Digital SIGNAL PROCESSING

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Discrete-Time Signals and Systems in the Time Domain

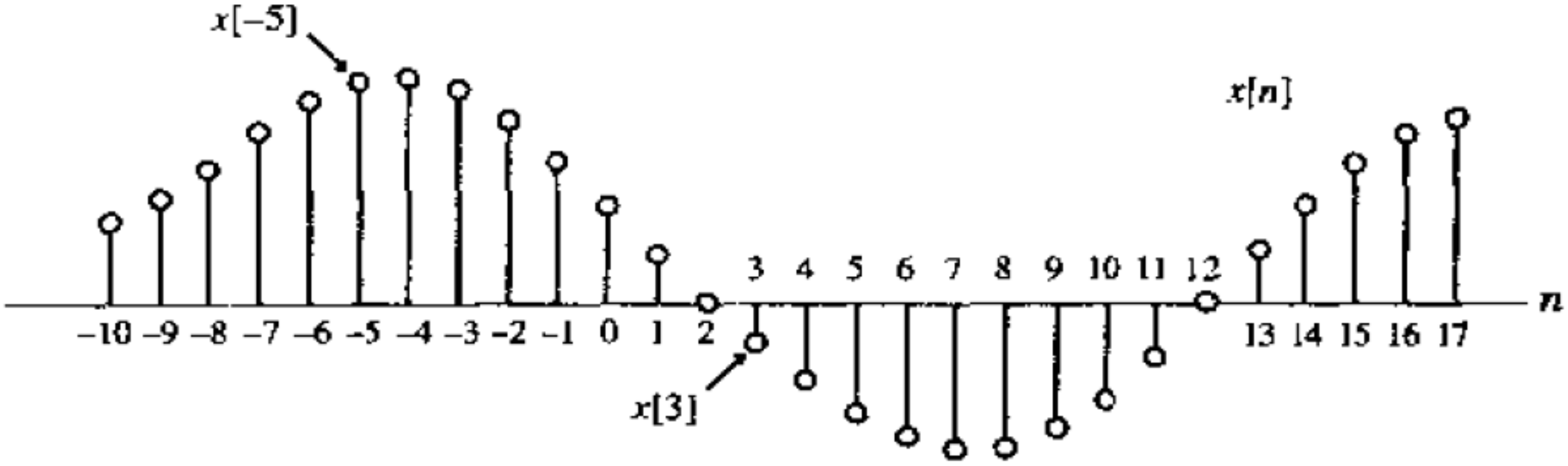


Figure 2.1: Graphical representation of a discrete-time sequence $\{x[n]\}$.

Discrete-Time Signals

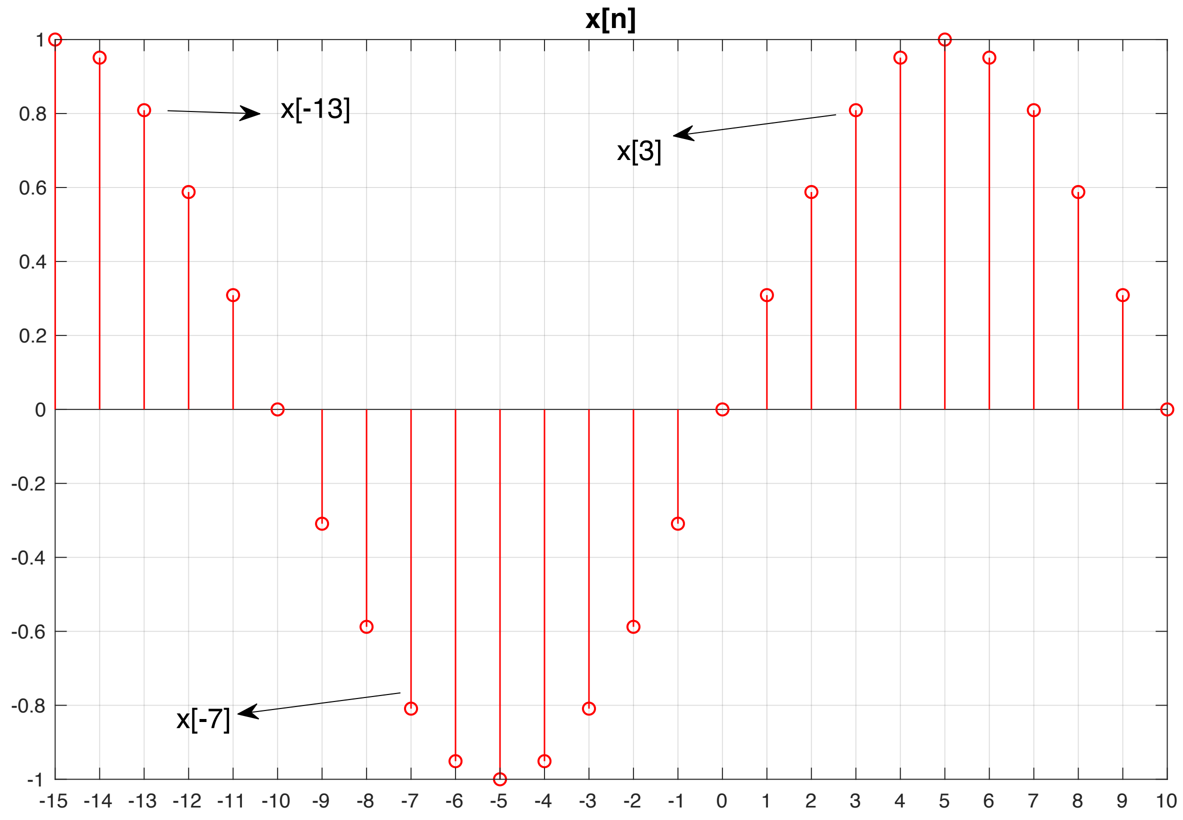
Time-Domain Representation

- Signal are represented as sequences of numbers called samples.
- $\{x[n]\} = \{\dots, 0.95, -0.2, \underset{\uparrow}{2.17}, 1.1, 0.2, -3.67, 2.9, -0.8, 4.1, \dots\}$ (2.1)
- The arrow $\uparrow$ indicates the sample value of $x[n]$ at $n = 0$.
- The samples to its right are for positive values of n, and the sample values to its left are for negative values of n.
- For example: $x[-1] = -0.2, x[0] = 2.17, x[1] = 1.1$, and so on.

Discrete-Time Signals

Time-Domain Representation

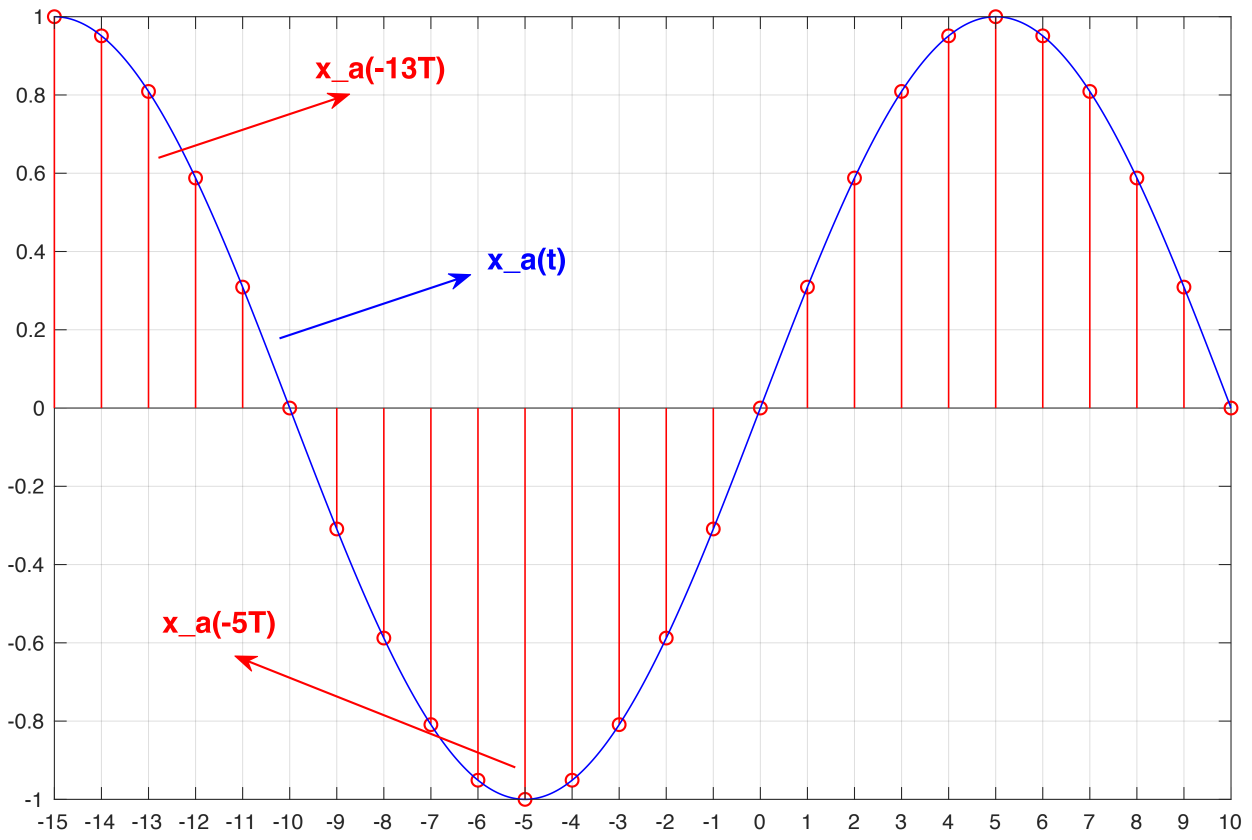
- The graphical representation of a sequence $\{x[n]\}$, with real-valued sample is shown in following figure.



Discrete-Time Signals

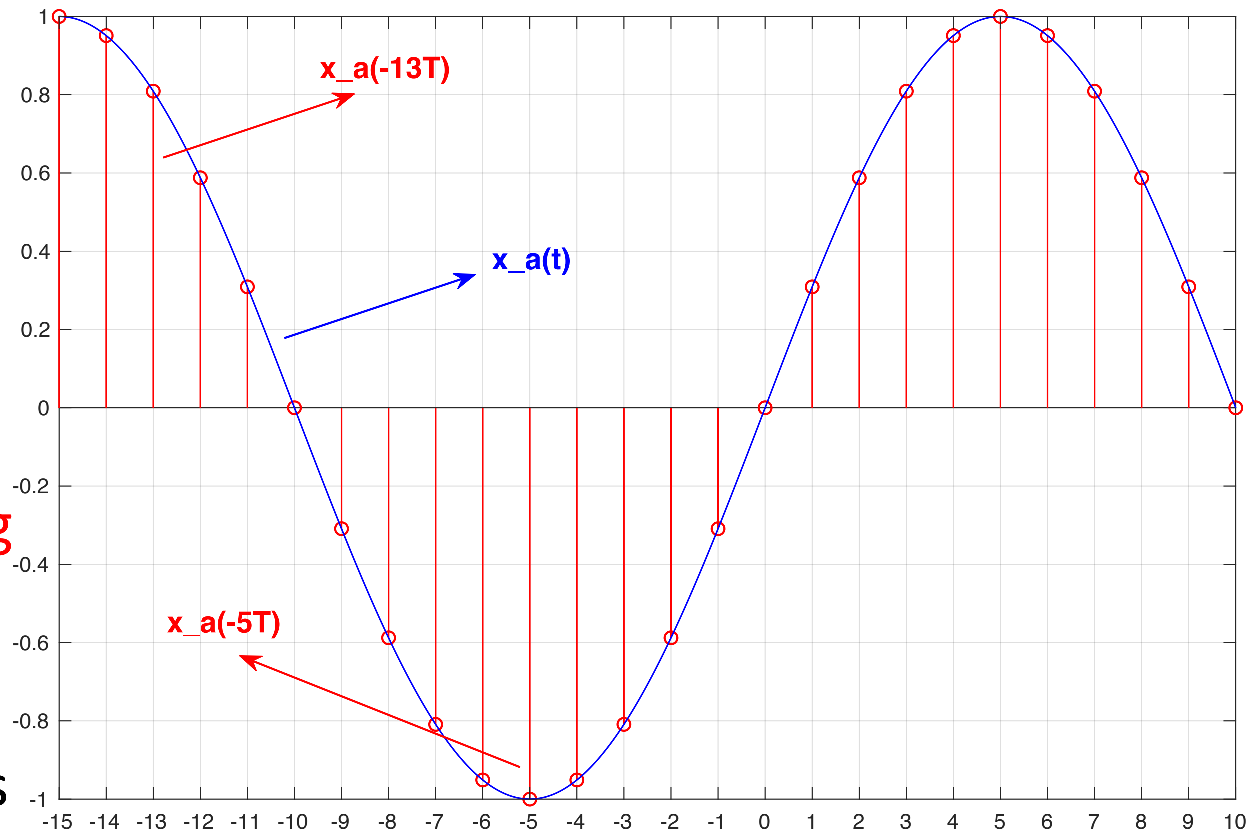
Time-Domain Representation

- A discrete-time sequence $\{x[n]\}$ is generated by periodically sampling a continuous time signal $x_a(t)$ at uniform intervals:
- $x[n] = x_a(t)|_{t=nT} = x_a(nT)$ (2.2)
- Where, $n = \cdots, -2, -1, 0, 1, 2, \ldots$
- This procedure is shown in Figure.



Discrete-Time Signals
Time-Domain Representation

- Sampling period:
  - The spacing T between two consecutive samples is called 'sampling period' or 'sampling interval'.
- Sampling Frequency:
  - The reciprocal of the sampling interval T, denoted by F_T, is called the sampling frequency
  - $F_T = \frac{1}{T}$ (2.3)
  - The unit of sampling frequency is cycles per second or Hertz (Hz).



Discrete-Time Signals

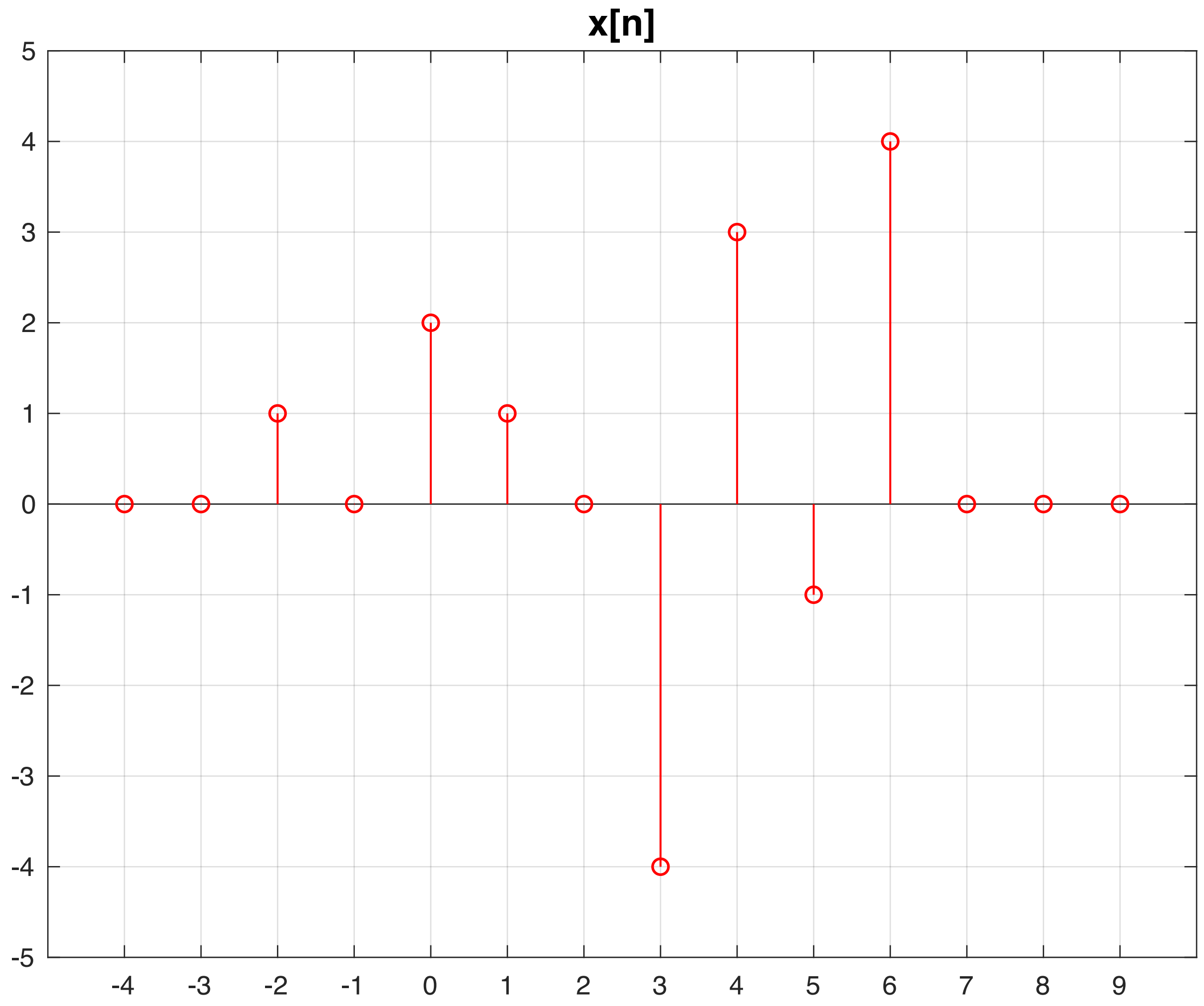
Time-Domain Representation

- nth sample:
  - The quantity $x[n]$ is called the nth sample of the sequence $\{x[n]\}$.
- Real Sequence:
  - If $x[n]$ is real for all values of n, then $\{x[n]\}$, is a real sequence.
- Complex Sequence:
  - If the nth sample value is complex for one or more values of n, then it is a complex sequence.
  - By separating the real and imaginary parts of $x[n]$, a complex sequence $\{x[n]\}$ can be written as:
  - $\{x[n]\} = \{x_{re}[n]\} + j\{x_{im}[n]\} \qquad (2.4)$
  - For a real sequence, $x_{im}[n] = 0$ for all values of n.
  - For a purely imaginary sequence, $x_{re}[n] = 0$ for all values of n.
  - The complex conjugate $\{x^*[n]\}$ of a complex sequence $\{x[n]\}$ is written as:
  - $\{x^*[n]\} = \{x_{re}[n]\} - j\{x_{im}[n]\} \qquad (2.5)$

Discrete-Time Signals

Time-Domain Representation

- Digital Signal:
  - A signal in which samples are discrete-valued.
  - Digital signals are obtained by quantizing the sample values either by rounding or truncation.
  - For example, the digital signal $\{\hat{x}[n]\}$ obtained by rounding the sample values of the discrete-time sequence $x[n]$ of Eq. (2.1) to the nearest integer values is given by:
  - $\{x[n]\} = \{\ldots, 1, 0, \underset{\uparrow}{2}, 1, 0, -4, 3, -1, 4, \ldots\}$



Discrete-Time Signals

Length of a Discrete-Time Signal

- The discrete-time signal may be a finite-length or an infinite-length sequence.
- Finite-length (Finite-duration or finite-extent) sequence:
  - It is defined only for finite time interval:
  - $N_1 \leq n \leq N_2 \qquad (2.5)$
  - Where, $\infty < N_1$ and $N_2 < \infty$ with $N_1 \leq N_2$.
  - The length or duration N of the above finite-length sequence is
  - $N = N_2 - N_1 + 1. \qquad (2.6)$
  - A length-N discrete-time sequence consists of N samples and is often referred as an N-point sequence.

