R})$  denote the set of  $3 \times 3$  magic squares with entries from  $\mathbb{R}$ .

- (a) Show that  $M$  is a subspace of  $\mathbb{R}^{3 \times 3}$ .

**Solution.**  $M$  is a subspace of  $\mathbb{R}^{3 \times 3}$  since

$$0^{3 \times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M$$

by the definition of magic squares with every column, row, and diagonal sum having the same value of 0. Then  $\forall a, b \in \mathbb{R}$  and  $\forall A, B \in M, aA + bB \in M$  since, for any row, column, or diagonal sum  $S_A$  for magic square  $A$  and the corresponding row, column, or diagonal sum  $S_B$  for magic square  $B$ , the row, column, and diagonal sums  $aS_A + bS_B$  for the sum of the scalar multiple of the magic squares  $A$  and  $B$  will be the same, maintaining the defining property for a  $3 \times 3$  magic square.  $\square$

- (b) Find a basis for  $M$ .

**Solution.** A basis for  $M$  is given by

$$S = \left\{ \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -2/3 & 1/3 & 4/3 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 0 \end{pmatrix} \right\}$$

where the scalar for the first matrix is the row, column, and diagonal sum for the magic square.  $\square$

- (c) What is the dimension of  $M$ ?

**Solution.** The dimension of  $M$  is the size of its basis given above, which is  $\dim_{\mathbb{R}} M = 3$ .  $\square$