Chapter 2

Mathematical Fundamentals

If the only tool you have is a hammer, you tend to treat everything as if it were a nail.

Abraham Maslow

This chapter reviews some mathematical concepts essential for understanding MRI principles. We will begin with a brief description of vector quantities and then review the definitions of some commonly used mathematical functions and two integral transforms: the Fourier transform and the Radon transform.

Before we proceed, a word on the notation is in order. Throughout this text, we will use $f(x), g(x), h(x)$, and so on, to represent continuous-variable functions, and $f[n], g[n], h[n]$, or f_n, g_n, h_n, and the like to represent discrete-variable functions (or sequences) with the understanding that n is an integer variable. In many situations when there is no confusion, we will also use $f[n]$ or f_n to represent data samples from $f(x)$ such that $f_n = f[n] = f(n\Delta x)$ for some Δx. Multivariable functions are sometimes expressed in vector notation such that $f(\boldsymbol{r})$ may refer to $f(x, y)$ or $f(x, y, z)$, depending on the context.

2.1 Vectors

Many physical quantities, such as velocity, magnetic field, and angular momentum, have both magnitude and direction. These quantities, called *vectors*, are represented in this book by two types of symbols: symbols with an overhead arrow such as $\vec{A}$ (called *explicit vector notations*) and boldface symbols such as $\boldsymbol{A}$ (called *implicit vector notations*). To understand the subtle difference between these two notations, consider a three-dimensional vector $\vec{A}$ shown in Fig. 2.1. In explicit form,

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} \tag{2.1}$$

and in implicit form

$$\boldsymbol{A} = (A_x, A_y, A_z) \tag{2.2}$$

where A_x, A_y, and A_z are the x-, y-, and z-components of $\vec{A}$, and $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit directional vectors along the x-, y-, and z-axes, respectively. By definition,

$$\begin{cases} \vec{i} \sim (1,0,0) \\ \vec{j} \sim (0,1,0) \\ \vec{k} \sim (0,0,1) \end{cases} \tag{2.3}$$

Within a single reference frame, both types of notations uniquely specify a vector, and we will use them interchangeably when there is no confusion. However, when multiple reference frames are used, as is often the case in describing MRI techniques, it is important to know which reference frame is used for vectors in the implicit notation.

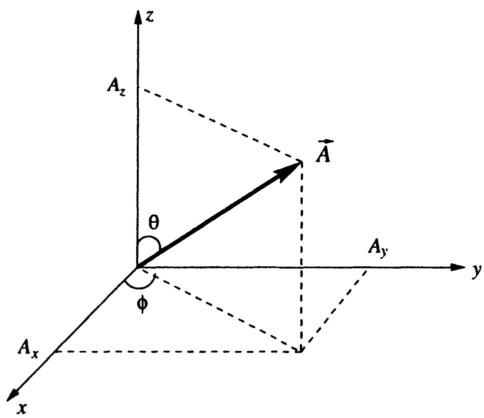


Figure 2.1 Graphical representation of a vector.

The magnitude and direction of $\boldsymbol{A}$ are represented by $|\boldsymbol{A}|$ and $\boldsymbol{\mu}_A$, respectively, such that

$$\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{\mu}_A \tag{2.4}$$

where $\boldsymbol{\mu}_A$ is the unit directional vector of $\boldsymbol{A}$. $|\boldsymbol{A}|$ and $\boldsymbol{\mu}_A$ can be determined from the components of $\boldsymbol{A}$ using the following formulas:

$$|\boldsymbol{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{2.5}$$

and

$$\boldsymbol{\mu}_A = \frac{1}{\sqrt{A_x^2 + A_y^2 + A_z^2}}(A_x, A_y, A_z) \tag{2.6}$$

Alternatively, $\boldsymbol{\mu}_A$ can be expressed in terms of the polar and azimuthal angles as follows:

$$\boldsymbol{\mu}_A = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.7}$$

where

$$\theta = \tan^{-1}\left(\frac{\sqrt{A_x^2 + A_y^2}}{A_z}\right) \tag{2.8}$$

and

$$\phi = \arctan\left(\frac{A_y}{A_x}\right) \tag{2.9}$$

For planar vectors, a complex notation is often used in the MR literature. Specifically, let $\boldsymbol{A} = (A_x, A_y)$. We regard $\boldsymbol{A}$ as equivalent to A if A is constructed as

$$A \triangleq A_x + iA_y \tag{2.10}$$

where $i \triangleq \sqrt{-1}$. In the polar (or exponential) form, A can be expressed as

$$A = |A|e^{i\phi} = |A|(\cos\phi + i\sin\phi) \tag{2.11}$$

where $|A| = \sqrt{A_x^2 + A_y^2}$, and ϕ is given in Eq. (2.9). Geometrically, ϕ is the smaller angle between $\boldsymbol{A}$ and the x-axis (real axis).

The exponential form in Eq. (2.11) is used frequently at various places in this book. A special example is

$$A(t) = A_0 e^{i\omega_0 t} \tag{2.12}$$

which represents a vector of length A_0 rotating counterclockwise at an angular speed ω_0.

Vector quantities can be used in various operations. The simplest one is vector addition, which is defined as follows:

$$\vec{A} + \vec{B} \triangleq (A_x + B_x, A_y + B_y, A_z + B_z) \tag{2.13}$$

Vector addition can be performed graphically using the so-called *parallelogram rule* shown in Fig. 2.2. It is obvious that vector addition obeys the commutative and associative laws:

Commutative law: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ (2.14a)

Associative law: $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ (2.14b)

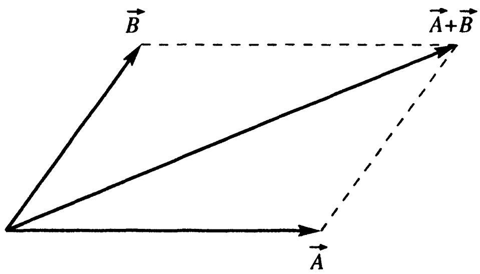


Figure 2.2 Parallelogram rule for vector addition.

Vector multiplication exists in two forms: scalar (or dot) product and vector (or cross) product.[1] The dot product of two vectors $\vec{A}$ and $\vec{B}$, denoted by $\vec{A}\cdot\vec{B}$, is a scalar that is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the (smaller) angle between them. Let ϕ_{AB} be the smaller angle between $\vec{A}$ and $\vec{B}$. Then,

$$\vec{A}\cdot\vec{B} \triangleq |\vec{A}||\vec{B}|\cos\phi_{AB} \tag{2.15}$$

It is easy to see that the dot product is commutative and distributive:

$$\text{Commutative law:} \quad \vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} \tag{2.16a}$$

$$\text{Distributive law:} \quad \vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C} \tag{2.16b}$$

The associative law, however, does not apply to the dot product. Using the distributive property, one can show that

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{2.17}$$

The cross product of two vectors, $\vec{A}$ and $\vec{B}$, is defined by

$$\vec{A}\times\vec{B} \triangleq |\vec{A}||\vec{B}||\sin\phi_{AB}|\vec{\mu}_{AB} \tag{2.18}$$

where $\vec{\mu}_{AB}$ is determined according to the ***right-hand rule***. That is, $\vec{\mu}_{AB}$ takes the direction of the thumb of the right hand when the fingers rotate from $\vec{A}$ to $\vec{B}$ through the angle ϕ_{AB}, as shown in Fig. 2.3.

The cross product obeys the distributive law:

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C} \tag{2.19}$$

However,

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A} \tag{2.20}$$

[1] It is important to note that while addition of planar vectors is equivalent to addition of the corresponding complex numbers, vector multiplication (dot or cross product) does not correspond to complex multiplication. Therefore, the complex notation of planar vectors should be used with caution.

and

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C} \tag{2.21}$$

Therefore, the cross product is neither commutative nor associative.

Based on the distributive property, one can show that

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\vec{i} + (A_z B_x - A_x B_z)\vec{j} + (A_x B_y - A_y B_x)\vec{k} \tag{2.22}$$

or in determinant form

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{2.23}$$

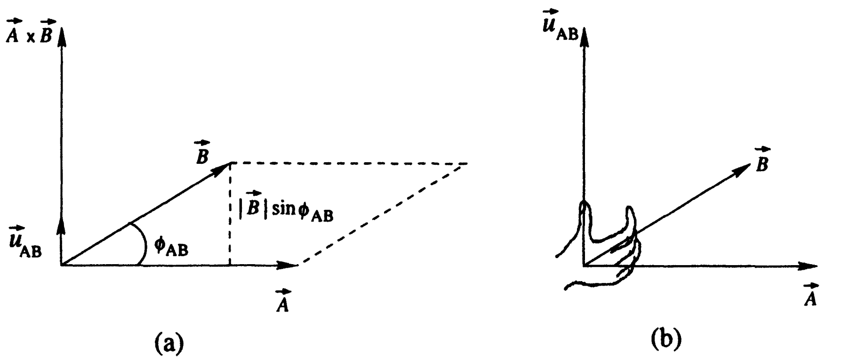


Figure 2.3 (a) $\vec{A} \times \vec{B}$, and (b) the right-hand rule. Note that μ_{AB} is orthogonal to the plane containing $\vec{A}$ and $\vec{B}$.

The following identities for the unit directional vectors of the x-, y- and z-axes directly follow from the definition of dot and cross products:

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \tag{2.24a}$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0 \tag{2.24b}$$

$$\vec{i} \times \vec{j} = \vec{k}; \quad \vec{j} \times \vec{k} = \vec{i}; \quad \vec{k} \times \vec{i} = \vec{j} \tag{2.24c}$$

2.2 Basic Concepts of Matrix Algebra

The (implicit) vector notation is often used to represent one-dimensional data sequences. Specifically, a column vector containing N elements is denoted as

$$\boldsymbol{v} \triangleq \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \tag{2.25}$$

Similarly, a row vector of size N is denoted as

$$\boldsymbol{v} \triangleq [v_1, v_2, \cdots, v_N] \tag{2.26}$$

A column vector of size N is also called an $N \times 1$ vector and is written as

$$\boldsymbol{v} = \{v_n\}_{N\times 1} \tag{2.27}$$

Likewise, a row vector of size N is called a $1 \times N$ vector and written as

$$\boldsymbol{v} = \{v_n\}_{1\times N} \tag{2.28}$$

Two-dimensional data sequences can be represented by matrices, which are represented by boldface Roman symbols. A matrix of size $M \times N$ has M rows and N columns and is written as

$$\mathbf{W} = \{w_{mn}\}_{M\times N} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N} \\ w_{21} & w_{22} & \cdots & w_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ w_{M1} & w_{M2} & \cdots & w_{MN} \end{bmatrix} \tag{2.29}$$

Some basic matrix definitions are summarized as follows:

- Identity matrix: $\mathbf{I} = \{\delta(m-n)\}_{M\times M}$
- Determinant: $|\mathbf{W}| = |\{w_{mn}\}_{M\times M}|$
- Inverse, $\mathbf{W}^{-1}$: $\mathbf{W}^{-1}\mathbf{W} = \mathbf{W}\mathbf{W}^{-1} = \mathbf{I}$
- Transpose: $\mathbf{W} = \{w_{mn}\}_{M\times N} \longrightarrow \mathbf{W}^T = \{w_{nm}\}_{N\times M}$
- Complex conjugate: $\mathbf{W} = \{w_{mn}\}_{M\times N} \longrightarrow \mathbf{W}^* = \{w_{mn}^*\}_{M\times N}$
- Hermitian: $\mathbf{W} = \{w_{mn}\}_{M\times N} \longrightarrow \mathbf{W}^H = \{w_{nm}^*\}_{N\times M}$
- Matrix addition: $\mathbf{A} + \mathbf{B} \triangleq \{a_{mn} + b_{mn}\}_{M\times N}$
- Scalar multiplication: $c\mathbf{W} = \{cw_{mn}\}_{M\times N}$
- Matrix multiplication: $\mathbf{C}_{M\times N} \triangleq \mathbf{A}_{M\times L}\mathbf{B}_{L\times N} \longrightarrow c_{mn} = \sum_{\ell=1}^{L} a_{m\ell}b_{\ell n}$
- Eigenvalues, λ: Roots of $|\mathbf{W} - \lambda\mathbf{I}|$
- Eigenvectors, $\boldsymbol{v}$: Nonzero solutions of $\mathbf{W}\boldsymbol{v} = \lambda\boldsymbol{v}$
- Orthogonal matrix: $\mathbf{W}^{-1} = \mathbf{W}^T$ or $\mathbf{W}\mathbf{W}^T = \mathbf{W}^T\mathbf{W} = \mathbf{I}$
- Unitary matrix: $\mathbf{W}^{-1} = \mathbf{W}^H$ or $\mathbf{W}\mathbf{W}^H = \mathbf{W}^H\mathbf{W} = \mathbf{I}$

2.3 Some Commonly Used Functions

This section reviews the definitions of several commonly used functions.

