Digital SIGNAL PROCESSING

Discrete-Time Signals in the Frequency Domain

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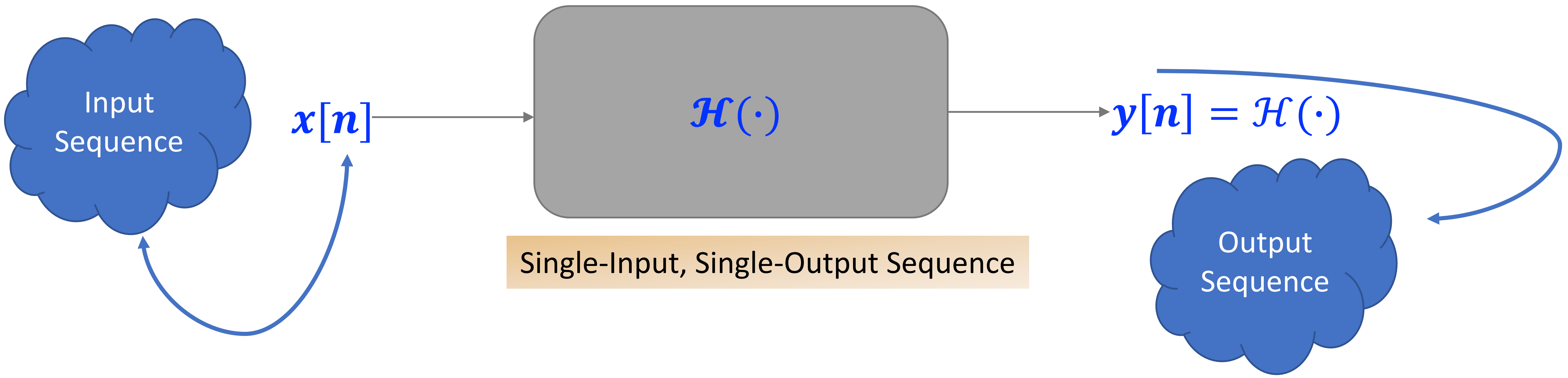
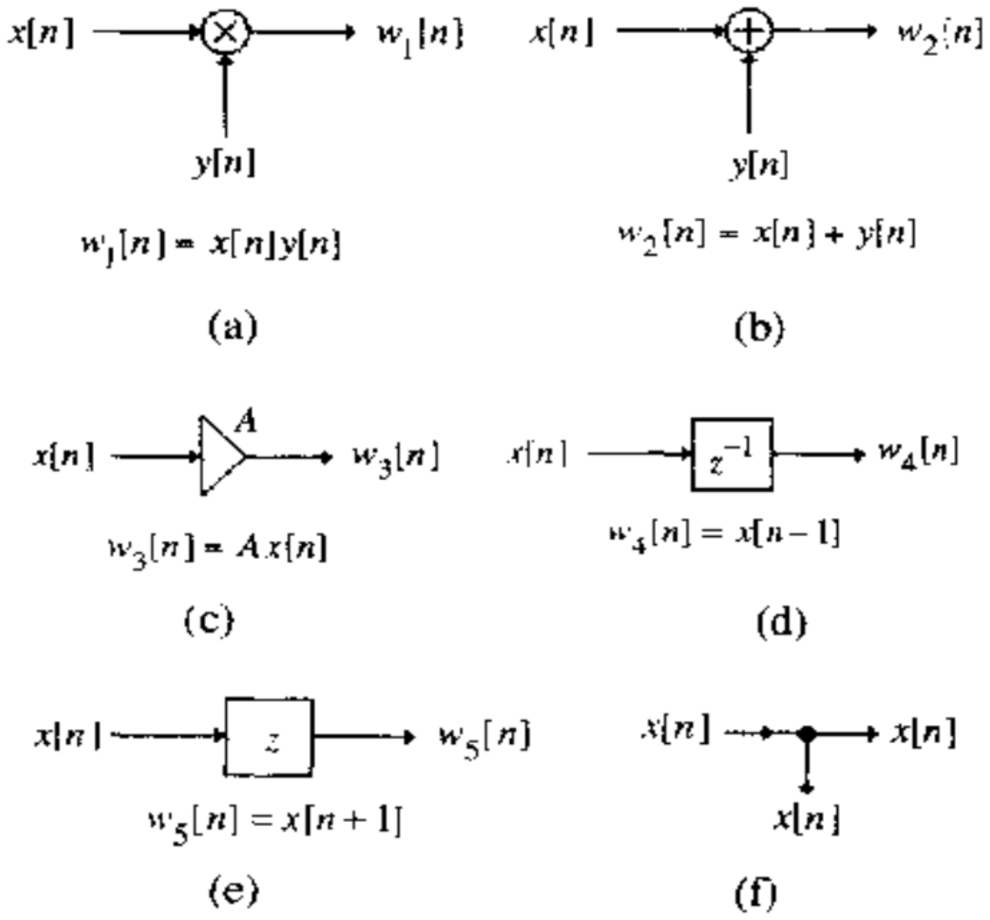
Discrete-Time Systems

- The function of a discrete-time system is to process agiven sequence, called the *input sequence*, to generate another sequence, called the *output sequence*, with more desirable properties or to extract certain information about the input signal.



Discrete-Time Systems
Discrete-Time System Examples

- A discrete-time system is a device that implements the basic operations, such as modulation, addition, etc.
- Following are the few examples of Discrete-time systems:
  - Accumulator
  - Moving-Average Filter
  - Exponentially Weighted Running Average Filter
  - Linear Interpolator
  - Median Filter



Discrete-Time Systems

Discrete-Time System Examples: *Accumulator*

- The accumulator can be defined by the input-output relation:

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l] \ldots\ldots\ldots\ldots (4.1)$$

- The output $y[n]$ at time instant n is the sum of the input sample value $x[n]$ at time instant n and all the past values.
- The accumulator can be considered as a discrete-time equivalent of a continuous-time integrator.
- The equation (1) can be written in the form

$$y[n] = \sum_{l=-\infty}^{n-1} x[l] + x[n] = y[n-1] + x[n] \ldots\ldots\ldots\ldots (4.2)$$

This $y[n-1]$ indicates that it will sum $-\infty \; to \; n-1$

- or alternatively another variation is given by

$$y[n] = \sum_{l=-\infty}^{-1} x[l] + \sum_{l=0}^{n} x[l] = y[-1] + \sum_{l=0}^{n} x[l] \qquad n \geq 0 \ldots\ldots\ldots\ldots (4.3)$$

This $y[-1]$ indicates that it will sum $-\infty \; to \; -1$

Discrete-Time Systems

Discrete-Time System Examples: *Moving-Average Filter*

- Often data cannot be measured very accurately because of random variation in the measurements, and in the case of the data being corrupted by an additive noise, the $n-th$ sample of the measured data $x[n]$ is modeled as $x[n] = s[n] + d[n]$, where $s[n]$ and $d[n]$ denote the $n-th$ samples of the data and the noise, respectively.
- If multiple measurements of the same set of data samples are available, a reasonably good estimate of the uncorrupted data vector can be found by evaluating the ensemble average.
- In applications where data measurements cannot be repeated, a commonly used estimate of the data sample $x[n]$ at instant n from M measurements of the noise-corrupted data sample x[l] available for the range $n - M + 1 \leq l \leq n$ is the M-point average or mean $y[n]$ given by

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l] \ldots\ldots\ldots (4.4)$$

The discrete-time system implementating Eq. (4.4) usually is called the M-point moving-average filter .

Discrete-Time Systems

Discrete-Time System Examples: *Moving-Average Filter*

- An estimate of the spread of the mean value y[n] from the actual value s[n] usually is given by the standard deviation defined by

$$\sigma[n] = \sqrt{\frac{\sum_{l=0}^{M-1}(x[n-l]-y[l])^2}{M}}$$

- The M-point moving-average filter can be implemented using

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l] \ldots\ldots\ldots (4.4)$$

- The implementation of a M-point moving-averging filter involves $M-1$ additions, one multiplication by a factor with value $1/M$, and storage of $M-1$ past input data samples.
- A more efficient is possible (see next slide)

Discrete-Time Systems

Discrete-Time System Examples: *Moving-Average Filter*

- The M-point moving-average filter can be implemented using

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l] \ldots\ldots\ldots (4.4)$$

$$y[n] = \frac{1}{M}\left(\sum_{l=0}^{M-1} x[n-l] + x[n-M] - x[n-M]\right)$$

$$y[n] = \frac{1}{M}\left(\sum_{l=0}^{M} x[n-l] - x[n-M]\right)$$

$$y[n] = \frac{1}{M}\left(\sum_{l=1}^{M} x[n-l] + x[n] - x[n-M]\right)$$

$$y[n] = \frac{1}{M}\left(\sum_{l=0}^{M-1} x[n-1-l] + x[n] - x[n-M]\right)$$

$$y[n] = \frac{1}{M}\left(\sum_{l=0}^{M-1} x[n-1-l]\right) + \frac{1}{\mathrm{M}}(+x[n] - x[n-M])$$

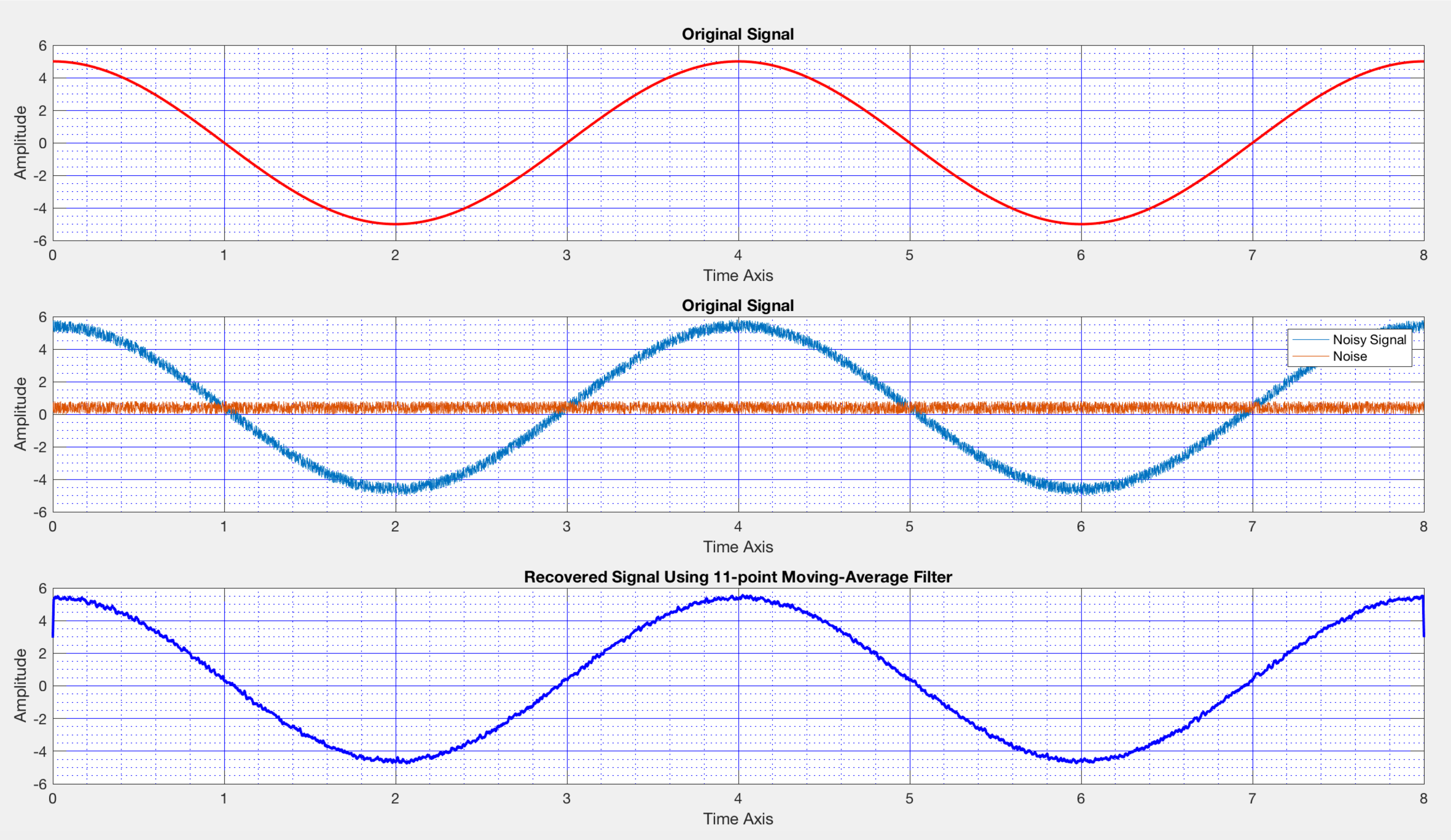
$$y[n] = y[n-1] + \frac{1}{\mathrm{M}}(+x[n] - x[n-M]) \ldots\ldots\ldots (4.6)$$

- The implementation of a M-point moving-averging filter using equation (4.6) involves *two* additions, one multiplication by a factor with value $\frac{1}{M}$.

Discrete-Time Systems

Discrete-Time System Examples:

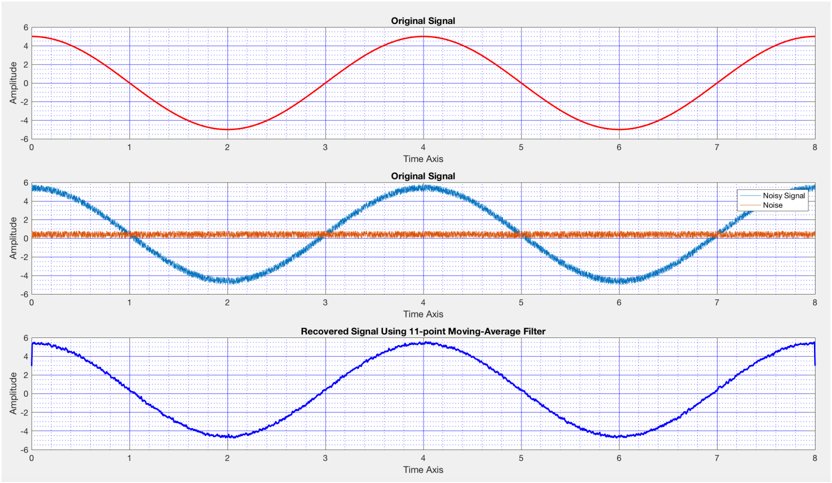
Moving-Average Filter



Discrete-Time Systems

Discrete-Time System Examples: *Moving-Average Filter*

- Some facts about Moving-Average Filter
  - The moving-average filter acts like a lowpass filter and, as a result, smoooths the input data by removing the high-frequency components.
  - Most random noise has frequency components through out the frequency range $0 \leq \omega < \pi$, and hence, some low-frequency components of the noise will also be present in the output of the moving-average filte.
  - An increase in the number of points M used for averaging decreases the passband width of the lowpass filter and may result in an overly smoothed output by removing some mid-frequency components of the original uncorrupted signal.
  - The proper choice of M depends on the nature of the noise corrupting the original signal.
    - In some applications, higher quality smoothed output may be obtained by using a cascase of identical movering average filters with a smaller value of M to process the noise-corrupted signal.



Remember that the frequencies in the neighborhood of $2\pi k$ are low frequencies, whereas, in the neighborhood of $\pi(k+1)$ are higher frequencies

Discrete-Time Systems

Discrete-Time System Examples: *Exponentially Weighted Running Average Filter*

- The moving-average filter gives emphases on all M data sample.
- In some applications, it may be necessary to place more emphasis on data samples near the time instant n and less emphasis on the data samples that are further away in dertermining the average.
- Sucha an average filter is called the exponentially weighted running average filter.
- An exponentially weighted running average filter can be implemented using

$$y[n] = \alpha y[n-1] + x[n] \qquad 0 < \alpha < 1 \ldots\ldots\ldots (4.7)$$

- The computation required only one addition and one multiplication with the factor α. Moreover, it doesn’t require any storage.
- From Equation (4.7), it is clear that exponentially weighted running average filter gives more emphasis on current data, and less emphasis on past values.

Discrete-Time Systems

Discrete-Time System Examples: *Exponentially Weighted Running Average Filter*

- Example: Apply an exponentially weighted running average filter of order 3 on the inpust sequence $x[n] = [1, 2, 3, 4, 5]$ for $\alpha = 0.5$.
- Solution:
- The size of the output $y[n]$ would be equal to the size of input sequence $x[n]$; that is, 5 in given case.
- We have implementation as

$$y[n] = \alpha y[n-1] + x[n]$$

- For $n = 0$

$$y[0] = 0.5 \times y[-1] + x[0]$$

- Here, suppose $y[-1] = 0$ is the initial condition

$$y[0] = 1$$

- For $n = 1$

$$y[0] = 0.5 \times y[0] + x[1]$$

$$y[0] = 0.5 \times 1 + 2 = 2.5$$

- For $n = 2$

$$y[0] = 0.5 \times y[1] + x[2] = 4.25$$

- For $n = 3$

$$y[0] = 0.5 \times y[2] + x[3] = 6.125$$

- For $n = 4$

$$y[0] = 0.5 \times y[3] + x[4]$$

$$= 8.0625$$

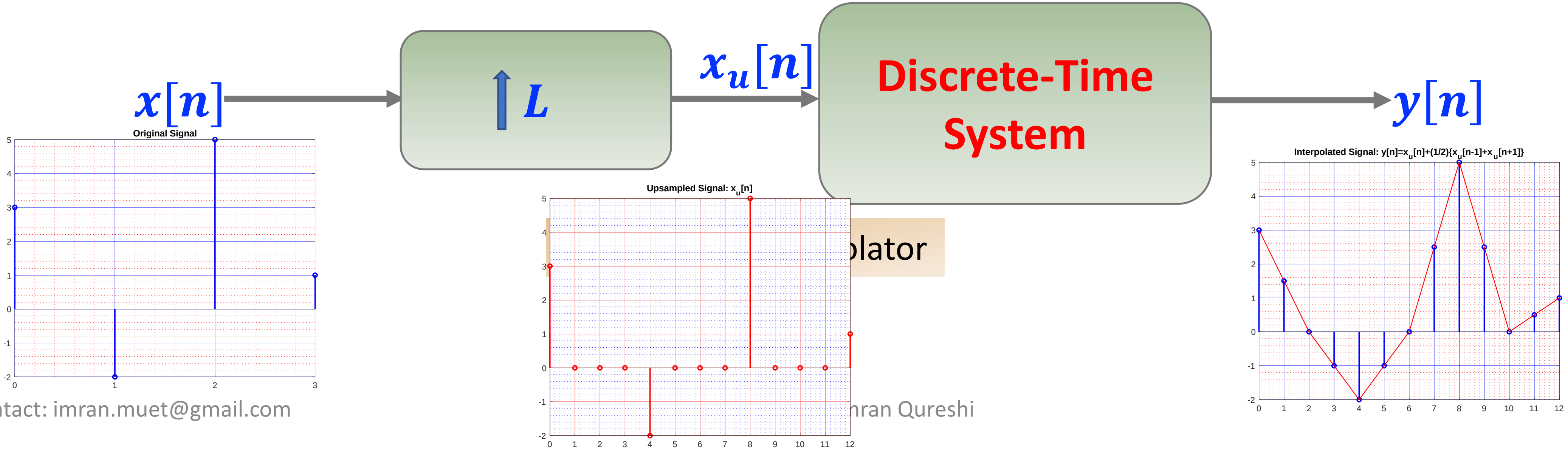
The output $y[n]$ of exponentially weighted running average filter is

$$y[n] = [1, 2.5, 4.25, 6.125, 8.0625]$$

Discrete-Time Systems

Discrete-Time System Examples: *Linear Interpolator*

- Linear interpolator is another example of discrete-time system.
- The linear interpolator is implemented by:
- First passing the input sequence $x[n]$ through an up-sampler to get an output $x_u[n]$
- Second passed the up-sampled output $x_u[n]$ through a discrete-time system to fill in the zero-valued samples with the values obtained by linear-interpolation.



Discrete-Time Systems

Discrete-Time System Examples: *Linear Interpolator*

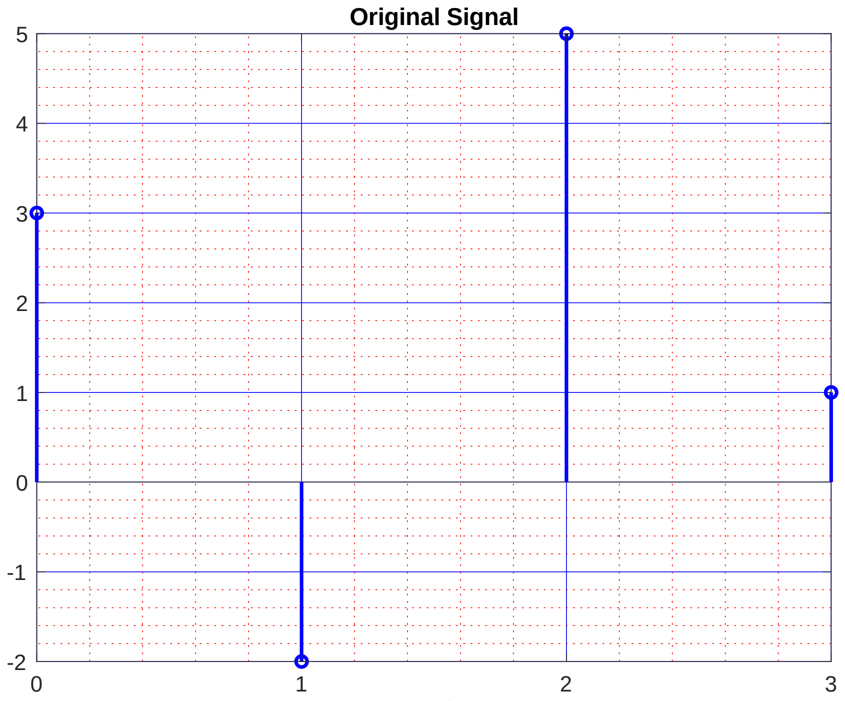
- **Linear Factor-of-2-Interpolator:**
- If $x_u[n]$ is a zero-valued sample inserted between a pair of input samples, it is replaced with the average of the two original input samples, $x_u[n-1]$ and $x_u[n+1]$:

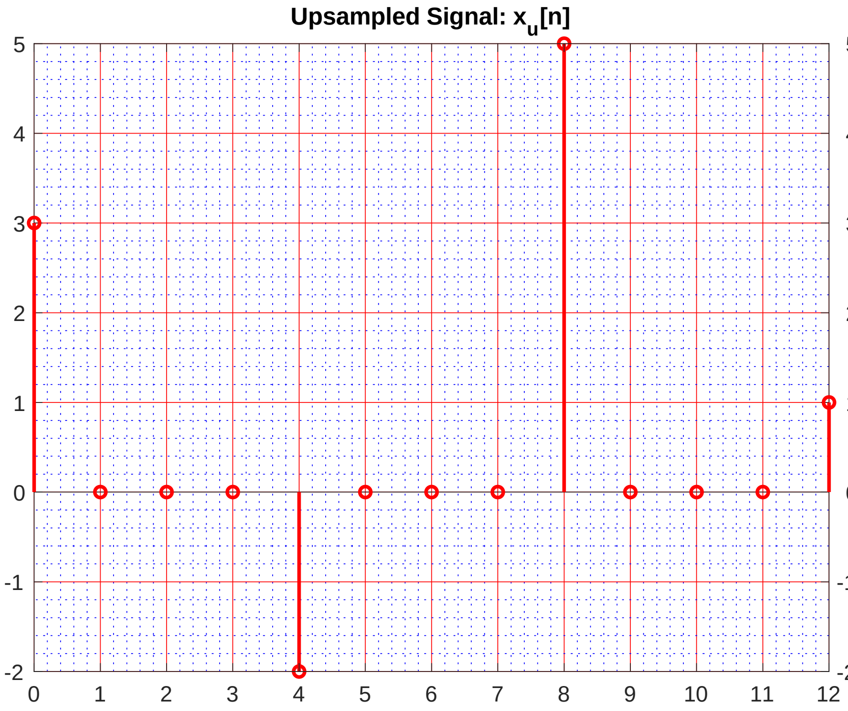
$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1]) \quad \ldots\ldots\ldots (4.8)$$

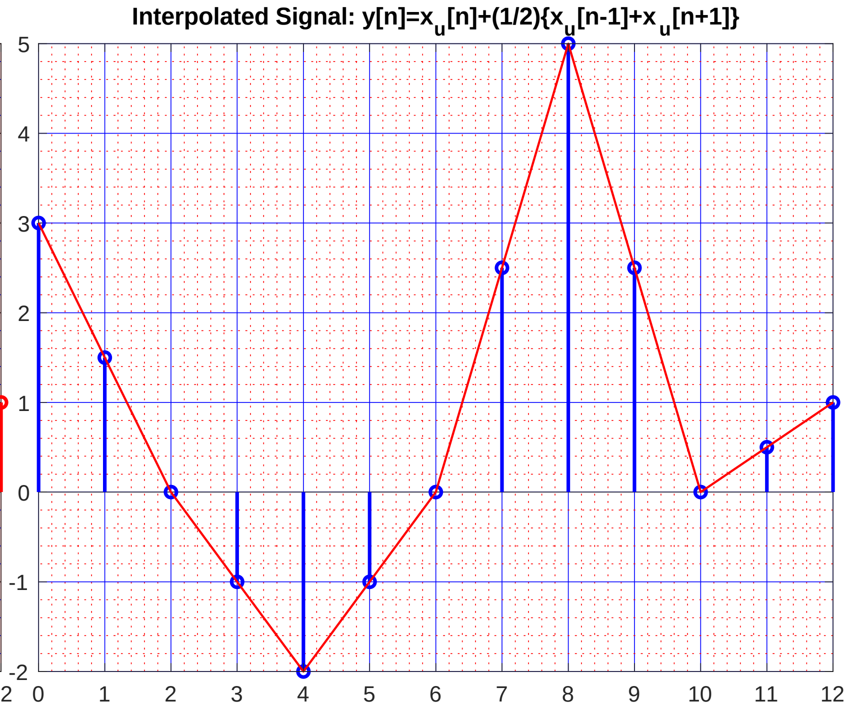
- The above interpolator is also known as bilinear interpolation.

