Daily Assignment 1

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1. Consider the set $\mathbb{Z}_7 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$ of residue classes of integers mod 7.

   (a) Construct the multiplication table for the group $\left(\mathbb{Z}_7 \backslash \{\overline{0}\}, \cdot\right)$ where $\cdot$ is defined using representatives: $\overline{m} \cdot \overline{n} = \overline{mn}$.
   Answer:

$\cdot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{1}$	$\overline{3}$	$\overline{5}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{2}$	$\overline{5}$	$\overline{1}$	$\overline{4}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{6}$	$\overline{3}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{3}$	$\overline{1}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{6}$	$\overline{0}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

   (b) Use part (a) to find the multiplicative inverse of every nonzero element of $\mathbb{Z}_7$.
   Answer:

$$(\overline{1})^{-1} = \overline{1}, (\overline{2})^{-1} = \overline{4}, (\overline{3})^{-1} = \overline{5}, (\overline{4})^{-1} = \overline{2}, (\overline{5})^{-1} = \overline{3}, (\overline{6})^{-1} = \overline{6}$$

2. Let $\mathbf{V}$ be a vector space over a field $\mathbf{F}$. Using only the definitions, prove the following for all $v \in V, a \in F$:

   (a) $0v = \mathbf{0}$
   Proof: This is saying that $0v$ is equal to the additive identity $\mathbf{0}$ in V. To see this, notice that by distributivity of F over V, $0v + 0v = (0+0)v = 0v$. Then by adding $-(0v)$ to both sides, we get $0v = \mathbf{0}$ as desired. ■

   (b) $(-a)v = -(av)$
   Proof: This is saying that $(-a)v$ equals the unique additive inverse $-(av)$ of av. To see this, notice that by distributivity of V over F, $(-a)v + av = (-a+a)v = 0v = \mathbf{0}$. Therefore $(-a)v$ is indeed the unique additive inverse of av in V, so $(-a)v = -(av)$. ■

   (c) $a\mathbf{0} = \mathbf{0}$
   Proof: This is saying that $a\mathbf{0}$ is the additive identity in V. To see this, notice that by distributivity of V over F, $a\mathbf{0} + a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) = a\mathbf{0}$. Then by adding $-(a\mathbf{0})$ to both sides, we get $a\mathbf{0} = \mathbf{0}$. ■

   (d) If $av = \mathbf{0}$, then $a = 0$ or $v = \mathbf{0}$.
   Proof: Suppose that $av = \mathbf{0}$, so av is the unique additive identity in V. Now suppose that $v \neq \mathbf{0}$.