Discrete-Time Signals

Length of a Discrete-Time Signal

- Infinite-length sequence:
  - A finite-length sequence can be made infinite-length sequence by assigning zero values to samples, outside the above range.
  - The process of lengthening a sequence by adding zero-valued samples is called appending with zeros or zero-padding.
  - There are three types of infinite-length sequences:
    1. Right-sided sequence
    2. Left-sided sequence
    3. Two-sided sequence

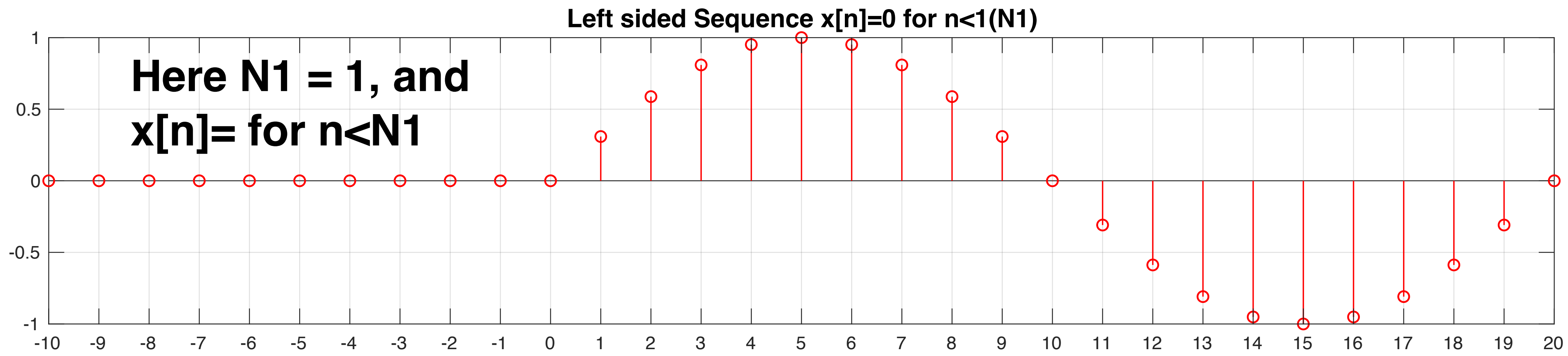
Discrete-Time Signals

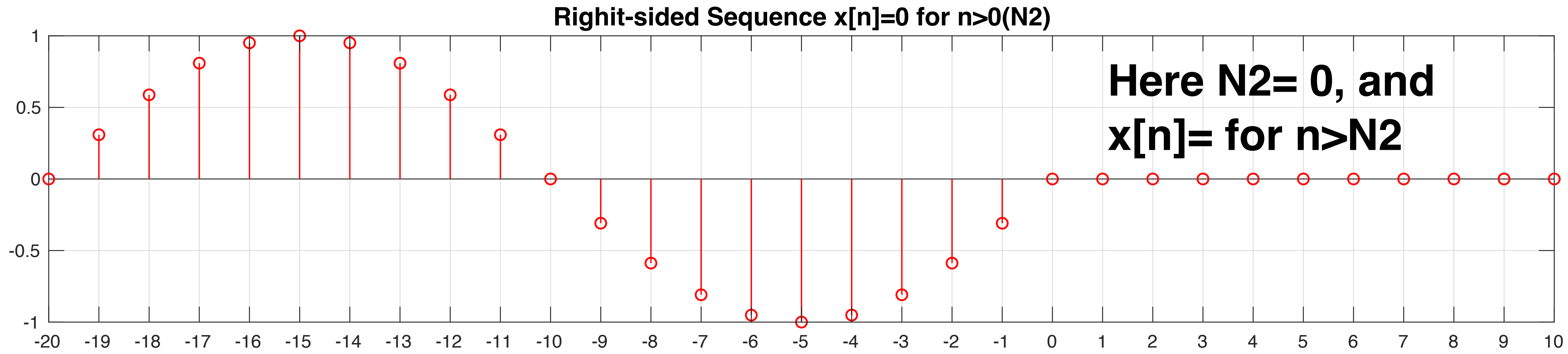
Length of a Discrete-Time Signal

- There are three types of infinite-length sequences:
  1. Right-sided sequence
     - A right-sided sequence $x[n]$ has zero-valued samples for $n < N_1$
     - $x[n] = 0, \quad for\ n < N_1 \quad (2.7)$
     - Where, N_1 is a positive integer that can be positive or negative.
     - If $N_1 \geq 0$, a right-sided sequence is called a causal sequence.
  2. Left-sided sequence
     - A left-sided sequence $x[n]$ has zero-valued samples for $n > N_2$
     - $x[n] = 0, \quad for\ n > N_2 \quad (2.8)$
     - Where, N_2 is a positive integer that can be positive or negative.
     - If $N_2 \leq 0$, a left-sided sequence is called a anticausal sequence.
  3. Two-sided sequence

Discrete-Time Signals

Length of a Discrete-Time Signal

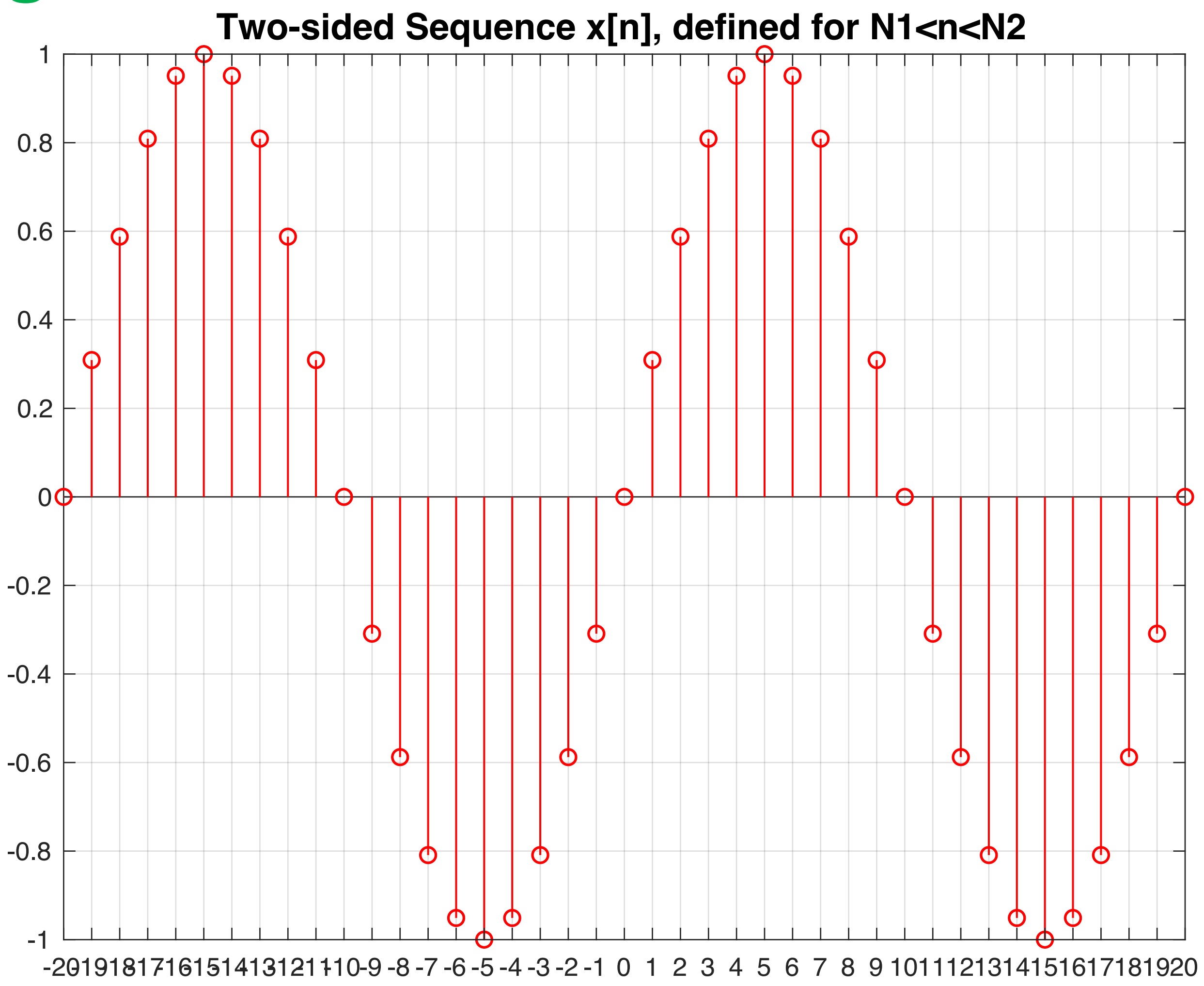




Discrete-Time Signals

Length of a Discrete-Time Signal

- There are three types of infinite-length sequences:
    1. Right-sided sequence
    2. Left-sided sequence
    3. Two-sided sequence
        - A two-sided sequence $x[n]$ is defined for both positive and negative values of n.



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Discrete-Time Signals

Strength of a Discrete-Time Signal

- The strength of a discrete-time signal is given by its norm.
- The $\mathcal{L}_p - norm$ of a sequence $\{x[n]\}$ is defined by
- $\|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x[n]|^p\right)^{1/p}$ (2.9)
- Where, p is a positive integer.
- The values of p is typically 1 or 2 or ∞.
- Peak absolute value
  - Peak absolute value of $\{x[n]\}$ is given by $\mathcal{L}_p - norm$. i.e.
  - $\|x\|_p = |x|_{max}$ (2.10)
- Root Mean Squared Value (RMS) Value:
  - For a sequence $\{x[n]\}$ of length N is given by:
  - $RMS\ Value = \frac{\|x\|_2}{\sqrt{N}}$

Discrete-Time Signals

Strength of a Discrete-Time Signal

- **Mean Squared Value:**
  - For a sequence $\{x[n]\}$ of length N, mean square value is given by:
  - $Mean\ Square\ Value = \frac{\|x\|_1}{N}$
- One of the application of norms is to calculate the error in the approximation of a discrete-time signal by another discrete-time signal.
- **Mean-Squared Error (MSE):**
  - Consider a sequence $x[n], 0 \leq n \leq N$, represents a signal corrupted by additive noise, and $y[n], 0 \leq n \leq N$, represents a signal obtained as a result of removing noise from $x[n]$. The estimation of the quality (similarity between original $x[n]$, and recovered signal $y[n]$) of the approximation (i.e. the signal $y[n]$) is given by MSE
  - $MSE = \frac{1}{N}\sum_0^{N-1}(|y[n] - x[n]|)^2 = \frac{1}{N}(\|y[n] - x[n]\|_2)^2.$ (2.11)
  - The Eq. (2.11) is the $\mathcal{L}_2 - norm$ of the difference signal divided by the length of the signals.

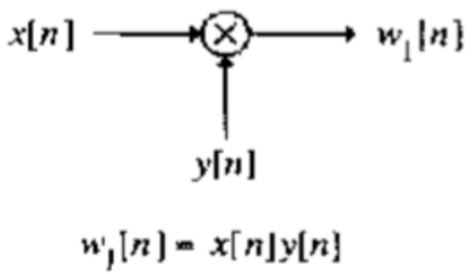
Discrete-Time Signals

Strength of a Discrete-Time Signal

- Relative Error (MSE):
  - Relative error is another metric use to measure the quality of the recovered signal.
  - It is the ratio of $\mathcal{L}_2 - norm$ of the difference signal and $\mathcal{L}_2 - norm$ of the original signal:
  - $E_{rel} = \left(\frac{\sum_{n=0}^{N-1}|y[n]-x[n]|^2}{\sum_{n=0}^{N-1}|x[n]|^2}\right)^{\frac{1}{2}} = \frac{\|y[n]-x[n]\|_2}{\|x[n]\|_2} \qquad (2.1)$
  - For $\mathcal{L}_2 - norm$ in MATLAB, command norm(x) or norm(x,2) can be used.
  - For $\mathcal{L}_2 - norm$ in MATLAB, command norm(x,1) can be used.
  - For $\mathcal{L}_\infty - norm$ in MATLAB, command norm(x, inf) can be used.

Operations on Sequences

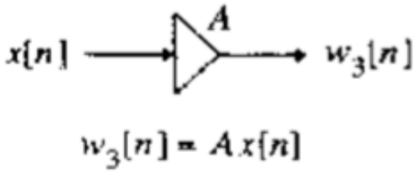
Elementary Operations



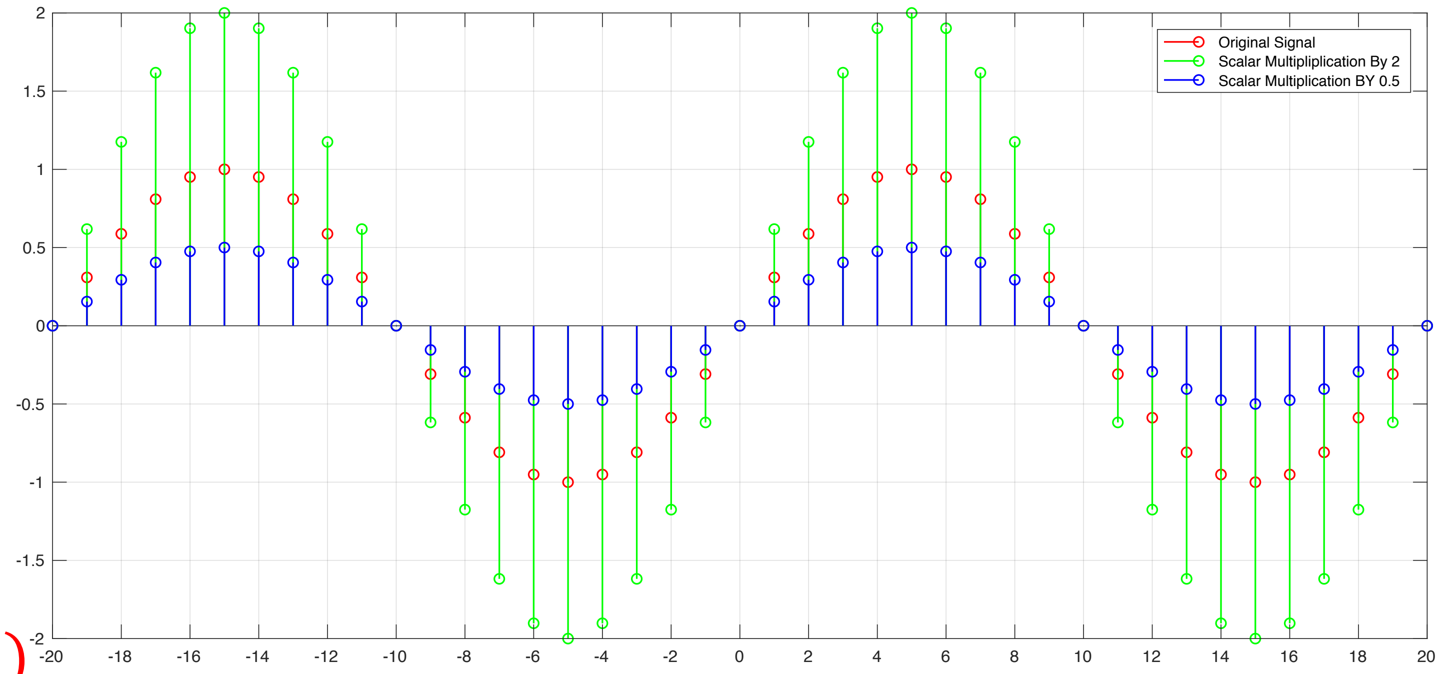
- Let $x[n]$ and $y[n]$ be two known sequences.
- Product of Two Sequences (Modulations):
  - The product of the sample values of these two sequences at each instant, to form a new sequence $w_1[n]$:
  - $w_1[n] = x[n].\, y[n]$ (2.13)
  - This operation is also called modulation, and the devices used to perform modulation is called modulator.
  - Two Applications of Product of Two Sequences:
    1. Shown in Example 2.14
    2. A finite-length sequence can be formed from an infinite-length sequence by multiplying the latter with a finite-length sequence called a window sequence.

Operations on Sequences

Elementary Operations



- Let $x[n]$ and $y[n]$ be two known sequences.
- Scalar Multiplication:
  - A new sequence $w_2[n]$, is formed by multiplying a sequence $x[n]$ with a scalar quantity A:
  - $w_1[n] = Ax[n]$ (2.14)
  - The device implementing this operation is called a multiplier.



Operations on Sequences
Elementary Operations

- Let $x[n]$ and $y[n]$ be two known sequences.
- Addition:
  - A new sequence $w_3[n]$, is formed by adding above two sequences :
  - $w_1[n] = x[n] + y[n]$ (2.15)
  - The device implementing this operation is called a adder.
  - Ensemble Averaging (Application of Addition Operation):
    - The signal corrupted by random noise can be recovered by addition operation (by adding multiple times and then taking average, called ensemble average).
    - Let $\boldsymbol{d}_i$ denote the noise vector corrupting the ith measurement of the uncorrupted data vector $\boldsymbol{s}$.
    - $\boldsymbol{x}_i = \mathrm{s} + \boldsymbol{d}_i$

Operations on Sequences
Elementary Operations

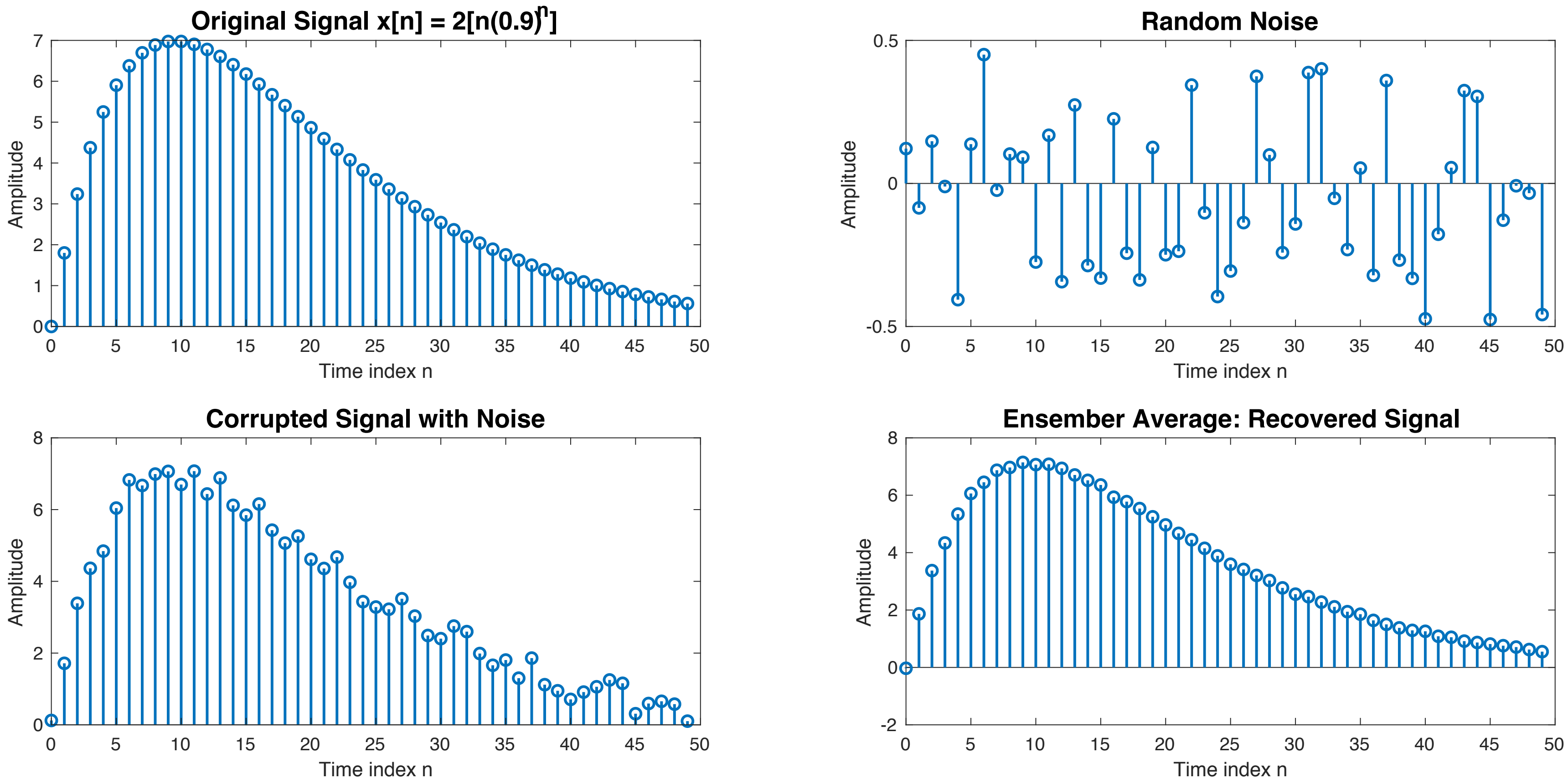
- Addition:
  - Ensemble Averaging (Application of Addition Operation):
    - The signal corrupted by random noise can be recovered by addition operation (by adding multiple times and then taking average, called ensemble average).
    - Let $\boldsymbol{d}_i$ denote the noise vector corrupting the ith measurement of the uncorrupted data vector $\boldsymbol{s}$.
    - $\boldsymbol{x}_i = \mathrm{s} + \boldsymbol{d}_i$
    - The average data vector, called the ensemble average, obtained by after K measurements is then given by
    - $\boldsymbol{x}_{ave} = \frac{1}{K}\sum_{i=1}^{K}(x_i) = \frac{1}{K}\sum_{i=1}^{K}(s + \boldsymbol{d}_i) = s + \frac{1}{K}\left(\sum_{i=1}^{K}\boldsymbol{d}_i\right)$
    - For a very large value of K, $\boldsymbol{x}_{ave}$ is usually a reasonable replica of the desired data vector s.

