636 Homework

1.) $A_z(r) = \mu I_0 \, dl \, \frac{e^{-jkr}}{4\pi r}$ $\qquad \nabla^2 A_z + k^2 A_z = 0$

$\nabla^2 A_z(r) + k^2 A_z = 0$ $\qquad \omega^2 \mu \varepsilon (\mu I_0 \, dl) \frac{e^{-jkr}}{4\pi r} + \frac{1}{r^2}\frac{d}{\partial r}\left(r^2 \frac{dA(r)}{\partial r}\right) = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \mu_0 I_0 dl\left(-\frac{e^{-jkr}}{4\pi r^2} - \frac{jk}{4\pi r}e^{-jkr}\right)\right) \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(\mu_0 I dl\left(-\frac{e^{-jkr}}{4\pi} - \frac{jkr}{4\pi}e^{-jkr}\right)\right)$$

$$\frac{1}{r^2}\mu_0 I dl\left(\cancel{+\frac{e^{-jkr}}{4\pi}} - \cancel{\frac{jk}{4\pi}}e^{-jkr} + \frac{j^2k^2}{4\pi}re^{-jkr}\right) = \frac{-\mu_0 I dl k^2}{4\pi r}e^{-jkr} = -k^2 A(r)$$

$-k^2 A(r) + k^2 A(r) = 0 \qquad 0 = 0$ ✓

2) $$A_z(r) = \mu I_0 \, dl \, \frac{e^{-jkr}}{4\pi r} \qquad r = 0$$

$$\nabla^2 A_z + k^2 A_z = -\mu I_0 \, dl \, \delta(r)$$

Note that $\hat{J}(r) = I_0 \, dl \, a_z \, \delta(0)$ is the point source equivalent of a z-directed ideal dipole

$$\nabla^2 A_z + k^2 A_z = -\mu I_0 \, dl \, \delta(r)$$

$$\nabla^2\left(\mu I_0 \frac{e^{-jkr}}{4\pi r}\right) + k^2\left(\mu_0 I_0 \, dl \, \frac{e^{-jkr}}{4\pi r}\right) = -\mu I_0 \, dl \, \delta(r)$$

$$\int_0^{2\pi}\int_0^{\pi}\int_{-a}^{a}\left(\frac{1}{r^2}\frac{\partial}{\partial r^2} r^2 \frac{\mu I_0 e^{-jkr}}{4\pi r} + k^2 \mu_0 I_0 \, dl \, \frac{e^{-jkr}}{4\pi r}\right) r^2 \, dr \, d\theta \sin\theta \, d\phi = -\mu I_0 \, dl$$

$$2\pi \cdot 2 \cdot \int_{-a}^{a} \frac{\partial}{\partial r}\left(\frac{\mu_0 I_0 \, dl}{4\pi}\left(e^{-jkr} - jkr e^{-jkr}\right)\right) + k^2 \frac{\mu I_0 \, dl}{4\pi}\left(r e^{-jkr}\right) dr = -\mu_0 I_0 \, dl$$

$$2\pi 2 \cdot \int_{-a}^{a} \frac{\mu I_0 \, dl}{4\pi}\left(-jk e^{-jkr} - jk e^{-jkr} - \cancel{k^2 r e^{-jkr}} + \cancel{k^2 r e^{-jkr}}\right) dr = -\mu I_0 \, dl$$

$$2\pi \cdot 2 \int_{-a}^{a} \frac{\mu I_0 \, dl}{4\pi}\left(2(-jk) e^{-jkr}\right) dr$$

$$\mu_0 I_0 \, dl \left[2 e^{-jkr}\right]_{-a}^{a} = -\mu_0 I_0 \, dl$$

this should be the same

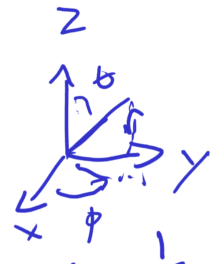
$$\lim_{a \to 0} e^{-jka} - e^{+jka} = -\frac{1}{2}$$

