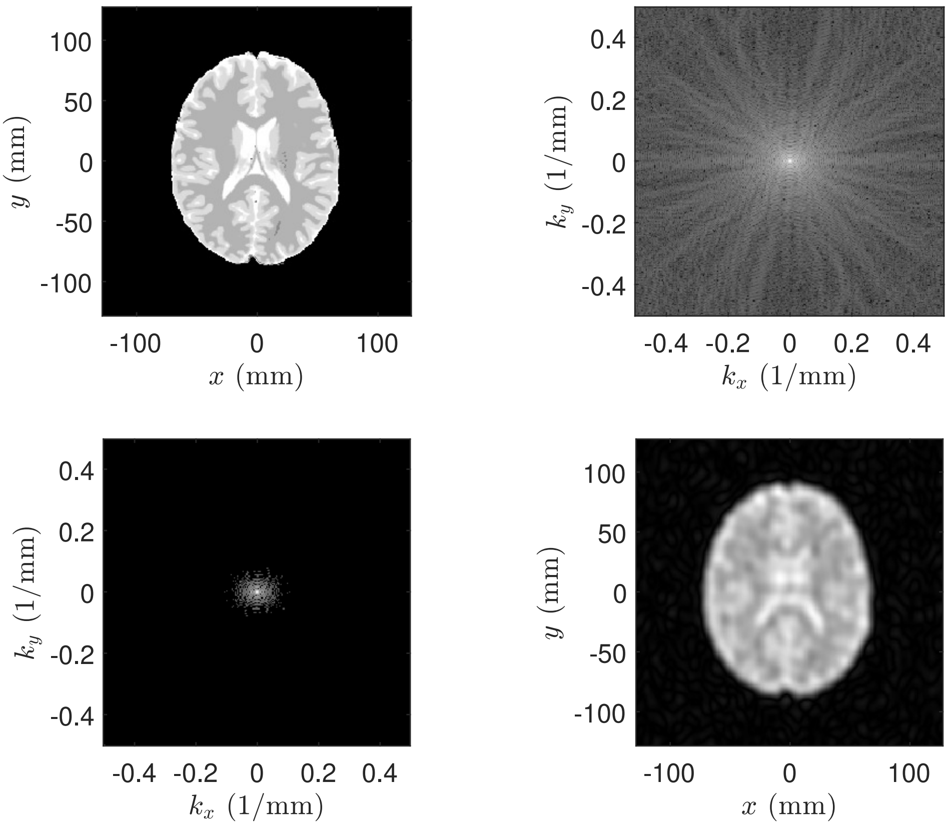
Coursework 3



University College London

Medical Physics & Biomedical Engineering department

MRes - Coursework 3

COMP0121 - Computational MRI

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This report along with the zip file containing animations make up the coursework 3 submission.

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1 Problem 1: Digital Fourier Transform 1-D

1.1 Task 1

A Field Of View (FOV) of 8 mm corresponds, in k-space, to a k-space spacing (Δk) of $\frac{1}{8}$ 1/mm. There are four important factors that should be taken into account when determining the k-space locations and number. 1) In order to use the Inverse Fast Fourier Transform IFFT with no padding, there should be 2^p points sampled. 2) The same amount of points used in k-space should be used in image space to ensure an efficient number of k-space values are obtained. 3) The peak signal where k=0 should be captured. 4) The k-space values should be approximately symmetric across axes. These factors act as constraints the values of k-space sampled. The points sampled given our constraints are:

```
k = linspace(Δ k × N/2, Δ k × (N/2-1), N)
```

Where N is the number of points 128(=2^7), and Δk is $\frac{1}{8}$ 1/mm

1.2 Task 2

As we are prescribing data in the frequency domain and wanting to ascertain what this data is in the spatial domain we use the discrete inverse fourier transform (maps from frequency to spatial domains). We utilise the inverse fast fourier transform (IFFT) which is very computationally efficient. The IFFT takes a discrete inverse fourier transform using only a vector of magnitudes of each frequency (i.e. the signal at each frequency), not the frequency and magnitude which are vectors 'k' and 'signal' respectively in the code. Thus the information of value of the frequency is lost. Instead the IFFT assumes the zeroth frequency is the first value of the vector, linearly increasing to the final value in the vector. As the initial value of the vector is the point we specified as $\Delta k \times \frac{N}{2}$ this effectively shifts k-space by $\delta(k + \Delta k \times \frac{N}{2})$. Therefore we need to pre-shift then re-shift the domain, firstly to format for the IFFT then secondly to shift in image domain. This can be done by:

```
x = fftshift(ifft(ifftshift(signal)))
```

With an impulse of 1 at k=0, if there was no shifting a frequency of $\rho(x) = \frac{1}{N}\Delta k e^{i2\pi\Delta k\frac{N}{2}x}$ would be observed. The, correct, shifted image is a constant of $\rho = 1$ across the spatial domain. This is visualised in Fig. 1.

