Homework 2

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1. If true, show that a group is abelian if every subgroup is normal. Otherwise, give a counterexample.

   Counterexample: The quaternion group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ with $i^2 = j^2 = k^2 = -1, (-1)^2 = 1$ is an example of a *non*abelian group whose subgroups are all normal.

2. Suppose that N, M are two normal subgroups of G and that $N \cap M = \{e\}$. Show that for any $n \in N, m \in M, nm = mn$.

   Proof: Since N, M are normal subgroups of G, for any $g \in G, gN = Ng$ and $gM = Mg$. Then, since $N \subset G$, we also have for any $n \in N, nM = Mn$, or $nMn^{-1} = M$, so for any $m \in M, nmn^{-1} \in M$, and since M is a subgroup, we have $m^{-1} \in M$, so by closure, $\left(nmn^{-`1}\right)m^{-1} \in M$. Similarly, since $M \subset G$, we have for any $m \in M, mN = Nm$ or $mNm^{-1} = N$, and since N is a subgroup, for any $n \in N, n^{-1} \in N$, so $mn^{-1}m^{-1} \in N$, and by closure and associativity, $n\left(mn^{-1}m^{-1}\right) = \left(nmn^{-1}\right)m^{-1} \in N$. Therefore $\left(nmn^{-1}\right)m^{-1} \in N \cap M = \{e\}$, so $\left(nmn^{-1}\right)m^{-1} = e$, or $nm = mn$. ***QED***

3. If N is normal in G and $a \in G$ is of order $o(a)$, prove that the order, m, of Na in G/N is a divisor of $o(a)$.

   Proof: Let $s = o(a)$. Then we have $a^n = e$, so $Na^n = (Na^n) = Ne = N$. And since $ord(Na) = m$, we also have $(Na)^m = N$. Thus we must have $m \leq n$ since $ord(Na) = m$. By the division algorithm, let $n = qm + r$ for $q, r \in \mathbb{Z}$ with $0 \leq r < n$. We need to show that $r = 0$, so let $r = n - qm$ and suppose $r > 0$. Then $a^r = a^{n-qm} = a^n a^{-qm} = a^{-qm}$, which contradicts $r > 0$. Therefore, we must have $r = 0$, and hence m divides a. ***QED***

   (I'm unsure if this is correct. . .)

4. If G is a non-abelian group of order 6, prove that $G \approx S_6$ (G is isomorphic to S_6).

   Proof: Not sure how to approach this. . .

5. Let G_1 be group of all nonzero complex numbers under multiplication, and let G_2 be group of all real 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where not both a and b are 0 under matrix multiplication. Show that G_1 and G_2 are isomorphic.

   Answer: Not sure. . .