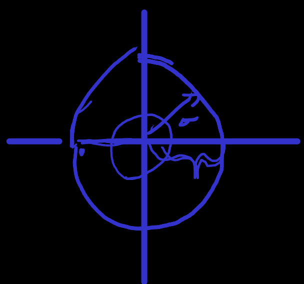
Problem 1: A dart is thrown at random at a wall. Let (x, y) denote the Cartesian coordinates of the point in the wall pierced by the dart. Suppose that x and y are statistically independent Gaussian random variables, each with mean zero and variance σ^2, i.e.,

$$p_x(a) = p_y(a) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2}}.$$

a) Find the probability that the dart will fall within the σ-radius circle centered at the point $(0, 0)$.
b) Find the probability that the dart will hit in the first quadrant $(x \geq 0,\ y \geq 0)$.
c) Find the conditional probability that the dart will fall within the σ-radius circle centered at $(0, 0)$ given that the dart hits the first quadrant.
d) Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ be the polar coordinates associated with (x, y). Find $\mathbb{P}[0 \leq r \leq R,\ 0 \leq \theta \leq \Theta]$ and obtain density $p_{r,\,\theta}(R, \Theta)$.

a) $P[\text{Dart in } \sigma \text{ circle}] = P[x^2+y^2 < \sigma^2] = P[y^2 < \sigma^2 - x^2]$



$P[0 < X < \sigma] = \Phi(1) -$

$\pi P(|X| < \sigma)^2 =$ in circle

b) Symmetric around the origin.

$P[\text{First Quadrant}] = \frac{1}{4}$

c) Same as a).

$$P_r(R<r) = P\left(\sqrt{x^2+y^2} < r\right) = P\left[x^2+y^2 < r^2\right]$$

$$P\left[y < \pm\sqrt{r^2-x^2}\right]$$

$$P\left[\Theta < \theta\right] = P\left[\tan^{-1}(y/x) < \theta\right] = P\left[y < x\tan\theta\right]$$

Problem 2: Let x_1 and x_2 be zero-mean jointly Gaussian random variables with covariance matrix

$$\Lambda_x = \begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix}$$

(a) Verify that Λ_x is a valid covariance matrix.
(b) Find the marginal probability density for x_1. Find the probability density for $y = 2x_1 + x_2$.
(c) Find a linear transformation defining two new variables

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so that x_1' and x_2' are statistically independent and so that $TT^t = I$.

a) $\Lambda_{xy} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$ $\quad \rho = \dfrac{E[xy]}{\sigma_x\sigma_y}$ $\quad \sigma_{x_1} = \sqrt{34} \quad \sigma_{x_2} = \sqrt{41}$

$E[X_1X_2] = 12$

Λ_x is symmetric

$\begin{vmatrix} 34-\lambda & 12 \\ 12 & 41-\lambda \end{vmatrix} = (34-\lambda)(41-\lambda) - 144 = 0$

Because both eigenvalues > 0

$\lambda^2 - 75\lambda + (34)41 - 144 = 0$

Λ_x is positive definite

$\Rightarrow$ positive semidefinite

$\Rightarrow$ it is a valid covariance matrix

$\lambda^2 - 75\lambda + 1230 + 164 - 144 = 0 \Rightarrow \lambda^2 - 75\lambda + 1250 = 0$

$(\lambda - 37.5)^2 + 1250 - (37.5)^2 = 0$

$\lambda_1 = \underbrace{\sqrt{-1250 + (37.5)^2}}_{12.5} + 37.5 \quad \lambda_2 = -\underbrace{\sqrt{-1250 + 37.5^2}}_{< 37.5} + 37.5$

$\lambda_1 > 0 \quad \lambda_2 > 0$

$\lambda_1 = 50 \quad \lambda_2 = 25$

B) Find marginal probability for X_1

$$f_{X_1}(x) = \frac{e^{-\frac{x^2}{2\sqrt{34}}}}{\sqrt{2\pi 34}} \qquad x \in \mathbb{R}$$

$$f_Y(y) = \frac{e^{-\frac{y^2}{2\sqrt{34+41+34}}}}{\sqrt{2\pi(34+41+34)}} \qquad y = 2X_1 + X_2 = \underbrace{X_1 + X_1}_{} + X_2$$

c) $\operatorname{diag} \Lambda_X \qquad \Lambda_X = \begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix}$

For λ_1, $(\Lambda_X - \lambda_1 I)\vec{v} = \begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$-16v_1 + 12v_2 = 0$

$12v_1 - 9v_2 = 0$

$4v_1 - 3v_2$

$v_1 = 3/4 \quad v_2 = 1$

For λ_2 $(\Lambda_X - \lambda_2 I)\vec{v} = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = 0$

$9v_3 + 12v_4 = 0$

$3v_3 + 4v_4$

$v_3 = 1 \quad v_4 = -\frac{3}{4}$

$$E = \begin{bmatrix} 3/4 & 1 \\ 1 & -3/4 \end{bmatrix}$$

Problem 3: Let $\underline{z}$ be an $N+M$ dimensional Gaussian random vector with mean vector $\underline{m}_z$ and covariance matrix Λ_z. The random vector $\underline{z}$ is partitioned into a N-vector $\underline{x}$ and an M-vector $\underline{y}$ as shown below

$$\underline{z} = \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix}, \quad \underline{m}_z = \begin{bmatrix} \underline{m}_x \\ \underline{m}_y \end{bmatrix}, \quad \Lambda_z = \begin{bmatrix} \Lambda_x & \Lambda_{xy} \\ \Lambda_{xy}^t & \Lambda_y \end{bmatrix}.$$

We wish to find an expression for $p_{\underline{x}/\underline{y}}(\underline{X}/\underline{Y})$. We shall do this in three steps. Express all your results in terms of $\underline{m}_x$, $\underline{m}_y$, Λ_x, Λ_y and Λ_{xy}. Assume Λ_x and Λ_y are nonsingular.

