Digital SIGNAL PROCESSING

Discrete-Time Signals in the Frequency Domain

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Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Definition)

- The frequency-domain representation of a continuous-time signal $x_a(t)$ is given by the continuous-time Fourier transform (CTFT):
- $X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt. \qquad (3.1)$
- The CTFT also referred as Fourier spectrum or sometimes simply spectrum of continuous-time signal.
- The continuous time signal can be recovered from tis CTFT $X_a(j\Omega)$ via the inverse Fourier transform
- $x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{-j\Omega t}d\Omega \qquad (3.2)$
- Where, Ω is real and dnotes the continuous-time angular frequency variable in radians per second.

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The Continuous-Time Fourier Transform (Definition)

- In Eq. (3.2), the inverse Fourier transform is a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi}e^{j\Omega t}d\Omega,$ weighted by the complex constant $X_a(j\Omega)$ over the angular frequency range form $-\infty\ to\ \infty.$
- $X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt.$ (3.1)
- Eq. (3.1) can be written in polar form as
- $X_a(j\Omega) = |X_a(j\Omega)|\mathrm{e}^{j\theta_a(\Omega)}$ where, $\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$
- The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum.
- The quantity $\theta_a(\Omega)$ is called phase spectrum.
- Both magnitude and phase spectrum are real function of Ω.

Discrete Time Signals in the Frequency Domain

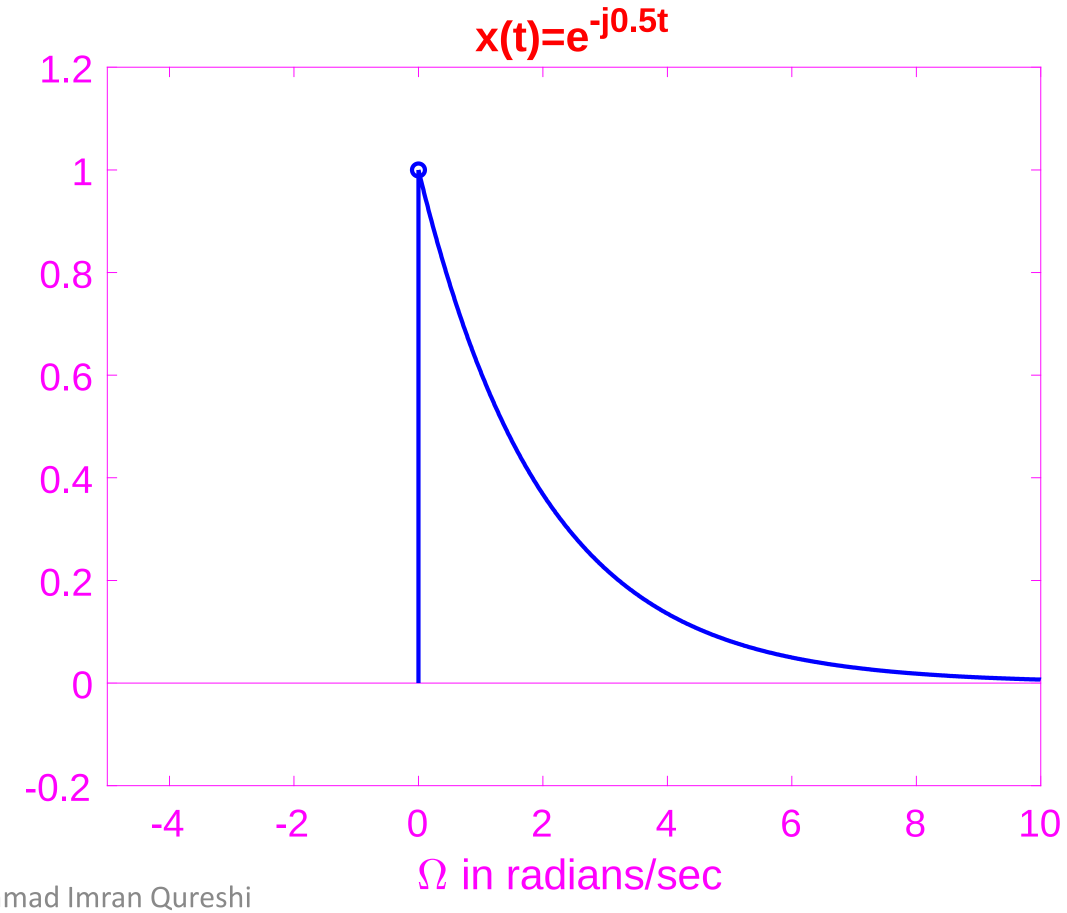
The Continuous-Time Fourier Transform (Definition)

- $X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt.$ (3.1)
- The CTFT $X_a(j\Omega)$ (defined in Eq. (3.1)) exists if the continuous-time signal $x_a(t)$ satisfies the Dirchlet conditions:
  1. The signal has a finite number of finite discontinuities and a finite number of maxima and minima in any finite interval.
  2. The signal is absolutely integrable; that is,
- $\int_{-\infty}^{\infty} |x_a(t)| dt < \infty$ (3.3)
- If the Dirichlet conditions are satisfied, the integral on the right-hand side of Eq. (3.2) converges to $x_a(t)$ at values of t where $x_a(t)$ has discontinuities.

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Examples)

- Example: Find CTFT of following real signal and also magnitude and phase spectra:
- $x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$
- where $0 < \alpha < \infty$



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The Continuous-Time Fourier Transform (Examples)

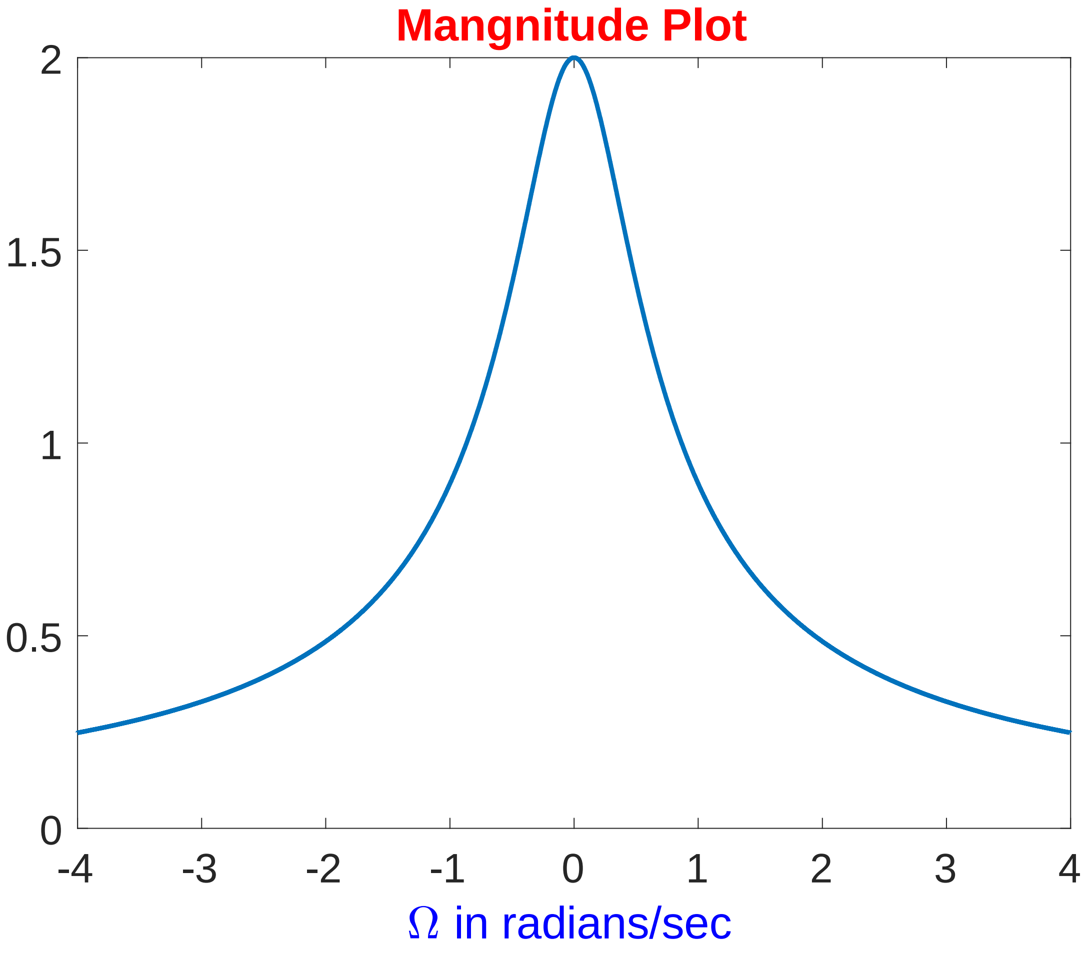
- Solution: Let $\alpha = 0.5$
- Before proceeding to solution, we will check Dirchlet condition (the function is absolutely integrable or not?)
- $\int_{\infty}^{\infty} |x_a(t)|dt = \int_{0}^{\infty} e^{-\alpha t} dt = -\frac{e^{-\alpha t}}{\alpha}|_0^{\infty} = \frac{1}{\alpha} < \infty$ (Integrable, hence CTFT is possible.
- $X_a(j\Omega) = \int_{\infty}^{\infty} x(t)e^{-j\Omega t} dt = \int_{0}^{\infty} e^{-\alpha t}e^{-j\Omega t} dt = \int_{0}^{\infty} e^{-(\alpha+j\Omega)t} dt$
- $X_a(j\Omega) = \left| -\frac{e^{-(\alpha+j\Omega)t}}{\alpha+j\Omega} \right|_0^{\infty} = \frac{1}{\alpha+j\Omega}$

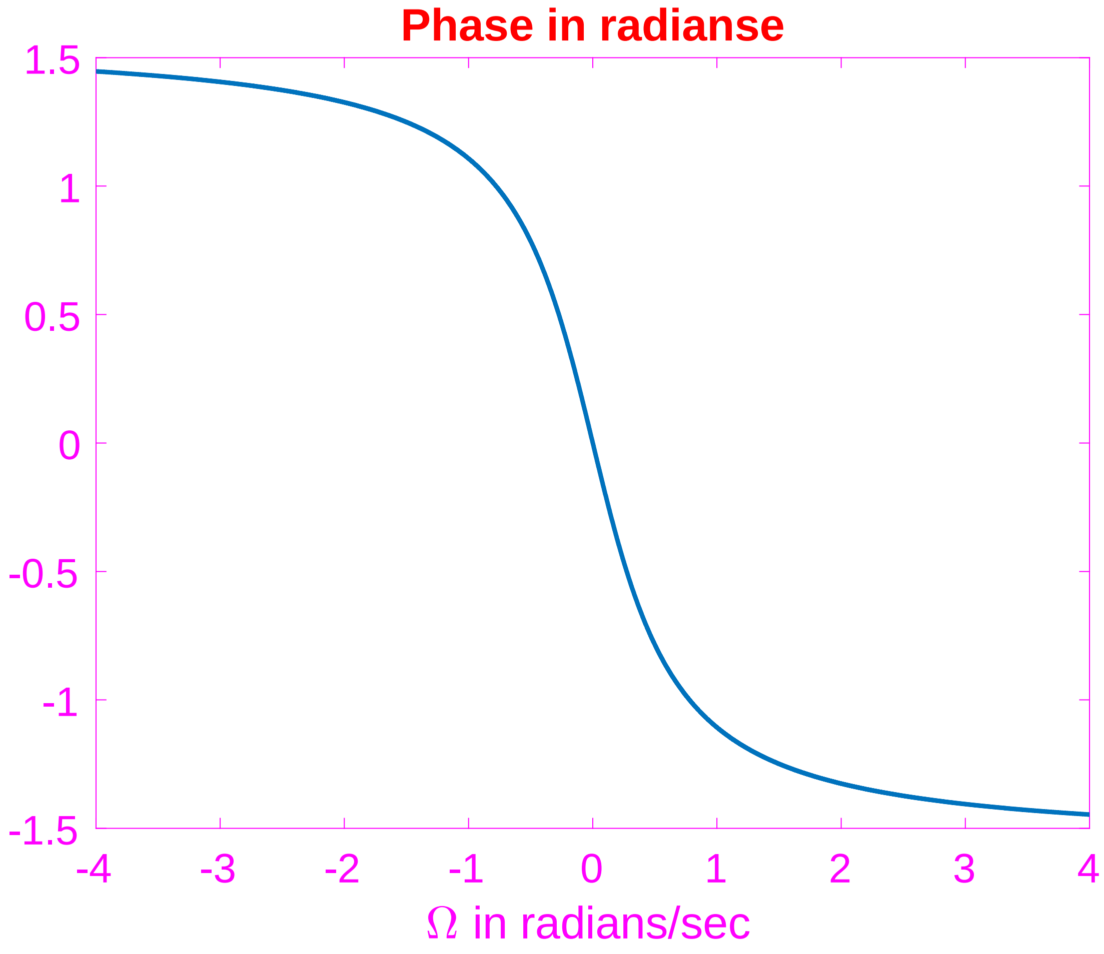
Magnitude: $|X_a(j\Omega) = \frac{1}{\sqrt{\alpha^2+\Omega^2}}$

Phase: $\theta_a(\Omega) = -\tan^{-1}\left(\frac{\Omega}{\alpha}\right)$

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Examples)





Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Examples)

- Example 3.2: Continuous-Time Fourier Transform of an Impulse Function
- Solution: Using the sampling property of the delta function, the CTFT $\Delta(j\Omega)$ of an ideal impulse $\delta(t)$ is obtained as:
- $\Delta(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = 1$

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Examples)

- Example 3.2: Continuous-Time Fourier Transform of a Shifted Impulse Function i.e. $x_a(t) = \delta(t - t_o)$
- Solution: The CTFT of a shifted impulse function is obtained as:
- $X_a(j\Omega) = \int_0^{\infty} \delta(t - t_o) e^{-j\Omega t} dt = e^{-j\Omega t_o}$
- $\Delta(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = 1$

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Important Points)

- An absolutely integrable continuous-time signal $x_a(t)$ with bounded amplitude always has finite energy; that is,
- $\int_0^\infty |x_a(t)|^2 dt < \infty \qquad (3.6)$
- The CTFT may exit for a finite-energy continuous-time signal that is not absolutely integrable.
- The CTFT can also be defined using ideal impulses for some functions that do not satisfy either integrable condition (i.e $Eq. (3.3)$) nor energy condition (i.e. $Eq. (3.4)$).

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Energy Density Spectrum)

- The total energy $\mathcal{E}_x$ of a finite-energy continuous-time complex signal $x_a(t)$ is given by
- $\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) dt \qquad (3.7)$
- From the definition of inverse CTFT, we know that,
- $x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$
- Taking complex conjugate of both side will lead us to
- $x_a^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} d\Omega$ put this equation in Eq. (3.7)
- $\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} d\Omega \right] dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) [\int_{\infty}^{\infty} x_a(t) e^{-j\Omega t} dt] d\Omega$
- $\mathcal{E}_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega \qquad (3.8)$
- $\int_{-\infty}^{\infty} |x_a(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega \qquad (3.9)$

$$\int_{-\infty}^{\infty} |x_a(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

This is known as Parseval's theorem for finite-energy continuous-time signals

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Energy Density Spectrum)

- Example 3.4: Find the total energy of given continuous-time signal
- $x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$
- where $0 < \alpha < \infty$
- Solution:
- As we have found its CTFT in Example 3.1, which is
- $X_a(j\Omega) = \frac{1}{\alpha + j\Omega}$
- So, the energy is given by
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega) X_a^*(j\Omega) d\Omega$
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{1}{\alpha + j\Omega}\right]\left[\frac{1}{\alpha - j\Omega}\right] d\Omega$
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \Omega^2} d\Omega$
- $\mathcal{E}_x = \frac{1}{2\pi}\left|\frac{1}{\alpha}\tan^{-1}\left(\frac{\Omega}{\alpha}\right)\right|_{-\infty}^{\infty}$
- $\mathcal{E}_x = \frac{1}{2\pi\alpha}\{\tan^{-1}(\infty) - \tan^{-1}(-\infty)\}$
- $\mathcal{E}_x = \frac{1}{2\pi\alpha}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\}$
- $\mathcal{E}_x = \frac{1}{2\alpha}$ (For $\alpha = 0.5$, $\mathcal{E}_x = 1$)

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Energy Density Spectrum)

- The total energy $\mathcal{E}_x$ of a finite-energy continuous-time complex signal $x_a(t)$ is given by
- $\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$ (3.9)
- The term $|X_a(j\Omega)|^2$ in above equation is called the energy density spectrum $\left(S_{xx}(\Omega)\right)$ of the continuous-time signal $x_a(t)$ i.e.,
- $S_{xx}(\Omega) = |X_a(j\Omega)|^2$
- The energy $\mathcal{E}_{x,r}$ over a specified range of frequencies $\Omega_a \le \Omega \le \Omega_b$ of the signal $x_a(t)$ is given by
- $\mathcal{E}_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega) d\Omega$

energy density spectrum $\left(S_{xx}(\Omega)\right)$ of the continuous-time signal $x_a(t)$ is given by
$S_{xx}(\Omega) = |X_a(j\Omega)|^2$

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Band-Limited Continuous-Time Signals)

- A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty < \Omega < \infty$.
- A band-limited signal, continuous-time signal has a spectrum occupying a limited portion of the frequency range $\Omega_a \leq \Omega \leq \Omega_b$.
- An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq \Omega \leq \Omega_b$: that is,
- $X_a(j\Omega) = \begin{cases} 0, & 0 \leq |\Omega| < \Omega_a \\ 0, & \Omega_b < |\Omega| < \infty \end{cases}$ i.e., the signal exisits only in range $\Omega_a \leq \Omega \leq \Omega_b$
- An ideal band-limited signal cannot be generated in practice.
- However, for practical purposes, it is ensured that the energy of band-limited signal outside the specified frequency range is sufficiently small.

Discrete Time Signals in the Frequency Domain

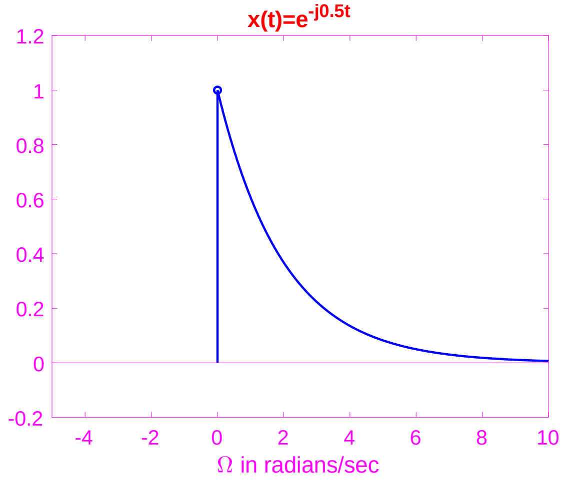
The Continuous-Time Fourier Transform (Band-Limited Continuous-Time Signals)

- **Lowpass Continuous-time Signal:**
  - A low pass continuous-time signal is defined as
  - $X_a(j\Omega) = \begin{cases} X_a(j\Omega), & 0 \le |\Omega| \le \Omega_p \\ 0, & \Omega_p < |\Omega| < \infty \end{cases}$ i.e., the signal exisits only in range $0 \le |\Omega| \le \Omega_p$
  - Where, Ω_p is called bandwidth of the signal and is less than ∞
- **Highpass Continuous-time Signal:**
  - A highpass continuous-time signal is defined as
  - $X_a(j\Omega) = \begin{cases} 0, & 0 \le |\Omega| < \Omega_p \\ X_a(j\Omega),, & \Omega_p \le |\Omega| < \infty \end{cases}$ i.e., the signal exisits only in range $\Omega_p \le |\Omega| < \infty$
  - The bandwidth of the signal is from $\Omega_p\ to\ \infty$
- **Bandpass Continuous-time Signal:**
  - A bandpass continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \le |\Omega| \le \Omega_H < \infty$
  - $X_a(j\Omega) = \begin{cases} 0, & 0 \le |\Omega| < \Omega_a \\ 0, & \Omega_b < |\Omega| < \infty \end{cases}$ i.e., the signal exisits only in range $\Omega_a \le \Omega \le \Omega_b$
  - The bandwidth of the signal is $\Omega_H - \Omega_L$.

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Band-Limited Continuous-Time Signals)

- Example 3.1: Find total energy fo the following signal for $\alpha = 0.5$, and determine its 80% bandwidth



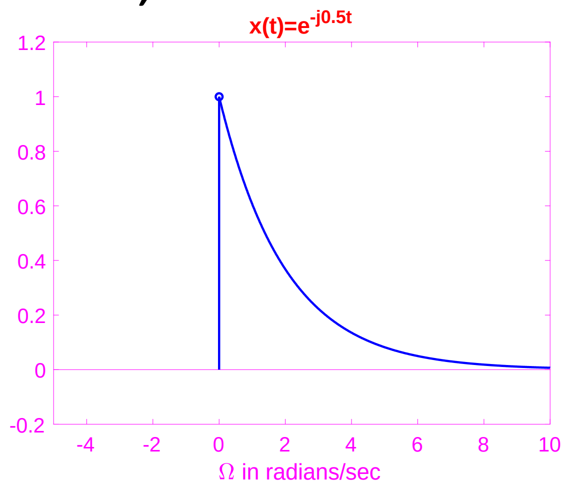
- $x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$
- Solution: (Method 1)
- The energy can be computed as:
- $\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_0^{\infty} |e^{-\alpha t}|^2 dt = \int_0^{\infty} e^{-2\alpha t} dt = \left| \frac{e^{-2\alpha t}}{-2\alpha} \right|_0^{\infty}$
- $\mathcal{E}_x = \left| \frac{e^{-2\alpha t}}{-2\alpha} \right|_0^{\infty} = \left(\frac{1}{-2 \times 0.5} \right) (0 - 1) = 1$

The total energy is 1, i.e., $\mathcal{E}_x = 1$

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Band-Limited Continuous-Time Signals)

- Example 3.1: Find total energy fo the following signal for $\alpha = 0.5$, and determine its 80% bandwidth



- $x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$
- Solution: (Method 2)
- The energy using Parseval's theorem can be computed as:
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X_a(j\Omega)|^2 d\Omega$ where, $X_a(j\Omega) = \frac{1}{\alpha + j\Omega}$ (calculated on slide 6)
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{\alpha+j\Omega}\right|^2 d\Omega = \frac{1}{2\pi}\int_{0}^{\infty}\frac{1}{\alpha^2+\Omega^2} d\Omega = 1$

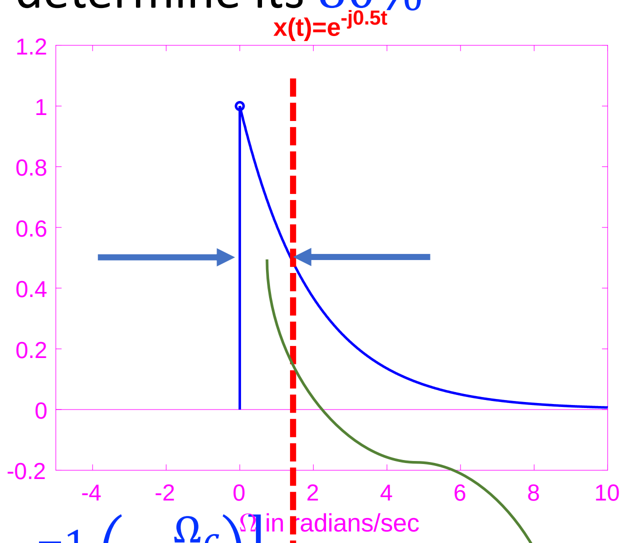
The total energy is 1, i.e.,
$\mathcal{E}_x = 1$

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Band-Limited Continuous-Time Signals)

- Example 3.1: Find total energy fo the following signal for $\alpha = 0.5$, and determine its 80% bandwidth
- $x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$

The bandwidth is
$0 \leq \Omega \leq 1.5388$ or
$0 \leq \Omega \leq 0.4898\pi$

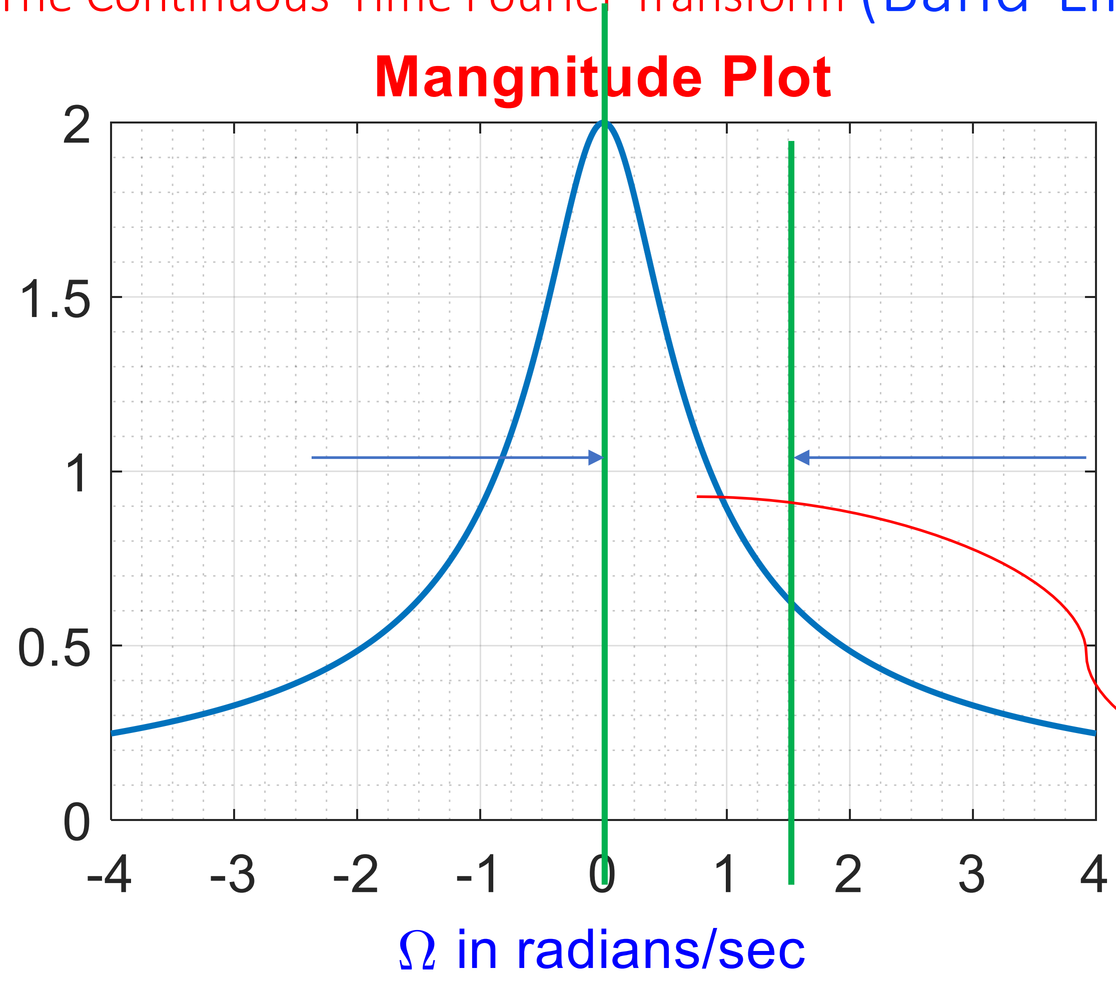


- Solution: (80% bandwidth)
- The 80% bandwidth Ω_c can be computed as:
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} \frac{1}{\alpha^2+\Omega^2} d\Omega$
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} \frac{1}{\alpha^2+\Omega^2} d\Omega = \frac{1}{2\pi}\left[\frac{1}{\alpha}\tan^{-1}\left(\frac{\Omega}{\alpha}\right)\right]_{-\Omega_c}^{\Omega_c} = \frac{1}{2\pi\alpha}\left[\tan^{-1}\left(\frac{\Omega_c}{\alpha}\right) - \tan^{-1}\left(-\frac{\Omega_c}{\alpha}\right)\right]$
- $\mathcal{E}_x = \frac{1}{2\pi\alpha}\left[\tan^{-1}\left(\frac{\Omega_c}{\alpha}\right) + \tan^{-1}\left(\frac{\Omega_c}{\alpha}\right)\right] = \frac{1}{\pi\alpha}\left[\tan^{-1}\left(\frac{\Omega_c}{\alpha}\right)\right]$ since $\alpha = 0.5$
- $\frac{2}{\pi}[\tan^{-1}(2\Omega_c)] = 0.8 \quad \Longrightarrow \quad \Omega_C = \frac{1}{2}\left\{\tan\left(\frac{0.8\pi}{2}\right)\right\} \quad \Longrightarrow \quad \Omega_C = 1.5388$