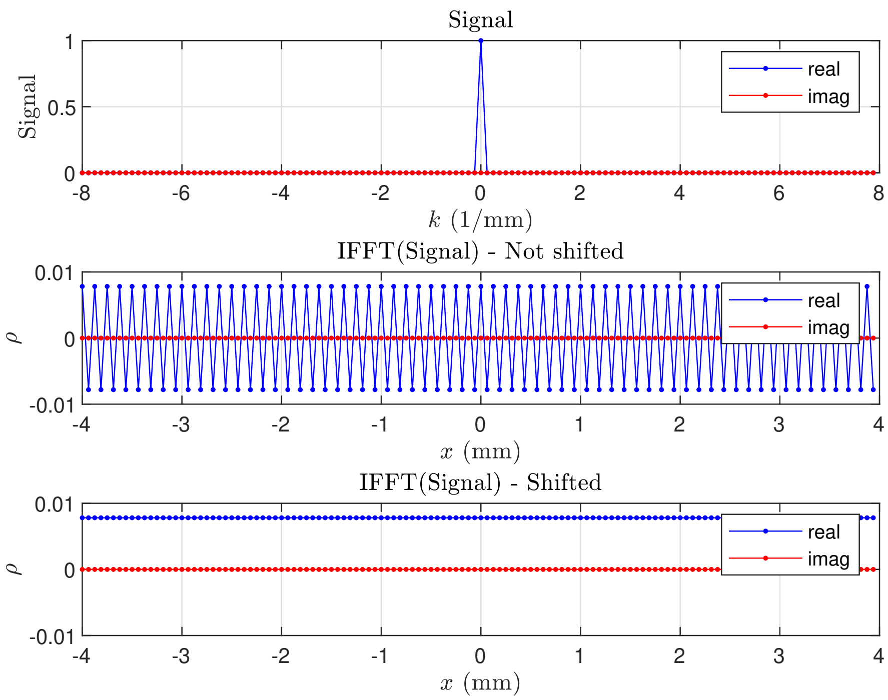

Figure 1: Problem 1 - task 2, effect of IFFT on not shifted and shifted on a unit impulse k-space at k=0. Top is k-space, middle is not shifted and bottom is shifted.

1.3 Task 3

This task further demonstrates the effect of shifting property of the discrete Fourier transform. The impulse is shifted such that $\delta(k-k_0)$, where $k_0 = n\Delta k$ and $-\frac{N}{2} \leq n \leq \frac{N}{2} - 1$. Therefore $\rho(x) = \frac{1}{N} e^{2\pi i k_0 x}$. As before by shifting the impulse further away from the origin, increasing $|k_0|$, it has the effect of increasing the frequency of spin density seen in the image domain. This is seen in animation 'T3_Animation'.

1.4 Task 4

Using the definition of a 1D Gaussian:

$$G(k) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(k-\mu)^2}{2\sigma^2}}$$

Where $\mu = 0$ as the Gaussian mean is $k = 0$, additionaly the variance is specified as $\sigma = 8\Delta k$. By applying a impulse function to the Gaussian, this results in the signal $G(k)\delta(k-k_0)$ where k_0 has the previous definition. It can be easily seen that the effect of the impulse is to pick out individual points of the Gaussian, specifically, the point sampled is at $G(k = k_0)$. Therefore we can say that the signal $G(k)\delta(k-k_0) = G(k_0)\delta(k-k_0)$. In animation 'T4_Animation'. The linearity plot shows that shifting and scaling the impulse $\rho(x) = G(k_0)\frac{1}{N}e^{2\pi i k_0 x}$ gives the same result as a brute force IFFT that directly computes $G(k)\delta(k-k_0)$. The linearity aspect is that $G(k_0)$ is effectively a scalar that can be moved outside of the discrete inverse Fourier transform due to linearity.

1.5 Task 5

As previously described the linearity of discrete inverse Fourier transform can be used to decompose problems. Further to the last task the discrete inverse Fourier transform of the whole Gaussian can be decomposed into a summation of scaled and shifted impulses, such that:

$$\rho(x) = \mathcal{F}^{-1}G(k) = \sum_{n=\frac{N}{2}}^{\frac{N}{2}-1} \mathcal{F}^{-1}G(n\Delta k)\delta(k-n\Delta k)$$

As previously discussed $G(n\Delta k)$ is effectively a scalar, this provides the scaling to the impulse, and can be brought outside the discrete inverse Fourier transform:

$$\sum_{n=\frac{N}{2}}^{\frac{N}{2}-1} \mathcal{F}^{-1}G(n\Delta k)\delta(k-n\Delta k) = \sum_{n=\frac{N}{2}}^{\frac{N}{2}-1} G(n\Delta k)\mathcal{F}^{-1}\delta(k-n\Delta k)$$

Furthermore the discrete inverse Fourier transform of a shifted impulse has a given solution:

$$\sum_{n=\frac{N}{2}}^{\frac{N}{2}-1} G(n\Delta k)\mathcal{F}^{-1}\delta(k-n\Delta k) = \sum_{n=\frac{N}{2}}^{\frac{N}{2}-1} G(n\Delta k)\frac{1}{N}e^{i2\pi\Delta knx}$$

Giving the solution:

$$\rho(x) = \mathcal{F}^{-1}G(k) = \sum_{n=\frac{N}{2}}^{\frac{N}{2}-1} G(n\Delta k)\frac{1}{N}e^{i2\pi\Delta knx}$$

A comparison of the summation of scaled and translated impulses and brute force IFFT can be seen in Fig. 2. There is no difference between the two methods, as expected.

