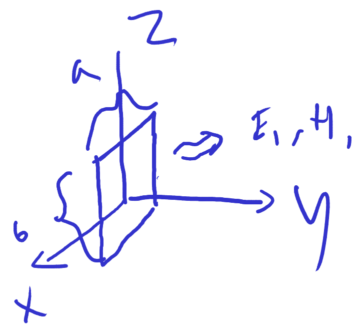
Ecen 636 Homework 2

1

1). Find the far fields of a rectangular aperture centered at the origin in the x-z plane. (Find the fields in the y>0 half-space.) The aperture field is:

$$\vec{E}_a(x,z) = \hat{a}_x E_o \quad |x| \le \frac{b}{2}, |z| \le \frac{a}{2}$$

Fill in all the details, as done in the notes.



Huygen plane
On the x, y Boundary
S, S' normal is in the y direction

$E_a(x,z) = \hat{a}_x E_o \quad |x| \le \frac{b}{2} \quad |z| \le \frac{a}{2}$

Away from the aperture the E field is 0

$\boxed{E_4}$ $M_s = -a_n \times E$ $J_s = ?$

PEC Replacement

$M_s = -2 a_n \times E_a(x,z)$

$J_s = 0$

Everywhere else $M_s = 0$ $J_s = ?$

$M_s = 2(a_y \times a_x) E \quad |x| \le \frac{b}{2} \quad |z| \le \frac{a}{2}$

$= 2E_o a_z \quad |x| \le \frac{b}{2} \quad |z| \le \frac{a}{2}$

FT

$\bar{A}(\bar{r}) = \mu G(r) \tilde{\tilde{J}}_s(k) = 0$

$\bar{F}(\bar{r}) = \varepsilon G(r) \tilde{\tilde{M}}_s(k)$

$\bar{F}(\bar{r}) = \varepsilon G(r) 2E_o \operatorname{sinc}\left(\frac{k_x b}{2}\right) \operatorname{sinc}\left(\frac{k_z a}{2}\right) a \cdot b \cdot \hat{a}_z$

$F\{M_s\} = \int_{-a/2}^{a/2}\int_{-b/2}^{b/2} 2E_o \hat{a}_z e^{jk_x x} e^{jk_z z}$

$= 2E_o a \cdot b \cdot \operatorname{sinc}\left(\frac{k_x b}{2}\right) \operatorname{sinc}\left(\frac{k_z a}{2}\right)$

$\int_{-a/2}^{a/2} c e^{jk_x x} dx = \left. \frac{c e^{jk_x x}}{jk_x} \right|_{-a/2}^{a/2} = \frac{c}{jk_x}\left(e^{jk_x \frac{a}{2}} - e^{-jk_x \frac{a}{2}}\right) = \frac{c}{k_x}\left(2\sin\left(\frac{k_x a}{2}\right)\right)$

$= \frac{2ca}{k_x a} \sin\left(\frac{k_x a}{2}\right) = c a \operatorname{sinc}\left(\frac{k_x a}{2}\right)$ 1D

unnormalized

$H_F(\bar{r}) = -j\omega F_T(r)$

$= -j\omega \varepsilon G(r) \tilde{\tilde{M}}_s^T$

$\cancel{z = r\cos\theta}$

$y = \cancel{r\sin\theta}\sin\psi$

$r^2 = x^2 + y^2 + z^2$

$x = r\sin\theta\cos\phi$

$\cancel{z\sin\theta = \hat{\theta}}$

$$\begin{bmatrix} r \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$X_{\hat{r}} = {}^{x}\hat{z}\cos\theta$

