Ecen 601 Homework 2

1. Let $\underline{x} = (x_1, \dots, x_n), \underline{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and consider the function ρ given by

$$\rho(\underline{x}, \underline{y}) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}.$$

Show that ρ is a metric. To be a metric, the function must satisfy

1. $d(x, y) \geq 0, \forall x, y \in X$ equality iff $x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x,y,z \in X$

Proof

1. $p(x,y) \geq 0$

$p(x,y) = \max\{|x_1 - y_1| \dots |x_n - y_n|\} = |x_m - y_m|$ where m is the integer that results in the max

≥ 0 from the definition of Absolute Value

the max Value $= 0$ only when $\forall i \;\; x_i = y_i$

2. $p(x,y) = p(y,x)$ $\qquad |x-y| = |y-x|$

$\downarrow$ Apply

$p(x,y) = \max(|x_1 - y_1| \dots |x_n - y_n|) = \max(|y_1 - x_1| \dots |y_n - x_n|) = p(y,x)$

3. $p(x,y) + p(y,z) \geq p(x,z)$

$i \in \{1 \cdots n\} \mid |x_i - z_i| = p(x,z)$

$P(x,z) = \max(|x_1 - z_1| \dots |x_n - z_n|) = |x_i - z_i| = |x_i - y_i + y_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$

$|x_i - y_i| \leq \max(|x_1 - y_1| \dots |x_n - y_n|) = p(x,y)$

$|y_i - z_i| \leq \max(|y_1 - z_1| \dots |y_n - z_n|) = p(y,z)$ $\quad \therefore |x_i - y_i| + |y_i - z_i| \leq p(x,y) + p(y,z)$

2. Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by

$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$

Show that $\bar{d}$ is also a metric.

because it is a metric

1. $\bar{d}(x,y) \geq 0$ $\bar{d}(x,y) = \min\{d(x,y), 1\} \geq 0$ $\to d(x,y) \geq 0$, $\to 1 \geq 0$

Check if 0 only holds for $x=y$

For any $x \neq y$ $\bar{d}(x,y) = \min\{d(x,y), 1\}$ because $d(x,y)$ is a metric and $x \neq y$ then $d(x,y) > 0$. Because 1 also $\neq 0$ then $\min(d(x,y), 1) > 0$

If $x=y$ $\bar{d}(x,y) = \min\{d(x,y), 1\}$ $d(x,y) = 0$ because $x=y$

$\bar{d}(x,y) = \min\{0, 1\} = 0$

2. $\bar{d}(x,y) = \bar{d}(y,x)$

$\bar{d}(x,y) = \min\{d(x,y), 1\} \overset{\text{bc } d(x,y) \text{ is a metric}}{=} \min\{d(y,x), 1\} = \bar{d}(y,x)$

3. $\bar{d}(x,y) + \bar{d}(y,z) \geq \bar{d}(x,z)$

Because $d(x,z)$ is a metric

$\bar{d}(x,z) = \min\{d(x,z), 1\}$ $\forall x,z \mid d(x,z) \leq 1$ $\bar{d}(x,z) = d(x,z) \leq d(x,y) + d(y,z)$

$d(x,z) \leq 1 + d(y,z)$ as $d(y,z) \geq 0$

similarily $d(x,z) \leq d(x,y) + 1$ and $d(x,z) \leq 1 \leq 2$

$\therefore \forall d(x,z) \leq 1$ $\bar{d}(x,z) \leq \min(d(x,y), 1) + \min(d(y,z), 1)$

$= \bar{d}(x,y) + \bar{d}(y,z)$

$\forall x,z \; d(x,z) > 1 \qquad \bar{d}(x,y) = 1$

$1 < d(x,y) \leq d(x,y) + d(y, \cancel{x})$

$\rightarrow \bar{d}(x,z) < d(x,y) + d(y,z)$

$\bar{d}(x,z) = 1 \leq 1 + d(y,z)$ as $d(y,z) > 0$

similarity $\bar{d}(x,z) = 1 \leq d(x,y) + 1$

$\min(d(x,y), 1) + \min(d(y,z), 1)$

$\bar{d}(x,z) = 1 \leq 2 \therefore \bar{d}(x,z) \leq \min(d(x,y), 1) + \min(d(y,z), 1)$

3. This problem outlines a proof that the Euclidean distance d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, \ldots, x_n+y_n) \qquad c\mathbf{x} = (cx_1, \ldots, cx_n),$$
$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + \cdots + x_ny_n \qquad \|\mathbf{x}\| = (\mathbf{x}\cdot\mathbf{x})^{1/2}.$$

(a) Show that $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{z})$.