2.3.1 Unit Step Function

The unit step function, denoted by $u(\cdot)$, is defined by

$$u(x) \triangleq \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.30}$$

2.3.2 Signum Function

The signum function, denoted by $\operatorname{sgn}(\cdot)$, is defined by

$$\operatorname{sgn}(x) \triangleq \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{2.31}$$

It can be expressed in terms of the unit step function as

$$\operatorname{sgn}(x) = 2u(x) - 1 \tag{2.32}$$

2.3.3 Rectangular Window Function

The unit-width rectangular window function, denoted as $\Pi(\cdot)$, is defined by

$$\Pi(x) \triangleq \begin{cases} 1 & |x| < 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{2.33}$$

Clearly,

$$\Pi(x) = u\left(x + \tfrac{1}{2}\right) - u\left(x - \tfrac{1}{2}\right) \tag{2.34}$$

2.3.4 Triangle Window Function

The unit triangle window function, denoted by $\Lambda(\cdot)$, is defined by

$$\Lambda(x) \triangleq \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.35}$$

2.3.5 Hamming Window Function

The Hamming window function is a raised cosine function defined by

$$H(x) \triangleq \begin{cases} 0.54 + 0.46\cos(2\pi x) & |x| \le 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{2.36}$$

2.3.6 Gaussian Function

The Gaussian window function is defined as

$$G(\mu, \sigma, x) \triangleq \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{2.37}$$

where μ and σ are the mean and standard deviation of the Gaussian distribution because of the following properties:

$$\mu = \int_{-\infty}^{\infty} xG(\mu, \sigma, x)dx \tag{2.38}$$

and

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 G(\mu, \sigma, x)dx \tag{2.39}$$

2.3.7 Dirac Delta Function

The Dirac delta function or impulse function, denoted by $\delta(x)$, is a mathematical abstraction for pulses that are so brief and intense that making them any briefer and more intense does not matter as long as their integral stays the same. Mathematically, $\delta(x)$ is a generalized function because we cannot define its value point by point as with ordinary functions. A formal definition of $\delta(x)$ is based on the distribution theory. In this definition, $\delta(x)$ is a functional such that

$$\int_{-\infty}^{\infty} \varphi(x)\delta(x)dx = \varphi(0) \tag{2.40}$$

for any $\varphi(x)$ that is continuous at the origin. The definition above implies the following properties:

(a) $\delta(x) = 0$ for $x \neq 0$

(b) $\delta(x)$ is unbounded at $x = 0$

(c) $\int_{-\infty}^{\infty} \delta(x)dx = 1$

Pictorially, $\delta(x)$ is represented by an upward arrow at the origin, as shown in Fig. 2.4, which signifies a pulse at the origin with an unbounded amplitude and zero duration.

In practice, $\delta(x)$ is taken to be the limit of a function sequence $g_n(x)$:

$$\delta(x) = \lim_{n\to\infty} g_n(x) \tag{2.41}$$

provided that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi(x)g_n(x)dx = \varphi(0) \tag{2.42}$$

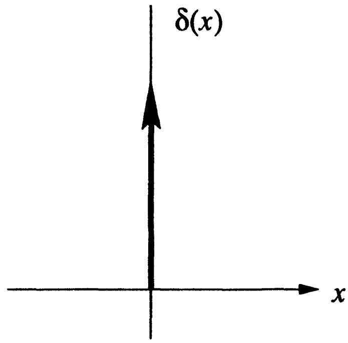


Figure 2.4 Pictorial representation of the delta function.

Many sequences with this property are available. Four well-known examples are as follows:

$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Pi\left(\frac{x}{\Delta x}\right) \tag{2.43a}$$

$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \text{sinc}\left(\frac{\pi x}{\Delta x}\right) \tag{2.43b}$$

$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Lambda\left(\frac{x}{\Delta x}\right) \tag{2.43c}$$

$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} e^{-\frac{\pi x^2}{\Delta x^2}} \tag{2.43d}$$

The impulse function $\delta(x)$ and the unit step function $u(x)$ are closely related to each other by

$$\frac{du(x)}{dx} = \delta(x) \tag{2.44}$$

This relationship can be proved from the distribution definition of $\delta(x)$ by noting that

$$\int_{-\infty}^{\infty} \frac{du(x)}{dx}\varphi(x)dx = u(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\frac{d\varphi(x)}{dx}dx$$

$$= \varphi(\infty) - [\varphi(\infty) - \varphi(0)] = \varphi(0)$$

The following properties of the impulse function $\delta(x)$ immediately follow from the defining integral:

(a) *Scaling property*:

$$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad a \neq 0 \tag{2.45}$$

(b) *Sampling property*:

$$\varphi(x)\delta(x - x_0) = \varphi(x_0)\delta(x - x_0) \tag{2.46a}$$

$$\int_{-\infty}^{\infty} \varphi(x)\delta(x - x_0)dx = \varphi(x_0) \tag{2.46b}$$

(c) *Derivative property*:

$$\int_{-\infty}^{\infty} \varphi(x)\delta^{(n)}(x - x_0)dx = (-1)^n \varphi^{(n)}(x_0) \tag{2.47}$$

2.3.8 Kronecker Delta Function

The discrete counterpart of the Dirac delta function is the Kronecker delta function, written as $\delta[n]$ or simply δ_n. Mathematically, it is defined as

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.48}$$

Note that $\delta[n]$ is not a generalized function, but it has similar effects on discrete functions as $\delta(x)$ does on continuous functions. For example, similarly to Eqs. (2.44) and (2.46), we have

$$\delta[n] = u[n] - u[n-1] \tag{2.49}$$

and

$$\sum_{n=-\infty}^{\infty} \varphi[n]\delta[n - n_0] = \varphi[n_0] \tag{2.50}$$

2.3.9 Comb Function

The comb function is a periodic function consisting of a sequence of equally spaced Dirac delta functions. A formal definition of $\mathrm{comb}(x)$ is

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n) \tag{2.51}$$

It is easy to show that

$$\sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) = \frac{1}{\Delta x}\mathrm{comb}\left(\frac{x}{\Delta x}\right) \tag{2.52}$$

2.3.10 Sinc Function

The sinc function is defined by

$$\text{sinc}(x) = \frac{\sin(x)}{x} \tag{2.53}$$

This function is rather popular in the imaging literature because it is directly tied to the Fourier transform of a rectangular window function, as discussed in Section 2.5. Inspection of both Eq. (2.53) and the plot in Fig. 2.5 reveals the following properties of the sinc function:

(a) It is an even function.

(b) It has a peak value 1 at $x = 0$ by the L'Hôpital rule.

(c) Its zero crossings are located at $\pm n\pi$. The region between $x = -\pi$ and $x = \pi$ is called the *main lobe*, and the regions between $x = -(n+1)\pi$ and $x = -n\pi$ or between $x = n\pi$ and $x = (n+1)\pi$ for $n > 1$ are called the *side lobes*.

(d) Since $\text{sinc}(x)$ is the product of an oscillating $\sin(x)$ and a monotonically decreasing function $1/x$, $\text{sinc}(x)$ exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.

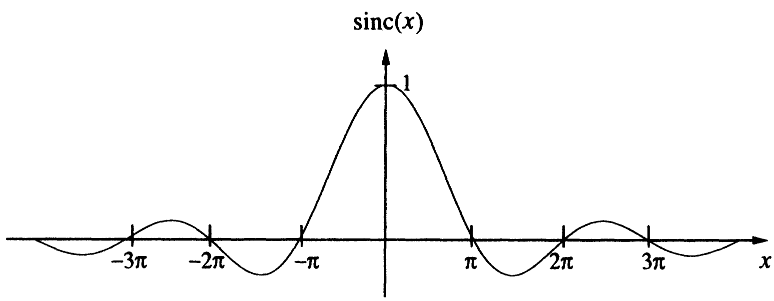


Figure 2.5 Pictorial representation of the sinc function.

2.3.11 Dirichlet Function

The Dirichlet function is defined as

$$\text{Dir}(N, x) = \frac{\sin(Nx)}{\sin(x)} \tag{2.54}$$

It is easy to show that $\text{Dir}(N, x)$ is a periodic function of period π for N odd but 2π for N even. Within the region $-\pi/2 < x < \pi/2$, $\text{Dir}(N, x)$ looks like a sinc function in the sense that it has a main lobe spanning over the region between $x = -\pi/N$ and $x = \pi/N$ and a sequence of side lobes spanning over the regions between $x = -(n+1)\pi/N$ and $x = -n\pi/N$ or between $x = n\pi/N$ and $x = (n+1)\pi/N$. Two examples of $\mathrm{Dir}(N, x)$ are shown in Fig. 2.6.

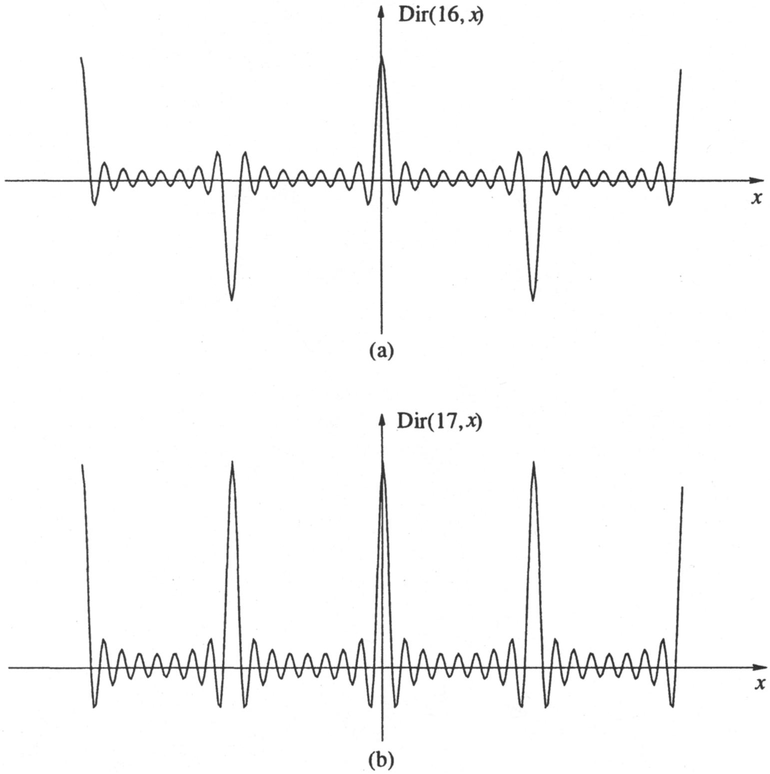


Figure 2.6 Plots of $\mathrm{Dir}(N, x)$ for (a) $N = 16$ and (b) $N = 17$, respectively.

2.3.12 Bessel Functions

Bessel functions of the first kind of integer order n, denoted by $J_n(x)$, are useful for describing some imaging effects. One definition of $J_n(x)$ is given in terms of the following integral:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n\theta - x \sin\theta)} d\theta \tag{2.55}$$

A special case is the zero-order Bessel function of the first kind, which, according to Eq. (2.55), is defined by

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta}d\theta \tag{2.56}$$

Equivalently, we have

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\cos\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\cos\theta}d\theta \tag{2.57}$$

Some useful relations of $J_n(x)$ are summarized below.

$$J_n(-x) = (-1)^n J_n(x) \tag{2.58a}$$

$$J_{-n}(x) = (-1)^n J_n(x) \tag{2.58b}$$

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta} \tag{2.58c}$$

$$e^{ix\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(x)e^{in\theta} \tag{2.58d}$$

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos 2n\theta \tag{2.58e}$$

$$\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n+1}(x)\sin(2n+1)\theta \tag{2.58f}$$

Plots of $J_0(x)$, $J_1(x)$, and $J_2(x)$ are given in Fig. 2.7. As can be seen, the Bessel functions oscillate (but are not exactly periodic), and their amplitudes decay gradually (asymptotically as $1/\sqrt{x}$).