Discrete-Time Systems

Discrete-Time System Examples: *Linear Interpolator*







Original Signal

Signal Obtained from factor-of-3-up-sampler

Signal obtained as a result of Factor-of-2-interpolator

Discrete-Time Systems

Discrete-Time System Examples: *Linear Interpolator*

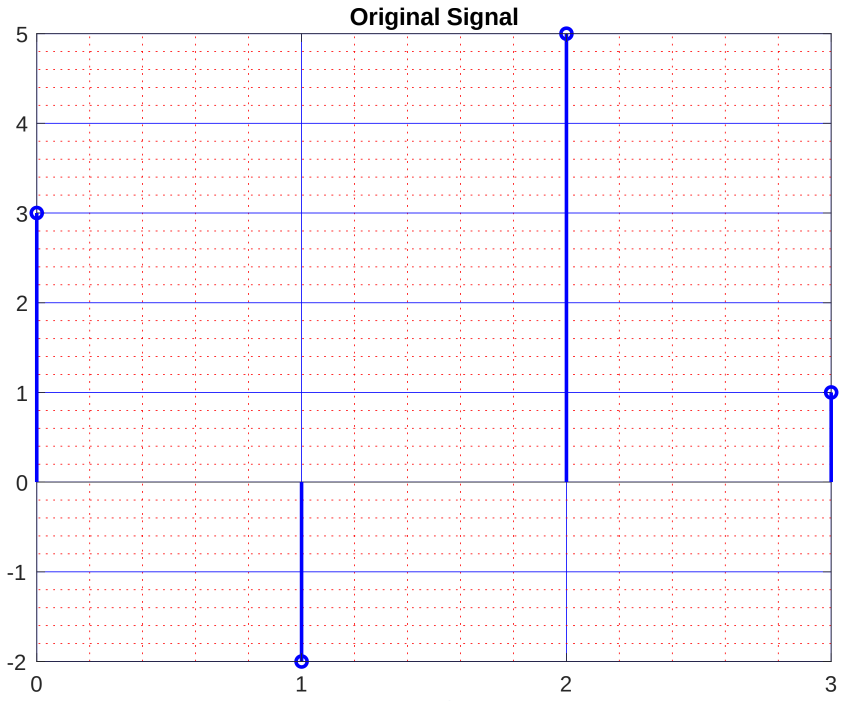
- **Linear Factor-of-3-Interpolator:**
- For a factor-of-3-interpolator, the discrete-time system following the factor-of-3up-sampler is charachterized by the input-output relation given by

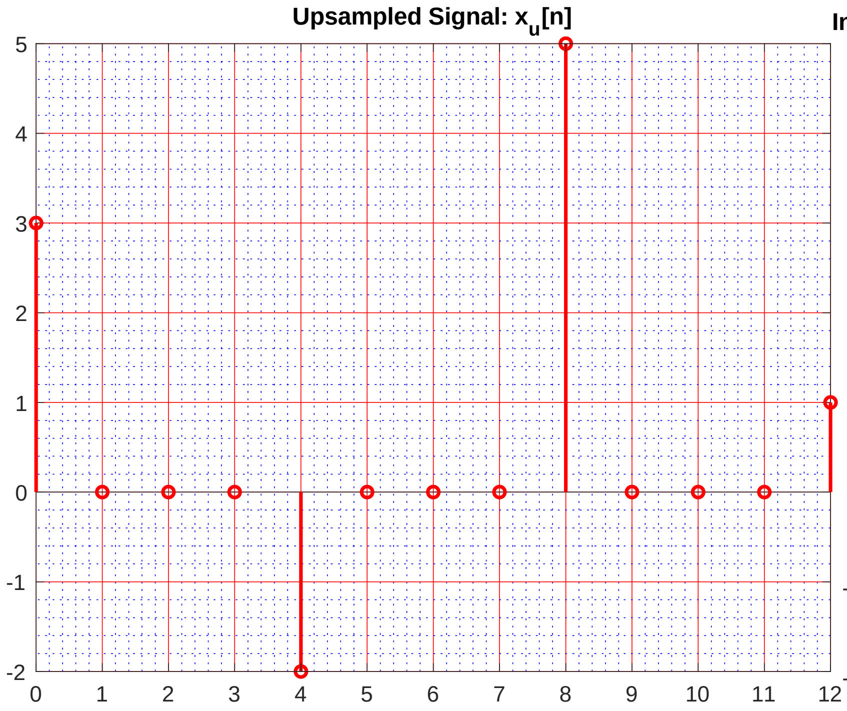
$$y[n] = x_u[n] + \frac{2}{3}(x_u[n-1] + x_u[n+1]) + \frac{1}{3}(x_u[n-2] + x_u[n+2] \quad \ldots (4.9)$$

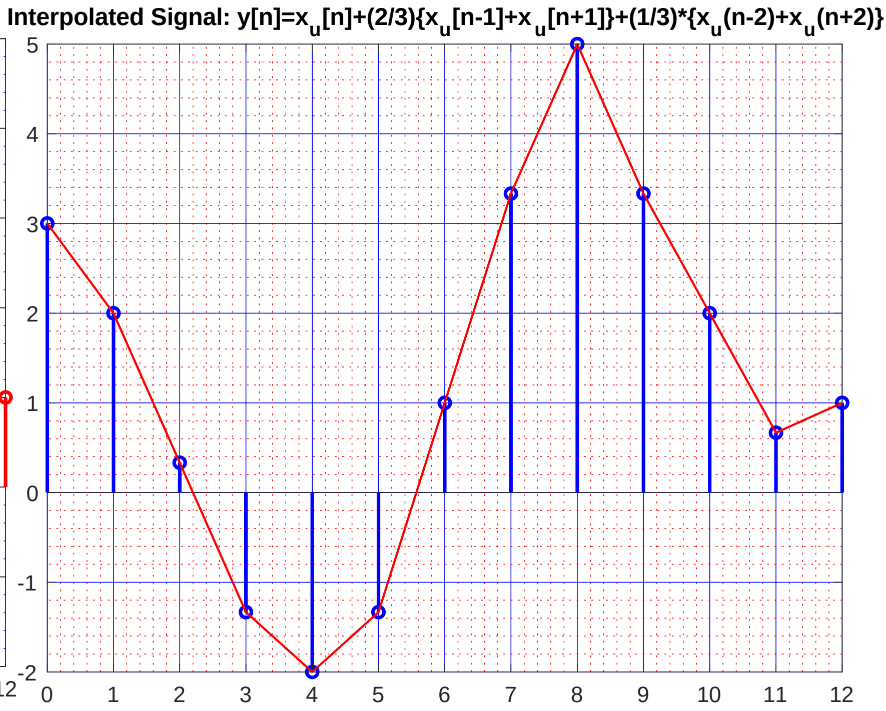
- Applications:
  - In Image Zooming

Discrete-Time Systems

Discrete-Time System Examples: *Linear Interpolator*







Original Signal

Signal Obtained from factor-of-3-up-sampler

Signal obtained as a result of Factor-of-3-interpolator

Discrete-Time Systems

Discrete-Time System Examples: *Median Filtering*

- The median filter is implemented by sliding a window of odd length over the input sequence $x[n]$ one sample at a time.
- At any instant, the output of the filter is the median value of the input samples inside the window.
- More specifically, the output sample $\mathrm{y}[\mathrm{n}]$ at the nth instant of the median filter with a window of length $(2K+1)$ is given by

$$y[n] = med\{x[n-K], \dots, x[n-1], x[n], x[n+1], \dots, x[n+K]\} \dots (4.10)$$

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Discrete-Time Systems

Discrete-Time System Examples: *Median Filtering*

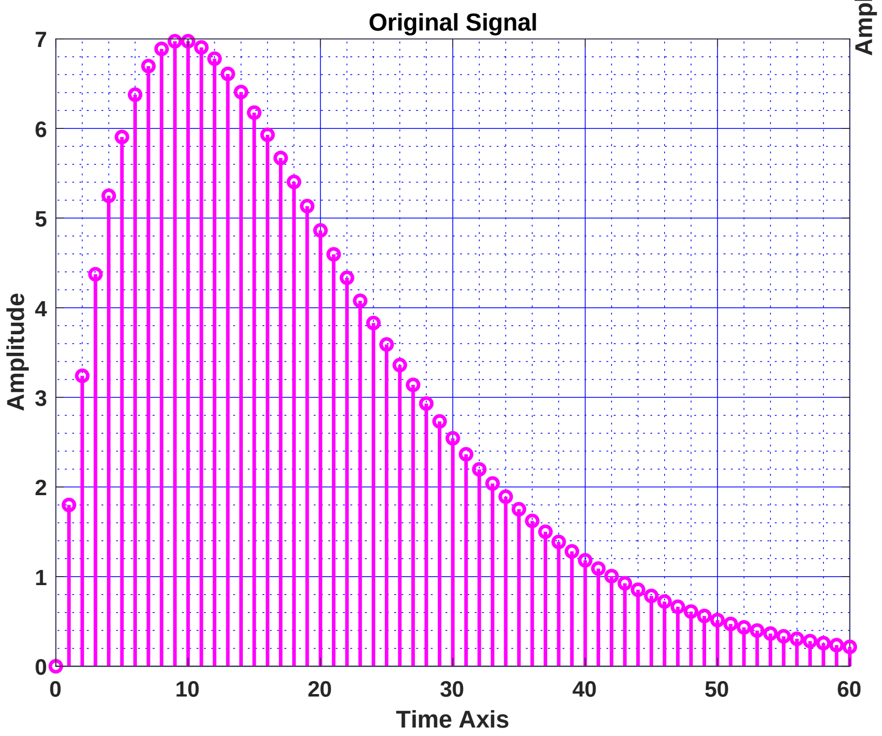
- For a sequence $x[n]$ of length N, when passes through a median filter of length M, where $N > M$, $\frac{M-1}{2}$ zeros are padded before and after $x[n]$ to creat a new sequence $x_e[n]$ of length $N + M - 1$.

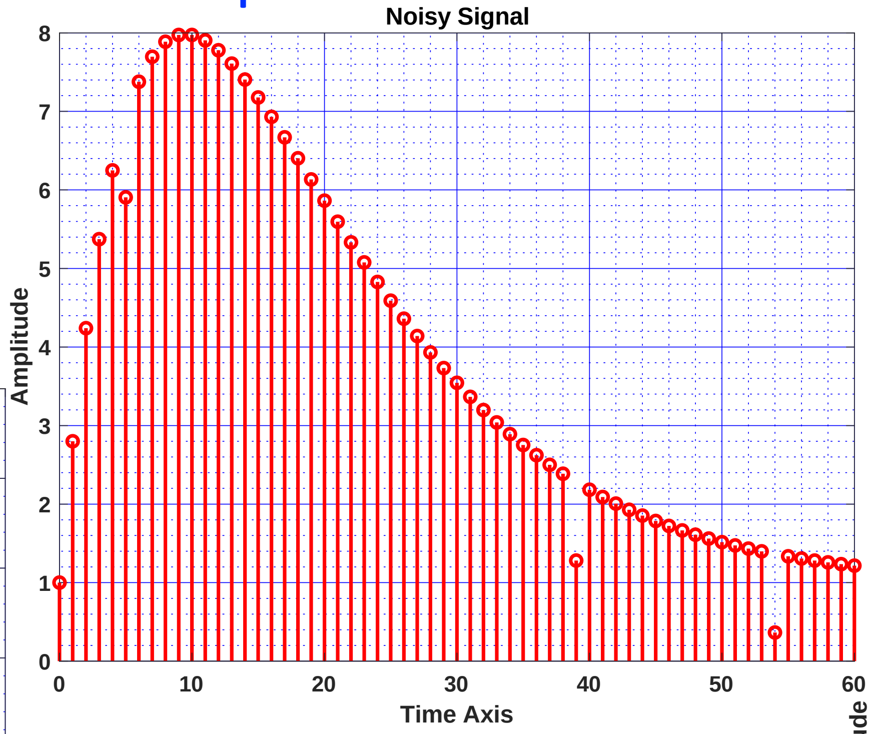
$$x_e[n] = \begin{cases} 0, & -\dfrac{M-1}{2} \leq n \leq 1 \\ x[n], & 0 \leq n \leq N-1 \\ 0 & N \leq n \leq N-1+\dfrac{M-1}{2} \end{cases}$$

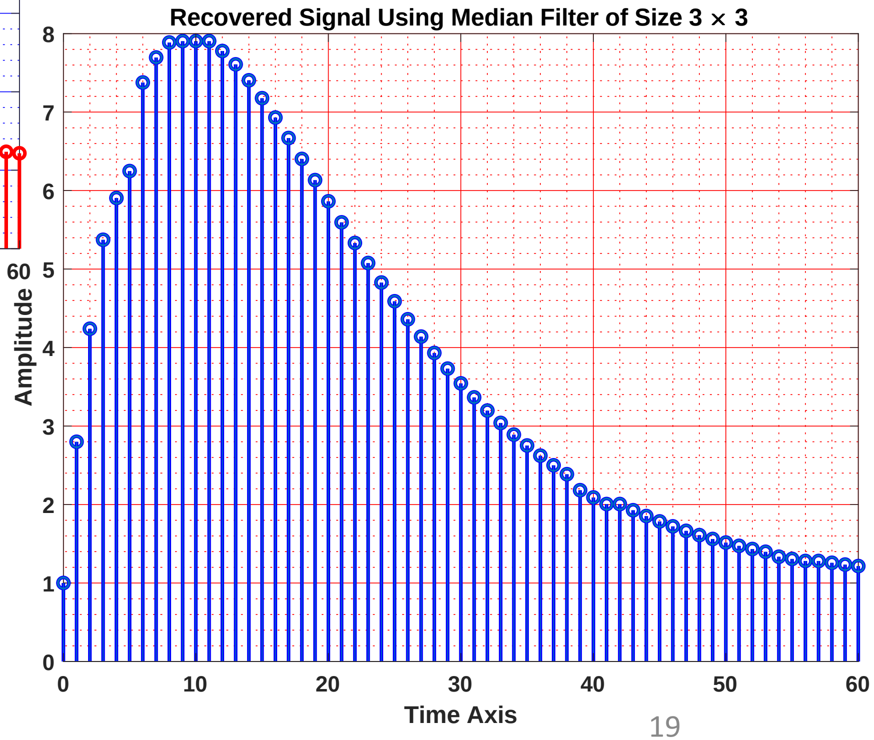
- Then the sequence $x_e[n]$ is passed through median window to get the output $y[n]$

Discrete-Time Systems

Discrete-Time System Examples: *Median Filtering*



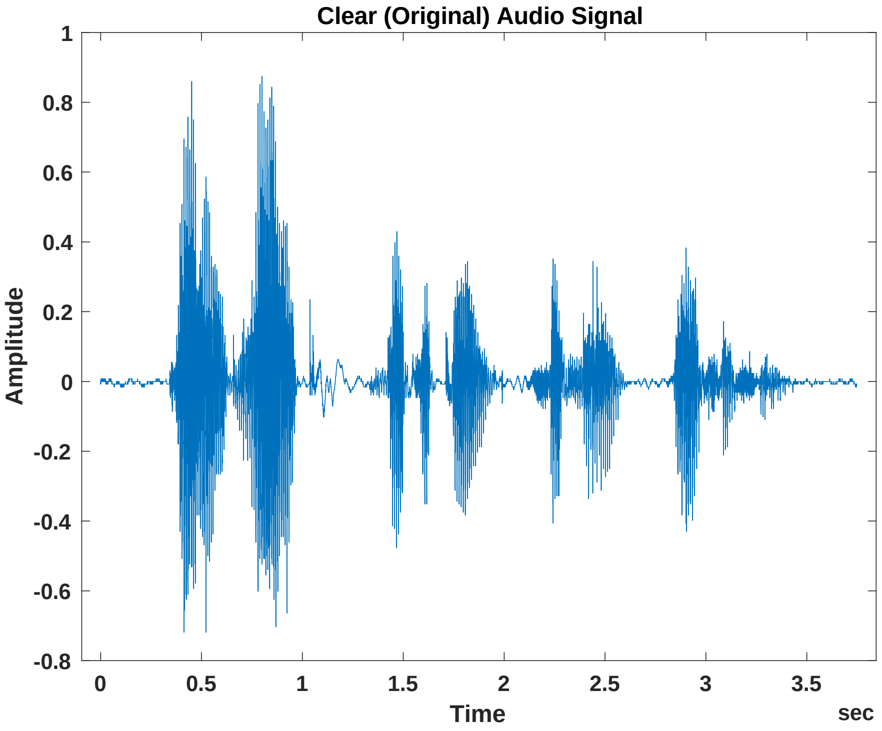


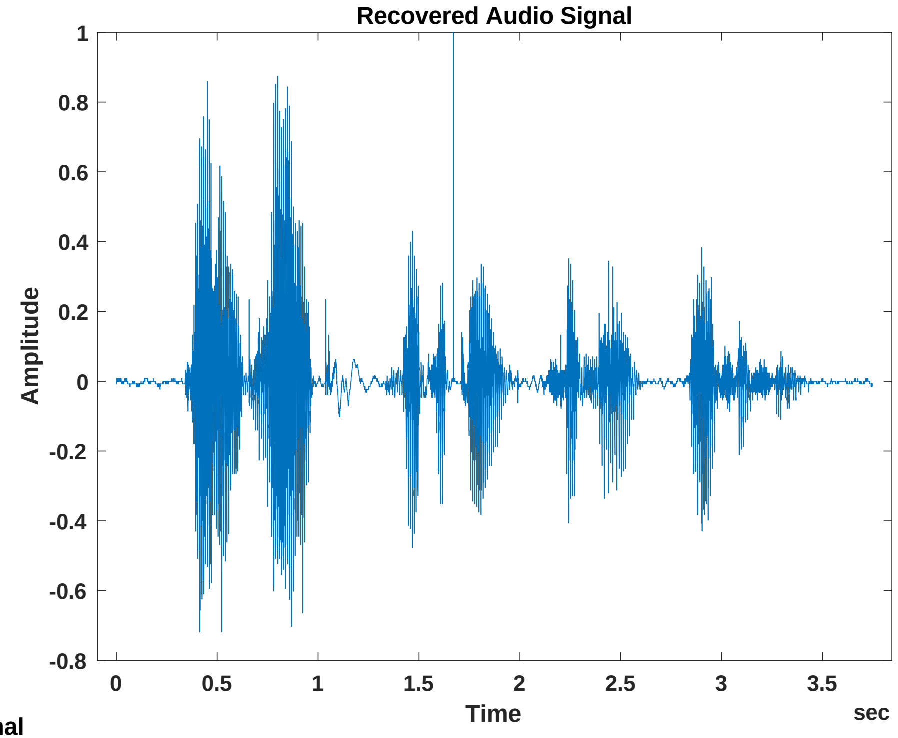


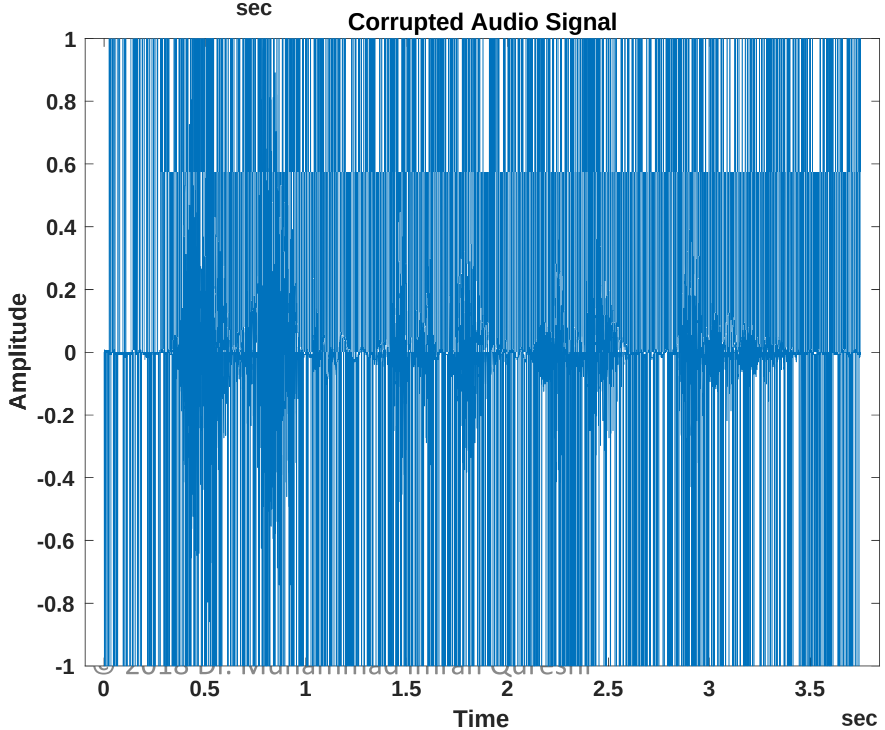
Discrete-Time Systems

Discrete-Time System Examples:

Median Filtering (DEMO)







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Discrete-Time Systems

Classification of Discrete-Time Systems

- Baed on the input-output relation, discrete-time systems can be classified as
  - Linear Systems
  - Shift-Invariant System
  - Causal System
  - Stable System
  - Passive and Lossless Systems

Discrete-Time Systems

Classification of Discrete-Time Systems: Linear System

- For a linear discrete-time system, if $y_1[n]$ and $y_2[n]$ are the responses to the input sequences $x_1[n]$ and $x_2[n]$, respectively, then for an input
$$x[n] = \alpha x_1[n] + \beta x_2[n],$$
- The response is given by
$$y[n] = \alpha y_1[n] + \beta y_2[n]$$
- The above property makes it very easy to compute the response of a linear discrete-time system to a complicated sequence that can be decomposed as a weighted combination of some simple sequences, such as the unit sample sequences or the complex exponential sequences.