(a) For any $N \times M$ matrix A it is clear that the transformation from $\underline{z}$ to $\underline{z}'$ given by

$$\underline{z}' = \begin{bmatrix} \underline{x} - A\underline{y} \\ \underline{y} \end{bmatrix} = \begin{bmatrix} \underline{x}' \\ \underline{y}' \end{bmatrix}$$

is $1:1$, i.e., knowledge of $\underline{x}$ and $\underline{y}$ is equivalent to knowledge of $\underline{x}'$ and $\underline{y}$. Find the matrix A which makes $\underline{x}'$ and $\underline{y}$ statistically independent random variables.

(b) Using the matrix A found in part a) find the marginal density $p_{\underline{x}'}(\underline{X}')$.

(c) Use the transformation of part a) to write $p_{\underline{x},\underline{y}}(\underline{X},\underline{Y})$ in terms of $p_{\underline{x}',\underline{y}}(\underline{X}',\underline{Y})$. Now use the statistical independence of $\underline{x}'$ and $\underline{y}$ and the result of part b) to find $p_{\underline{x}/\underline{y}}(\underline{X}/\underline{Y})$.

a) $\underline{x} - A\underline{y}$ independent y

$\mathrm{Cov}(\underline{x} - A\underline{y}, \underline{y})$

$= \mathrm{Cov}[x y^T] - \mathrm{Cov}(-A y y^T)$

$= \Lambda_{xy^T} + A\Lambda_y = 0$

$$\boxed{A = -\Lambda_{xy}\Lambda_y^{-1}}$$

b) $p_{x'}(X') = \bar{e}$

$$\lambda = \frac{(\Lambda_x + \Lambda_y) \pm \sqrt{(\Lambda_x + \Lambda_y)^2 - 4 \cdot (\Lambda_x \Lambda_y - I)}}{2}$$

$$(\Lambda_x - \lambda_+) V_1 + \Lambda_{x,y} V_2 = 0$$

$$(\Lambda_x - \lambda_-) V_3 + \Lambda_{x,y} V_4 = 0$$

$$\underbrace{\begin{bmatrix} V_1 & V_3 \\ V_2 & V_4 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} =$$

x $\Lambda_{x,y}$ y $\begin{bmatrix} \omega \\ u \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -V_2 & V_4^{-1} \end{bmatrix}}\begin{bmatrix} \omega \\ u \end{bmatrix}$$

$$u = V_2 x + V_4 y$$

$\Lambda_{x,y}$

b)

Problem 4: Let Θ be a random variable uniformly distributed on $[0, 2\pi]$. Show that $X = \sin\Theta$ and $Y = \cos\Theta$ are uncorrelated.

$$E\Big[\big(\sin\theta - E[\sin\theta]\big)\big(\cos\theta - E[\cos\theta]\big)\Big]$$

$$= E\Big[\sin\theta\cos\theta - \cancel{E[\sin\theta]\cos\theta} - \cancel{\sin\theta E[\cos\theta]} + \cancel{E[\cos\theta]E[\sin\theta]}\Big]$$

$$E[\sin\theta] = \int_0^{2\pi} \frac{1}{2\pi}\sin\theta\, d\theta = \frac{-\cos\theta}{2\pi}\Big|_0^{2\pi} = 0 \qquad E[\cos\theta] = \int_0^{2\pi}\frac{1}{2\pi}\cos\theta\, d\theta$$

$$= \frac{\sin\theta}{2\pi}\Big|_0^{2\pi} = 0$$

$$= E[\sin\theta\cos\theta] = \int_0^{2\pi} \sin\theta\cos\theta\, d\theta \qquad u = \sin\theta \quad du = \cos\theta\, d\theta$$

$$= \int_0^0 u\, du = \boxed{0}$$ uncorrelated.

Problem 5: Show that for all random variables X and Y with finite variance, and for all real numbers a, b, c, d

$$\rho_{aX+b,\, cY+d} = \rho_{X,\, Y}\ sgn(ac)$$

where $sgn(ac) = 1$ if $ac > 0$, -1 if $ac < 0$, and 0 if $ac = 0$.

$$P_{X,Y}(X,Y) = \frac{E[XY] - E[X]E[Y]}{\sqrt{E[X^2] - E[X]^2}\sqrt{E[Y^2] - E[Y]^2}}$$

$$\rho_{aX+b,\,cY+d} = \frac{E[(aX+b)(cY+d)] - E[aX+b]E[cY+d]}{\sqrt{E[(aX+b)^2] - E[aX+b]^2}\sqrt{E[(cY+d)^2] - E[cY+d]^2}}$$

$$= \frac{acE[X,Y] + bcE[Y] + daE[X] + db - (acE[X]E[Y] + bcE[Y] + daE[X] + bd)}{\sqrt{a^2E[x^2] + 2abE[x] + b^2 - a^2E[x]^2 - 2abE[x] - b^2}\sqrt{c^2E[y^2] + 2cdE[y] + d^2 - c^2E[y]^2 - 2cdE[y] - d^2}}$$

$$= \frac{ac(E[X,Y] - E[X,Y])}{\sqrt{a^2(E[x^2] - E[x]^2)}\sqrt{c^2(E[y^2] - E[y]^2)}} = \left(\frac{ac}{a^2c^2}\right)\rho_{X,Y} = \boxed{sign(ac)\,\rho_{X,Y}}$$