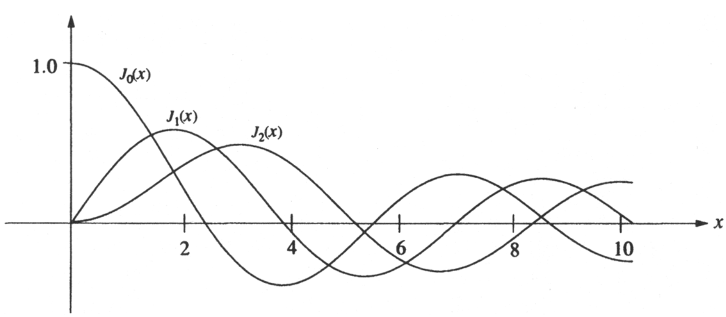


Figure 2.7 Plots of Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$.

2.4 Convolution

Convolution is an important mathematical operation between two functions. For two continuous-variable functions $g(x)$ and $h(x)$, their convolution is defined as

$$f(x) = \int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau = \int_{-\infty}^{\infty} g(x-\tau)h(\tau)d\tau \tag{2.59}$$

Symbolically, Eq. (2.59) is often written as

$$f(x) = g(x) * h(x) \tag{2.60}$$

For discrete-variable functions (or data sequences), the integration in Eq. (2.59) is replaced by a summation such that

$$f[n] = g[n] * h[n] = \sum_{m=-\infty}^{\infty} g[m]h[n-m] \tag{2.61}$$

The following is a summary of some useful properties of convolution:

(a) *Commutativity*:

$$g * h = h * g \tag{2.62}$$

(b) *Associativity*:

$$f * (g * h) = (f * g) * h \tag{2.63}$$

(c) *Distributivity*:

$$f * (g + h) = f * g + f * h \tag{2.64}$$

(d) *Differentiation property*:
For continuous convolution, it is easy to show that

$$\frac{d}{dx}(g * h) = \frac{dg}{dx} * h = g * \frac{dh}{dx} \tag{2.65}$$

(e) *Shifting property*:
If $g * h = f$, then

$$g(x - x_0) * h(x) = g(x) * h(x - x_0) = f(x - x_0) \tag{2.66}$$

(f) *Width property*:
If $g(x)$ and $h(x)$ are functions of finite widths W_g and W_h, respectively, then $g(x) * h(x)$ is another finite duration function with width $W_g + W_h$.

The width property can be extended to discrete convolution, but it takes a slightly different form. Suppose that $g[n]$ and $h[n]$ are sequences of finite widths N_g and N_h, respectively; then $g[n] * h[n]$ is another finite duration sequence with width $N_g + N_h - 1$. This property can be understood by considering the simple example, where $g[n] = h[n] = \{1, 1, 1\}$ are sequences of width 3, while $g[n] * h[n] = \{1, 2, 3, 2, 1\}$ is a sequence of width 5.

Theorem 2.1 (Central Limit Theorem) *When a function $h(x)$ is convolved with itself n times, in the limit $n \to \infty$, the convolution product is a Gaussian function with a variance that is n times the variance of $h(x)$, provided the area, mean, and variance of $h(x)$ are finite.*

One may have seen this theorem in the statistical literature. It can be interpreted here as saying that convolution is a smoothing process. Therefore, it is often appropriate to say that an image obtained from a practical imaging system is a smoothed version of the true image function.

■ Example 2.1

This example evaluates the convolution of two exponential functions from the definition.

$$
\begin{aligned}
ae^{-\alpha x}u(x) * be^{-\beta x}u(x) &= \int_{-\infty}^{\infty} ae^{-\alpha\tau}u(\tau)be^{-\beta(x-\tau)}u(x-\tau)d\tau \\
&= ab\left[\int_0^x e^{-(\alpha-\beta)\tau-\beta x}d\tau\right]u(x) \\
&= \frac{ab}{\beta-\alpha}(e^{-\alpha x} - e^{-\beta x})u(x)
\end{aligned}
$$

Note that both exponential functions have a step discontinuity at the origin, but their convolution is smooth throughout the entire x-axis, as shown in Fig. 2.8.

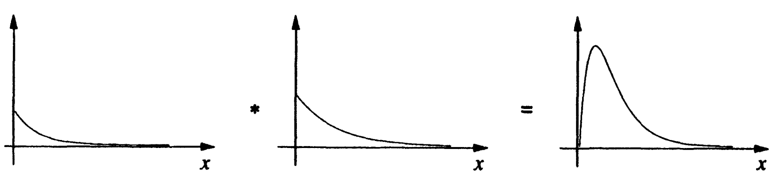


Figure 2.8 Convolution of two exponential functions.

Example 2.2

This example calculates $\Pi(x) * \Pi(x)$ using the derivative property.

First, the derivative property gives

$$\begin{aligned}\frac{d}{dx}[\Pi(x) * \Pi(x)] &= \left[\frac{d}{dx}\Pi(x)\right] * \Pi(x) \\ &= [\delta(x+\tfrac{1}{2}) - \delta(x-\tfrac{1}{2})] * \Pi(x) \\ &= \Pi(x+\tfrac{1}{2}) - \Pi(x-\tfrac{1}{2})\end{aligned}$$

Then,

$$\begin{aligned}\Pi(x) * \Pi(x) &= \int_{-\infty}^{x} [\Pi(\tau+\tfrac{1}{2}) - \Pi(\tau-\tfrac{1}{2})]\, d\tau \\ &= \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

which is a triangular pulse, as shown in Fig. 2.9. Note that the resulting triangular pulse is wider and smoother than the rectangular pulse. This result is to be expected from the central limit theorem.

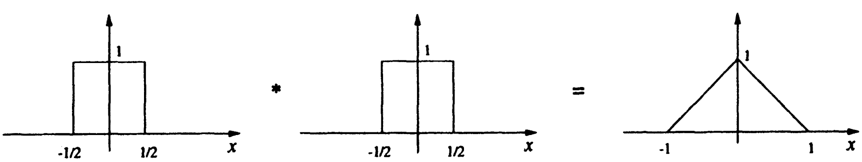


Figure 2.9 Convolution of two identical rectangular pulses yields a triangular pulse.

2.5 The Fourier Transform

The Fourier transform is of fundamental importance in MRI as will become evident in the subsequent chapters. This section reviews its definition and some of its properties relevant to MRI. A number of excellent books provide in-depth discussions of the theory and applications of the Fourier transform. For engineering students, Bracewell [6] and Papoulis [51] are especially suitable.

2.5.1 Definition

The Fourier transform of a spatial function $\rho(x)$, denoted by $\mathcal{F}\{\rho(x)\}$, or $\{\mathcal{F}\rho\}(k)$, or simply $S(k)$, is defined as

$$S(k) = \mathcal{F}\{\rho(x)\} = \mathcal{F}\rho = \int_{-\infty}^{\infty} \rho(x)e^{-i2\pi kx}dx \tag{2.67}$$

where k is a spatial frequency variable having the unit of cycles per unit distance.

In signal processing, $S(k)$ is called the *frequency spectrum* of $\rho(x)$. In general, $S(k)$ is a complex-valued function of k and is often conveniently written as

$$S(k) = |S(k)|e^{i\varphi(k)} \tag{2.68}$$

where $|S(k)|$ is the magnitude spectrum and $\varphi(k)$ is the phase spectrum. In MRI, $S(k)$ are the experimental data measured in the Fourier space (often called *k-space*), while $\rho(x)$ is the desired image function representing, for example, the spin density function. Variables x and k are termed *conjugate variables* of the Fourier transform. Another pair of conjugate variables frequently mentioned in the MRI literature consists of time t and spectral frequency f.

Given $S(k)$, we can recover $\rho(x)$ using the *inverse* Fourier transform:

$$\rho(x) = \mathcal{F}^{-1}\{S(k)\} = \int_{-\infty}^{\infty} S(k)e^{i2\pi kx}dk \tag{2.69}$$

The functions $\rho(x)$ and $S(k)$ are said to constitute a Fourier transform pair. A shorthand notation "$\longleftrightarrow$" is often used to signify this pairing relationship such that

$$\rho(x) \overset{\mathcal{F}}{\longleftrightarrow} S(k) \quad \text{or simply} \quad \rho(x) \longleftrightarrow S(k) \tag{2.70}$$

The definitions above can be extended to higher dimensions. In vector notation, the forward and inverse Fourier transforms can be written as

$$S(\boldsymbol{k}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \rho(\boldsymbol{r})e^{-i2\pi \boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{r} \tag{2.71}$$

and

$$\rho(\boldsymbol{r}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} S(\boldsymbol{k})e^{i2\pi \boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k} \tag{2.72}$$

These formulas apply to functions of any dimensionality if $\boldsymbol{k}$ and $\boldsymbol{r}$ are defined appropriately. For instance, in two dimensions, $\boldsymbol{k} = (k_x, k_y)$ and $\boldsymbol{r} = (x, y)$; then

$$S(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, y)e^{-i2\pi(k_x x + k_y y)}dxdy \tag{2.73}$$

and

$$\rho(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k_x, k_y)e^{i2\pi(k_x x + k_y y)}dk_x dk_y \tag{2.74}$$

An important property of higher-dimensional Fourier transforms is that they can be expressed as sequential one-dimensional transforms along each dimension. For example, the two-dimensional forward Fourier transform can be written as

$$\{\mathcal{F}\rho\}(k_x, k_y) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \rho(x, y) e^{-i2\pi k_x x} dx \right\} e^{-i2\pi k_y y} dy \tag{2.75}$$

This property makes it possible to treat higher-dimensional Fourier transform imaging problems in the one-dimensional setting. In general, the following recursive relationship exists for expressing the n-dimensional Fourier transform in terms of lower-dimensional transforms:

$$\mathcal{F}_n = \mathcal{F}_m \mathcal{F}_{n-m} \tag{2.76}$$

where $\mathcal{F}_n$ is the n-dimensional Fourier transform operator. As an example, we have $\mathcal{F}_4 = \mathcal{F}_3\mathcal{F}_1 = \mathcal{F}_2\mathcal{F}_2 = \mathcal{F}_1\mathcal{F}_1\mathcal{F}_1\mathcal{F}_1$, which means that the four-dimensional Fourier transform can be decomposed into one three-dimensional transform cascaded with a one-dimensional transform, or into two cascaded two-dimensional transforms, or into four cascaded one-dimensional transforms.

2.5.2 Properties

Some properties of the Fourier transform relevant to MRI are summarized below for easy reference. For notational simplicity, we assume $\rho(x) \overset{\mathcal{F}}{\longleftrightarrow} S(k)$ whenever appropriate. All the formulas listed are based on the one-dimensional Fourier transform; extension to higher dimensions is in most cases straightforward and left to the reader.

(a) *Uniqueness*:

$$\rho_1(x) = \rho_2(x) \longrightarrow S_1(k) = S_2(k) \tag{2.77}$$

(b) *Linearity*:

$$a\rho_1(x) + b\rho_2(x) \longleftrightarrow aS_1(k) + bS_2(k) \tag{2.78}$$

(c) *Shifting theorem*:

$$\rho(x - x_0) \longleftrightarrow S(k)e^{-i2\pi k x_0} \tag{2.79a}$$

$$e^{i2\pi k_0 x}\rho(x) \longleftrightarrow S(k - k_0) \tag{2.79b}$$

(d) *Modulation*:

$$\rho(x)\cos(2\pi k_0 x) \longleftrightarrow \tfrac{1}{2}[S(k + k_0) + S(k - k_0)] \tag{2.80a}$$

$$\rho(x)\sin(2\pi k_0 x) \longleftrightarrow \tfrac{1}{2}i[S(k + k_0) - S(k - k_0)] \tag{2.80b}$$

(e) *Conjugate symmetry*:

$$\rho^*(x) \longleftrightarrow S^*(-k) \tag{2.81}$$

Specifically, if $\rho(x)$ is a real-valued function, that is, $\rho(x) = \rho^*(x)$, then $S(k) = S^*(-k)$.

