Daily Assignment 1

Rohan Kapur, Math 117 Summer 2023

August 1, 2023

2. Consider the set $\mathbb{Z}_7 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$ of residue classes of integers mod 7.

   (a) Construct the multiplication table for the group $\left(\mathbb{Z}_7 \backslash \{\overline{0}\}, \cdot\right)$ where $\cdot$ is defined using representatives: $\overline{m} \cdot \overline{n} = \overline{mn}$.
   Answer:

$\cdot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{1}$	$\overline{3}$	$\overline{5}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{2}$	$\overline{5}$	$\overline{1}$	$\overline{4}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{6}$	$\overline{3}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{3}$	$\overline{1}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{6}$	$\overline{0}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

   (b) Use part (a) to find the multiplicative inverse of every nonzero element of $\mathbb{Z}_7$.
   Answer:

$$(\overline{1})^{-1} = \overline{1}, (\overline{2})^{-1} = \overline{4}, (\overline{3})^{-1} = \overline{5}, (\overline{4})^{-1} = \overline{2}, (\overline{5})^{-1} = \overline{3}, (\overline{6})^{-1} = \overline{6}$$

3. Let V be a vector space over a field F. Using only the definitions, prove the following for all $v \in V, a \in F$:

   (a) $0v = \mathbf{0}$
   Proof: This is saying that $0v$ maps to the additive identity $\mathbf{0}$ in V. To see this, notice that by distributivity of F over V, $0v + 0v = (0+0)v = 0v$. Then by adding $-(0v)$ to both sides, we get $0v = \mathbf{0}$ as desired. ■

   (b) $(-a)v = -(av)$
   Proof: This is saying that $(-a)v$ maps to the unique additive inverse $-(av)$ of av. To see this, notice that by distributivity of V over F, $(-a)v + av = (-a+a)v = 0v = \mathbf{0}$. Therefore $(-a)v$ is indeed the unique additive inverse of av in V, so $(-a)v = -(av)$. ■

   (c) $a\mathbf{0} = \mathbf{0}$
   Proof: This is saying that $a\mathbf{0}$ is equivalent to the additive identity $\mathbf{0} \in V$. To see this, notice that by distributivity of V over F, $a\mathbf{0} + a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) = a\mathbf{0}$. Then by adding $-(a\mathbf{0})$ to both sides, we get $a\mathbf{0} = \mathbf{0}$. ■

   (d) If $av = \mathbf{0}$, then $a = 0$ or $v = \mathbf{0}$.
   Proof: Suppose that $av = \mathbf{0}$, so av is the additive identity in V, and suppose for

contradiction that $a \neq 0$ and $v \neq \mathbf{0}$. Then there must exist a multiplicative inverse $a^{-1} \in F$ of a such that $aa^{-1} = 1$, so $a^{-1}av = (a^{-1}a)v = 1v = v = a^{-1}\mathbf{0}$, and then by part (b), $v = a^{-1}\mathbf{0} = \mathbf{0}$. However, this contradicts $v \neq \mathbf{0}$, and therefore we must have $a = 0$ or $v = \mathbf{0}$ to avoid the contradiction. ∎

4. Let $C(\mathbb{R})$ be the real vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Determine which of the following are subspaces of $C(\mathbb{R})$, and justify your reasoning.

   (a) $A = \{f : f \text{ is twice differentiable and } f''(x) - 2f'(x) + 3f(x) = 0 \text{ for all } x \in \mathbb{R}\}$
   Answer: A **is a subspace** of $C(\mathbb{R})$ since $\mathbf{0}_{C(\mathbb{R})}$, which is the function $f(x) = 0$ for all $x \in \mathbb{R}$, is in A since $f(x) = 0$ is twice-differentiable, and $f''(x) - 2f'(x) + 3f(x) = 0$. Also, $\forall a, b \in \mathbb{R}$, we have $\forall f_1, f_2 \in A$ that $f_3 = af_1 + bf_2 \in A$ since, by linearity of differentiation,

$$
\begin{aligned}
&f_3''' - 2f_3' + 3f_3 \\
&= af_1''' + bf_2''' - 2af_1' - 2bf_2' + 3af_1 + 3bf_2 \\
&= a(f_1''' - 2f_1' + 3f_1) + b(f_2''' - 2f_2' + 3f_2) \\
&= a0 + b0 = \boxed{0}
\end{aligned}
$$

   Therefore, both subgroup criteria are satisfied. ∎

   (b) $B = \{g : g \text{ is twice differentiable and } g''(x) = g(x) + 1 \text{ for all } x \in \mathbb{R}\}$
   Answer: B **is *not* a subspace** since $\forall x \in \mathbb{R}, f(x) = 0$, the additive identity of $C(R)$, is *not* in B since $f''(x) = 0 \neq f''(x) + 1 = 1$. ∎

   (c) $C = \{h : h \text{ is twice differentiable and } h''(0) = 2h(1)\}$
   Answer: C **is a subspace** of $C(\mathbb{R})$ since, firstly, the function $\forall x \in \mathbb{R}, f(x) = 0$ is in $C(\mathbb{R})$ as $f''(0) = 0 = 2f(1) = 0$, and also $\forall a, b \in \mathbb{R}$ and $\forall f_1, f_2 \in C$, we have $af_1 + bf_2 \in C$ since $f_1''(0) = 2f_1(1)$ and $f_2''(0) = 2f_2(1)$, so by linearity of differentiation, $(af_1 + bf_2)'''(0) = af_1'''(0) + bf_2'''(0) = 2af_1(1) + 2bf_2(1) = 2(af_1 + bf_2)(1)$, satisfying both subgroup criteria. ∎