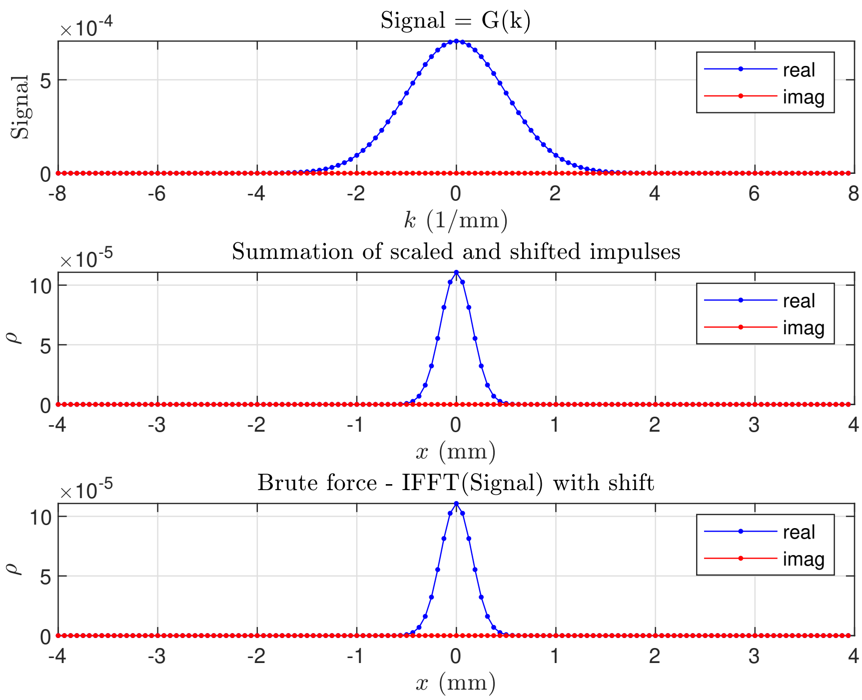

Figure 2: Problem 1 - task 5, inverse Fourier transform via the summation of scaled and shifted impulses and brute force IFFT.

1.6 Task 6

This task requires the addition of a rect function to act as a filter to remove high frequency components of the signal. The width of the rect function is $16\Delta k$. The point spread function (PSF) of a rect function is a summation of the frequency values over which the value of the rect is 1. More formally, $PDF = \sum_{p=-16}^{16} \frac{1}{N} e^{i2\pi\Delta kpx}$. The PSF is approximately a sinc function, and it can be noted that leads Gibbs ringing. Comparing Figs. 2 and 3, the effect of the rect filter is to diffuse the spin density across the spatial dimension (blurring).

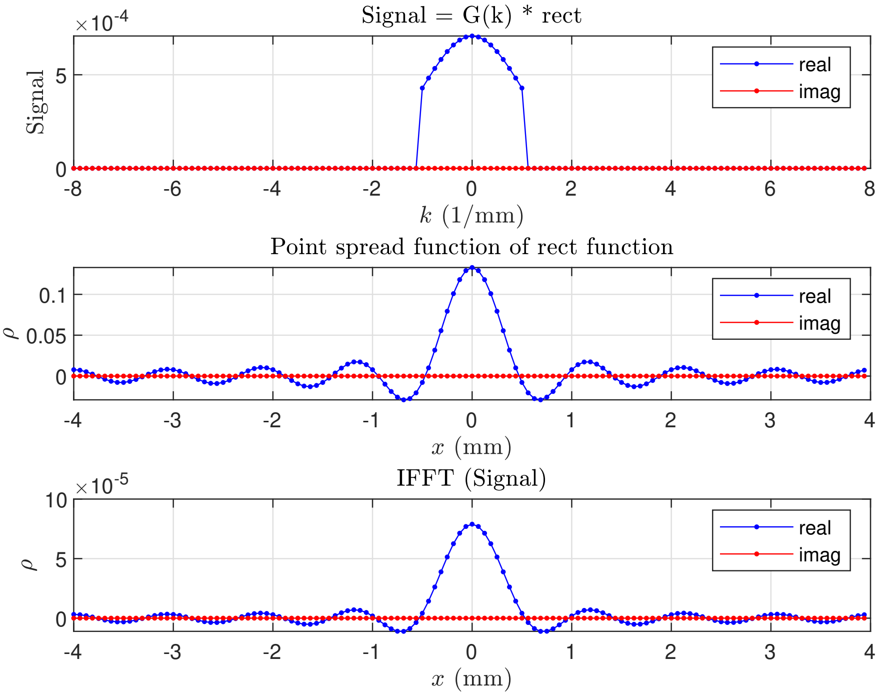

Figure 3: Problem 1 - task 6, filtered Gaussian by rect function, the point spread function of rect and the IFFT of the signal.