80% (i.e 0.8 of 1) is here

Discrete Time Signals in the Frequency Domain

The Continuous-Time Fourier Transform (Band-Limited Continuous-Time Signals)



The bandwidth is
$0 \le \Omega \le 1.5388$ or
$0 \le \Omega \le 0.4898\pi$

80% (i.e 0.8 of 1) is here

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Definition)

- The discrete-time Fourier transform $X(e^{jw})$ of a sequence $x[n]$ is defined by
- $\mathcal{F}\{x[n]\} = X\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \qquad (3.10)$
- Where, w is the real normalized frequency variable.
- Equation (3.10) is called analysis equation.
- Example 3.5: Discrete-Time Fourier Transform of the Unit Sample Sequence
- $\Delta\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-jwn} = 1 \qquad (3.11)$
- Example 3.5: Discrete-Time Fourier Transform of the Shifted Unit Sample Sequence
- $X(e^{jw}) = \sum_{n=-\infty}^{\infty} \delta[n - n_o]e^{-jwn} = e^{-jwn_o} \qquad (3.11)$

Discrete Time Signals in the Frequency Domain

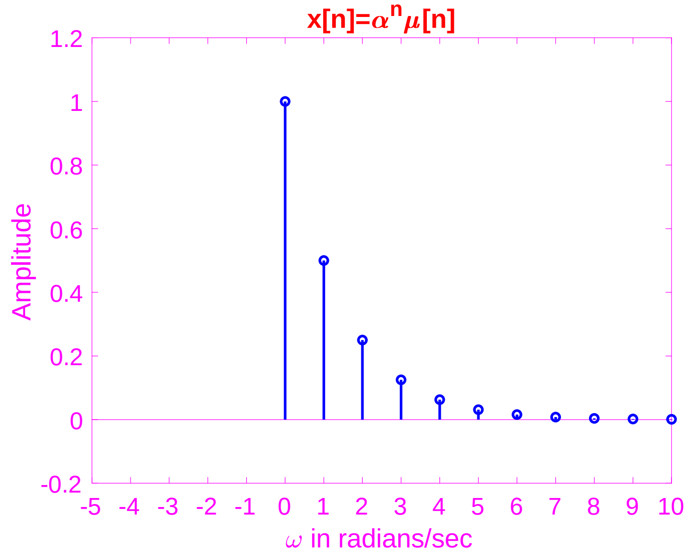
The Discrete-Time Fourier Transform (Definition)

- Example 3.6: Discrete-Time Fourier Transform of the Exponential Sequence
- Find the DTFT of an exponential sequence given
- $x[n] = \alpha^n u[n], \qquad |\alpha| < 1, \qquad (3.12)$
- Solution:
- $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-jwn} = \sum_{n=0}^{\infty} \alpha^n e^{-jwn}$
- $X(e^{jw}) = \sum_{n=0}^{\infty} (\alpha e^{-jw})^n = 1 + \alpha e^{-jw} + (\alpha e^{-jw})^2 + \cdots = \frac{1}{1-\alpha e^{-jw}}$

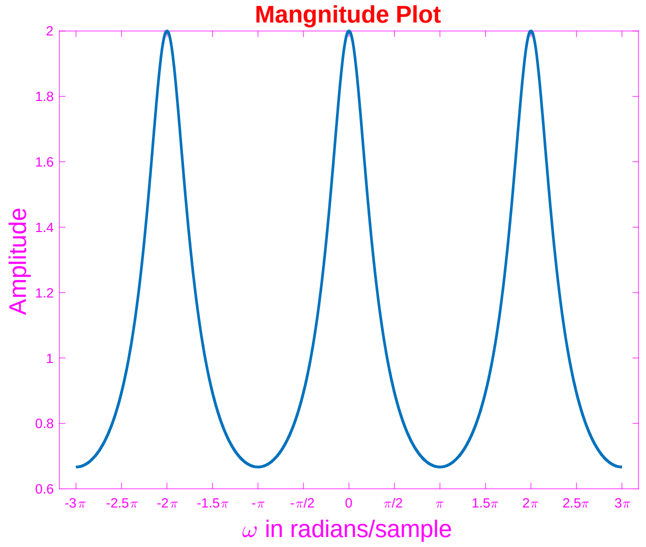
This is geometric series and $|\alpha e^{-jw}| < 1,$
Because $\alpha = 0.5$
We will use the formula $\frac{a}{1-r}$ (for convergent geometric series)

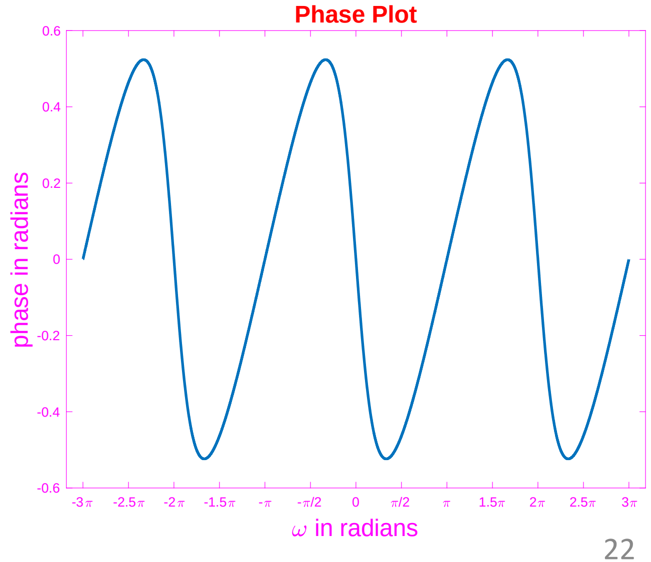
Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Definition)



- Find the DTFT of an exponential sequence given
- $x[n] = \alpha^n u[n], \qquad |\alpha| < 1, \qquad (3.12)$
- Solution:
- $X(e^{j\omega}) = \dfrac{1}{1-\alpha e^{-j\omega}}$
- Magnitude:
- $|X(e^{j\omega})| = \dfrac{1}{\sqrt{1-2\alpha\cos\omega+\alpha^2}}$
- Phase:
- $\theta(\omega) = \tan^{-1}\left(\dfrac{\alpha\sin\omega}{1-\alpha\cos\omega}\right)$

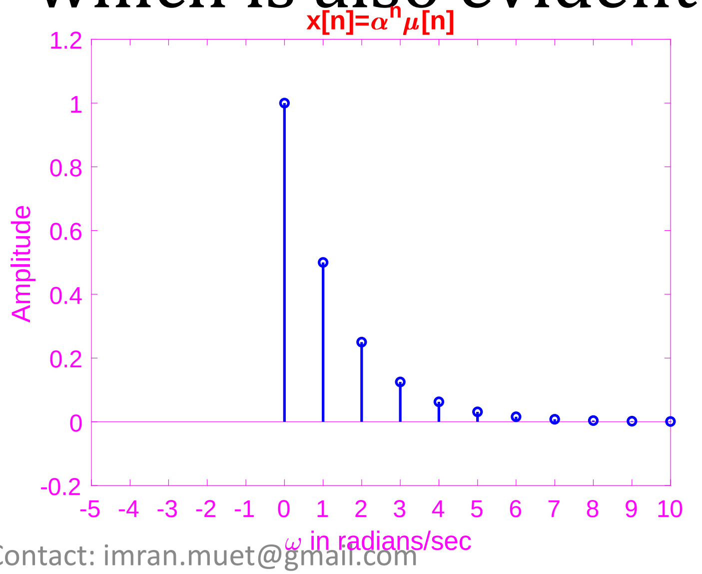


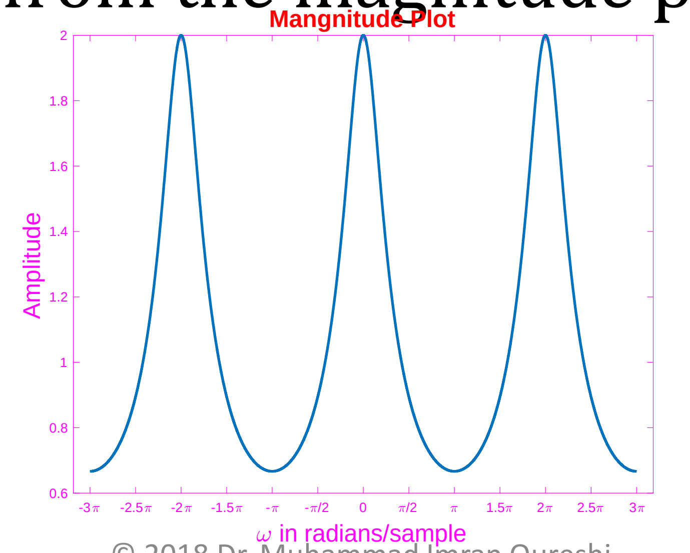


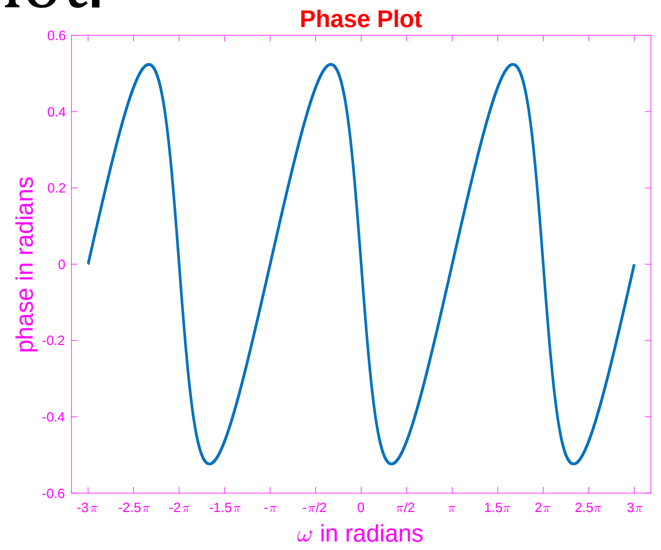
Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Definition)

- The DTFT $X(e^{j\omega})$ of a discrete-time signal $x[n]$ is a function of the normalized angular frequency ω.
- It should be noted that $x[n] = \alpha^n u[n]$ is aperiodic signal, whereas, its DTFF $X(e^{j\omega})$ is periodic function in ω with a period of 2π, which is also evident from the magnitude plot.







Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Definition)

- The DTFT $X(e^{j\omega})$ of a discrete-time signal $x[n]$ is a function of the normalized angular frequency ω.
- It should be noted that $x[n] = \alpha^n u[n]$ is aperiodic signal, whereas, its DTFF $X(e^{j\omega})$ is periodic function in ω with a period of 2π, which is also evident from the magnitude plot.
- The **periodicity property** can be proved here.
- $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$
- Replacing $\omega = \omega + 2\pi k$, where k is an integer, the above expression reduces to
- $X(e^{j(w+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(w+2\pi k)n}$
- $X(e^{j(w+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(w+2\pi k)n}$
- $= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi kn}$
- As we know that $e^{-j2\pi kn} = 1$, for all values of k and n, so the above expression reduces to
- $X(e^{j(w+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = X(e^{j\omega})$
- It means that For DTFT of any function $x[n]$ repeats after 2π

Periodicity Property

$$X(e^{j(w+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = X(e^{j\omega})$$

DTFF $X(e^{j\omega})$ is periodic function in ω with a period of 2π

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Definition)

- The inverse discrete-Fourier transform can be computed from $X(e^{j\omega})$ using following expression
- $\mathcal{F}^{-1}\{X(e^{j\omega})\} = x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad (3.14)$
- Equation (3.14) is called **synthesis equation**.
- Integration can be carrid out over any interval of duration 2π, however, it is common practice to choose the interval $[-\pi, \pi]$.
- A discrete-time Fourier transform pair will be denoted as

$$x[n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{j\omega})$$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Definition)

- The inverse discrete-Fourier transform can be computed from $X(e^{j\omega})$ using following expression
- $\mathcal{F}^{-1}\{X(e^{j\omega})\} = x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad (3.14)$
- Where $X(e^{jw}) = \sum_{l=-\infty}^{\infty} x[l]e^{-jwl}$, putting this in above equation will result
- $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{\sum_{l=-\infty}^{\infty} x[l]e^{-jwl}\}e^{j\omega n}d\omega,$
- If $X(e^{jw})$ exists i.e. the summation converges, then the order of integration and summation can be interchanged
- $x[n] = \sum_{l=-\infty}^{\infty} x[l]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-l)}d\omega\right) = \sum_{l=-\infty}^{\infty} x[l]\left(\frac{1}{2\pi}\left|\frac{e^{jw(n-l)}}{j(n-l)}\right|_{-\pi}^{\pi}\right)$
- $x[n] = \sum_{l=-\infty}^{\infty} x[l]\left\{\frac{1}{2\pi j(n-l)}\left(e^{j\pi(n-l)} - e^{-j\pi(n-l)}\right)\right\}$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Definition)

- $x[n] = \sum_{l=-\infty}^{\infty} x[l] \left\{ \frac{1}{2\pi j(n-l)} \left(e^{j\pi(n-l)} - e^{-j\pi(n-l)} \right) \right\}$
- We know $\sin\theta = \frac{e^{j\theta} - e^{\wedge} - j\theta}{2j}$, putting this in above formula will result
- $x[n] = \sum_{l=-\infty}^{\infty} x[l] \left\{ \frac{\sin \pi(n-l)}{\pi(n-l)} \right\} = \sum_{l=-\infty}^{\infty} x[l] \, sinc\,(n-l)$
- Since, $sinc\ \phi = \frac{\sin \pi\phi}{\pi\phi}$
- Case 1: For $n \neq l, \sin \pi(n-l) = 0$, and as a result $sinc\ (n-l) = 0$.
- Case 2: For $n = l, sinc\ (n-l) = \frac{\sin \pi(n-l)}{\pi(n-l)} = \frac{\sin \pi(0)}{\pi(0)} = \frac{0}{0}\ (undefined)$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Definition)

- As, $sinc\ (n) = \sin\frac{(\pi n)}{\pi n}$ is the sampled version of contuous-time function $\frac{\sin(\pi t)}{\pi t}$ for $t = n$; i.e. $\frac{\sin(\pi t)}{\pi t} = \left.\frac{\sin(\pi t)}{\pi t}\right|_{t=n}$
- We will use l'Hopital's rule, i.e.
- $\lim_{t \to 0} \frac{\sin(\pi t)}{\pi t} = \lim_{t \to 0} \frac{\pi \cos(\pi t)}{\pi} = 1$
- Therefore,
- $sinc\ (n-l) = \begin{cases} 1, & n = l, \\ 0, & n \neq l, \end{cases}$ in other words $sinc\ (n-l) = \delta[n-l]$
- Hence,
- $\sum_{l=-\infty}^{\infty} x[l]\, sinc\ (n-l) = \sum_{l=-\infty}^{\infty} x[l]\, \delta[n-l] = x[n]$

Hence, proved that the integration

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega,$$

gives the original discrete-time signal

$x[n]$

back from its DTFT $X(e^{j\omega})$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Basic Properties)

- The Fourier Transform Decomposition into Real and Imaginary Parts:
  - The Fourier transform $X(e^{j\omega})$ is a complex function of the real variable ω.
  - The Fourier transform $X(e^{j\omega})$ can be decomposed into its real $X_{re}(e^{j\omega})$ and imaginary $X_{im}(e^{j\omega})$ parts
  - $X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega}) \qquad (3.17)$
    - Where, both $X_{re}(e^{j\omega})$ and $X_{im}(e^{j\omega})$ are functions of real variable ω.
  - $X_{re}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{j\omega})\} \qquad (3.18a)$
  - $X_{im}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{j\omega})\} \qquad (3.18b)$
    - Where, $X^*(e^{j\omega})$ is complex conjugate of $X(e^{j\omega})$.

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Basic Properties)

- The Fourier Transform Polar Form Representation:
  - The Fourier transform $X(e^{j\omega})$ can be represented in the polar form as
  - $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$ (3.19)
  - Where $X(e^{j\omega})$ is called the Fourier spectrum
  - $|X(e^{j\omega})|$ is a magnitude function, also called magnitude specturm, and is a function ω.
  - $\theta(\omega)$ is a phase function, called phase specturem, and is also function of real variable ω.

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Basic Properties)

- Relationship Between Rectangular and Polar Form Representation:
  - The rectangular and polar form are related as follows:
  - $X_{re}(e^{j\omega}) = |X(e^{j\omega})| \cos\theta(\omega)$ $\quad(3.21a)$
  - $X_{im}(e^{j\omega}) = |X(e^{j\omega})| \sin\theta(\omega)$ $\quad(3.21b)$
  - The magnitude spectrum is given as
  - $|X(e^{j\omega})| = \sqrt{X_{re}^2(e^{j\omega}) + X_{im}^2(e^{j\omega})}$ $\quad(3.21c)$
  - The phase sepcturm is given as
  - $\tan\theta(\omega) = \frac{X_{im}(e^{j\omega})}{X_{re}(e^{j\omega})}$ $\quad(3.21d)$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Basic Properties)

- Principal Value:
  - The phase function is not uniquely specified for the discrete-time Fourier transform.
  - $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$
  - Replacing $\theta(\omega)$ with $\theta(\omega) + 2\pi k$ (where k is an integer), the above equation reduces to
  - $X(e^{j\omega}) = |X(e^{j\omega})|e^{j(\theta(\omega)+2\pi k)}$
  - $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$
  - The above expression indicates that changing phase does not change the Fourier transform.
  - The phase function $\theta(\omega)$ is restricted to following range of values, called principal value
  - $-\pi \leq \theta(\omega) < \pi$

$$e^{j(\theta(\omega)+2\pi k)} = \cos(\theta(\omega) + 2\pi k) + j\sin(\theta(\omega) + 2\pi k)$$
$$\cos(\theta(\omega) + 2\pi k) = \cos\big(\theta(\omega)\big)\cos 2\pi k \; - sin\,\big(\theta(\omega)\big)\sin 2\pi k$$
$$\sin(\theta(\omega) + 2\pi k) = \cos\big(\theta(\omega)\big)\sin 2\pi k \; + \; \sin\big(\theta(\omega)\big)\cos 2\pi k$$
$$\cos 2\pi k \; = 1$$
$$\sin 2\pi k \; = 0$$
$$e^{j(\theta(\omega)+2\pi k)} = \cos\big(\theta(\omega)\big) + j\sin(\theta(\omega)) = e^{j\theta(\omega)}$$

Euler's Forumla

$$e^{j\theta} = \frac{\cos\theta + j\sin\theta}{2}$$
$$e^{-j\theta} = \frac{\cos\theta - j\sin\theta}{2j}$$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Symmetry Relations)

- Symmetry relations of the discrete-time Fourier transform of the real signals:

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + \mathrm{j}X_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$X_{od}(e^{j\omega})$
Symmetry Relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$
	$X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$
	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

We know that the magnitude spectrum is obtained as

$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})X^*(e^{j\omega})$$

For real singlas

$$X^*(e^{j\omega}) = X(e^{-j\omega})$$

So, the magnitude spectrum reduces to

$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})X(e^{-j\omega})$$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Symmetry Relations)

- Symmetry relations of the discrete-time Fourier transform of the complex signals:

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x_{re}[n]$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$jx_{im}[n]$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

Here, $X_{cs}(e^{j\omega})$ and $X_{ac}(e^{j\omega})$ are conjugate-symmetric and conjugate-antisymmetric part of $X(e^{j\omega})$.
Likewise, $x_{cs}[n]$ and $x_{ca}[n]$ are conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$.

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Symmetry Relations)

- Example 3.7: Find DTFT, real & imaginary, and magnitude & phase functions of following discrete-time real:
- $x[n] = \alpha^n u[n], \qquad |\alpha| < 1,$
- Solution:
- Discrete-Time Fourier Transform
- We have found its DTFT (slide 20) of this function; that is,
- $X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}}$

- Real & Imaginary Functions
- $X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} \times \frac{1-\alpha e^{j\omega}}{1-\alpha e^{j\omega}}$
- $X(e^{jw}) = \frac{1-\alpha e^{j\omega}}{1-\alpha(e^{-j\omega}+e^{j\omega})+\alpha^2}$
- $X(e^{jw}) = \frac{1-\alpha(\cos\omega+j\sin\omega)}{1-2\alpha\cos\omega+\alpha^2}$
- $X(e^{jw}) = \frac{1-\alpha\cos\omega}{1-2\alpha\cos\omega+\alpha^2} - \frac{j\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}$
- $X(e^{jw}) = X_{re}(e^{jw}) + X_{im}(e^{jw})$
- So, $X_{re}(e^{jw}) = \frac{1-\alpha\cos\omega}{1-2\alpha\cos\omega+\alpha^2}$
- $X_{im}(e^{jw}) = -\frac{j\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Symmetry Relations)

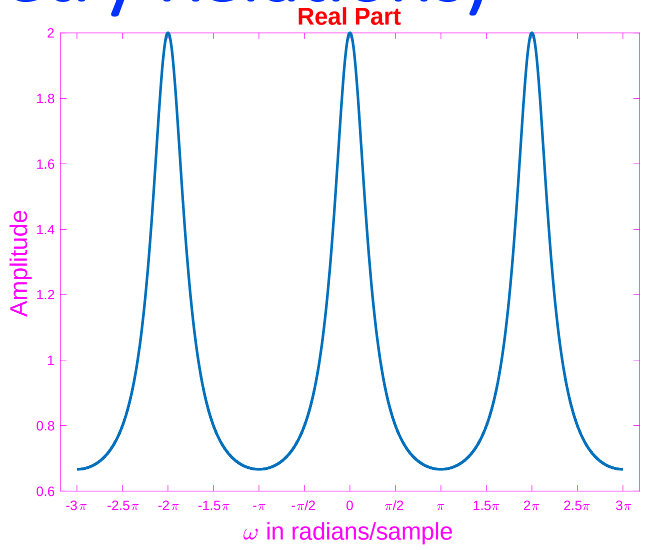
- Real & Imaginary Functions
  - $X_{re}\left(e^{jw}\right) = \dfrac{1-\alpha\cos\omega}{1-2\alpha\cos\omega+\alpha^2}$
  - $X_{im}\left(e^{jw}\right) = -\dfrac{j\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}$

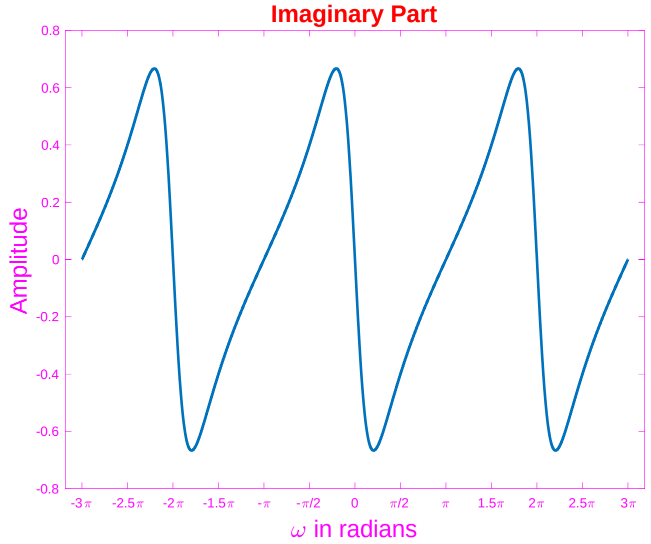
Since $\cos\omega$ and $\sin\omega$ are periodic function of ω with period of 2π, as a result both $X_{re}\left(e^{jw}\right)$ and $X_{im}\left(e^{jw}\right)$ are also periodic of function ω with period of 2π.

Since $\cos\omega$ and $\sin\omega$ are even and odd function, respectively, as a result $X_{re}\left(e^{jw}\right)$ and $X_{im}\left(e^{jw}\right)$ are also even and odd, respectively.

Real function is an **even function** and also **periodic** of period 2π.

Imaginary function is an **odd function** and also **periodic** of period 2π.





Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Symmetry Relations)

- Magnitude Function
- $\left|X(e^{jw})\right| = \sqrt{X_{re}(e^{jw})^2 + X_{im}(e^{jw})^2}$

OR

- $\left|X(e^{jw})\right| = \sqrt{X(e^{jw}).X^*(e^{jw})}$
- $\left|X(e^{jw})\right| = \sqrt{\frac{1}{1-\alpha e^{-jw}} \times \frac{1}{1-\alpha e^{jw}}}$
- $\left|X(e^{jw})\right| = \sqrt{\frac{1}{1-\alpha\left(e^{-j\omega}+e^{j\omega}\right)+\alpha^2}}$
- $\left|X(e^{jw})\right| = \sqrt{\frac{1}{1-2\alpha\cos\omega+\alpha^2}}$

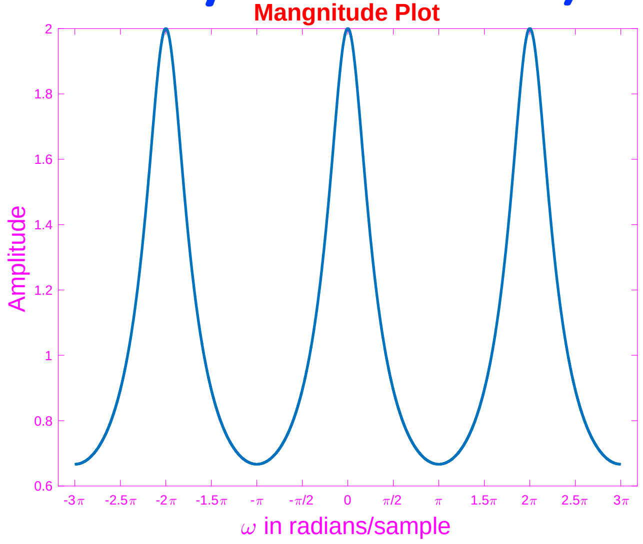
- Phase Function
- $\theta(\omega) = \tan^{-1}\left(\frac{X_{im}(e^{jw})}{X_{re}(e^{jw})}\right)$
- $\theta(\omega) = \tan^{-1}\left(\frac{-\frac{\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}}{\frac{1-\alpha\cos\omega}{1-2\alpha\cos\omega+\alpha^2}}\right)$
- $\theta(\omega) = \tan^{-1}\left(\frac{-\alpha\sin\omega}{1-\alpha\cos\omega}\right)$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Symmetry Relations)

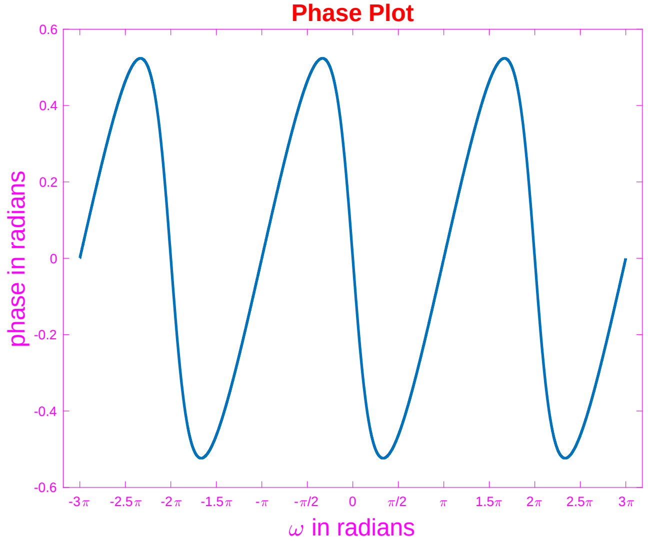
- Magnitude & Phase Functions
  - $\left|X(e^{jw})\right| = \sqrt{\dfrac{1}{1-2\alpha\cos\omega+\alpha^2}}$
  - $\theta(\omega) = \tan^{-1}\left(\dfrac{-\alpha\sin\omega}{1-\alpha\cos\omega}\right)$

Magnitude function $\left|X(e^{jw})\right|$ is an even function



Both $\left|X(e^{jw})\right|$ and $\theta(\omega)$ are periodic functions of ω with a period of 2π.

Phase function $\boldsymbol{\theta(\omega)}$ is an odd function



Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

- The Fourier transform $X(e^{jw})$ of a discrete-time sequence $x[n]$ exists if the following series converges.
- $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$ (3.10)
- Let Eq. (3.24) denote the partial sum of the weighted complex exponentials in Eq. (3.10).
- $X_K(e^{jw}) = \sum_{n=-K}^{K} x[n]e^{-jwn}$ (3.24)
- The uniform convergence of $X(e^{jw})$ is given by
- $\lim_{k\to\infty} X_K(e^{jw}) = X(e^{jw})$
- Now, if $x[n]$ is an absolutely summable sequence; that is, if
- $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ (3.25)
- Then
- $|X(e^{jw})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \forall\ \omega$
- Gaurantees the existence of $X(e^{jw})$.

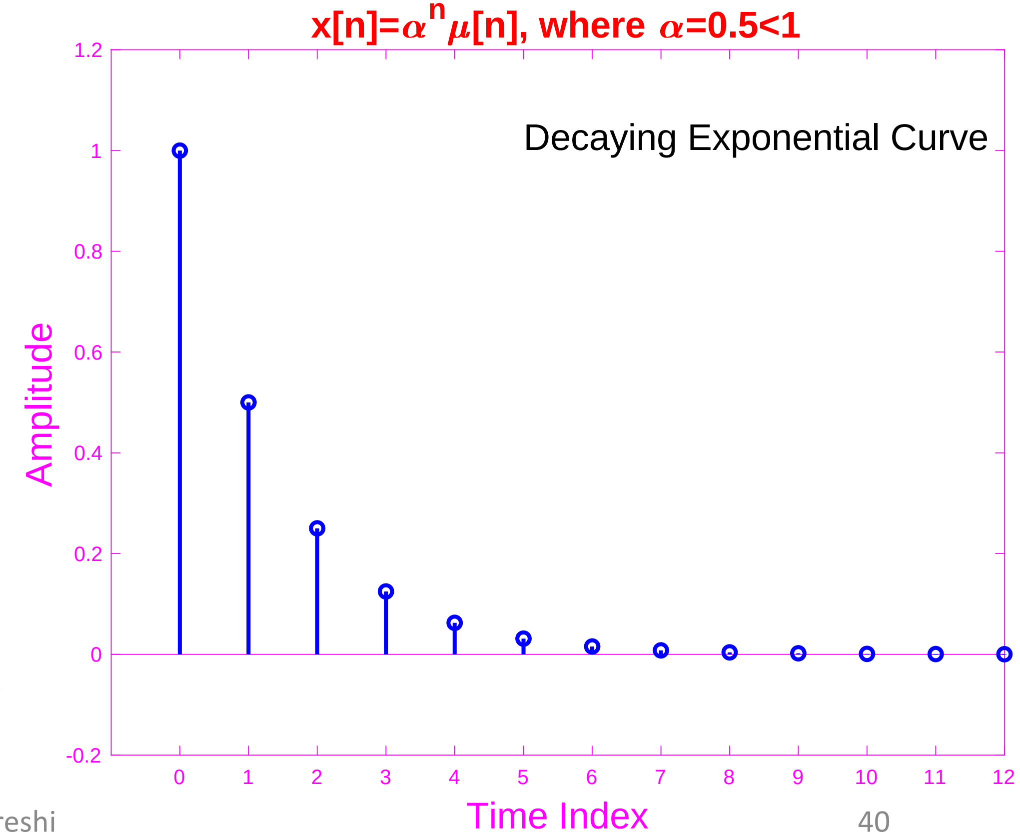
> - Thus, the condition: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ is sufficient for the existence of the Fourier transform $X(e^{jw})$ of the sequence $x[n]$.
> - In other words, for an absolutely summable sequence, the Fourier transform converges for all values of ω

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

- Example: Find out whether the sequence given below is absolutely summable or not. Its Fourier transform is possible, if yes, does that converges too?
- $x[n] = \alpha^n \mu[n] \qquad \alpha < 1$
- Solution:
- $\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=-\infty}^{\infty} |\alpha^n \mu[n]| =$
- $= \sum_{n=0}^{\infty} |\alpha^n| = 1 + |\alpha| + |\alpha^2| + \cdots$
- Since $\alpha < 1$, it means the series will converge, and we will apply the formula $S = \frac{a}{1-r}$, the above expression reduces to
- $\sum_{n=-\infty}^{\infty} |x[n]| = \frac{1}{1-|\alpha|} < \infty$ Hence the sequence is absolutely summable.

(the presence of $\mu[n]$ here indicates that limit is from 0 to ∞ $i.e. n = 0 \rightarrow \infty$)



Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

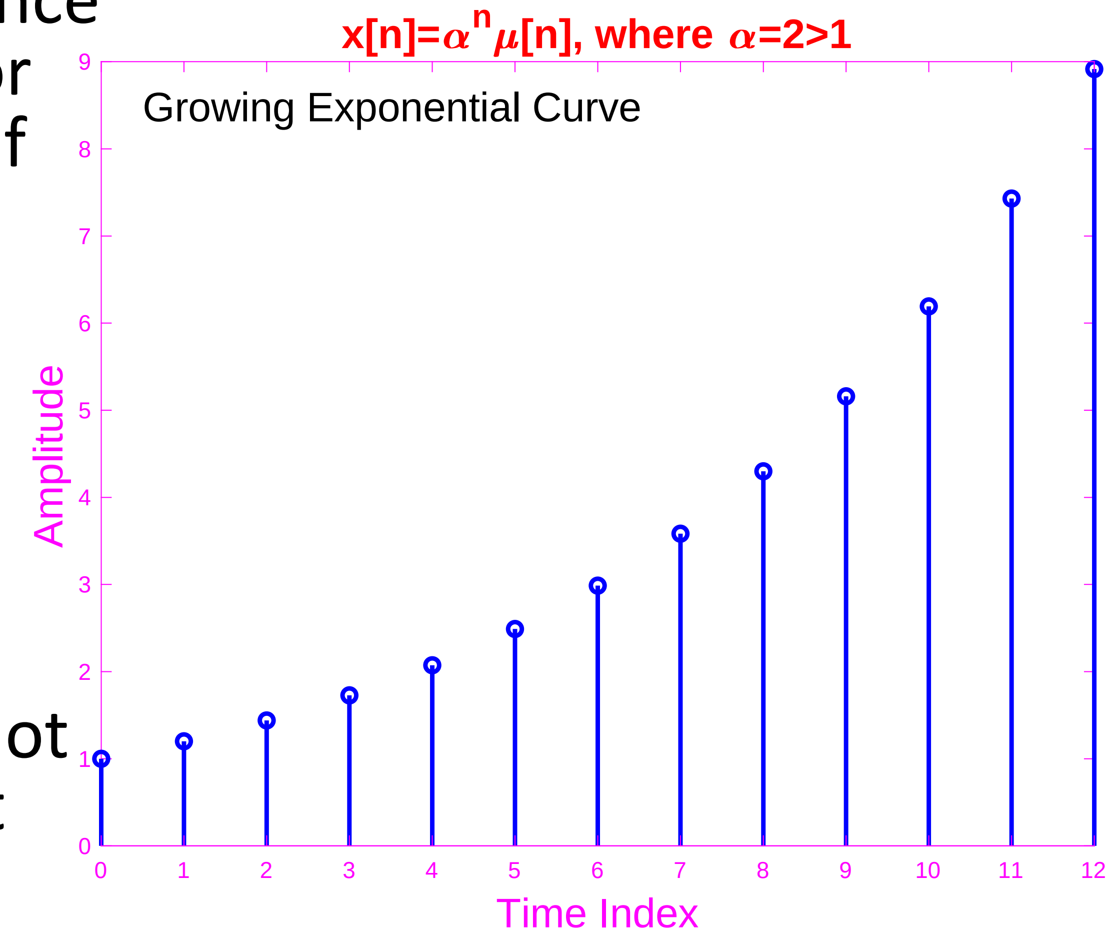
- Example: Find out whether the sequence given below is absolutely summable or not. Its Fourier transform is possible, if yes, does that converges too?
- $x[n] = \alpha^n \mu[n] \qquad \alpha > 1$
- Solution:
- $\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=-\infty}^{\infty} |\alpha^n \mu[n]| =$
- $= \sum_{n=0}^{\infty} |\alpha^n| = 1 + |\alpha| + |\alpha^2| + \cdots$
- Since $\alpha < 1$, it means the series will converge, and we will apply the formula $S = \frac{a}{1-r}$, the above expression reduces to
- $\sum_{n=-\infty}^{\infty} |x[n]| = \frac{1}{1-|\alpha|} < \infty$ Hence the sequence is absolutely summable.

- Its discrete-time Fourier transform is possible and it converges.
- $X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}}$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

- Example: Find out whether the sequence given below is absolutely summable or not. Its Fourier transform is possible, if yes, does that converges too?
- $x[n] = \alpha^n \mu[n] \qquad \alpha < 1$
- Solution:
- $\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=-\infty}^{\infty} |\alpha^n \mu[n]| =$
- $= \sum_{n=0}^{\infty} |\alpha^n| = 1 + |\alpha| + |\alpha^2| + \cdots$
- Since $\alpha > 1$, it means the series will not converge and hence its DTFT does not exit.



Discrete Time Signals in the Frequency Domain

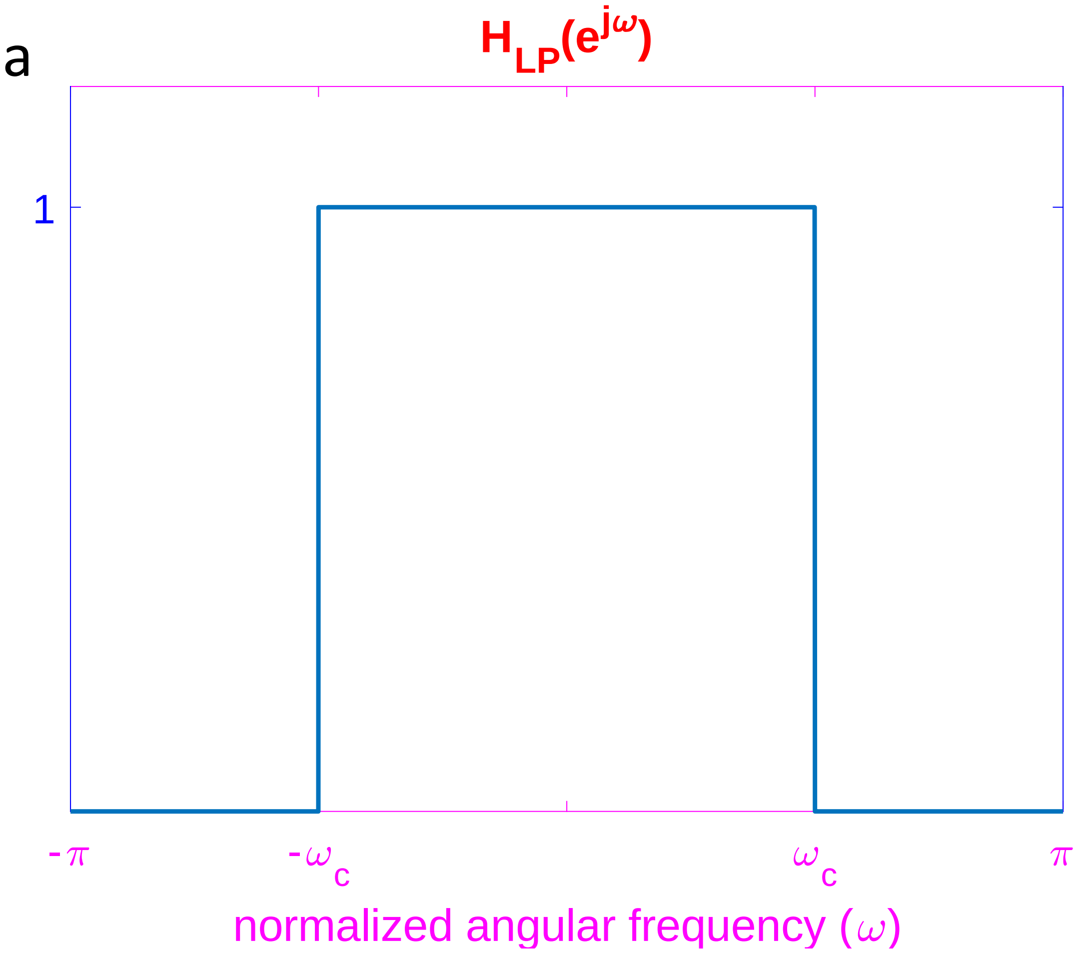
The Discrete-Time Fourier Transform (Convergence Condition)

- Mean-Square Convergence
  - An absolutely summable sequence always has a finite energy.
  - However, a finite-energy signal is not necessarily absolutely summable.
  - To represent such sequence by a discrete-time Fourier transform, it is necessary to consider a mean-square convergence of $X(e^{jw})$, in which case the total energy of the error $\mathcal{E}(\omega) = X(e^{jw}) - X_K(e^{jw})$ must approach 0 at each value of ω as K goes to ∞; that is,
  - $\lim_{K\to\infty} \int_{-\pi}^{\pi} \left|X(e^{jw}) - X_K(e^{jw})\right|^2 d\omega = 0 \qquad (3.26)$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

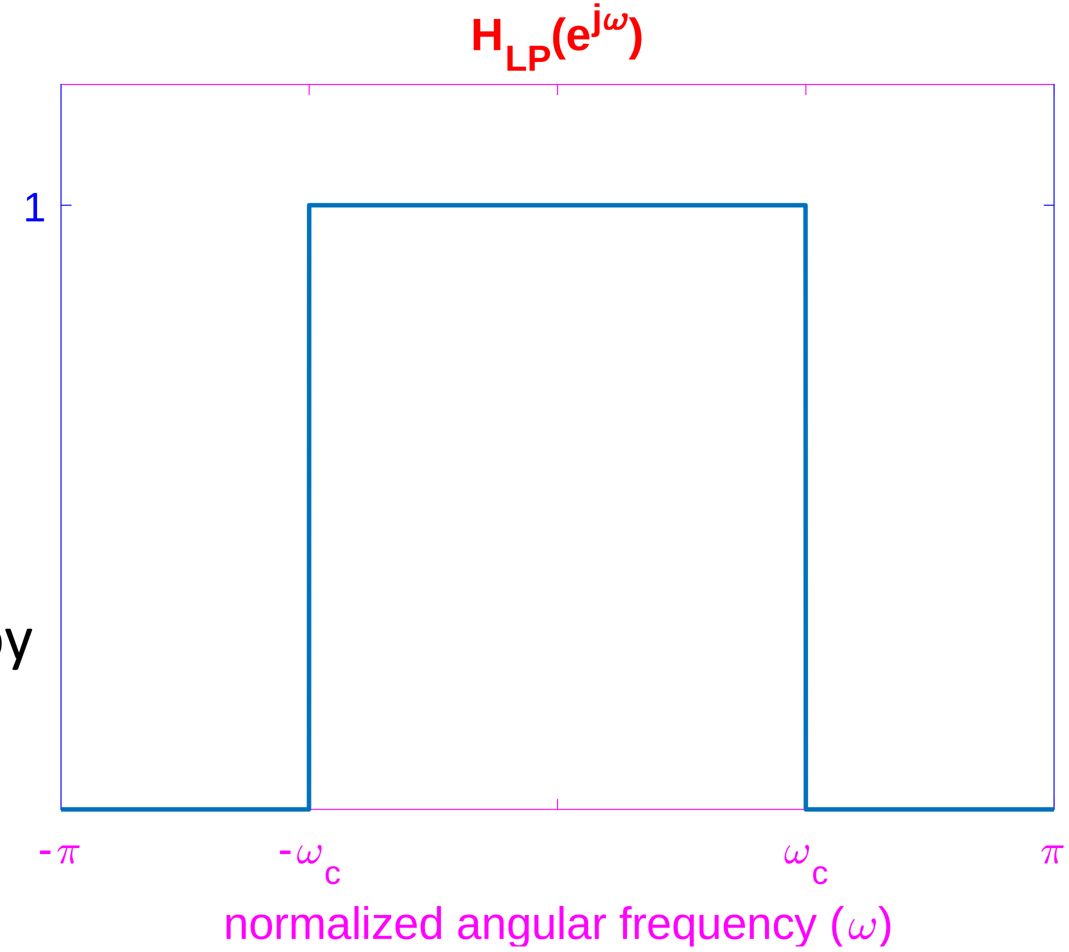
- Example: Consider the Fourier transform of a lowpass filter, shown in figure,
- $X_{LP}(e^{jw}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases} \quad (3.27)$
- Solution:
- The inverse DTFT of $X_{LP}(e^{j\omega})$ is given by
- $h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}d\omega$
- $= \frac{1}{2\pi}\int_{\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right)$
- $= \frac{1}{\pi n}\left(\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right)$
- $= \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty \; n \neq 0. \quad (3.28)$



Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

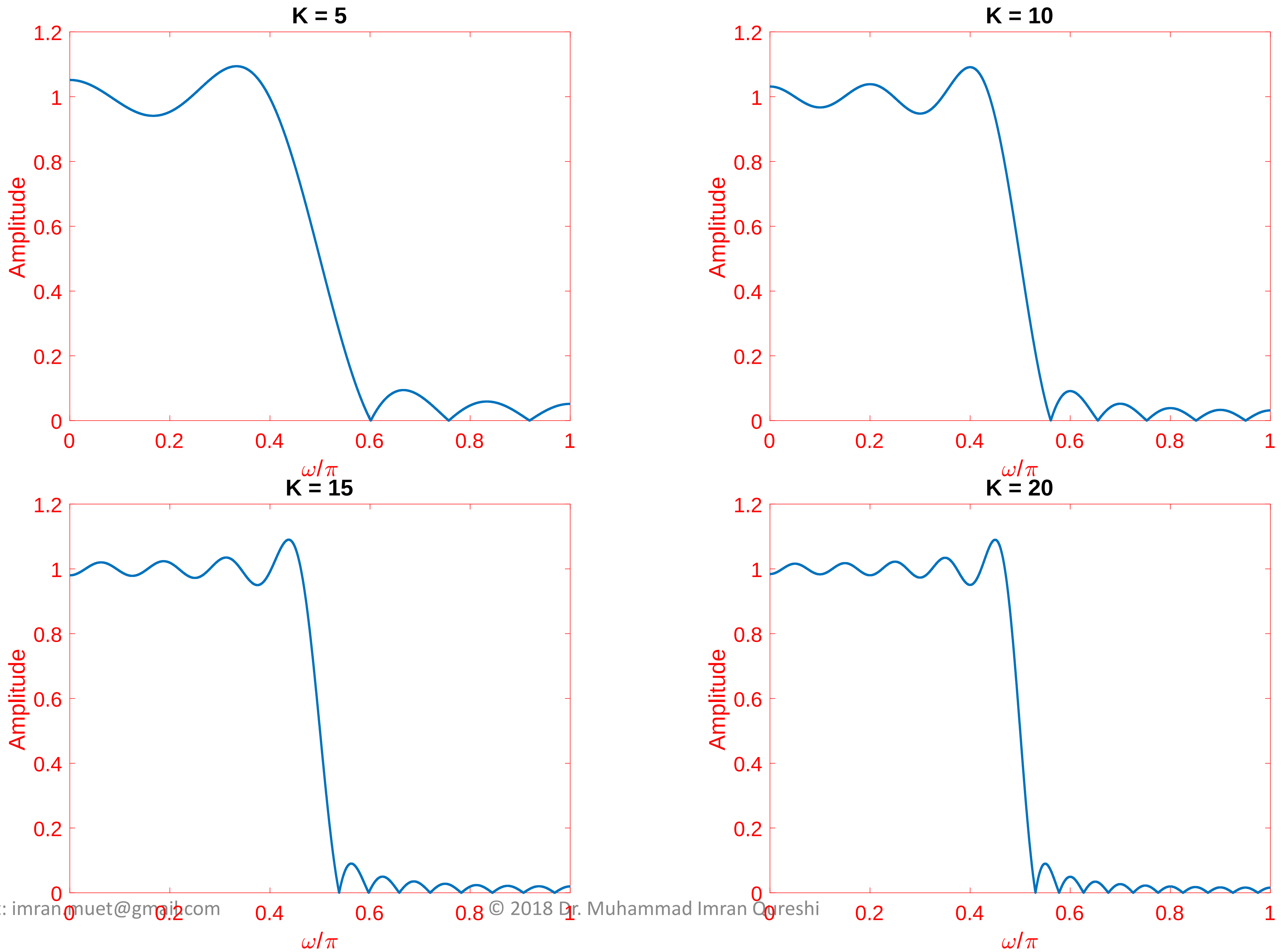
- $h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}d\omega$
- $= \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty \; n \neq 0. \quad (3.28)$
- For $n = 0,$ the inverse DTFT is given by
- $h_{LP}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega(0)}d\omega$
- $h_{LP}[0] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} \qquad (3.29)$
- The overall function for $h_{LP}[n]$ is obtained by combining (3.28) and (3.29)
- $h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & n = 0, \\ \frac{\sin \omega_c n}{\pi n}, & n \neq 0. \end{cases} \qquad (3.30)$



Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

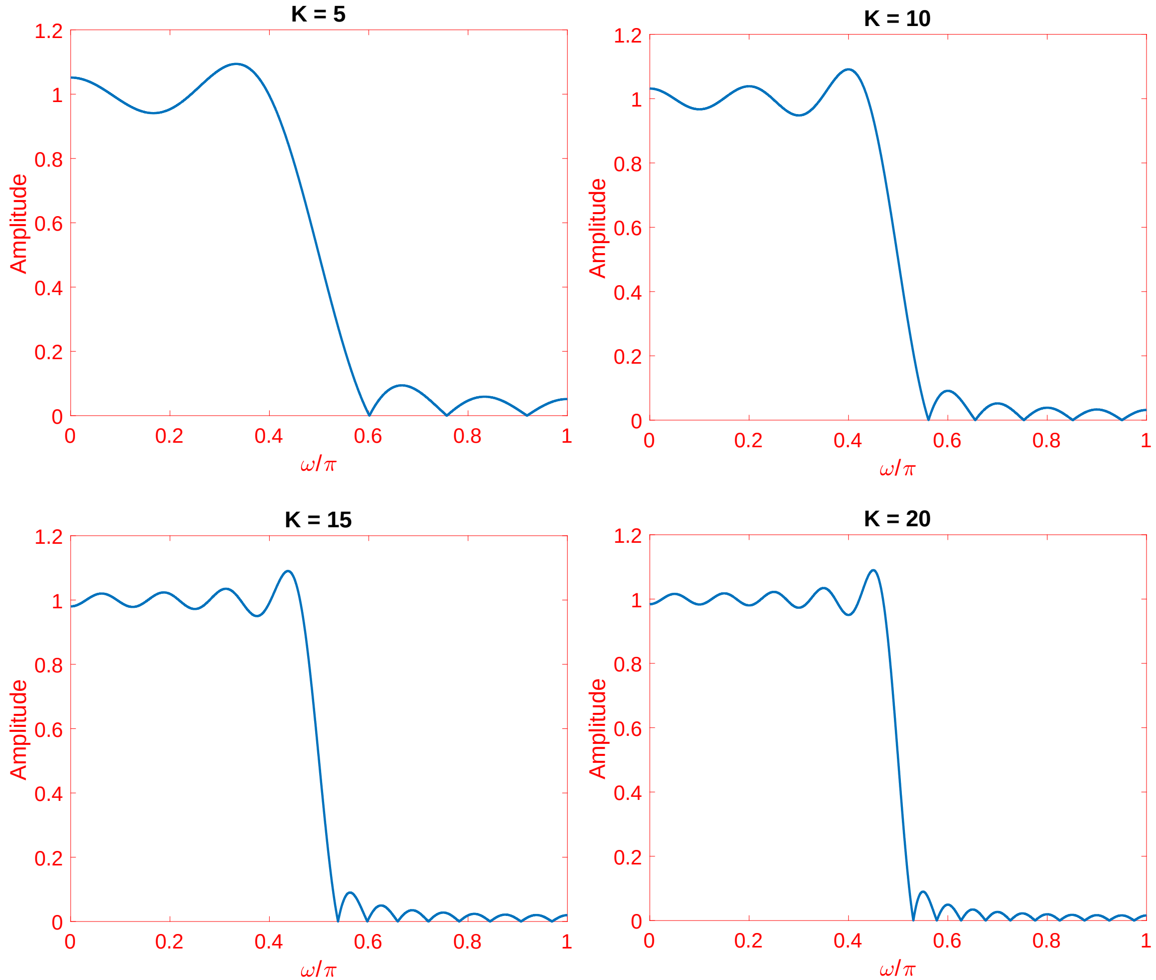
- $$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & n = 0, \\ \frac{\sin \omega_c n}{\pi n}, & n \neq 0. \end{cases} \qquad (3.30)$$
- The above function is often written as shown in Eq. (3.31) with the tacit assumption that at $n = 0, h_{LP}[n] = \frac{\omega_c}{\pi}$.
- $$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty \qquad (3.31)$$
- The energy of the above sequence $h_{LP}[n]$ is $\frac{\omega_c}{\pi}$ and it converges, where the sequene $h_{LP}[n]$ is not absolutely summable. As a result its DTFT $X_{LP}(e^{j\omega})$, shown in (3.27) does not converge uniformly for all values of ω, but converges in mean-square sense.