Operations on Sequences

Elementary Operations

- **Ensemble Averaging (Application of Addition Operation):**

Here the original signal is corrupted with random additive noise. The corrupted signal is added 50 times and then taken average to recover original signal. It can be seen that the recovered signal is almost same as original signal.



Operations on Sequences

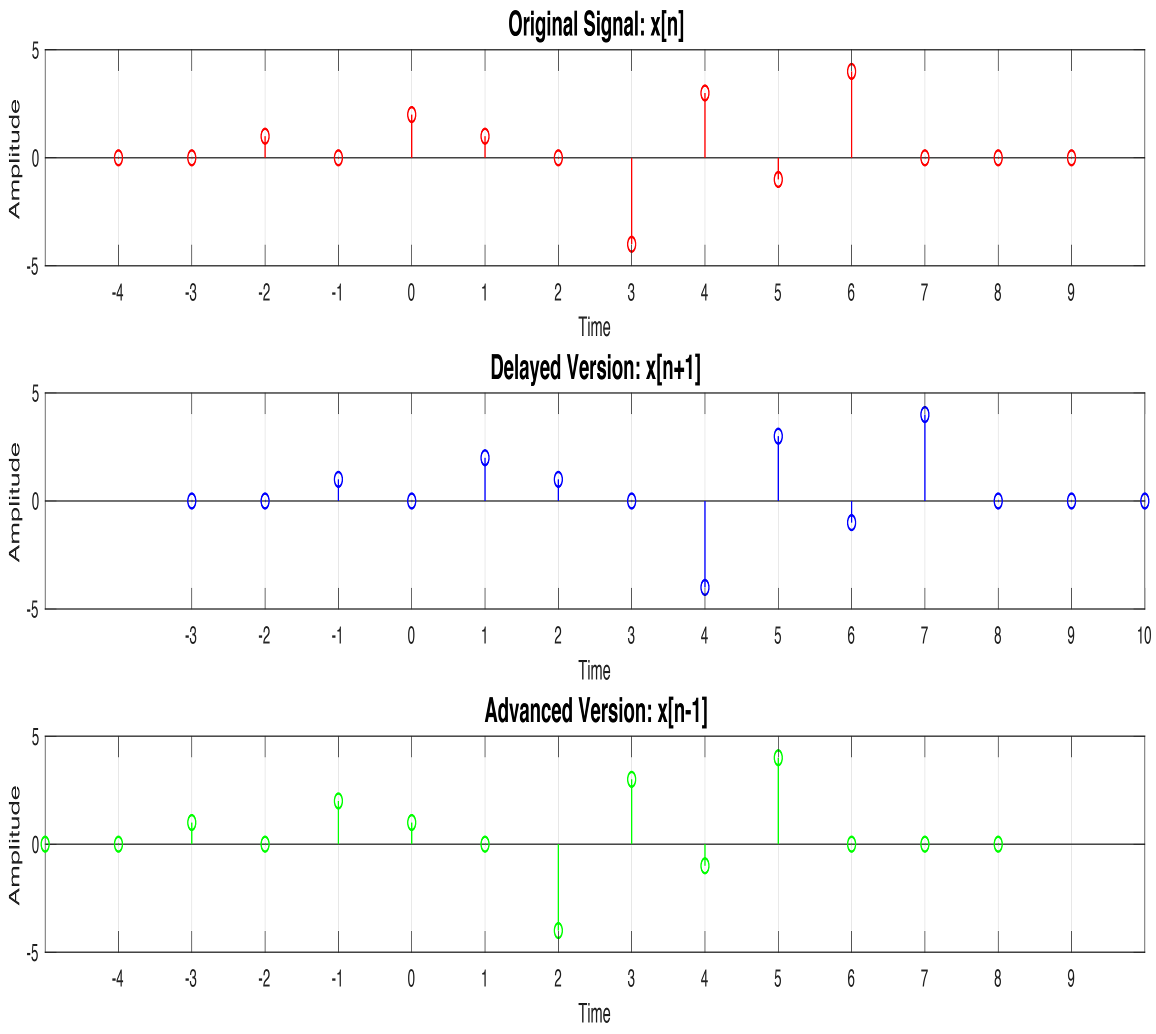
Elementary Operations

- Time-Shifting (Delay and Advance):
  - The time-shifting operation, shown in Eq. (2.17), shows the relation between $x[n]$ and its time-shifted version $w_4[n]$:
  - $w_4[n] = x[n - N] \qquad (2.17)$
  - Where N is an integer.
  - Delaying Operation:
    - If $N > 0$, it is a delaying operation.
  - Advancing Operation:
    - If $N < 0$, it is an advancing operation.

Operations on Sequences

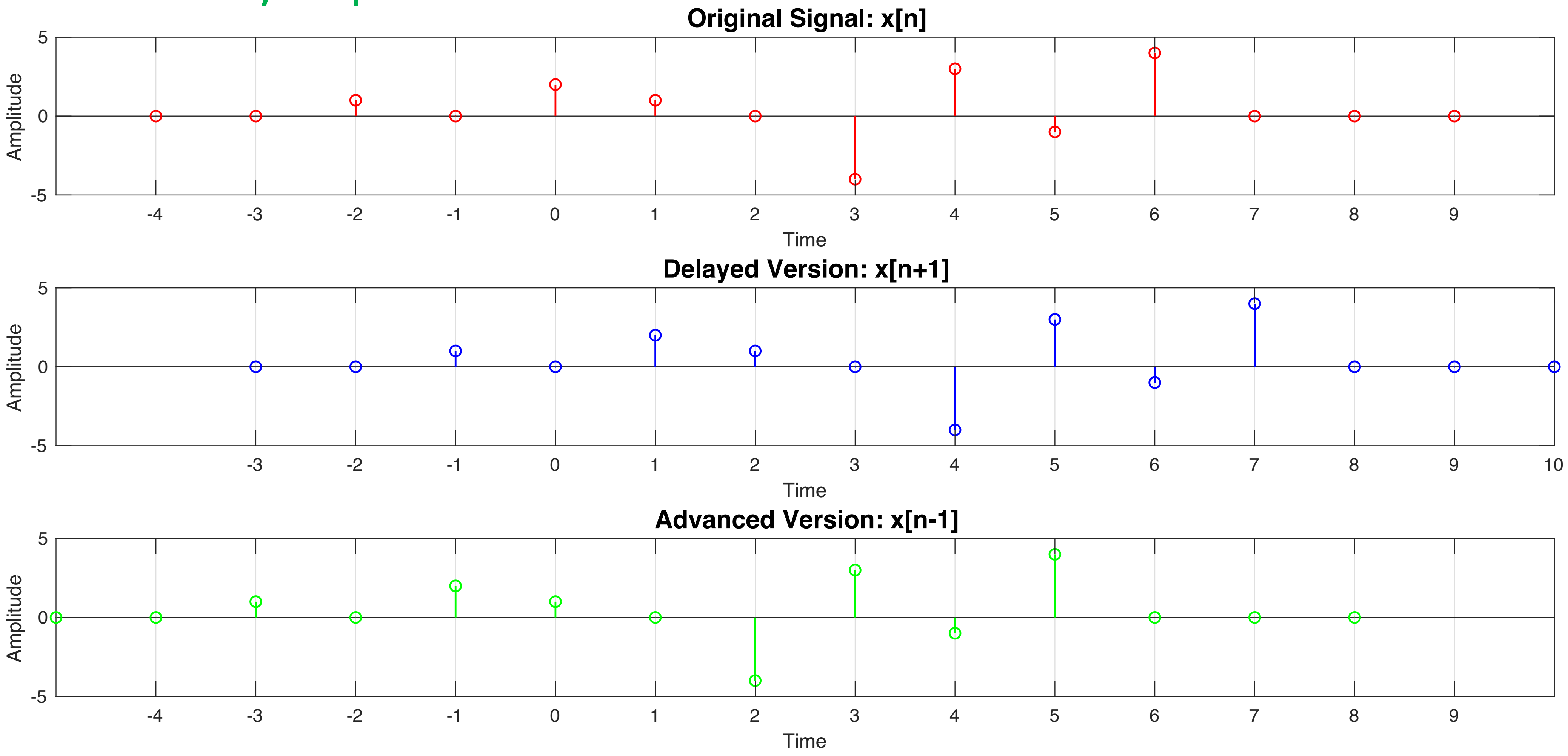
Elementary Operations

- Time-Shifting (Delay and Advance):
  - $\{x[n]\} = \{\ldots, 1, 0, 2, 1, 0, -4, 3, -1, 4, \ldots\}$
  - Delaying Operation:
  - $\{x[n+1]\} = \{\ldots, 1, 0, 2, 1, 0, -4, 3, -1, 4, \ldots\}$
  - Advancing Operation:
  - $\{x[n-1]\} = \{\ldots, 1, 0, 2, 1, 0, -4, 3, -1, 4, \ldots\}$



Operations on Sequences

Elementary Operations



Operations on Sequences

Elementary Operations

- Time-Reversal (Folding) Operation:
  - A folding operation for a sequence $x[n]$ is given below:
  - $w_6[n] = x[-n] \qquad (2.18)$

Operations on Sequences

Elementary Operations

- Example 2.2: Consider two sequences: $c[n]$ & $d[n]$, defined for $0 \leq n \leq 4$.
- $c[n] = \{3.2, 41, 36, -9.5, 0\}$,
- $d[n] = \{1.7, -0.5,\ \ 0,\ \ 0.8, 1\}$.
- Product of Two Sequences:
- $w_1[n] = c[n].d[n] = \{5.44, -20.5, 0, -7.6, 0]$
- Addition:
- $w_2[n] = c[n] + d[n] = \{4.9, 40.5, 36, -8.7, 1\}$
- Scalar Multiplication
- $w_3[n] = \frac{7}{2}c[n] = \{11.2, 143.5, 126, -33.25, 0\}$

Operations on Sequences

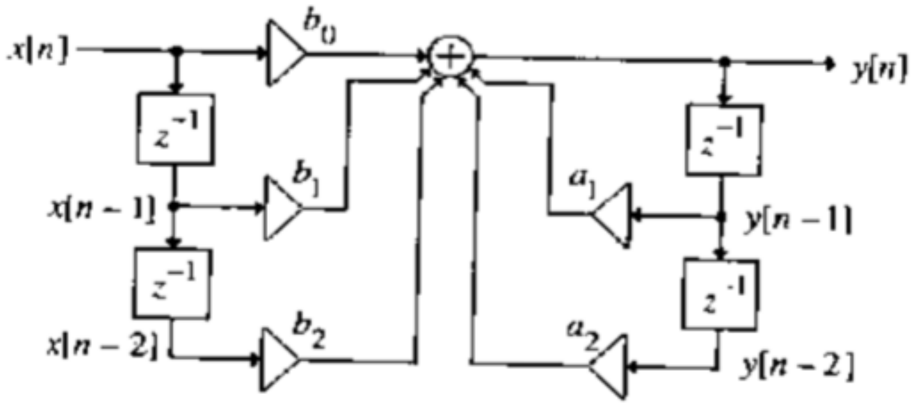
Combination of Elementary Operations

- Example 2.3: Consider two sequences: $c[n]$ defined for $0 \leq n \leq 4$, and $g[n]$ defined for $0 \leq n \leq 2$
- $c[n] = \{3.2, 41, 36, -9.5, 0\}$,
- $g[n] = \{-21, 1.5, 3\}$.
- Product of Two Sequences:
  - In order to multiply two sequences, the sequence with lower length will be added zeros
  - $g[n] = \{-21, 1.5, 3, 0, 0\}$.
  - $w_1[n] = c[n]. g[n] = \{-67.2, 61.5, 108, 0, 0]$
- Addition:
  - In order to multiply two sequences, the sequence with lower length will be added zeros
  - $g[n] = \{-21, 1.5, 3, 0, 0\}$.
  - $w_2[n] = c[n] + g[n] = \{-17.8, 42.5, 39, -9.5, 0\}$

Operations on Sequences

Basic Operation of Sequences of Unequal Lengths

- Example 2.4:
  - In some applications, a sequence is generated by a weighted combination of another sequence and its past values, along with a weighted combination of past values of the sequence being generated as shown below:
  - $y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$ (2.19)



Operations on Sequences

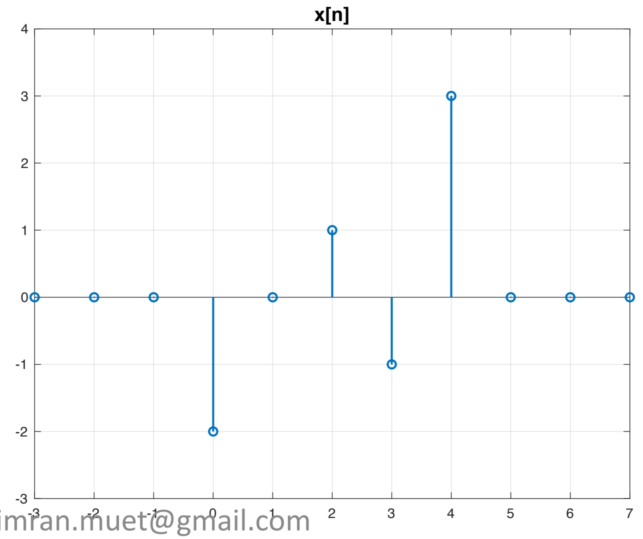
Convolution Sum

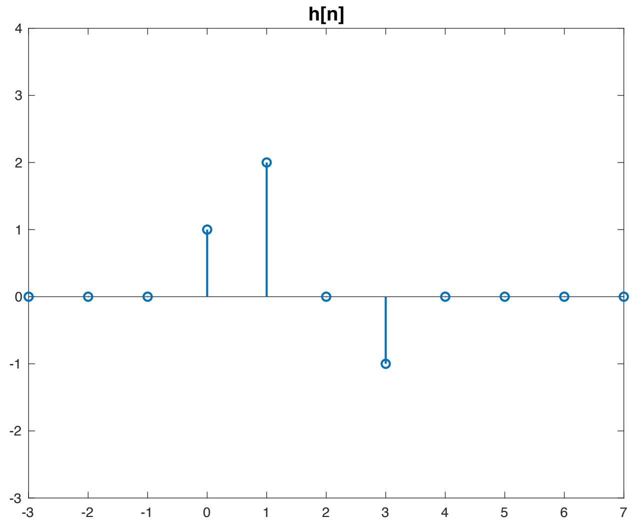
- The convolution sum between two sequences: $x[n]$ and $h[n]$, will yield another sequence $y[n]$, is given by:
- $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (2.20a)$
- Or alternatively
- $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] \qquad (2.20b)$
- Example: Find Convolution sum between two sequences:
- $x[n] = \{\dots, 0, \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \dots\} \quad for \;\; 0 \le n \le 4$
- $h[n] = \{\dots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \dots\} \qquad for \;\; 0 \le n \le 3$

Operations on Sequences

Convolution Sum

- Example: Find Convolution sum between two sequences:
- $x[n] = \{\dots, 0, \ -2, 0, 1, -1, 3, 0 \dots\} \quad for \ \ 0 \le n \le 4$
- $h[n] = \{\dots, 0, 1, 2, 0, -1, 0, \dots\} \qquad for \ \ 0 \le n \le 3$

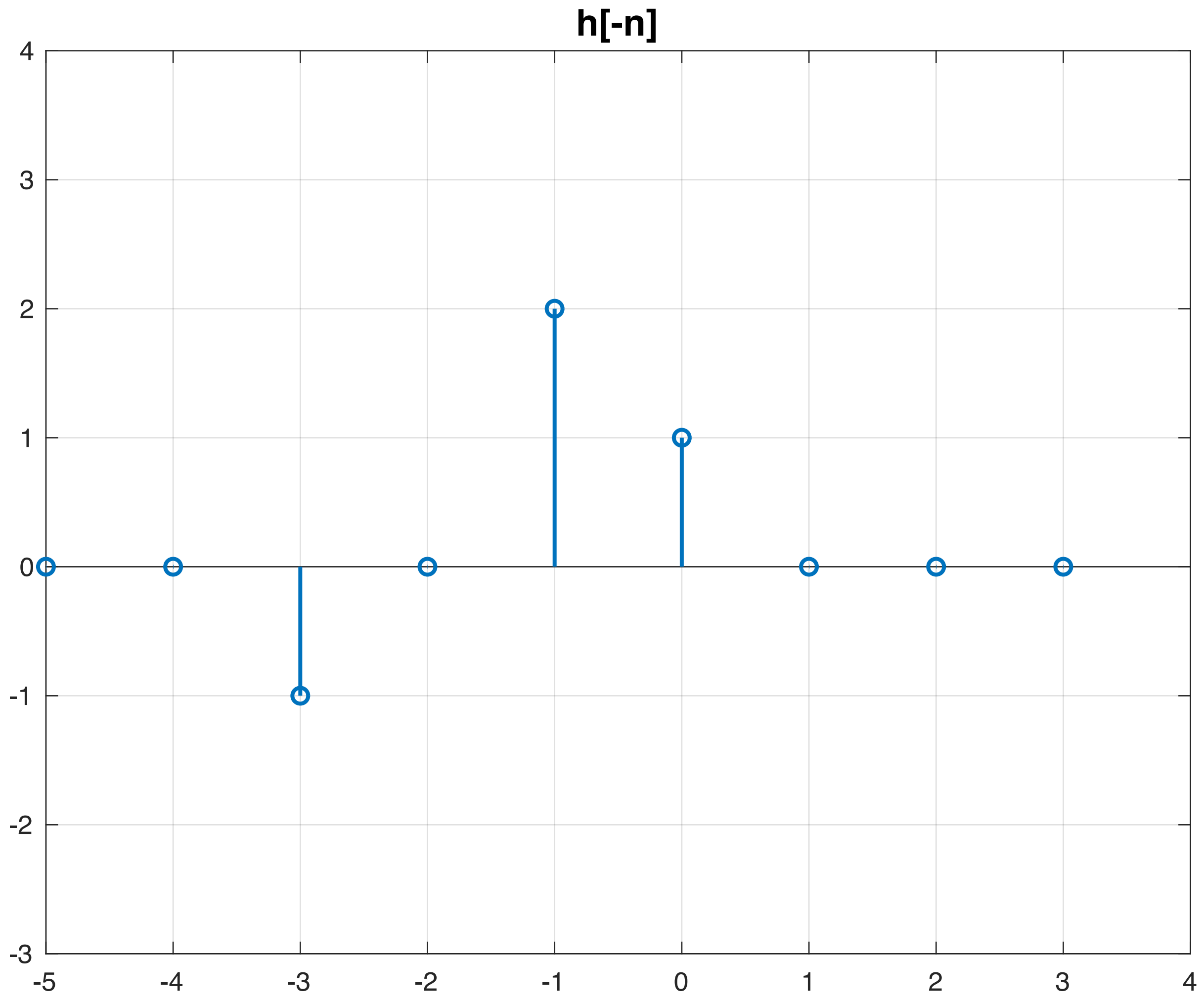




Operations on Sequences

Convolution Sum

- Example: Find Convolution sum between two sequences:
- $x[n] = \{\dots, 0, \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \dots\}$
- $h[n] = \{\dots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \dots\}$
- Step 1: Fold any one, we will fold $h[n]$ here to get $h[-n]$
- $h[-n] = \{\dots, 0, -1, 0, 2, \underset{\uparrow}{1}, 0, \dots\}$