Discrete-Time Systems

Classification of Discrete-Time Systems: Linear System

- Example: Find whether given discrete-time system (Accumulator) is linear or non-linear?

$$y[n] = \sum_{l=-\infty}^{n} x[n]$$

- Solution:
- The response of above system for input $\alpha x_1[n]$ is given by

$$y_1[n] = \alpha \sum_{l=-\infty}^{n} x_1[n] \ldots \ldots (1)$$

- The response of above system for input $\beta x_2[n]$ is given by

$$y_2[n] = \beta \sum_{l=-\infty}^{n} x_2[n] \ldots \ldots (2)$$

- Adding Eq. (1) and Eq. (2)

$$y[n] = y_1[n] + y_2[n] \ldots (3)$$

- Now the response of above system for input $x[n] = \alpha x_1[n] + \beta x_2[n]$ is given by

$$y[n] = \sum_{l=0}^{n} (\alpha x_1[n] + \beta x_2[n])$$

$$y[n] = \alpha \sum_{l=0}^{n} x_1[n] + \beta \sum_{l=0}^{n} x_2[n] \ldots (4)$$

- Comparing Eq(3) and Eq (4), we get that givn is system is linear

Discrete-Time Systems

Classification of Discrete-Time Systems: Shift-Invariant System

- For a shift-invariant discrete-time system, if $y_1[n]$ is the responses to the input sequences $x_1[n]$, then the response to an input $x[n] = x_1[n - n_0]$ is simply

$$y[n] = y_1[n - n_0]$$

- Where n_0 is any positive or negative integer.
- The time-invariant property ensures that for a specified input, the output of the system is independent of the time the input is being applied.
- Median Filter is a non-linear discrete-time system.
- Median filter is a time-invariant discrete-time system.

Discrete-Time Systems

Classification of Discrete-Time Systems: Shift-Invariant System

- Example: Find whether the given system (up-sampler) is time-invariant or time-variant?

$$x_{1,u}[n] = \begin{cases} x_1\left[\frac{n}{L}\right], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & otherwise. \end{cases}$$

- Solution:
- Let introduce a delay in output

$$x_{1,u}[n] = \begin{cases} x_1\left[\frac{n}{L} - n_0\right], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & otherwise. \end{cases} \quad \dots (1)$$

- Equation (1) can be written as

$$x_{1,u}[n] = \begin{cases} x_1\left[\frac{n - Ln_0}{L}\right], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & otherwise. \end{cases} \quad \dots (2)$$

- Now apply a delayed input; that is, $x_u[n - n_0]$, the response is
- $x_u[n - n_0] =$

$$\begin{cases} x_1\left[\frac{n-n_0}{L}\right], & n = n_0, n_0 \pm L, n_0 \pm 2L, \dots, \\ 0, & otherwise. \end{cases} \quad \dots (3)$$

- From Equation (2) and (3), we get that
- $x_{1,u}[n] \neq x_u[n - n_0]$
- Hence given system is time varying

Discrete-Time Systems

Classification of Discrete-Time Systems: Shift-Invariant System

- Example: Find whether the given system (up-sampler) is time-invariant or time-variant?

$$x_{1,u}[n] = \begin{cases} x_1\left[\frac{n}{L}\right], & n = 0, \pm L, \pm 2L, \ldots, \\ 0, & otherwise. \end{cases}$$

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- Now apply a delayed input; that is, $x_u[n - n_0]$, the response is
- $x_u[n - n_0] = \begin{cases} x_1\left[\frac{n-n_0}{L}\right], & n = n_0, n_0 \pm L, n_0 \pm 2L, \ldots, \\ 0, & otherwise. \end{cases} \quad \ldots (3)$
- From Equation (2) and (3), we get that
- $x_{1,u}[n] \neq x_u[n - n_0]$
- Hence given system is time varying

Discrete-Time Systems

Classification of Discrete-Time Systems: Causal System

- In a **causal discrete-time system**, the n_0th output sample $y[n_0]$ depends only on the input sequence $x[n]$ for $n \leq n_0$ and does not depend on the input samples for $n > n_0$.
- Thus, if $y_1[n]$ and $y_2[n]$ are the responses of a causal discrete-time system to the inputs $u_1[n]$ and $u_2[n]$, respectively, then

$$u_1[n] = u_2[n] \quad for \quad n > N \quad implies\ also\ that \quad y_1[n] = y_2[n] \quad for\ n < N$$

- Oherwise system will be called noncausal discrete-time system.

Discrete-Time Systems

Classification of Discrete-Time Systems: Stable System

- A **discrete-time system** to be **stable** if and only if for every bounded input, the output is also bounded.
- If the response to $x[n]$ is $y[n]$ and if

$$|x[n]| < B_x \qquad \forall\, n, \qquad then \qquad |y[n]| < B_y$$

- For all values of n where B_x and B_y are finite positive constants.
- This type of stability is called bounded-input, bounded-output (BIBO) stability.

Discrete-Time Systems

Classification of Discrete-Time Systems: Stable System

- Find whether given system is stable or not?

$$y[n] = \alpha^{\{n+1\}} y[-1] + \sum_{l=0}^{n} \alpha^{i} \qquad for \qquad n \geq 0, \qquad and \;\; \alpha < 1$$

- Solution:
- For $\alpha < 1$, the output $y[n]$ will become smaller as n increases, and the system will be stable.
- For $\alpha > 1$, the output $y[n]$ will become larger as n increases, and the system will be an unstable system.

Discrete-Time Systems

Classification of Discrete-Time Systems:

Passive and Lossless Systems

- A discrete-time system is said to be **passive** if, for every finite energy input sequence $x[n]$, the output sequence $y[n]$ has, at most, the same energy; that is,

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n}^{\infty} |x[n]|^2 < \infty \qquad (4.13)$$

- If the energy of the output $y[n]$ is exactly same as the energy of input sequence $x[n]$, then the system is called a **lossless system.**

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n}^{\infty} |x[n]|^2 < \infty$$

- A lossless system is always a passive system, but a passive system may not necessarily be a lossless system.

Discrete-Time Systems

Classification of Discrete-Time Systems:

Passive and Lossless Systems

- Example: Consider the discrete-time system defined by $y[n] = \alpha x[n-N]$, with N a positive integer.
- Solution:
- The energy of input sequence $x[n]$ is given by

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$\mathcal{E}_x = \alpha^2 \sum_{n=-\infty}^{\infty} |x[n]|^2 \ldots\ldots(1)$$

- The energy of the output sequence $y[n]$ is

$$\mathcal{E}_y = \sum_{n=-\infty}^{\infty} |y[n]|^2$$

$$\mathcal{E}_y = \alpha^2 \sum_{n=-\infty}^{\infty} |x[n-N]|^2 \ldots(2)$$

- Comparing Eq (1) and (2)
- $\mathcal{E}_x \leq \mathcal{E}_y \quad for \quad \alpha \leq 1$ **(Passive System)**
- However, if $\alpha = 1$, $\mathcal{E}_x = \mathcal{E}_y$, system is Lossless.

Discrete-Time Systems

Classification of Discrete-Time Systems: Impulse and Step Responses

- **Impulse Response:**
  - The response of a digital filter to a unit sample sequence $\delta[n]$ is called the unit sample or the impulse response, and is denoted as $h[n]$.
- **Step Response:**
  - The response of a discrete-time system to a unit step sequence $\mu[n]$, denoted as $s[n]$, is its unit step response or step response.
- A linear time-invariant (LTI) digital filter is completely charachterized in the time domain by its impulse response or its step response.

Discrete-Time Systems

Classification of Discrete-Time Systems: Impulse and Step Responses

- **Example: Determine the impulse response of a discrete-time defined by**

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3] \dots \dots (4.14)$$

- Solution:
- The impulse response is obtained by setting $x[n] = \delta[n]$ in above equation

$$h[n] = \alpha_1 \delta[n] + \alpha_2 \delta[n-1] + \alpha_3 \delta[n-2] + \alpha_4 \delta[n-3]$$

- Thus impulse response is thus a finite-length sequence of length 4 given by

$$h[n] = [\alpha_1, \alpha_2, \alpha_3, \alpha_4] \qquad 0 \leq n \leq 4$$

Discrete-Time Systems

Classification of Discrete-Time Systems: Impulse and Step Responses

- **Example:** Determine the impulse response of accumulator defined by

$y[n] = \sum_{l=-\infty}^{n} x[l]$

- **Solution:**
- The impulse response is obtained by setting $x[n] = \delta[n]$ in above equation

$$h[n] = \sum_{l=-\infty}^{n} \delta[l]$$

- The above one is the unit step response as well.

Discrete-Time Systems

Classification of Discrete-Time Systems: Impulse and Step Responses

- **Example:** Determine the impulse response of a factor-of-2-interpolator defined by

$$y[n] = x[n] + \frac{1}{2}\{\delta[n-1] + \delta[n+1]\}$$

- **Solution:**
- The impulse response is obtained by setting $x[n] = \delta[n]$ in above equation

$$h[n] = \delta[n] + \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1]$$

- Thus impulse response is thus a finite-length sequence of length 3 given by

$$h[n] = \left[\frac{1}{2}, 1, \frac{1}{2}\right] \qquad -1 \leq n \leq 1$$

To know the outpout of an interpolator just convolve the impulse response of interpoltor with any input you want to find the output for.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

- A linear time-invariant (LTI) discrete-time system is a system that satisfies both: linearity and time-invariant properties.
- A consequence of the linear, time-invariance property is that as LTI discret-time system is completely characterized by its impulse response; that is, knowing the impulse response we can compute the output of the system to any arbitrary input (by convolving that input with the impulse response).

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Input-Output Relationship

- Let $h[n]$ denote the impulse response of the LTI discrete-time system; that is, the response to an impulse input $\delta[n]$.
- Let compute the response of this filter to the following input

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + \delta[n-4] + 0.75\delta[n-6]$$

- The response of this system to $\delta[n+2]$ is $h[n+2], \delta[n-1]$ is $h[n-1], \delta[n-2]$ is $h[n-2]$, and to $\delta[n-4]$ is $h[n-4]$.
- Since the filter is linear time-invariant, so the reponses can be added

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + h[n-4] + 0.75h[n-6]$$

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Input-Output Relationship

- Since the filter is linear time-invariant, so the reponses can be added

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + h[n-4] + 0.75h[n-6]$$

- It follows from the above result that an arbitrary input sequence $x[n]$ can be expressed as a weighted linear combination of delayed and advanced unit sample sequences in the form

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \ldots (4.15)$$

- The response of the LTI discrete-time system to the sequence $x[k]\delta[n-k]$ is $x[k]h[n-k]$.
- As a result, the response $y[n]$ of the discrete-time system to $x[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \ldots (4.16)$$

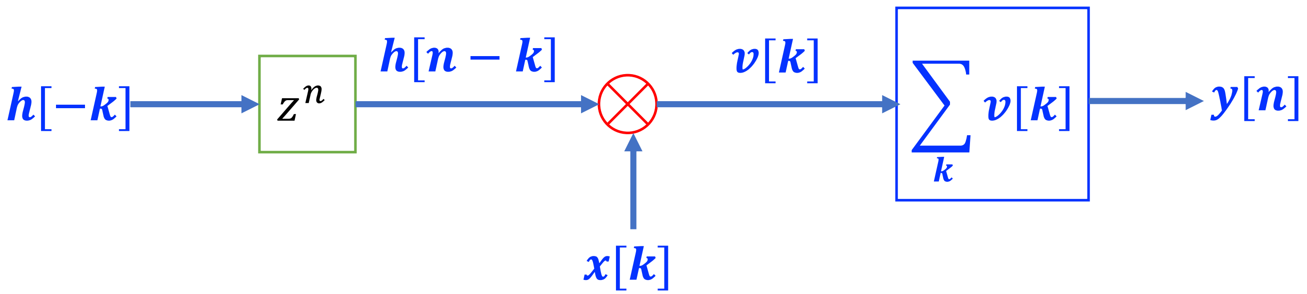
- The above relation is called the convolution sum.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Input-Output Relationship

- The convolution obeys commutative, associative, and distributive laws:
- $x_1[n] * x_2[n] = x_2[n] * x_1[n]$
- $x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$
- $x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$



Schemetic Representation of Convolution Sum

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Input-Output Relationship

- **Example:** Consider a causal LTI discrete-time system with an impulse response $h[n] = \beta^n \mu[n]$, where $|\beta| < 1$. Find the output $y[n]$ of that system for input sequence $x[n] = \alpha^n \mu[n]$, where, $|\alpha| < 1$.
- **Solution:**
- We use the convolution sum here

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^k \mu[k] \beta^{n-k} \mu[n-k]$$

- Since both $h[n]$ and $x[n]$ are causal as a result $y[n]$ will be causal. The output is given by.

$$y[n] = \sum_{k=0}^{\infty} \alpha^k \beta^{n-k}$$

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Stability Condition in Terms of the Impulse Response

- An LTI digital filter is BIBO stable if and only if its impulse response sequence $h[n]$ is absolutely summable; that is,

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- If impulse response of an LTI digital filter is absolutely summable (bounded); that is, $\mathcal{S} < \infty$, it means the system is stable.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Stability Condition in Terms of the Impulse Response

- If the impulse response is bounded, then the output sequence $y[n]$ of that system will also be bounded.
- Proof:
- Consider a bounded input sequence; that is, $|x[n]| \le B_x < \infty$, then the output amplitude at time instant n is

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right|$$

- As the convolution obey commutative law, so the above expression can be written as

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \le B_x \sum_{k=-\infty}^{\infty} |h[k]| = B_x \mathcal{S} < \infty$$

- Thus, $\mathcal{S} < \infty$ implies $|y[n]| \le B_y = B_x \mathcal{S} < \infty$, indicating that the sequence $y[n]$ is also bounded.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Stability Condition in Terms of the Impulse Response

- If the impulse response is unbounded, then the output sequence $y[n]$ of that system will also be unbounded.
- Proof:
- Consider a bounded input sequence; that is, $|x[n]| \leq B_x < \infty$, then the output amplitude at time instant n is

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right|$$

- As the convolution obey commutative law, so the above expression can be written as

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]| = B_x \mathcal{S} < \infty$$

- Thus, $\mathcal{S} < \infty$ implies $|y[n]| \leq B_y = B_x \mathcal{S} < \infty$, indicating that the sequence $y[n]$ is also bounded.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Stability Condition in Terms of the Impulse Response

- Find the stability of following causal first-order LTI discrete-time system defined by

$$h_1[n] = \alpha^n \mu[n]$$

- Solution:
- For this system

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} |\alpha^n \mu[n]| = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1 - |\alpha|} \qquad for \quad |\alpha| < 1$$

- Therefore, $S < \infty$ if $|\alpha| < 1$ for which the system is BIBO stable.
- If $|\alpha| \geq 1$, the infinite series $\sum_{n=0}^{\infty} |\alpha|$ does not converge, then the above causal system is not BIBO stable.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Stability Condition in Terms of the Impulse Response

- Examine the stability of an LTI discrete-time system defined by following anticausal impulse response

$$h_1[n] = -\beta^n \mu[-n-1]$$

- Solution:
- For this system

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} |\beta^n \mu[-n-1]| = \sum_{n=-\infty}^{-1} |\beta|^n = \frac{1}{1-|\alpha|}$$

$$= \sum_{m=1}^{\infty} |\beta|^{-m} = \beta^{-1} \sum_{n=0}^{\infty} |\beta|^{-m} = \frac{|\beta|^{-1}}{1-|\beta|^{-1}} \qquad for \quad |\beta| > 1$$

- Therefore, $S < \infty$ if $|\beta| > 1$ for which the system is BIBO stable.
- If $|\beta| \leq 1$, the infinite series $\sum_{m=0}^{\infty} |\beta|^m$ does not converge, then the above causal system is not BIBO stable.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Stability Condition in Terms of the Impulse Response

- Examine the stability of an LTI discrete-time system defined by following finite impulse response

$$h[n] = \begin{cases} \alpha^n, & N_1 \leq n \leq N_2, \\ 0, & otherwise. \end{cases} \qquad (4.24)$$

- Solution:
- The above impulse response is absolutely summable independent of the value of α as long as it is not infinite. Hence the above system is BIBO stable for finite values of α.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Causality Condition in Terms of the Impulse Response

- Consider two inputs: $x_1[n]$ and $x_2[n]$, and assume

$$x_1[n] = x_2[n] \;\; for\; n \leq n_0 \; \ldots (1)$$
$$x_1[n] \neq x_2[n] \; for\; n < n_0 \;\; \ldots (2)$$

- Let $y_1[n]$ and $y_2[n]$ are the responses for above two inputs for a system with impulse response $h[n]$

$$y_1[n] = \sum_{k=-\infty}^{\infty} x_1[k]h[n-k]$$

$$y_2[n] = \sum_{k=-\infty}^{\infty} x_2[k]h[n-k]$$

- The output samples at $n = n_0$ is given by

$$y_1[n_0] = \sum_{k=-\infty}^{\infty} x_1[k]h[n_0-k]$$

$$y_2[n_0] = \sum_{k=-\infty}^{\infty} x_2[k]h[n_0-k]$$

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Causality Condition in Terms of the Impulse Response

- An LTI discrete-time system is **causal** if and only if its impulse response $h[n]$ is a causal sequence satisfying the following condition

$$h[k] = 0 \qquad for \qquad k < 0 \qquad\qquad (4.27)$$

- Example: Examine the causality of an LTI discrete-time system with following impulse response

$$h[n] = [\alpha_1, \alpha_2, \alpha_3, \alpha_4] \qquad 0 \leq n \leq 4$$

- Solution:
- The above impulse response is $h[k] = 0$ for $k < 0$, hence given system is causal.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Causality Condition in Terms of the Impulse Response

- Example: Examine the causality of an LTI discrete-time system; accumulator with following impulse response

$$h[n] = \sum_{l=-\infty}^{n} \delta[l]$$

- Solution:
- The above impulse response is $h[k] = 0$ for $k < 0$, because the delta function is $\delta[n] = 1$ when $n = 0$ and $\delta[n] = 0$ when $n \neq 0$, hence given system is causal.

Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

Causality Condition in Terms of the Impulse Response

- Example: Examine the causality of an LTI discrete-time system (factor-of-2-interpolator) with following impulse response

$$h[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$$

$$or$$

$$h[n] = \left[\frac{1}{2}, 1, \frac{1}{2}\right] \qquad -1 \le n \le 1$$

- Solution:
- The above impulse response is $h[k] \neq 0$ for $k < 0$, as $h[-1] = \frac{1}{2}$, hence given system is noncausal.
- Sometimes a noncuasal system can be made causal by introducing a proper delay in the output

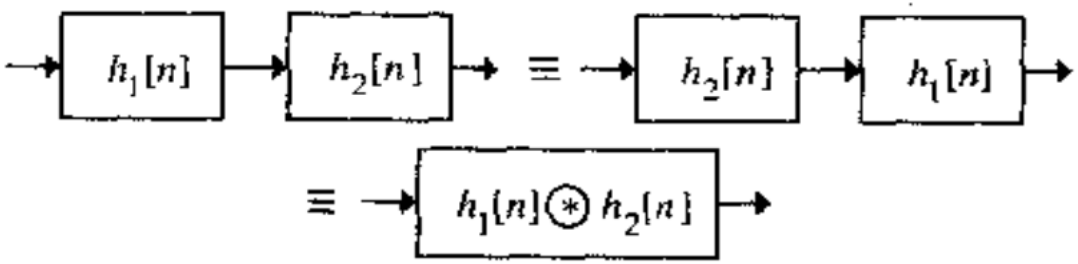
$$y[n] = \frac{1}{2}x_u[n] + x_u[n-1] + \frac{1}{2}x_u[n-2] = \{\underset{\uparrow}{0.5}, \quad 1, \quad 0.5\}$$

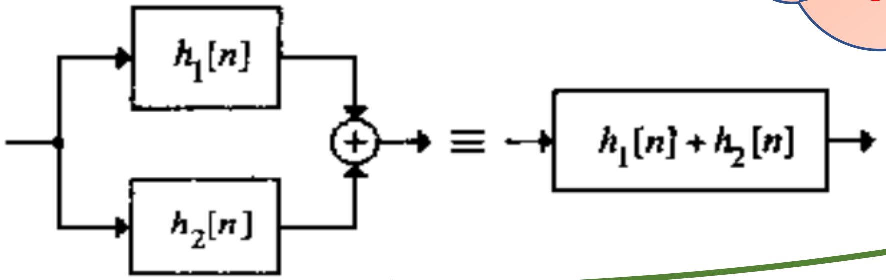
- The above system is causal now.

Discrete-Time Systems
Simple Interconnection Schemes

- A more complex LTI discrete-time system is usually developed by combining simple LTI discrete-time systems.

Two widely used schems:
Cascade and Parallel Connection





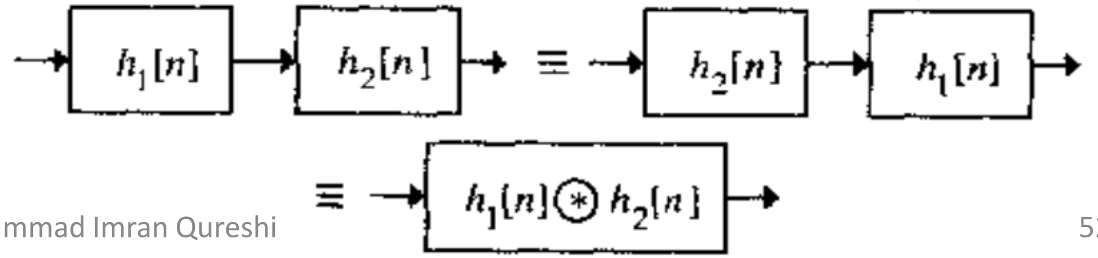
Discrete-Time Systems

Simple Interconnection Schemes: Cascade Connection

- A more complex LTI discrete-time system is usually developed by combining simple LTI discrete-time systems.
- The overall impulse response $h[n]$ of the cascade of the two systems with the impulse response $h_1[n]$ and $h_2[n]$, respectively, is given by their linear convolution; that is

$$h[n] = h_1[n] * h_2[n]$$

- The cascade connection of stable system is stable.
- The cascade connection of passive (lossless) systems is passive (lossless).



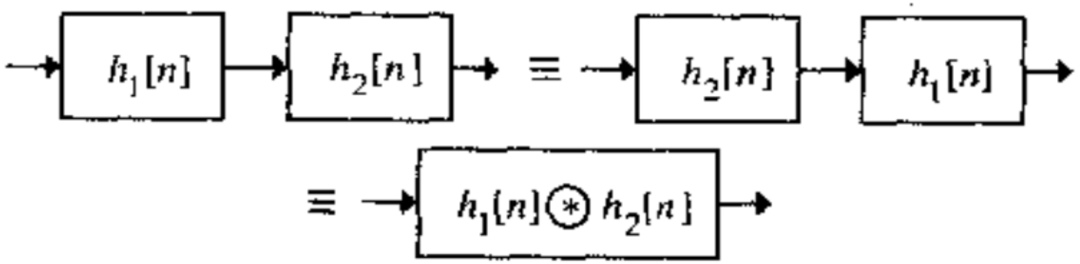
Discrete-Time Systems

Simple Interconnection Schemes: Cascade Connection

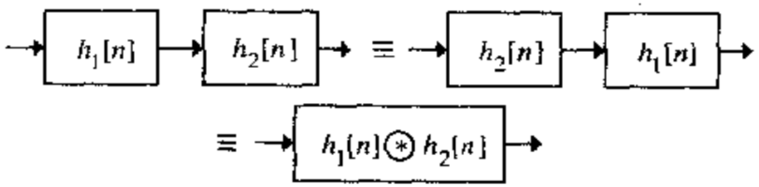
- An application of the cascade connection of scheme is in the development of an inverse system.
- If two LTI systems in the casade connection are such that

$$h_1[n] * h_2[n] = \delta[n]$$

- Then the LTI system $h_2[n]$ is said to be the inverse of the LTI system $h_1[n]$, and vice versa.