$X_{\hat{\theta}} = -{}^{x}\hat{z}\sin\theta$

$$\bar{\tilde{M}}_\phi = 0$$

$$\bar{\tilde{M}}_r = 2E_0\,\mathrm{sinc}\left(\frac{k_x b}{2}\right)\mathrm{sinc}\left(\frac{k_y a}{2}\right)a.b\cos\theta$$

$$\bar{\tilde{M}}_\theta = -2E_0\,\mathrm{sinc}\left(\frac{k_x b}{2}\right)\mathrm{sinc}\left(\frac{k_y a}{2}\right)a.b\cdot\sin\theta$$

$$H_F = -j\omega\varepsilon G(r)\left(M_r\hat{r} + M_\theta\hat{\theta}\right)$$

$$E_F = -\eta\bar{H}\times a_r = -\eta\left(-j\omega\varepsilon G(r)M_\theta(-\hat{\phi})\right) = -j\eta\omega\varepsilon\, G(r)M_\theta\tilde{\hat{\phi}}$$

4)

4) Show that $\vec{E} = -j\omega\vec{A}_T$ can be obtained from $\vec{E} = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{A})$ (1)

You can assume that we have a z directed current if you wish.

$$A = \hat{a}_r A_r(r,\theta,\phi) + \hat{a}_\theta A_\theta(r,\theta,\phi) + \hat{a}_\phi A_\phi(r,\theta,\phi) \simeq \left[\hat{a}_r A_r'(\theta,\phi) + \hat{a}_\theta A_\theta'(\theta,\phi) + \hat{a}_\phi A_\phi'(\theta,\phi)\right]\frac{e^{-jkr}}{r} \quad (2)$$

Neglect terms of a higher order than $\frac{1}{r}$

Substituting (2) into (1)

$$E = -j\omega\left[\hat{a}_r A_r'(\theta,\phi) + a_\theta A_\theta'(\theta,\phi) + a_\phi A_\phi'(\theta,\phi)\right]\frac{e^{-jkr}}{r} - \frac{j}{\omega\mu\varepsilon}\nabla\left(\nabla\cdot\left(\left[\hat{a}_r A_r'(\theta,\phi) + \hat{a}_\theta A_\theta'(\theta,\phi) + \hat{a}_\phi A_\phi'(\theta,\phi)\right]\frac{e^{-jkr}}{r}\right)\right)$$

$$= \cdots - \frac{j}{\omega\mu\varepsilon}\nabla\left(\frac{1}{r^2}\frac{\partial r e^{-jkr}}{\partial r}A_r'(\theta,\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}A_\theta'\sin\theta\frac{e^{-jkr}}{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi'(\theta,\phi)\frac{e^{-jkr}}{r}\right)$$

$$= \cdots - \frac{j}{\omega\mu\varepsilon}\nabla\left(\frac{1}{r^2}\left(\cancel{e^{-jkr}}^{\,0} - jkre^{-jkr}\right)A_r'(\theta,\phi) + \cancel{\frac{e^{-jkr}}{r^2\sin\theta}\frac{\partial}{\partial\theta}A_\theta\sin\theta}^{\,0} + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}A_\phi(\theta,\phi)e^{-jkr}}^{\,0}\right)$$

Our Approximation

$$= -j\omega\left[\hat{a}_r A_r(\theta,\phi) + \bar{a}_\theta A_\theta(\theta,\phi) + \bar{a}_\phi A_\phi(\theta,\phi)\right]\frac{e^{-jkr}}{r} - \frac{j}{\omega\mu\varepsilon}\frac{\partial}{\partial r}\left(-jk\frac{e^{-jkr}}{r}A_r(\theta,\phi)\right) = \cdots - \frac{j}{\omega\mu\varepsilon}\left(\cancel{+\frac{jk\,e^{-jkr}}{r^2}}^{\,0} - \frac{k^2e^{-jkr}}{r}\right)A_r(\theta,\phi)$$

$$= -j\omega\left[a_r A_r(\theta,\phi) + a_\theta A_\theta(\theta,\phi) + a_\phi A_\phi(\theta,\phi)\right]\frac{e^{-jkr}}{r} + \frac{jk^2}{\omega\mu\varepsilon}\frac{e^{-jkr}}{r}A_r(\theta,\phi)$$

$K = \omega^2\mu\varepsilon$

$$= -j\omega\left[\hat{a}_\theta A_\theta(\theta,\phi) + \hat{a}_\phi A_\phi(\theta,\phi)\right] = \boxed{-j\omega A_T}$$