(f) *Scaling property*:

$$\rho(ax) \longleftrightarrow \frac{1}{|a|} S\left(\frac{k}{a}\right) \tag{2.82}$$

(g) *Parseval's formula*:

$$\int_{-\infty}^{\infty} \rho_1(x)\rho_2^*(x)dx = \int_{-\infty}^{\infty} S_1(k)S_2^*(k)dk \tag{2.83}$$

Setting $\rho_1(x) = \rho_2(x) = \rho(x)$ yields

$$\int_{-\infty}^{\infty} |\rho(x)|^2 dx = \int_{-\infty}^{\infty} |S(k)|^2 dk \tag{2.84}$$

which states that energy is conserved in both the space and frequency domains.

(h) *Derivative property*:

$$(-i2\pi x)^n \rho(x) \longleftrightarrow \frac{d^n S(k)}{dk^n} \tag{2.85a}$$

$$\frac{d^n \rho(x)}{dx^n} \longleftrightarrow (i2\pi k)^n S(k) \tag{2.85b}$$

(i) *Convolution theorem*:

$$\rho_1(x) * \rho_2(x) \longleftrightarrow S_1(k)S_2(k) \tag{2.86}$$

$$\rho_1(x)\rho_2(x) \longleftrightarrow S_1(k) * S_2(k) \tag{2.87}$$

(j) *Analyticity*:
Functions that we deal with in imaging applications have nonzero values only in a finite spatial region. Functions of this type are called spatially *support-limited*, and their Fourier transform are analytic over the entire k-space. In the one-dimensional case, this property states that

$$\frac{d^n S(k)}{dk^n} \quad \text{exists for all } n \text{ and } k$$

(k) *Asymptotic property*:
If $\rho(x)$ and all of its derivatives up to order n exist and are bounded, then $S(k)$ decays as fast as $1/k^{n+1}$ as $|k| \to \infty$.

2.5.3 Examples

This section presents several examples of Fourier transform calculations based on the definition or properties.

■ Example 2.3

This example analyzes the Fourier transform of a rectangular pulse with unit width and amplitude. From the definition,

$$\int_{-\infty}^{\infty} \Pi(x)e^{-i2\pi kx}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i2\pi kx}dx$$

$$= \frac{1}{-i2\pi k}e^{-i2\pi kx}\bigg|_{x=-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \text{sinc}(\pi k) \tag{2.88}$$

Several properties of the Fourier transform can be observed from the above result.

(a) *Analyticity*: $\mathcal{F}\{\Pi(x)\} = \text{sinc}(\pi k)$ is analytic along the entire k-axis because $\Pi(x)$ is support-limited.

(b) *Asymptotic property*: $\mathcal{F}\{\Pi(x)\}$ decays as fast as $1/k$ as $|k|$ approaches infinity because $\Pi(x)$ has zero-order discontinuities.

(c) By the *scaling property*,

$$a\Pi\left(\frac{x}{W_x}\right) \longleftrightarrow aW_x\Pi(\pi W_x k) \tag{2.89}$$

(d) Based on the *Parseval formula*,

$$\int_{-\infty}^{\infty} \text{sinc}^2(\pi k)dk = \int_{-\infty}^{\infty} \Pi^2(x)dx = 1 \tag{2.90}$$

Also of interest is that

$$\int_{-\infty}^{\infty} \text{sinc}(\pi k)dk = \lim_{x\to 0}\int_{-\infty}^{\infty} \text{sinc}(\pi k)e^{i2\pi kx}dk = \Pi(0) = 1 \tag{2.91}$$

■ **Example 2.4**

This example illustrates the use of the properties for simplifying Fourier transform calculations.

Consider the unit triangular pulse $\Lambda(x)$ defined in Section 2.3. Noting that

$$\frac{d\Lambda(x)}{dx} = \Pi\left(x + \tfrac{1}{2}\right) - \Pi\left(x - \tfrac{1}{2}\right)$$

and the derivative property, we have

$$\begin{aligned}
\{\mathcal{F}\Lambda\}(k) &= \frac{1}{i2\pi k}\mathcal{F}\left\{\frac{d\Lambda(x)}{dx}\right\} \\
&= \frac{1}{i2\pi k}\left[\mathcal{F}\left\{\Pi\left(x + \tfrac{1}{2}\right)\right\} - \mathcal{F}\left\{\Pi\left(x - \tfrac{1}{2}\right)\right\}\right] && (2.92) \\
&= \frac{1}{i2\pi k}\left[\operatorname{sinc}(x)e^{i\pi k} - \operatorname{sinc}(x)e^{-i\pi k}\right] \\
&= \operatorname{sinc}^2(\pi k) && (2.93)
\end{aligned}$$

■ **Example 2.5**

The Gaussian function has an interesting Fourier transform relationship. For simplicity, consider the normalized Gaussian function ($\mu = 0$ and $\sigma = 1$). Its Fourier transform is given by

$$\begin{aligned}
\mathcal{F}\left\{\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right\}(k) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}e^{-i2\pi kx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x/\sqrt{2}+i\pi\sqrt{2}k)^2-(\sqrt{2}\pi k)^2}\,dx \\
&= \frac{1}{\sqrt{\pi}}e^{-(\sqrt{2}\pi k)^2}\int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= e^{-(\sqrt{2}\pi k)^2} && (2.94)
\end{aligned}$$

which is another Gaussian pulse of variances $1/(4\pi^2)$.

■ Example 2.6

In this example, we derive the following transform pairs:

$$\delta(x) \xleftrightarrow{\mathcal{F}} 1 \tag{2.95a}$$

$$\operatorname{sgn}(x) \xleftrightarrow{\mathcal{F}} \frac{1}{i\pi k} \tag{2.95b}$$

$$u(x) \xleftrightarrow{\mathcal{F}} \tfrac{1}{2}\delta(k) + \tfrac{1}{i2\pi k} \tag{2.95c}$$

First, Eq. (2.95a) can easily be obtained from the definition. That is,

$$\int_{-\infty}^{\infty} \delta(x) e^{-i2\pi kx} dx = 1$$

Second, Eq. (2.95b) can be derived as follows.

$$\begin{aligned}
\mathcal{F}\{\operatorname{sgn}(x)\} &= \lim_{a\to 0}\left[\int_0^{\infty} e^{-ax} e^{-i2\pi kx} dx - \int_{-\infty}^{0} e^{ax} e^{-i2\pi kx} dx\right] \\
&= \lim_{a\to 0}\left[\frac{1}{a+i2\pi k} - \frac{1}{a-i2\pi k}\right] \\
&= \frac{1}{i\pi k}
\end{aligned}$$

Incidentally, the preceding expression implies that $\operatorname{sgn}(x)$ has an unbounded dc term; in fact, $\mathcal{F}\{\operatorname{sgn}(x)\} = 0$ for $k = 0$ because $\operatorname{sgn}(x)$ is an odd function. However, this subtlety is widely ignored in the signal processing texts.

Finally, the Fourier transform of $u(x)$ can be derived from the forgoing results. Specifically, noting that

$$u(x) = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn}(x)$$

and

$$\mathcal{F}\{1\} = \delta(k)$$

we have

$$\mathcal{F}\{u(x)\} = \mathcal{F}\left\{\tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn}(x)\right\} = \tfrac{1}{2}\delta(k) + \tfrac{1}{i2\pi k}$$

■ **Example 2.7**

In this example, we derive the Fourier transform of the comb function defined in Section 2.3.

First, recognizing that $\mathrm{comb}(x)$ is a periodic function of period 1, we can express it in terms of a Fourier series as

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx} \tag{2.96}$$

where the coefficients c_n are determined by

$$c_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{comb}(x) e^{-i2\pi nx} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(x) e^{-i2\pi nx} dx = 1$$

Second, applying the Fourier transform to both sides of Eq. (2.96) yields

$$\begin{aligned} \mathcal{F}\{\mathrm{comb}(x)\} &= \sum_{n=-\infty}^{\infty} \mathcal{F}\left\{e^{i2\pi nx}\right\} \\ &= \sum_{n=-\infty}^{\infty} \delta(k-n) = \mathrm{comb}(k) \end{aligned} \tag{2.97}$$

Using the scaling property of the delta function and that of the Fourier transform, we have the following more general result (shown in Fig. 2.10):

$$\sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) \overset{\mathcal{F}}{\longleftrightarrow} \Delta k \sum_{n=-\infty}^{\infty} \delta(k - n\Delta k) \tag{2.98}$$

where $\Delta k = 1/\Delta x$.

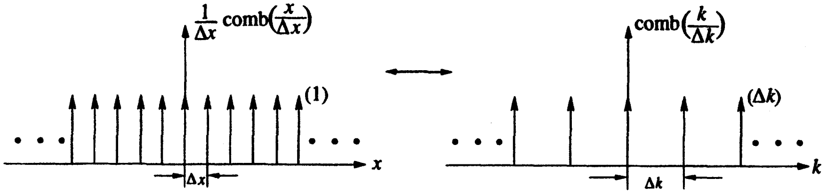


Figure 2.10 The Fourier transform of a delta function train.

2.6 The Radon Transform

The Radon transform is known as the mathematical basis for tomographic imaging from projection (line, plane, or hyperplane integral) data. Unlike the case of the Fourier transform, the popularity and growth of the Radon transform have been closely tied to tomographic imaging [16, 32]. In fact, the basic Radon transform theory was ignored for nearly half a century following its invention in 1917 until its first successful use for generating X-ray tomographic images in 1972. For this reason, this transform has been reinvented several times in different forms and in different areas. In this section, we first present its mathematical definition and then discuss some of its fundamental properties. To bring out the concept, we begin with the definition of the two-dimensional Radon transform. Then we will define higher-dimensional Radon transforms and partial Radon transforms. Finally, we describe the projection-slice theorem and other properties of the Radon transform. The inverse Radon transform is discussed in Chapter 6 in the context of image reconstruction from projection data.

2.6.1 Two-Dimensional Radon Transforms

The two-dimensional Radon transform is simply a line integral, as shown in Fig. 2.11. More specifically, for an arbitrary function $\rho(x, y)$, its Radon transform, denoted as $\mathcal{R}\{\rho(x, y)\}$, or $\{\mathcal{R}\rho\}(p, \phi)$, or simply $P(p, \phi)$, is the integration of $\rho(x, y)$ along a line (ray) L,

$$P(p, \theta) = \mathcal{R}\{\rho(x, y)\} = \int_L \rho(x, y) dl \tag{2.99}$$

where the integral path L is defined by

$$x \cos\phi + y \sin\phi = p \tag{2.100}$$

In some medical imaging literature, $\{\mathcal{R}\rho\}(p, \phi)$ is called a *raysum* since the line integral is accomplished physically by passing a ray through an object. For a fixed ϕ, $\{\mathcal{R}\rho\}(p, \phi)$, as a function of p, is a projection of $\rho(x, y)$ along L, and ϕ is referred to as the *projection angle*. It is important to note, however, that the projection angle is defined as the angle between the x-axis and the line normal to the ray path (not the orientation angle of the ray path itself).

Mathematically, $\mathcal{R}\{\rho(x, y)\}$ can be written in several equivalent forms. For example, the line integral in Eq. (2.99) can be converted to a one-dimensional integration as

$$\{\mathcal{R}\rho\}(p, \phi) = \int_{-\infty}^{\infty} \rho(p\cos\phi - q\sin\phi, p\sin\phi + q\cos\phi) dq \tag{2.101}$$

Equation (2.101) is obtained by transforming the (x, y) coordinate system to the rotated coordinate system (p, q), shown in Fig. 2.12. The required transformation

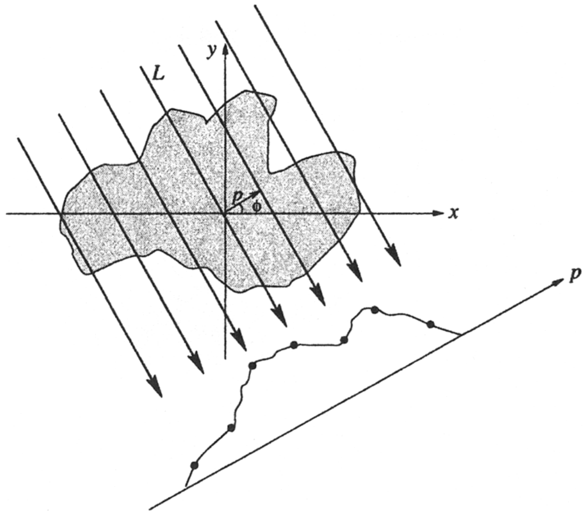


Figure 2.11 Two-dimensional Radon transform as line integrals.