1.7 Task 7

The Hanning function is a smooth filter. It also referred to as the cosine bell as it takes a shifted and scaled period of the cosine function, where the gradient goes from zero -> positive -> zero -> negative -> zero. The PSF is defined by $PSF = \sum_{p=-16}^{16} Hann(\Delta kp) \frac{1}{N} e^{i2\pi\Delta kpx}$. It can be seen that the PDF of the Hanning function is non-zero from approximately -1 to 1. Comparing Figs. 2 and 4, it can be said that applying the filter to the Gaussian results in blurring the image. Comparing it with the Fig. 3, the period nature is not observed i.e. no Gibbs ringing, blurring is more concentrated for the Hanning filter and the peak of the ρ is decreased a lot more.

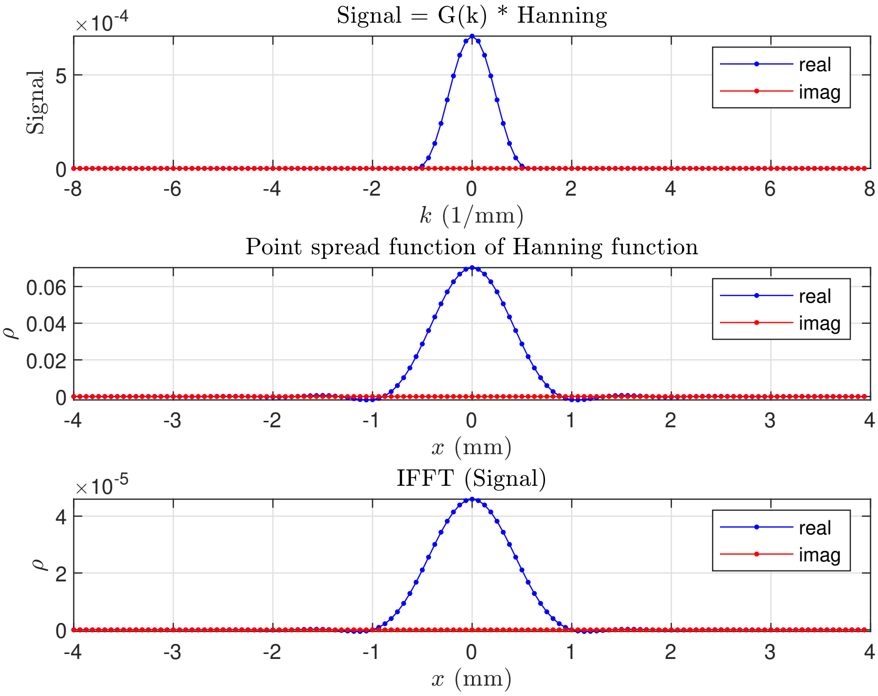

Figure 4: Problem 1 - task 7, filtered Gaussian by Hanning function, the point spread function of Hanning and the IFFT of the signal.

2 Problem 2: Digital Fourier Transform in 2-D

2.1 Task 1

The brain data was manipulated to ensure that larger y values are above lower, and larger x values are to the right of smaller. This gave the following image Fig. 5, which is in-line with what is found in Figure 11.1c on pg 210 of the course reference textbook.

For all subsequent figures spatial images (brain data), is presented with a colour map that is linear with black being the lowest pixel intensity and white being the largest, and the absolute values of brain data are used.

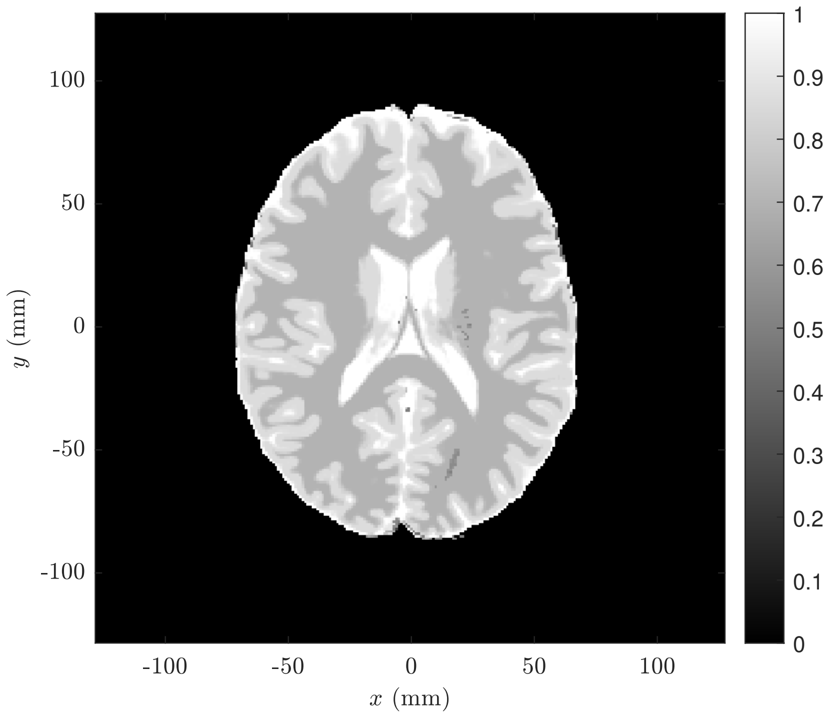

Figure 5: Problem 2 - task 1, spatial image of axial slice of brain in the correct orientation