K = 5
K = 10
K = 15
K = 20
Amplitude
ω/π
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

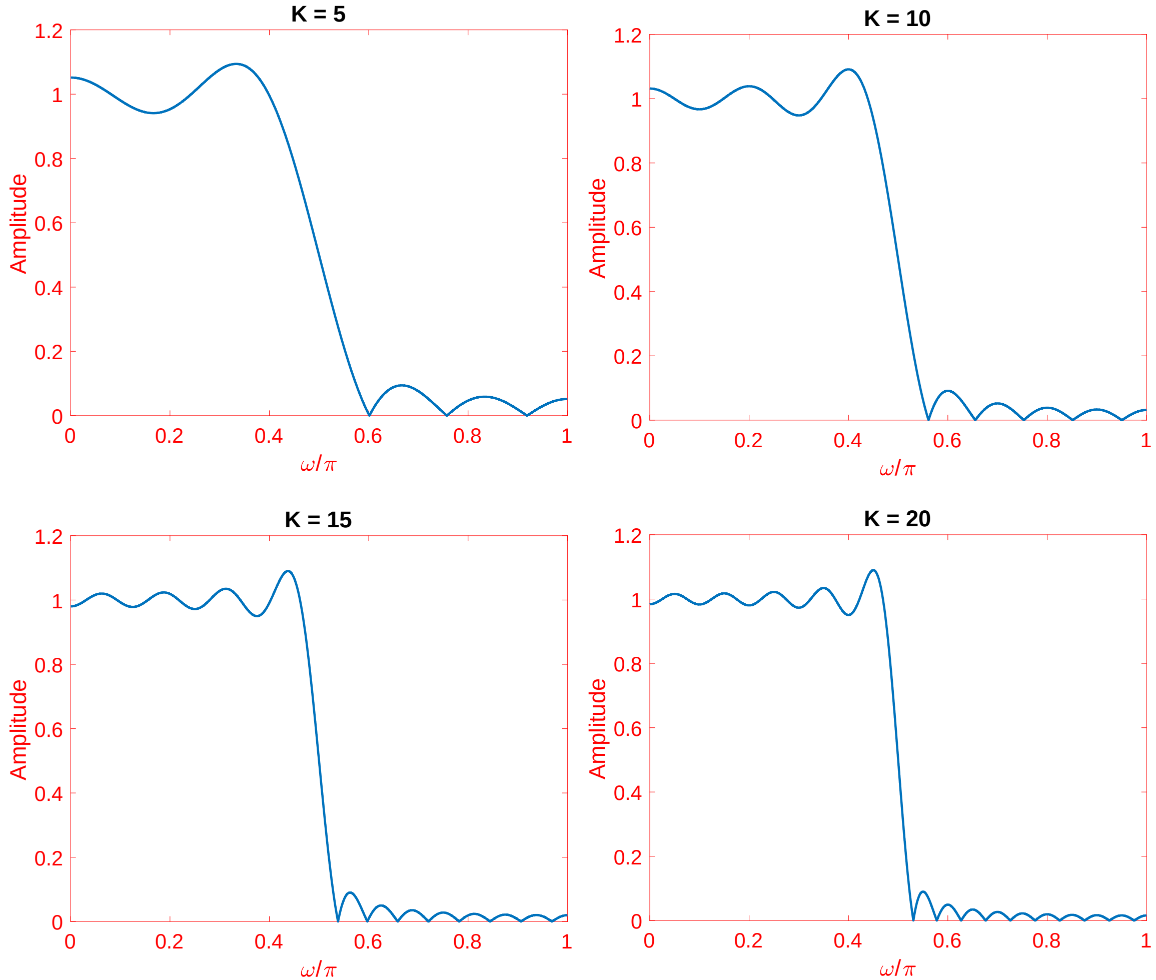
- The mean-square convergence property of the sequence $h_{LP}[n]$ cn be further illustrated by examining the plot of the function
- $H_{LP,K} = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$ (3.32)
- For various values of K.
- The plots for various values of K are shown in the figures.



Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

- The number of ripples increases as K increases.
- The height of the largest ripple remains the same for all values of K.
- The oscillatory behavior in the plot of $H_{LP,K}(e^{j\omega})$ approximating a Fourier transform $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity, known as Gibbs phenomenon.



Discrete Time Signals in the Frequency Domain

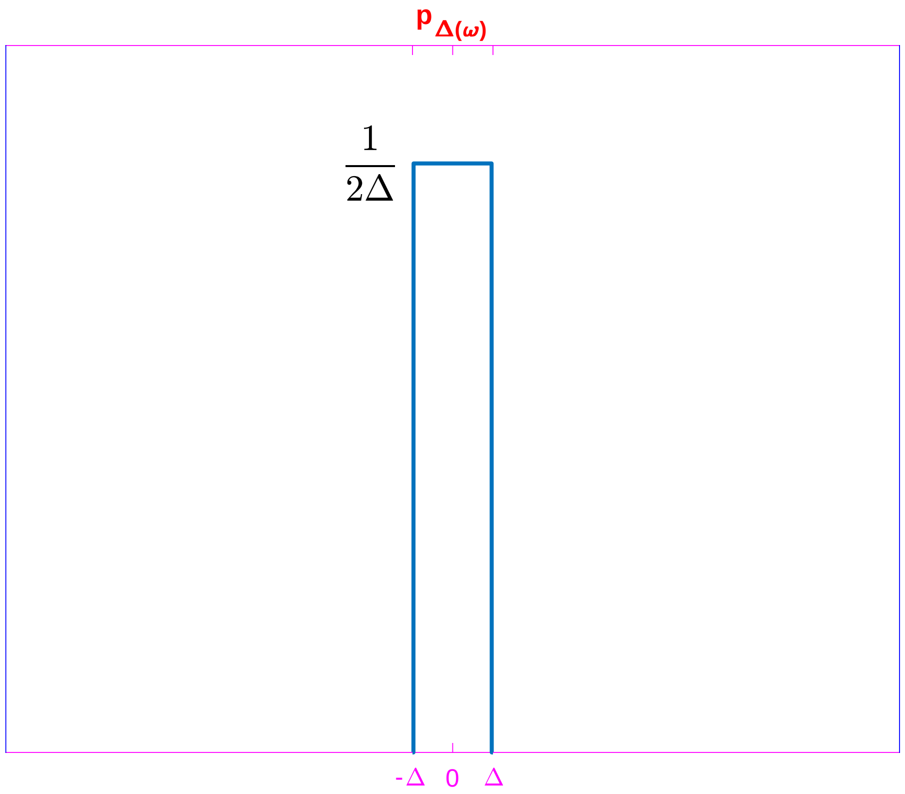
The Discrete-Time Fourier Transform (Convergence Condition)

- There are sequences that are neither absolutely summable nor square-summable. For examples:
  1. Unit Step Sequence:
     - It is denoted by $\mu[n]$ and is defined by
     - $\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \qquad (2.46)$
  2. Sinusoidal Sequence:
     - A real sinusoidal sequence with constant amplitude (A) has the form
     - $x[n] = A\cos(w_o n + \phi) \quad -\infty \leq n \leq \infty \qquad (2.48)$
  3. Complex Exponential Sequence
     - $x[n] = A\alpha^n \quad -\infty \leq n \leq \infty \quad (2.51)$
     - Where, $\alpha = e^{(\sigma_o + jw_o)}$
- The Fourier transform for such signals (singals neither absolutely summable nor square-summable) is possible by Dirac delta functions.

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

- Dirac delta Function or Ideal Impulse Function $\delta(\omega)$:
- It is a function of normalized angular frequency ω, having following features:
  1. Infinite height
  2. Zero width
  3. Unit area
- It is defined by the equation
- $\int_{-\infty}^{\infty} \delta(\omega) d\omega = 1, \quad \delta(\omega) = 0 \; when \; \omega = 0. \quad (3.33)$
- In terms of limiting form of a unit area pulse function $p_{\Delta(\omega)}$, shown figure, can be defined as:
- $\delta(\omega) = \lim_{\Delta \to \infty} p_{\Delta(\omega)},$
- Where,
- $\int_{-\infty}^{\infty} p_{\Delta(\omega)} d\omega = 1, \;\; p_{\Delta(\omega)} = 0 \; when \; \omega = 0. \quad (3.33)$
- The sampling property of the delta function is given by

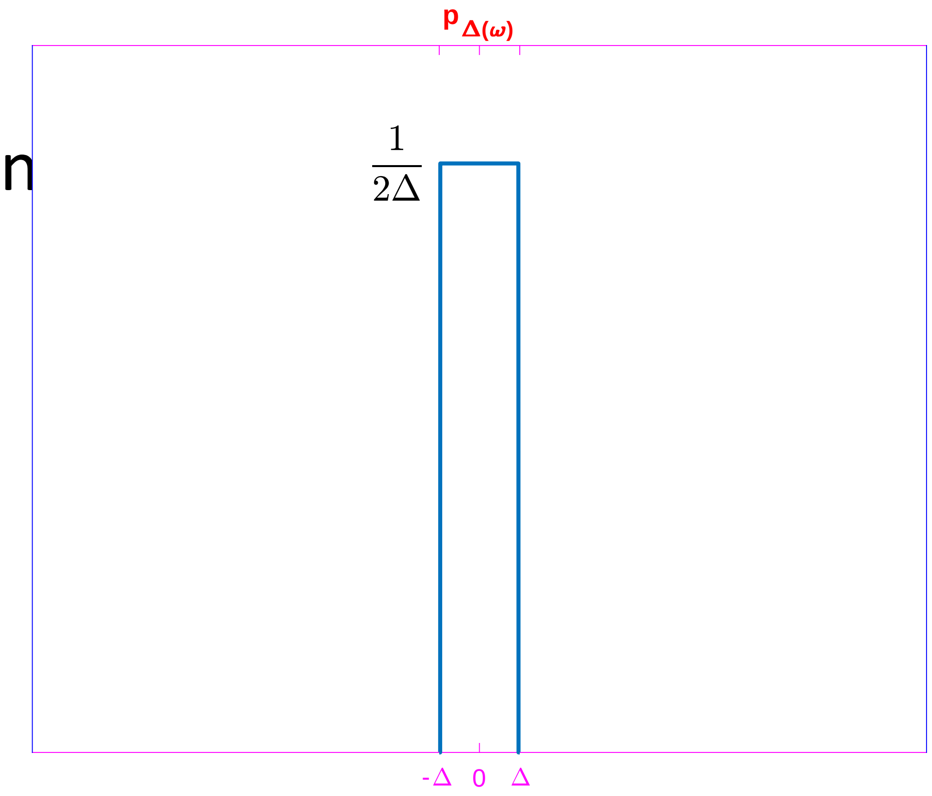


Unit Area Pulse Function

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

- Dirac delta Function or Ideal Impulse Function $\delta(\omega)$:
- The sampling property of the delta function is given by
- $\int_{-\infty}^{\infty} D(\omega)\delta(\omega-\omega_o)d\omega = D(\omega_o)$
- Where $D(\omega_o)$ is an arbitrary function of ω that is continuous ω_o.
- The Fourier transforms resulting from the use of Dirac delta functions are not continuous functions of ω.



Unit Area Pulse Function

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform (Convergence Condition)

- Example 3.9: Find the Fourier transform representation of following complex exponential sequence
- $x[n] = e^{j\omega_o n}$
- Where, ω_o is real.
- Solution:
- The Fourier transform is given by
- $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$ (3.34)
- Where $\delta(\omega)$ is an impulse function of ω and $-\pi \leq \omega_o \leq \pi$.

- The term $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$ is periodic function of ω with a period of 2π and is called a periodic impulse train.

Commonly Used Discrete-Time Fourier transform pairs

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
$1\ (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\frac{1}{1 - e\wedge - j\omega} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\alpha^n\mu[n], \quad (\lvert\alpha\rvert < 1)$	$\frac{1}{1 - \alpha e^{-j\omega}}$
$(n+1)\alpha^n\mu[n], \quad (\lvert\alpha\rvert < 1)$	$\frac{1}{(1 - \alpha e^{-j\omega})^2}$
$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} \quad (-\infty < n < \infty)$	$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & n = 0, \\ \frac{\sin \omega_c n}{\pi n}, & n \neq 0. \end{cases}$

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Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Linearity Theorem)

- Let $g[n]$ and $h[n]$ are two discrete-time sequences, and $\mathcal{F}\{g[n]\} = G(e^{j\omega})$ and $\mathcal{F}\{h[n]\} = H(e^{j\omega})$ then
- $\alpha g[n] + \beta h[n] \overset{\mathcal{F}}{\leftrightarrow} \alpha G\left(e^{j\omega}\right) + \beta H\left(e^{j\omega}\right) \qquad (3.39)$
- Where α and β are two constants.

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Time-Reversal Theorem)

- The Fourier transform of the delayed sequence $g[-n]$ is given by $G(e^{-j\omega})$; that is,
- $g[-n] \overset{\mathcal{F}}{\leftrightarrow} H\left(e^{-j\omega}\right)$ (3.40)
- Proof:
- The Fourier transform of the sequence $g[k]$ is given by
- $\mathcal{F}\{g[k]\} = \sum_{k=-\infty}^{\infty} x[k]e^{-jwk} = G\left(e^{j\omega}\right)$
- Let $k = -n$, the above expression reduces to
- $\mathcal{F}\{g[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]e^{-jwn} = G\left(e^{-j\omega}\right)$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Time-Shifting Theorem)

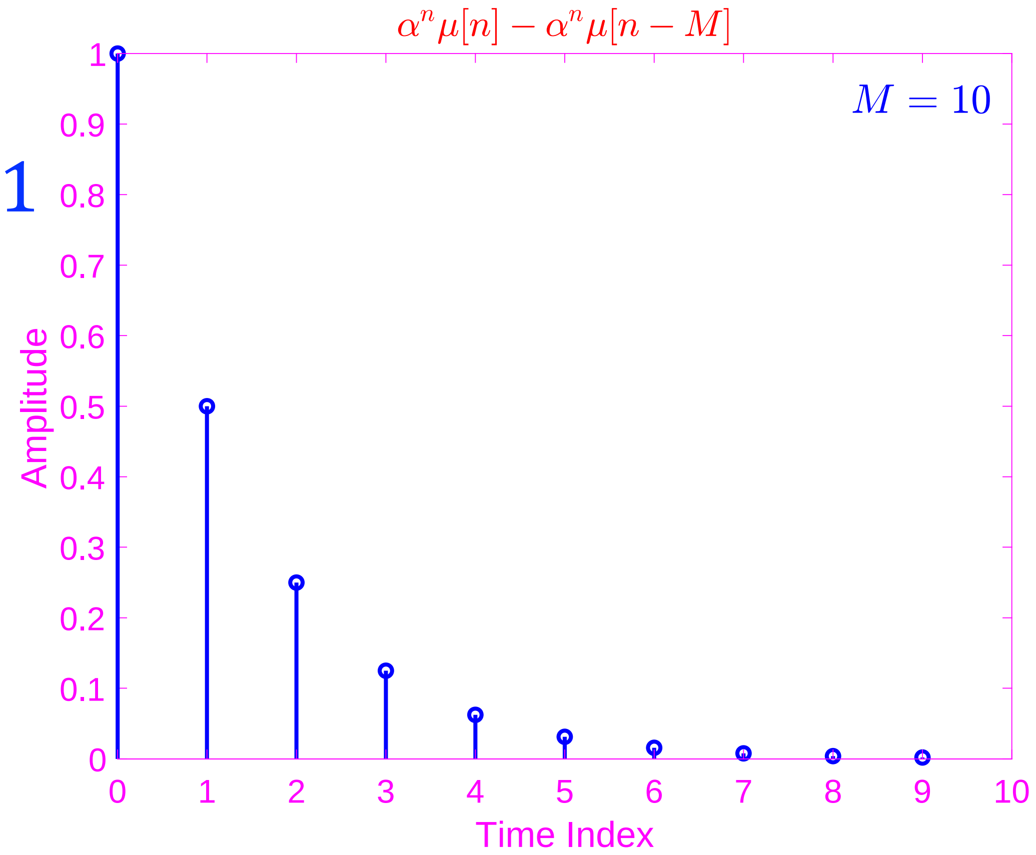
- The Fourier transform of the time-reversed sequence $x[n] = g[n - n_o]$ is given by $\mathrm{e}^{-\mathrm{j}\omega n_\mathrm{o}} G(e^{j\omega})$; that is,
- $g[n - n_o] \overset{\mathcal{F}}{\leftrightarrow} \mathrm{e}^{-\mathrm{j}\omega n_\mathrm{o}} G\left(e^{j\omega}\right)$ (3.41)
- Proof:
  - The Fourier transform of the sequence $g[n - n_o]$ is given by
  - $\mathcal{F}\{g[n - n_o]\} = \sum_{n=-\infty}^{\infty} x[n - n_o] e^{-jwn} = G\left(e^{j\omega}\right)$
  - Let $k = n - n_o$, the above expression reduces to
  - $\mathcal{F}\{g[n - n_o]\} = \sum_{k=-\infty}^{\infty} x[k] e^{-jw(k+n_o)} =$
  - $= e^{-j\omega n_o} \sum_{k=-\infty}^{\infty} x[k] e^{-jwk} = e^{-j\omega n_o} G\left(e^{j\omega}\right)$
- Similarly,
- $g[n + n_o] \overset{\mathcal{F}}{\leftrightarrow} \mathrm{e}^{\mathrm{j}\omega n_\mathrm{o}} G\left(e^{j\omega}\right)$

Since
$\left|e^{-j\omega n_o}\right| = 1$,
thereby
$\left|G\left(e^{j\omega}\right)\right| = \left|X\left(e^{j\omega}\right)\right|$;
that is , the magnitude of the spectrum is unchanged by shifting a signal in time.

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Time-Shifting Theorem)

- Find the Fourier transform of a Finite-length Exponential Sequence
- $y[n] = \begin{cases} \alpha^n, & 0 \le n \le M-1, \\ 0, & otherwise, \end{cases} \quad |\alpha| < 1$
- Solution:
- The above sequence can be written as
- $y[n] = \alpha^n \mu[n] - \alpha^n \mu[n-M]$

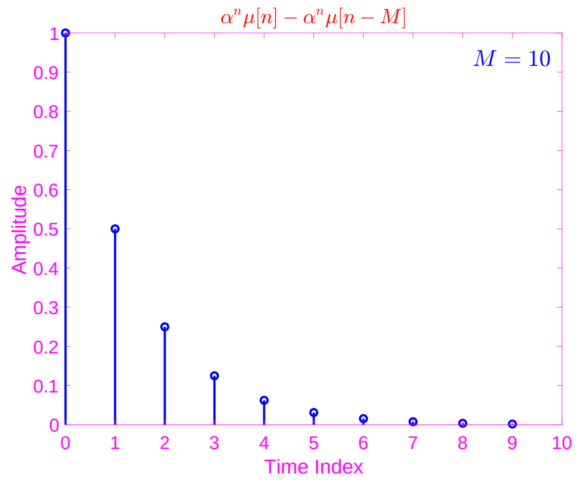


Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Time-Shifting Theorem)

- Example 3.10: Find the Fourier transform of a Finite-length Exponential Sequence
- $y[n] = \begin{cases} \alpha^n, & 0 \le n \le M-1, \\ 0, & otherwise, \end{cases} \quad |\alpha| < 1$
- Solution: (Method 1)
- $y[n] = 1 + \alpha^1 + \alpha^2 + \cdots + \alpha^{M-1}.$
- The Fourier transform is obtained as
- $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-jwn}$
- $Y(e^{j\omega}) = \sum_{n=0}^{M-1} \alpha^n e^{-jwn}$
- $Y(e^{j\omega}) = 1 + (\alpha e^{-j\omega}) + \cdots + (\alpha e^{-j\omega})^{M-1}$

- Using Geometric series formula $S_n = \frac{a_1(1-r^n)}{1-r}$ (for a finite-length convergent series of total n terms)
- $Y(e^{j\omega}) = \frac{1-(\alpha e^{-j\omega})^M}{1-\alpha e^{-j\omega}}$



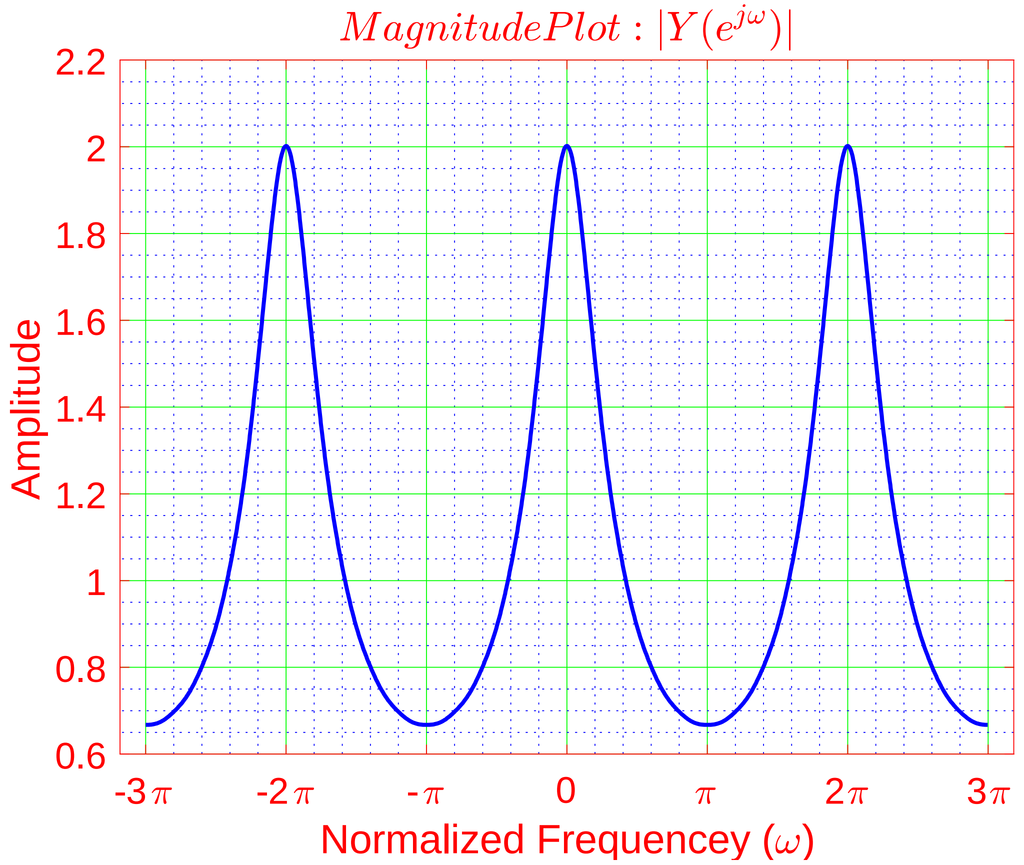
Discrete Time Signals in the Frequency Domain

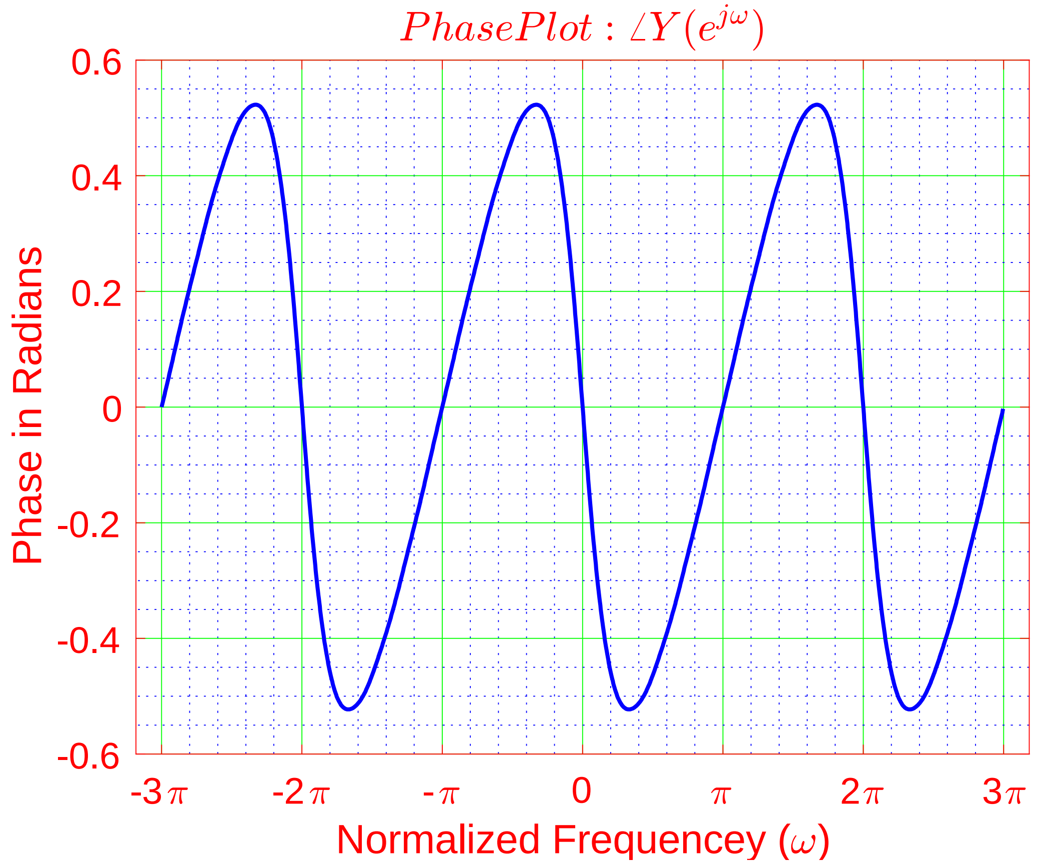
The Discrete-Time Fourier Transform Theorems (Time-Shifting Theorem)

- Example 3.10: Find the Fourier transform of a Finite-length Exponential Sequence
- $y[n] = \begin{cases} \alpha^n, & 0 \le n \le M-1, \\ 0, & otherwise, \end{cases} \quad |\alpha| < 1$
- Solution: **(Method 2)**
- The above sequence can be written as
- $y[n] = \alpha^n \mu[n] - \alpha^n \mu[n-M]$
- The above sequence can be written as
- $y[n] = \alpha^n \mu[n] - \alpha^M \alpha^{n-M} \mu[n-M]$
- The DTFT for $\alpha^n \mu[n]$ was found on slide # 34; that is,
- $\mathcal{F}\{\alpha^n \mu[n]\} = \frac{1}{1-\alpha e^{-j\omega}}$
- Using the time-shifting property the DTFT for $\alpha^{n-M}\mu[n-M]$ is obtained as
- $\mathcal{F}\{\alpha^{n-M}\mu[n-M]\} = e^{-j\omega M}\frac{1}{1-\alpha e^{-j\omega}}$
- The overall DTFT of $y[n]$ is obtained as
- $Y(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} + \alpha^M \frac{e^{-j\omega M}}{1-\alpha e^{-j\omega}}$
- $Y(e^{j\omega}) = \frac{1+\alpha^M e^{-j\omega M}}{1-\alpha e^{-j\omega}}$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Time-Shifting Theorem)





Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Time-Shifting Theorem)

- Example 3.11: Find the Fourier transform of a sequence defined by a difference equation
- $d_o v[n] + d_1 v[n-1] = p_o \delta[n] + p_1 \delta[n-1], \; \left|\frac{d_1}{d_2}\right| < 1.$
- Solution:
- The DTFT of $\delta[n]$ is 1; that is,
- $V(e^{j\omega}) = 1,$
- The DTFT of $v[n-1] = \delta[n-1]$ is obtained by using shifting property
- $e^{-j\omega} V(e^{j\omega}) = e^{-j\omega}$
- So the overall DTFT is obtained as
- $d_o V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_o + p_1 e^{-j\omega}$
- $V(e^{j\omega}) = \frac{p_o + p_1 e^{-j\omega}}{d_o + d_1 e^{-j\omega}}$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Freqeuncy-Shifting Theorem)

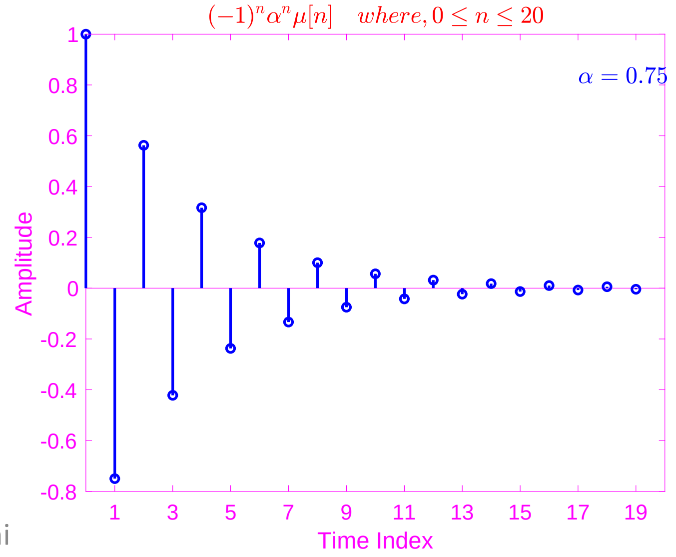
- The Fourier transform of the sequence $x[n] = e^{j\omega_o n} g[n]$ is given by $\mathrm{X}\left(\mathrm{e}^{j\omega}\right) = G\left(e^{j(\omega-\omega_o)}\right)$; that is,
- $e^{j\omega_o n} g[n] \overset{\mathcal{F}}{\leftrightarrow} G\left(e^{j(\omega-\omega_o)}\right) \qquad (3.45)$
- Proof:
- $\mathcal{F}\{g[n]\} = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} = G\left(e^{j\omega}\right)$ ..... (A)
- Similarly, the DTFT of $e^{j\omega_o n} g[n]$ is obtained as
- $\mathcal{F}\left\{e^{j\omega_o n} g[n]\right\} = \sum_{n=-\infty}^{\infty} e^{j\omega_o n} g[n]e^{j\omega n}$
- $\mathcal{F}\left\{e^{j\omega_o n} g[n]\right\} = \sum_{n=-\infty}^{\infty} g[n]e^{-j(\omega-\omega_o)n} \ldots .(B)$
- Comparing (A) and (B) will result
- $\mathcal{F}\left\{e^{j\omega_o n} g[n]\right\} = \sum_{n=-\infty}^{\infty} g[n]e^{-j(\omega-\omega_o)n} = G\left(e^{j(\omega-\omega_o)}\right)$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Freqeuncy-Shifting Theorem)

- Example 3.12: Find the Fourier transform of a Finite-length Exponential Sequence
- $y[n] = (-1)^n \alpha^n \mu[n] \quad |\alpha| < 1$
- Solution: (Method 1)
- $y[n] = 1 - \alpha^1 + \alpha^2 - \cdots.$
- The Fourier transform is obtained as
- $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-jwn}$
- $Y(e^{j\omega}) = \sum_{n=0}^{\infty} (-1)^n \alpha^n e^{-jwn}$
- $Y(e^{j\omega}) = 1 + (-\alpha e^{-j\omega}) + (-\alpha e^{-j\omega})^2 + \cdots$

- Using Geometric series formula $S_n = \frac{a_1}{1-r}$ (for a infinite-length convergent series)
- $Y(e^{j\omega}) = \frac{1}{1-(-\alpha e^{-j\omega})} = \frac{1}{1+\alpha e^{-j\omega}}$



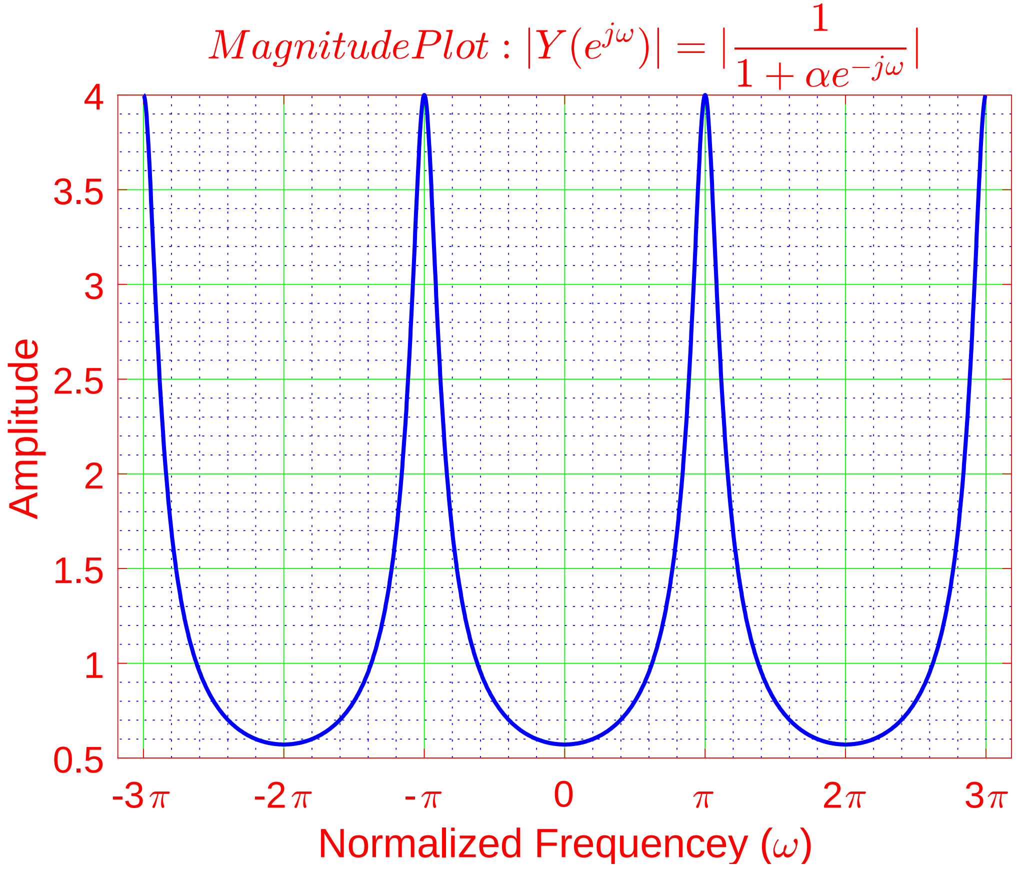
Discrete Time Signals in the Frequency Domain

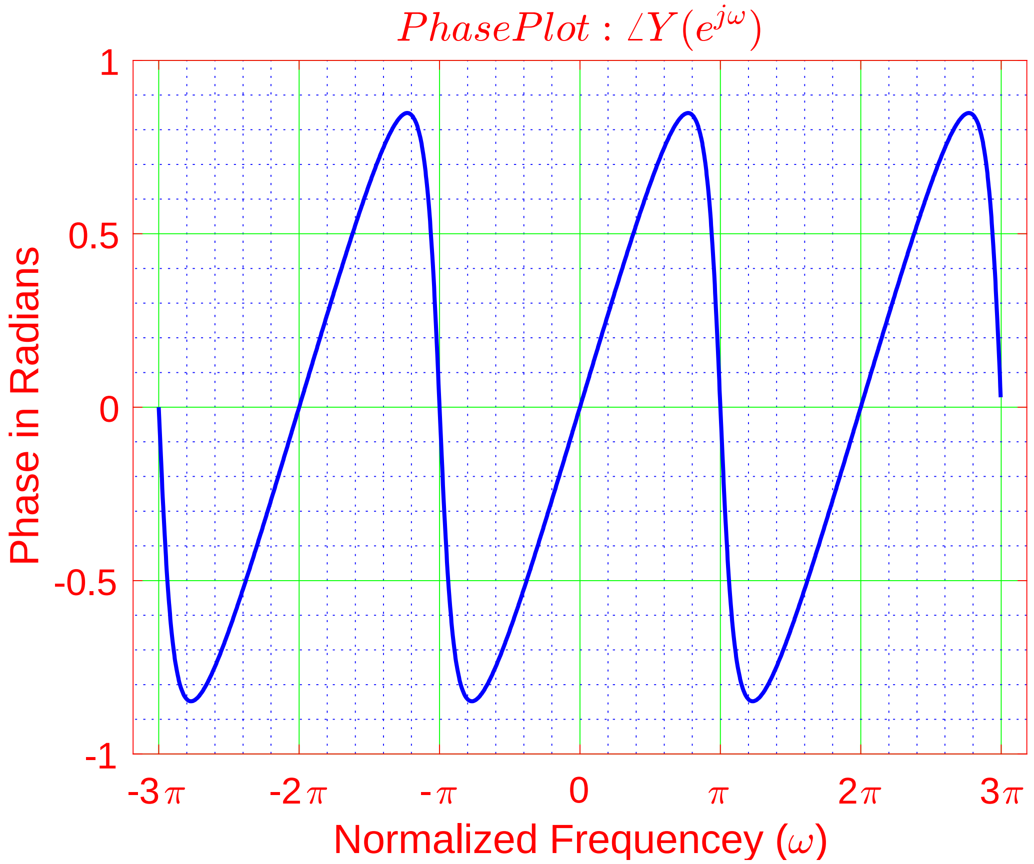
The Discrete-Time Fourier Transform Theorems (Freqeuncy-Shifting Theorem)

- Example 3.12: Find the Fourier transform of a Finite-length Exponential Sequence
- $y[n] = (-1)^n \alpha^n \mu[n] \quad |\alpha| < 1$
- Solution: (Method 2)
- The above sequence can be written as
- $y[n] = e^{j\pi n} \alpha^n \mu[n]$ (because $e^{j\pi n} = (-1)^n$)
- The DTFT of $\alpha^n \mu[n]$ is (as we found in previous slides)
- $\mathcal{F}\{\alpha^n \mu[n]\} = \frac{1}{1-\alpha e^{-j\omega}}$
- Applying frequency-shifting theorem, the DTFT of $y[n]$ is obtained as:
- $\mathcal{F}\{e^{j\pi n} \alpha^n \mu[n]\} = \frac{1}{1-\alpha e^{-j(\omega-\pi)}}$
- Where, $e^{-j(\omega-\pi)} = -e^{-j\omega}$
- The above expression reduces to
- $\mathcal{F}\{e^{j\pi n} \alpha^n \mu[n]\} = \frac{1}{1+\alpha e^{-j\omega}}$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Frequency-Shifting Theorem)





It can be seen that the spectrum of the sequence $(-1)^n\alpha^n\mu[n]$ is same as that of $\alpha^n\mu[n]$ (shown on slide 22), except the spectrum of $e^{j\pi n}\alpha^n\mu[n]$ is shifted by π radians

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Differentiation-in-Frequency-Theorem)

- The Fourier transform of the sequence $x[n] = ng[n]$ is given by $\mathrm{X}(\mathrm{e}^{j\omega}) = \frac{jd}{d\omega} G(e^{j\omega})$; that is,
- $ng[n] \overset{\mathcal{F}}{\leftrightarrow} j\frac{dG(e^{j\omega})}{d\omega}$ (3.46)

Proof:

The DTFT of sequence $g[n]$ is obtained as

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$$

Differentiating both sides with respect to ω.

$$\frac{d}{d\omega} G(e^{j\omega}) = \frac{d}{d\omega} \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$$

$$\frac{d}{d\omega} G(e^{j\omega}) = (-j) \sum_{n=-\infty}^{\infty} ng[n]e^{-j\omega n}$$

Multiplying both sides by j

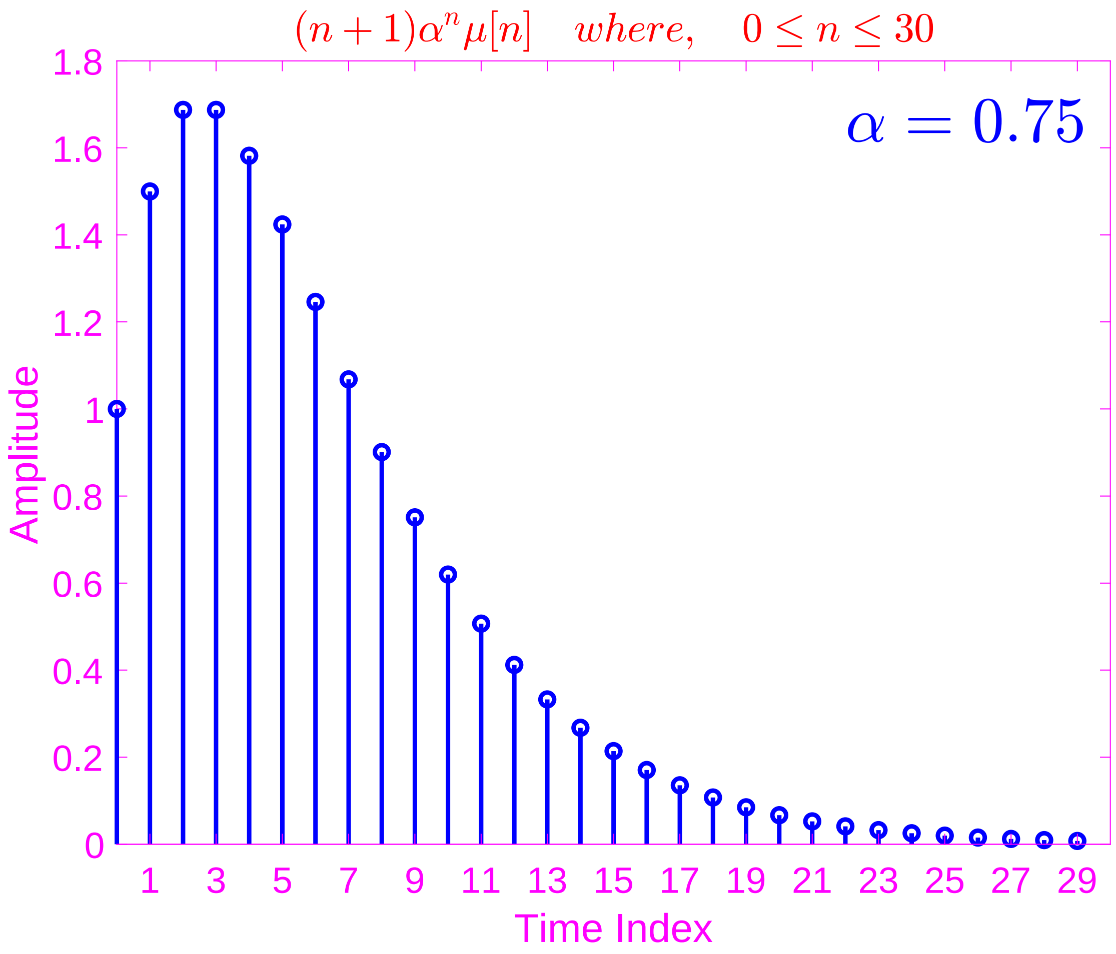
$$\frac{jd}{d\omega} G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} ng[n]e^{j\omega n}$$

$$\frac{jd}{d\omega} G(e^{j\omega}) = \mathcal{F}\{ng[n]\}$$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Differentiation-in-Freqeuncy-Theorem)

- Example 3.13: Find out the discret-time Fourier transform representation of following sequence
- $y[n] = (n+1)\alpha^n \mu[n], \quad |\alpha| < 1$



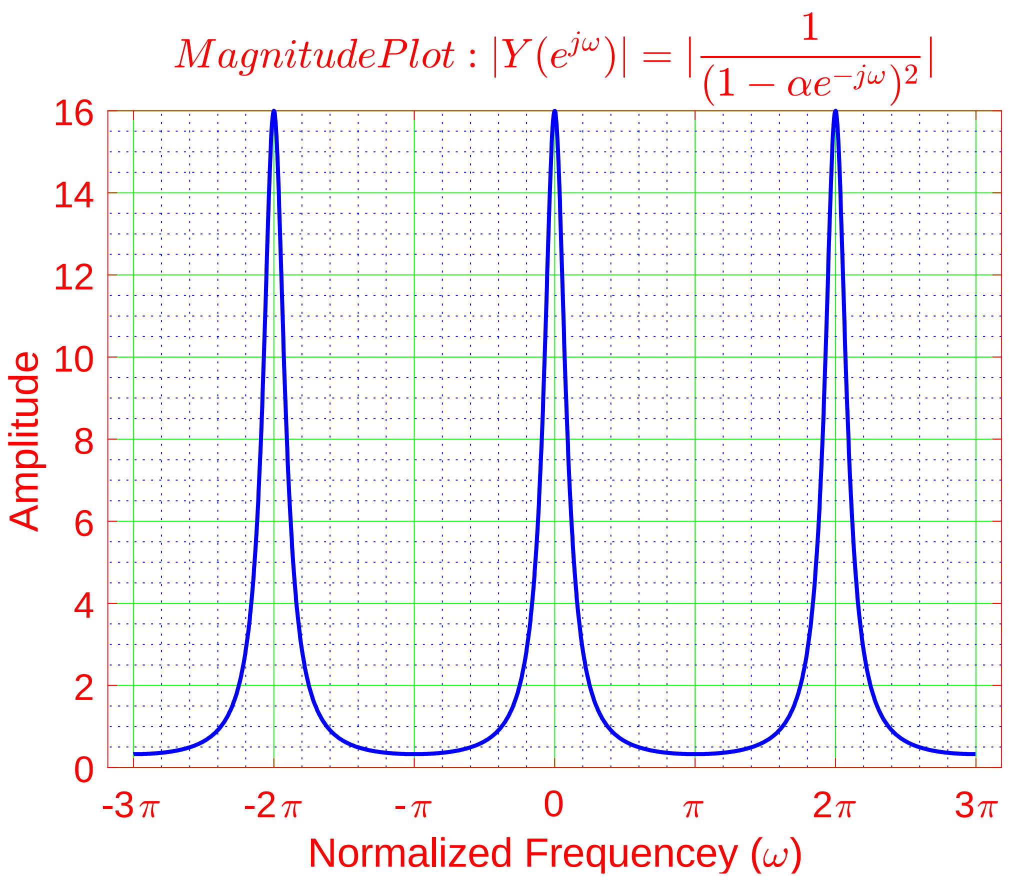
Discrete Time Signals in the Frequency Domain

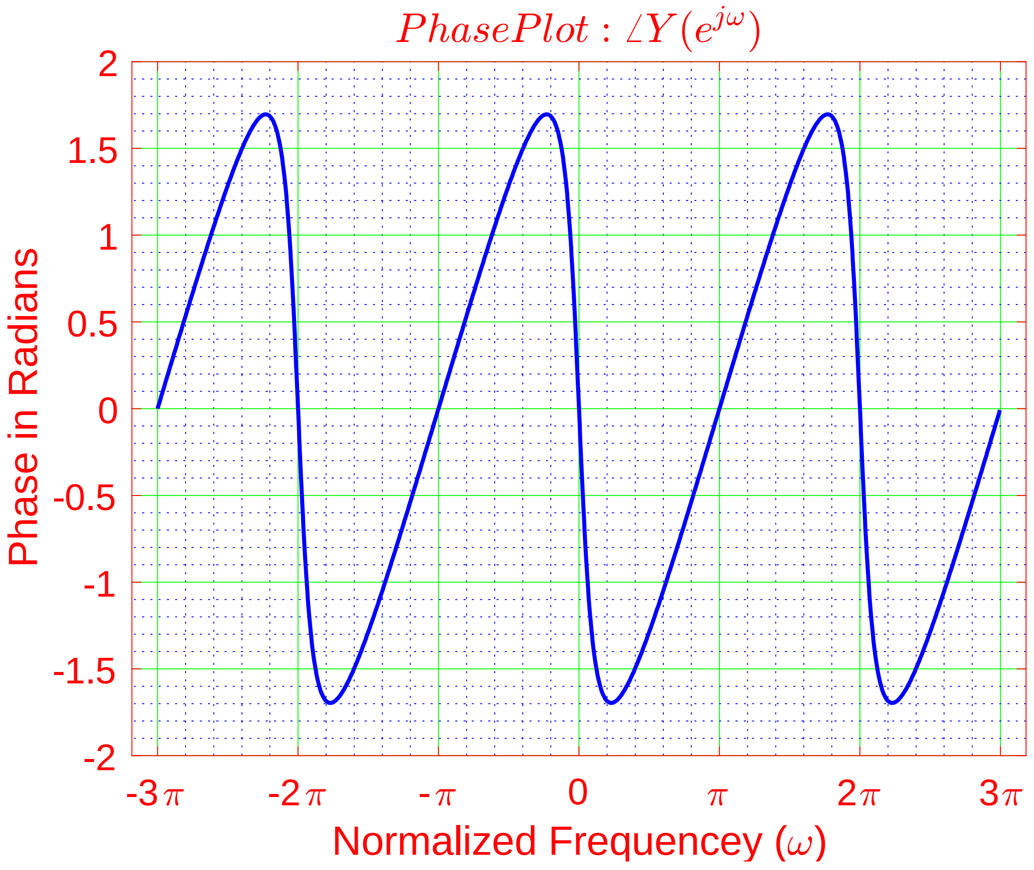
The Discrete-Time Fourier Transform Theorems (Differentiation-in-Freqeuncy-Theorem)

- Example 3.13: Find out the discrete-time Fourier transform representation of following sequence
- $y[n] = (n+1)\alpha^n \mu[n], \quad |\alpha| < 1$
- Solution:
- $y[n] = n\alpha^n \mu[n] + \alpha^n \mu[n] \ldots (A)$
- We have found discrete-time Fourier transform for $\alpha^n \mu[n]$
- $\mathcal{F}\{\alpha^n \mu[n]\} = \frac{1}{1-\alpha e^{-j\omega}} \ldots \ldots (B)$
- The discrete-time Fourier transform for $n\alpha^n \mu[n]$ can be found using differentiation theorem of DTFT
- $\mathcal{F}\{n\alpha^n \mu[n]\} = \frac{jd}{d\omega} \mathcal{F}\{\alpha^n \mu[n]\}$
- $\mathcal{F}\{n\alpha^n \mu[n]\} = \frac{jd}{d\omega} \left\{\frac{1}{1-\alpha e^{-j\omega}}\right\}$
- $\mathcal{F}\{n\alpha^n \mu[n]\} = j\left(\frac{-j\alpha e^{-j\omega}}{\left(1-\alpha e^{-j\omega}\right)^2}\right)$
- $\mathcal{F}\{n\alpha^n \mu[n]\} = \frac{\alpha e^{-j\omega}}{\left(1-\alpha e^{-j\omega}\right)^2} \ldots (C)$
- Using (B) and (C) in (A), will results
- $Y\left(e^{j\omega}\right) = \frac{1}{\left(1-\alpha e^{-j\omega}\right)^2}$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Differentiation-in-Freqeuncy-Theorem)





Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Convolution Theorem)

- The Fourier transform $\mathrm{Y}(\mathrm{e}^{j\omega})$ of the convolution sum of two sequence $y[n] = g[n] * h[n]$, is given by the product of their Fourier transforms $\mathrm{G}(\mathrm{e}^{j\omega})\ \mathrm{H}(\mathrm{e}^{j\omega})$; that is,
- $g[n] * h[n] \overset{\mathcal{F}}{\leftrightarrow} G(e^{j\omega})H(e^{j\omega}) \quad (3.46)$
- Proof:
- The convolution of $y[n] = g[n] * h[n]$, is given by
- $y[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$
- Taking DTFT of above equation
- $\mathrm{Y}(\mathrm{e}^{j\omega}) = \sum_{n=-\infty}^{\infty}\{\sum_{k=-\infty}^{\infty} g[k]h[n-k]\}\, e^{-j\omega n}$
- $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[k]\{\sum_{k=-\infty}^{\infty} h[n-k]e^{-j\omega n}\}$
- Let $m = n - k \implies n = m + k$
- $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[k]\{\sum_{k=-\infty}^{\infty} h[m]e^{-j\omega(m+k)}\}$
- $Y(e^{j\omega}) = \sum_{n}^{\infty} g[k]\{\sum_{k=-\infty}^{\infty} h[m]e^{-j\omega m}\}e^{-j\omega k}$
- $Y(e^{j\omega}) = \sum_{n}^{\infty} g[k]\{H(e^{j\omega})\}e^{-j\omega k}$
- $Y(e^{j\omega}) = \sum_{n}^{\infty} g[k]e^{-j\omega k}\ H(e^{j\omega})$
- $Y(e^{j\omega}) = G(e^{j\omega})H(e^{j\omega})$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Convolution Theorem)

- Example 3.14: Find the convolution sum of following two sequences using DTFT
- $x[n] = \alpha^n \mu[n] \quad |\alpha| < 1$ and
- $h[n] = \beta^n \mu[n] \quad |\beta| < 1$
- $y[n] = x[n] * h[n]$
- Solution:
- The DTFT of $x[n]$ and $y[n]$ of the two sequences are obtained as follows:
- $X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$
- $H(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$
- The convolution is obtained as
- $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- $Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$
- Using partial fraction by parts above expression can be written as
- $Y(e^{j\omega}) = \frac{\frac{\alpha}{\alpha-\beta}}{1-\alpha e^{-j\omega}} - \frac{\frac{\beta}{\alpha-\beta}}{1-\beta e^{-j\omega}}$
- After taking IDFT, we have
- $y[n] = \frac{\alpha}{\alpha-\beta}\alpha^n \mu[n] + \frac{\beta}{\alpha-\beta}\beta^n \mu[n]$
- $y[n] = \left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\mu[n]$

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Modulation Theorem)

- The Fourier transform $\mathrm{Y}(\mathrm{e}^{j\omega})$ of the product of two sequence $y[n] = g[n]h[n]$, is given by the convolution integral of their Fourier transforms $\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{G}(\mathrm{e}^{j\theta})\, \mathrm{H}(\mathrm{e}^{j(\omega-\theta)})d\theta$; that is,
- $g[n]h[n] \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{G}(\mathrm{e}^{j\theta})\, \mathrm{H}(\mathrm{e}^{j(\omega-\theta)})d\theta \quad (3.50)$
- Proof:
- The DTFT of $y[n] = g[n]h[n]$, is given by
- $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]h[n]e^{-j\omega n}$
- Taking inverse DTFT of $g[n]$ is given below
- $g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})e^{j\theta n}d\theta$ put this in above equation will result
- $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})e^{j\theta n}d\theta\right\} h[n]e^{-j\omega n}$
- $Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})\{\sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega-\theta)n}\}d\theta$
- Where
- $\mathrm{H}(\mathrm{e}^{j(\omega-\theta)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega-\theta)n}$
- Putting in above equation will reduce to
- $Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})\mathrm{H}(\mathrm{e}^{j(\omega-\theta)})d\theta$

Modulation theorem is also called Windowing Theorem, used in amplitude modulation and FIR filter design

Discrete Time Signals in the Frequency Domain

The Discrete-Time Fourier Transform Theorems (Parseval's Theorem)

- The Parseval's theorem states that the sum of sample-by-sample product of two complex sequences in terms of an integral of the product of their Fourier transforms.
- $\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{G}(\mathrm{e}^{j\omega})\,\mathrm{H}^*(\mathrm{e}^{j\omega})d\omega \quad (3.51)$
- Proof:
- $\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \sum_{n=-\infty}^{\infty} g[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})e^{-j\omega n}d\omega\right)$
- $= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})\left(\sum_{n=-\infty}^{\infty} g[n]\,e^{-j\omega n}\right)d\omega$
- $\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})G(e^{j\omega})d\omega$

The Parseval's theorem is used in the computation of the energy of a finite-energy sequene.