Discrete-Time Systems



Simple Interconnection Schemes: Cascade Connection

- Example: Consider a system with impulse response $h_1[n] = \mu[n]$, find its inverse of that system with impulse response $h_2[n]$.
- Solution
- The impulse response $h_2[n]$ should be inverse of the impulse response $h_1[n] = \mu[n]$; that is,

$$h_1[n] * h_2[n] = \delta[n]$$
$$\mu[n] * h_2[n] = \delta[n] \dots (1)$$

- It follows from Eq (1), that $h_2[n] = 0$ from $n < 0$, as the step function is zero (0) for $n < 0$
- $h_2[0] = 1$ as delta and step function are one (1) for $n = 0.$
- $h_2[l] = \sum_{l=0}^{\infty} h_2[l] = 0 \qquad n \geq 1$
- For $l = 1$

$$h_2[1] = h_2[0] + h_2[1] = 0$$
$$h_2[0] + h_2[1] = 0$$
$$1 + h_2[1] = 0$$
$$h_2[1] = -1$$

- For $n \geq 2, \quad h[n] = 0$
- Hence the impulse response is

$$h_2[n] = h[0] - h[n-1]$$

Or

$$h_2[n] = \delta[0] - \delta[n-1]$$

- The above system is called a backward difference system.

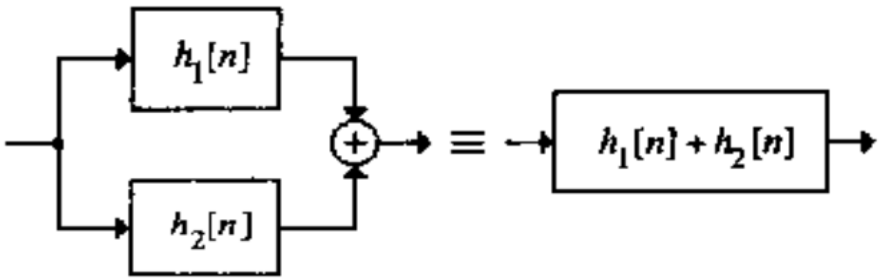
Discrete-Time Systems

Simple Interconnection Schemes : Parallel Connections

- The overall impulse response $h[n]$ of the parallel of the two systems with the impulse response $h_1[n]$ and $h_2[n]$, respectively, is given by their sum; that is,

$$h[n] = h_1[n] + h_2[n]$$

- The parallel connection of stable system is stable.
- The parallel connection of passive (lossless) systems may or may not be passive (lossless).

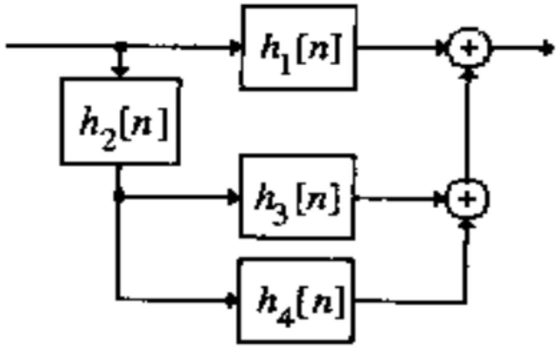


Discrete-Time Systems

Simple Interconnection Schemes

For k<0, $\delta[k] = 0$,
For k>0, $\delta[k] = 0$,
And
For k=0, $\delta[k] = 1$,
So, when we put k=0, we get
$\boldsymbol{\delta[k]\delta[n-k] = \delta[0]\delta[n] = \delta[n]}$

- Example: Consider a discrete-time system shown below



Where,

$$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$
$$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]$$
$$h_3[n] = 2\delta[n]$$
$$h_4[n] = -2\left(\frac{1}{2}\right)^n \mu[n]$$

- **Solution:**

$$h[n] = h_1[n] + h_2[n] * h_3[n] + h_2[n] * h_4[n] \ldots (1)$$

- We will first solve the part $h_2[n] * h_3[n]$

$$h_1[n] * h_3[n] = \left(\frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]\right) * 2\delta[n]$$

$$h_1[n] * h_3[n] = \left(\delta[n] * \delta[n] - \frac{1}{2}\delta[n-1] * \delta[n]\right) \ldots (2)$$

- $\delta[n] * \delta[n] = \sum_{k=-\infty}^{\infty} \delta[k]\delta[n-k] = \delta[n]$
- $\delta[n-1] * \delta[n] = \sum_{k=-\infty}^{\infty} \delta[k-1]\delta[n-k] = \delta[n-1]$
- Put above equations in (2), we get

$$h_1[n] * h_3[n] = \delta[n] - \frac{1}{2}\delta[n-1] \ldots (3)$$

Discrete-Time Systems

Simple Interconnection Schemes

$\mu[n]-\mu[n-1]=\delta[n]$

Since, $\delta[n]=1\ for\ only\ n=0$

that's why

$-\left(\frac{1}{2}\right)^{n}\{\mu[n]-\mu[n-1]\}$

$=-\delta[n]$

- We will first solve the part $h_2[n]*h_4[n]$

$$h_2[n]*h_4[n]$$
$$=\left(\frac{1}{2}\delta[n]-\frac{1}{4}\delta[n-1]\right)*\left(-2\left(\frac{1}{2}\right)^{n}\mu[n]\right)$$

$$\delta[n]*\left(\frac{1}{2}\right)^{n}\mu[n]$$
$$=\sum_{k=-\infty}^{\infty}\delta[k]\left(\frac{1}{2}\right)^{n-k}\mu[n-k]=\left(\frac{1}{2}\right)^{n}\mu[n]$$

$$\delta[n-1]*\left(\frac{1}{2}\right)^{n}\mu[n]$$
$$=\sum_{k=-\infty}^{\infty}\delta[k-1]\left(\frac{1}{2}\right)^{n-k}\mu[n-k]=\left(\frac{1}{2}\right)^{n-1}\mu[n-1]$$

- Put above equations in (2), we get

$$h_2[n]*h_4[n]=-\left(\frac{1}{2}\right)^{n}\mu[n]+\frac{1}{2}\left\{\left(\frac{1}{2}\right)^{n-1}\mu[n-1]\right\}$$

$$=-\left(\frac{1}{2}\right)^{n}\mu[n]+\left(\frac{1}{2}\right)^{n}\mu[n-1]$$
$$=-\left(\frac{1}{2}\right)^{n}\{\mu[n]-\mu[n-1]\}$$
$$=-\delta[n]\ldots(4)$$

- Putting Eq. (3) and (4) in Eq. (1), we get

$$h[n]$$
$$=\delta[n]+\frac{1}{2}\delta[n-1]+\delta[n]$$
$$-\frac{1}{2}\delta[n-1]-\delta[n]$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems

- An important subclass of LTI discrete-time system is characterized by a linear constant coefficient difference equation of the form

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k] \dots\dots (4.32)$$

- Where $x[n]$, and $y[n]$ are input and output, respectively, and d_k and p_k are constants.
- The order of the above system is given by $\max(N, M)$
- If we assume above system to be causal, then the output can be calculated by

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k] \dots (4.33)$$

- provided $d_0 \neq 0$
- The output $y[n]$ can be computed for all $n \geq n_0$, knowing $x[n]$ and the initial conditions $y[n_0-1], y[n_0-2], \dots, y[n_0-N]$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Total Solution Calculation

- In the case of the discrete-time system defined by

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k] \ldots\ldots (4.32)$$

- The output response $y[n]$ consists of two components: $y_c[n]$ and $y_p[n]$.

$$y[n] = y_c[n] + y_p[n] \ldots (4.39)$$

- The component $y_c[n]$, called complementary solution, is the solution of Eq. (4.32) with the input $x[n] = 0$; that is

$$\sum_{k=0}^{N} d_k y[n-k] = 0$$

- The component $y_p[n]$, called particular solution, is the solution of Eq. (4.32) with the input $x[n] \neq 0$.
- The sum of $y_c[n]$ and $y_p[n]$ is called the total solution.

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Total Solution Calculation (Method of Computing $\boldsymbol{y_c[n]}$)

- Let's assume
- $y_c[n] = \lambda^n$
- Put above value in Eq. (4.32)

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{N} d_k \lambda^{n-k}$$
$$= d_0\lambda^n + d_1\lambda^{n-1} + \cdots + d_N\lambda^{n-N}$$
$$= \lambda^{n-N}(d_0\lambda^N + d_1\lambda^{N-1} + \cdots + d_N)$$
$$= 0 \ldots (4.42)$$

- The above polynomial is called the characteristic polynomial of the discrete-time system defined Eq. (4.32).
- Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ denote N roots of given system.
- If these roots are disctinct, then the $y_c[n]$ is given by
  $$y_c[n] = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_N\lambda_N^n$$
- Where, $\alpha_1, \alpha_2, \ldots, \alpha_N$ are constants determined by the initial conditions.
- In case of multiple roots, the $y_c[n]$ takes different form.
- If λ_1 is of multiplicity of L and remaining $N-L$ roots $(\lambda_2, \lambda_3, \ldots, \lambda_{N-L})$ are distinct, the $y_c[n]$ takes the form

$$y_c[n]$$
$$= \alpha_1\lambda_1^n + \alpha_2 n\lambda_1^n + \alpha_3 n^2\lambda_1^n + \cdots$$
$$+ \alpha_L n^{L-1}\lambda_1^n + \alpha_{L+1}\lambda_2^n + \cdots + \alpha_N\lambda_{N-L}^n$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Total Solution Calculation (Method of Computing $\boldsymbol{y_p[n]}$)

- To calculate $y_p[n]$, assume the particular solution is also of the same form as the specified input $x[n]$.
- If $x[n]$ is a constant, then $y_p[n]$ is also assumed to be constant.
- If $x[n]$ is a sinusoidal sequence, then $y_p[n]$ is also assumed to be a sinusoidal sequence.

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Total Solution Calculation (Total Solution Calculation of an LTI System for a Constant Input)

- Example: Find the total solution for $n \geq 0$ of a discrete-time system defined by
  $y[n] + y[n-1] - 6y[n-2] = x[n] \ldots (*)$
- With initial conditions $y[-1] = 1$ and $y[-2] = -1$.
- Solution:
- For complementary solution, we will set $x[n] = 0$ and $y_c[n] = \lambda^n$ in above equation
  $$\lambda^n + \lambda^{n-1} - 6\lambda^{n-2} = 0$$
- $= \lambda^{n-2}\,(\lambda^2 + \lambda - 6)$
- The roots of characteristic polynomial $(\lambda^2 + \lambda - 6)$ are $\lambda = -3$ and $\lambda = 2$.
- The $y_c[n]$ is obtained as
  $$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n \ldots (1)$$
- For the particular solution, we assume
- $y_p[n] = \beta \ldots (2)$
- Substituting above in Eq. (*), we get
- $\beta + \beta - 6\beta = 8\mu[n]$
- $-4\beta = 8\mu[n]$
- $\beta = -2\mu[n]$ or $\beta = -2$ for $n \geq 0$
- The total solution is calculated
- $y[n] = y_c[n] + y_p[n]$
- $y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2 \ldots (2)$

Here $\mu[n]$ is only to show that the signal if defined for $n \geq 0$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Total Solution Calculation (Total Solution Calculation of an LTI System for a Constant Input)

- From initial conditions we know that $y[-1] = 1$ put in Eq. (2)

$$y[-1] = \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} - 2 = 1$$

$$2\alpha_1 + 3\alpha_2 = 13 \ldots (3)$$

- The second initial condition is $y[-2] = -1$ put in Eq. (2), we get

$$y[-2] = \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} - 2 = -1$$

$$4\alpha_1 + 9\alpha_2 = 73 \ldots (4)$$

- By solving Eq. (3) and (4), we get

$\alpha_1 = -1.8$ and $\alpha_2 = 4.8$

- Put above values in Eq. (2), we get

$$\boldsymbol{y[n] = -1.8(-3)^n + 4.8(2)^n - 2}$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Total Solution Calculation (Total Solution Calculation of an LTI System for an Exponential Input)

- Example: Find the total solution for $n \geq 0$ of a discrete-time system defined by
  $y[n] + y[n-1] - 6y[n-2] = x[n] \dots (*)$
- Where, $x = 8\mu[n]$ with initial conditions $y[-1] = 1$ and $y[-2] = -1$.
- Solution:
- For complementary solution, we will set $x[n] = 0$ and $y_c[n] = \lambda^n$ in above equation
  $$\lambda^n + \lambda^{n-1} - 6\lambda^{n-2} = 0$$
  $$= \lambda^{n-2}\,(\lambda^2 + \lambda - 6)$$
- The roots of characteristic polynomial $(\lambda^2 + \lambda - 6)$ are $\lambda = -3$ and $\lambda = 2$.
- The $y_c[n]$ is obtained as
  $$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n \dots (1)$$
- The complementary solution contains the term $\alpha_2(2)^n$ which is same as the specified input; that is, $x[n] = 2^n\mu[n]$.
- Therefore, we need to select a form for the particular solution that is different and does not contain any terms similar to those contained in the complementary solution.
- For the particular solution, we assume
- $y_p[n] = \beta n(2)^n \dots (2)$
- Substituting above in Eq. (*), we get
- $\beta n(2)^n + \beta(n-1)(2)^{n-1} - 6\beta(n-2)(2)^{n-2} = (2)^n\mu[n]$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Total Solution Calculation (Total Solution Calculation of an LTI System for an Exponential Input)

- The complementary solution contains the term $\alpha_2(2)^n$ which is same as the specified input; that is, $x[n] = 2^n\mu[n]$.
- Therefore, we need to select a form for the particular solution that is different and does not contain any terms similar to those contained in the complementary solution.
- For the particular solution, we assume
- $y_p[n] = \beta n(2)^n \ldots (2)$
- Substituting above in Eq. (*), we get
- $\beta n(2)^n + \beta(n-1)(2)^{n-1} - 6\beta(n-2)(2)^{n-2} = (2)^n\mu[n]$
- $\beta(2)^n(n + (n-1)(2)^{-1} - 6(n-2)(2)^{-2}) = (2)^n\mu[n]$
- $\beta\,(n + (n-1)(2)^{-1} - 6(n-2)(2)^{-2}) = \mu[n]$
- $\beta n + 0.5\beta n - 0.5\beta - 1.5\beta n + 3\beta = \mu[n]$
- $2.5\beta = \mu[n]$
- $\beta = 0.4\mu[n]$
- $\beta = 0.4\,\mu[n]$ or $\beta = 0.4$ for $n \geq 0$

Here $\mu[n]$ is only to show that the signal if defined for $n \geq 0$

- The total solution is calculated
- $y[n] = y_c[n] + y_p[n]$
- $y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 0.4n(2)^n \ldots (2)$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Total Solution Calculation (Total Solution Calculation of an LTI System for an Exponential Input)

- From initial conditions we know that $y[-1] = 1$ put in Eq. (2)

$$y[-1] = \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} - 0.4(-1)(2)^{-1} = 1 \ldots (3)$$

- The second initial condition is $y[-2] = -1$ put in Eq. (2), we get

$$y[-2] = \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} - 0.4(-2)(2)^{-2} = -1 \ldots (4)$$

- By solving Eq. (3) and (4), we get

$\alpha_1 = -5.04$ and $\alpha_2 = -0.96$

- Put above values in Eq. (2), we get

$$\boldsymbol{y[n] = -5.04(-3)^n - 0.96(2)^n - 0.4n(2)^n}$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Zero-Input Response and Zero-State Response

- An alternate approach to determining the total solution $y[n]$ of the difference equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k] \ldots (4.32)$$

- is by computing its zero-input response $(y_{zi}[n])$, and zero-state response $(y_{zs}[n])$.
- The component $y_{zi}[n]$ is obtained by solving Eq. (4.32) by setting the input $x[n] = 0$.
- The component $y_{zs}[n]$ is obtained by solving Eq. (4.29) by applying the specified input with all initial conditions set to zero.

$$h_1[n] * h_2[n] = \delta[n] \ldots (4.29).$$

- The total solution is then given by

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Zero-Input Response and Zero-State Response (Total Solution Computation from Zero-Input and Zero-State Responses)

- Example: Find the total solution for $n \geq 0$ of a discrete-time system defined by
  $$y[n] + y[n-1] - 6y[n-2] = x[n] \dots (*)$$
  By computing the zero-input response $y_{zi}[n]$ and the zero-state response $y_{zs}[n]$, with initial conditions $y[-1] = 1$ and $y[-2] = -1.$
- Solution:
- The first step in computing $y_{zi}[n]$ and $y_{zs}[n]$ is to calculate the total solution using complementary and particular solution
- For complementary solution, we will set $x[n] = 0$ and $y_c[n] = \lambda^n$ in above equation
  $$\lambda^n + \lambda^{n-1} - 6\lambda^{n-2} = 0$$
  $$= \lambda^{n-2}\,(\lambda^2 + \lambda - 6)$$
- The roots of characteristic polynomial $(\lambda^2 + \lambda - 6)$ are $\lambda = -3$ and $\lambda = 2.$
- The $y_c[n]$ is obtained as
  $$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n \dots (1)$$
- For the particular solution, we assume
  $$y_p[n] = \beta \dots (2)$$
- Substituting above in Eq. (*), we get
  $$\beta + \beta - 6\beta = 8\mu[n]$$
  $$-4\beta = 8\mu[n]$$
  $$\beta = -2\mu[n] \text{ or } \beta = -2 \text{ for } n \geq 0$$
- The total solution is calculated
  $$y[n] = y_c[n] + y_p[n]$$
  $$y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2 \dots (2)$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Zero-Input Response and Zero-State Response (Total Solution Computation from Zero-Input and Zero-State Responses)

- For zero-input response $y_{zi}[n]$, the input is zeros; that is $x[n] = 0$. The Eq. (*) reduces to

$$y[n] + y[n-1] - 6y[n-2] = 0$$
$$y[n] = -y[n-1] + 6y[n-2]$$

- For $n = 0$

$$y[0] = -y[-1] + 6y[-2] = -7$$

- For $n = 1$

$$y[1] = -y[0] + 6y[-1] = 13$$

- The $y_c[n]$ is obtained as

$$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n$$

- For $n = 0$

$$y[0] = \alpha_1(-3)^0 + \alpha_2(2)^0$$
$$\alpha_1 + \alpha_2 = -7 \ldots (3)$$

- For $n = 1$

$$y_c[1] = \alpha_1(-3)^1 + \alpha_2(2)^1$$
$$-3\alpha_1 + 2\alpha_2 = 13 \ldots (4)$$

- By solving Eq. (3) and (4), we get

$$\alpha_1 = -5.4 \text{ and } \alpha_2 = -1.6$$

- As a result, the $y_{zi}[n]$ reduces to

$$y_{zi}[n] = -5.4(-3)^n - 1.6(2)^n \ldots (5)$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Zero-Input Response and Zero-State Response (Total Solution Computation from Zero-Input and Zero-State Responses)

- For zero-state response $y_{zs}[n]$, the input is zeros; that is, $x[n]=0$. The Eq. (*) reduces to

$$y[n]+y[n-1]-6y[n-2]=x[n]$$
$$y[n]=-y[n-1]+6y[n-2]+x[n]$$

- For $n=0$

$$y[0]=-y[-1]+6y[-2]+8\mu[0]$$

- For zero-state response $y_{zs}[n]$, the intial conditions are assumed zero; tht is, $y[-1]=y[-2]=0$

$$y[0]=8$$

- For $n=1$

$$y[1]=-y[0]+6y[-1]+8\mu[n]=0$$

- The total solution is

$$y[n]=\alpha_1(-3)^n+\alpha_2(2)^n-2$$

- For $n=0$

$$y[0]=\alpha_1(-3)^0+\alpha_2(2)^0-2$$
$$\alpha_1+\alpha_2=10\ldots(6)$$

- For $n=1$

$$y[1]=\alpha_1(-3)^1+\alpha_2(2)^1-2$$
$$-3\alpha_1+2\alpha_2=2\ldots(7)$$

- By solving Eq. (6) and (7), we get

$$\alpha_1=3.6 \text{ and } \alpha_2=6.4$$

- As a result, the $y_{zs}[n]$ reduces to

$$y_{zs}[n]=3.6(-3)^n+6.4(2)^n-2\ldots(8)$$

- The total solution $y[n]$ is obtained as

$$y[n]=y_{zi}[n]+y_{zs}[n]$$
$$\boldsymbol{y[n]=-1.8(-3)^n+4.8(2)^n-2 \quad n\geq 0}$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems:

Impulse Response Calculation

Because the input signal $x[n]$ does not exisit for $n > 0$, so no particular solution will exist.

- The impulse response $h[n]$ of a causal LTI discrete-time system is the output observed with input $x[n] = \delta[n]$.
- Thus, it is simply the zero-state response with $x[n] = \delta[n]$.
- It means $x[n] = 0$ for $n > 0$, and thus, the particular solution is zero; that is, $y_p[n] = 0$.
- Hence, the impulse response can be computed from the complementary solution $y_c[n]$, where

$$y_c[n] = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_N \lambda_N^n$$

- In the case of simple roots of the characteristic equation by determining the constants α_i to satisfy the zero initial condition.

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Impulse Response Calculation (Impulse Response Computation from Zero-State Response)

- Example: Find the **impulse response** for $n \geq 0$ of a discrete-time system defined by
$$y[n] + y[n-1] - 6y[n-2] = x[n] \ldots (*)$$
with initial conditions $y[-1] = 1$ and $y[-2] = -1$.
- Solution:
- The complementary solution $y_c[n]$ for above system is given below (as it was calculated in previous examples)
$$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n \ldots (1)$$
- The impulse response $h[n]$ is the response for impulse input, and is given by
$$h[n] = \alpha_1(-3)^n + \alpha_2(2)^n \ldots (2)$$
- For $n = 0$
$$h[0] = \alpha_1(-3)^0 + \alpha_2(2)^0$$
$$h[0] = \alpha_1 + \alpha_2 \ldots (3)$$
- For $n = 1$
$$h[1] = \alpha_1(-3)^1 + \alpha_2(2)^1$$
$$h[1] = -3\alpha_1 + 2\alpha_2 = 2 \ldots (4)$$

- The discrete-time system is defined by
$$y[n] + y[n-1] - 6y[n-2] = x[n]$$
- The impulse response is obtained by putting $x[n] = \delta[n]$
$$h[n] = -y[n-1] + 6y[n-2] + x[n]$$
- For $n = 0$
$$h[0] = -h[-1] + 6h[-2] + \delta[0] = 1$$
- The initial conditions are assumed zero; that is, $h[-1] = h[-2] = 0.$
- For $n = 1$
$$h[1] = -h[0] + 6h[-1] + \delta[n] = -1$$
- After subtiuting the values of $h[0] = 1$ and $h[1] = 1$, in Eq. (3) and (4), we get the values $\alpha_1 = 0.6$ and $\alpha_2 = 0.4.$
- The total solution is obtained by putting α_1 and α_2 in Eq. (2).
$$h[n] = 0.6(-3)^n + 0.4(2)^n$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Impulse Response Calculation (Impulse Response Computation from Zero-State Response)

- Example: A causal LTI discrete-time system with an **impulse response** satisfies the following difference equation:

$$h[n] - ah[n-1] = \delta[n] \ldots (*)$$

  Determine:

  1. Close form expression for $h[n]$
  2. The input-output relation of the above system.

- Solution:
- The total solution of the difference equation is given by

$$h[n] = h_c[n] + h_p[n]$$

- For complementary solution, we will set $x[n] = 0$ and $h_c[n] = \lambda^n$ in above equation

$$\lambda^n - a\lambda^{n-1} = 0$$
$$\lambda^{n-1}(-a + \lambda) = 0$$

- The root of characteristic polynomial is $\lambda = a$.
- The $y_c[n]$ is obtained as

$$h_c[n] = a^n \ldots (1)$$

- For particular solution, we will set $x[n] = \beta$ and $h_p[n] = \beta$, the total solution is obtained as

$$h[n] = h_c[n] + h_p[n]$$
$$h[n] = a^n + \beta \ldots (2)$$

- If we put $n = 0$ in Eq. (*), we get

$$h[0] - ah[-1] = \delta[0] \Longrightarrow h[0] = 1 \ldots (3)$$

- If we put $n = 0$ in Eq. (2), we get
- $h[0] = a^0 + \beta$ put value of $h[0] = 1$, we get $\beta = 0$
- Putting the value of β in Eq. (2) we get

$$h[n] = a^n$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Impulse Response Calculation (Impulse Response Computation from Zero-State Response)

- The impulse response is obtained as

$$h[n] = a^n$$

OR

$$h[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- To determine the general input-output relation of the above discrete-time system, we convolove both sides of Eq. (4.50) with $x[n]$

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] - a \sum_{k=-\infty}^{\infty} x[k]h[n-k-1] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] - ay[n-1] = x[n]$$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems: Impulse Response Calculation (A Causal Stable LTI Discrete-Time System with No Difference Equation)

- The impulse response of a finite-dimensional LTI system characterized by a differene equation is of infinite length.
- However, there exist LTI discrete-time systems with an infinite impulse response that **cannot** be characterized by the difference equation.
- The system defined by the impulse response

$$h[n] = \frac{1}{n^2}\mu[n-1]$$

- does not have a representation in the form of a linear constant coefficient difference equation.
- It should be noted that the above system is causal and BIBO stable.