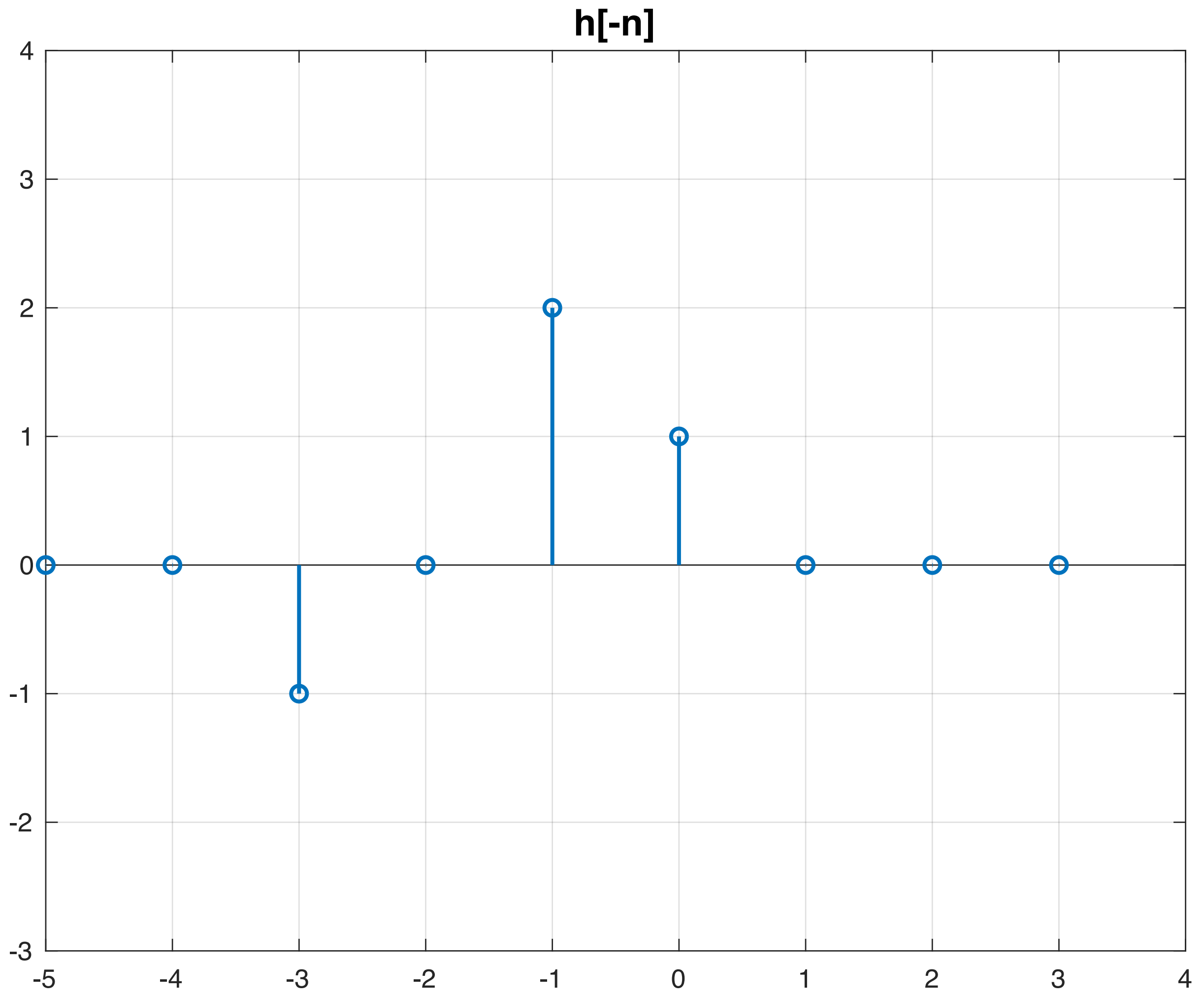
Operations on Sequences

Convolution Sum

- $x[k] = \{\ldots, 0, \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \ldots\}$

- $h[k] = \{\ldots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \ldots\}$

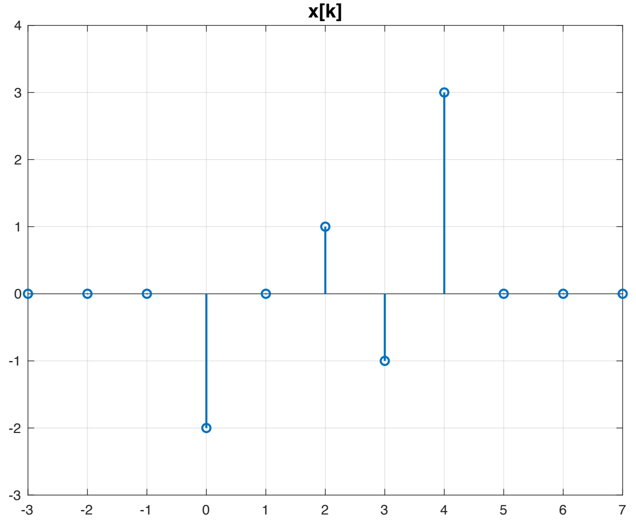
- Step 2: Since both $x[n]$ is defined for $n < 0 < 4$ and $h[n]$ is defined for $0 < n < 3$, so the limit of n will be from 0 to 7.

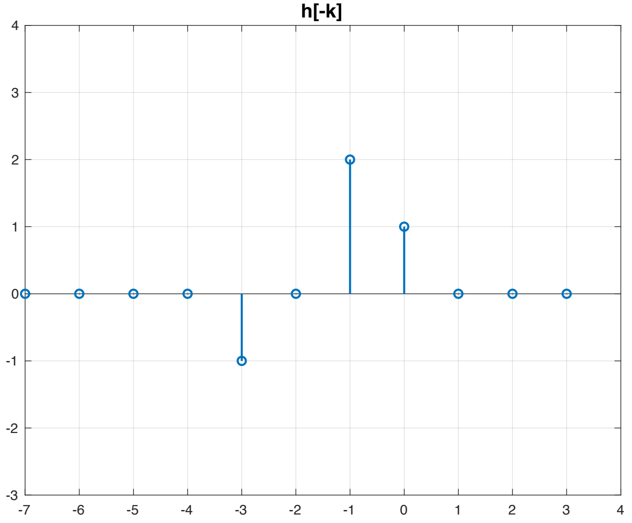


Operations on Sequences

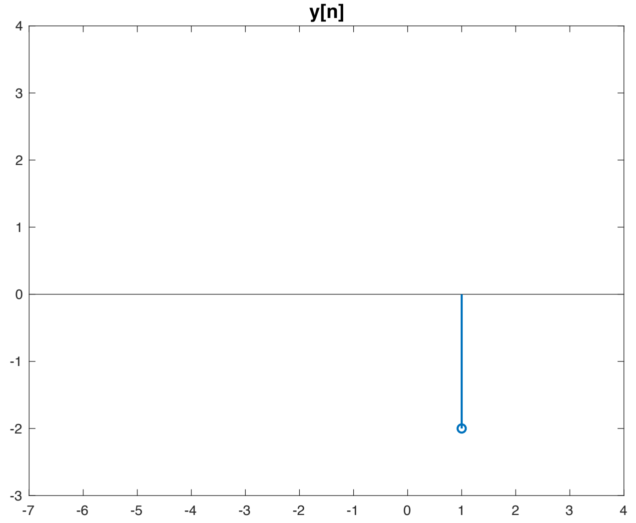
Convolution Sum

- $x[k] = \{\ldots, 0, \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \ldots\}$
- $h[k] = \{\ldots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \ldots\}$





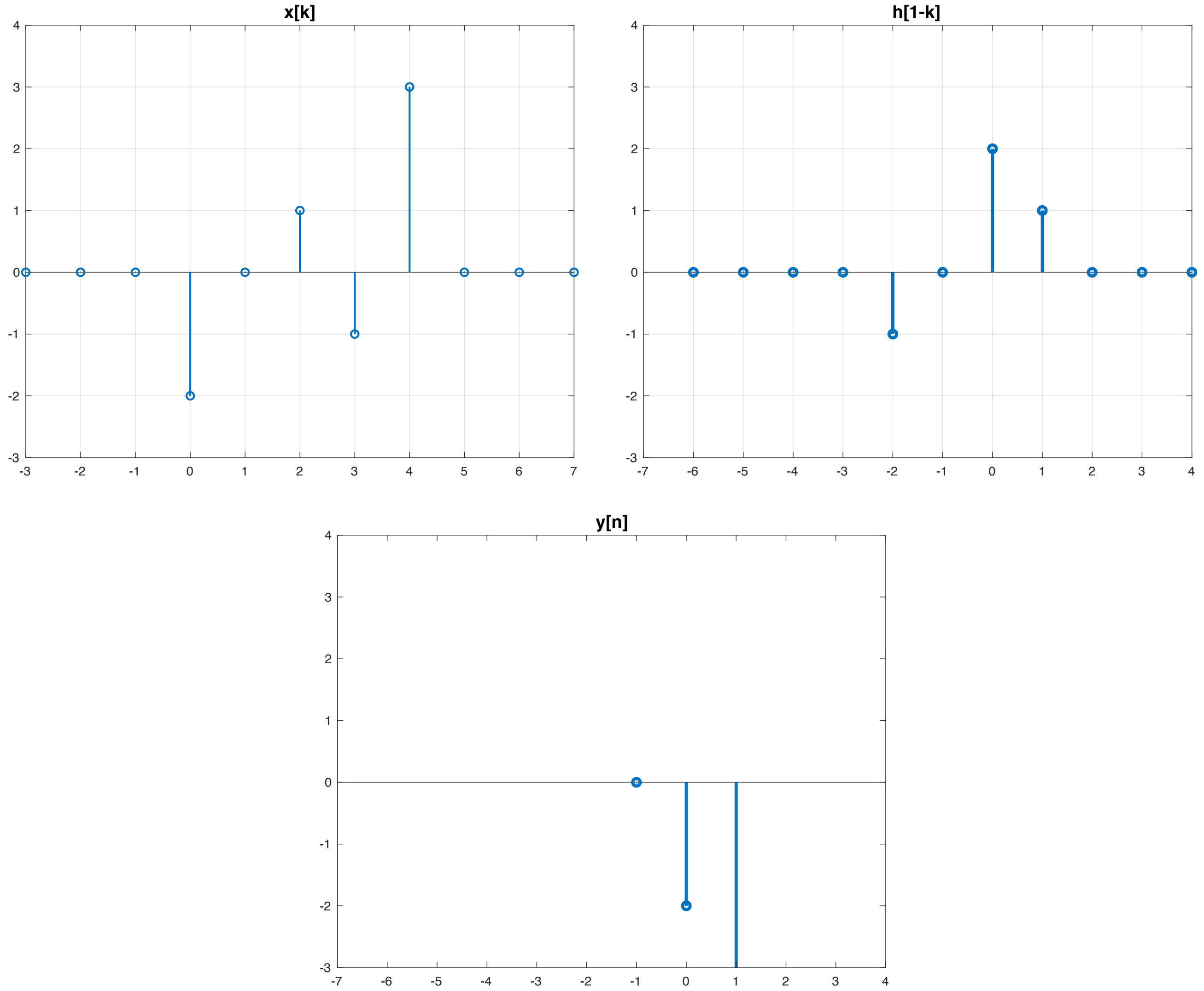
- Step 3: For $n = 0$
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- $y[0] = x[0]h[0] = -2$



Operations on Sequences

Convolution Sum

- $x[k] = \{\dots, 0, \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \dots\}$
- $h[n] = \{\dots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \dots\}$



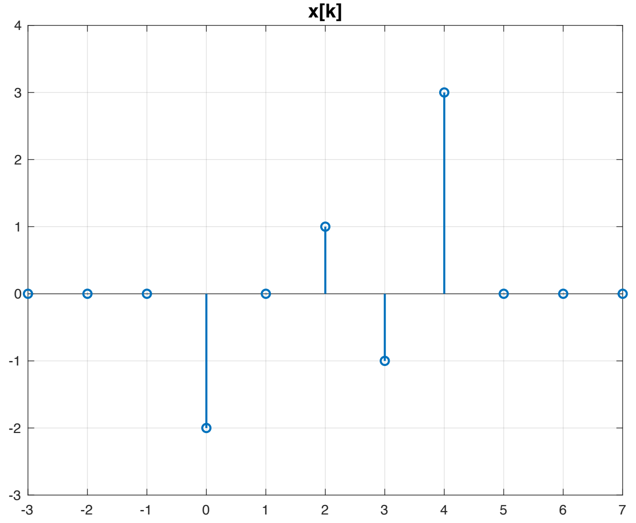
- Step 3: For $n = 1$
- $y[1] = \sum_{k=0}^{1} x[k]h[1-k]$
- $y[1] = x[0]h[1] + x[1]h[0] = -2\times2 + 0\times1 = -4$

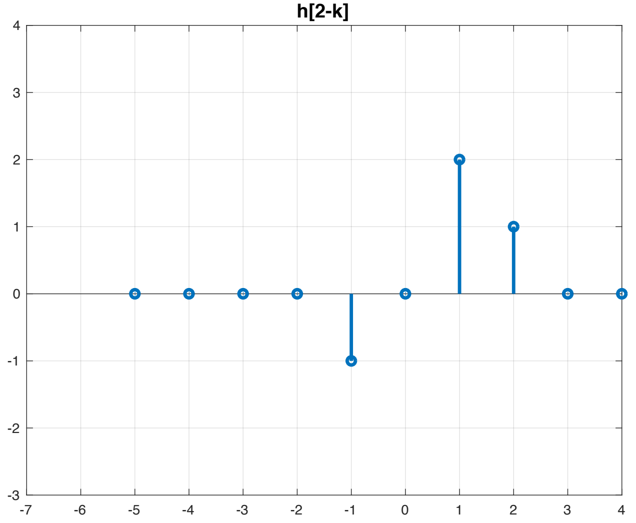
Operations on Sequences

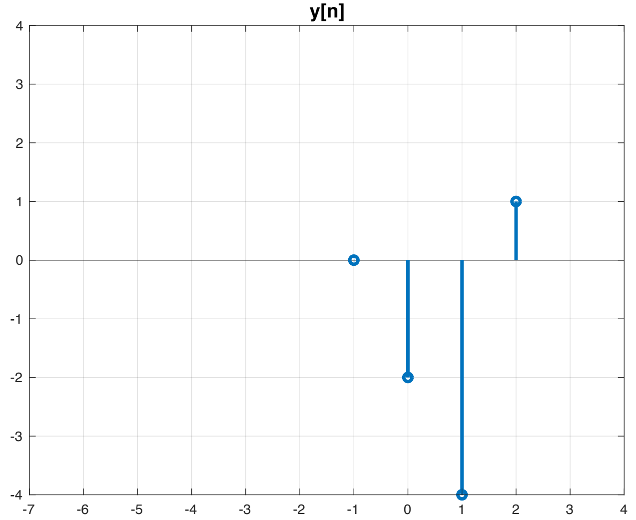
Convolution Sum

- $x[k] = \{\ldots, 0, \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \ldots\}$

- $h[n] = \{\ldots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \ldots\}$







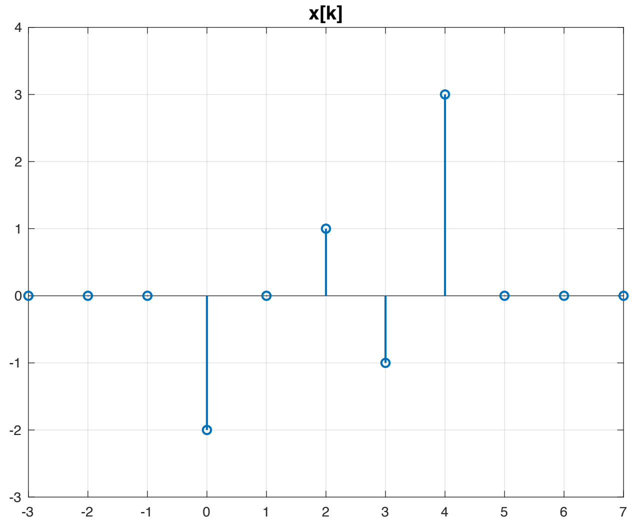
- Step 4: For $n = 2$
- $y[1] = \sum_{k=0}^{2} x[k]h[2-k]$
- $y[1] = x[0]h[2] + x[1]h[1] + x[2]h[0] = -2\times0 + 0\times2 + 1\times1 = 1$

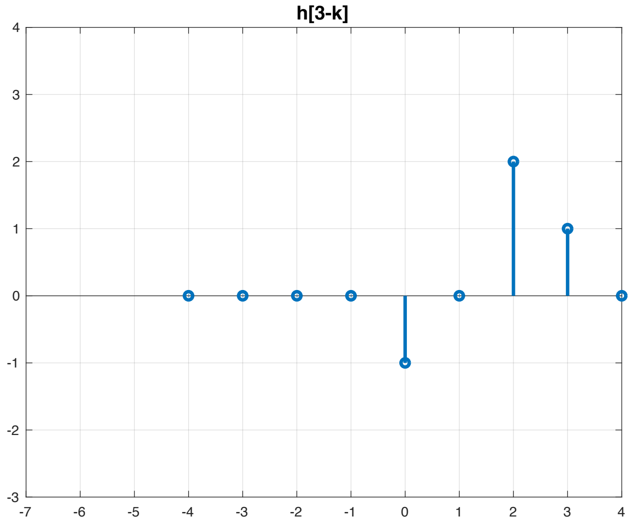
Operations on Sequences

Convolution Sum

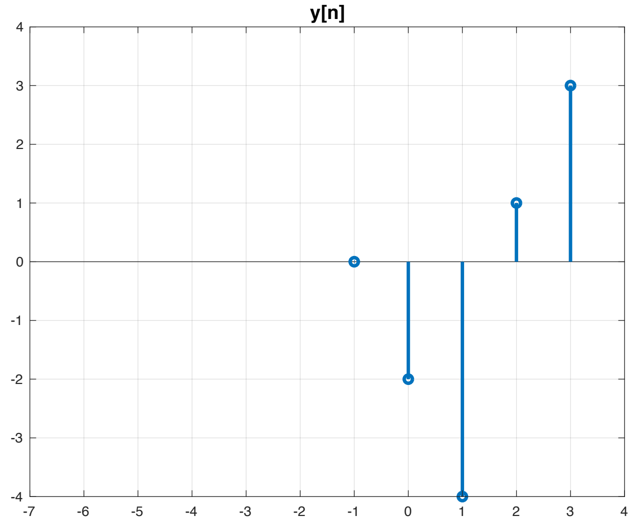
- $x[k] = \{\ldots, 0, \; \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \ldots\}$

- $h[n] = \{\ldots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \ldots\}$





- Step 5: For $n = 3$
- $y[1] = \sum_{k=0}^{3} x[k]h[3-k]$
- $y[1] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] = -2\times-1 + 0\times0 + 1\times2 - 1\times1 = 3$