Discrete Time Signals in the Frequency Domain

Energy Density Spectrum of a Discrete-Time Sequence

- The total energy of a finite-energy sequence $g[n]$ is given by
- $\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2$
- $\mathcal{E}_g = \sum_{n=-\infty}^{\infty} g[n] g^*[n]$
- From Paseval's theorem, we know that

$$\sum_{n=-\infty}^{\infty} g[n] h^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{G}(\mathrm{e}^{\mathrm{j}\omega})\, \mathrm{H}^*(\mathrm{e}^{\mathrm{j}\omega}) d\omega,$$

so the above expression reduces o

- $\mathcal{E}_g = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{G}(\mathrm{e}^{\mathrm{j}\omega})\, \mathrm{G}^*(\mathrm{e}^{\mathrm{j}\omega}) d\omega$
- $\mathcal{E}_g = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\mathrm{G}(\mathrm{e}^{\mathrm{j}\omega})\right|^2 d\omega \qquad (3.52)$
- $S_{gg}(e^{j\omega}) = \left|G(e^{j\omega})\right|^2 \qquad (3.53)$

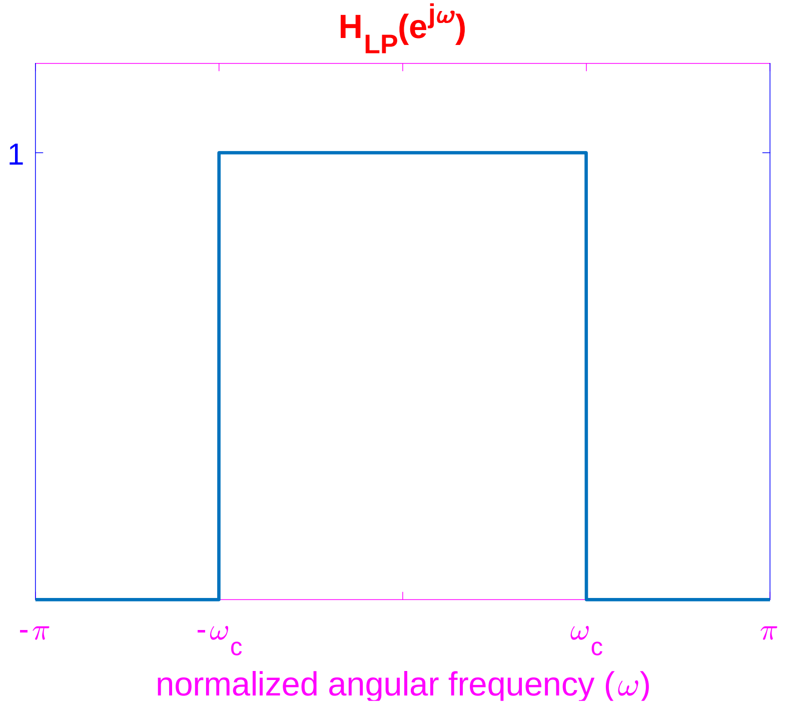
The quantity
$S_{gg}(e^{j\omega}) = \left|G(e^{j\omega})\right|^2$
Is called the **energy density spectrum** of the sequence.

The area under this curve $\left|G(e^{j\omega})\right|^2$ in the range $-\pi \le \omega < \pi$ divided by 2π is the energy of the sequence

Discrete Time Signals in the Frequency Domain

Energy Density Spectrum of a Discrete-Time Sequence

- Example 3.15: Find the energy of a lowpass filter, shown in figure,
- $X_{LP}(e^{jw}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases} \quad (3.27)$
- Solution:
- $\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\mathrm{H_{LP}}(\mathrm{e}^{j\omega})\right|^2 d\omega$
- $\mathcal{E}_g = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{1}{2\pi}|\omega|_{-\omega_c}^{\omega_c} = \frac{\omega_c}{\pi}$
- $\mathcal{E}_g = \frac{\omega_c}{\pi} < \infty$ Hence, $h_{LP}[n]$ is a finite-energy sequence



Discrete Time Signals in the Frequency Domain

Energy Density Spectrum of a Discrete-Time Sequence

- Example 3.6: Find the energy of following causal exponential sequence
- $x[n] = \alpha^n u[n], \qquad |\alpha| < 1,$
- Solution:
- The energy is given by
- $\mathcal{E}_g = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{X}(\mathrm{e}^{\mathrm{j}\omega})\right|^2 d\omega$
- $\mathcal{E}_g = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{X}(\mathrm{e}^{\mathrm{j}\omega})\mathrm{X}^*(\mathrm{e}^{\mathrm{j}\omega})d\omega$
- Where,
- $X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}}, X^*(e^{jw}) = \frac{1}{1-\alpha e^{jw}}$
- So,
- $\mathcal{E}_g = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1-\alpha e^{-jw}}\frac{1}{1-\alpha e^{jw}}d\omega$
- $\mathcal{E}_g = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1+\alpha^2-2\alpha\cos\omega}d\omega$
- Using MATLAB command we have following integration
- >> syms a w
- >> f=1/(1+a^2-2*a*cos(w));
- >> int(f)
- For $\alpha = 0.5$
- $\mathcal{E}_g = 0.8488\tan^{-1}\left\{3\tan\left(\frac{\pi}{2}\right)\right\}$
- $\mathcal{E}_g = 1.333333$

Discrete Time Signals in the Frequency Domain

Energy Density Spectrum of a Discrete-Time Sequence

- $\frac{1}{1+\alpha^2-2\alpha\cos\omega}=\frac{1}{1+\alpha^2-2\alpha\left\{\cos^2\left(\frac{\omega}{2}\right)-\sin^2\left(\frac{\omega}{2}\right)\right\}}=\frac{1}{1+\alpha^2-2\alpha\cos^2\left(\frac{\omega}{2}\right)+2\alpha\sin^2\left(\frac{\omega}{2}\right)}$
- $\frac{1}{1+\alpha^2-2\alpha\cos\omega}=\frac{1}{\left\{\cos^2\left(\frac{\omega}{2}\right)+\sin^2\left(\frac{\omega}{2}\right)\right\}\alpha^2+\left\{\cos^2\left(\frac{\omega}{2}\right)+\sin^2\left(\frac{\omega}{2}\right)\right\}-2\alpha\cos^2\left(\frac{\omega}{2}\right)+2\alpha\sin^2\left(\frac{\omega}{2}\right)}$
- $\frac{1}{1+\alpha^2-2\alpha\cos\omega}=$
$\frac{1}{\left\{\cos^2\left(\frac{\omega}{2}\right)\alpha^2-2\alpha\cos^2\left(\frac{\omega}{2}\right)+\cos^2\left(\frac{\omega}{2}\right)\right\}+\left\{\sin^2\left(\frac{\omega}{2}\right)\alpha^2+2\alpha\sin^2\left(\frac{\omega}{2}\right)+\sin^2\left(\frac{\omega}{2}\right)\right\}}$
- $\frac{1}{1+\alpha^2-2\alpha\cos\omega}=\frac{1}{\cos^2\left(\frac{\omega}{2}\right)\{\alpha-1\}^2+\sin^2\left(\frac{\omega}{2}\right)\{\alpha+1\}^2}$

Discrete Time Signals in the Frequency Domain

(Band-Limited Discrete-Time Signals)

- The spectrum of a discrete-time signal is a periodic function of ω with a period of 2π $(\text{i.e } 0 \leq \omega < 2\pi \quad or \quad -\pi \leq \omega \leq \pi)$.
- A full-band, discrete-time signal has a spectrum occupying a whole frequency range $-\pi \leq \omega \leq \pi$.
- A band-limited, discrete-time signal has a spectrum occupying a limited portion of the above frequency range $-\pi \leq \omega_a \leq \omega \leq \omega_b \leq \pi$.
- An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $0 \leq \omega_a \leq \omega \leq \omega_b < \pi$: that is,
- $X(e^{j\omega}) = \begin{cases} 0, & 0 \leq |\omega| < \omega_a \\ 0, & \omega_b < |\omega| < \pi \end{cases}$ i.e., the signal exisits only in range $\omega_a \leq \omega \leq \omega_b$
- An ideal band-limited signal cannot be generated in practice.
- However, for practical purposes, it is ensured that the energy of band-limited signal outside the specified frequency range is sufficiently small.

Discrete Time Signals in the Frequency Domain

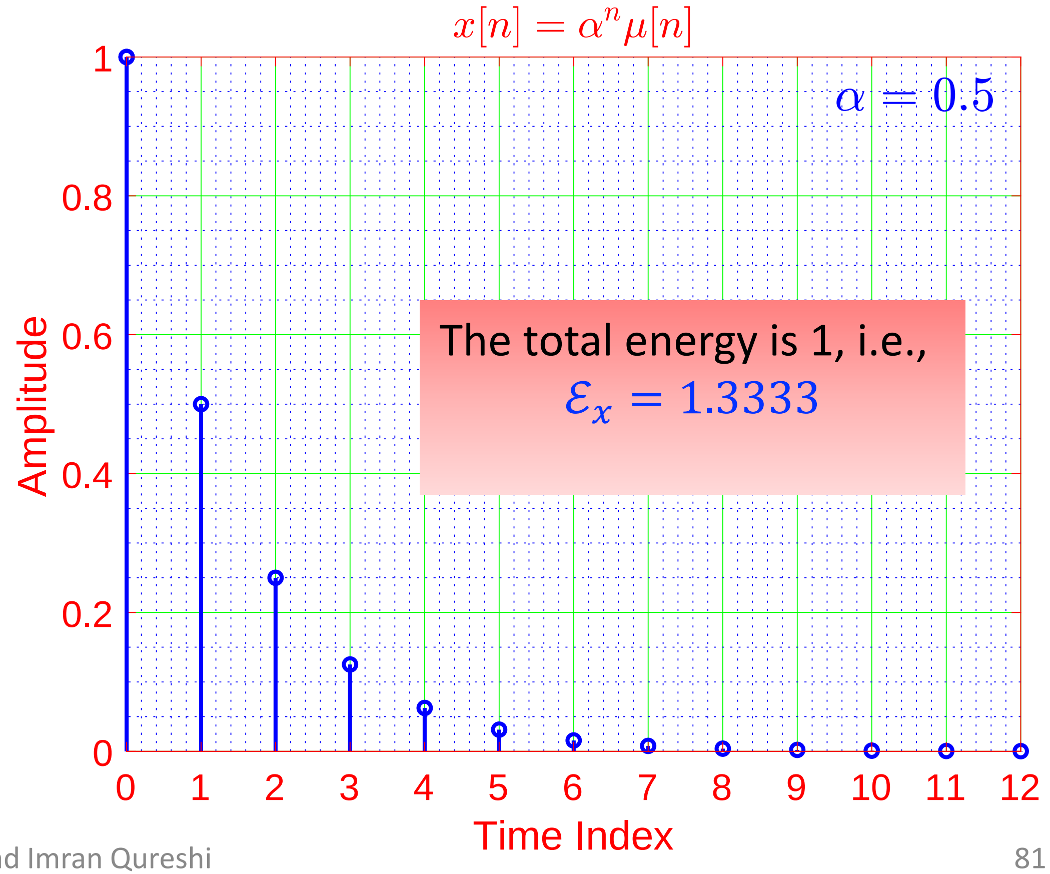
(Band-Limited Discrete-Time Signals)

- Lowpass Discrete-time Signal:
  - A low pass discrete-time signal exists only in the range $0 \leq \omega \leq \omega_p < \pi$, is defined as
  - $X(e^{j\omega}) = \begin{cases} |X(e^{j\omega})|, & 0 \leq |\omega| \leq \omega_p \\ 0, & \omega_p < |\omega| < \pi \end{cases}$ i.e., the signal exisits only in range $0 \leq |\omega| \leq \omega_p$
  - Where, ω_p is called bandwidth of the signal.
- Highpass Discrete-time Signal:
  - A highpass discrete-time signal exists in the range $\omega_p \leq \omega < \pi,$ and is defined as
  - $X(e^{j\omega}) = \begin{cases} 0, & 0 \leq |\omega| < \omega_p \\ |X(e^{j\omega})|, & \omega_p \leq |\omega| < \pi \end{cases}$ i.e., the signal exisits only in range $\omega_p \leq |\omega| < \pi$
  - The energy of the signal is split evenly between the positive and negative frequency range, and $\pi - \omega_p$ is the bandwidth of the signal.
- Bandpass Discrete-time Signal:
  - A bandpass discrete-time signal has a spectrum occupying the frequency range $0 < \omega_L \leq |\omega| \leq \omega_H < \pi$
  - $X(e^{j\omega}) = \begin{cases} 0, & 0 \leq |\omega| < \omega_a \\ 0, & \omega_b < |\omega| < \infty \end{cases}$ i.e., the signal exisits only in range $\omega_a \leq \omega \leq \omega_b$
  - The bandwidth of the signal is $\omega_H - \omega_L$.
  - A bandwidth signal with a bandwidth much smaller than $\frac{\omega_H+\omega_L}{2}$ is referred **as narrow-band** signal.

Discrete Time Signals in the Frequency Domain

(Band-Limited Discrete-Time Signals)

- Example : Find total energy of the following signal for $\alpha = 0.5$, and determine its 80% bandwidth
- $x[n] = \alpha^n \mu[n]$
- Solution: (Total Energy)
- The energy can be computed as:
- $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} \alpha^{2n}$
- $\mathcal{E}_x = 1 + \alpha^2 + \alpha^4 + \cdots$
- $\mathcal{E}_x = \frac{1}{1-\alpha^2} = \frac{1}{1-0.25} = 1.333$

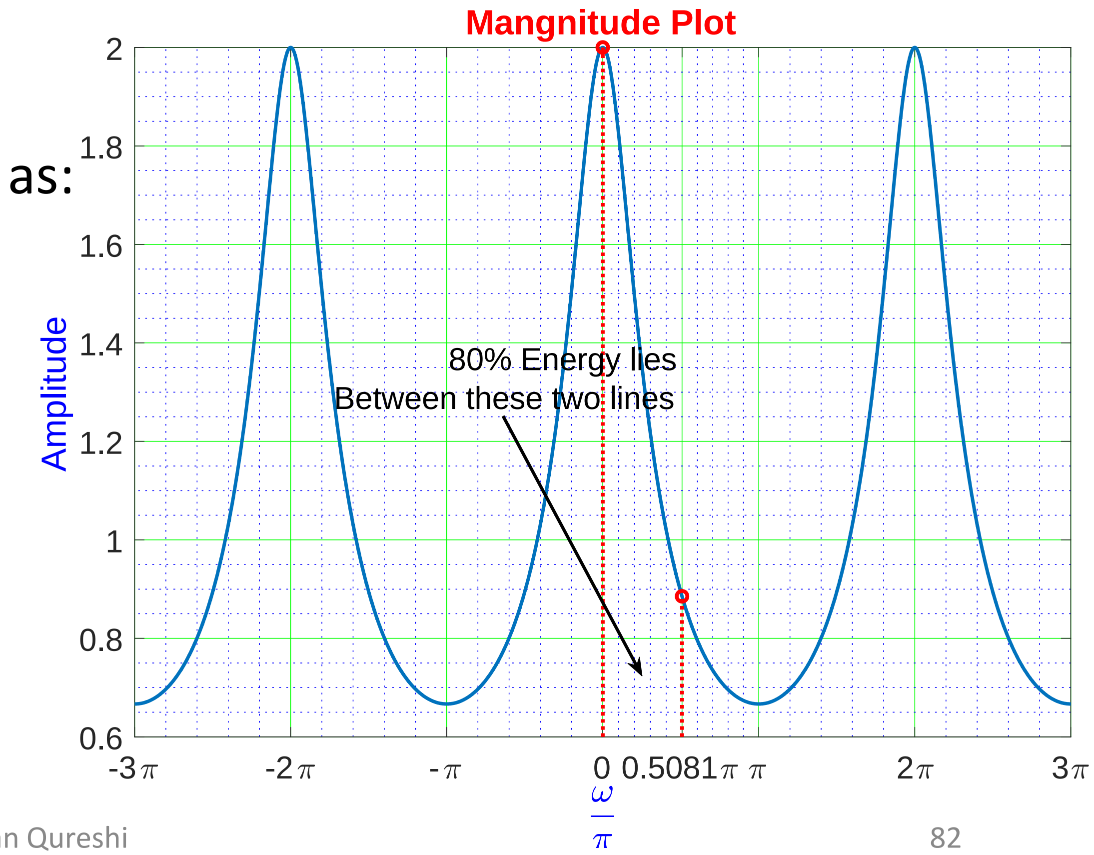


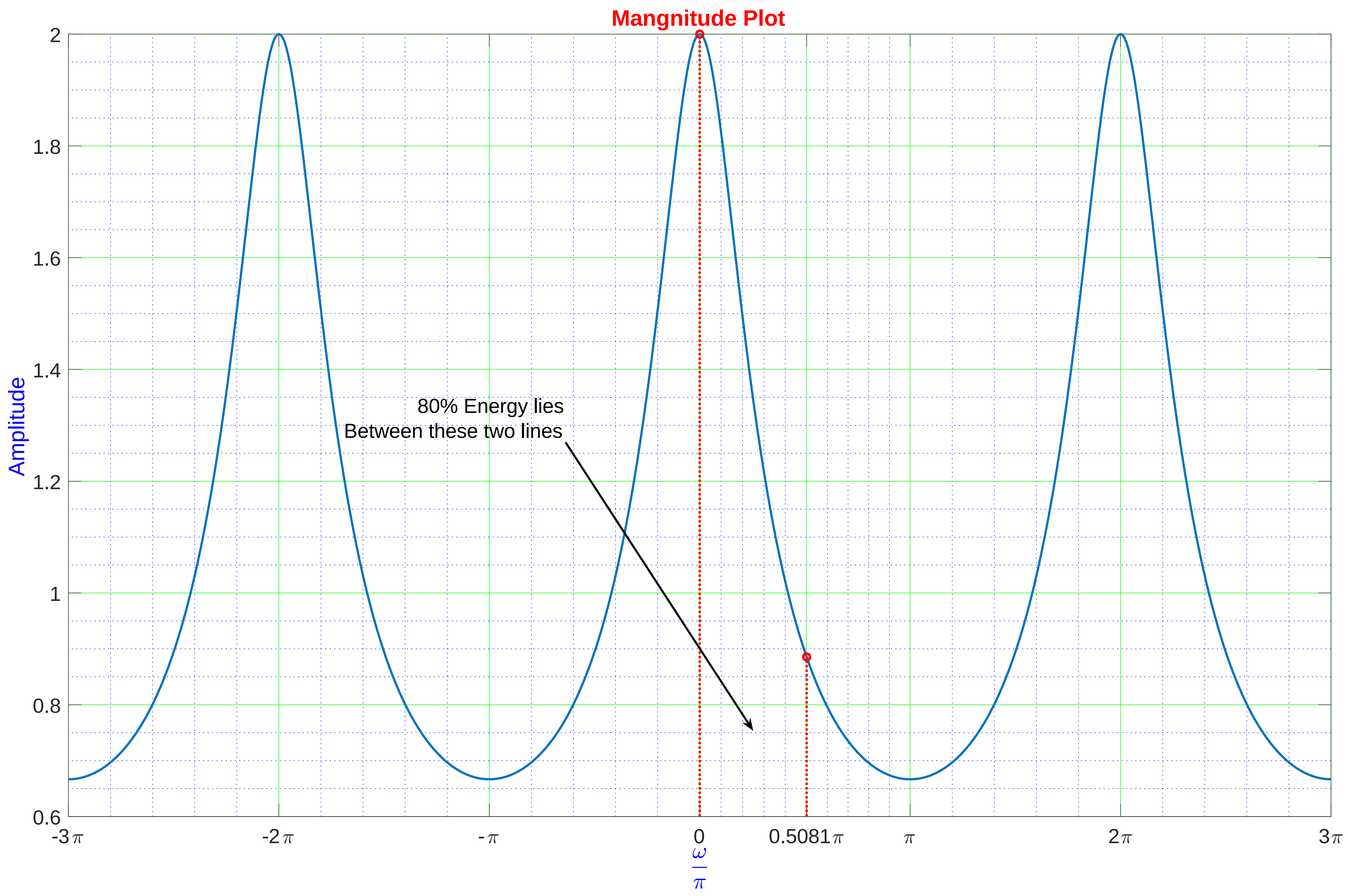
Discrete Time Signals in the Frequency Domain

(Band-Limited Discrete-Time Signals)

- Example 3.1: Find total energy of the following signal for $\alpha = 0.5$, and determine its 80% bandwidth
- $x[n] = \alpha^n \mu[n]$
- Solution: (80% Bandwidth)
- The energy can be computed Parseval's theorem as:
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$
- $\mathcal{E}_x = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \left|\frac{1}{1-\alpha e^{-jw}}\right|^2 d\omega$
- $0.8\left(\frac{4}{3}\right) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \frac{1}{1+\alpha^2-2\alpha\cos\omega} d\omega$
- $\omega_c \leq 0.5081 \times \pi$

The bandwidth is
$0 \leq \omega \leq 1.5962$ radians or
$0 \leq \omega \leq 0.5081\pi$ radians

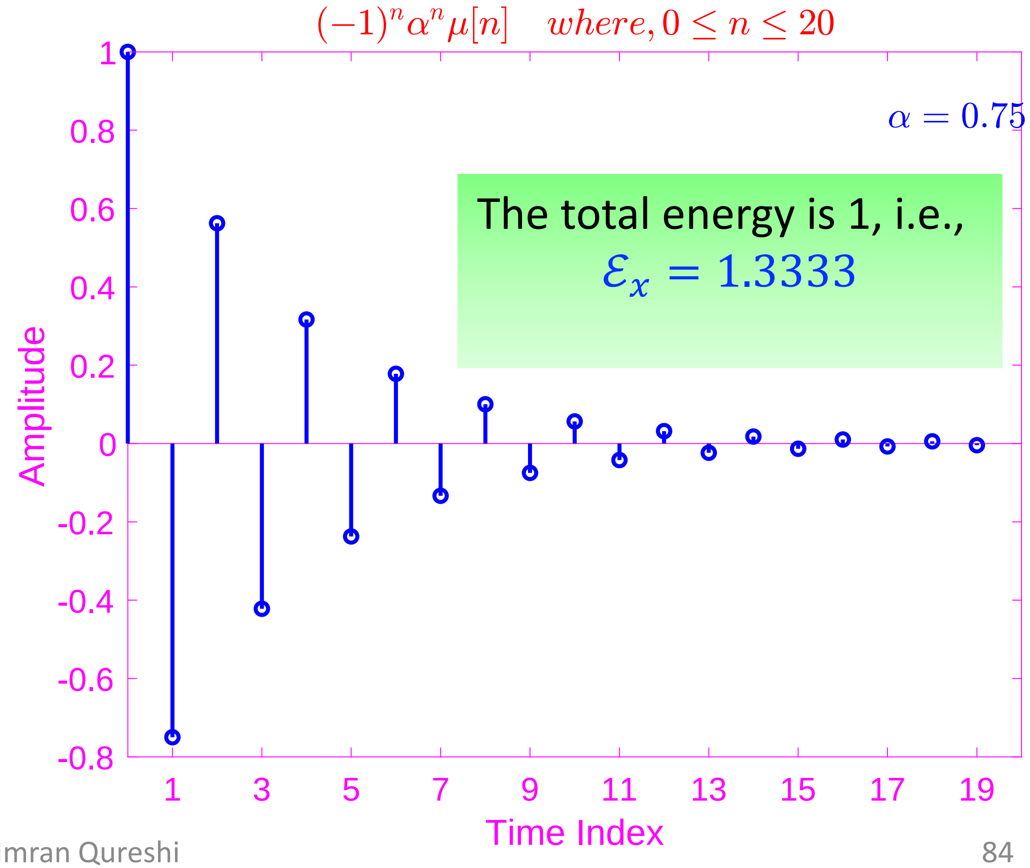



Mangnitude Plot
Amplitude
2
1.8
1.6
1.4
1.2
1
0.8
0.6
-3π
-2π
-π
0
0.5081π
π
2π
3π
ω/π
80% Energy lies
Between these two lines

Discrete Time Signals in the Frequency Domain

(Band-Limited Discrete-Time Signals)

- Example : Find total energy of the following signal for $\alpha = 0.5$, and determine its 80% bandwidth
- $y[n] = (-1)^n \alpha^n \mu[n] \quad |\alpha| < 1$
- Solution: (Total Energy)
- The energy can be computed as:
- $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} \alpha^{2n}$
- $\mathcal{E}_x = 1 + \alpha^2 + \alpha^4 + \cdots$
- $\mathcal{E}_x = \frac{1}{1-\alpha^2} = \frac{1}{1-0.25} = \frac{4}{3} = 1.333$



Discrete Time Signals in the Frequency Domain

(Band-Limited Discrete-Time Signals)