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems:

Impulse Response Calculation (Total Solution Calculation Using MATLAB)

- We have two matlab commands to find totoal solution from a finite constant difference equation; that is,

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

The d here represent the coefficient of $y[n-k]$

y = filter(p, d, x)

This is the input sequence

The p here represent the coefficient of $x[n-k]$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems:

Impulse Response Calculation (Total Solution Calculation Using MATLAB)

- We have two matlab commands to find totoal solution from a finite constant difference equation; that is,

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

These are the initial conditions

The d here represent the coefficient of $y[n-k]$

[y, sf] = filter(p, d, x, si)

These are the final conditions

This is the input sequence

The p here represent the coefficient of $x[n-k]$

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems:

Impulse Response Calculation (Total Solution Calculation Using MATLAB)

- Example: Find the **total solution using MATLAB** for $n \geq 0$ of a discrete-time system defined by

$$y[n] + y[n-1] - 6y[n-2] = x[n] \ldots (*)$$

- Where, $x = 8\mu[n]$ with initial conditions $y[-1] = 1$ and $y[-2] = -1$.
- Sol:
- The above equation can be written as
- $y[n] = x[n] + s_1[n-1]$
- $s_1[n] = s_2[n-1] - y[n]$
- $s_2[n] = 6y[n]$
- $s_2[-1] = 6y[-1] = 6$
- $s_2[-2] = 6y[-2] = -6$
- $s_1[-1] = s_2[-2] - y[-1] = -7$
- We will use the matlab command
- %%%%%%%%%%%%%%%%%%%%%
- clear; close; clc; workspace;
- p = 1;
- d = [1,1,-6];
- x = 8*ones(1,8);
- si = [-7, 6];
- [y, sf]=filter(p,d,x,si)
- %%%%%%%%%%%%%%%%%%%%%
- y = [1, 13, 1, 85, -71, 589, -1007, 4549]

Discrete-Time Systems

Finite-Dimensioinal LTI Discrete-Time Systems:

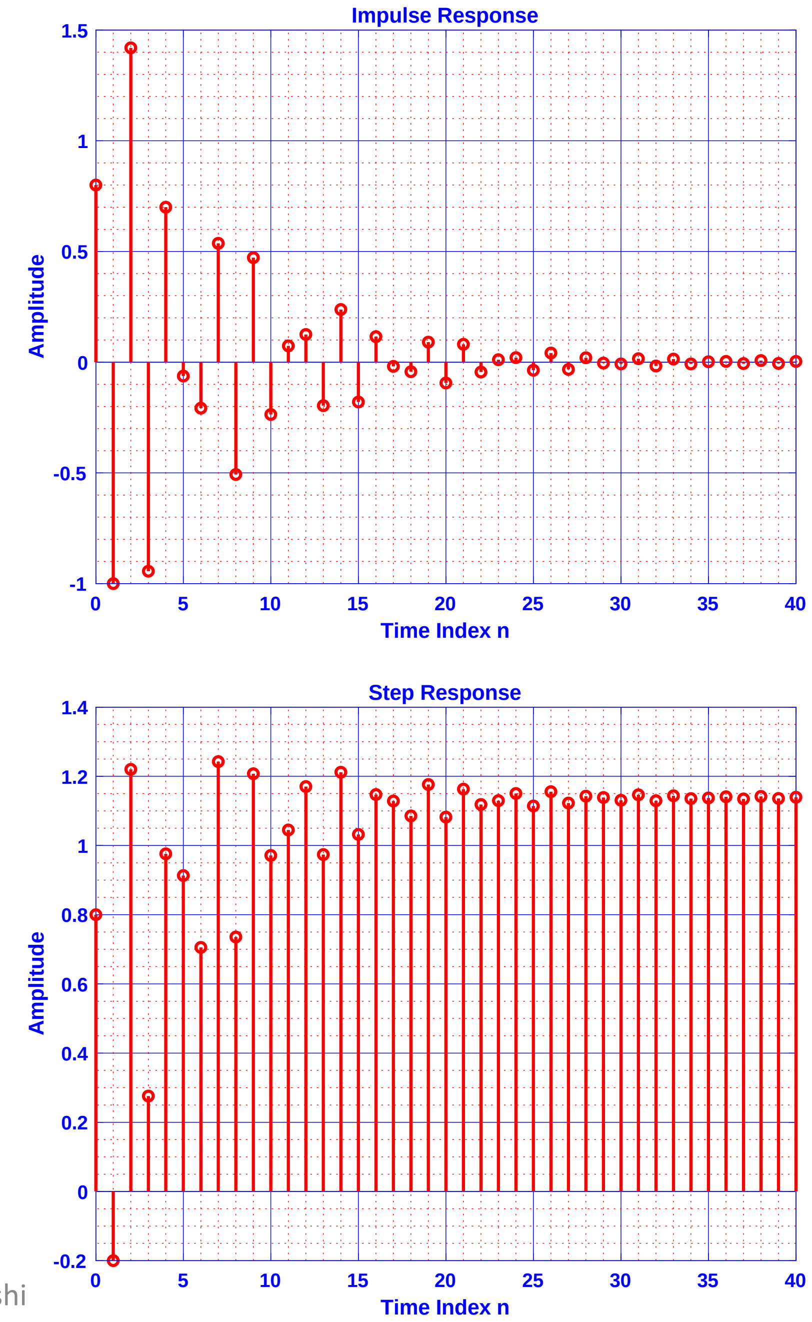
Impulse Response Calculation (Total Solution Calculation Using MATLAB)

- We will use the matlab command
- %%%%%%%%%%%%%%%%%%%%%
- clear; close; clc; workspace;
- p = 1;
- d = [1,1,-6];
- x = 8*ones(1,8);
- si = [-7, 6];
- [y, sf]=filter(p,d,x,si)
- %%%%%%%%%%%%%%%%%%%%%
- y = [1, 13, 1, 85, -71, 589, -1007, 4549]

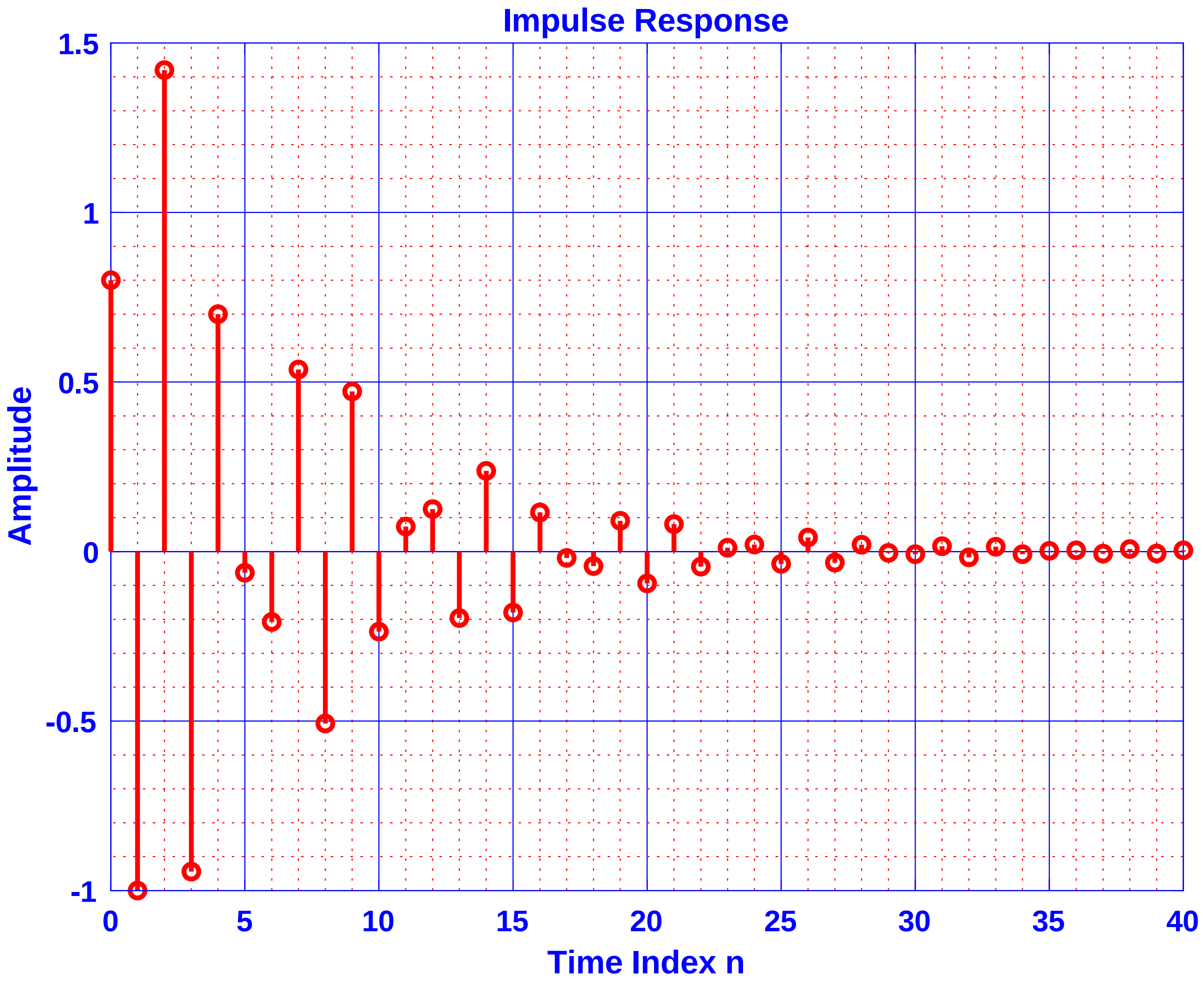
- If we compute the output using the total solution calculation for $0 \leq n \leq 7$, we will get same answer.
- %%%%%%%%%%%%%%%%%%%%%%
- n = 0:7;
- y1 = -1.8*(-3).^n+4.8*(2).^n-2
- %%%%%%%%%%%%%%%%%%%%%%
- y = [1, 13, 1, 85, -71, 589, -1007, 4549]

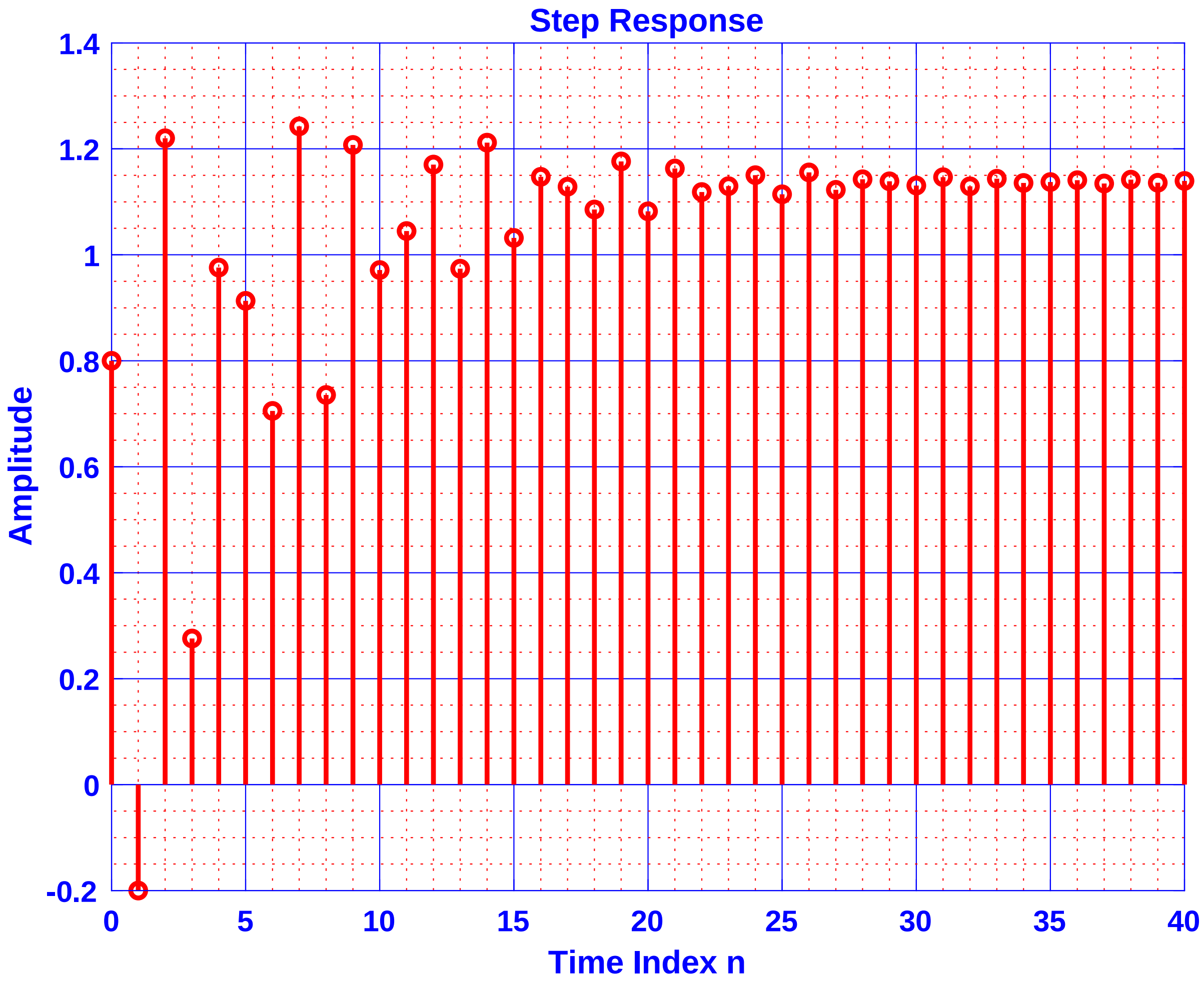
Impulse and Step Response Computation Using MATLAB

- Determine the first 41 samples of the impulse and step responses of the causal LTI system defined by
- $y[n] + 0.7y[n-1] - 0.45y[n-2] - 0.6y[n-3] = 0.8x[n] - 0.44x[n-1] + 0.36x[n-2] + 0.02x[n-3]$
- MATLAB Code:
- p = [0.8, -0.44, 0.36, 0.02];
- d = [1, 0.7, -0.45, -0.6];
- [h,m] = impz(p,d,41);
- [h,m] = stepz(p,d,41);



Impulse and Step Response Computation Using MATLAB

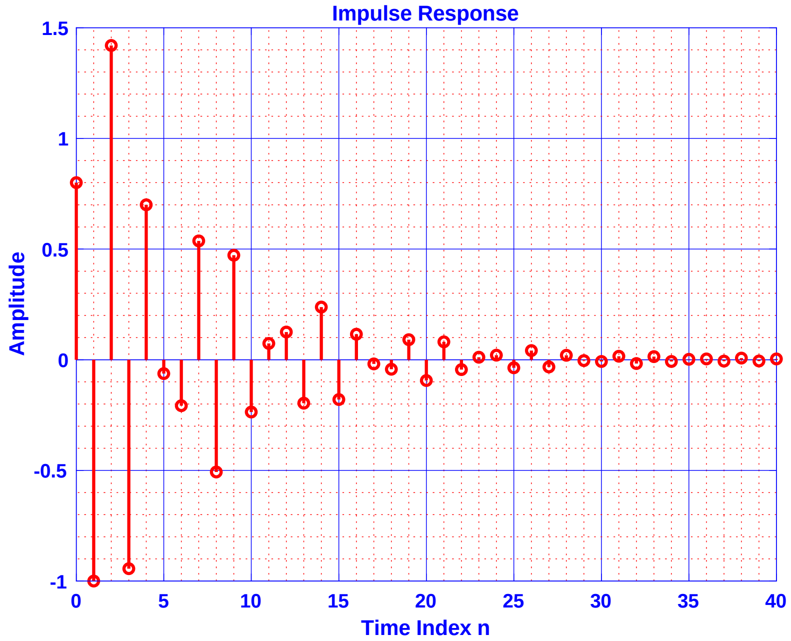




Discrete-Time Systems

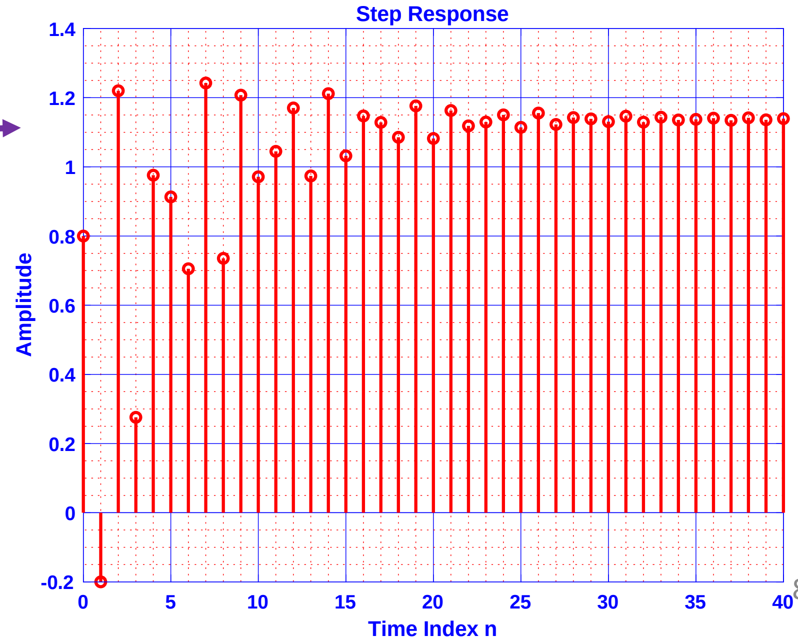
Finite-Dimensioinal LTI Discrete-Time Systems:

Location of Roots of Chateristics Polynomial for BIBO Stability



The impulse response samples of a stable LTI system decay to zero values as the time index n becomes very large

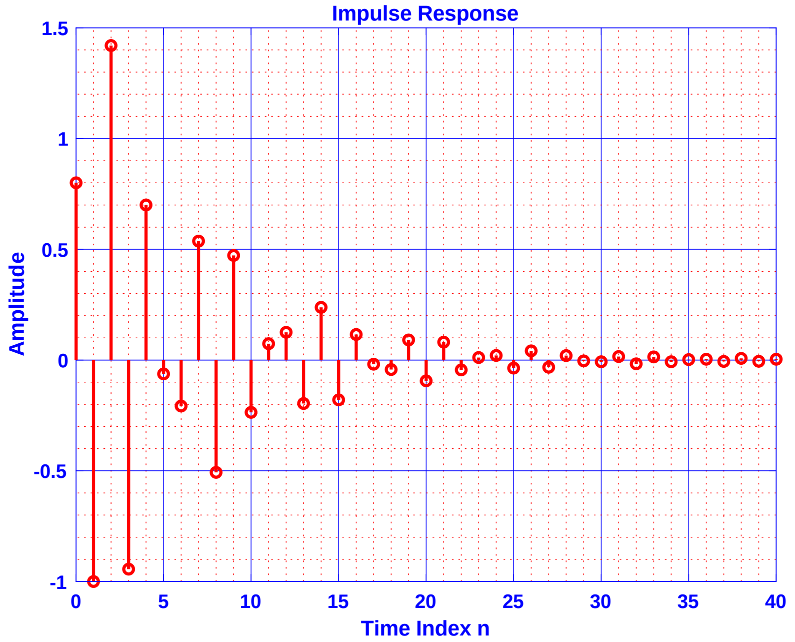
The step response samples of a stable LTI system approache a constant value as the time index n becomes very large

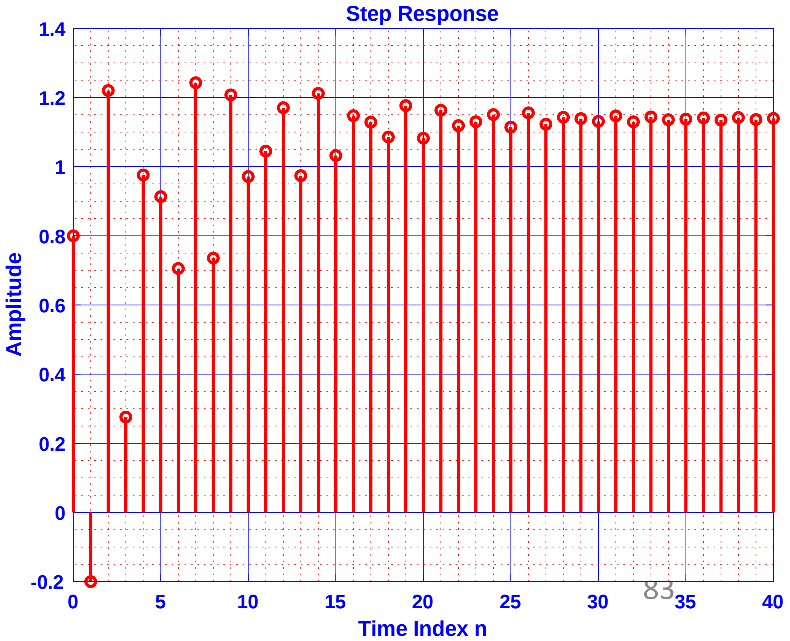


Discrete-Time Systems:

Finite-Dimensioinal LTI Discrete-Time Systems: Location of Roots of Chateristics Polynomial for BIBO Stability

- From looking at the impulse and step response, we can say that the system is BIBO stable.
- However, it is impossible to check the stability of a system just by examining only a finite segment of its impulse or step response.
- The BIBO stability of a causal LTI system characterized by a constant coefficient difference equation an be inferred from the values of the roots λ_i of its characteristic polynomial.
- If $|\lambda_i| < 1, \forall\, i$, then the impulse response is absolutely summable, implying BIBO stability of the causal LTI discrete-time system.
- If $|\lambda_i| > 1$, then system is not BIBO stable.
- A causal LTI system characterized by a linear constant coefficient difference equation is BIBO stable if the magnitude of each of the roots of its characteristic equation is less than one.





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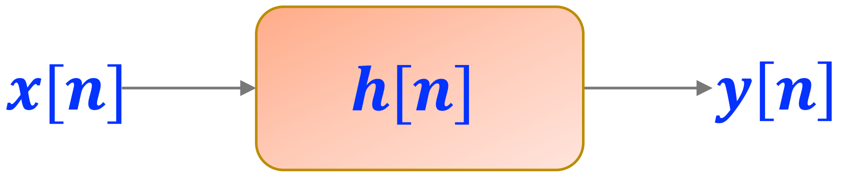
Discrete-Time Systems

Classification of LTI Discrete-Time Systems

- LTI discrete-time systems are usually classified:
  - Either according to the length of their impulse response
    1. Finite Impulse Response (FIR) Discrete-Time System
    2. Infinite Impulse Response (IIR) Discrete-Time System
  - Or according to the method of calculation employed to determine the output samples
    1. Nonrecursive Discrete-Time Systems
    2. Recursive Discrete-Time Systems
    3. Moving Average (MA) Model
    4. Autoregressive (AR) Model
    5. Autoregressive Moving Average (ARMA) Model



An LTI Discrete-Time System

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on Impulse Response Length

- **Finite Impulse Response (FIR) Discrete-Time System:**
  - A discrete system with finite impulse response $h[n]$; that is,

$$h[0] = 0 \;\; for \;\; n < N_1 \quad and \quad n > N_2 \quad with \quad N_1 < N_2 \ldots (4.59)$$

  - is known as a finite impulse response (FIR) discrete-time system.
  - The convolution sum reduces to

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k] \ldots (4.60)$$

- Following are the few examples of an FIR discrete-time Systems

1. Moving Average Filter

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l] \ldots \ldots \ldots (4.4)$$

2. Linear Interpolator

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1]) \quad \text{OR}$$

$$y[n] = x_u[n] + \frac{2}{3}(x_u[n-1] + x_u[n+1]) + \frac{1}{3}(x_u[n-2] + x_u[n+2]$$

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on Impulse Response Length

- **Infinite Impulse Response (IIR) Discrete-Time System:**
  - A discrete system with infinite impulse response $h[n]$ is known as a infinite impulse response (IIR) discrete-time system.
  - For a causal IIR system with a causal input $x[n]$, the convolution sum reduces to

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k]$$

  - For increasing n, the computational complexity to compute the output sample increases as the number of products to be summed also increases.
- The class of IIR filters we will discuss in our lectures will be causal characterized by linear constant coefficient difference equation, involving a finite sum of terms for all values of n.