2.2 Task 2

The pixel size is given as 1×1 mm this is equivalent to saying that the spatial spacing is 1 mm for both dimensions such that $\Delta x = \Delta y = 1$ mm. Additionally the number of pixels is 256 in both dimensions ($N = N_x = N_y = 256$) thus the field of view (FOV) is equal to $FOV = \Delta x.N = 256$ mm. As the domain is square the k-space spacing $\Delta k = \Delta k_x = \Delta k_y = 1/FOV$ which is 1/256 1/mm. Furthermore for four factors presented for problem 1, task 1, can be used when defining these k-space values, such that:

k_x = k_y = `linspace(`Δ `k` $\times$ $\frac{N}{2}$`,` Δ `k` $\times$ `(`$\frac{N}{2}$`-1), N)`

The above k-space values ensure that $signal(k = 0)$ is captured and that there are an even number of points captured. When the FFT is used shift is necessary to correctly transform, additionally the resulting signal is complex. To aid with visualisation only the magnitude of the signal is visualised and a logarithmic scale is used for colour. The visualisation can be seen in Fig. 6

For all subsequent figures frequency images (k-space data), is presented with a colour map that is logarithmic, and the absolute values of k-space data are used.

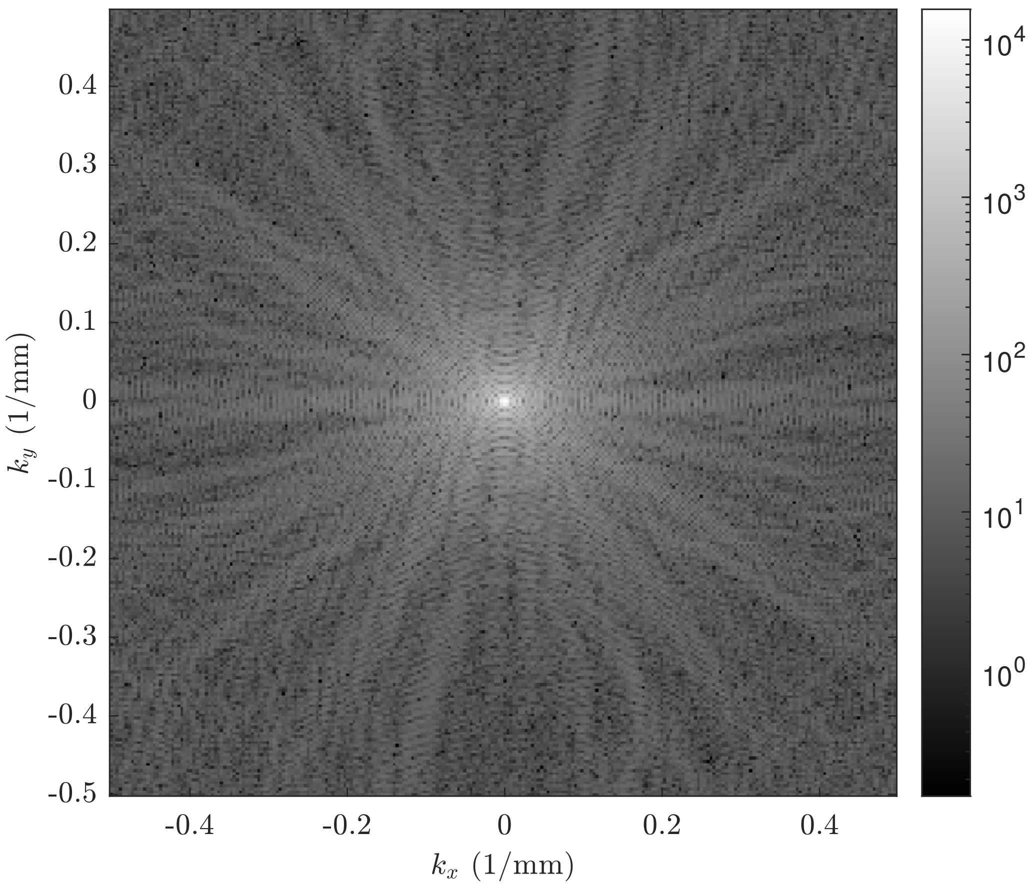

Figure 6: Problem 2 - task 2, frequency image of absolute values of k-space data (—k-space data—) with a logarithmic scale.

2.3 Task 3

A low-pass filter is defined on the image as a rect function centered around $k_x = k_y = 0$ and extended a width of $32\Delta k$ in each dimension. This was applied to the k-space data. This can be seen in Fig. 7. The bottom right image is most interesting and shows the Gibbs ringing artifact. The edges are blurred and only the low-spatial frequencies of the image are kept. For most images of biological tissue there are typically more low-spatial frequencies than high present, i.e. majority of signal is captured at the origin of k-space.