Contact: imran.muet@gmail.com

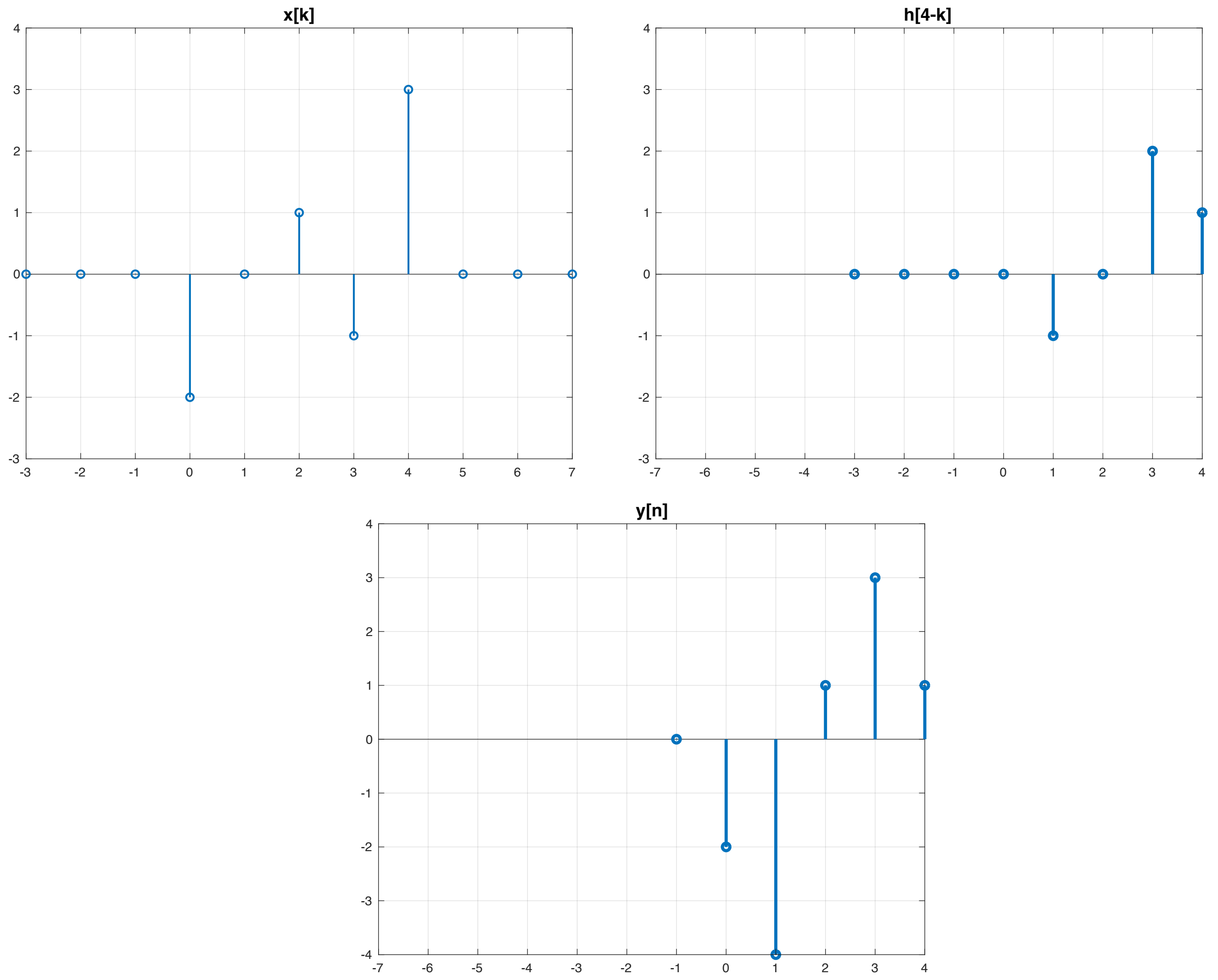
Operations on Sequences

Convolution Sum

- $x[k] = \{\ldots, 0, \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \ldots\}$

- $h[n] = \{\ldots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \ldots\}$

- Step 6: For $n = 4$
- $y[4] = \sum_{k=0}^{4} x[k]h[4-k]$
- $y[4] = x[0]h[4] + x[1]h[3] + x[2]h[2] + x[3]h[1] + x[4]h[0] = -2\times 0 + 0\times -1 + 1\times 0 - 1\times 2 + 3\times 1 = 1$

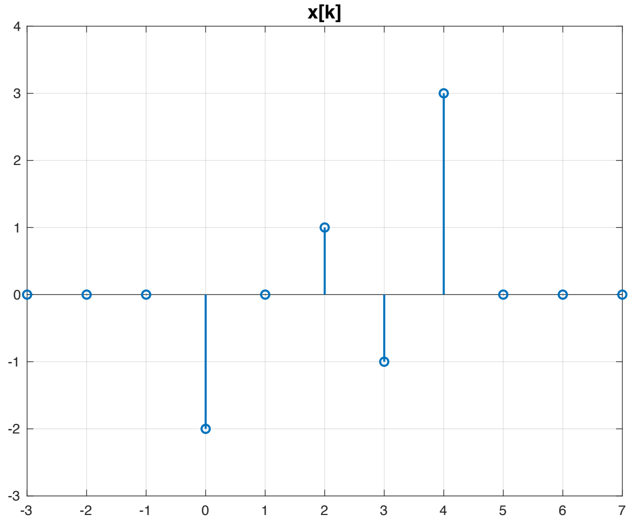


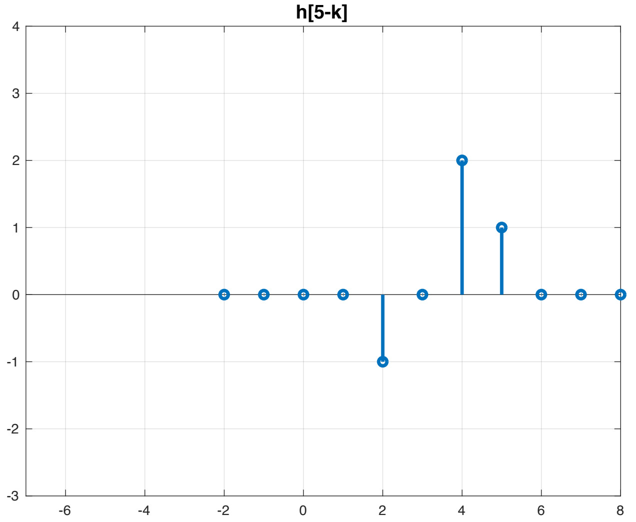
Operations on Sequences

Convolution Sum

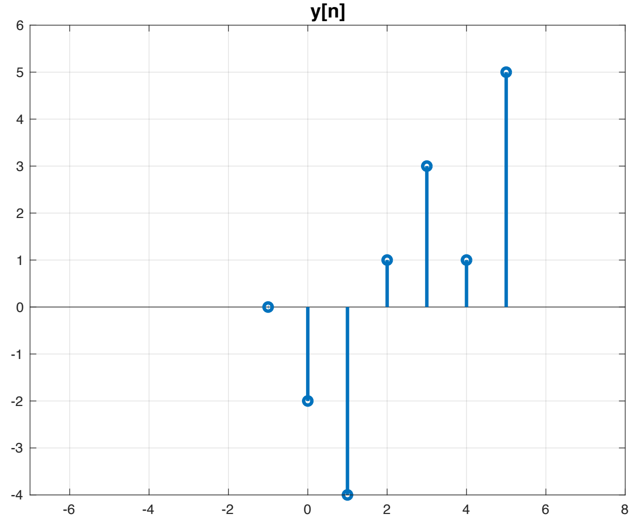
- $x[k] = \{\dots, 0, \ -2, 0, 1, -1, 3, 0 \dots\}$

- $h[n] = \{\dots, 0, 1, 2, 0, -1, 0, \dots\}$





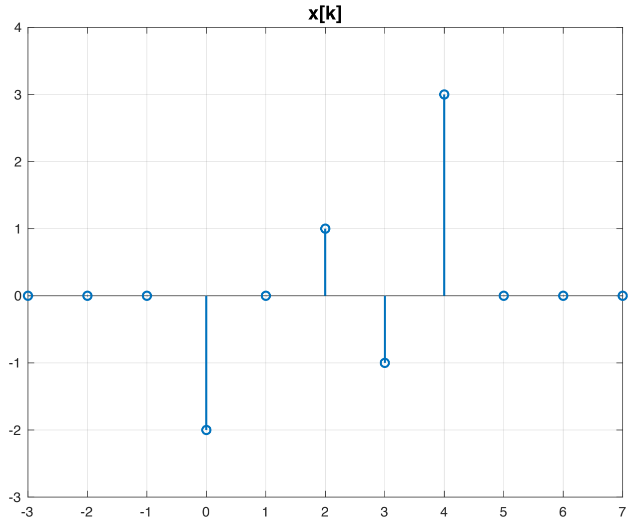
- Step 6: For $n = 5$
- $y[5] = \sum_{k=0}^{5} x[k]h[5-k]$
- $y[5] = x[0]h[5] + x[1]h[4] + x[2]h[3] + x[3]h[2] + x[4]h[1] + x[5]h[0] = -2\times 0 + 0\times 0 + 1\times -1 - 1\times 0 + 3\times 2 + 0\times 1 = 5$

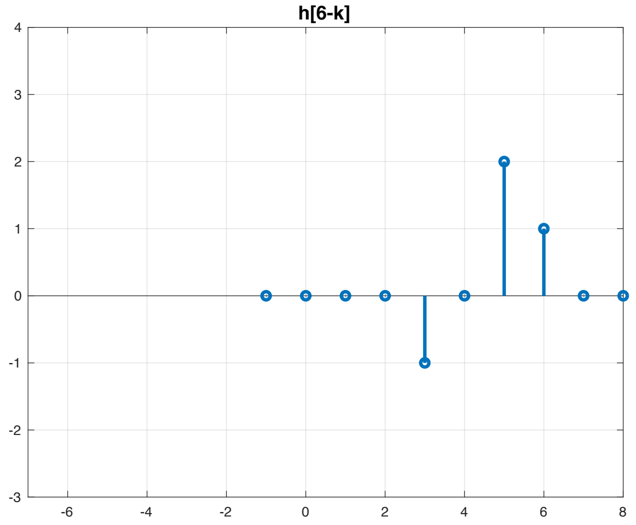


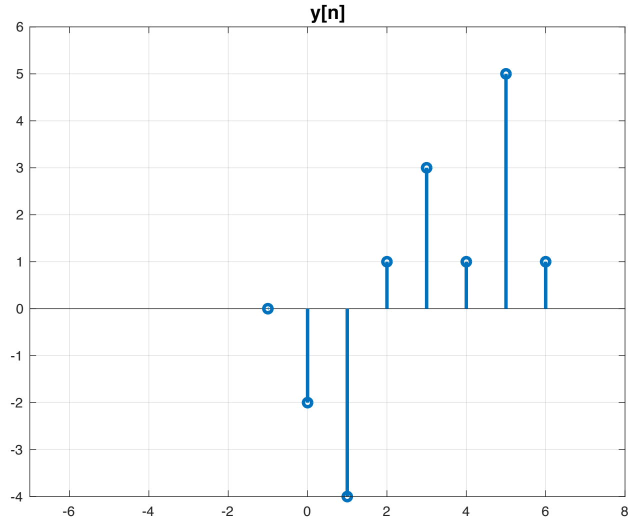
Operations on Sequences

Convolution Sum

- $x[k] = \{\ldots, 0, \; -2, 0, 1, -1, 3, 0 \ldots\}$
- $h[n] = \{\ldots, 0, 1, 2, 0, -1, 0, \ldots\}$
- Step 6: For $n = 6$
- $y[6] = \sum_{k=0}^{6} x[k]h[6-k]$
- $y[6] = x[0]h[6] + x[1]h[5] + x[2]h[4] + x[3]h[3] + x[4]h[2] + x[5]h[1] + x[6]h[0] = -2\times0 + 0\times0 + 1\times0 - 1\times-1 + 3\times0 + 0\times2 + 0\times1 = 1$



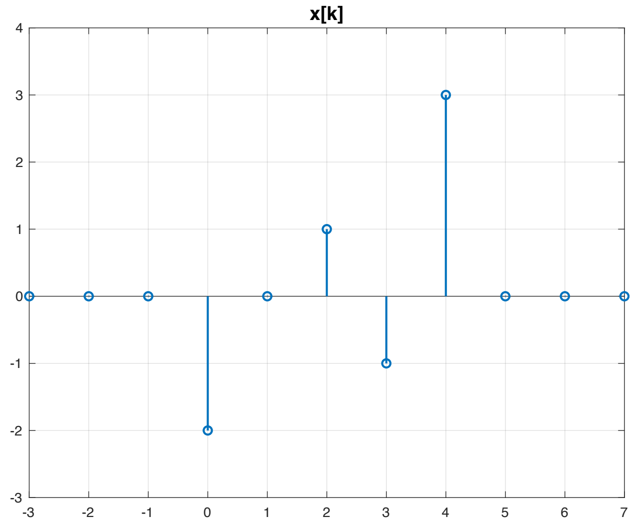


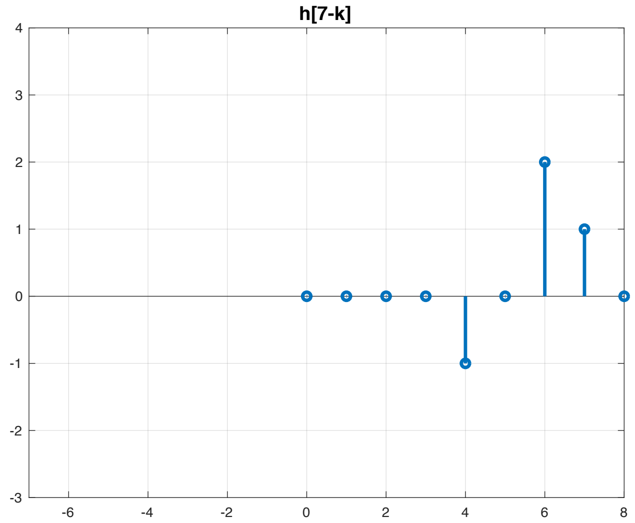


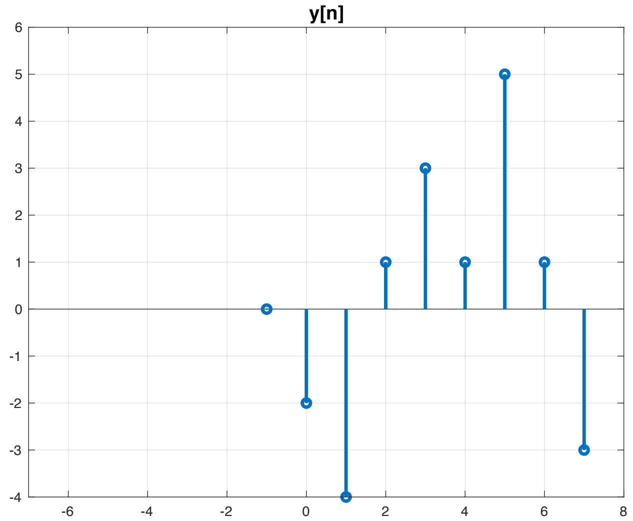
Operations on Sequences

Convolution Sum

- $x[k] = \{\ldots, 0,\ -2, 0, 1, -1, 3, 0\ \ldots\}$
- $h[n] = \{\ldots, 0, 1, 2, 0, -1, 0, \ldots\}$
- Step 6: For $n = 7$
- $y[7] = \sum_{k=0}^{7} x[k]h[7-k]$
- $y[7] = x[0]h[7] + x[1]h[6] + x[2]h[5] + x[3]h[4] + x[4]h[3] + x[5]h[3] + x[6]h[1] + x[7]h[0] = -2\times 0 + 0\times 0 + 1\times 0 - 1\times 0 + 3\times -1 + 0\times 0 + 0\times 2 + 0\times 0 = -3$







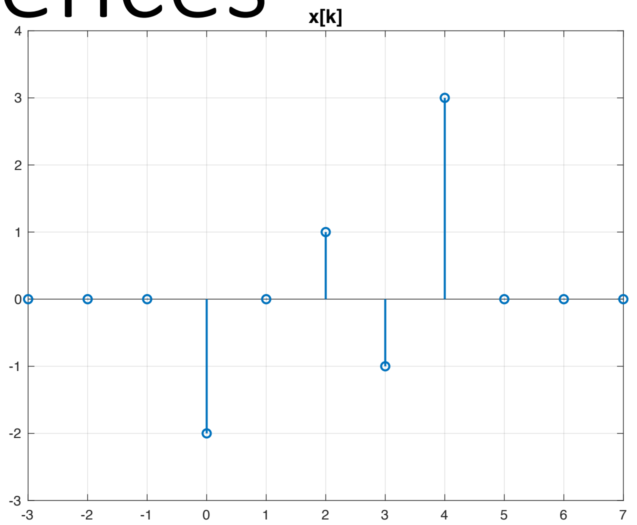
Operations on Sequences

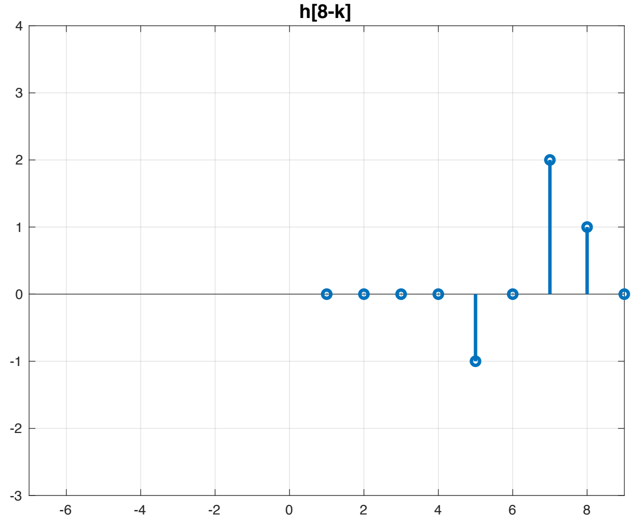
Convolution Sum

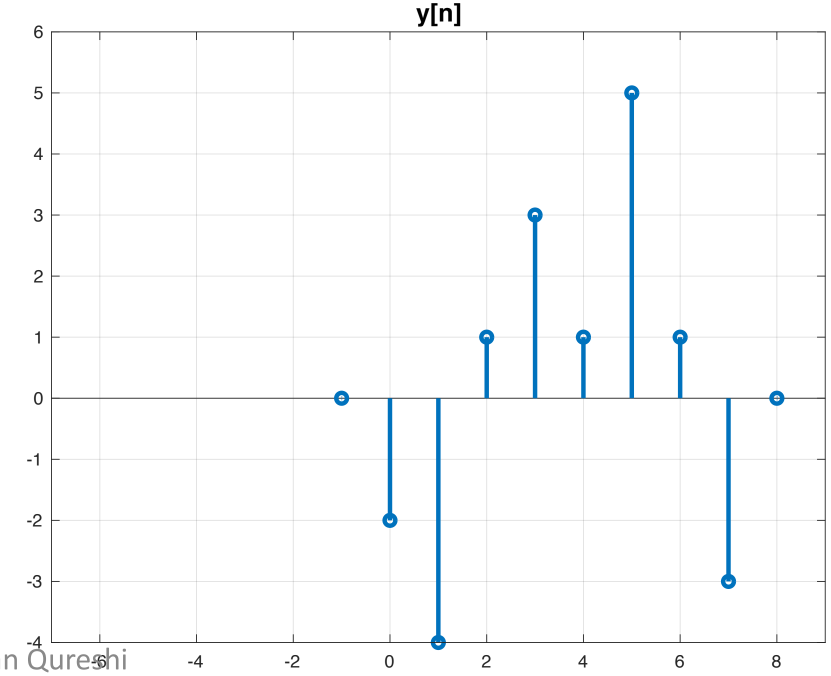
- $x[k] = \{\dots, 0, \underset{\uparrow}{-2}, 0, 1, -1, 3, 0 \dots\}$