(b) Show that $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$. [Hint: If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.]

(c) Show that $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [Hint: Compute $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})$ and apply previous result.]

(d) Verify that the Euclidean distance $d(\mathbf{x}, \mathbf{y})$ is a metric.

a) $x\cdot(y+z) = (x\cdot y) + (x\cdot z)$

$x\cdot(y+z) = x\cdot(y_1+z_1, \ldots, y_n+z_n) = (x_1(y_1+z_1), \ldots x_n(y_n+z_n))$

$= (x_1y_1 + x_1z_1, \ldots x_ny_n + x_nz_n) = x\cdot y + x\cdot z$

$(x+y)(x+y) = x\cdot x + 2x\cdot y + y\cdot y$

b) $\|ax \pm by\| = ((ax \pm by) \cdot (ax \pm by))^{1/2}$ Let $a = \frac{1}{\|x\|}$ $b = \frac{1}{\|y\|}$

$= (a^2x^2 \pm 2abxy + b^2y^2)^{1/2} \geq 0$ Because $\sqrt{\ }$ is monotonically increasing

$a^2x^2 \pm 2abx \cdot y + b^2y^2 \geq 0$

$a^2x^2 + b^2y^2 \geq \mp 2abxy$

$a^2\|x\|^2 + b^2\|y\|^2 \geq \mp 2abxy$

$\frac{\|x\|^2}{\|x\|^2} + \frac{\|y\|^2}{\|y\|^2} \geq \mp 2abxy$

$2 \geq \mp 2abxy$

$-$ $+$

$\frac{-1}{ab} \leq xy$ | $\frac{1}{ab} \geq xy$

Take the absolute value of both sides

$\left|\frac{-1}{ab}\right| \geq |x \cdot y|$ $\left|\frac{1}{ab}\right| \geq |x \cdot y|$

$\|x\|\|y\| \geq \|x \cdot y\|$

c) $||x+y|| \le ||x|| + ||y||$

$$||x+y|| = \sqrt{(x+y)\cdot(x+y)} = \sqrt{x\cdot x + 2xy + y\cdot y}$$

$$= \sqrt{||x||^2 + 2xy + ||y||^2} \le \sqrt{||x||^2 + 2||x||\,||y|| + ||y||^2}$$

inner product always positive

using result from 6 and $\sqrt{\ }$ is monotonically increasing

$$= \sqrt{(||x|| + ||y||)^2} = ||x|| + ||y||$$

d)

Euclidean Distance $d(v,w) = \sqrt{(v_1 - w_1)^2 + \dots + (v_n - w_n)^2}$

$= \|(v-w)\| = ((v-w)\cdot(v-w))^{1/2}$

Metric Requirements

1a)

$d(v,w) \geq 0$

$d(v,w) = \|(v-w)\| \geq 0$ By definition of the norm

1b)

$\|(v-w)\| = 0$ iff $v - w = 0$ $v = w$

2)

$d(v,w) = d(w,v)$

$d(v,w) = \|(v-w)\| = ((v-w)\cdot(v-w))^{1/2} = \sqrt{(v_1-w_1)^2 + \dots (v_n - w_n)^2}$

$i \in \{1, \dots, n\}$

$(v_i - w_i)^2 = (v_i - w_i)\cdot(v_i - w_i) = -1\cdot(v_i - w_i)\cdot(-1)(v_i - w_i) = (w_i - v_i)^2$