3) $A(r) = a_z \mu I_0 dl \frac{e^{-jkr}}{4\pi r}$

find the E and H for an elemental dipole

$J(r) = I_0 dl\, \delta(r) a_z$

$A_z(r) = a_z \mu I_0 dl \frac{e^{-jkr}}{4\pi r}$



$r\cos\theta = z \qquad r = \frac{z}{\cos\theta}$

$r = \sqrt{x^2+y^2+z^2}$

$A_r = a_r \cdot a_z\, \mu I_0 dl \frac{e^{-jkr}}{4\pi r} = \mu I_0 dl \frac{e^{-jkr}}{4\pi r} \frac{1}{\cos\theta}$

$\bar{h}(\theta,\phi) = \int_V \bar{J}(\bar{r}')\, e^{j}$

$A_\theta = a_\theta \cdot a_z\, \mu I_0 dl \frac{e^{-jkr}}{4\pi r} = -\mu I_0 dl \frac{e^{-jkr}}{4\pi} \sin\theta$

$A_\phi = 0$

$E = -j\omega A - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla\cdot A) \qquad H = \frac{1}{\mu}\nabla\times A$

$= -j\omega\mu I_0 dl \frac{e^{-jkr}}{4\pi r} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot A) = -j\omega\mu I_0 dl \frac{e^{-jkr}}{4\pi r}(\cos\theta\, \hat{a}_r - \sin\theta\, \hat{a}_\theta)$

$\boxed{(\cos\theta a_r - \sin\theta a_\theta)}$

$\nabla = \frac{d}{dx}a_x + \frac{d}{dy}a_y + \frac{d}{dz}a_z$

$= \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$

$-\frac{j}{\omega\mu\varepsilon}\nabla^2 A$

0

$-\frac{j}{\omega\mu\varepsilon}\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2} r^2 A + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}A + \frac{1}{r^2\sin\theta}\frac{\partial^2 A}{\partial\phi^2}\right)$

$= \frac{1}{4\pi r^2}\left(-\frac{j}{\varepsilon} I_0 dl \frac{\partial}{\partial r}\left(e^{-jkr} - jkre^{-jkr}\right)\right)(\cos\theta\hat{a}_r - \sin\theta\hat{a}_\theta)$

$-\frac{1}{r\sin\theta}\frac{I_0 dl}{\varepsilon}\frac{e^{-jkr}}{4\pi r}\frac{\partial}{\partial\theta}\sin\theta(-\sin\theta\hat{a}_r - \cos\theta\hat{a}_\theta)$

$= -\frac{I_0 dl}{4\pi r^2\varepsilon}\left(-2jke^{-jkr} - k^2 r e^{-jkr}\right)(\cos\theta - \sin\theta)$

$-\frac{I_0 dl}{r\sin\theta\,\varepsilon}\frac{e^{-jkr}}{4\pi r}\left(-2\sin\theta\cos\theta\hat{a}_r + (\sin^2\theta - \cos^2\theta)\hat{a}_\theta\right)$

$H = \frac{1}{\mu}\nabla\times A = \frac{1}{\mu}\nabla\times\left(\mu I_0 dl \frac{e^{-jkr}}{4\pi r}(\cos\theta\hat{a}_r - \sin\theta a_\theta)\right)$

$\nabla\times A = \frac{1}{r^2\sin\theta}\begin{vmatrix} a_r & r a_\theta & r\sin\theta a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

$= \mu\frac{1}{r^2\sin\theta}\left(0\cdot a_r - (0)\cdot\hat{a}_\theta\right) + \left(\frac{\partial}{\partial\theta}A_r - \frac{\partial}{\partial r}rA_\theta\right)r\sin\theta a_\phi)$