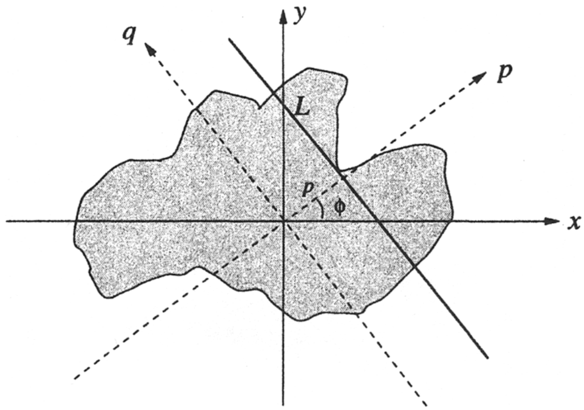


Figure 2.12 Ray path L relative to the original and rotated coordinates.

is given by

$$\begin{cases} p = x\cos\phi + y\sin\phi \\ q = -x\sin\phi + y\cos\phi \end{cases} \quad \text{or} \quad \begin{cases} x = p\cos\phi - q\sin\phi \\ y = p\sin\phi + q\cos\phi \end{cases} \tag{2.102}$$

Another more convenient form of the two-dimensional Radon transform is

$$\{\mathcal{R}\rho\}(p,\phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x,y)\delta(x\cos\phi + y\sin\phi - p)dxdy \tag{2.103}$$

The equivalence between Eqs. (2.101) and (2.103) can be justified using the coordinate transformation defined in Eq. (2.102). Specifically, upon performing the variable substitution

$$\begin{cases} x = \hat{p}\cos\phi - \hat{q}\sin\phi \\ y = \hat{p}\sin\phi + \hat{q}\cos\phi \end{cases} \tag{2.104}$$

Eq. (2.103) becomes

$$\{\mathcal{R}\rho\}(p,\phi) = \iint \rho(\hat{p}\cos\phi - \hat{q}\sin\phi, \hat{p}\sin\phi + \hat{q}\cos\phi)\delta(\hat{p}-p)|\mathbf{J}|d\hat{p}d\hat{q} \tag{2.105}$$

Equation (2.101) immediately follows by making use of the property of the delta function and noting that

$$|\mathbf{J}| = \begin{vmatrix} \dfrac{\partial x}{\partial \hat{p}} & \dfrac{\partial x}{\partial \hat{q}} \\ \dfrac{\partial y}{\partial \hat{p}} & \dfrac{\partial y}{\partial \hat{q}} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} = 1 \tag{2.106}$$

■ **Example 2.8**

This examples calculates $\{\mathcal{R}\rho\}(p, 0^\circ)$ for $\rho(x,y) = \Pi(\frac{x}{a})\Pi(\frac{y}{b})$. Based on the definition, we have

$$\begin{aligned}
\{\mathcal{R}\rho\}(p,0^\circ) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)\delta(x\cos 0^\circ + y\sin 0^\circ - p)dxdy \\
&= \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \delta(x-p)dxdy \\
&= b\int_{-a/2}^{a/2} \delta(x-p)dxdy \\
&= \begin{cases} b & -a/2 \le p \le a/2 \\ 0 & \text{otherwise} \end{cases} \\
&= b\Pi\left(\frac{p}{a}\right)
\end{aligned}$$

2.6.2 Higher-Dimensional Radon Transforms

The Radon transform of higher-dimensional functions can be defined by extending Eq. (2.103) to higher dimensions. First, we rewrite Eq. (2.103) in vector form with the dimension of the transform operator and the vector variables being made explicit:

$$\{\mathcal{R}_2\rho\}(p,\boldsymbol{\mu}_2) = \int_{\mathbb{R}^2} \rho(\boldsymbol{r}_2)\delta(p - \boldsymbol{\mu}_2 \cdot \boldsymbol{r}_2)d\boldsymbol{r}_2 \tag{2.107}$$

where $\boldsymbol{\mu}_2 = (\cos\phi, \sin\phi)$, $\boldsymbol{r}_2 = (x, y)$, and $d\boldsymbol{r}_2 = dxdy$. Extending Eq. (2.107) to n dimensions gives

$$\{\mathcal{R}_n\rho\}(p,\boldsymbol{\mu}_n) = \int_{\mathbb{R}^n} \rho(\boldsymbol{r}_n)\delta(p - \boldsymbol{\mu}_n \cdot \boldsymbol{r}_n)d\boldsymbol{r}_n \tag{2.108}$$

which is called the *nth-dimensional Radon transform* of $\rho(\boldsymbol{r}_n)$.

To gain a better understanding of Eq. (2.108), consider the three-dimensional case. In the spherical coordinate system,

$$\boldsymbol{\mu}_3 = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.109}$$

and

$$\boldsymbol{\mu}_3 \cdot \boldsymbol{r}_3 = x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta \tag{2.110}$$

where θ and ϕ are the polar and azimuthal angles, respectively. Substituting Eq. (2.110) into Eq. (2.108) yields

$$\{\mathcal{R}_3\rho\}(p,\boldsymbol{\mu}_3) = \iiint \rho(x,y,z)\delta(x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta - p)dxdydz \tag{2.111}$$

Noting that

$$x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta = p \tag{2.112}$$

defines a plane as shown in Fig. 2.13, it is clear that Eq. (2.111) is a plane integral in contrast to the line integral given by Eq. (2.103) for the two-dimensional case.

For $n > 3$, it is convenient to employ the hyperspherical polar coordinates $(r, \theta_1, \cdots, \theta_{n-2}, \phi)$ with

$$\begin{cases} r_1 = r\cos\theta_1 \\ r_2 = r\sin\theta_1\cos\theta_2 \\ \quad\vdots \\ r_{n-2} = r\sin\theta_1\cdots\sin\theta_{n-3}\cos\theta_{n-2} \\ r_{n-1} = r\sin\theta_1\cdots\sin\theta_{n-2}\cos\phi \\ r_n = r\sin\theta_1\cdots\sin\theta_{n-2}\sin\phi \end{cases} \tag{2.113}$$

for $0 \leq \theta_\ell \leq \pi$, $0 \leq \phi \leq 2\pi$, and $r \geq 0$. The unit vector in Eq. (2.108) is now given by

$$\boldsymbol{\mu}_n = (\cos\theta_1, \sin\theta_1\cos\theta_2, \cdots, \sin\theta_1\cdots\sin\theta_{n-2}\sin\phi) \tag{2.114}$$

and the volume element is

$$d\boldsymbol{r} = r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}(\sin\theta_2)d\theta_1 d\theta_2\cdots d\theta_{n-2}d\phi \tag{2.115}$$

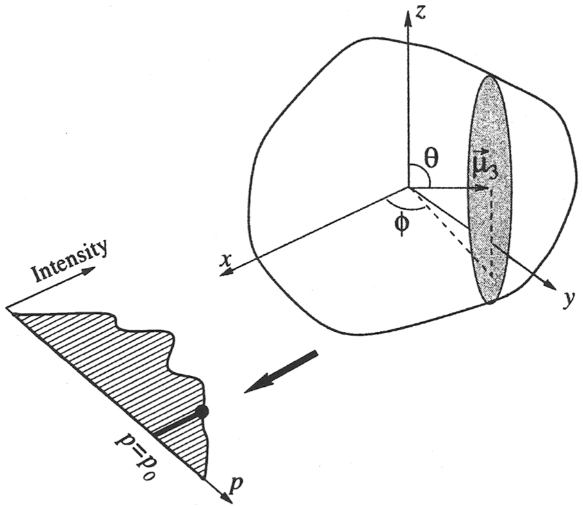


Figure 2.13 Three-dimensional Radon transform as plane integrals.

2.6.3 Partial Radon Transforms

For functions of more than two dimensions, it is sometimes useful to define a *partial Radon transform*, which is the mathematical foundation for the multistage projection reconstruction algorithm to be described in Chapter 6. Before introducing the general definition, let us first consider the three-dimensional Radon transform. From Eq. (2.111), we have

$$
\begin{aligned}
\mathcal{R}_3\rho &= \iiint \rho(x,y,z)\delta(x\cos\phi\sin\theta + y\sin\phi\sin\theta + z\cos\theta - p)dxdydz \\
&= \iiint \rho(x,y,z)\delta[(x\cos\phi + y\sin\phi)\sin\theta + z\cos\theta - p]dxdydz \\
&= \iiiint \rho(x,y,z)\delta(q\sin\theta + z\cos\theta - p) \\
&\qquad \delta(x\cos\phi + y\sin\phi - q)dqdxdydz \\
&= \iint \left[\iint \rho(x,y,z)\delta(x\cos\phi + y\sin\phi - q)dxdy\right] \\
&\qquad \delta(q\sin\theta + z\cos\theta - p)dqdz
\end{aligned}
\tag{2.116}
$$

The double integral inside the brackets is a line integral in the (x, y)-plane. Treating $\rho(x, y, z)$ as a function of x and y with z being a free parameter, the bracketed double integral can be viewed as a two-dimensional Radon transform and written as

$$\{\mathcal{R}_2\rho\}(q,\phi;z) = \iint \rho(x,y,z)\delta(x\cos\phi + y\sin\phi - q)dxdy \tag{2.117}$$

where $\boldsymbol{\mu}_2 = (\cos\phi, \sin\phi)$. Substituting Eq. (2.117) into Eq. (2.116) yields

$$\{\mathcal{R}_3\rho\}(p,\phi,\theta) = \iint \mathcal{R}_2\rho(q,\phi;z)\delta(q\sin\theta + z\cos\theta - p)dqdz \tag{2.118}$$

We refer to $\{\mathcal{R}_2\rho\}(q,\phi;z)$ as a *partial* Radon transform of $\rho(x,y,z)$ because the function is only partially transformed (along the x- and y-directions). As with Eq. (2.117), we also have other partial transforms such as $\{\mathcal{R}_2\rho\}(q,\phi;x)$ and $\{\mathcal{R}_2\rho\}(q,\phi;y)$, as well as those not in the cardinal directions. In general, we define a partial Radon transform as a lower-dimensional transform of a higher-dimensional function. A more formal definition is as follows.

Definition 2.1 *For an n-dimensional function $\rho(\boldsymbol{r})$, its m-dimensional partial Radon transform along the first m cardinal directions $r_1, r_2, \ldots, r_m$ with $m \leq n$ is defined as*

$$\mathcal{R}_m\rho(p,\boldsymbol{\mu}_m; r_{m+1},\ldots,r_n) = \int_{\mathbb{R}^m} \rho(\boldsymbol{r}_m; r_{m+1},\ldots,r_n)\delta(p - \boldsymbol{\mu}_m \cdot \boldsymbol{r}_m)d\boldsymbol{r}_m \tag{2.119}$$

where $\boldsymbol{r}_m = (r_1, r_2, \ldots, r_m)$ and $\boldsymbol{\mu}_m$ is a unit directional vector in $\mathbb{R}^m$ defining the projection direction.

It is clear from this definition and from Fig. 2.14 that a notable distinction between partial and full Radon transforms is that, for a given projection angle, the Radon transform reduces a function $\rho(\boldsymbol{r})$ to a one-dimensional projection profile, while the partial Radon transforms are planar or hyperplanar projections of $\rho(\boldsymbol{r})$. Specifically, for an n-dimensional function, its m-dimensional partial Radon transform has $(n-m)$ untransformed spatial dimensions which, combined with distributions along the p-axis, form an $(n-m+1)$-dimensional projection of the function for $2 \leq m \leq n$. For the special case $m = n$, the partial Radon transform becomes the (full) Radon transform.