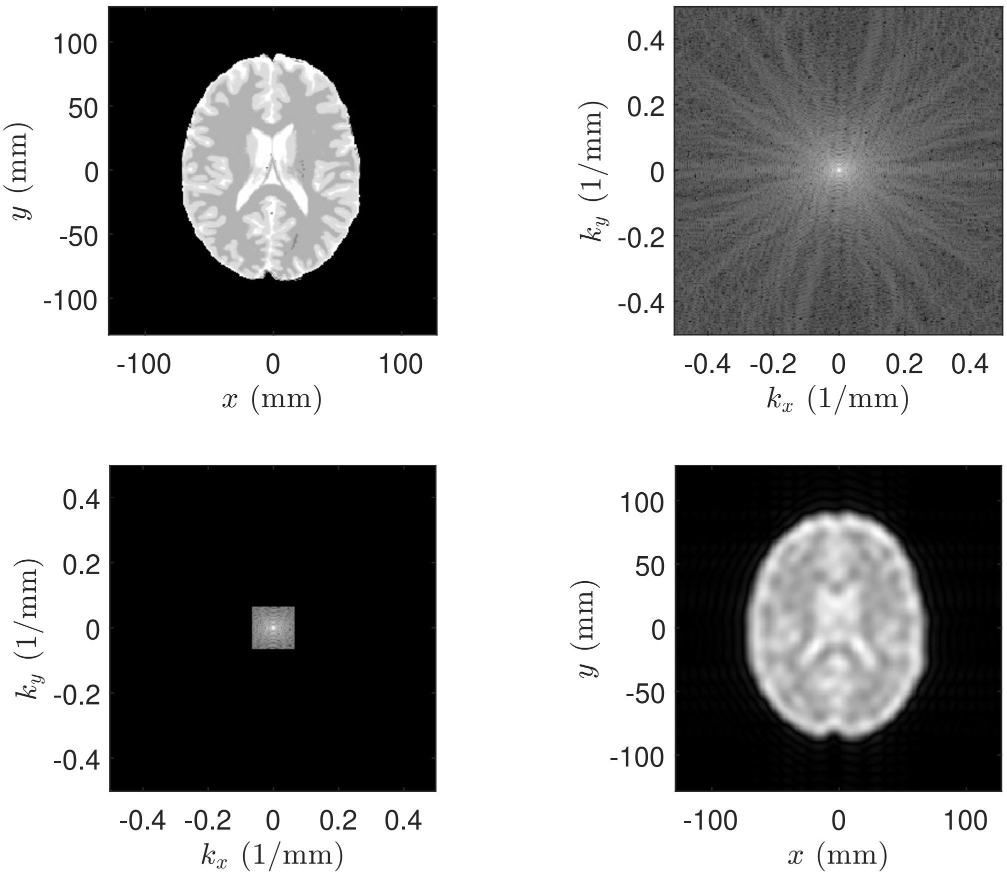

Figure 7: Problem 2 - task 3. Upper row is the inverse Fourier transform going from spatial to frequency image. Bottom row is the Fourier transform going from frequency to spatial image. Upper left, initial brain data. Upper right, k-space data. Bottom left, low-pass filter applied to k-space data of brain data. Bottom right, spatial image of low-pass filter applied to k-space data.

2.4 Task 4

A high-pass filter is defined on the image as the reciprocal of the low-pass filter, i.e. only keeping the high frequency components. This was applied to the k-space data. This can be seen in Fig. 8. The bottom right image is most interesting and shows the Gibbs ringing artifact again. Additionally, the high-frequencies are typically associated with edges of the image and therefore the edges of the brain data are accentuated.