$= \frac{1}{r}\hat{a}_\phi\left(+\frac{\mu}{\mu}I_0 dl\frac{e^{-jkr}}{4\pi r}\sin\theta + \frac{\mu}{\mu}I_0 dl\, jk\frac{e^{-jkr}}{4\pi}\sin\theta\right)$

$$\boxed{H_\phi = \frac{I_0\, dl\, e^{-jkr}}{4\pi r}\sin\theta\left(\frac{1}{r}+jk\right)}$$

$$\boxed{H_\theta = H_r = 0}$$

$\nabla \times H = j\omega\varepsilon E$ $\quad k = \omega\sqrt{\mu\varepsilon}$

$\eta = 120\pi = \sqrt{\ }$

$$E = \frac{1}{j\omega\varepsilon}\nabla\times H = \frac{1}{j\omega\varepsilon}\frac{1}{r^2\sin\theta}\begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ H_r & rH_\theta & r\sin\theta H_\phi \end{vmatrix} = \frac{1}{j\omega\varepsilon}\frac{1}{r^2\sin\theta}\left(+\hat{a}_r\frac{\partial}{\partial\theta} r\sin\theta H_\phi - \hat{a}_\theta\frac{\partial}{\partial r} r\sin\theta H_\phi\right)$$

$$= \frac{1}{j\omega\varepsilon}\frac{1}{r^2\sin\theta}\left(\hat{a}_r\frac{\partial}{\partial\theta}\left(\frac{I_0\, dl\, e^{-jkr}}{4\pi}\sin^2\theta\left(\frac{1}{r}+jk\right)\right) - \hat{a}_\theta\frac{\partial}{\partial r}\left(\frac{I_0\, dl\, e^{-jkr}}{4\pi}\sin^2\theta\left(\frac{1}{r}+jk\right)\right)\right)$$

$$= \frac{1}{j\omega\mu}\frac{1}{r^2\sin\theta}\frac{I_0\, dl}{4\pi}\left(\hat{a}_r 2\sin\theta\cos\theta\, e^{-jkr}\left(\frac{1}{r}+jk\right)\right)$$

$$-\hat{a}_\phi \sin^2\theta\left(-jk e^{-jkr}\left(\frac{1}{r}+jk\right) - e^{-jkr}\left(r^{-2}\right)\right)\Big)$$

$$\boxed{= \frac{I_0\, dl}{j\omega\mu\, 4\pi}\left(\hat{a}_r 2\cos\theta\, e^{-jkr}\left(\frac{1}{r^3}+\frac{jk}{r^2}\right) - \hat{a}_\phi\sin\theta\left(-jk\, e^{-jkr}\left(\frac{1}{r^3}+\frac{jk}{r^2}-1\right)\right)\right)}$$

$$E_r = \eta\frac{I_0\, dl}{4\pi}2\cos\theta\, e^{-jkr}\left(\frac{1}{r^2}+\frac{1}{jkr^3}\right) \qquad E_\theta = \frac{I_0\, dl}{4\pi}\eta\sin\theta(jk)e^{-jkr}\left(\frac{1}{r^3}+\frac{jk}{r^2}-1\right)$$

4 Radiated power from the elemental dipole $J(r) = I_0\, dl\, \delta(r)\, \hat{a}_z$ $d\Omega = \sin\theta\, d\theta\, d\phi$

$$P_{rad} = \frac{1}{2}\frac{k^2\eta}{(4\pi)^2}\int_0^{2\pi}\int_0^{\pi} |\bar{h}_t(\theta,\phi)|^2\, d\Omega$$

$$= \frac{1}{2}\frac{k^2\eta}{(4\pi)^2}\int_0^{2\pi}\int_0^{\pi} -I_0 dl \sin\theta\, d\Omega$$

$$= \frac{1}{2}\frac{k^2\eta}{(4\pi)^2}\int_0^{2\pi}\int_0^{\pi} -I_0 dl \sin\theta\, d\theta\, d\phi$$