- $h[n] = \{\dots, 0, \underset{\uparrow}{1}, 2, 0, -1, 0, \dots\}$

- Step 6: For $n > 7$
- $y[n] = \sum_{k=0}^{n} x[k]h[n-k]$
- $y[n] = 0$







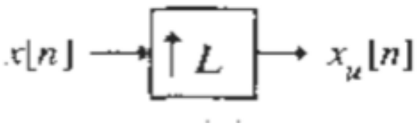
Operations on Sequences

Sampling Rate Alteration

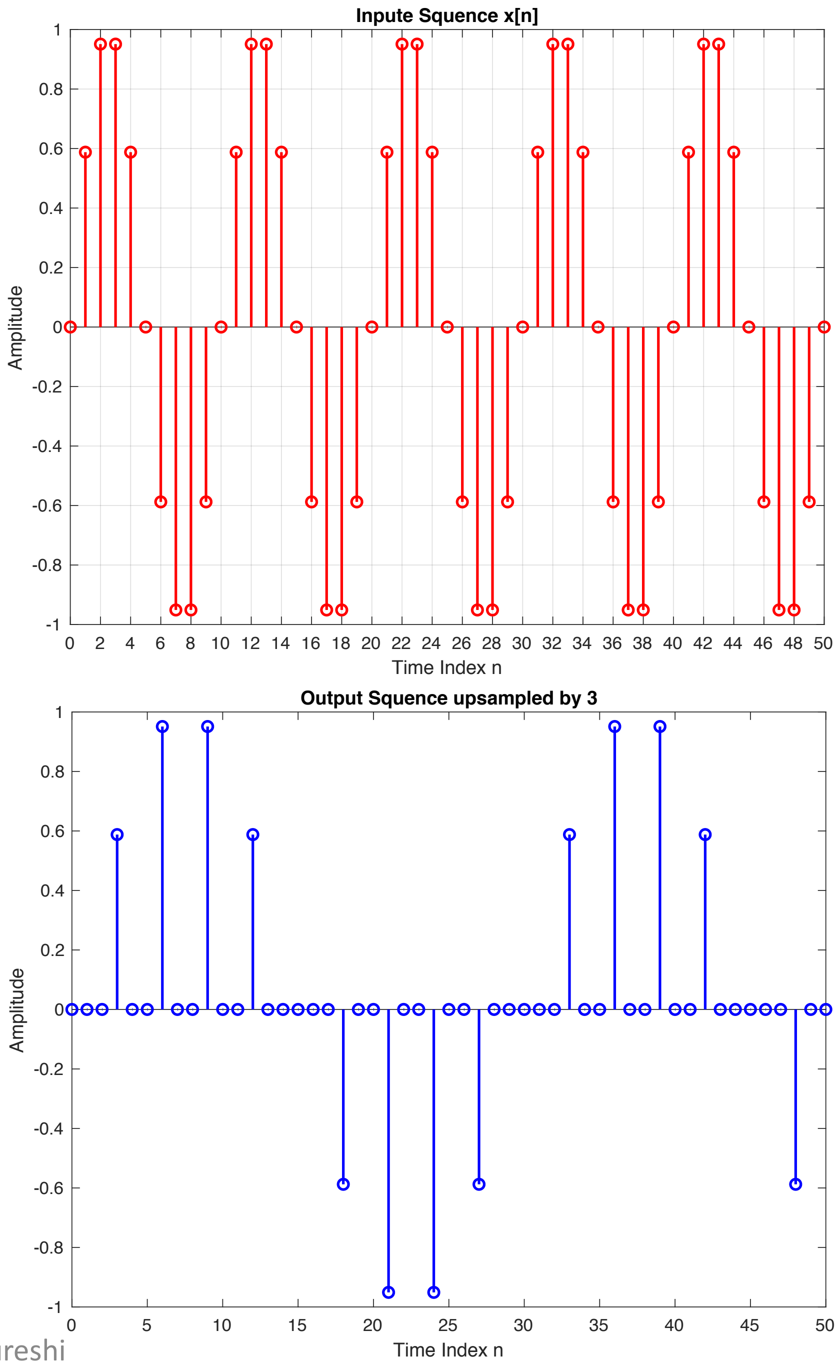
- If $x[n]$ is a sequence with a sampling rate of F_T Hz.
- Another, $y[n]$ is generated from $x[n]$ using sampling rate F'_T Hz.
- The sampling rate alteration ratio is given by:
- $R = \frac{F'_T}{F_T}$ (2.22)
- Interpolation:
  - If $R > 1$ $(i.e.\ F'_T > F_T)$, then the resulting $y[n]$ sequence will result with a higher sampling rate.
  - The operation is called interpolation.
  - The device used is called interpolator.
- Decimation:
  - If $R < 1$ $(i.e.\ F'_T < F_T)$, then the resulting $y[n]$ sequence will result with a lower sampling rate.
  - The operation is called decimation.
  - The device used is called decimator.

Operations on Sequences

Sampling Rate Alteration



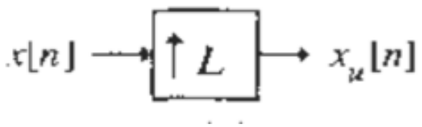
- **Upsampling:**
  - In up-sampling by an integer factor $L > 1, L - 1$ equidistant zero-valued samples are inserted between each two consecutive samples of the input sequence $x[n]$ to develop an output sequence $x_u[n]$ according to the relation
  - $x_u[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = 0, \pm L, \pm 2L, \dots, \\ 0, & otherwise. \end{cases}$
  - The sampling rate of $x_u[n]$ is L times larger than that of the original sequence $x[n]$.
  - Figure shows the upsampling for $L = 3$.

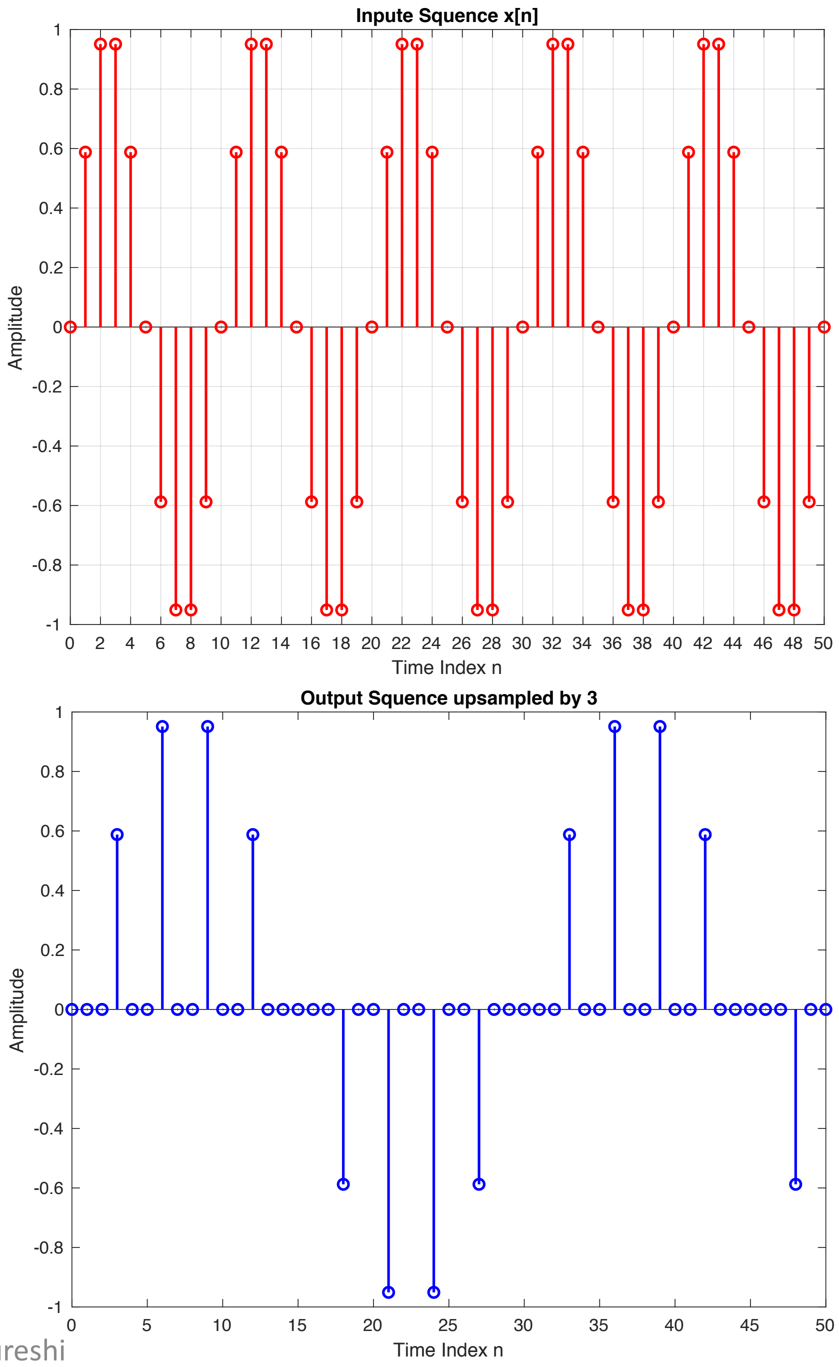


Operations on Sequences

Sampling Rate Alteration

- **Upsampling:**
  - The interpolator consists of an up-sampler followed by a discrete-time system that replaces the inserted zero-valued samples of $x_u[n]$ with more appropriate values given by a linear combination of samples of $x[n]$.

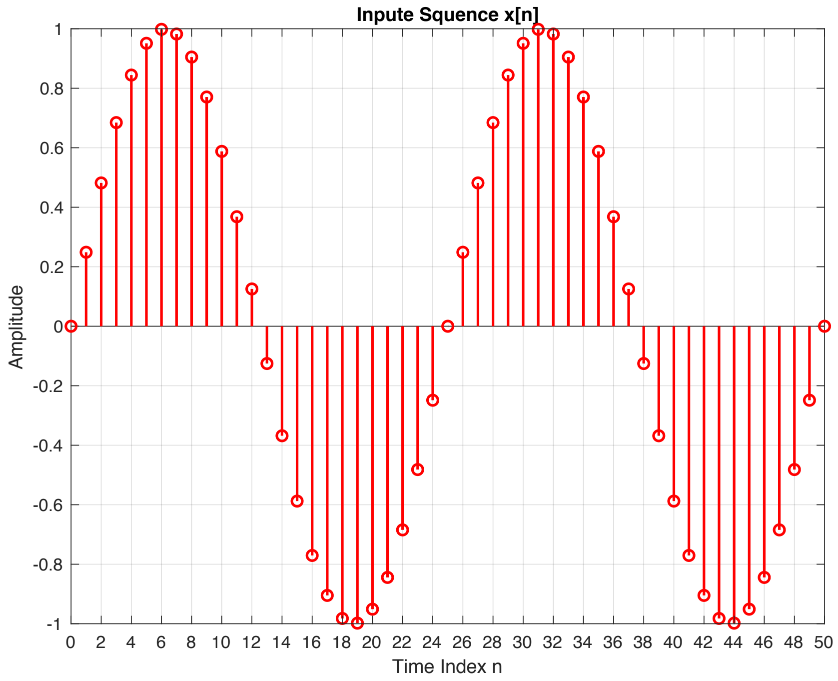


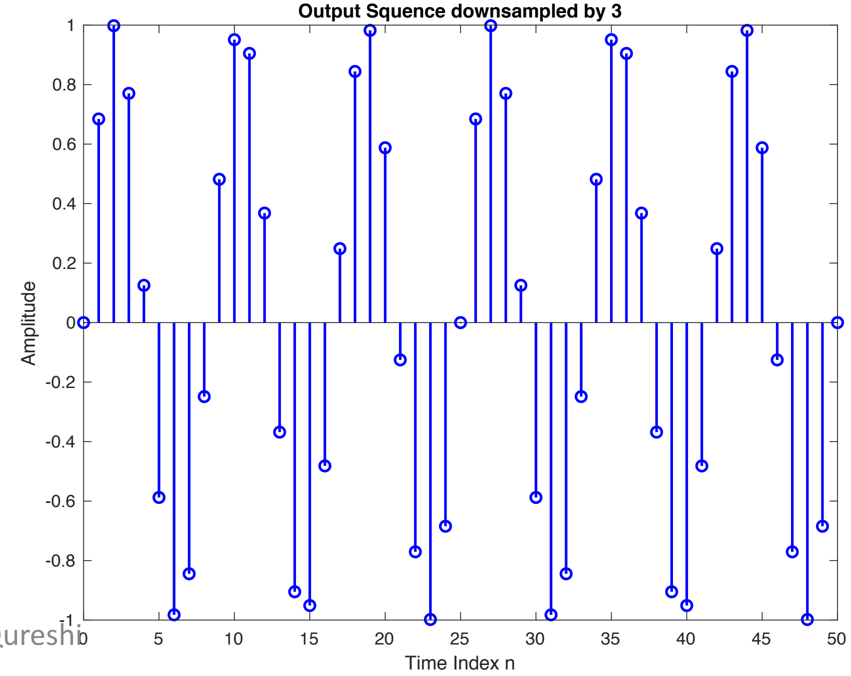


Operations on Sequences

Sampling Rate Alteration

- Down-Sampling:
  - The down-sampling operation by an integer factor $M > 1$ on a sequence $x[n]$ consists of keeping every Mth sample of $x[n]$ and removing $M - 1$ between samples, generating an output sequence $y[n]$ according to the relation.
  - $y[n] = x[nM] \qquad (2.24)$
  - This results in a sequence $y[n]$ whose sampling rate is $\left(\frac{1}{M}\right)th$ that $x[n]$.
  - Figure shows the operation of down-sampling for $M = 3$.





Operations on Sequences

Sampling Rate Alteration

- Down-Sampling:
  - The down-sampling operation by an integer factor $M > 1$ on a sequence $x[n]$ consists of keeping every Mth sample of $x[n]$ and removing $M - 1$ between samples, generating an output sequence $y[n]$ according to the relation.
  - $y[n] = x[nM] \qquad (2.24)$
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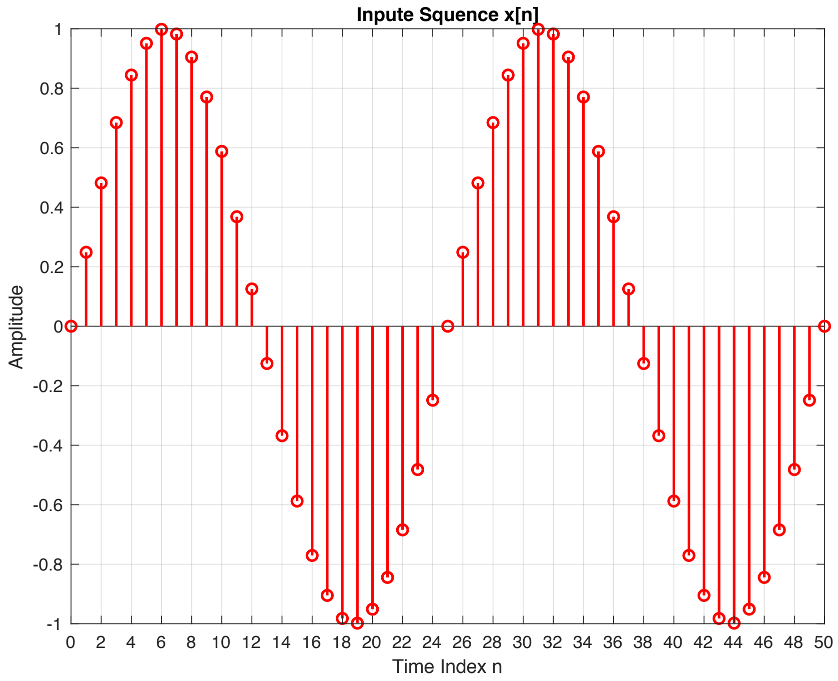
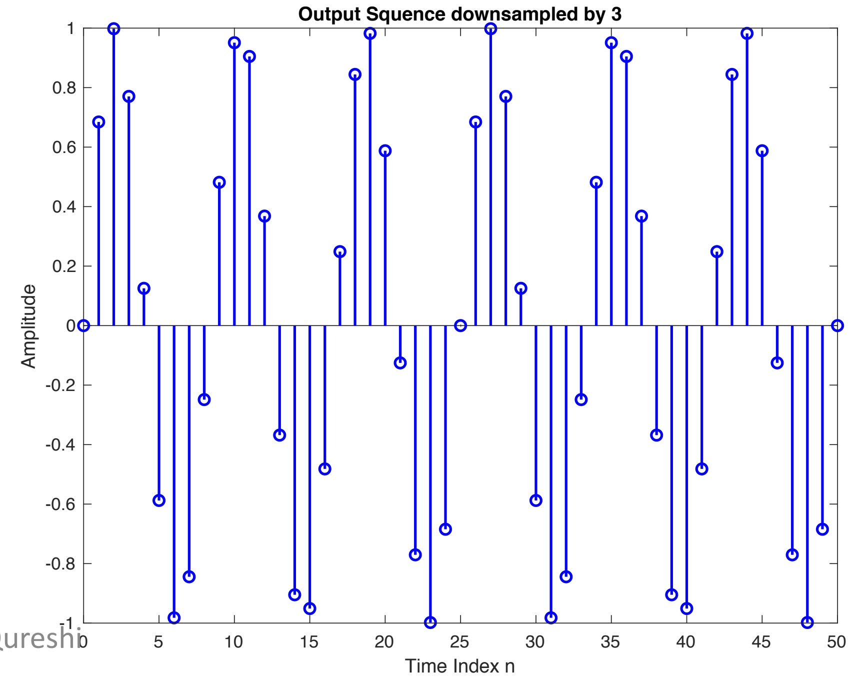
  - Figure shows the operation of down-sampling for $M = 3$.

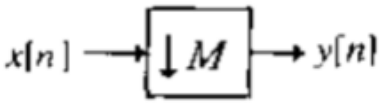


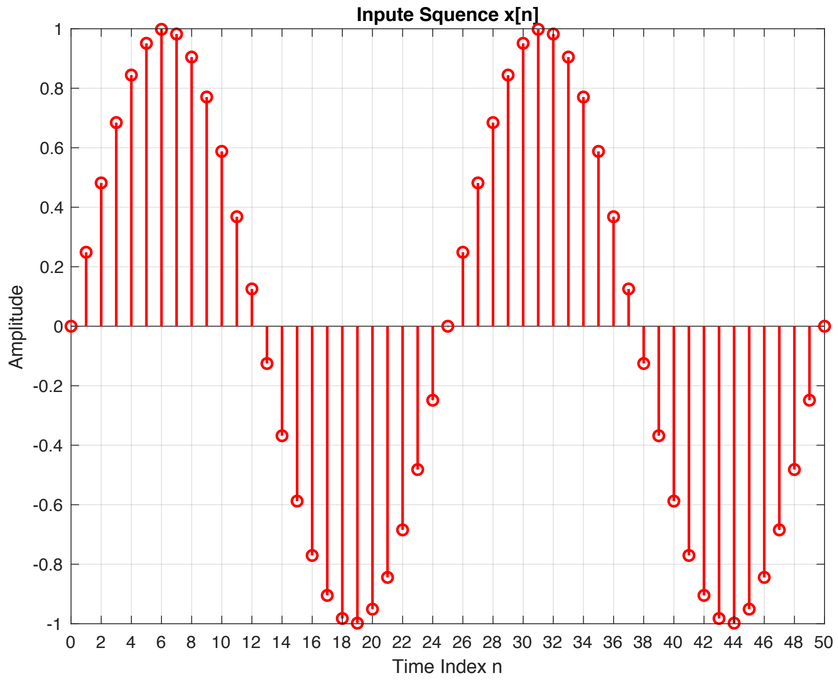


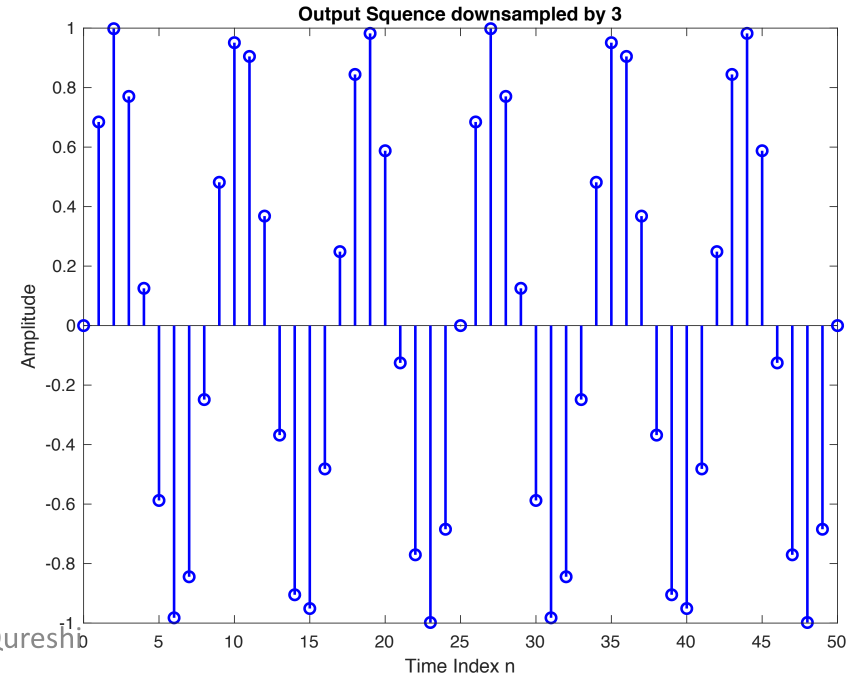
Operations on Sequences

Sampling Rate Alteration

- Down-Sampling:
  - The decimator consists of a discrete-time system followed by a down-sampler.
  - The discrete-time system preceding the down-sampler ensures that the input signal x[n] is appropriately band-limited to prevent aliasing that is caused by the down-sampling operations.