$\therefore \sqrt{(v_1-w_1)^2 + \dots + (v_n - w_n)^2} = \sqrt{(w_1 - v_1)^2 + \dots + (w_n - v_n)^2} = \|(w-v)\| = d(w,v)$

3) $d(x,y) + d(y,z) \geq d(x,z)$ $\|x-y\| + \|y-z\| \geq \|x-z\|$

$d(x,z) = \|x-z\| = \|x-y+y-z\|$

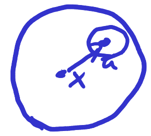
$\leq \|x-y\| + \|y-z\|$ Using part c from th.3 problem

$= d(x,y) + d(y,z)$

4. Suppose $a \in B_d(x, \epsilon)$ with $\epsilon > 0$. Find an explicit $\delta > 0$ such that the open ball $B_d(a, \delta)$ centered at a is contained in $B_d(x, \epsilon)$.

Def $B_d(x, \varepsilon) = \{ y \in X \mid d(x, y) < \varepsilon \}$

$a \in B_d(x, \varepsilon)$ st $d(x, a) < \varepsilon$ let $\delta = \varepsilon - d(x, a)$



then there is a ball $B_d(a, \delta)$ such that $\forall y \in B_d(a, \delta)$ $d(a, y) < \varepsilon - d(x, a)$

$d(x, a) + d(a, y) \geq d(x, y)$

$d(x, a) + d(a, y) < d(x, a) + \varepsilon - d(x, a) = \varepsilon$

$\varepsilon > d(x, a) + d(a, y) \geq d(x, y)$ $\quad \varepsilon > d(x, y) \therefore y \in B_d(x, \varepsilon) \; \forall y \in B_d(a, \delta)$

$B_d(a, \delta) \subset B_d(x, \varepsilon)$ $\square$

x_{lp} iff $f(x)_{lp}$

5. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

let $f: X \to Y$ that is continuous. let x be a limit point of A which is a subset of X

Because x is a limit point of A, $\forall \delta > 0$ the set $\{a \in A \mid d(x,a) < \delta\}$ contains some point besides x

Because f is continuous on x (as $x \in X$)
for any $\varepsilon > 0$ there exists
a $\delta > 0$ such that $\forall a \in A$ satisfying
$d_X(x,a) < \delta$ $\quad d_Y(f(x), f(a)) < \varepsilon$

$\therefore$ $f(x)$ is a limit point of $f(A)$
as $\forall \varepsilon > 0$ $\{f(a) \in f(A) \mid d(f(x), f(a)) < \varepsilon\}$

Optional Problems:

1. The image of a function applied to a set-valued argument is defined by $f(A) \triangleq \{f(x)|x \in A\}$ and $f^{-1}(B) \triangleq \{x \in X|f(x) \in B\}$. Let $f: X \to Y$, $A \subseteq X$, and $B \subseteq Y$.

   (a) Show that $f^{-1}(f(A)) \supseteq A$ and that equality holds if f is injective.

   (b) Show that $f(f^{-1}(B)) \subseteq B$ and that equality holds if f is surjective.

a) $f: X \to Y \quad A \subseteq X \quad B \subseteq Y$

$f(A) \triangleq \{f(x) | x \in A\} \qquad f^{-1}(B) \triangleq \{x \in X | f(x) \in B\}$

$f^{-1}(f(A)) = \{x \in X | f(x) \in f(A)\} = \{x \in A | f(x) \in f(A)\} \cup \{x \in X/\{A\} | f(x) \in f(A)\}$

$= A \cup \{x \in X/\{A\} | f(x) \in f(A)\} \supseteq A$

If f is injective (aka one to one) $x, x' \in X$ if $f(x) = f(x')$ then $x = x'$

this means that $\{x \in X/\{A\} | f(x) \in f(A)\} = \emptyset$

$\therefore A \cup \emptyset = A \qquad f^{-1}(f(A)) = A$

b) $f(f^{-1}(B)) \subseteq B$

$f(f^{-1}(B)) \qquad \exists x \in f^{-1}(B) | f(x) = y \qquad y \in B$

$\therefore f(f^{-1}(B)) \subseteq B$