Similarly to the Fourier transform, a higher-dimensional Radon transform can be expressed as cascaded lower-dimensional (partial) Radon transforms. This is demonstrated by Eq. (2.118), in which the three-dimensional Radon transform is equivalent to two cascaded two-dimensional Radon transforms. It is also easy to visualize that a four-dimensional Radon transform can be expressed as three cascaded two-dimensional Radon transforms, or as one two-dimensional Radon transform followed by a three-dimensional Radon transform, or vice versa. In general, the following recursive relationship exists for the partial Radon transform operator:

$$\mathcal{R}_n = \mathcal{R}_{n-m+1}\mathcal{R}_m \qquad 2 \leq m \leq n-1 \text{ and } n > 2 \tag{2.120}$$

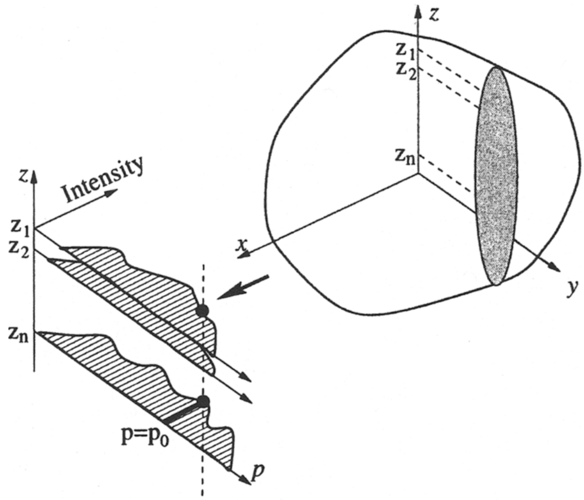


Figure 2.14 Graphical representation of two-dimensional partial Radon transforms of a three-dimensional object.

where $\mathcal{R}_n$ represents the n-dimensional (partial) Radon transform operator. To prove the decomposition in Eq. (2.120), we rewrite Eq. (2.108) as

$$\{\mathcal{R}_n\rho\}(p,\boldsymbol{\mu}_n) = \int_{\mathbb{R}^n} \rho(\boldsymbol{r}_n)\delta(p-\boldsymbol{\mu}'_m\cdot\boldsymbol{r}_m-\boldsymbol{\mu}'_{n-m}\cdot\boldsymbol{r}_{n-m})d\boldsymbol{r}_m d\tilde{\boldsymbol{r}}_{n-m} \quad (2.121)$$

where it is assumed that

$$\boldsymbol{\mu}'_m = (\mu_1,\mu_2,\ldots,\mu_m) \quad (2.122a)$$
$$\boldsymbol{r}_m = (r_1,r_2,\ldots,r_m) \quad (2.122b)$$
$$\tilde{\boldsymbol{\mu}}'_{n-m} = (\mu_{m+1},\mu_{m+2},\ldots,\mu_n) \quad (2.122c)$$
$$\tilde{\boldsymbol{r}}_{n-m} = (r_{m+1},r_{m+2},\ldots,r_n) \quad (2.122d)$$

such that

$$\boldsymbol{\mu}_n = (\boldsymbol{\mu}'_m,\tilde{\boldsymbol{\mu}}'_{n-m}) \quad (2.123a)$$
$$\boldsymbol{r}_n = (\boldsymbol{r}'_m\tilde{\boldsymbol{r}}_{n-m}) \quad (2.123b)$$

The prime superscripts on $\boldsymbol{\mu}'_m$ and $\tilde{\boldsymbol{\mu}}'_{n-m}$ are used to indicate that they are not unit vectors. Let $q = \frac{\boldsymbol{\mu}'_m}{|\boldsymbol{\mu}'_m|}\cdot\boldsymbol{r}_m$. Then

$$\{\mathcal{R}_n\rho\}(p,\boldsymbol{\mu}_n) = \int_{\mathbb{R}^{n-m+1}} \left[\int_{\mathbb{R}^m} \rho(\boldsymbol{r}_m,\tilde{\mathbf{r}}_{n-m})\delta\left(q-\frac{\boldsymbol{\mu}'_m}{|\boldsymbol{\mu}'_m|}\cdot\boldsymbol{r}_m\right)d\boldsymbol{r}_m\right]$$
$$\delta(p-|\boldsymbol{\mu}'_m|q-\tilde{\boldsymbol{\mu}}'_{n-m}\cdot\tilde{\boldsymbol{r}}_{n-m})dq d\tilde{\boldsymbol{r}}_{n-m} \quad (2.124)$$

Equation (2.120) immediately follows from Eq. (2.124) by noting that $\frac{\mu'_m}{|\mu'_m|}$ is a unit directional vector in $\mathbb{R}^m$ and $(|\mu'_m|, \mu'_{n-m})$ forms a unit directional vector in $\mathbb{R}^{n-m+1}$.

2.6.4 Basic Properties

The Radon transform possesses many useful properties. Some of the straightforward ones for the two-dimensional case are listed below; others are discussed subsequently in detail. Most of them can be extended to higher dimensions. For notational simplicity, we assume that $\rho(x, y) \overset{\mathcal{R}}{\longleftrightarrow} P(p, \phi)$.

(a) *Linearity*:

$$a\rho_1(x, y) + b\rho_2(x, y) \longleftrightarrow aP_1(p, \phi) + bP_2(p, \phi) \tag{2.125}$$

(b) *Symmetry*:

$$P(p, \phi) = P(-p, \phi \pm \pi) \tag{2.126}$$

(c) *Periodicity*:

$$P(p, \phi) = P(p, \phi + 2n\pi) \qquad \text{for integer } n \tag{2.127}$$

(d) *Shifting property*:

$$\rho(x - x_0, y - y_0) \longleftrightarrow P(p - x_0 \cos\phi - y_0 \sin\phi, \phi) \tag{2.128}$$

(e) *Rotation by ϕ_0*:

$$\rho(x\cos\phi_0 - y\sin\phi_0, x\sin\phi_0 + y\cos\phi_0) \longleftrightarrow P(p, \phi + \phi_0) \tag{2.129}$$

(f) *Scaling property*:

$$\rho(ax, ay) \longleftrightarrow \frac{1}{|a|} P(ap, \phi) \tag{2.130}$$

(g) *Energy conservation*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x, y)dxdy = \int_{-\infty}^{\infty} P(p, \phi)dp \tag{2.131}$$

2.6.5 Sinogram

The two-dimensional Radon transform maps the spatial domain (x, y) to the Radon domain (p, ϕ). This mapping exhibits some interesting properties. For example, each point in the Radon space corresponds to a straight line in the spatial domain; in other words, the data value at a particular point in the Radon space receives contributions from data points along a line in the spatial domain. On the other hand, a point in the spatial domain is mapped to a sinusoid in the Radon space. The first point is clear from the definition of the Radon transform. The latter point can be understood by considering a point source located at $\boldsymbol{r}_0 = (x_0, y_0)$ with intensity A. Its Radon transform is

$$\begin{aligned}\mathcal{R}\{A\delta(\boldsymbol{r}-\boldsymbol{r}_0)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\delta(\boldsymbol{r}-\boldsymbol{r}_0)\delta(p-\boldsymbol{\mu}\cdot\boldsymbol{r})d\boldsymbol{r} \\ &= A\delta(p-\boldsymbol{\mu}\cdot\boldsymbol{r}_0) \\ &= A\delta(x_0\cos\phi + y_0\sin\phi - p) \end{aligned} \tag{2.132}$$

which is a sheet of impulses supported on a sinusoidal curve in the (p, ϕ)-plane, as shown in Fig. 2.15, defined by

$$p = x_0\cos\phi + y_0\sin\phi = r_0\cos(\phi - \phi_0) \tag{2.133}$$

where $r_0 = \sqrt{x_0^2 + y_0^2}$ and $\phi_0 = \arctan(y_0/x_0)$. For this reason, the two-dimensional function formed by stacking up all the projections taken sequentially along the angular direction is called a *sinogram*.

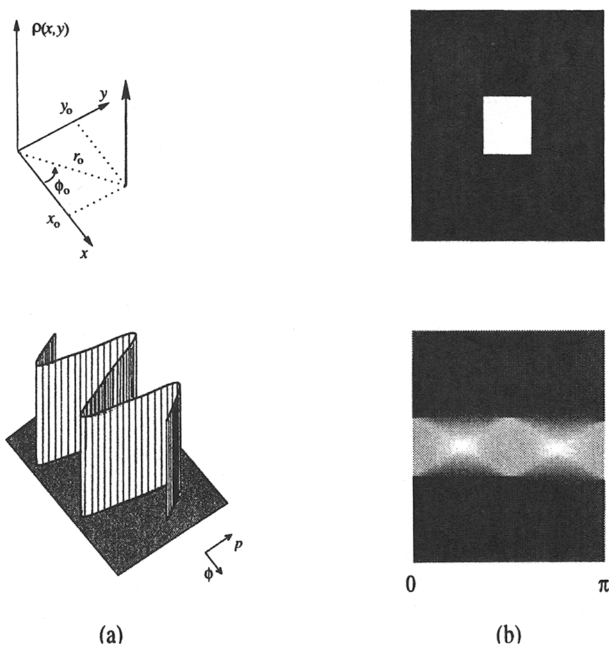


Figure 2.15 Sinograms of (a) a point source and (b) a rectangular function.

For an arbitrary function $\rho(x, y)$, the sinogram may be more complicated. Based on the sampling property of the delta function, we can rewrite $\rho(x, y)$ as

$$\rho(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x_0, y_0)\delta(x - x_0, y - y_0)dx_0 dy_0 \tag{2.134}$$

Since the Radon transform operation is linear, by superposition the sinogram for $\rho(x, y)$ is a continuous sum of sinusoidal sheets with intensity $\rho(x_0, y_0)$, amplitude $r_0 = \sqrt{x_0^2 + y_0^2}$, and initial phase $\phi_0 = \arctan(y_0/x_0)$.

2.6.6 The Projection-Slice Theorem

The Radon transform is closely related to the Fourier transform by the famous projection-slice theorem, which is the theoretical basis for several image reconstruction algorithms. This theorem actually exists in two forms, although only the first form is popularly known. We will describe them both here. The first form of this theorem is for the (full) Radon transform, which relates a one-dimensional projection to a line of data in k-space, while the second form is for the partial Radon transform, which connects planar or hyperplanar projections to the k-space data.

Theorem 2.2 (Projection-Slice Theorem) *For an n-dimensional function $\rho(\boldsymbol{r})$, the one-dimensional Fourier transform of $\{\mathcal{R}\rho\}(p, \boldsymbol{\mu})$ along the p-axis for a fixed projection angle $\boldsymbol{\mu}$ is identical to the n-dimensional Fourier transform of $\rho(\boldsymbol{r})$ evaluated along a line passing through the origin with the same orientation angle in the Fourier space. Mathematically, this theorem can expressed as*

$$\mathcal{F}_p\{\{\mathcal{R}\rho\}(p, \boldsymbol{\mu})\} = \{\mathcal{F}\rho\}(k\boldsymbol{\mu}) \tag{2.135}$$

where $\mathcal{F}_p$ represents one-dimensional Fourier transform along the p-axis.