$$\bar{h}(\theta,\phi) = \int_V \bar{J}(\bar{r}')\, e^{j\Omega}\, dv'$$

$$= \int_V I_0\, dl\, \delta(\bar{r})\, a_z\, e^{j\Omega}\, dv$$

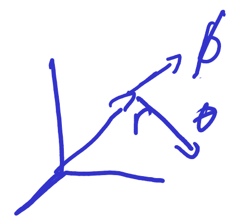
$$= I_0\, dl\, \bar{a}_z$$

$$h_r = I_0\, dl \cos\theta$$

$$h_\theta = -I_0\, dl \sin\theta$$

$$h_\phi = 0$$

$$\bar{h}_t(\theta,\phi) = h_\theta \hat{a}_\theta + h_\phi \hat{a}_\phi$$



Poynting vector

$$\bar{P} = \frac{1}{2}(E \times H^*) = \frac{1}{2}(E_r \hat{a}_r + E_\theta \hat{a}_\theta) \times H_\phi^* a_\phi = \frac{1}{2}\left(E_\theta H_\phi^* \hat{a}_r - \hat{a}_\theta E_r H_\phi^*\right)$$

$$\frac{1}{2}\left(-\frac{I_0 dl}{j\omega\mu\, 4\pi}\sin\theta\left(-jk\, e^{-jkr}\left(-1+\frac{1}{r^3}+\frac{jk}{r^2}\right)\right)\left(\frac{I_0\, dl\, e^{-jkr}}{4\pi r}\sin\theta\left(\frac{1}{r}+jk\right)\right)^* \hat{a}_r\right.$$

$$\left. - \hat{a}_\theta E_r H_\phi^*\right)$$

Total Power Radiated $P_{rad} = \oint_S \bar{P} \cdot d\bar{s}$

Only the radial part will contribute

$$\begin{array}{l}\text{Power}\\ \text{overall}\end{array} = \int_0^{2\pi}\int_0^{\pi} \frac{1}{2} E_\theta H_\phi^* \hat{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} \frac{1}{2} j\eta k \left(\frac{I\,dl}{4\pi r}\right)^2 \sin^2\theta \left[1 + \frac{1}{jkr} - \frac{1}{k^2 r^2}\right]\left(\frac{1}{r} - jk\right) r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\pi} \frac{1}{16} j\eta k \left(I\,dl\right)^2 \frac{1}{\pi r^2} \cdot \left[r^2 + \frac{r}{jk} - \frac{1}{k}\right]\left[\frac{1}{r} - jk\right] \sin^3\theta \, d\theta$$

$$= \int_0^{\pi} \sin^3\theta \, d\theta \left(\frac{1}{16} j\eta k \left(I\,dl\right)^2 \cdot \frac{1}{\pi r^2}\left[\cancel{r} + \cancel{\frac{1}{jk}} - \frac{1}{rk} - jkr^2 - \cancel{r} + \cancel{\frac{1}{k}}\right]\right.$$

$$= \frac{4}{3}\pi \cdot \frac{1}{16} j\eta k \left(I\,dl\right)^2 \cdot \frac{1}{\pi}\left[-jk - \frac{1}{r^3 k}\right] = \frac{\pi}{3}\eta\left(\frac{I\,dl}{\lambda}\right)^2\left(1 - \frac{j}{(\beta r)^3}\right)$$

$$\text{Radiated Power} = \text{Real}\left(\begin{array}{c}\text{overall}\\ \text{Power}\end{array}\right) = \boxed{\frac{\pi}{3}\eta\left(\frac{I\,dl}{\lambda}\right)^2}$$

5) Derive an expression for the radiation resistance of the elemental dipole

Overall Power $= \frac{1}{2} R_r I^2$ $\quad R_r = \frac{2 P_{over}}{I^2} = \frac{2 \cdot \frac{\pi}{3} \eta \left(\frac{I\,dl}{\lambda} \right)^2}{I^2}$

$$\boxed{= \frac{2\pi}{3} \eta \left(\frac{dl}{\lambda} \right)^2}$$