Operations on Finite-Length Sequences

- Let $x[n]$ is a sequence defined for $0 \le n \le N-1.$
- A time reversal operation on $x[n]$ will result in another sequence $x[-n]$, which is defined for $-(N-1) \le n \le 0$.
- **The sequence $x[-n]$ obtained from $x[n]$, is no longer define for the same duration**.
- Likewise, a linear time-shift applied on $x[n]$ to get $x[n+M]$, where $M > 1$, where $x[n+M]$ is no longer defined for the same interval i.e. $0 \le n \le N-1$.
- Similarly, a convolution sum of a length-N sequence x[n] defined for $0 \le n \le N-1$ with another length-N sequence $h[n]$ defined for $0 \le n \le N-1$ will generate a sequence of length $2N-1$ defined for $0 \le n \le 2N-2$.
- We need another type of shifting or convolution that produces the output defined for the same interval as input sequence i.e. $0 \le n \le N-1$

Operations on Finite-Length Sequences

Circular Time-Reversal Operation

- The time—reversed version $\{y[n]\}$ of the length-N sequence $\{x[n]\}$ defined for $0 \leq n \leq N-1$ is given by:
- $\{y[n]\} = \{x[\langle -n \rangle_N]\}$
- Where $\langle -n \rangle_N$ is modulo operation, explained below with examples:
- $\langle 25 \rangle_7 = \frac{25}{7} = 4\ (if\ we\ divide\ 25\ by\ 7, the\ remainder\ is\ 4\ because\ 25 - 7 * 3 = 4).$
- $\langle -15 \rangle_7 = -\frac{15}{7} = 6\ (if\ we\ divide\ -15\ by\ 7,\ the\ remainder\ is\ 6\ \ because\ 7 * 3 - 15 = 6)$
- $\langle -1 \rangle_5 = -\frac{1}{5} = 4\ (if\ we\ divide\ -1\ by\ 5,\ the\ remainder\ is\ 4\ \ because\ 5 * 1 - 1 = 4)$
- $\langle 15 \rangle_7 = \frac{15}{7} = 1\ (if\ we\ divide\ 15\ by\ 7,\ the\ remainder\ is\ 1\ \ because\ 15 - 7 * 2 = 1)$

Operations on Finite-Length Sequences

Circular Time-Reversal Operation

- Consider the length-5 sequence $\{x[n]\} = \{x[0], x[1], x[2], x[3], x[4]\}$.
- Perform circular time-reversal operations?
- Solution:
- Since the length of sequence is 5 (there are five samples), so the circular time-reversed sequence is obtained as:
- $\{y[n]\} = \{x[\langle -n \rangle_5]\}$

- The 5 samples of $\{y[n]\}$ are obtained as:
- $\{y[0]\} = \{x[\langle 0 \rangle_5]\} = x[0] = 1$
- $\{y[1]\} = \{x[\langle -1 \rangle_5]\} = x[4] = 5$
- $\{y[2]\} = \{x[\langle -2 \rangle_5]\} = x[3] = 4$
- $\{y[3]\} = \{x[\langle -3 \rangle_5]\} = x[2] = 3$
- $\{y[4]\} = \{x[\langle -4 \rangle_5]\} = x[1] = 2$

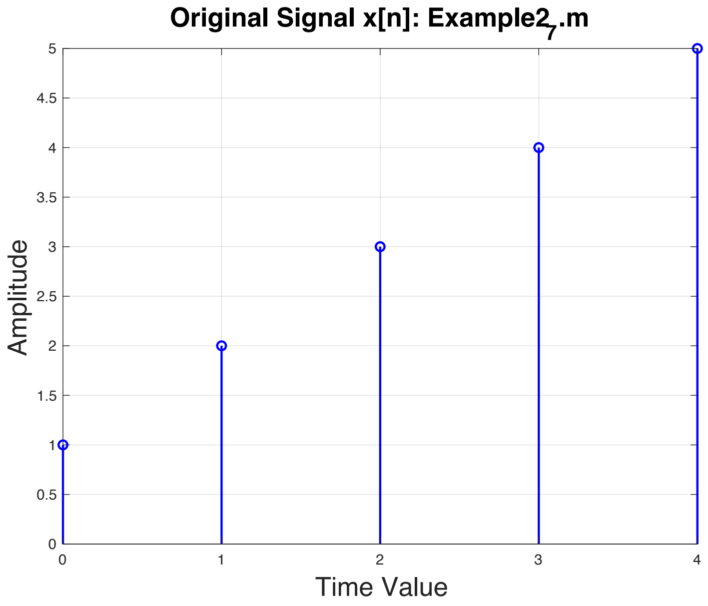
- $\{y[n]\} = \{x[0], x[1], x[2], x[3], x[4]\}$

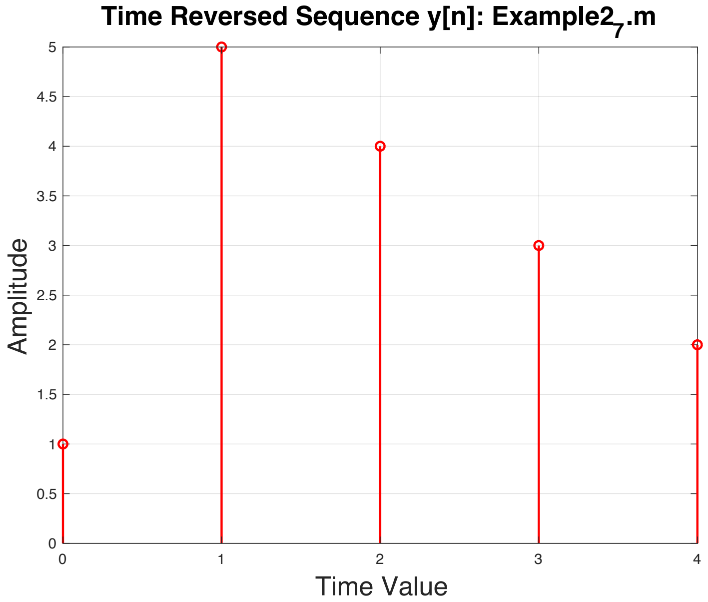
Operations on Finite-Length Sequences

Circular Time-Reversal Operation

- The 5 samples of $\{y[n]\}$ are obtained as:
- $\{y[0]\} = \{x[\langle 0 \rangle_5]\} = x[0] = 1$
- $\{y[1]\} = \{x[\langle -1 \rangle_5]\} = x[4] = 5$
- $\{y[2]\} = \{x[\langle -2 \rangle_5]\} = x[3] = 4$
- $\{y[3]\} = \{x[\langle -3 \rangle_5]\} = x[2] = 3$
- $\{y[4]\} = \{x[\langle -4 \rangle_5]\} = x[1] = 2$

- $\{y[n]\} = \{x[0], x[1], x[2], x[3], x[4]\}$





Operations on Finite-Length Sequences

Circular Shift of a Sequence

- The sequence obtained as a result of circular-time shift will be defined for same length as the original sequence.
- The circular shift of a length-N sequence $x[n]$ by an amount n_0 sample periods is defined by the equation
- $x_c[n] = x[\langle n - n_0 \rangle_N]$ (2.25)
- Where $x_c[n]$ is of same length as $x[n]$.
- If $n_0 > 0$, it is a right circular shift.
- If $n_0 < 0$, it is a left circular shift.

Operations on Finite-Length Sequences

Circular Shift of a Sequence

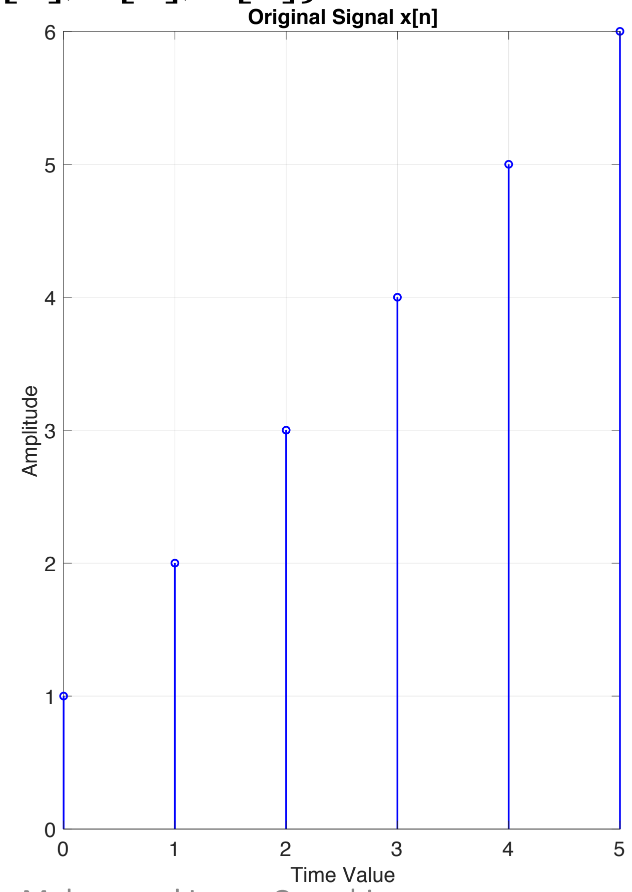
- The circular shift for a finite-length sequence can be implemented using the following equation for $n_0 > 0$:
- $x_c[n] = \begin{cases} x[n-n_0], & for\ n_0 \le n \le N-1 \\ x[N-n_0+n], & for\ 0 \le n < n_0 \end{cases}$ (2.26)
- Consider a sequence, shown below. Apply circular shift operation for $n_0 = 1, n_0 = -4, n_0 = 4, and\ n_0 = -5$
- $x[n] = \{1,2,3,4,5,6\} = \{x[0], x[1], x[2], x[3], x[4], x[5]\}$
- Solution:
- For $n_0 = 1$
- $x_c[n] = \begin{cases} x[n-1], & for\ 1 \le n \le 5 \\ x[5+n], & for\ 0 \le n < 1 \end{cases}$

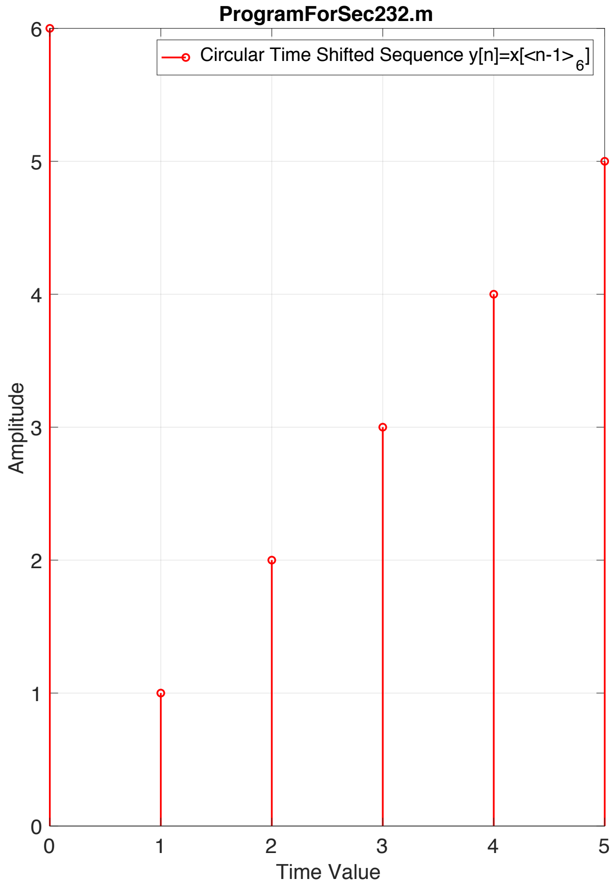
Operations on Finite-Length Sequences

Circular Shift of a Sequence

- Consider a sequence, shown below. Apply circular shift operation for $n_0 = 1, n_0 = -4, n_0 = 4, and\ n_0 = -5$
- $x[n] = \{1,2,3,4,5,6\} = \{x[0], x[1], x[2], x[3], x[4], x[5]\}$
- Solution:
- For $n_0 = 1$
- $\boldsymbol{x_c[n] = \begin{cases} x[n-1], & for\ 1 \le n \le 5 \\ x[5+n], & for\ 0 \le n \le 1 \end{cases}}$
- $\{y[0]\} = x[5] = 6$
- $\{y[1]\} = x[0] = 1$
- $\{y[2]\} = x[1] = 2$
- $\{y[3]\} = x[2] = 3$
- $\{y[4]\} = x[3] = 4$
- $\{y[5]\} = x[4] = 5$

The sequence shifted to right by 1 sample or equivalently, shifted left by 5 samples because $x[\langle n-1\rangle_6] = x[\langle n+5\rangle_6]$





Operations on Finite-Length Sequences

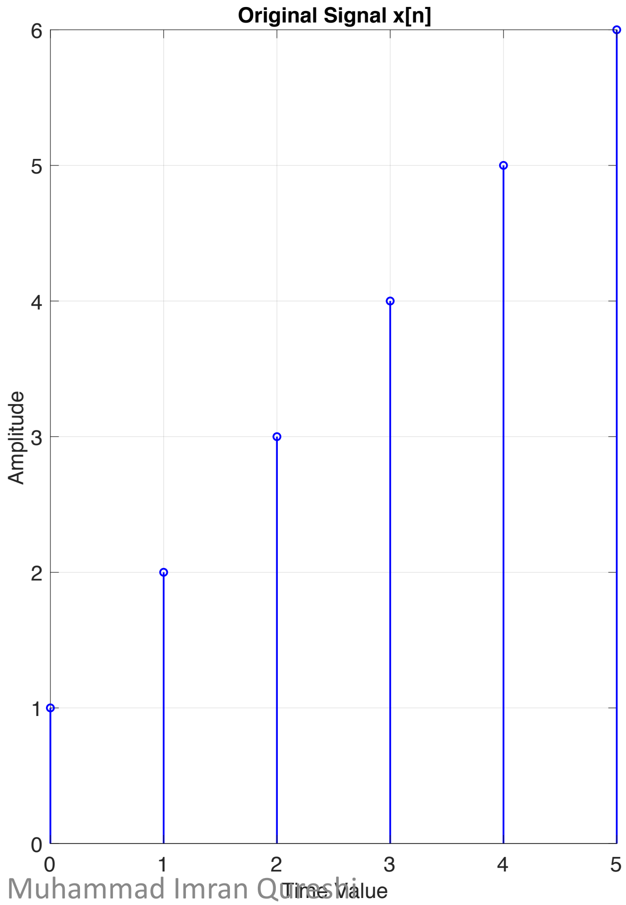
Formula (n0>0)

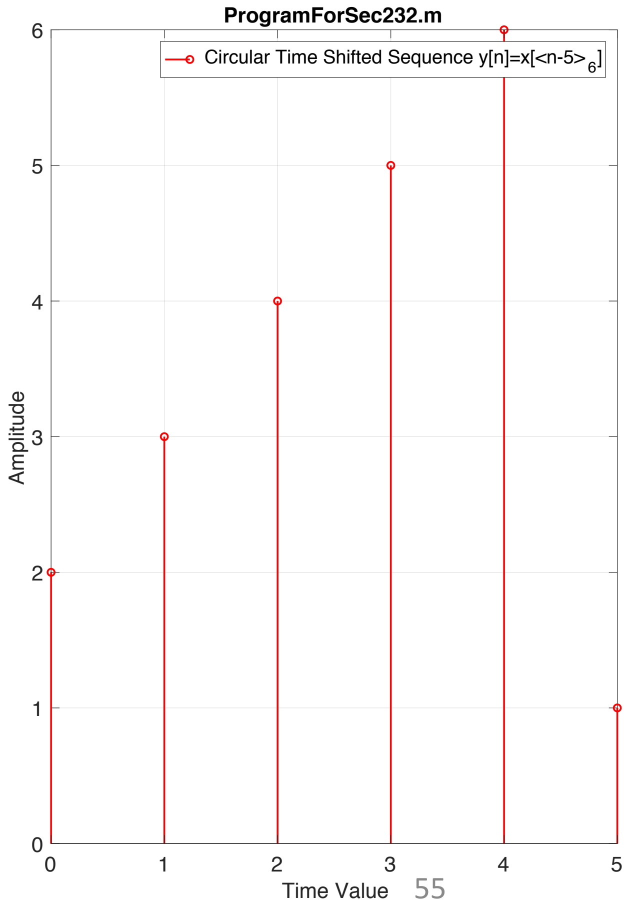
$$x_c[n] = \begin{cases} x[n-n_0], & for\ n_0 \le n \le N-1 \\ x[N-n_0+n], & for\ 0 \le n < n_0 \end{cases}$$

Circular Shift of a Sequence

- Consider a sequence, shown below. Apply circular shift operation for $n_0 = 1, n_0 = -4, n_0 = 4, and\ n_0 = -5$
- $x[n] = \{1,2,3,4,5,6\} = \{x[0], x[1], x[2], x[3], x[4], x[5]\}$
- Solution:
- For $n_0 = 5$
- $\boldsymbol{x_c[n] = \begin{cases} x[n-5], & for\ 5 \le n \le 5 \\ x[1+n], & for\ 0 \le n < 5 \end{cases}}$
- $\{y[0]\} = x[1] = 2$
- $\{y[1]\} = x[2] = 3$
- $\{y[2]\} = x[3] = 4$
- $\{y[3]\} = x[4] = 5$
- $\{y[4]\} = x[5] = 6$
- $\{y[5]\} = x[0] = 1$

The sequence shifted to right by 5 sample or equivalently, shifted left by 1 samples because $x[\langle n-5\rangle_6] = x[\langle n+1\rangle_6]$



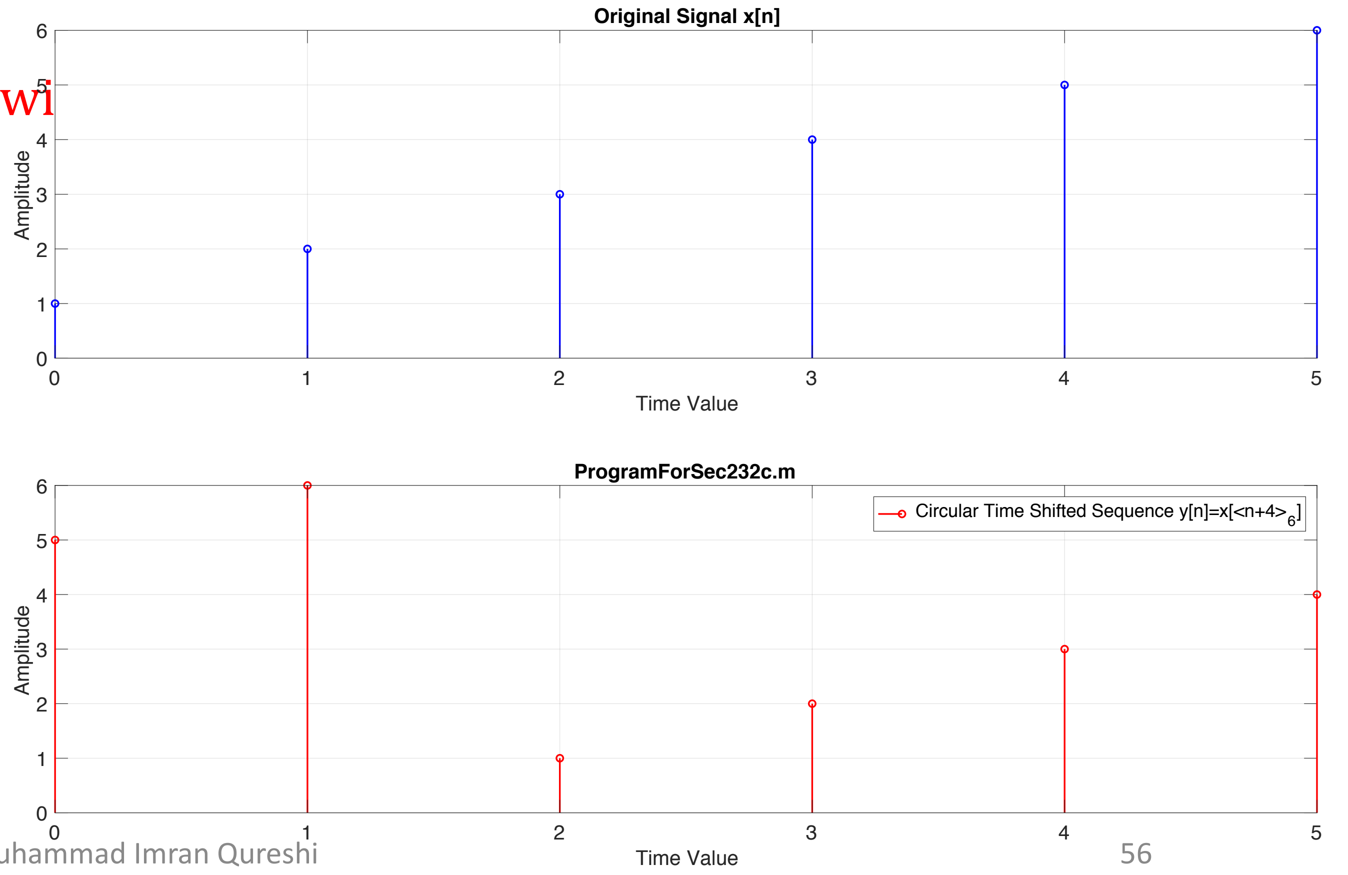


Operations on Finite-Length Sequences

Circular Shift of a Sequence

- Consider a sequence, shown below. Apply circular shift operation for $n_0 = 1, n_0 = -4, and\ n_0 = -5$
- $x[n] = \{1,2,3,4,5,6\} = \{x[0], x[1], x[2], x[3], x[4], x[5]\}$
- Solution:
- For $n_0 = -4$ (since $n_0 < 0,$ we will use followi
- $\boldsymbol{x_c[n] = x[\langle n - n_0\rangle_N]}$
- $\{y[0]\} = x[\langle 0+4\rangle_6] = x[\langle 4\rangle_6] = x[4] = 5$
- $\{y[1]\} = x[\langle 1+4\rangle_6] = x[\langle 5\rangle_6] = x[5] = 6$
- $\{y[2]\} = x[\langle 2+4\rangle_6] = x[\langle 6\rangle_6] = x[0] = 1$
- $\{y[3]\} = x[\langle 3+4\rangle_6] = x[\langle 7\rangle_6] = x[1] = 2$
- $\{y[4]\} = x[\langle 4+4\rangle_6] = x[\langle 8\rangle_6] = x[2] = 3$
- $\{y[5]\} = x[\langle 5+4\rangle_6] = x[\langle 9\rangle_6] = x[3] = 4$