This theorem can be proven easily from the definition as follows:

$$\begin{aligned}
\mathcal{F}_p\{\{\mathcal{R}\rho\}(p, \boldsymbol{\mu})\} &= \int_{-\infty}^{\infty} \{\mathcal{R}\rho\}(p, \boldsymbol{\mu}) e^{-i2\pi kp} dp \\
&= \int_{-\infty}^{\infty} \left[\int_{\boldsymbol{r}\in R^n} \rho(\boldsymbol{r})\delta(p - \boldsymbol{\mu}\cdot\boldsymbol{r})d\boldsymbol{r}\right] e^{-i2\pi kp} dp \\
&= \int_{\boldsymbol{r}\in R^n} \rho(\boldsymbol{r}) \left[\int_{-\infty}^{\infty} \delta(p - \boldsymbol{\mu}\cdot\boldsymbol{r}) e^{-i2\pi kp} dp\right] d\boldsymbol{r} \\
&= \int_{\boldsymbol{r}\in R^n} \rho(\boldsymbol{r}) e^{-i2\pi k\boldsymbol{\mu}\cdot\boldsymbol{r}} d\boldsymbol{r} \\
&= \{\mathcal{F}\rho\}(k\boldsymbol{\mu})
\end{aligned}$$

For a better appreciation of the theorem, we take a closer look at it for the two- and three-dimensional cases. In two dimensions, we have $\boldsymbol{\mu} = (\cos\phi, \sin\phi)$ and, consequently,

$$\mathcal{F}_p\{\{\mathcal{R}\rho\}(p,\phi)\} = \{\mathcal{F}\rho\}(k\cos\phi, k\sin\phi) \tag{2.136}$$

which is depicted in Fig. 2.16. It is clear from this example that projections of $\rho(x,y)$ correspond to slices of its Fourier transform $\{\mathcal{F}\rho\}(k_x,k_y)$, thus the name *projection-slice theorem.*

In three dimensions, $\boldsymbol{\mu} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and the projection-slice theorem states that

$$\mathcal{F}_1\{\{\mathcal{R}\rho\}(p,\phi,\theta)\} = \{\mathcal{F}\rho\}(k\sin\theta\cos\phi, k\sin\theta\sin\phi, k\cos\theta) \tag{2.137}$$

Note that

$$\begin{cases} k_x = k\sin\theta\cos\phi \\ k_y = k\sin\theta\sin\phi \\ k_z = k\cos\theta \end{cases} \tag{2.138}$$

or equivalently,

$$\frac{k_x}{\sin\theta\cos\phi} = \frac{k_y}{\sin\theta\sin\phi} = \frac{k_z}{\cos\theta} \tag{2.139}$$

defines a line along the direction of $\boldsymbol{\mu}$ in the three-dimensional k-space.

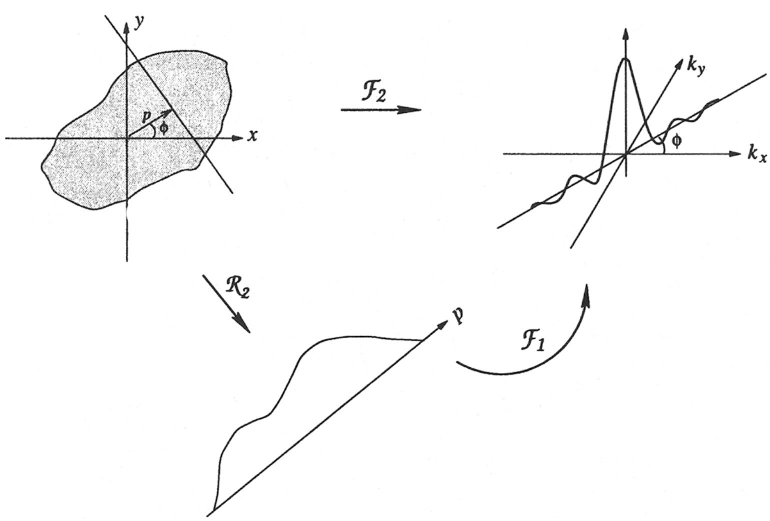


Figure 2.16 Pictorial representation of the projection-slice theorem in two dimensions.

■ **Example 2.9**

This example calculates the Radon transform of the object shown in Fig. 2.17.

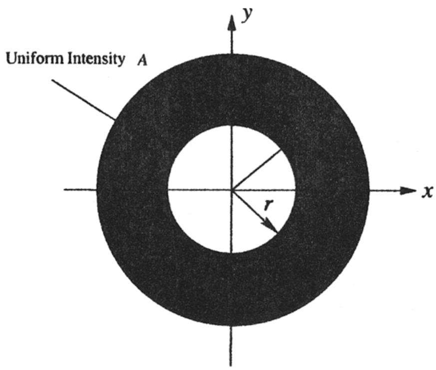


Figure 2.17 Graphical representation of $\rho(x, y)$.

First, we express the object function as

$$\rho(x,y) = \begin{cases} A & r^2 \leq x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

Second, noting the object has a circular symmetry, it suffices to calculate $\{\rho\}(p, \phi)$ for only one projection angle, say $\phi = 0°$.

Third, let $\rho(x, y) = \rho_1(x, y) - \rho_2(x, y)$, where

$$\rho_1(x,y) = \begin{cases} A & x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases} \qquad \rho_2(x,y) = \begin{cases} A & x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

By definition,

$$\begin{aligned} \{\mathcal{R}\rho_1\}(p, \phi = 0°) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_1(x,y)\delta(p - x)dxdy \\ &= \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} A\delta(p - x)dydx \\ &= \int_{-R}^{R} 2A\sqrt{R^2 - x^2}\delta(p - x)dx \\ &= \begin{cases} 2A\sqrt{R^2 - p^2} & -R \leq p \leq R \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Similarly, we have

$$\{\mathcal{R}\rho_2\}(p, \phi = 0°) = \begin{cases} 2A\sqrt{r^2 - p^2} & -r \le p \le r \\ 0 & \text{otherwise} \end{cases}$$

Finally, based on the linearity of the Radon transform, we have

$$\begin{aligned}\mathcal{R}\{\rho(x, y)\} &= \{\mathcal{R}\rho_1\}(p, \phi = 0°) - \{\mathcal{R}\rho_2\}(p, \phi = 0°) \\ &= \begin{cases} 2A\sqrt{R^2 - p^2} & -R \le p \le -r, r \le p \le R \\ 2A\left[\sqrt{R^2 - p^2} - \sqrt{r^2 - p^2}\right] & -r \le p \le r \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

■ **Example 2.10**

This example calculates the Fourier transform of $\mathcal{R}\{\rho\}(p, 45°)$ for a square object defined by $\rho(x, y) = \Pi(x/2)\Pi(y/2)$. We will do so directly and by the projection-slice theorem.

First, we evaluate $\{\mathcal{R}\rho\}(p, 45°)$ for the given object. It is clear from Fig. 2.18 that the projection is taken along the diagonal of the square. Simple geometric analysis shows that the result is a triangular pulse. Namely,

$$\{\mathcal{R}\rho\}(p, 45°) = 2\sqrt{2}\Lambda\left(\frac{p}{\sqrt{2}}\right)$$

Based on the result in Example 2.4 and the scaling property of the Fourier transform, we have

$$\mathcal{F}\{\{\mathcal{R}\rho\}(p, 45°)\} = \mathcal{F}\left\{2\sqrt{2}\Lambda\left(\frac{p}{\sqrt{2}}\right)\right\} = 4\,\mathrm{sinc}^2(\sqrt{2}\pi k)$$

We next derive the result by using the projection-slice theorem. We first find the two-dimensional Fourier transform of $\rho(x, y)$, which is

$$\{\mathcal{F}\rho\}(k_x, k_y) = 4\,\mathrm{sinc}(2\pi k_x)\mathrm{sinc}(2\pi k_y)$$

Then, from the projection-slice theorem, we immediately get the following result.

$$\begin{aligned}\mathcal{F}\{\{\mathcal{R}\rho\}(p, 45^\circ)\} &= \{\mathcal{R}\rho\}(k\cos 45^\circ, k\sin 45^\circ)\\ &= 4\,\mathrm{sinc}(\sqrt{2}\pi k)\mathrm{sinc}(\sqrt{2}\pi k)\\ &= 4\,\mathrm{sinc}^2(\sqrt{2}\pi k)\end{aligned}$$

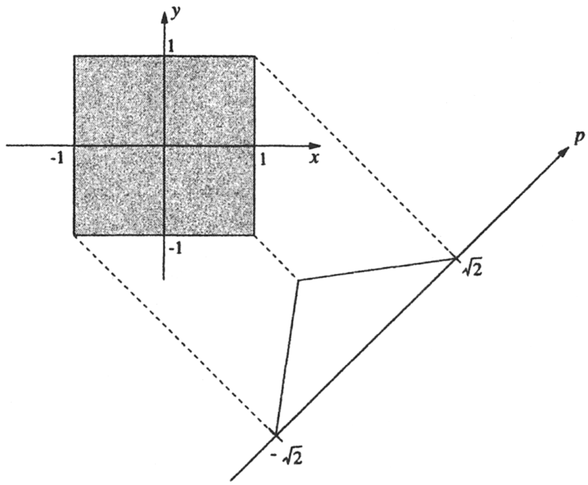


Figure 2.18 Projection of a square object.

Theorem 2.3 (Generalized Projection-Slice Theorem) *For an n-dimensional function $\rho(\boldsymbol{r})$, the Fourier transform of its partial Radon transform $\mathcal{R}_m\rho$ is related to its Fourier transform $\mathcal{F}\rho$ by the following relationship:*

$$\{\mathcal{F}_{n-m+1}\mathcal{R}_m\rho\}(k, k_{m+1}, \ldots, k_n) = \{\mathcal{F}\rho\}(k\boldsymbol{\mu}_m, k_{m+1}, \ldots, k_n) \tag{2.140}$$

where it is understood that $\mathcal{R}_m\rho$ is a function of $(p, \boldsymbol{\mu}_m; x_{m+1}, \ldots, x_n)$ such that p and k, and $(x_{m+1}, \ldots, x_n)$ and $(k_{m+1}, \ldots, k_n)$ are conjugate variable pairs of $\mathcal{F}_{n-m+1}$. The function on the right-hand side of Eq. (2.140) should be interpreted as $\{\mathcal{F}_n\rho\}(k_1, k_2, \ldots, k_n)$ evaluated at $(k_1, k_2, \ldots, k_m) = k\boldsymbol{\mu}_m$.

We call Eq. (2.140) the *generalized projection-slice theorem* because it includes the basic projection-slice theorem as a special case with $m = n$. This theorem can be proven following the same procedure used to prove the basic projection-slice theorem as follows.

$$
\begin{aligned}
&\{\mathcal{F}_{n-m+1}\mathcal{R}_m\rho\}(k, k_{m+1}, \ldots, k_n) \\
&= \int_{R^{n-m}} \int_{-\infty}^{\infty} \left[\int_{R^m} \rho(\boldsymbol{r}_m; x_{m+1}, \ldots, x_n)\delta(p - \boldsymbol{\mu}_m \cdot \boldsymbol{r}_m) d\boldsymbol{r}_m \right] \\
&\qquad\qquad e^{-i2\pi kp} e^{-i2\pi(k_{m+1}x_{m+1}+\cdots+k_n x_n)} dp dx_{m+1} \cdots dx_n \\
&= \int_{R^n} \rho(\boldsymbol{r}_m; x_{m+1}, \ldots, x_n) e^{-i2\pi(k\boldsymbol{\mu}_m \cdot \boldsymbol{r}_m + k_{m+1}x_{m+1}+\cdots+k_n x_n)} \\
&\qquad\qquad d\boldsymbol{r}_m dx_{m+1} \cdots dx_n \\
&= \{\mathcal{F}\rho\}(k\boldsymbol{\mu}_m, k_{m+1}, \ldots, k_n)
\end{aligned}
$$

■ **Example 2.11**

In this example, we take a look at the generalized projection-slice theorem for a special case with $n = 3$ and $m = 2$. Specifically, if we choose the projection direction such that $\boldsymbol{\mu}_2 = (\cos 90°, \sin 90°) = (0, 1)$, we have

$$
\begin{aligned}
\{\mathcal{R}_2\rho\}(p = y, z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, y, z)\delta(p - y) dx dy \\
&= \int_{-\infty}^{\infty} \rho(x, y, z) dx
\end{aligned}
$$

which is a two-dimensional projection of the object along the x-axis. Based on the foregoing result, we easily get

$$
\begin{aligned}
\{\mathcal{F}_2\mathcal{R}_2\rho\}(k_y, k_z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, y, z) e^{-i2\pi(k_y y + k_z z)} dx dy dz \\
&= \mathcal{F}\rho(0, k_y, k_z)
\end{aligned}
$$

which is exactly what the generalized projection-slice theorem predicts noting that $k\boldsymbol{\mu}_2 = (0, k) = (0, k_y)$.