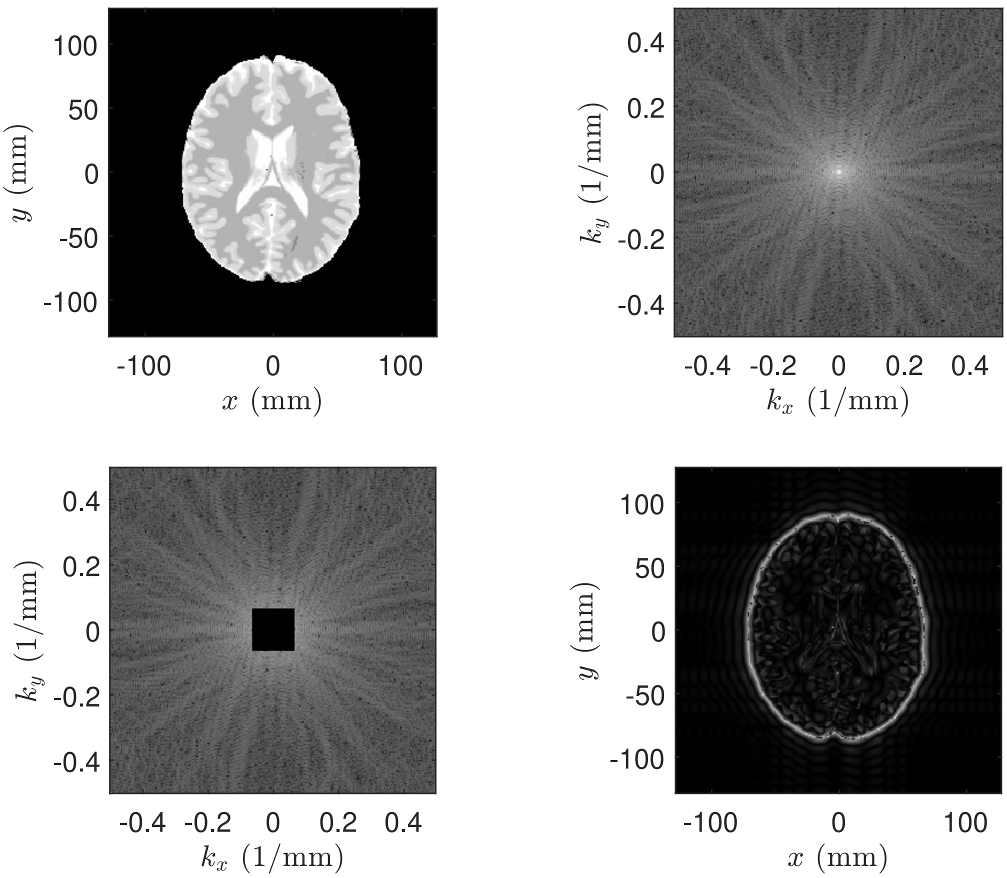

Figure 8: Problem 2 - task 4. Upper row is the inverse Fourier transform going from spatial to frequency image. Bottom row is the Fourier transform going from frequency to spatial image. Upper left, initial brain data. Upper right, k-space data. Bottom left, high-pass filter applied to k-space data of brain data. Bottom right, spatial image of high-pass filter applied to k-space data.

2.5 Task 5

A threshold was used to remove 20% of the smallest k-space values (weakest signal). This can be seen in Fig. 9.

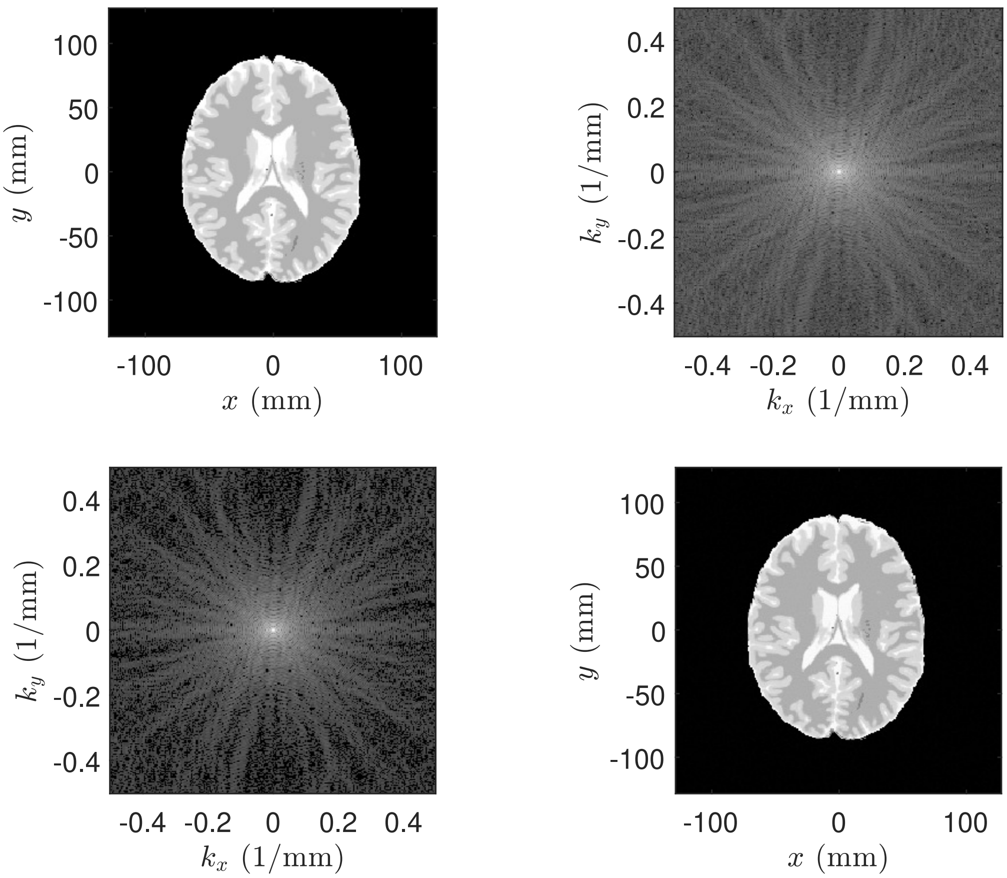

Figure 9: Problem 2 - task 5. Upper row is the inverse Fourier transform going from spatial to frequency image. Bottom row is the Fourier transform going from frequency to spatial image. Upper left, initial brain data. Upper right, k-space data. Bottom left, removed 20% of the smallest k-space values. Bottom right, image resulting from removal of smallest k-space.