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on Impulse Response Length

- **Infinite Impulse Response (IIR) Discrete-Time System:**
- Following are the few examples of an IIR discrete-time Systems
  1. Accumulator

$$y[n] = \sum_{l=-\infty}^{n-1} x[l] + x[n] = y[n-1] + x[n]$$

  2. Exponentially Weighted Running Average Filter

$$y[n] = \alpha y[n-1] + x[n]$$

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on the Output Calculation Process

- **Nonrecursive Discrete-Time System:**
  - If the output sequence can be calculated sequentially, knowing only the present and past input samples, the filter is said to be nonrecursive discrete-time system.
  - Following are the examples of nonrecursive discrete-time systems:
    1. A simple FIR discrete-time system

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

    1. M-point Averaging Filter (That is also an FIR discrete-time system)

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l] \ldots \ldots \ldots (4.4)$$

  - The recursive implementation of a nonrecursive FIR system is alos possible.
  - Following is the recursive implementation of a nonrecursive system, shown in Eq. (4.4)

$$y[n] = y[n-1] + \frac{1}{\mathrm{M}}(x[n] - x[n-M])$$

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on the Output Calculation Process

- **Recursive Discrete-Time System:**
  - If the output sequence can be calculated sequentially, knowing the past output values in addition to the present and past input samples, the filter is said to be recursive discrete-time system.
  - Following is the examples of a recursive discrete-time systems:

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k] \ldots (4.33)$$

  - The nonrecursive implementation of a recursive IIR system is alos possible.

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on the Output Calculation Process

- Based on the form of the linear constant coefficient difference equation, causal finite-dimensional LTI discrete-time system can be classified as:
  - Moving Average (MA) Model
  - Autoregressive (AR) Model
  - Autoregressive Moving Average (ARMA) Model

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on the Output Calculation Process

- Moving Average (MA) Model
  - A moving average (MA) model that is also an FIR discrete-time system is shown below

$$y[n] = \sum_{k=0}^{M} p_k x[n-k]$$

  - It is a generalization of the M-point average filter with different weights assigned to input samples.

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on the Output Calculation Process

- Autoregressive (AR) Model
  - The simplest IIR system, called the autoregressive (AR) model, is characterized by the input-output relation

$$y[n] = x[n] - \sum_{k=1}^{N} d_k y[n-k]$$

- Autoregressive Moving Average (ARMA) Model
  - The second type IIR system, called the autoregressive moving average (ARMA) model, is characterized by the input-output relation

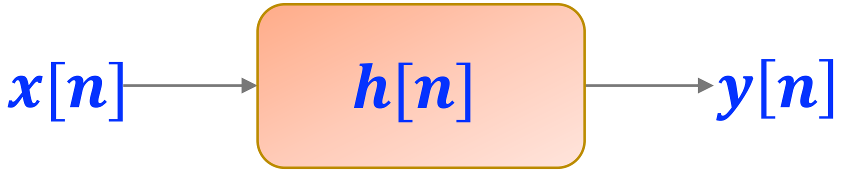
$$y[n] = \sum_{k=0}^{M} p_k\, x[n-k] - \sum_{k=1}^{N} d_k y[n-k]$$

Discrete-Time Systems

Classification of LTI Discrete-Time Systems:

Classification Based on the Impulse Response Coefficient

- Real Discrete-Time System
  - A discrete-time system with real-valued impulse response is defined as real discrete-time system.
- Complex Discrete-Time System
  - A discrete-time system with complex-valued impulse response is defined as complex discrete-time system.



An LTI Discrete-Time System

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems:

- Most discrete-time signals can be represented as a linear combination of a very large, may be infinite, number of sinusoidal discrete-time signals of different angular frequencies.
- Thus, knowing the response of the LTI system to a single sinusoidal, we can determine its response to more complicated signals by making use of superposition property of the system.
- A sinusoidal signal can be expressed in terms of an exponential signal, the response of the LTI system to an exponential input is of practical interest.

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems: Frequency Response

- The output of an LTI discrete-time system with an impulse response $h[n]$ for the input $x[n]$ is given by convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \ldots (4.66)$$

A function when multiplied with input, the output is that function multiplied with some complex constant is called an eigenfunction.

- Let assume

$$x[n] = e^{j\omega n} \qquad -\infty < n < \infty \ldots (4.67)$$

- The above equation reduces to

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{j\omega n}$$

Here $e^{j\omega n}$ is an eigenfunction.

$$y[n] = H(e^{j\omega})e^{j\omega n}$$

Here $H(e^{j\omega})$ is a complex constant.

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems: Frequency Response (Continued…)

- The output of an LTI discrete-time system with an impulse response $h[n]$ for the input $x[n]$ is given by

$$y[n] = H\left(e^{j\omega}\right)e^{j\omega n} \quad \dots (4.69)$$

The output is also a complex sinusoidal due to a complex sinusoidal input, of same angular frequency as that of input $e^{j\omega w}$

- Where, $H\left(e^{j\omega}\right)$ is the Fourier transform of $h[n]$, and is given by

$$H\left(e^{j\omega n}\right) = \sum_{k}^{\infty} h[k]e^{-j\omega n} \quad \dots . (4.70)$$

- It is clear from above equation that for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H\left(e^{j\omega n}\right)$.
- The signal $e^{j\omega n}$ is, therefore, an **eigenfunction** of the system.
- The quantity $H\left(e^{j\omega n}\right)$ defined above is called the frequency response of the LTI discrete-time system, and it provides a frequency-domain description of the system.
- The quantity $H\left(e^{j\omega n}\right)$ exists if $h[n]$ is absolutely summable; that is, the system is BIBO stable.

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems: Frequency Response (Continued…)

- $H(e^{j\omega n})$ completely characterizes the LTI discrete-time system in the frequency domain.
- $H(e^{j\omega n})$ is also a complex function of ω with a period 2π and can be expressed as follows

Phase Response

$$H(e^{j\omega n}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega})$$

$$H(e^{j\omega n}) = |H(e^{j\omega})|e^{j\theta(\omega)} \quad \dots\dots(4.71)$$

- Magnitude function can be specified in decibels as well

$$\mathcal{G}(\omega) = 20\log_{10}|H(e^{j\omega})|\ dB \quad \dots\dots(4.72)$$

Magitude Response

- Where $\mathcal{G}(\omega)$ is called the gain function.
- The negative of gain function, $\mathcal{A}(\omega) = -\mathcal{G}(\omega)$, is called the attenuation or the loss function.

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems: Frequency Response (Continued…)

- The magnitude and phase functions are real functions of ω.
- The frequency response is a complex function of ω.
- $H_{re}\left(e^{j\omega}\right)$ is an even function.
- $H_{im}\left(e^{j\omega}\right)$ is an odd function.
- $\left|H\left(e^{j\omega}\right)\right|$ is an even function.
- $\theta(\omega) = \arg\left\{H\left(e^{j\omega}\right)\right\}$ is an odd function.

Frequency Response
$H\left(e^{j\omega}\right) = H_{re}\left(e^{j\omega}\right) + \mathrm{j}H_{im}\left(e^{j\omega}\right)$
$H\left(e^{j\omega}\right) = H^*\left(e^{-j\omega}\right)$
$H_{re}\left(e^{j\omega}\right) = H_{re}\left(e^{-j\omega}\right)$
$H_{im}\left(e^{j\omega}\right) = -H_{im}\left(e^{-j\omega}\right)$
$\left|H\left(e^{j\omega}\right)\right| = \left|H\left(e^{-j\omega}\right)\right|$
$\arg\left\{H\left(e^{j\omega}\right)\right\} = -\arg\left\{H\left(e^{-j\omega}\right)\right\}$

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems: Frequency-Domain Characterization of the LTI Discrete-Time System

- The output of a stable LTI discrete-time system with impulse response $h[n]$ for an input $x[n]$ is obtained by convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Taking Fourier transform of both sides

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right) \dots (4.74)$$

- It follows from above equation that the output has the same angular frequency as the input's angular frequency.
- However, if for some reason, the steady-state output contain any sinusoidal component with frequencies that are not present in the input sinusoidal component, then the system is unstable, nonlinear, time-varying or any combination of these properties.
- The frequency response of the LTI system is defined as

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} \dots (4.75)$$

For a stable system, the steady-state output cannot contain sinusoidal components of frequencies that are not present in the input of the system.

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems: Frequency Response of the LTI Discrete-Time System

- **Frequency Response of LTI FIR Discrete-Time Systems**
  - The LTI FIR discrete-time systems are characterized by an input-output relation of the form

$$y[n]=\sum_{k=-N_1}^{N_2} h[k]x[n-k]$$

$$OR$$

$$y[n]=\sum_{k=-N_1}^{N_2} x[k]h[n-k]$$

  - Taking Fourier transform of both sides

$$\sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n}=\sum_{n=-\infty}^{\infty}\sum_{k=-N_1}^{N_2} h[k]x[n-k]\,e^{-j\omega n}$$

$$Y(e^{j\omega})=\sum_{k=N_1}^{N_2}\sum_{k=-\infty}^{\infty} x[n-k]\,e^{-j\omega(n-k)}h[k]e^{-j\omega k}$$

$$Y(e^{j\omega})=\sum_{k=-N_1}^{N_2} h[k]e^{-j\omega k}\,X(e^{j\omega})$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\sum_{k=-N_1}^{N_2} h[k]e^{-j\omega k}$$

$$H(e^{j\omega})=\sum_{k=-N_1}^{N_2} h[k]e^{-j\omega k}$$

- The frequency response of an LTI FIR Discrete-time system is obtained using above equation.

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems: Frequency Response of the LTI Discrete-Time System

- **Frequency Response of LTI IIR Discrete-Time Systems**
  - The LTI IIR discrete-time systems are characterized by an input-output relation of the form

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

  - Taking Fourier transform of both sides

$$\sum_{n=-\infty}^{\infty} \sum_{k=0}^{N} d_k y[n-k]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{M} p_k x[n-k]\, e^{-j\omega n}$$

$$\sum_{k=0}^{N} \sum_{n=-\infty}^{\infty} d_k y[n-k]\, e^{-j\omega(n-k)} e^{-j\omega k} = \sum_{k=0}^{M} \sum_{n=-\infty}^{\infty} p_k x[n-k]\, e^{-j\omega(n-k)} e^{-j\omega k}$$

$$\left(\sum_{k=0}^{N} d_k e^{-j\omega k}\right) Y\left(e^{j\omega}\right) = \left(\sum_{k=0}^{N} p_k e^{-j\omega k}\right) X\left(e^{j\omega}\right)$$

  - The frequeny respone of an LTI IIR Discrete-Time system is given by

$$\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\left(\sum_{k=0}^{N} p_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} d_k e^{-j\omega k}\right)}$$

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems:

Frequency Response Computation Using MATLAB

- Example: Compute the frequency of the Moving-Average Filter defined by
- $y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l] \ldots\ldots\ldots (4.4)$
- Solution:
- The impulse response of above system is obtained by putting $x[n] = \delta[n]$ in above equation
- $h[n] = \frac{1}{M}\sum_{l=0}^{M-1} \delta[n-l]$
- The above equation can be written as
- $h[n] = \begin{cases} \frac{1}{M}, & 0 \leq n \leq M \\ 0, & otherwise \end{cases}$

- The frequency response of above system is obtained by the following equation
- $H(e^{j\omega}) = \sum_{k=-N_1}^{N_2} h[k] e^{-j\omega k}$
- $= \sum_{k=0}^{M-1} \frac{1}{M} e^{-j\omega k}$
- Using the formula for summation
- $S_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$
- We get
- $H(e^{j\omega}) = \frac{1}{M}\left\{\frac{1-e^{-j\omega M}}{1-e^{-j\omega}}\right\}$

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems:

Frequency Response Computation Using MATLAB

- $H(e^{j\omega}) = \frac{1}{M}\left\{\frac{1-e^{-j\omega M}}{1-e^{-j\omega}}\right\}$
- $= \frac{1}{M}\left\{\frac{e^{\frac{j\omega M}{2}}-e^{-\frac{j\omega M}{2}}}{e^{\frac{j\omega}{2}}-e^{-\frac{j\omega}{2}}}\right\}\left(\frac{e^{-\frac{j\omega M}{2}}}{e^{-\frac{j\omega}{2}}}\right)$
- Using the Euler's formula
- $Sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}, \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$
- We get
- $H(e^{j\omega}) = \frac{1}{M}\left\{\frac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}\right\} e^{-\frac{j\omega}{2}(M-1)}$
- $H(e^{j\omega}) = \frac{1}{M}\left\{\frac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}\right\}\left(\cos\left(\frac{\omega(M-1)}{2}\right) - j\sin\left(\frac{\omega(M-1)}{2}\right)\right)$
- The magnitude response is given by
- $|H(e^{j\omega})| = \sqrt{\frac{1}{M^2}\left\{\frac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}\right\}^2 (\cos^2\phi + \sin^2\phi)}$
- $|H(e^{j\omega})| = \frac{1}{M}\left\{\frac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}\right\}$
- Where, $\phi = \frac{\omega(M-1)}{2}$

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems:

Frequency Response Computation Using MATLAB

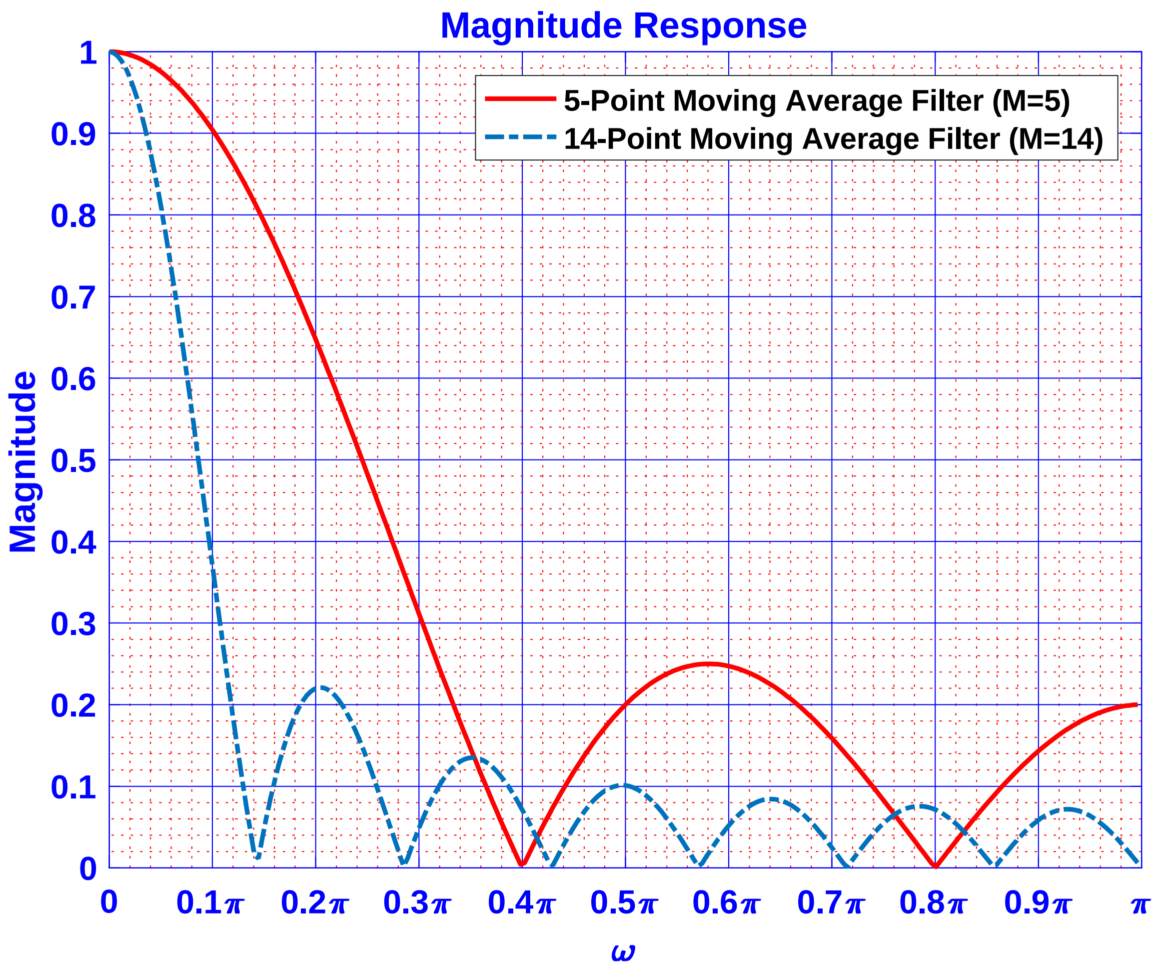
- The phase response is given by
- $\theta(\omega) = \arctan\left[\dfrac{\frac{1}{M}\left\{\dfrac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}\right\}\cos\phi}{-\frac{1}{M}\left\{\dfrac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}\right\}\sin\phi}\right]$
- $\theta(\omega) = \arctan\left[-\dfrac{\cos\phi}{\sin\phi}\right]$
- $\theta(\omega) = \arctan[-\tan\phi]$
- The arctan ($\tan^{-1}$) and tan cancels each other
- $\theta(\omega) = -\phi$
- Where, $\phi = \dfrac{\omega(M-1)}{2}$

- The phase response is given by
- $\theta(\omega) = -\dfrac{\omega(M-1)}{2}$
- The magnitude response is given by
- $|H(e^{j\omega})| = \dfrac{1}{M}\left\{\dfrac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}\right\}$

Discrete-Time Systems

Frequency-Domain Representation of LTI Discrete-Time Systems:

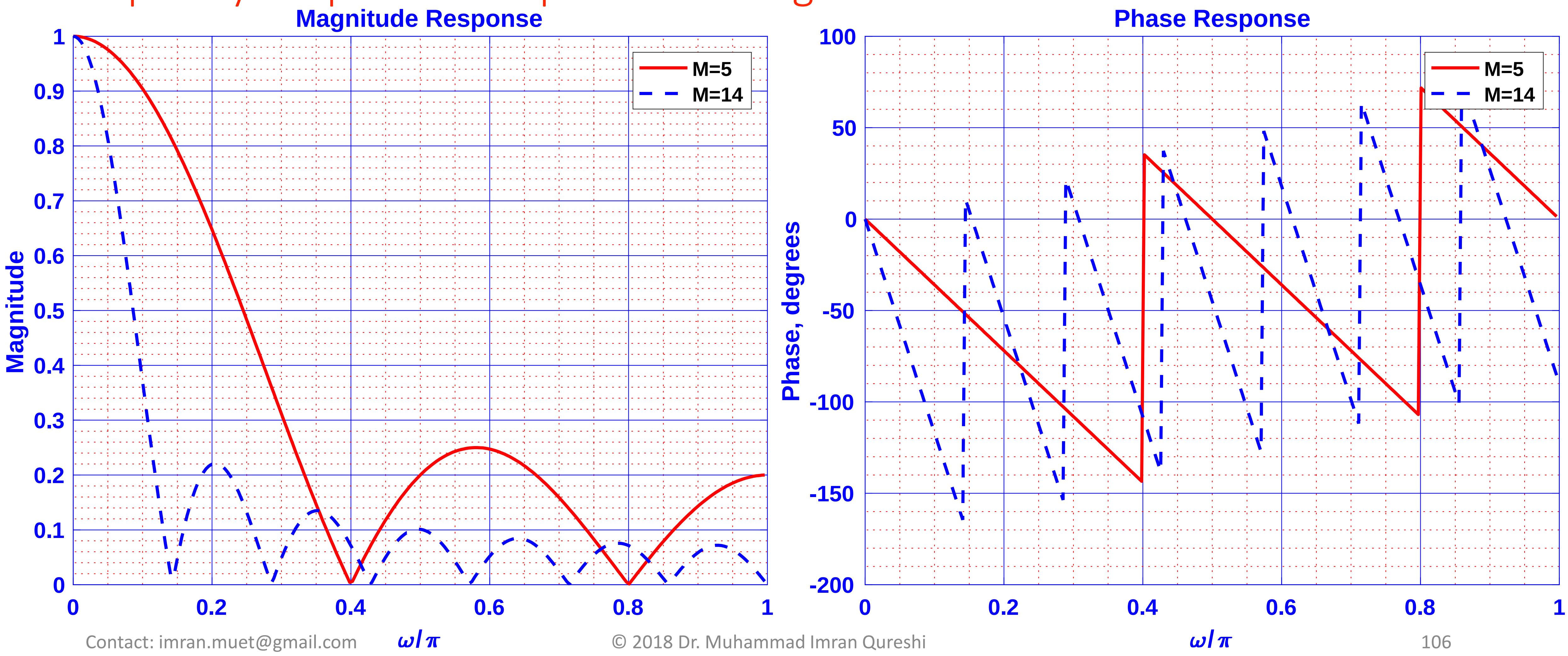
Frequency Response Computation Using MATLAB



Discrete-Time Systems

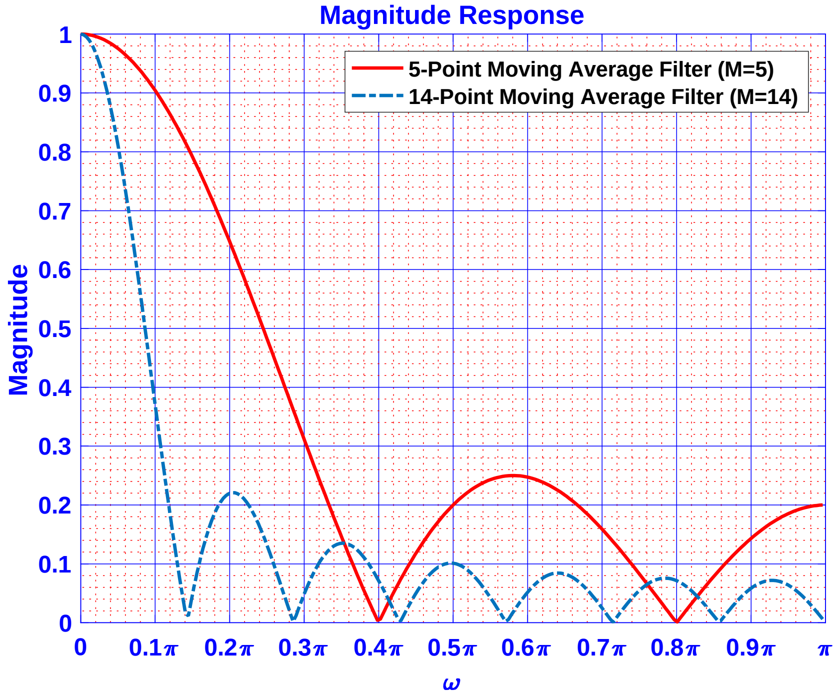
Frequency-Domain Representation of LTI Discrete-Time Systems:

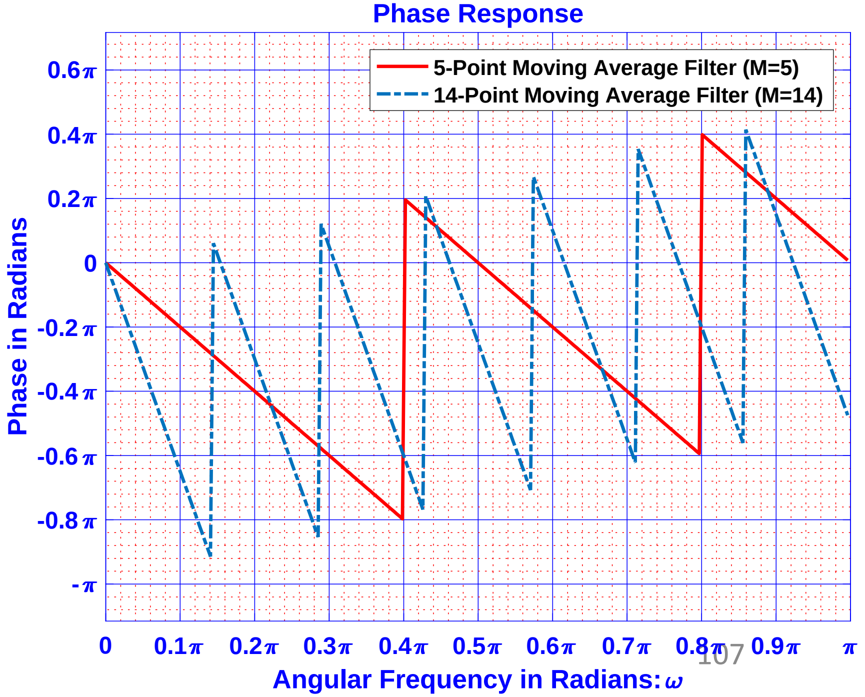
Frequency Response Computation Using MATLAB



Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: Frequency Response Computation Using MATLAB

- In the range $\omega = 0$ to $\omega = \pi$, the magnitude has a maximum value of unity at $\omega = 0$, and has zero values at $\omega = \frac{2\pi k}{M}$ with $k = 1,2, \dots \left\lfloor \frac{M}{2} \right\rfloor$.
- The phase function exhibits discontinuities of π at each zero of $H(e^{j\omega})$ and is linear elsewhere with a slope of $-\frac{M-1}{2}$.
- Both the magnitude and phase functions are periodic in ω with a period of 2π.





Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: Steady-State and Transient Responses

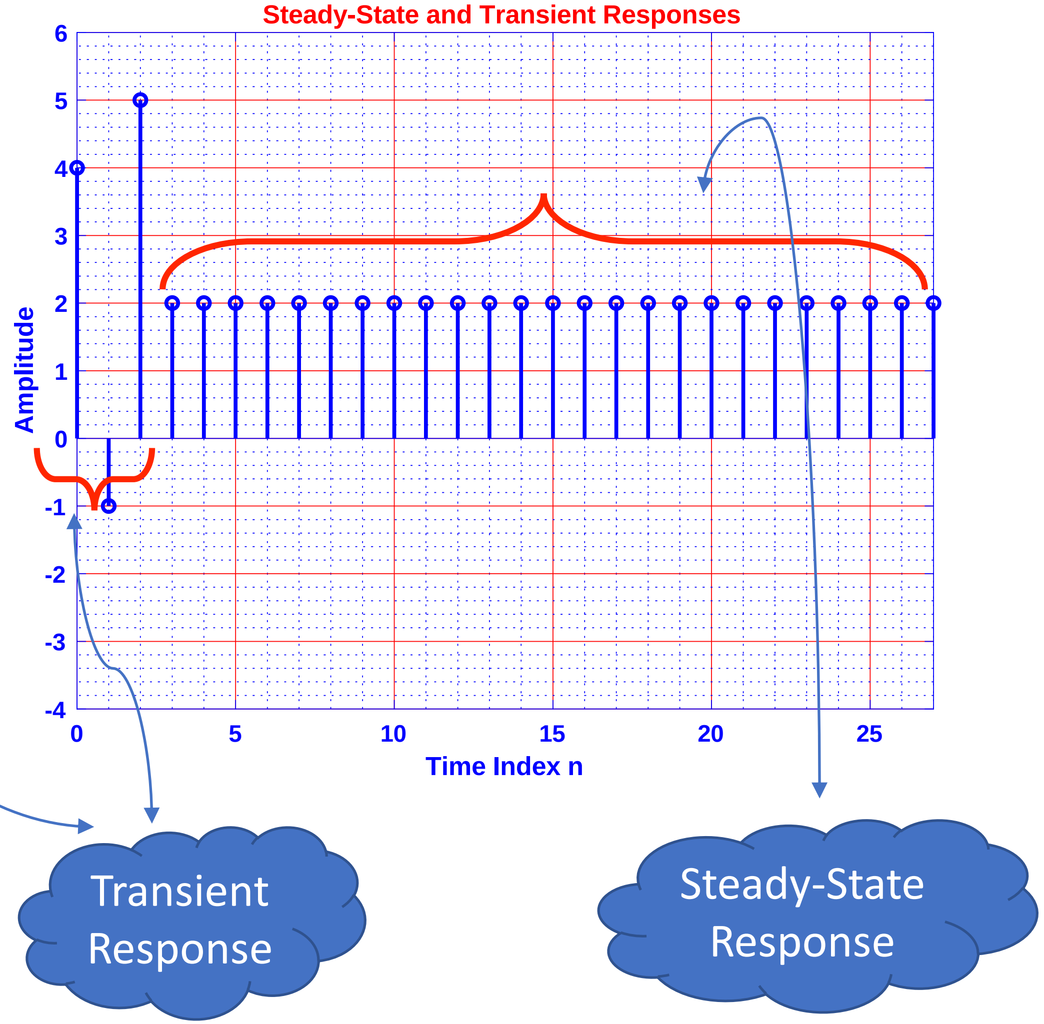
- The complementary solution $y_c[n]$ is of the form
- $y_c[n] = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_N\lambda_N^n$
- Where $\alpha_1, \alpha_2, \dots, \alpha_N$ are constants determined from the specified initial conditions of the discrete-time systems.
- $\lambda_1, \lambda_2, \dots, \lambda_N$ are the roots of the characteristic polynomial.
- For a stable system, $|\lambda_i| < 1$, and as a result, the complementary solution $y_c[n]$ decays to zero for a very large values of n.
- The complementary solution part $y_c[n]$ of the output is also referred to as the transient response.

Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: Steady-State and Transient Responses

- In order to understand the steady-state response consider an input sequence with a constant amplitude starting at some point, say n_0, and then continuing forever afterwards.
- The output of a causal stable LTI discrete-time system will then be composed of a steady-state response (the particular solution), which is also contant amplitude sequence and a transient response with zero-valued samples after some time instant $n_1 > n_0$, resulting in a constant amplitude output after the time instant n_1

Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: Steady-State and Transient Responses

- Example: A causal FIR discrete-time system characterized by an impulse response $h[n] = [4, -5, 6, -3], n \le 3$. Calculate its output $y[n]$ for a constant input $x[n] = \mu[n]$.
- Solution:
- The output can be computed using
- $y[n] = 4x[n] - 5x[n-1] + 6x[n-2] - 3x[n-3]$
- For $n = 0$
- $y[0] = 4x[0] = 4$
- $y[1] = 4x[1] - 5x[0] = -1$
- $y[2] = 4x[2] - 5x[1] + 6x[0] = 5$
- $y[3] = 4x[3] - 5x[2] + 6x[1] - 3x[0] = 2$
- $y[4] = 4x[4] - 5x[3] + 6x[2] - 3x[1] = 2$
- $y[5] = 4x[5] - 5x[4] + 6x[3] - 3x[2] = 2$



Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: Steady-State and Transient Responses

- Similarly, the output of a causal stable LTI system with

Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: Response to a Causal Exponential Sequence

- Consider an input that is causal exponential sequence applied at $n = 0$; that is,

$$x[n] = e^{j\omega n}\mu[n]$$

- Since $x[n] = 0$, for $n < 0$, we have $y[n] = 0$, for $n < 0$.
- For $n \geq 0$, the output response is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]e^{j\omega(n-k)}\mu[n-k]$$

$$= \left(\sum_{k=0}^{n} h[k]e^{-j\omega k}\right)e^{j\omega n} \quad as \quad \mu[n-k] = 0 \ \ for \ \ k > n.$$

$$y[n] = \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

Discrete-Time Systems: Frequency-Domain Representation of LTI

Discrete-Time Systems: Response to a Causal Exponential Sequence

$$y[n] = \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

$$y[n] = H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

- The first term is called the steady-state response is given by

$$y_{sr}[n] = H\left(e^{j\omega}\right)e^{j\omega n}$$

- The second term is called the transient response and is given by

$$y_{tr}[n] = -\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: Response to a Causal Exponential Sequence

- For a causal and stable IIR LTI discrete-time system, the impulse response is absolutely summable, and as a result, the transient response $y_{tr}[n]$ is a bounded sequence; that is,

$$|y_{tr}[n]| = \left| \sum_{k=n+1}^{\infty} h[k]e^{-j\omega(n-k)} \right| \leq \sum_{k=n+1}^{\infty} |h[k]| \leq \sum_{k=n+1}^{\infty} |h[k]|$$

- Moreover, as $n \to \infty$, $\sum_{k=n+1}^{\infty} |h[k]| \to 0$, and hence, the transient response decays to zero as n gets very large.
- In most cases, transient response becomes negligibly small after some finite amount of time.
- It should be noted that transients will ocuur whenever an input signal is applied or changed.

Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: The Concept of Filtering

- **Digital Filter**: An LTI discrete-time system that passes certain frequency components in an input sequence without any distortion and block other frequency components.
- By appropriating choosing the magnitude function of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these sinusoidal sequences can be selectively heavily attenuated or filtered with respect ot the others.
- To understand the mechanism behind the design of such a system, consider a ral coefficient LTI discrete-time system characterized by a magnitude function

$$\left|H(e^{j\omega})\right| = \begin{cases} 1, & 0 \le |\omega| \le \omega_c, \\ 0, & \omega_c < |\omega| < \pi. \end{cases} \qquad (4.90)$$

- We apply an input $x[n] = A\cos\omega_1 n + B\cos\omega_2 n$ to this system, where $0 < \omega_1 < \omega_c < \omega_2 < \pi$.
- The output $y[n]$ of this system is of the form

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1)) + B\left|H(e^{j\omega_2})\right|\cos(\omega_2 n + \theta(\omega_2))$$

- Making use of Eq. (4.90), we get

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$

- Indicating the LTI discrete-time system reject ω_2 frequencies and acts like a lowpass filter.

Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: The Concept of Filtering

- Example 4.33: An FIR filter of length 3 is defined by a symmetric impulse response, i.e., $h[0] = h[2]$. Let the input to be a sum of two cosine sequences of angular frequencies $0.1\ \ rad/samples$, and $0.4\ rad/samples$, respectively. Determine the impulse response coefficients so that the filter passes only the high-frequency component of the input.
- Solution:
- Step 1: Since the filter is symmetric and of length 3, let the impulse response would be
- $h[0] = h[2] = \alpha_0, \quad h[1] = \alpha_1$
- Step 2: The digital filter is applied using the following equation
- $y[n] = \sum_{k=0}^{2} h[k]x[n-k]$
- $y[n] = h[0]x[0] + h[1]x[n-1] + h[2]x[n-2]$
- $y[n] = \alpha_0 x[0] + \alpha_1 x[n-1] + \alpha_0 x[n-2] \quad (4.92)$

- Please Note: The task here is to find the values of α_0 and α_1 such that the filter output contains the cosine waveform of frequency $0.4\ rad/samples$ only.
- Step 3: The frequency response of above FIR filter is obtained by using Eq. (4.77)
- $H\left(e^{j\omega}\right) = \sum_{k=-N_1}^{N_2} h[k]e^{-j\omega k} \ldots (4.77)$
- $H\left(e^{j\omega}\right) = \sum_{k=0}^{2} h[k]e^{-j\omega k}$
- $H\left(e^{j\omega}\right) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$
- $H\left(e^{j\omega}\right) = \alpha_0 + \alpha_1 e^{-j\omega} + \alpha_0 e^{-j2\omega}$

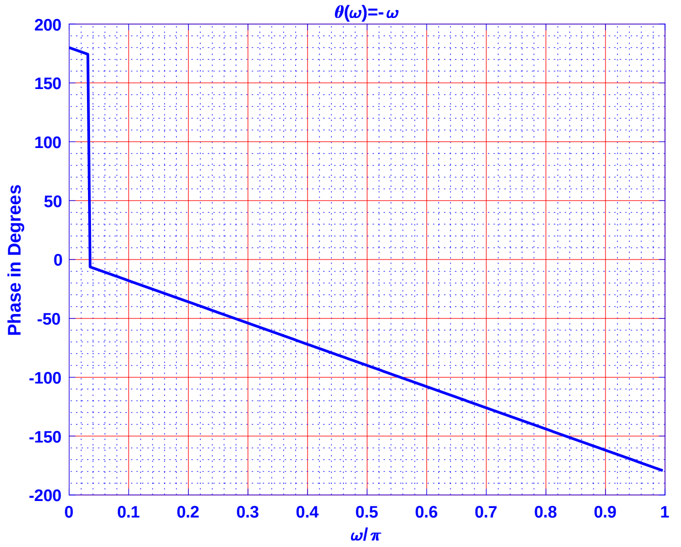
Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: The Concept of Filtering

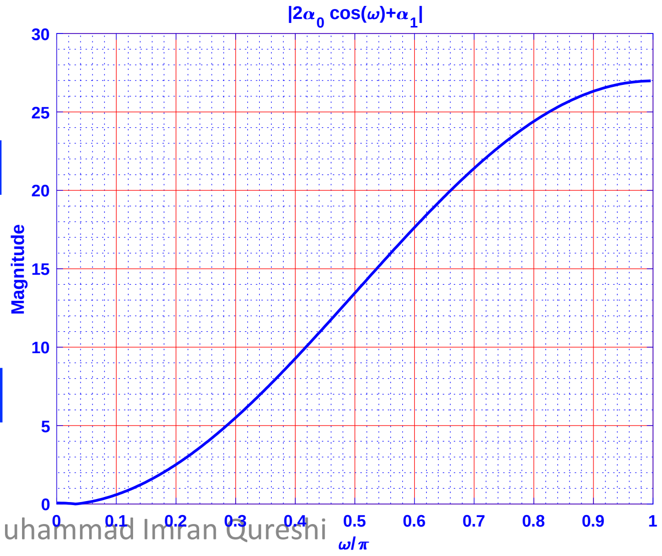
- $H(e^{j\omega}) = \alpha_0 + \alpha_1 e^{-j\omega} + \alpha_0 e^{-j2\omega}$
- $H(e^{j\omega}) = 2\alpha_0 \left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right) e^{-j\omega} + \alpha_1 e^{-j\omega}$
- $H(e^{j\omega}) = 2\alpha_0 \cos\omega\, e^{-j\omega} + \alpha_1 e^{-j\omega}$
- $H(e^{j\omega}) = (2\alpha_0 \cos\omega + \alpha_1) e^{-j\omega}$
- $H(e^{j\omega}) = (2\alpha_0 \cos\omega + \alpha_1)\cos\omega - j(2\alpha_0 \cos\omega + \alpha_1)\sin\omega$
- Step 4: Calculate the magnitude response
- $|H(e^{j\omega})| = |2\alpha_0 \cos\omega + \alpha_1| \ldots (4.94)$
- Step 4: Calculate the phase response
- $\theta(\omega) = -\omega + \beta \ldots (4.95)$
- Where $\beta = 0$ when $2\alpha_0 \cos\omega + \alpha_1 > 1$ and $\beta = \pi$ when $2\alpha_0 \cos\omega + \alpha_1 < 1$
- Step 5: In order to stop the low-frequency component, the magnitude should be zero at frequency $\omega = 0.1\ \mathrm{rad/samples}$
- $2\alpha_0 \cos\omega + \alpha_1 = 0 \ldots (\mathrm{a})$
- Step 6: In order to pass the high-frequency component without any distortion, the magnitude function at $\omega = 0.4\ rad/samples$ should be equal to 1; that is,
- $2\alpha_0 \cos\omega + \alpha_1 = 1 \ldots (\mathrm{b})$

Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: The Concept of Filtering

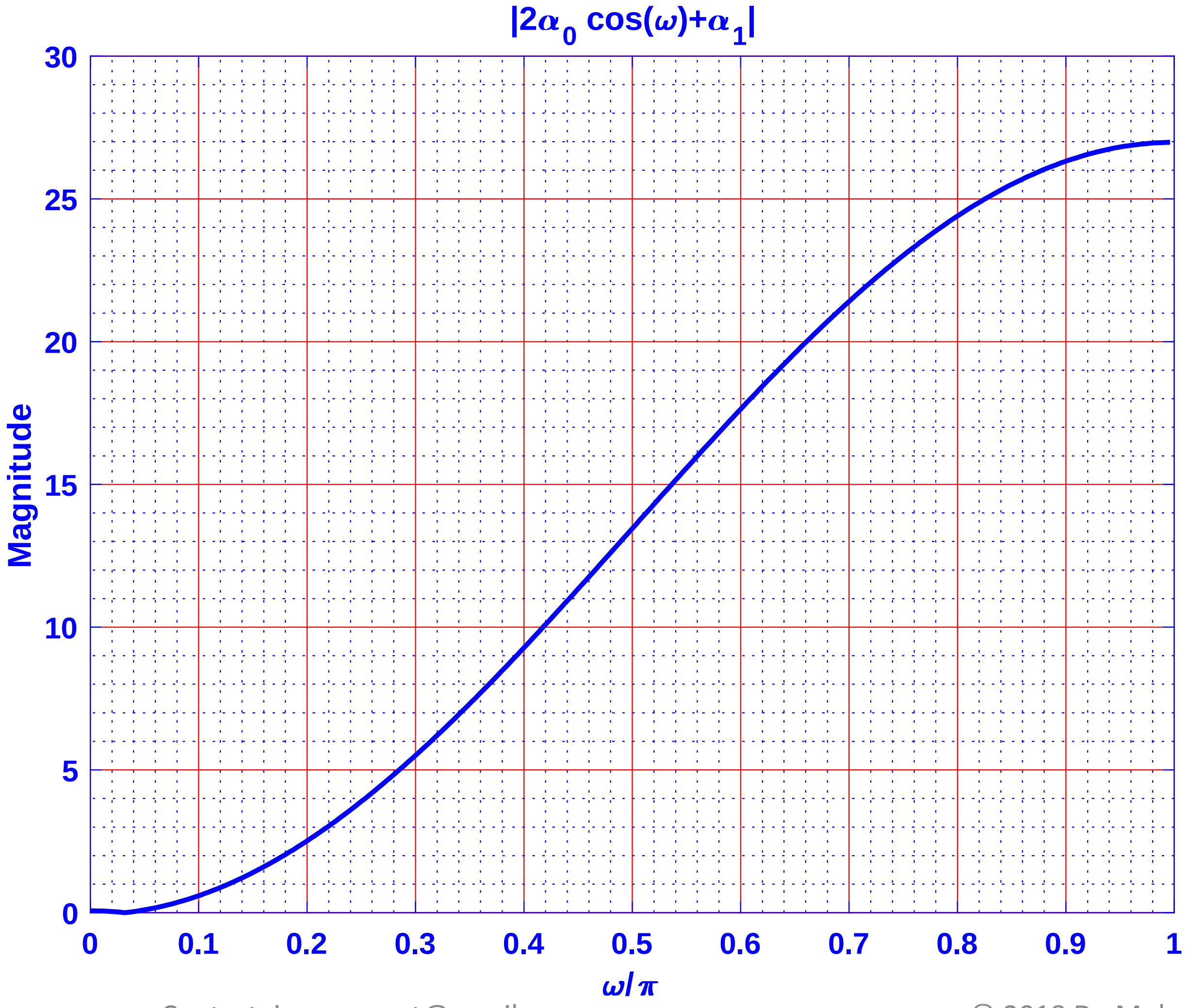
- MATLAB Code for solving Eq. (a) and (b) is as follows:
- >> syms a0 a1 w
- >> [a0 a1]=solve(2*a0*cos(0.1)+a1==0, 2*a0*cos(0.4)+a1==1);
- By solving Eq. (a) and (b), we get
- $\alpha_0 = -6.76195$ and $\alpha_1 = 13.456335$
- Put these values in Eq. (4.92)
- $y[n] = -6.76195x[0] + 13.456335x[n-1] - 6.76195x[n-2]$
- The impulse response is given by
- $h[n] = -6.76195h[0] + 13.456335h[n-1] - 6.76195h[n-2]$

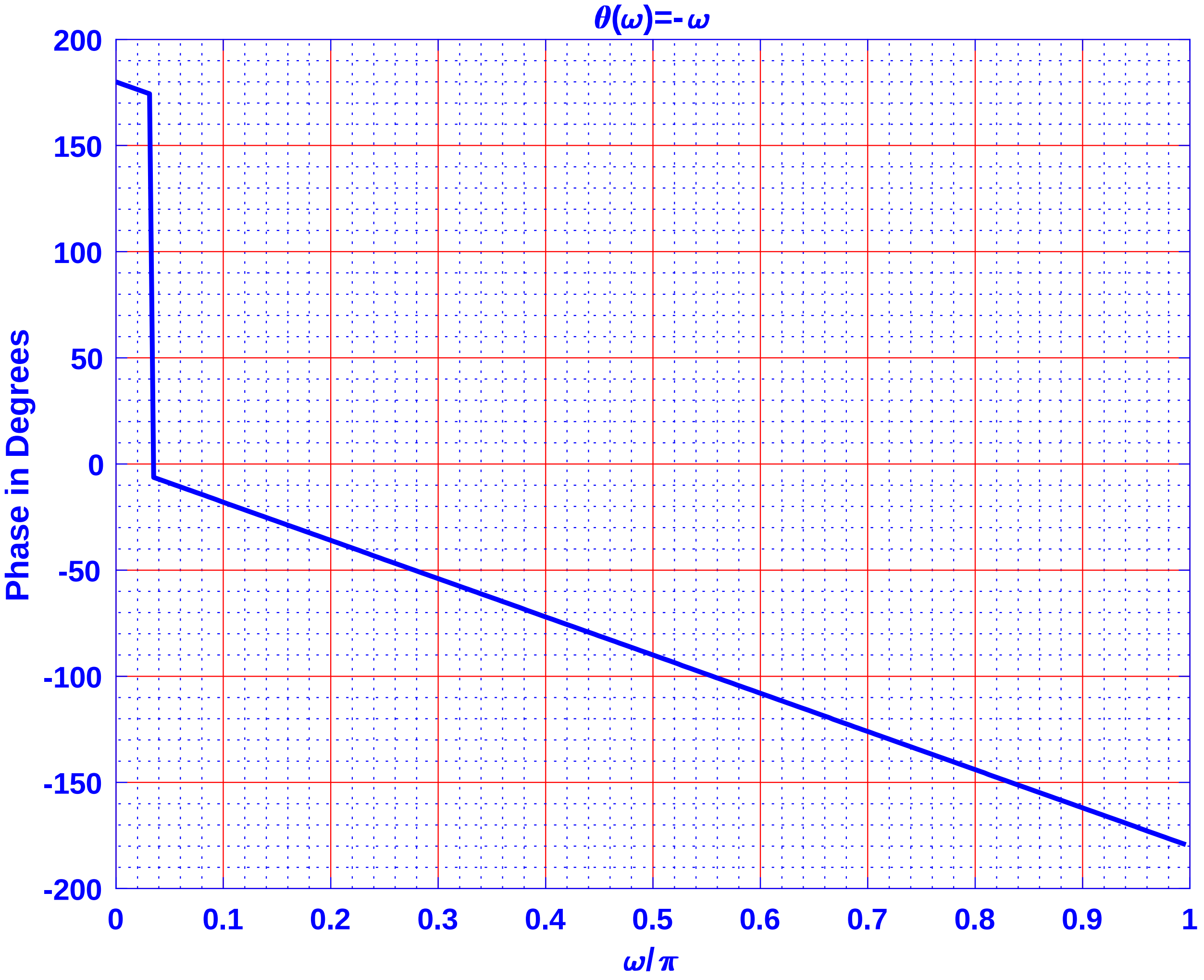
- The input is
- $x[n] = \{\cos(0.1n) + \cos(0.4n)\}\mu[n]$



