Jake Ruff Homework 6 Econ 646

Problem 8.4

a) Erlang order 13 — Sum of 13 independent exponentials

$$\boxed{f(t) = \frac{\lambda(\lambda t)^{k}}{k!} e^{-\lambda t}}$$

b) $P\{N_t = k\} = P\{13k \le N_t < 13(k+1)\}$

Poisson with parameter lambda

$$= \sum_{a=13k}^{13k+13} \frac{(\lambda t)^a e^{-\lambda t}}{a!} = e^{-\lambda t} \sum_{a=13k}^{13k+13} \frac{(\lambda t)^a}{a!}$$

8.10

$$P[N_t=k] = \sum_{i\in E} \pi(i) \frac{e^{-\lambda(i)t}(\lambda(i)t)^k}{k!}$$

$$P[N_t(a)=k \mid a=i] = \frac{e^{-\lambda(i)t}(\lambda(i)t)^{k!}}{k!}$$

$$P[N_t=k] = \sum_{i\in E} P[N_t(a)=k \mid a=i]\, P[i]$$

$$= \boxed{\sum_{i\in E} \frac{\pi(i)\, e^{\lambda(i)t}(\lambda(i)t)^{k!}}{k!}}$$

b) Show that it doesn't have independent increments

$$P[N_{t+s}=0] \neq P[N_t=0]\, P[N_{t+s}-N_t=0]$$

$$P[N_{t+s}=0] = \sum_{i\in E} \pi(i)\, e^{-\lambda(i)(t+s)}$$

$$P[N_t=0] = \sum_{i\in E} \pi(i)\, e^{-\lambda(i)t}$$

$$P[N_{t+s}-N_t=0] = \sum_{i\in E}$$

Problem 3 $[0,T]$ n points placed independently and randomly

$X_1 \dots X_k$ the number of points between $a_j\ b_j$

Prob point i is in interval k $\left(\frac{1}{T}(b_k - a_k)\right) = p_j$

X_1

$$P(X_1 = x_1 \dots X_k = x_k) = \frac{n!}{x_1! \dots x_k!\,(n - x_1 - \dots x_k)!} \prod_i^k \left(\frac{1}{T}(b_i - a_i)\right)^{x_i}$$

$$= \frac{n!}{x_1! \dots x_k!\,(n - x_1 \dots - x_k)!} \left(\frac{1}{T}\right)^{x_1 + \dots + x_k} \prod_i^k (b_i - a_i)^{x_i}$$

$$= \frac{n^{x_1 + x_2 + \cancel{x_3} \dots + x_k}}{x_1! + \dots x_k!} \left(\frac{1}{T}\right)^{x_1 + \dots + x_k} \prod_i^k (b_i - a_i)^{x_i}$$

$$= \frac{(\lambda)^{x_1 + x_2 + \dots + x_k}}{x_1! \dots x_k!} \prod_i^k (b_i - a_i)^{x_i}$$

Problem 4.

a) $P[T_{x1} < T_{y1}] = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

b) $P[N_{x60} + N_{y60} = 4] = P[N_{Z60} = 4] = \boxed{\frac{e^{-(\lambda_1+\lambda_2)60}((\lambda_1+\lambda_2)60)^4}{4!}}$

Z is poisson $\{\lambda = \lambda_1 + \lambda_2\}$

c) $P[N_{x60} = 4 \mid N_{x60} + N_{60} = 4] = \boxed{\frac{e^{-\lambda_1 60}(\lambda_1 60)^4}{e^{-(\lambda_1+\lambda_2)60}((\lambda_1+\lambda_2)60)^4}}$

d) T denote to a first customer store 2

$$P[X_T = k] = P[X_T = k \mid T]\, P[T]$$

$$= \frac{e^{-T\lambda_1}(\lambda_1 T)^k}{k!}$$

Problem 5 The random process $X(t)$ is mean square continuos at point t_0 if as $t \to t_0$ $E[(X(t)-X(t_0))^2] = 0$

random process N

$$\lim_{t \to t_0} E[(N(t)-N(t_0))^2] = \lim_{t \to t_0} E[N(t)^2 - 2N(t)N(t_0) + N(t_0)^2]$$

$$= \lim_{t \to 0} E[N(t)^2] - 2E[N(t)N(t_0)] + E[N(t_0)^2]$$

Problem 6 Generating a poisson random variable

$U_1, U_2 \dots (0,1)$ independent uniform random variables

a) $X_i = -(\log U_i)/\lambda$ $\quad P[X_i < x] = P\left[\frac{-\log U_i}{\lambda} < x\right]$

$$= P[U_i > e^{-\lambda x}] = 1 - P[U_i < e^{-\lambda x}]$$

$$= 1 - \int_0^{e^{-\lambda x}} du = \boxed{1 - e^{-\lambda x} = F_x(X_i)}$$

b) $\prod_{i=1}^{n} U_i \geq e^{-\lambda} > \prod_{i=1}^{n+1} U_i \xrightarrow[\log]{\text{Apply}} \sum_{i=1}^{n} \log U_i \geq -\lambda > \sum_{i=1}^{n+1} \log U_i \longrightarrow$

$\longrightarrow \sum_{i=1}^{n} \frac{-\log U_i}{\lambda} \leq 1 < \sum_{i=1}^{n+1} -\frac{\log U_i}{\lambda} \xrightarrow[\text{Sub part a}]{} \sum_{i=1}^{n} X_i \leq 1 < \sum_{i=1}^{n+1} X_i$

N is the value of n such that the sum of n exponential random variables is less than 1 while n+1 is greater than 1

The arrival intervals times are exponential N is poisson