The effect of removing 20% of the smallest values is negligible to the image quality. As the largest magnitude of the signal is $\mathcal{O}(10^4)$ whilst the 20th percentile of magnitude of signal is $\mathcal{O}(10^0)$. The smallest components of the signal could be due to measurement noise etc, and when they are removed there is no noticeable change in image quality. To show an extreme case only the 99th percentile of signal intensity was kept. This is shown on the front cover. Curiously, with only 1% of the data and zero padding everywhere else, a brain is somewhat visible. This is a curious result and begs the question that with this level of redundancy in the data, could data be represented in a sparser way? Furthermore can we sub-sample k-space and still reconstruct an accurate image?

3 Problem 3: MR Imaging Parameters

3.1 Problem 12.7

Assume the following 2D imaging parameters: $L_x = L_y = 256$ mm; $N_x = N_y = 256$; $TH = 5$ mm; $T_R = 600$ ms. Assume that $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the read, phase encoding, and slice select directions, respectively. Also suppose that $T_s = 5.12$ ms and $\tau_{PE} = 2.56$ms and that the rf excitation bandwidth BW_{rf} is 2 kHz. See Ch. 10 for the sequence diagrams.

a) Find the readout bandwidth, BW_{read}, the Nyquist sampling interval Δt in the read direction, the Nyquist sampling interval Δk_x in the k_x direction, and the strength of the read gradient (G_x) used.

$$\Delta t = T_s/N_x = 20\mu s \tag{1}$$
$$BW_{read} = 1/\Delta t = 50kHZ \tag{2}$$
$$\Delta k_x = 1/L_x = 3.91m^{-1} \tag{3}$$
$$G_x = 2\pi BW_{read}/(\gamma L_x) = 4.58mT/m \tag{4}$$

b) What is the Nyquist sampling interval Δk_y in the ky direction? What is the gradient step size ΔG_y in the phase encoding table? What is the strength of the maximum value $G_{y,max} = N_y \Delta G_y/2$ of the phase encoding gradient?

$$\Delta k_y = 1/L_y = 3.91m^{-1} \tag{5}$$
$$\Delta G_y = 2\pi \Delta k_y/(\gamma \tau_{PE}) = 3.59\mu T/m \tag{6}$$
$$G_{y,max} = N_y \Delta G_y/2 = 4.59mT/m \tag{7}$$

c) What is the slice select gradient G_{ss} used?

$$G_{ss} = 2\pi BW_{rf}/(\gamma TH) = 9.4mT/m \tag{8}$$

d) What is the total imaging time T_{acq}?

$$T_{acq} = N_y T_R = 153.6s \tag{9}$$

e) What happens to G_x, $G_{y,max}$ and G_{ss} N_x and N_y are doubled at the same time that TH is halved while all other quantities are unchanged?

If N_x is doubled then G_x doubles as $G_x \propto N_x/T_s$.
If N_y is doubled then $G_{y,max}$ doubles as $G_{y,max} = N_y \Delta G_y/2$.
If TH is halved then G_{ss} doubles as $G_{ss} \propto 1/TH$

f) How does G_x change when L_x is halved while all other parameters are held fixed? How does it change if, instead, Ts is changed to 2.56ms while L_x is unchanged from 256mm and N_x is fixed at 256?

If L_x is halved then G_x doubles as $G_x \propto 1/L_x$
If T_s is halved then G_x doubles as $G_x \propto N_x/T_s$

g) Suppose the imaging is performed as a 3D imaging experiment with $L_z \equiv TH$ = 32mm and N_z = 16, with all other parameters the same as in the original problem statement (the partition encoding and phase encoding gradient times are the same: $\tau_z = \tau_{PE}$ = 2.56ms).

i) What is G_{ss}? (The z-axis is now the 'slab' selection axis and the partition encoding gradient is referred to as G_z.)

$$G_{ss,g} = 2\pi BW_{rf}/(\gamma TH) = 1.5mT/m \tag{10}$$

ii) What is ΔG_z and what is $G_{z,max}$?

$$\Delta G_z = 2\pi/(L_z \gamma \tau_z) = 286 \mu T/m \quad (11)$$
$$G_{z,max} = \Delta G_z N_z/2 = 2.3 mT/m \quad (12)$$

iii) What is the total imaging time if (i) T_R were 600 ms (ii) T_R were 60 ms? How does T_{acq} in either case compare with the imaging time in the 2D imaging experiment?

$$i)\ T_{acq,i} = N_y N_z T_{R,i} = 2457.6 s \quad (13)$$
$$ii)\ T_{acq,ii} = N_y N_z T_{R,ii} = 245.76 s \quad (14)$$

T_{acq} for the 3D case is longer than 2D, but by decreasing the T_R this effect can be mitigated.

iv) How does G_{ss} compare between the 2D and 3D imaging experiments?

The slab selection gradient in 2D is 6.4 times ($G_{ss,g}/G_{ss}$) stronger than in the 3D case.

A Scripts for simulation:

Full scripts can be found on Github: `https://github.com/Imraj-Singh/COMP0121`.