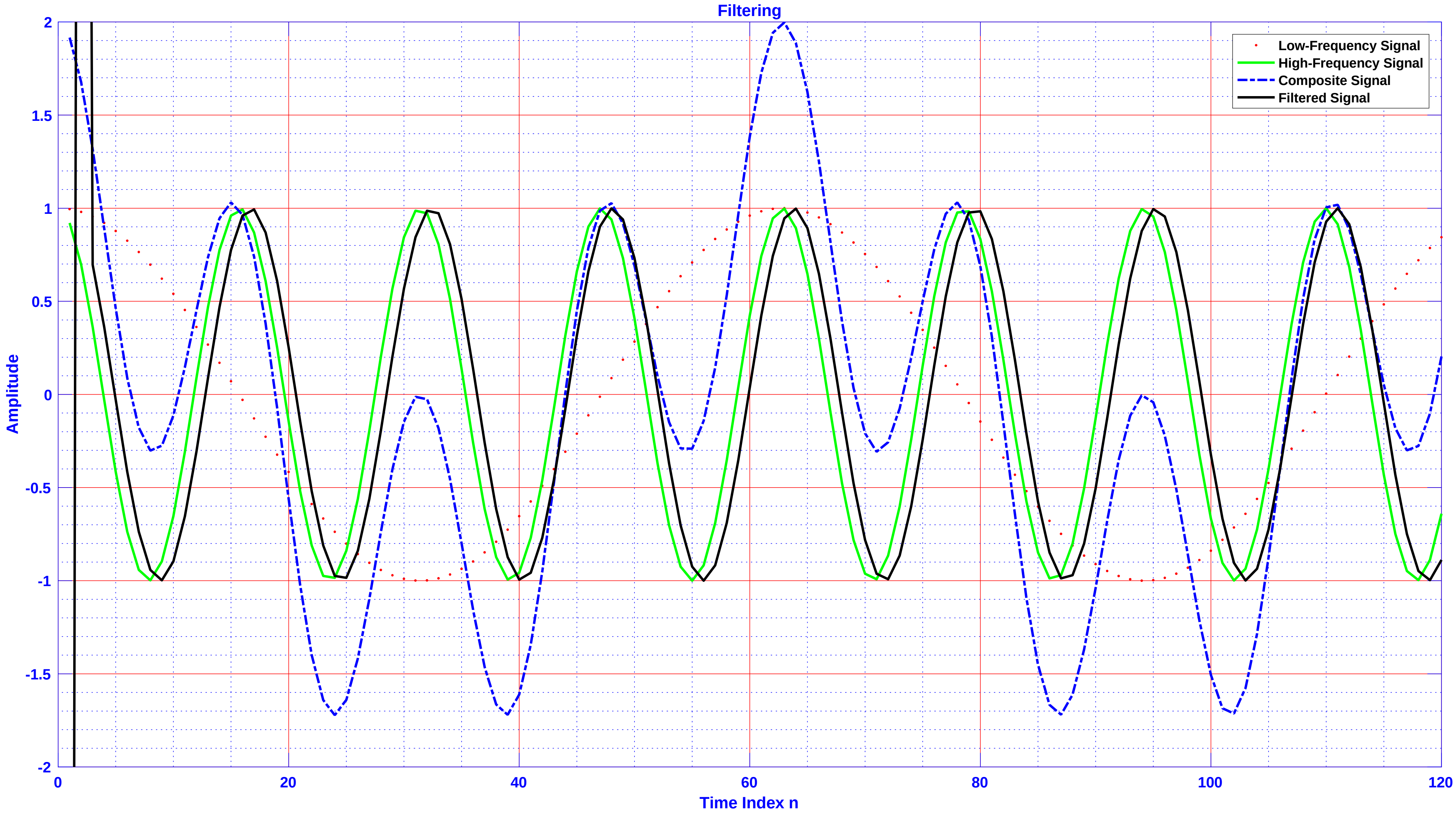


Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: The Concept of Filtering



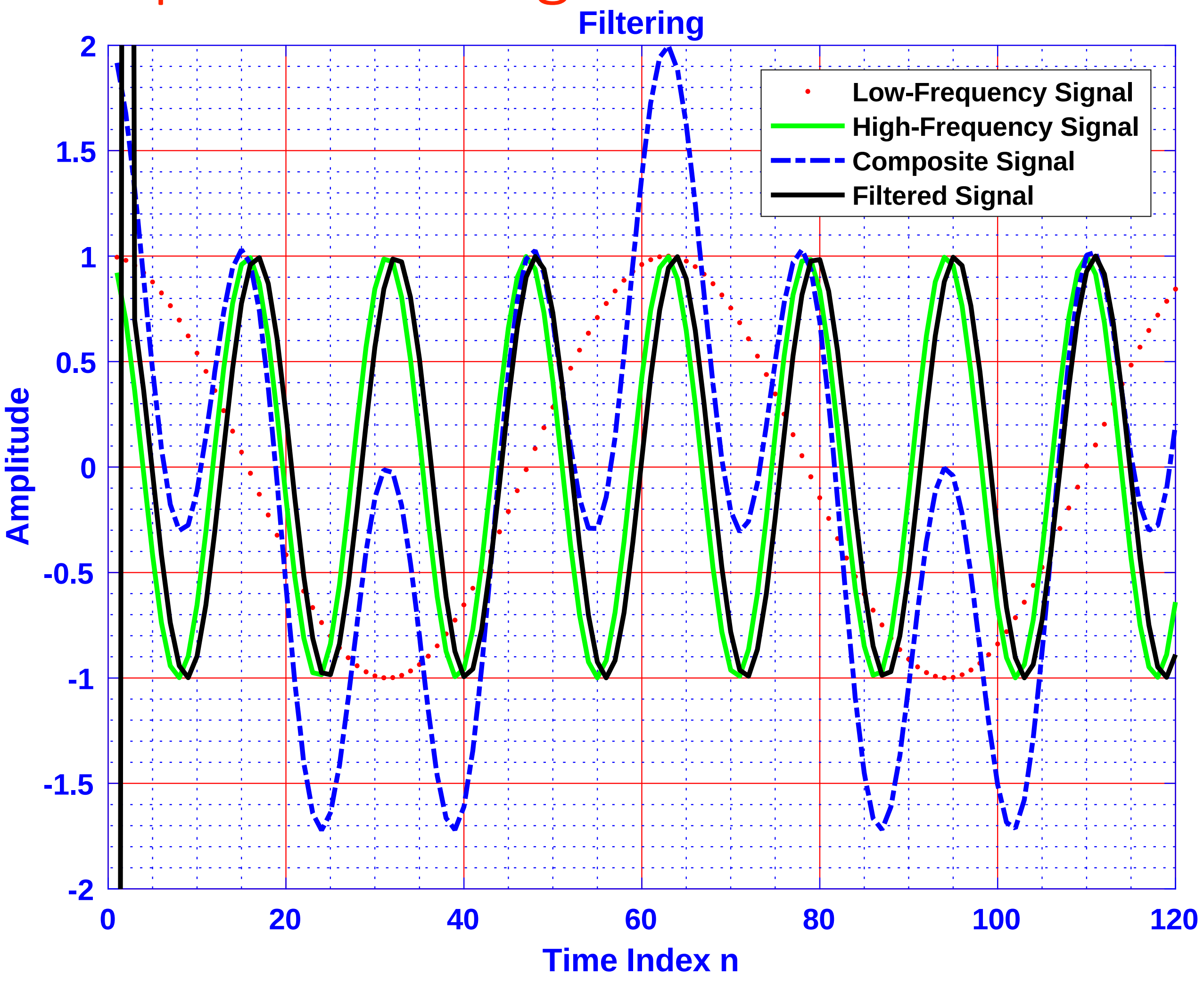


Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: The Concept of Filtering

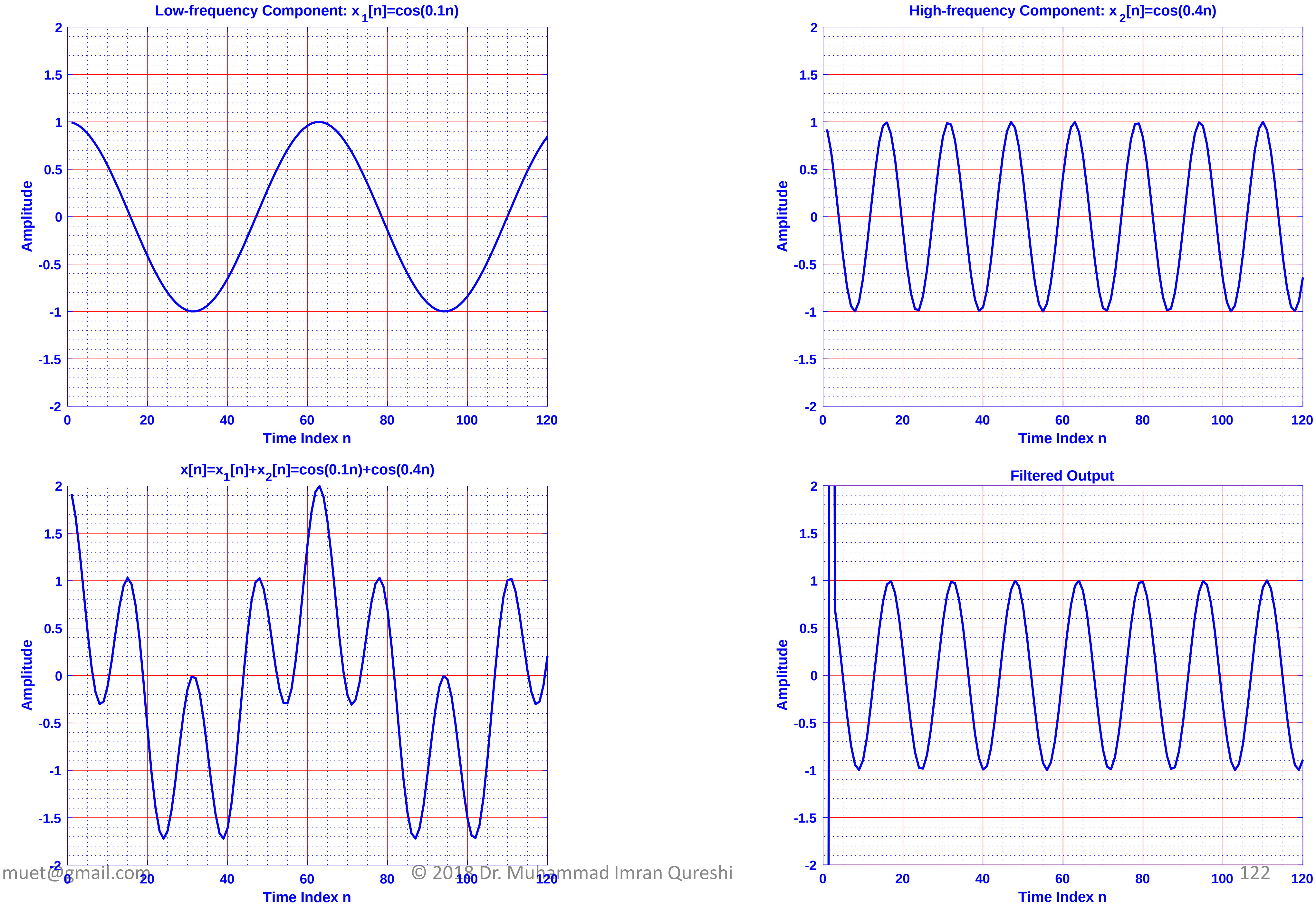


Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: The Concept of Filtering

- Note that in the computation of the present value of output requires the knowledge of the present and two previous input samples.
- Hence, the first two output samples are the result of the assumed zero inputs sample values at $n = -1$ and $n = -2$ and, therefore, include the transient part of the output in addition to the steady-state part.
- Since the impulse response is of length $N + 1 = 3$, the steady-state is reached at $n = N = 2$.
- In addition, the output is delayed version of the high-frequeny component $\cos(0.4n)$ of the input, and the delay is one sample period.



Discrete-Time Systems: Frequency-Domain Representation of LTI Discrete-Time Systems: The Concept of Filtering



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Discrete-Time Systems: Phase and Group Delays

- There are two important additional parameters that characterize the form of the output response $y[n]$ of an LTI discrete-time system excited by an input signal $x[n]$ composed of a weighted linear combination of sinusoidal sequences.
- These two parameters are associated with the frequency response $H\left(e^{j\omega}\right)$ of the system.
- The steady-state response of a stable LTI system for a sinusoidal input has the same form as the input except it suffers a change in its magnitude determined by the value of the magnitude function $\left|H(e^{j\omega_0}\right|$ of the LTI system ot the frequency ω_0 of the input sinusoidal signal, and a phase difference relative to the input signal phase by an amount given by the value of the phase function $\theta(\omega_0) = \arg\left\{H\left(e^{j\omega_0}\right)\right\}$ of the LTI system at ω_0.
- For a narrow-band input signal, we can assume that the magnitude functionis essentially constant at all frequencies of the constituent sinusoidal signals comprising the input, and only the phase of each term in the output response relative to the phase of its corresponding components in the input affects the behavior of the output signal.

Discrete-Time Systems: Phase and Group Delays: Phase Delay

- If the input is a sinusoidal signal of frequency ω_0 is given by

$$x[n] = A\cos(\omega_0 n + \phi) \qquad -\infty < n < \infty$$

- The output is also a sinusoidal signal of the same frequency ω_0 but lagging in phase by $\theta(\omega_0)$ as determined by the following equation

$$y[n] = A\left|H(e^{j\omega})\right|\cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \phi\right)$$

$$y[n] = A\left|H(e^{j\omega})\right|\cos\left(\omega_0\left(n - \tau_p(\omega_0)\right) + \phi\right) \ldots (4.99)$$

- Where

$$\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0} \ldots (4.100)$$

The minus sign indicates phase lag

Discrete-Time Systems: Phase and Group Delays: Group Delay

- When the input signal contains many sinusoidal components with different frequencies that are not harmonically related, each component goes through a different phase delays when processed by a frequency selective LTI discrete-time system, and the signal delay is is called group delay and is defined below:

$$\tau_g(\omega_0) = -\frac{d\theta(\omega)}{d\omega} \dots (4.101)$$

- It has been assumed that the phase function is unwrapped so that its derivative exists.

Discrete-Time Systems: Phase and Group Delays

- Example 4.34: Find the phase and group delay defined by $y[n] = -x[n-1]$ with unwrapped phase response is given by

$$\theta(\omega) = -\omega + \pi + 2K\pi$$

- Soltution:
- The phase delay is obtained as

$$\tau_p(\omega_0) = -1 - \frac{\pi}{\omega_0} + \frac{2K\pi}{\omega_0}$$

- The group delay is given by

$$\tau_g(\omega_0) = -\frac{d\theta(\omega)}{d\omega} = 1$$

Discrete-Time Systems: Phase and Group Delays

- Group delay of a high-frequency filter, whos phase function is given by

$$\theta(\omega) = -\omega + \beta$$

- Soltution:
- The group delay is given by

$$\tau_g(\omega_0) = -\frac{d\theta(\omega)}{d\omega} = 1$$

- It can be seen that there is a phase delay of 1 sample

- Group delay of a M-point moving average filter, whos phase function is given by

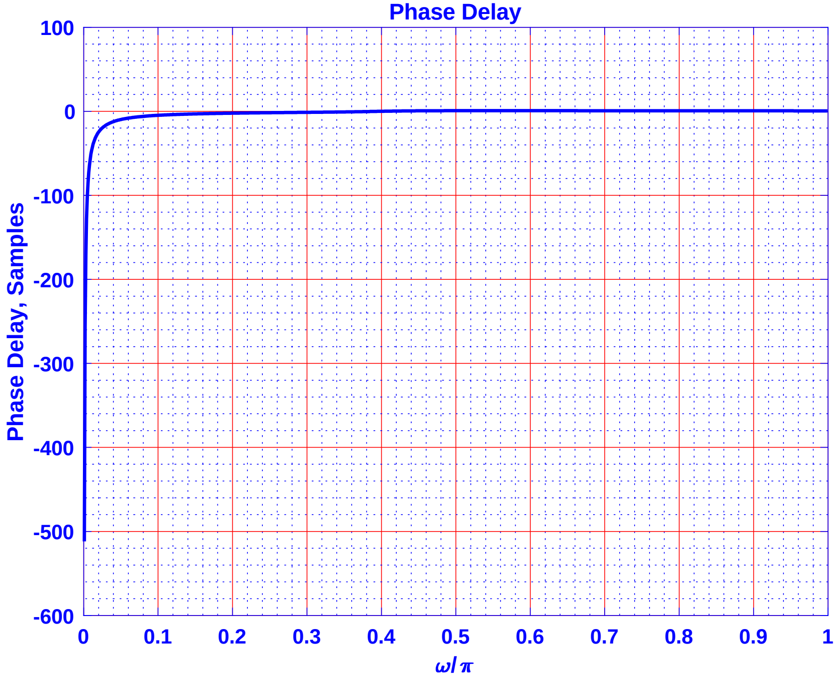
$$\theta(\omega) = -\frac{\omega(M-1)}{2}$$

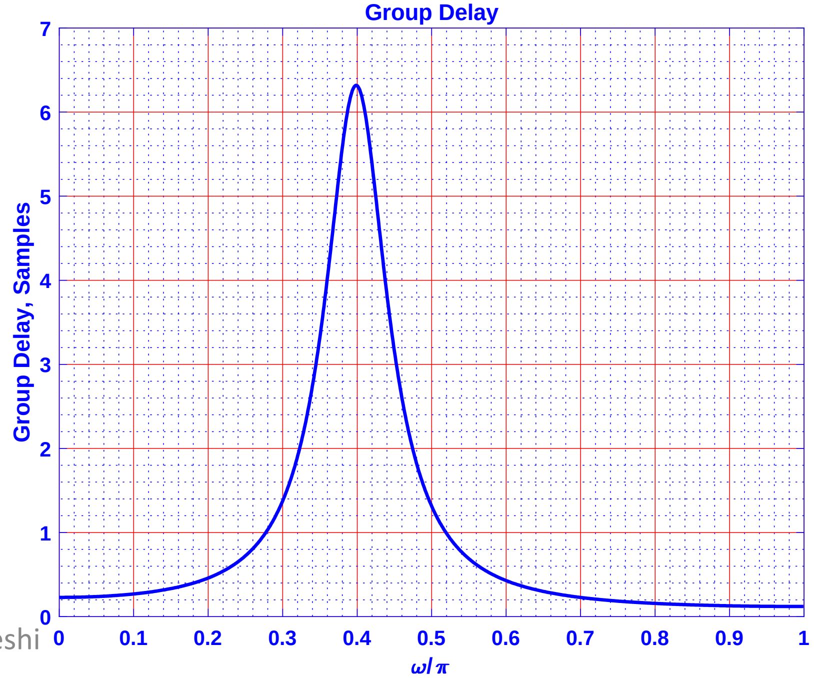
- Soltution:
- The group delay is given by

$$\tau_g(\omega_0) = \frac{M-1}{2}$$

Discrete-Time Systems: Phase and Group Delays Computation Using MATLAB

- Find phase delay and group delay of the digital filter characterized by a frequency response
- $\left|H\left(e^{j\omega}\right)\right| = \dfrac{0.136728736\left(1-e^{-j2\omega}\right)}{1-0.53353098e^{-j\omega}+0.726542528e^{-j2\omega}}$
- MATLAB code
- clear; close all; clc
- num = 0.136728736*[1, 0, -1];
- den = [1 -0.53353098 0.726542528];
- [phi, w] = phasedelay(num, den, 1024);
- [phi, w] = grpdelay(num, den, 1024);





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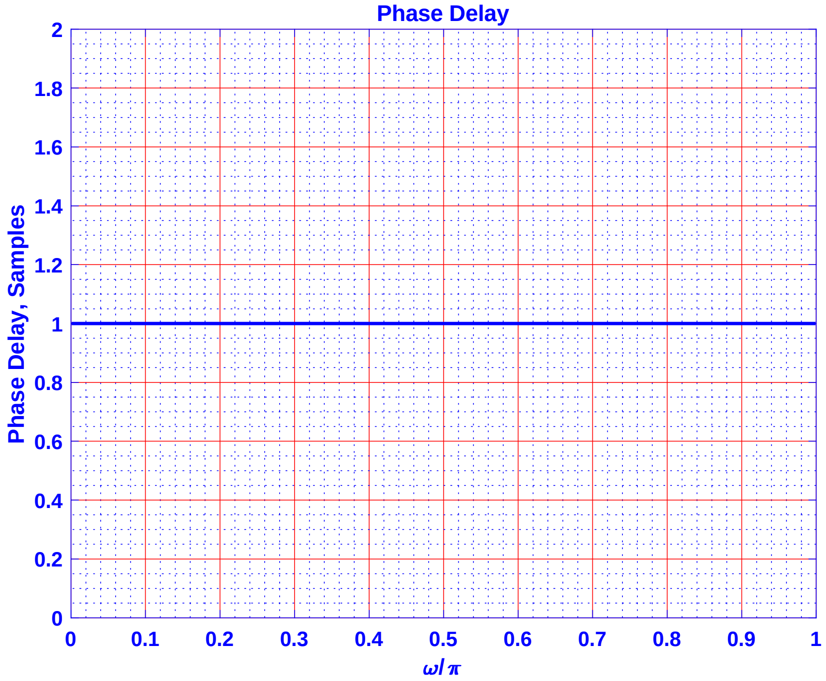
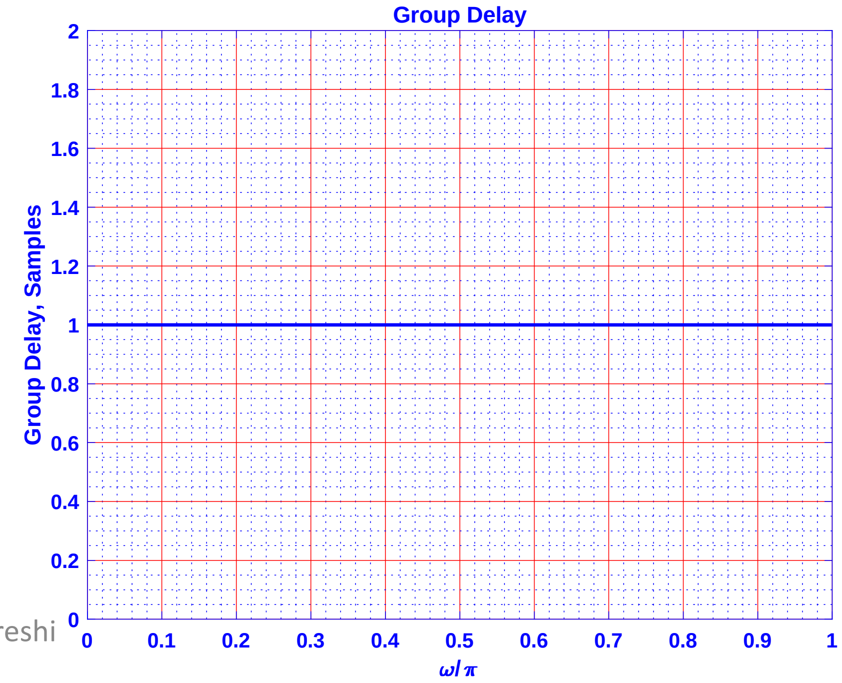
Discrete-Time Systems: Phase and Group Delays Computation Using MATLAB

- Find phase delay and group delay of the digital filter (discussed few slides back) characterized by a frequency response
- $h[n] = -6.76195h[0] + 13.456335h[n-1] - 6.76195h[n-2]$
- MATLAB code
- clear; close all; clc
- num = [-6.76195, 13.456335, -6.76195];
- den = 1;
- [phi, w] = phasedelay(num, den, 1024);
- [phi, w] = grpdelay(num, den, 1024);

It is clear from group delay that the delay is of 1 sample for all frequency component, also evident from the graphs of pervious slides





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