Problem 1: In the binary communication system shown on the next page, messages $m = 0$ and $m = 1$ occur with a priori probabilities 0.25 and 0.75, respectively. The random variable n takes on $-1,\ 0,\ 1$ with probabilities 0.125, 0.75, and 0.125 respectively.

Find the receiver which achieves the maximum probability of correct decision.

message

$m = 0$ 0.25

$m = 1$ 0.75

r.v n

$n = -1$ .125

0 .75

1 .125

$r = n + m$

$r = \begin{cases} -1 \\ 0 \\ 1 \\ 2 \end{cases}$

$n=-1\ m=0$ $1/32$

$n=0\ m=0$ $3/16$ + $n=-1\ m=1$ $3/32$

$n=1\ m=0$ $1/32$ $n=0\ m=1$ $9/16$

$n=1\ m=1$ $3/32$

$\boxed{\text{Source}} \xrightarrow[m=0 \text{ or } 1]{m} \oplus \ (n) \xrightarrow[\text{received sig}]{r} \boxed{\text{receiver}} \rightarrow$ output 0 or 1

If $r = -1$ pick $m = 0$ $r = 1$ pick $m = 1$

$r = 0$ pick $m = 0$ $r = 2$ pick $m = 1$

pick 1 if $f_{r|1} > f_{r|0}$

Problem 2: When a light beam of known intensity I is shined on a photomultiplier the number of photocounts observed in a T-second interval is a Poisson random variable with mean αI, where $\alpha > 0$ is a known constant. The number of photocounts observed in non-overlapping T-second intervals are statistically independent random variables. Suppose we shine a light beam of intensity I (I is a random variable) on a photomultiplier and measure

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \\ y_p \end{bmatrix},$$

where y_i is the number of counts in the ith of a set of p T-second nonoverlapping intervals. Then

$$Pr[y_i = k \mid I] = \frac{(\alpha I)^k e^{-\alpha I}}{k!} \text{ for } k = 0, 1, 2, \ldots$$

(a) Suppose $p_I(I) = (\overline{I})^{-1} e^{-I/\overline{I}}$ for $I \geq 0$. Find the Bayes least-squares estimate $\hat{I}_B$ of I and the resulting mean square estimation error.

(b) Show that $\overline{I} \to \infty$ the estimate of part (a) reduces to

$$\hat{I}_B = (\alpha p)^{-1} \left(\sum_{i=1}^{p} y_i + 1 \right)$$

$K = 1, 2, \ldots$

a) Bayes Least Squares Estimate $\quad$ Mean Square Estimation

$\hat{I}(y) - I \qquad e(\vec{a}, \hat{a}) = \|\vec{a} - \hat{a}\|^2$

$Pr[y = K \mid I] = \frac{(\alpha I)^K e^{-\alpha I}}{K!}$

$\hat{I}_{BLS}(b) = E[I \mid y = b]$

$p_I(I) = \bar{I}^{-1} e^{-I/\bar{I}}$ for $I \geq 0$

$$P(I, y) = \frac{(\alpha I)^y e^{-\alpha I}}{y!} (\bar{I})^{-1} e^{-\frac{I}{\bar{I}}}$$

$$P(I \mid \vec{y} = \vec{y}) = \frac{\prod_{j=1}^{p} \left(\frac{(\alpha I)^{y_j} e^{-\alpha I}}{y!} (\bar{I})^{-1} e^{-I/\bar{I}} \right)}{\int_0^\infty \prod_{i=1}^{p} \left(\frac{(\alpha I)^{y} e^{-\alpha I}}{y!} (\bar{I})^{-1} e^{-\frac{I}{\bar{I}}} \right) dI}$$

$$\prod_{j=1}^{p} \frac{\alpha^{y_j}}{y_j!} \int_0^\infty I^{y} e^{-I\left(\alpha+\frac{1}{\bar{I}}\right)} \, dI = \prod_{j=1}^{p} \frac{\alpha^{y_j}}{y_j!} \frac{y!}{\left(\alpha+\frac{1}{\bar{I}}\right)^{y_j+1}}$$

$$P(I \mid \vec{y} = y_j) = \prod_{j=1}^{p} \frac{I^{y_j} e^{-\alpha I}}{\left(\alpha+\frac{1}{\bar{I}}\right)^{y_j+1}} \cdot e^{-I/\bar{I}}$$

$$I_b = E[I \mid y] = \int_0^\infty I \prod_{j=1}^{p} \frac{I^{y_j} e^{-I\left(\alpha+\frac{1}{\bar{I}}\right)}}{\left(\alpha+\frac{1}{\bar{I}}\right)^{y_j+1}} \, dI$$