2.6.7 Convolution Theorem

Another useful relationship associated with the Radon transform is the convolution theorem, which states that the Radon transform of the convolution of two functions $\rho_1(\boldsymbol{r})$ and $\rho_2(\boldsymbol{r})$ is equal to the one-dimensional convolution of their Radon transforms. That is,

$$
\mathcal{R}\left\{ \int \rho_1(\hat{\boldsymbol{r}})\rho_2(\boldsymbol{r} - \hat{\boldsymbol{r}}) d\hat{\boldsymbol{r}} \right\} = \int_{-\infty}^{\infty} \{\mathcal{R}\rho_1\}(q, \boldsymbol{\mu})\{\mathcal{R}\rho_2\}(p - q, \boldsymbol{\mu}) dq \tag{2.141}
$$

Equation (2.141) can be obtained directly from the definition of convolution and the Radon transform. Specifically,

$$
\begin{aligned}
\mathcal{R}\{\rho_1 * \rho_2\}(p, \boldsymbol{\mu}) &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \rho_1(\hat{\mathbf{r}})\rho_2(\mathbf{r} - \hat{\mathbf{r}})d\hat{\mathbf{r}} \right] \delta(p - \boldsymbol{\mu} \cdot \mathbf{r})d\mathbf{r} \\
&= \int_{\mathbb{R}^n} \rho_1(\hat{\mathbf{r}}) \left[\int_{\mathbb{R}^n} \rho_2(\mathbf{r} - \hat{\mathbf{r}})\delta(p - \boldsymbol{\mu} \cdot \mathbf{r})d\mathbf{r} \right] d\hat{\mathbf{r}} \\
&= \int_{\mathbb{R}^n} \rho_1(\hat{\mathbf{r}}) \left[\int_{\mathbb{R}^n} \rho_2(\hat{\mathbf{r}}_1)\delta[(p - \boldsymbol{\mu} \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu} \cdot \hat{\mathbf{r}}_1]d\hat{\mathbf{r}}_1 \right] d\hat{\mathbf{r}} \\
&= \int_{\mathbb{R}^n} \rho_1(\hat{\mathbf{r}})\{\mathcal{R}\rho_2\}(p - \boldsymbol{\mu} \cdot \hat{\mathbf{r}}, \boldsymbol{\mu})d\hat{\mathbf{r}} \\
&= \int_{\mathbb{R}^n} \rho_1(\hat{\mathbf{r}}) \left[\int_{\infty}^{\infty} \{\mathcal{R}\rho_2\}(p - q, \boldsymbol{\mu})\delta(q - \boldsymbol{\mu} \cdot \hat{\mathbf{r}})dq \right] d\hat{\mathbf{r}} \\
&= \int_{\infty}^{\infty} \{\mathcal{R}\rho_2\}(p - q, \boldsymbol{\mu}) \left[\int_{\mathbb{R}^n} \rho_1(\hat{\mathbf{r}})\delta(q - \boldsymbol{\mu} \cdot \hat{\mathbf{r}})d\hat{\mathbf{r}} \right] dq \\
&= \int_{\infty}^{\infty} \{\mathcal{R}\rho_1\}(q, \boldsymbol{\mu})\{\mathcal{R}\rho_2\}(p - q, \boldsymbol{\mu})dq
\end{aligned}
$$

Exercises

2.1 Let $\boldsymbol{v} = (1, -2, 3)$. Determine $|\boldsymbol{v}|$ and $\boldsymbol{\mu}_v$, and graph $\boldsymbol{v}$.

2.2 Let $\boldsymbol{A} = (1, 1, 0)$ and $\boldsymbol{B} = (1, 2, -2)$.

(a) Graph $\boldsymbol{A}$ and $\boldsymbol{B}$.

(b) Evaluate and graph $\boldsymbol{A} + \boldsymbol{B}$, $\boldsymbol{A} \cdot \boldsymbol{B}$, and $\boldsymbol{A} \times \boldsymbol{B}$.

(c) Determine the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

2.3 Derive Eq. (2.17) from Eq. (2.15).

2.4 Derive Eq. (2.22) from Eq. (2.18).

2.5 Consider the following two matrices:

$$\mathbf{W}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{W}_2 = \begin{bmatrix} \sqrt{2} & i \\ -i & \sqrt{2} \end{bmatrix}$$

(a) Determine if they are orthogonal or unitary matrices.

(b) Evaluate $\mathbf{W}_1\mathbf{W}_2$.

(c) Determine $\mathbf{W}_1^{-1}$ and $\mathbf{W}_2^{-1}$.

2.6 Sketch the following functions:

(a) $\Pi(\frac{x}{2})$

(b) $\Pi(2x - 10)$

(c) $\Lambda(2x + 5)$

(d) $\Lambda(-2x + 5)$

(e) $\operatorname{sinc}(\frac{x-3}{2})$

(f) $\operatorname{sinc}(\frac{x}{5})\Pi(\frac{x}{10\pi})$

(g) $n^2\{u[n] - u[n-4]\}$

(h) $u[n+3] - u[n-5]$

(i) $u[-n+5]u[n+3]$

(j) $u[-n]u[n-1]$

2.7 For the Gaussian function defined in Eq. (2.37), show that

$$\mu = \int_{-\infty}^{\infty} xG(\mu, \sigma, x)dx$$

and

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 G(\mu, \sigma, x)dx$$

Hint:

$$\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}dx = \sqrt{2\pi}\sigma$$

2.8 Prove the following properties of the Dirichlet function $\mathrm{Dir}(N, x)$ defined in Eq. (2.54):

(a) $\mathrm{Dir}(N, x)$ is a periodic function of period π for N odd but 2π for N even.

(b)

$$\sum_{n=0}^{N-1} e^{i2nx} = \mathrm{Dir}(N, x)e^{i(N-1)x}$$

(c)

$$\int_{-\pi}^{\pi} \mathrm{Dir}(N, x)dx = \begin{cases} 2\pi & N \text{ odd} \\ 0 & N \text{ even} \end{cases}$$

2.9 Use the distribution definition to show that

$$\int_{-\infty}^{\infty} \varphi(x)\delta'(x)dx = -\varphi'(0)$$

2.10 Justify the formula given in Eq. (2.43a).

2.11 Show that

$$\int_0^{\pi} \delta(x\cos\phi + y\sin\phi)d\phi = \frac{1}{\sqrt{x^2+y^2}}$$

Hint: $x\cos\phi + y\sin\phi = \sqrt{x^2+y^2}\cos(\phi-\phi_0)$ where $\phi_0 = \arctan(y/x)$.

2.12 Based on Eq. (2.55), show that

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$$

2.13 Show that

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\cos\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\cos\theta}d\theta$$

2.14 Show that

$$e^{ix\sin\theta} = J_0(x)+2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta)+i2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin[(2n-1)\theta]$$

2.15 Show that $\frac{1}{\Delta x}\text{comb}(x/\Delta x)$ is a comb function of periodicity Δx, namely,

$$\frac{1}{\Delta x}\text{comb}(x/\Delta x) = \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$

2.16 Calculate the following convolutions and sketch the resulting functions:

(a) $\Pi(x/a) * \Pi(x/b)$

(b) $\Pi(x/a) * \Lambda(x/b)$

(c) $\frac{1}{x} * \frac{1}{x}$

(d) $G(\mu_1, \sigma_1, x) * G(\mu_2, \sigma_2, x)$

(e) $\{\cdots, 0, 0.5, 0, 1, 0, 0.5, 0, \cdots\} * \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$

2.17 Show that

$$f_1(t)e^{-i2\pi f_0 t} * f_2(t)e^{-i2\pi f_0 t} = [f_1(t) * f_2(t)]e^{-i2\pi f_0 t}$$

2.18 Let $S(k) = \mathcal{F}\{\rho(x)\}$. Prove the following properties:

(a) *Hermitian symmetry*:
If $\rho(x)$ is a real function, then $S(k) = S^*(-k)$

(b) *Modulation property*:

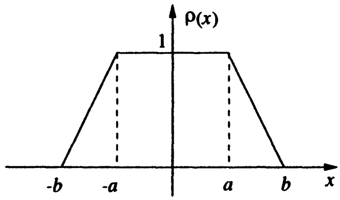
$$\mathcal{F}\{\rho(x)\cos(2\pi k_0 x)\} = \frac{1}{2}[S(k + k_0) + S(k - k_0)]$$

2.19 Prove the convolution theorem, namely,

$$\rho_1(x)\rho_2(x) \longleftrightarrow \{\mathcal{F}\rho_1\}(k) * \{\mathcal{F}\rho_2\}(k)$$

2.20 Calculate $\mathcal{F}\{\text{sinc}^2(\pi a x)\}$ based on the convolution theorem.

2.21 Calculate the Fourier transform of the trapezoidal pulse in the following figure using the derivative and shift theorems.



2.22 Calculate the Fourier transform of the following functions using the properties:

(a) $\Pi(\frac{t-t_0/2}{t_0})e^{-i2\pi f_0 t}$

(b) $\Pi(\frac{t-t_0/2}{t_0})e^{-i2\pi f_0 (t-t_0/2)}$

(c) $\mathrm{sinc}[\pi f_w(t-t_0)]e^{-i2\pi f_0 t}$

(d) $\mathrm{sinc}[\pi f_w(t-t_0)]e^{-i2\pi f_0 (t-t_0)}$

2.23 Show that any periodic function $f(t)$ of period T can be written as

$$f(t) = f_T(t) * \frac{1}{T}\mathrm{comb}\left(\frac{t}{T}\right)$$

where $f_T(t)$ is a period of $f(t)$.

2.24 Show the following Fourier transform relationship:

$$\mathcal{F}\left\{\frac{1}{\sqrt{x^2+y^2}}\right\} = \frac{1}{\sqrt{k_x^2+k_y^2}}$$

2.25 Calculate the following convolution:

$$\mathrm{sinc}(\pi a_1 x) * \mathrm{sinc}(\pi a_2 x) * \cdots * \mathrm{sinc}(\pi a_n x)$$

where it is assumed that $0 < a_1 < a_2 < \cdots < a_n$.

2.26 A function $\rho(x)$ can be expressed as a sum of its even and odd components as $\rho(x) = \rho_e(x) + \rho_o(x)$.

(a) If $\rho(x)$ is a real function, show that

$$\begin{aligned} \rho_e(x) &\longleftrightarrow \Re\{\mathcal{F}\rho(k)\} \\ \rho_o(x) &\longleftrightarrow i\Im\{\mathcal{F}\rho(k)\} \end{aligned}$$

(b) Verify the result in (a) with $\rho(x) = e^{-x}u(x)$.

2.27 Let $S(k)$ be the Fourier transform of a real function $\rho(x)$ with S_r and S_i being the real and imaginary parts of S, respectively. Find $\mathcal{F}^{-1}\{S_r(k)\}$ and $\mathcal{F}^{-1}\{S_i(k)\}$ and express the results in terms of $\rho(x)$.

2.28 Let $\rho(x, y)$ be a circularly symmetric function and let $S(k_x, k_y)$ be its Fourier transform. Show that

$$S(k\cos\phi, k\sin\phi) = 2\pi \int_0^{\infty} \rho(r\cos\phi, r\sin\phi) J_0(2\pi kr) r dr$$

2.29 Given that

$$\int_{\infty}^{-\infty} e^{-x^2} dx = \sqrt{\pi}$$

find the Radon transform of $e^{-\pi(x^2+y^2)}$.

2.30 Calculate the partial Radon transform $\{\mathcal{R}_2\rho\}(p, \phi; z)$ for the following function:

$$\rho(x, y, z) = \begin{cases} z & x^2 + y^2 \le 1 \text{ and } |z| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

2.31 Prove the properties of the Radon transform listed in Section 2.6.4.

2.32 Let $S(k_x, k_y)$ be the two-dimensional Fourier transform of $\rho(x, y)$ and $\tilde{S}(k)$ be a one-dimensional obtained from it by setting $k_x = k\cos\phi_0$ and $k_y = k\sin\phi_0 + k_0$. Namely, $\tilde{S}(k)$ is the value of $S(k_x, k_y)$ evaluated along the line $k_y = \tan\phi_0 k_x + k_0$ in the two-dimensional k-space, as shown in the following figure. Find $\mathcal{F}^{-1}\{\tilde{S}(k)\}$ and express the result in terms of $\rho(x, y)$.