Daily Assignment 2

Rohan Kapur, Math 117 Summer 2023

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1. Let F be any field. A *Fibonacci sequence* in F is a function $f : \mathbb{N} \to F$ defined recursively by letting $f(0), f(1)$ be elements of F and then setting $f(n+2) = f(n+1) + f(n)$ for all $n \in \mathbb{N}$. Show that the set $\mathcal{F}$ of all Fibonacci sequences in F is a subspace of $F^{\mathbb{N}}$. Then compute the dimension of this space.

   Solution. The zero element of $F^{\mathbb{N}}$, $\mathbf{0}_{F^{\mathbb{N}}}$, defined to be $g : \mathbb{N} \to F$ where $g(n) = 0$, is in $\mathcal{F}$ by letting $f(0) = f(1) = 0$ and noticing that $f(n+2) = f(n+1) + f(n)$ means that $f(n) = 0$ for all $n \in \mathbb{N}$. We also have that for any two elements $f, g \in \mathcal{F}$, and for any $\alpha, \beta \in F$, $\alpha f + \beta g \in \mathcal{F}$ since $\alpha f(n) + \beta g(n) = (\alpha f + \beta g)(n)$ for all $n \in \mathbb{N}$ under the operations of addition and scalar multiplication. Therefore $\mathcal{F}$ is a subspace of $F^{\mathbb{N}}$, and the dimension of this subspace is $\dim_F(\mathcal{F}) = 2$ since any Fibonacci sequence can be defined by any scalar multiple of the first two elements $f(0) = 1, f(1) = 1$ with basis $S = \{1, 1\}$. $\square$

2. For each of the following, determine whether the set of vectors S is a spanning set for the vector space V over the field F. If S is a spanning set, determine whether it's a basis, and justify your answers.

   (a) $F = \mathbb{Q}, V = Q(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}, S = \{1 - \sqrt{2}, 4\}$.
   Solution. Yes, S is a spanning set for V since $Span(S) = \{\alpha(1-\sqrt{2}) + \beta(4) : \alpha, \beta \in \mathbb{Q}\} = V$ where $\alpha(1-\sqrt{2}) + \beta(4) = (\alpha + \beta) - \alpha\sqrt{2} = c + d\sqrt{2}$, and $c = \alpha + \beta \in \mathbb{Q}$ by closure and $d = -\alpha \in \mathbb{Q}$. Then S is also a basis for V since $(\alpha+\beta) - \alpha\sqrt{2} = 0 = 0 + 0\sqrt{2}$ implies $\alpha = \beta = 0$, making the elements in S linearly independent. $\square$

   (b) $F = \mathbb{Z}_2, V = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in F^{2\times 2} : a_{11} + a_{22} = 0 \right\}, S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$
   Solution. Yes, S is a spanning set for V since

$$Span(S) = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$$

   where

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+b+c & b \\ c & a+b+c \end{pmatrix}$$

   and $a_{11} + a_{22} = 2(a+b+c) \equiv 0 \pmod 2$ for all $a, b, c \in \mathbb{Z}_2$, so $Span(S) = V$. Also, S is a basis for V since

$$\begin{pmatrix} a+b+c & b \\ c & a+b+c \end{pmatrix} = 0^{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   implies that $a + b + c = 0$ since $b = c = 0$, so the elements of S are linearly independent as well. $\square$

(c) $F = \mathbb{Z}_2, V = F_2[x] = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in F\}, S = \{1 + x, 1 + x^2\}$
Solution. Yes, $Span(S) = \{a(1 + x) + b(1 + x^2) : a, b \in \mathbb{Z}_2\} = V$ since

$$a(1 + x) + b(1 + x^2) = (a + b) + ax + bx^2 = c + ax + bx^2$$

where $c = a+b \in \mathbb{Z}_2$ by closure. Also, S is a basis since $(a+b)+ax+bx^2 = 0 = 0+0x+0x^2$ implies $a = b = 0$, making the S linearly independent. □

3. Let $M := \mathrm{Mag}_3(\mathbb{R})$ denote the set of 3×3 magic squares with entries from $\mathbb{R}$.

(a) Show that M is a subspace of $\mathbb{R}^{3\times 3}$.
Solution. M is a subspace of $R^{3\times 3}$ since

$$0^{3\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M$$

by the definition of magic squares with every column, row, and diagonal sum having the same value of 0. Then $\forall a, b \in \mathbb{R}$ and $\forall A, B \in M, aA+bB \in M$ since, for any row, column, or diagonal sum S_A for magic square A and the corresponding row, column, or diagonal sum S_B for magic square B, the row, column, and diagonal sums $aS_A + bS_b$ for the sum of the scalar multiple of the magic squares A and B will be the same, maintaining the defining property for a 3×3 magic square. □

(b) Find a basis for M.
Solution. A basis for M is given by

$$S = \left\{ \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -2/3 & 1/3 & 4/3 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 0 \end{pmatrix} \right\}$$

where the scalar for the first matrix is the row, column, and diagonal sum for the magic square. □

(c) What is the dimension of M?
Solution. The dimension of M is the size of its basis given above, which is $\dim_{\mathbb{R}} M = 3$.□