$$= \prod_{j=1}^{p} \int_0^\infty \frac{I^{y_j+\frac{1}{p}} e^{-I\left(\alpha+\frac{1}{\bar{I}}\right)}}{\left(\alpha+\frac{1}{\bar{I}}\right)^{y_j+1}} \, dI = \prod_{j=1}^{p} \frac{1}{\left(\alpha+\frac{1}{\bar{I}}\right)^{y_j+1}} \cdot \frac{\left(y_j+\frac{1}{p}\right)!}{\left(\alpha+\frac{1}{\bar{I}}\right)^{y_j+\frac{1}{p}+1}}$$

$$\lim_{\bar{I}\to\infty} \prod_{j=1}^{p} \frac{\left(y_j+\frac{1}{p}\right)!}{\alpha^{2y_j+2+\frac{1}{p}}} = \frac{1}{\alpha p}\left(\sum_{j=1}^{p} y_j + 1\right)$$

Problem 3: (a) Let $x(t)$ be an independent increments process on $t \geq 0$ whose covariance function is $K_{xx}(t,s)$, for $t,s \geq 0$. Show that

$$K_{xx}(t,s) = \text{Var}[x(\min(t,s))]$$

for $t,s \geq 0$. (b) Suppose $x(t)$ from (a) has stationary increments. Show that

$$m_x(t) \equiv E(x(t)) = at + b,\ t \geq 0$$

and

$$K_{xx}(t,s) = c\min(t,s) + d,\ t,s \geq 0$$

where a, b are constants and $c \geq 0,\ d \geq 0$ are nonnegative constants.

assume $t > s$

$$K_{xx}(t,s) = E\left[\left(X(t) - m_x(t)\right)\left(X(s) - m_x(s)\right)\right]$$

Because $X(t)$ has independent increments, $= E\left[\left(X(t-s) + X(s) - m_x(t-s) - m(s)\right)\left(X(s) - m(s)\right)\right]$

$$= E\left[\left(X(s) - m(s)\right)\left(X(t-s) + m(t-s)\right) + \left(\left(X(s) - m_x(s)\right)\left(X(s) - m_x(s)\right)\right)\right]$$

$$= 0 + E\left[\left(\left(X(s) - m_x(s)\right)\left(X(s) - m_x(s)\right)\right)\right]$$

$= \text{Var}(X(s))$ If $t < s$ then $= \text{Var}(X(t))$ $\square$

b) $X_t - X_s = X_{t-s}$

$$m_x(t) = E[X(t)] = E[X(t) - X(0)]$$

$$E[X(t)] = E[X(t)] - E[X(0)]$$

$$E[X(0)] = 0$$

$$E[X_t] - E[X_s] = E[X_{t-s}]$$

$f(t) - f(s) = f(t-s)$ $\leftarrow$ this must be a linear function

$\therefore$ $at + b - as - b = a(t-s) + b$ $\therefore$ $m_x = at + b$

$K_{xx}(t,s) = c\min(t,s) + d$

$K_{xx}(t,s) = Var(X(\min(t,s))) = E[X(\min(t,s))^2]$

$= E[Y(\min(t,s))]$

$Y = X^2$ will still be stationary & have independent increments

$\therefore K_{xx} = c\min(t,s) + d$.

Problem 4: Let X and Y be random vectors with known means and covariance matrices. Do not assume zero means. Find the best *purely linear* estimate of X based on Y; i.e., find the matrix A that minimizes $E[||X - AY||^2]$. Similarly, find the best constant estimate of X; i.e., find the vector b that minimizes $E[||X - b||^2]$.

$m_x \quad m_y \quad \Lambda_x \quad \Lambda_y \quad \Lambda_{xy} \quad \Lambda_{yx}$

$$\min_A \left\{ E\left[||X - AY||^2 \right] \right\} = \min_A E\left[(X - AY)(X - AY)^T \right]$$

$$= \min_A E\left[XX^T - AYX^T - X(AY)^T + (AY)(AY)^T \right]$$

$$= \min_A \Lambda_x - A\Lambda_{yx} - \Lambda_{xy} A^T + AA^T\Lambda_y$$

Solve for A after taking derivative

$$-\Lambda_{yx} - \Lambda_{xy} - 2A\Lambda_y = 0$$

$$A = \frac{\Lambda_{yx} + \Lambda_{xy}}{2\Lambda_y^{-1}}$$

$$\min_b E\left[||X - b||^2 \right] = \Lambda_{xx} - bm_x^T - m_x b^T + bb^T$$

$$\frac{d}{db} \to m_x^T - m_x + 2b = 0$$

$$\boxed{b = \frac{m_x^T + m_x}{2}}$$