- Example : Find total energy fo the following signal for $\alpha = 0.5$, and determine its 80% bandwidth
- $y[n] = (-1)^n \alpha^n \mu[n] \quad |\alpha| < 1$
- Solution: (Total Energy Using Parseval's Theorem)
- The energy can be computed as:
- $\mathcal{E}_y = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\mathrm{Y}(\mathrm{e}^{j\omega})\right|^2 d\omega$
- $\mathcal{E}_y = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\frac{1}{1+\alpha e^{-jw}}\right|^2 d\omega$
- $\mathcal{E}_y = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1+\alpha^2+2\alpha \cos \omega} d\omega$
- >> syms a w
- >> f = 1/(1+a^2+2*a*cos(w));
- >> F=int(f)
- $\mathcal{E}_y = \frac{1}{2\pi} \left| \frac{2}{|\alpha^2-1|} \tan^{-1}\left\{\frac{|\alpha-1|\tan\left(\frac{\omega}{2}\right)}{|\alpha+1|}\right\}\right|_{-\pi}^{\pi}$
- Put $\alpha = 0.5$
- $\mathcal{E}_y = \frac{4}{3\pi} \left|\tan^{-1}\left\{\frac{1}{3}\tan\left(\frac{\omega}{2}\right)\right\}\right|_{-\pi}^{\pi}$
- $\mathcal{E}_y = \frac{4}{3\pi} \left[\tan^{-1}\left\{\frac{1}{3}\tan\left(\frac{\pi}{2}\right)\right\} - \tan^{-1}\left\{\frac{1}{3}\tan\left(\frac{-\pi}{2}\right)\right\}\right]$
- $\mathcal{E}_x = \frac{4}{3\pi} \left[2\tan^{-1}\left\{\frac{1}{3}\tan\left(\frac{\pi}{2}\right)\right\}\right] = \frac{8}{3\pi}[1.5708]$
- $\mathcal{E}_x = 1.333 = \frac{4}{3}$

The total energy is 1, i.e., $\mathcal{E}_x = 1.3333$

Discrete Time Signals in the Frequency Domain

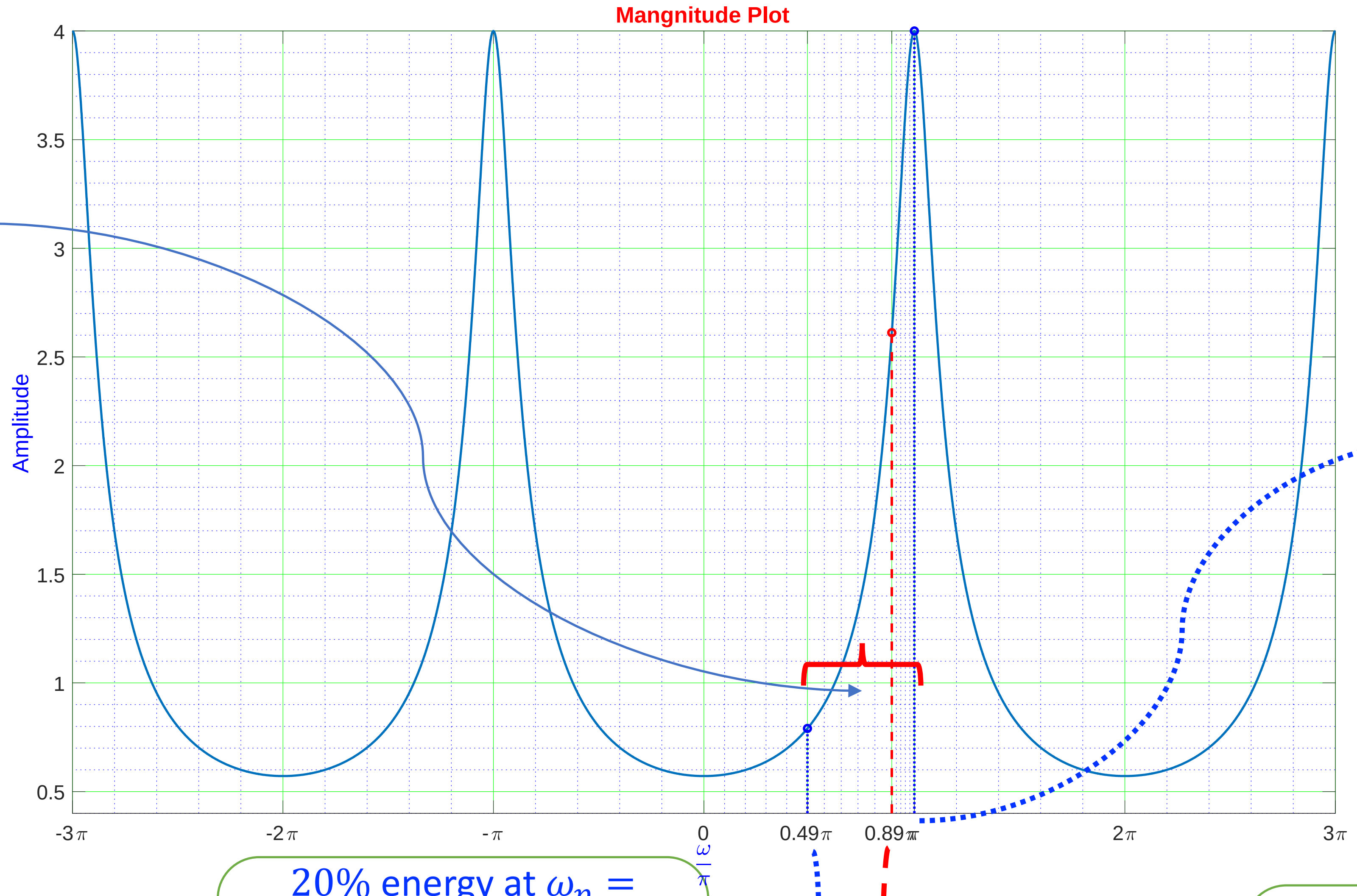
(Band-Limited Discrete-Time Signals)

- Example : Find total energy fo the following highpass signal for $\alpha = 0.5$, and determine its 80% bandwidth
- $y[n] = (-1)^n \alpha^n \mu[n] \quad |\alpha| < 1$
- Solution: (80% Bandwidth)
- The energy can be computed Parseval's theoremas:
- $\mathcal{E}_y = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\mathrm{Y}(\mathrm{e}^{\mathrm{j}\omega})\right|^2 d\omega$
- $\mathcal{E}_x = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \left|\frac{1}{1+\alpha e^{-\mathrm{j}\omega}}\right|^2 d\omega$
- $0.8\left(\frac{4}{3}\right) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{1}{1+\alpha^2+2\alpha\cos\omega} d\omega$
- $\frac{16}{15} = \frac{1}{2\pi} \left| \frac{2}{|\alpha^2-1|} \tan^{-1}\left\{ \frac{|\alpha-1|\tan\left(\frac{\omega}{2}\right)}{|\alpha+1|} \right\} \right|_{-\omega_c}^{\omega_c}$
- Put $\alpha = 0.5$
- $\frac{16}{15} = \frac{4}{3\pi} \left| \tan^{-1}\left\{ \frac{1}{3} \tan\left(\frac{\omega}{2}\right) \right\} \right|_{-\omega_c}^{\omega_c}$
- $\frac{4\pi}{5} = \left[2\tan^{-1}\left\{ \frac{1}{3} \tan\left(\frac{\omega_c}{2}\right) \right\} \right] = \frac{8}{3\pi}[1.5708]$
- $\mathcal{E}_x = 2 \times \tan^{-1}\left\{ 3\tan\left(\frac{2\pi}{5}\right) \right\} = 2.9258\ radians$

80% energy at $\omega_c = 2.958\ or\ 0.9313\pi\ radians$
20% energy at $\omega_p = 1.5452\ or\ 0.4919\pi\ radians$
70% energy at $\omega_p = 2.8051\ or\ 0.8929\pi\ radians$

The bandwidth of highpass signal is
$1.5452 \leq \omega \leq \pi$radiansor
$0.4919\pi \leq \omega \leq \pi$radians

The bandwidth is $1.5452 \le \omega \le \pi$ radians or $0.4919\pi \le \omega \le \pi$ radians



100% energy at $\omega = \pi$ radians

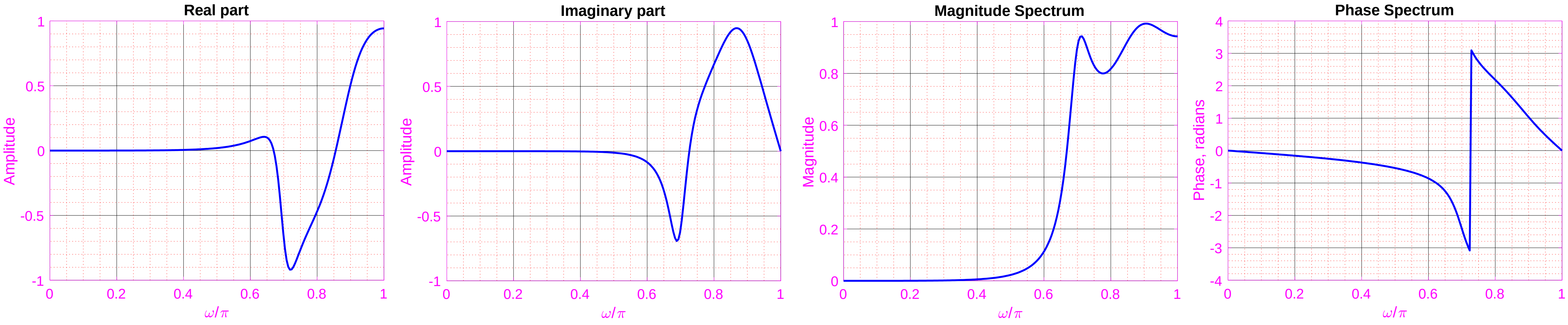
20% energy at $\omega_p = 1.5452$ or $0.4919\pi\ radians$

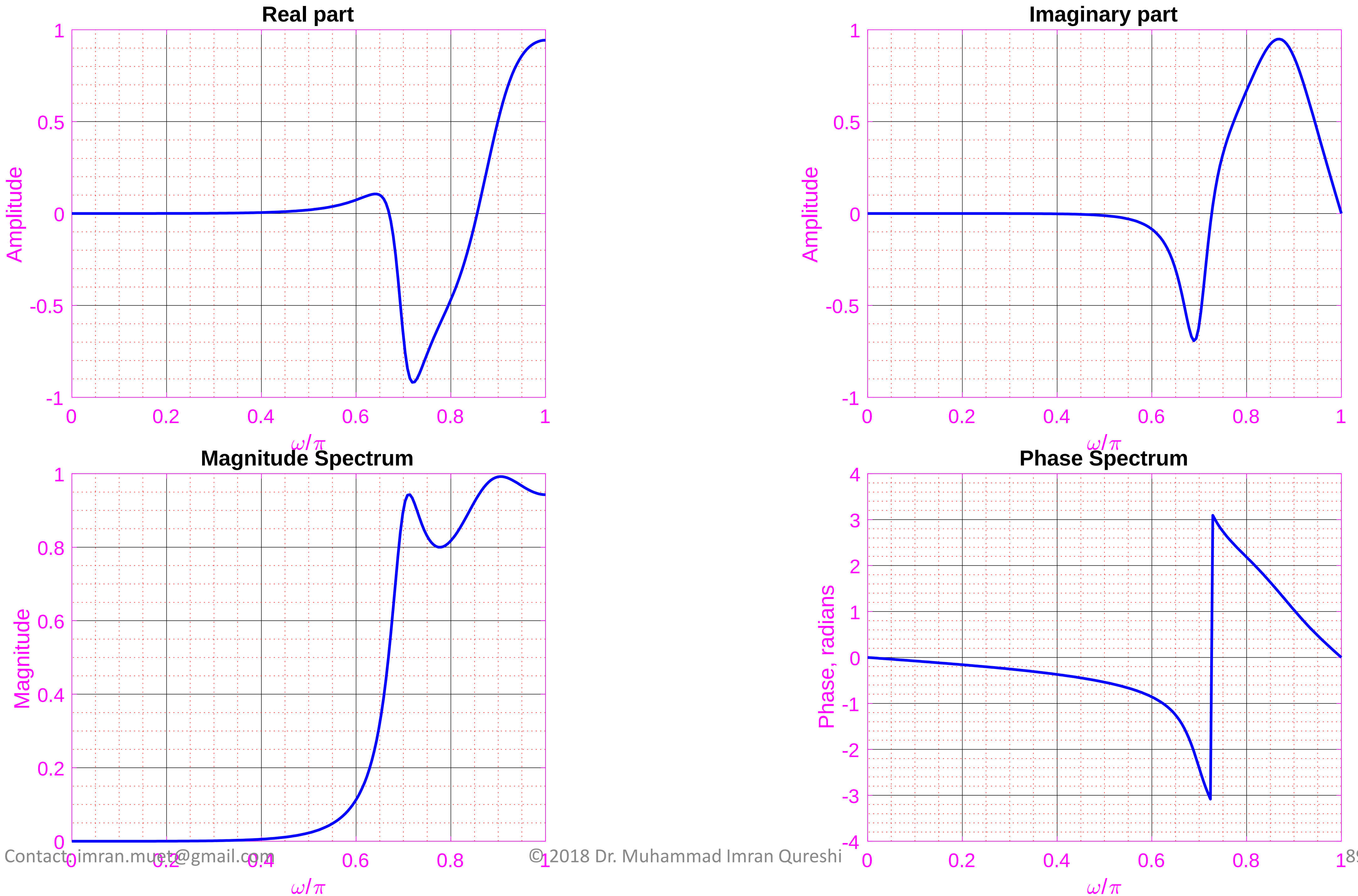
70% energy at $\omega_p = 2.8051$ or $0.8929\pi\ radians$

Discrete Time Signals in the Frequency Domain
(DTFT Compuation Using MATLAB)

- The function freqz can be used to calculate the values of the Fourier transform of a sequence, described as a rational function in $e^{j\omega}$ as shown below, at a prescribed set of discrete frequency points.

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.055e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

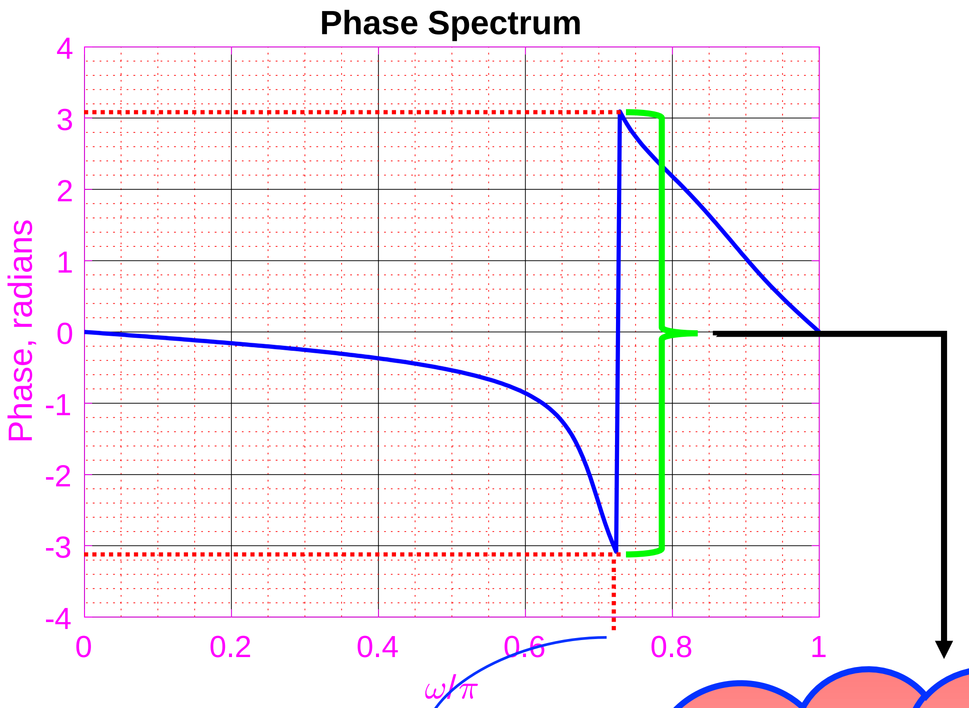



Real part
Imaginary part
Amplitude
Amplitude
1
0.5
0
-0.5
-1
0 0.2 0.4 0.6 0.8 1
ω/π
Magnitude Spectrum
Phase Spectrum
Magnitude
1
0.8
0.6
0.4
0.2
0
Phase, radians
4
3
2
1
0
-1
-2
-3
-4
ω/π

Discrete Time Signals in the Frequency Domain

(The Unwrapped Phase Function)

- In numerical computation, when the computed phase function is outised the range $[-\pi, \pi]$, the phase is computed modulo 2π to bring the computed value to this range.
- As a result, the phase functions of some sequence exhibit discontinuities of 2π radians in the plot.
- The discontinuity of 2π at around $\omega = 0.72$ can be seen in the graph.



Discontinuity at frequency $\omega = 0.72$

The is the discontinuity, as the phase reaches to $-\pi$, instead of going further onward, it jumps to π (modulo 2π) and starts over

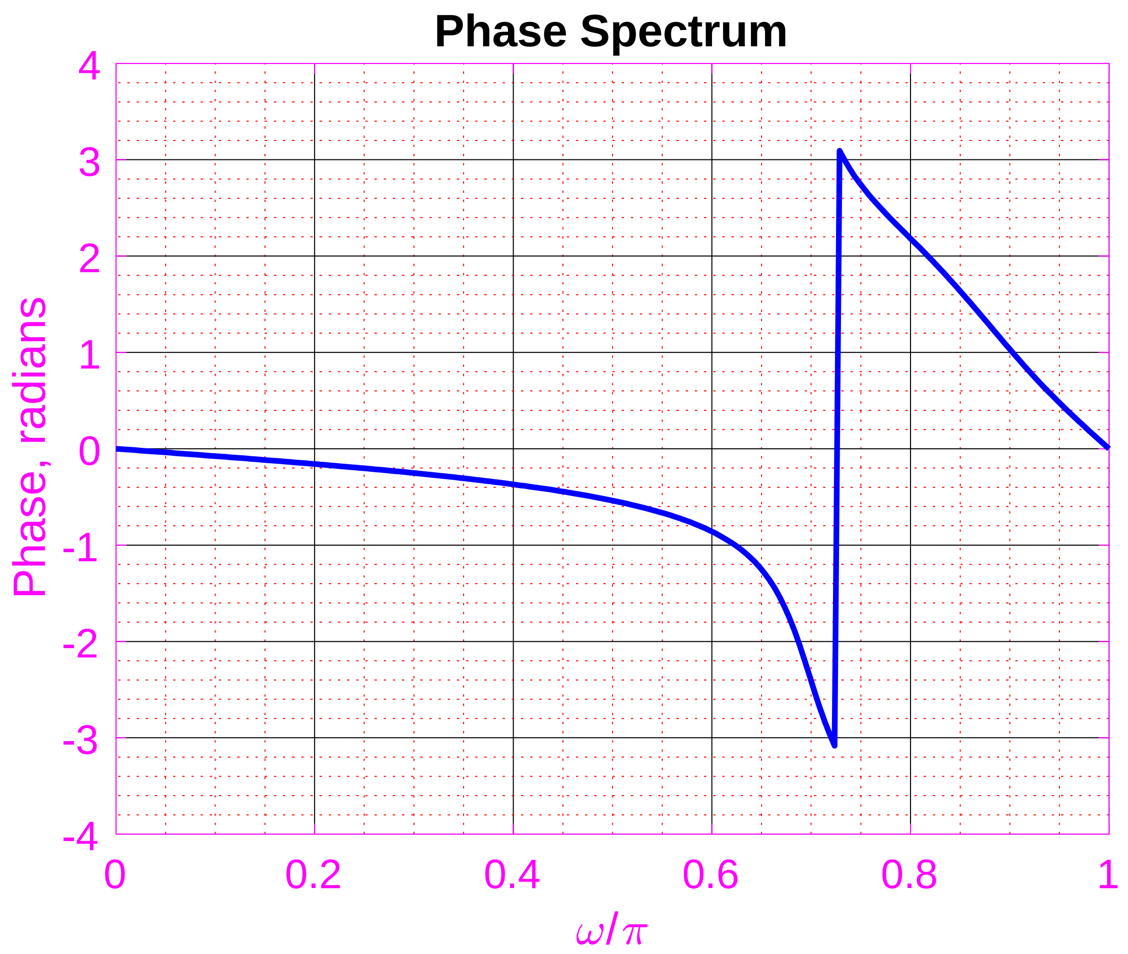
Discrete Time Signals in the Frequency Domain

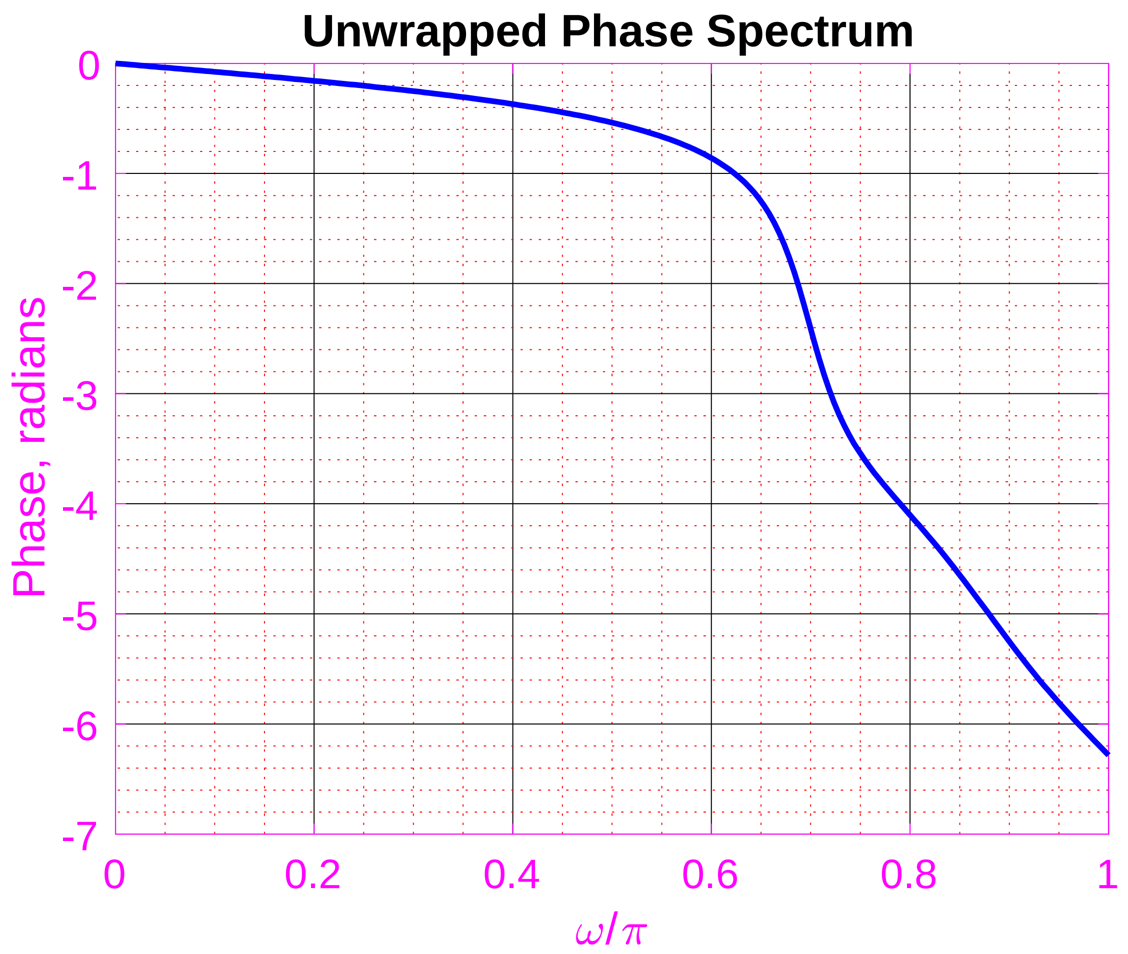
(The Unwrapped Phase Function)

- In numerical computation, when the computed phase function is outised the range $[-\pi, \pi]$, the phase is computed modulo 2π to bring the computed value to this range.
- As a result, the phase functions of some sequence exhibit discontinuities of 2π radians in the plot.
- The discontinuity of 2π at around $\omega = 0.72$ can be seen in the graph.
- Unwrapping the Phase: In such cases, an alternate type of phase function that is a continuous function of ω derived from the original phase function by removing the discontinuity of 2π.
- The new phase function $(\theta_c(\omega))$ is formed, and it is continuous function of ω.

Discrete Time Signals in the Frequency Domain

(The Unwrapped Phase Function)





Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals

- Most signals, such asspeech, music, and images, in real world are continuous in time.
- For processing by digital systems, such signals are needed to be converted to digital form (in binary form) using analog-to-digital converter.
- After processing the discrete-time signals are converted back to continuous-time signals using digital-to-analog converter.

Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals

- A/D converter and D/A converter are enough to convert analog to digital and digital to analog, respectively.
- In addition to A/D converter and D/A converter, we need several additional circuits.

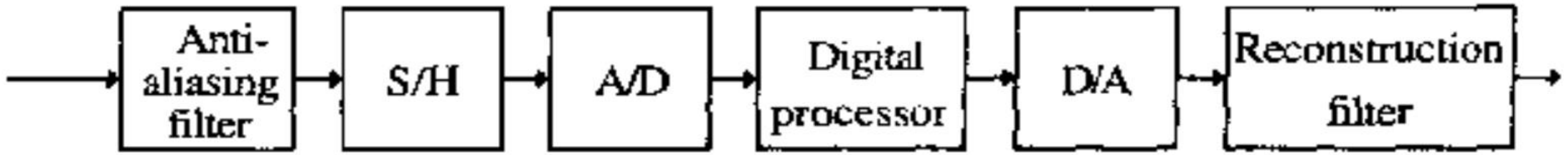


Figure 5.1: Block diagram representation of the discrete-time digital processing of a continuous-time signal.

Figure 5.2: A simplified representation of Figure 5.1.

Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals

- A/D converter and D/A converter are enough to convert analog to digital and digital to analog, respectively.
- In addition to A/D converter and D/A converter, we need several additional circuits.
  - Anti-aliasing Filter
  - Sample-and-hold (S/H) circuit:
  - Reconstruction (Smoothing) Filter

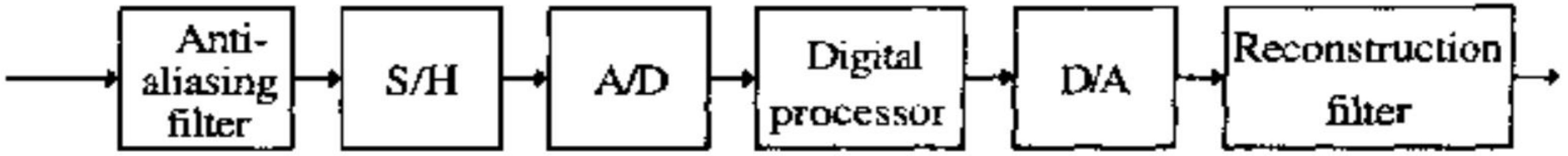


Figure 5.1: Block diagram representation of the discrete-time digital processing of a continuous-time signal.

Discrete Time Signals in the Frequency Domain
Digital Processing of Continuous-Time Signals

- Sample-and-hold (S/H) circuit:
  - It has dual purposes:
    1. It samples the input continuous-time signal at periodic intervals.
    2. Since the A/D conversion usually takes a finite amount of time, therefore the analog signal at the input of the A/D converter remains constant in amplitude until the conversion is complete to minimize the error in its representation. This is achieved by S/H circuit.
- Reconstruction (Smoothing) Filter:
  - The output of a D/A converter is a staircase-like waveform. In order to smooth that waveform an analog reconstruction(smoothing) filter is used.
- Anti-aliasing Filter
  - To prevent aliasing, an anti-aliasing filter is used before S/H circuit.

Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals

- In general, there exists an infinite number of continuous-time signals that, when sampled at the same sampling rate lead to the same discrete-time signal.
- However, under certain conditions,
  - it is possible to relate a unique continuous-time signal to a given discrete-time sequence.
  - It is possible to recover the original continuous-time signal from its sampled value.
- In following slides, we will study those conditions.

Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals:

Effect of Time-Domain Sampling in Frequency Domain

- Let $g_a(t)$ be a continuous-time signal.
- A discrete-time sequence $g[n]$ is obtained by sampling $g_a(t)$ uniformly at $t = nT$, where T is the sampling period, i.e.,
- $g[n] = g_a(nT), \qquad -\infty < n < \infty \qquad (3.61)$
- The reciprocal of sampling period (T) is called the sampling frequency (F_T); that is, $F_T = \frac{1}{T}$.
- The frequency-domain representation of $g_a(t)$ and $g[n]$ are given by continuous-time Fourier transform (CTFT) and discrete-time Fourier transform (DTFT), respectively:
- $G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t)e^{-j\Omega t}dt, \qquad (3.62)$
- $G(e^{j\omega}) = \sum_{n}^{\infty} g[n]e^{-j\omega n}. \qquad (3.63)$

Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals:

Effect of Time-Domain Sampling in Frequency Domain

- To establish the relation between $G_a(j\Omega)$ and $G(e^{j\omega})$, let's treat the sampling operation mathematically as a multiplication of the continuous-time signal $g_a(t)$ by a periodic image train $p(t)$:
- $g_p(t) = g_a(t)p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT).$ $\quad (3.65)$
- Where,
  - $p(t)$ is an impulse train with a period of T, such that
  - $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- It should be noted that the signal $g_p(t)$ consists of a train of uniformly spaced impulses with the impulse at $t = nT$ weighted by the sampled value $g_a(nT)$ of $g_a(t)$.

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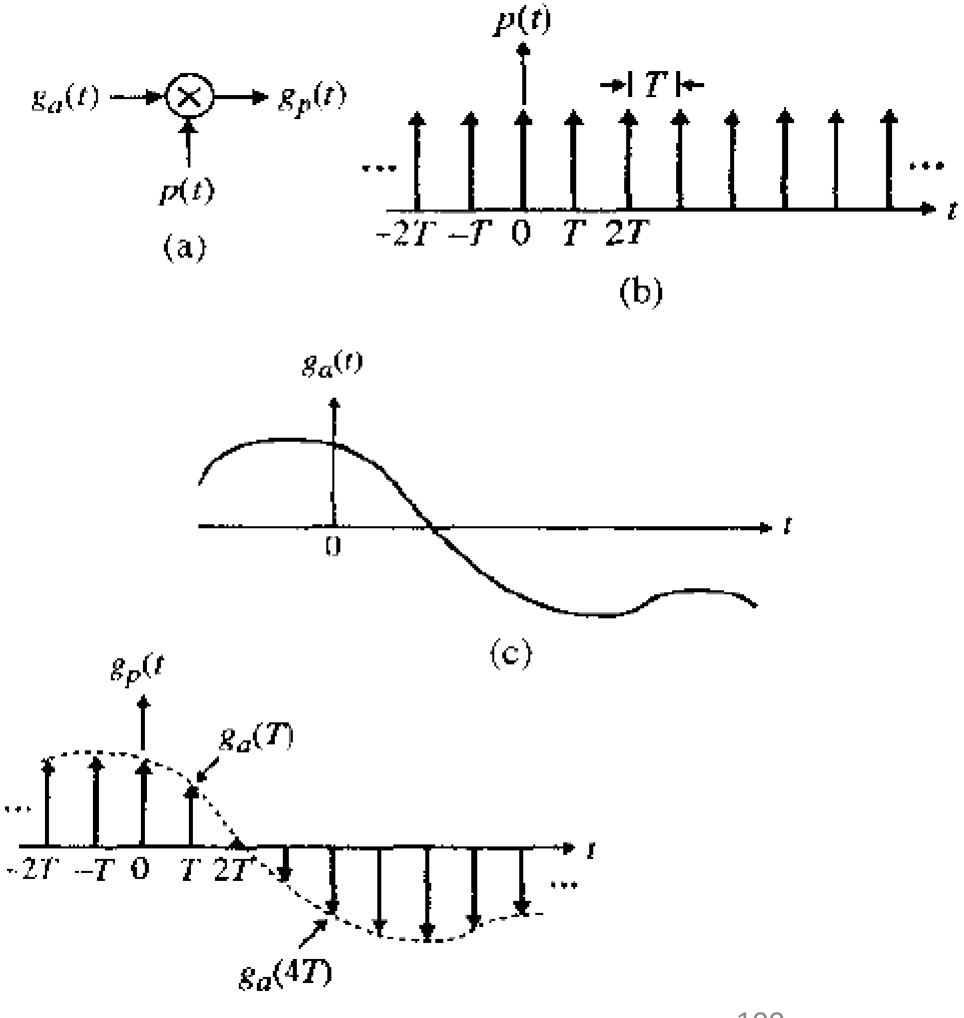

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Digital Processing of Continuous-Time Signals:

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There are two different forms of the continuous-time Fourier transform $G_p(j\Omega)$ **of** $g_p(t)$

- One form is obtained by taking CTFT of Eq. (3.65)
- $\mathcal{F}\{g_p(t)\} = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_a(nT)\delta(n - nT)e^{-j\Omega t}\,dt$
- $G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT) \int_{-\infty}^{\infty} \delta(n - nT)e^{-j\Omega t}\,dt$
- $G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT)\, e^{-j\Omega nT}$

- The second form, given below, is derived using Poisson's formula
- $G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\big(j(\Omega + k\Omega_T)\big)\ldots(3.70)$

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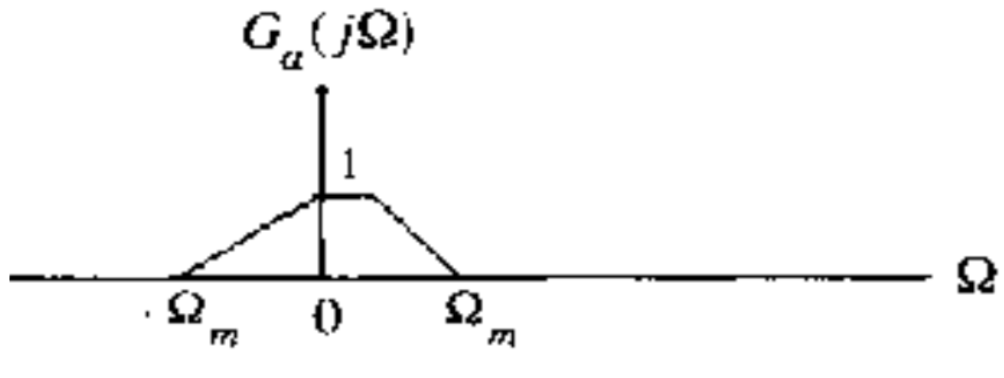
- $G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\big(j(\Omega + k\Omega_T)\big)$ (3.70)
  - $G_p(j\Omega)$ is a periodic function of frequency Ω.
- $G_p(j\Omega)$ consists of a sum of shifted (shifted by integer multiples of Ω_T) and scaled (scaled by $1/T$) replica of $G_p(j\Omega)$
- Putting $k = 0$, in Eq. (3.70) will give the baseband portion of $G_p(j\Omega)$. The remaining terms are the frequency-translated portions of $G_p(j\Omega)$.
- The frequency range $-\frac{\Omega_T}{2} \leq \Omega \leq \frac{\Omega_T}{2}$ is called the baseband or Nyquist band.

Discrete Time Signals in the Frequency Domain

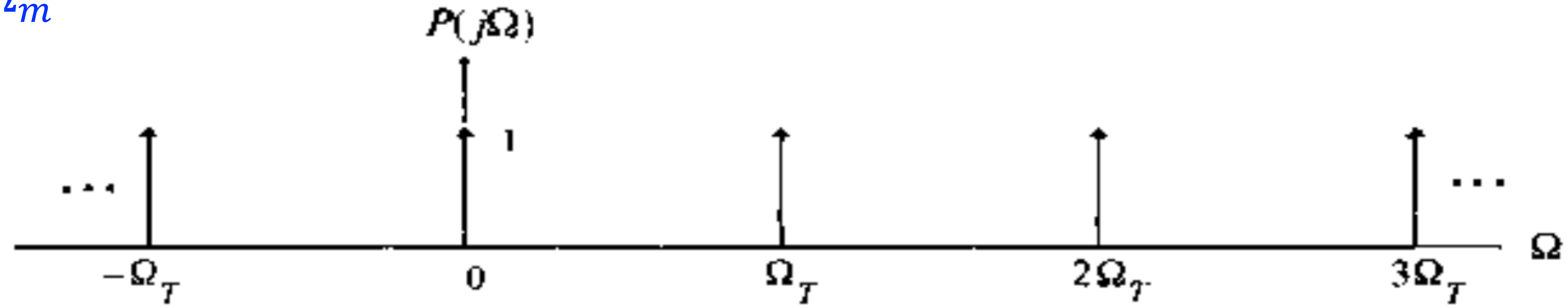
Digital Processing of Continuous-Time Signals:

Effect of Time-Domain Sampling in Frequency Domain

- $G_p(j\Omega)$, shown in Figure (a), is the frequency spectrum of a band-limited signal.
- Consider the spectrum $P(j\Omega)$ of the periodic impulse train $p(t)$ with a sampling period $T = \frac{2\pi}{\Omega_T}$, shown in Figure (b).
- The impulses are chosen in such a way that $\Omega_T \geq 2\Omega_m$



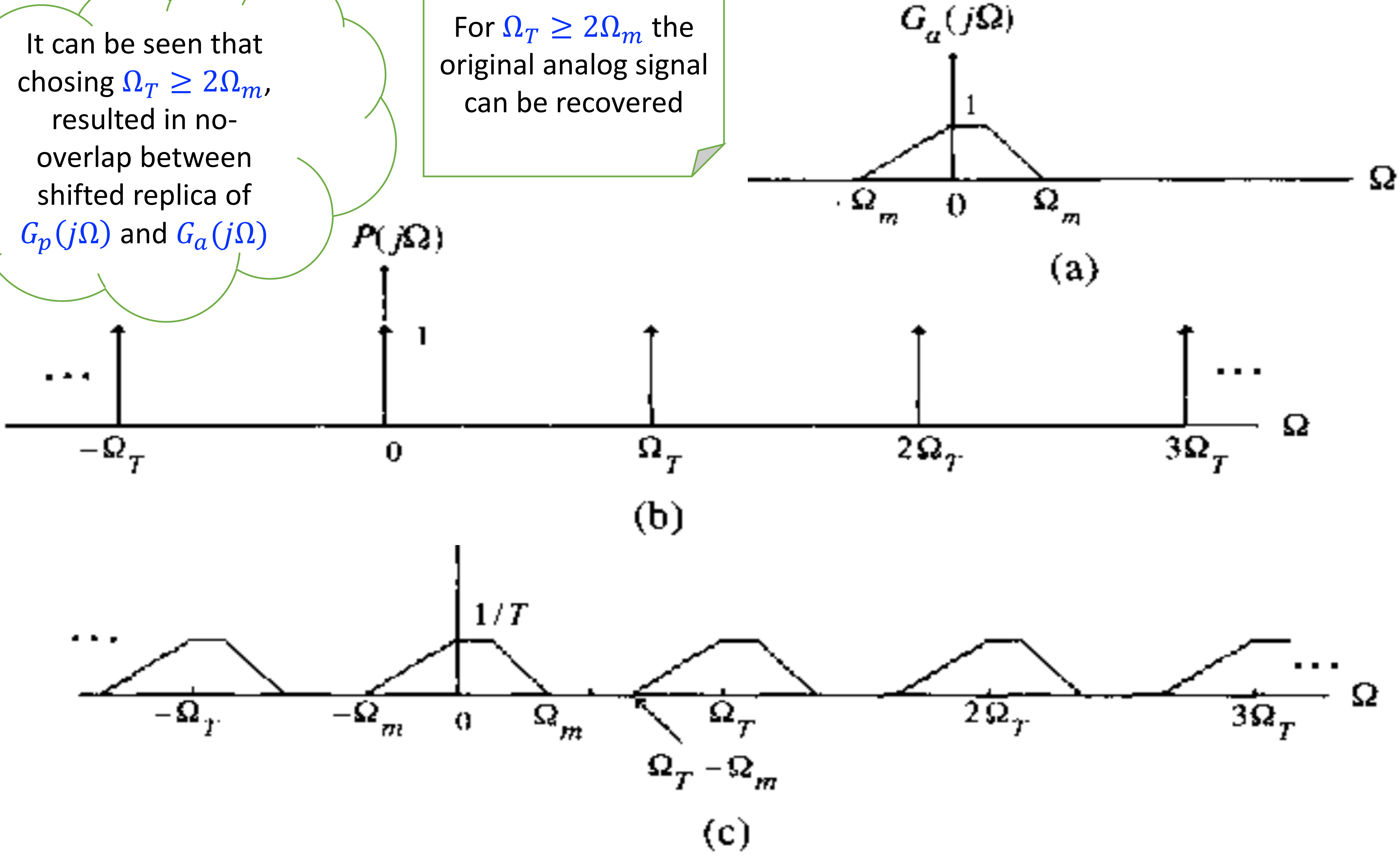
(a)



(b)

It can be seen that chosing $\Omega_T \geq 2\Omega_m$, resulted in no-overlap between shifted replica of $G_p(j\Omega)$ and $G_a(j\Omega)$

For $\Omega_T \geq 2\Omega_m$ the original analog signal can be recovered



Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals:

Effect of Time-Domain Sampling in Frequency Domain

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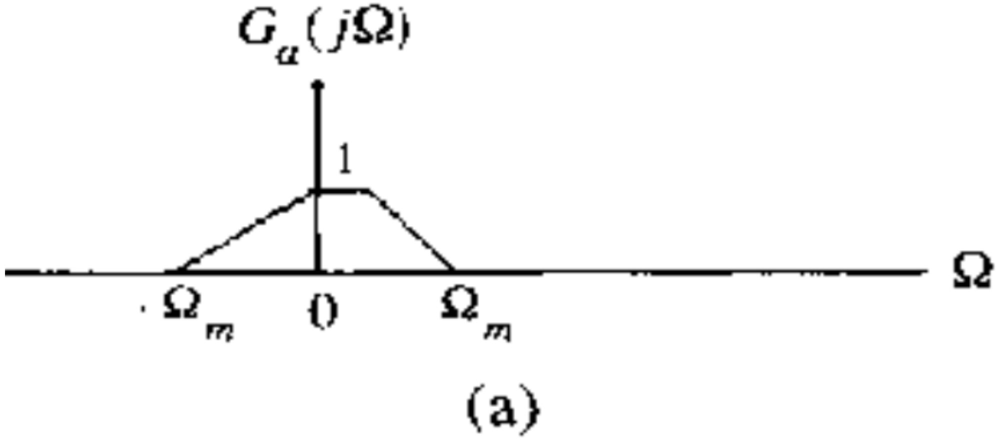
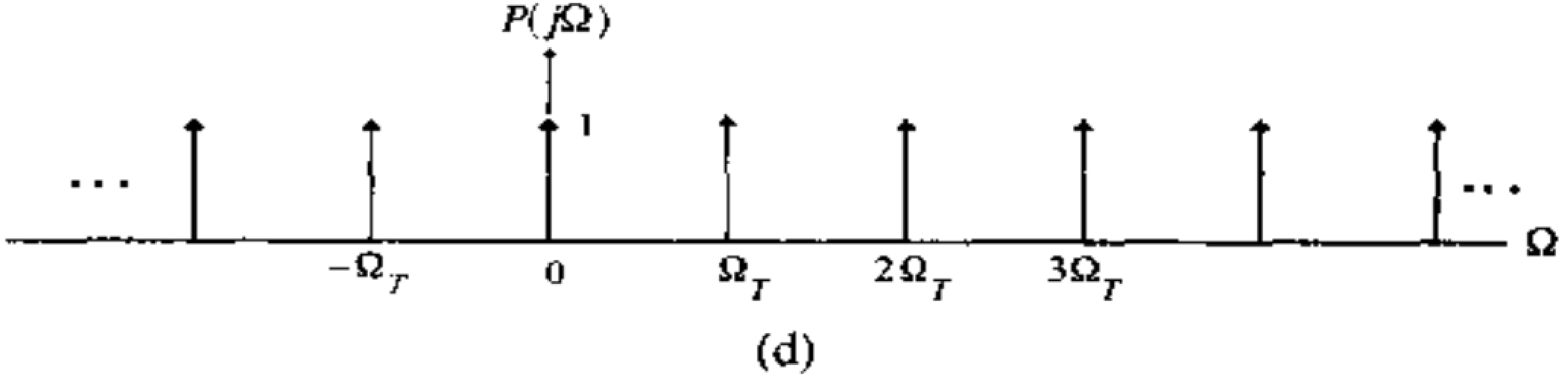
- Consider the spectrum $P(j\Omega)$ of the periodic impulse train $p(t)$ with a sampling period $T = \frac{2\pi}{\Omega_T}$, shown in Figure (b).
- The impulses are chosen in such a way that $\Omega_T < 2\Omega_m$



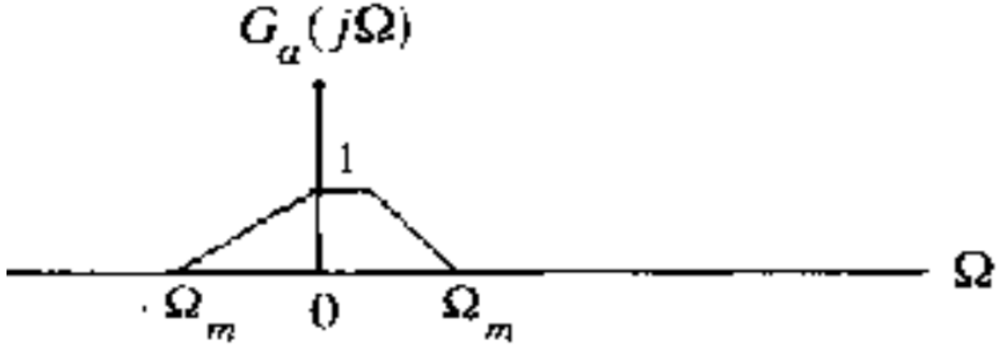
(a)



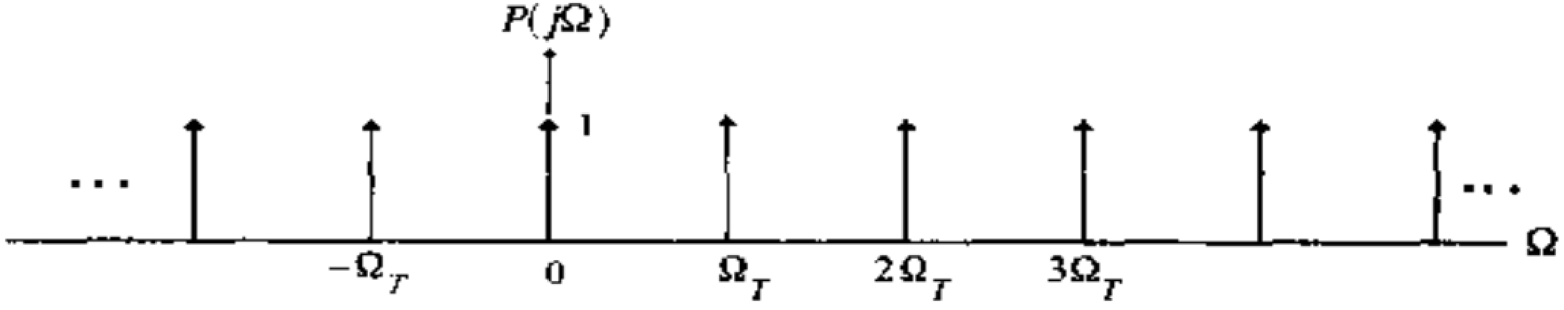
(d)

It can be seen that chosing $\Omega_T < \Omega_m$, resulted in an overlap between shifted replica of $G_p(j\Omega)$ and $G_a(j\Omega)$

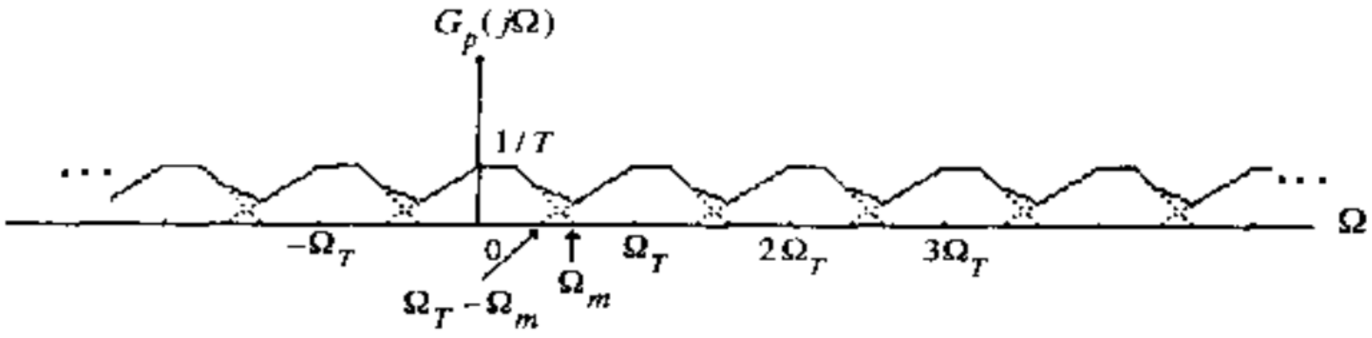
For $\Omega_T < 2\Omega_m$ the original analog signal cannot be recovered



(a)



(d)



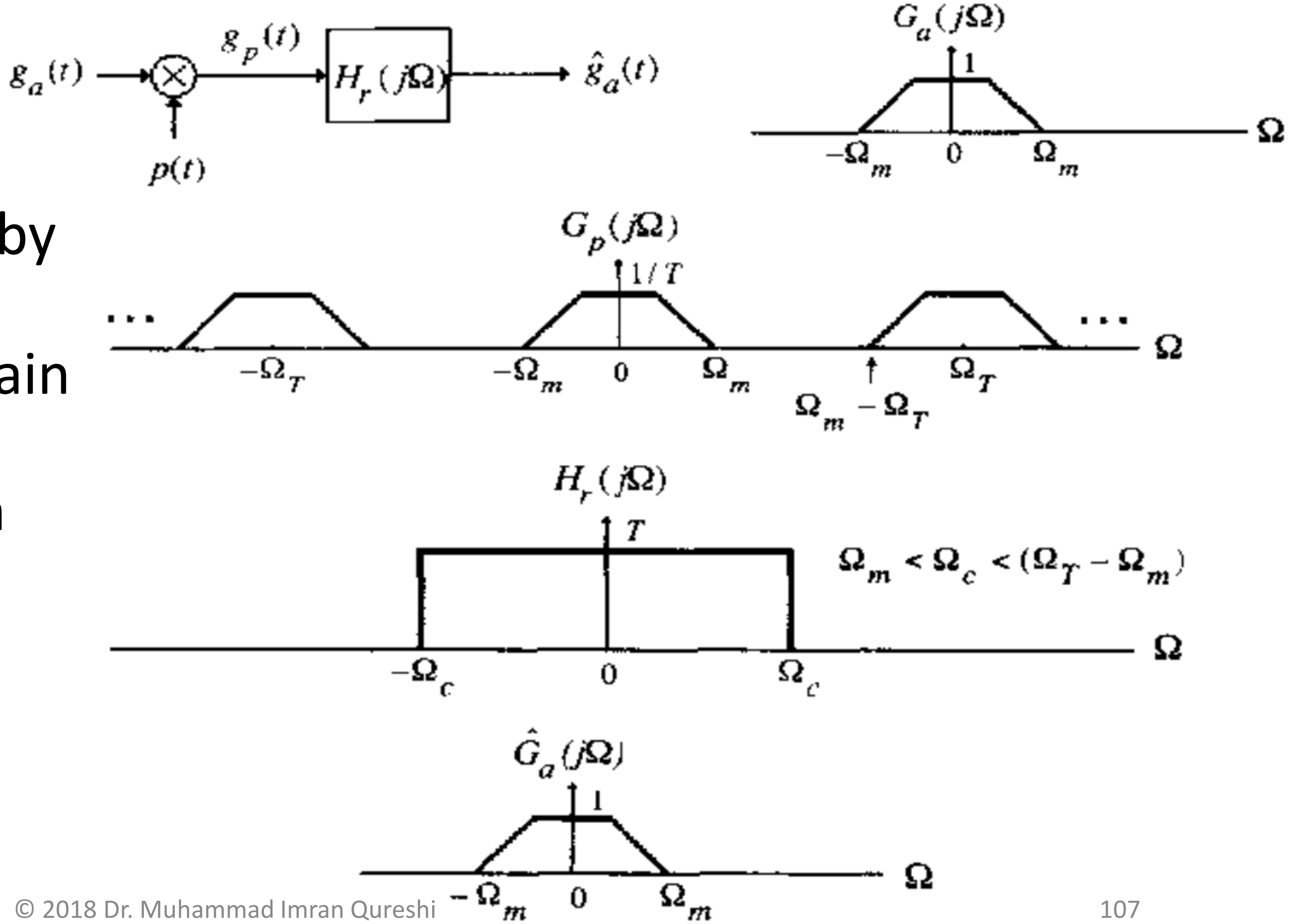
(e)

Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals:

Effect of Time-Domain Sampling in Frequency Domain

- If $\Omega_T \geq 2\Omega_m$, $g_a(t)$ can be recovered exactly from $g_p(t)$ by passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain T and a cutoff frequency Ω_c greater than Ω_m and less than $\Omega_T - \Omega_m$ as illustrated in the figure.



Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals:

Recovery of the Analog Signal

- If the discrete-time sequence $g[n]$ has obtained by uniformly sampling a band-limited continuous-time signal $g_a(t)$ with the highest frequency Ω_m at a rate $\Omega_T = \frac{2\pi}{T}$ satisfying the condition

$$\Omega_T \geq 2\Omega_m$$

Then the original continuous-time signal $g_a(t)$ can be fully recovered by passing the equivalent impulse train $g_p(t)$ through an ideal lowpass filter $H_r(j\Omega)$ with a cutoff frequency at Ω_c satisfying the condition

$$\Omega_m < \Omega_c < (\Omega_T - \Omega_m)$$

With a gain of T.

Discrete Time Signals in the Frequency Domain

Digital Processing of Continuous-Time Signals:

Recovery of the Analog Signal

- The frequency response $H_r(j\Omega)$ is given by
- $H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c, \\ 0, & |\Omega| > \Omega_c, \end{cases}$ (3.78)
- The impulse response $h_r(t)$ of the above ideal lowpass filter is obtained simply by taking the inverse continuous-time Fourier transform of (3.78).
- $h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}d\Omega$