The sequence shifted to left by 4 sample or equivalently, shifted right by 2 samples because $x[\langle n+4\rangle_6] = x[\langle n-2\rangle_6]$

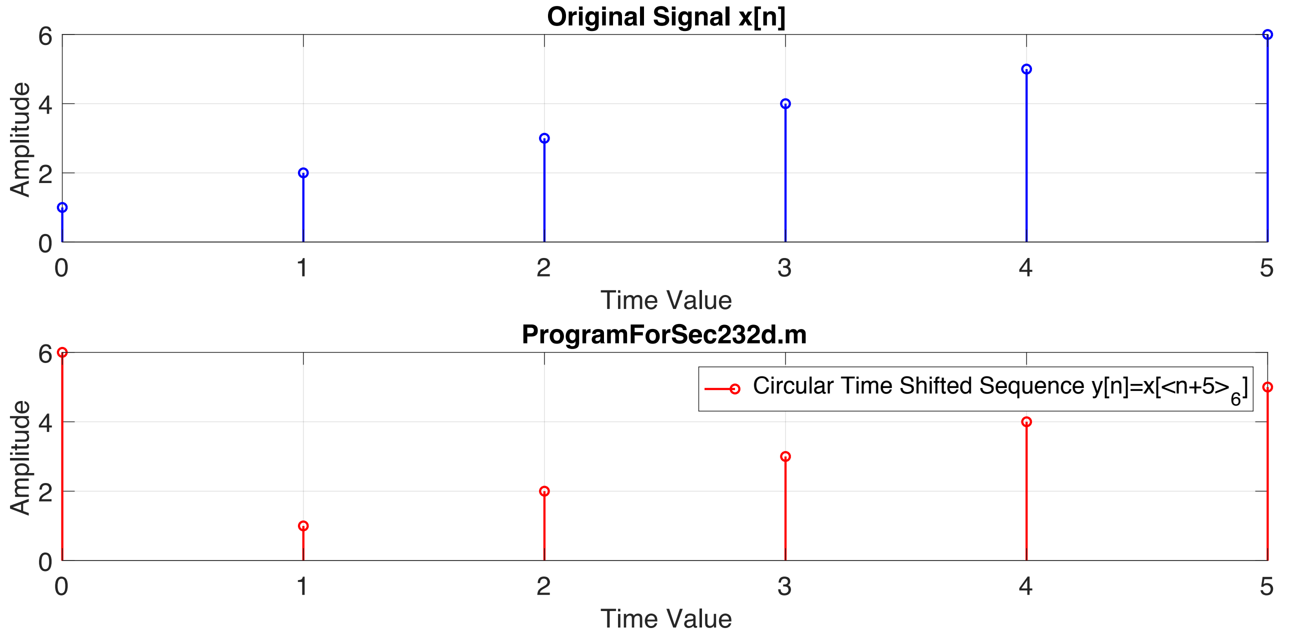


Operations on Finite-Length Sequences

Circular Shift of a Sequence

The sequence shifted to left by 5 sample or equivalently, shifted right by 1 samples because $x[\langle n+5\rangle_6] = x[\langle n-1\rangle_6]$

- Consider a sequence, shown below. Apply circular shift operation for $n_0 = 1, n_0 = 4, and\ n_0 = -5$
- $x[n] = \{1,2,3,4,5,6\} = \{x[0], x[1], x[2], x[3], x[4], x[5]\}$
- Solution:
- For $n_0 = -5$ (since $n_0 < 0$, we will use following formula:)
- $\boldsymbol{x_c[n] = x[\langle n - n_0\rangle_N]}$
- $\{y[0]\} = x[\langle 0+5\rangle_6] = x[\langle 5\rangle_6] = x[5] = 6$
- $\{y[1]\} = x[\langle 1+5\rangle_6] = x[\langle 6\rangle_6] = x[0] = 1$
- $\{y[2]\} = x[\langle 2+5\rangle_6] = x[\langle 7\rangle_6] = x[1] = 2$
- $\{y[3]\} = x[\langle 3+5\rangle_6] = x[\langle 8\rangle_6] = x[2] = 3$
- $\{y[4]\} = x[\langle 4+5\rangle_6] = x[\langle 9\rangle_6] = x[3] = 4$
- $\{y[5]\} = x[\langle 5+5\rangle_6] = x[\langle 10\rangle_6] = x[4] = 5$



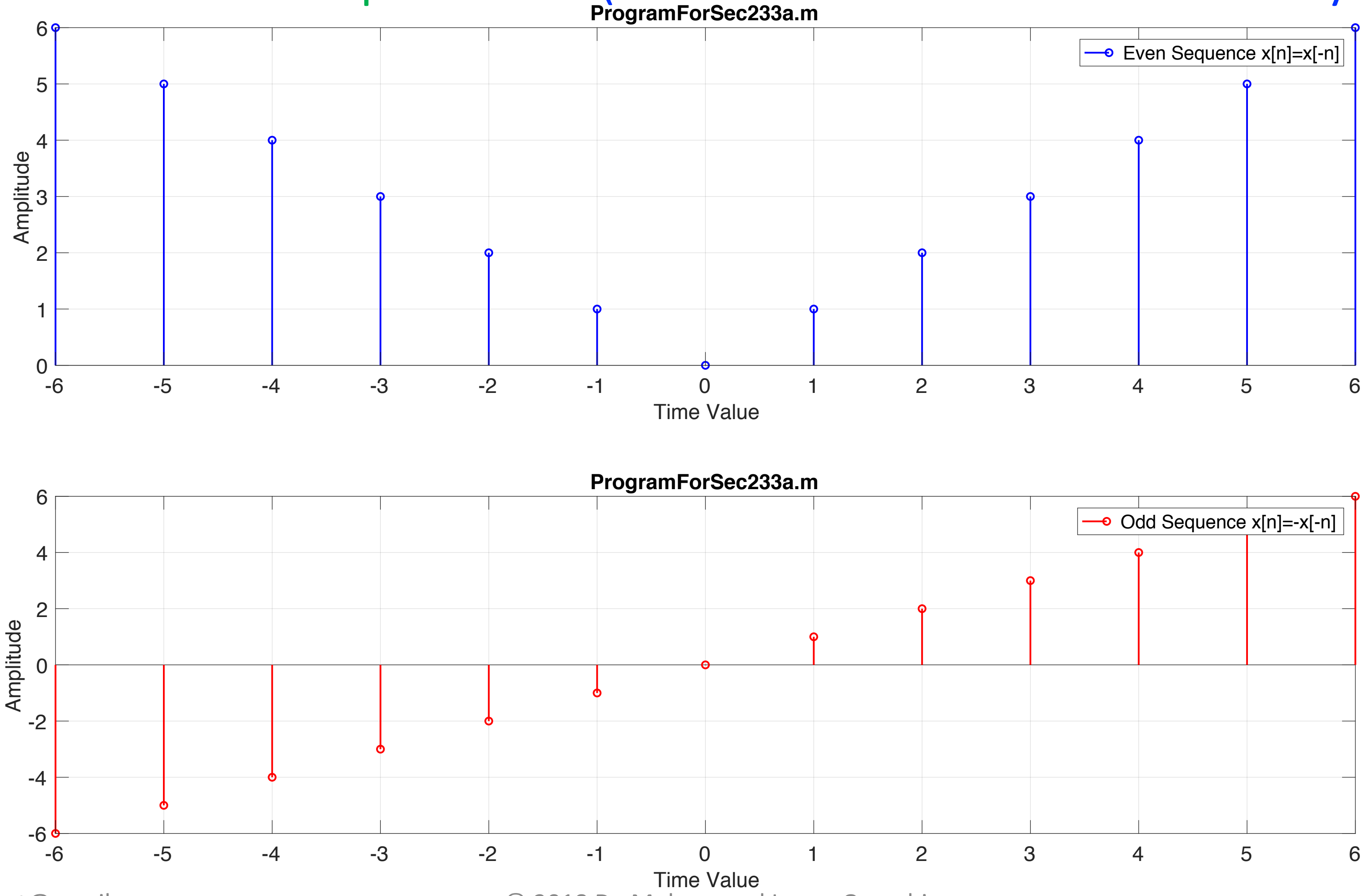
Operations on Finite-Length Sequences

Classification of Sequences (Classification Based on Symmetry)

- **Conjugate-Symmetric Sequence:**
  - A sequence $x[n]$ is called a conjugate-symmetric sequence if $x[n] = x^*[n]$.
  - A real conjugate-symmetric sequence is called an even sequence i.e. $x[n] = x[-n]$.
- **Conjugate-Antisymmetric Sequence:**
  - A sequence $x[n]$ is called a conjugate-antisymmetric sequence if $x[n] = -x^*[-n]$.
    - For a conjugate-antisymmetric sequence $x[n]$, the sample value at $n = 0$ must be purely imaginary.
  - A real conjugate-symmetric sequence is called an odd sequence $x[n] = -x[-n]$.
    - For an odd sequence $x[0] = 0$.

Operations on Finite-Length Sequences

Classification of Sequences (Classification Based on Symmetry)



Operations on Finite-Length Sequences

Classification of Sequences (Classification Based on Symmetry)

- Any complex sequence $x[n]$ can be expressed as a sum of its conjugate-symmetric part $x_{cs}[n]$, and its conjugate-antisymmetric part $x_{ca}[n]$:
- $x[n] = x_{cs}[n] + x_{ca}[n]$ (2.27)
- Where,
- $x_{cs}[n] = \frac{1}{2}\left[x[n] + x^*[-n]\right]$ (2.28a)
- $x_{ca}[n] = \frac{1}{2}\left[x[n] - x^*[-n]\right]$ (2.28b)
- A sequence can only be decomposed into conjugate-symmetric and conjugate antisymmetric sequences, if the parent sequence is of odd length (i.e. $-M \leq 0 \leq M$)

Operations on Finite-Length Sequences

Classification of Sequences (Classification Based on Symmetry)

- Any real sequence $x[n]$ can be expressed as a sum of its even part $x_{ev}[n]$, and its odd part $x_{od}[n]$:
- $x[n] = x_{ev}[n] + x_{od}[n]$ (2.29)
- Where,
- $x_{ev}[n] = \frac{1}{2}\left[x[n] + x[-n]\right]$ (2.30a)
- $x_{ca}[n] = \frac{1}{2}\left[x[n] - x[-n]\right]$ (2.30b)

Operations on Finite-Length Sequences

Classification of Sequences (Classification Based on Symmetry)

- Example 2.8: Decompose the following finite length sequence (of length 7) defined for $-3 \leq n \leq 3$ into conjugate symmetric and conjugate-antisymmetric parts
- $\{g[n]\} = \{0, 1 + 4j, -2 + j3, 4 - j2, -5 - j6, -j2, 3\}$
- Solution:
- Step 1: Perform the conjugate of above sequence
- $\{g^*[n]\} = \{0, 1 - 4j, -2 - j3, 4 + j2, -5 + j6, j2, 3\}$
- Step 2: Perform the time reversal of sequence obtained during step 1.
- $\{g^*[n]\} = \{3, j2, -5 + j6, 4 + j2, -2 - j3, 1 - 4j, 0\}$

Operations on Finite-Length Sequences

Classification of Sequences (Classification Based on Symmetry)

- Step 1: Perform the conjugate of above sequence
- $\{g^*[n]\} = \{0, 1 - 4j, -2 - j3, 4 + j2, -5 + j6, j2, 3\}$
$$\uparrow$$

- Step 2: Perform the time reversal of sequence obtained during step 1.
- $\{g^*[n]\} = \{3, j2, -5 + j6, 4 + j2, -2 - j3, 1 - 4j, 0\}$
$$\uparrow$$

- Step 3: the conjugate-symmetric sequence is obtained as $x_{cs}[n] = \frac{1}{2}[x[n] + x^*[-n]]$
- $\{g_{cs}[n]\} = \{1.5, 0.5 + j3, -3.5 + j4.5\ 4, -3.5 - j4.5, 0.5 - j3, 1.5\}$
$$\uparrow$$

Operations on Finite-Length Sequences

Classification of Sequences (Classification Based on Symmetry)

- Step 1: Perform the conjugate of above sequence
- $\{g^*[n]\} = \{0, 1 - 4j, -2 - j3, 4 + j2, -5 + j6, j2, 3\}$

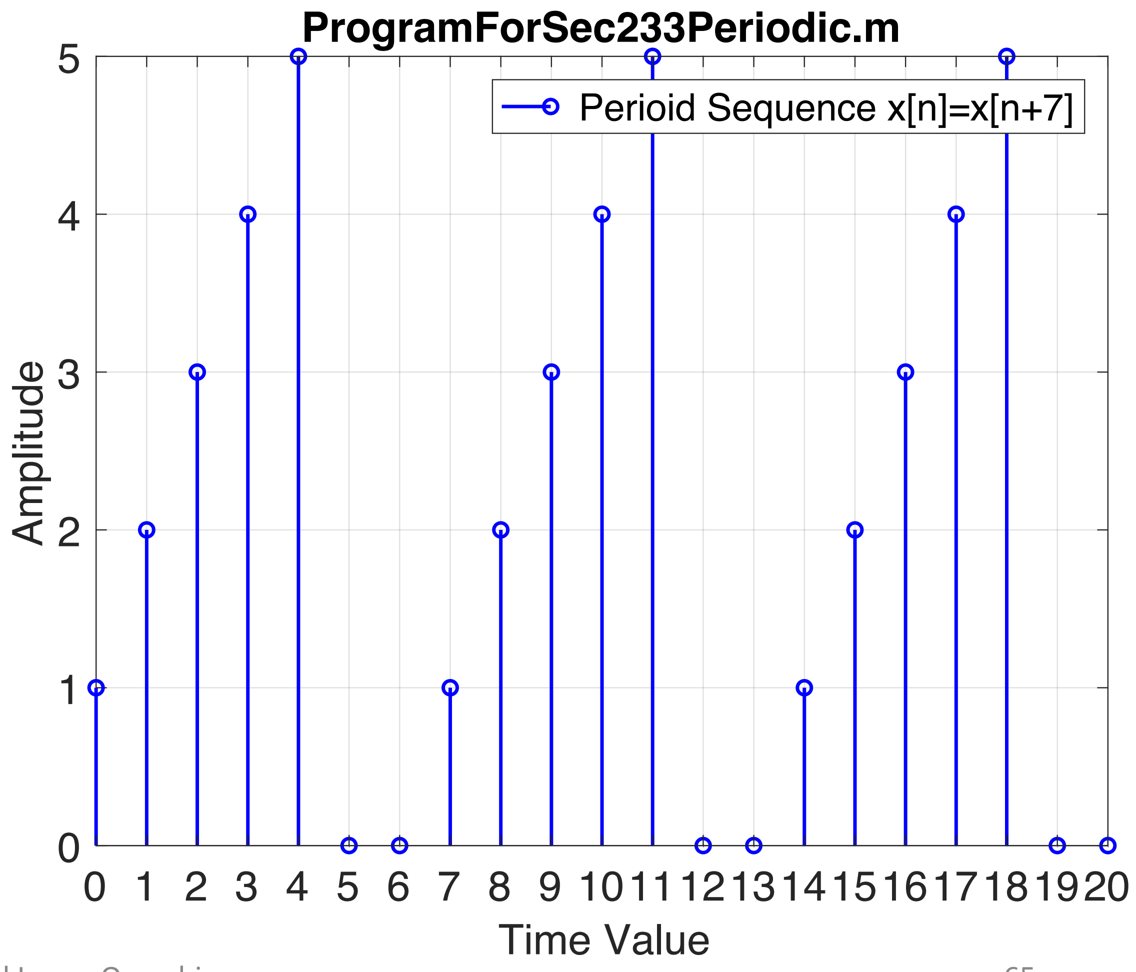
- Step 2: Perform the time reversal of sequence obtained during step 1.
- $\{g^*[n]\} = \{3, j2, -5 + j6, 4 + j2, -2 - j3, 1 - 4j, 0\}$

- Step 4: the conjugate-symmetric sequence is obtained as $x_{cs}[n] = \frac{1}{2}[x[n] - x^*[-n]]$
- $\{g_{ca}[n]\} = \{-1.5, 0.5 + j, 1.5 - j1.5, \ -j2, -1.5 - j1.5, 0.5 + j, 1.5\}$

Operations on Finite-Length Sequences

Classification of Sequences (Periodic and Aperiodic Signals)

- **Periodic Signals:**
  - A sequence $\tilde{x}[n]$ satisfying the following condition is called periodic sequence with period N.
  - $\tilde{x}[n] = \tilde{x}[n + kN] \qquad \forall n \qquad (2.31)$
  - Where N is a positive integer and k is any integer.
  - The fundamental period N_f of a periodic signal is the smallest value of N for which Eq. (2.31) holds.



Operations on Finite-Length Sequences

Classification of Sequences (Periodic and Aperiodic S

lcm and gcd command can be used in MATLAB for LCM and GCM computation, respectively.

- Aperiodic Signals:
  - A sequence is called aperiodic sequence if it is not a periodic sequence.
- The sum or product of two or more periodic sequence is also periodic sequence.
- If $\tilde{x}_a\,[n]$ is a periodic sequence with period N_a, and $\tilde{x}_b\,[n]$ is another periodic sequence with period N_b. Then the sequence $\tilde{y}[n] = \tilde{x}_a\,[n] + \tilde{x}_b\,[n]$ is also periodic with period N, where N can be calculated as follows:
- $N = LCM(N_a + N_b)$ (2.32)
- Similarly, another sequence $\tilde{y}[n] = \tilde{x}_a\,[n] \times \tilde{x}_b\,[n]$ can also be created with period N (which can also be computed using Eq. (2.32).

Few Basic Formulas For Geometric Series

- For a finite series, such as, $S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$
- The summation can be obtained using the following formula:
- $S_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$ (where r is the ratio of two consecutive terms)
- For an infinite series:
- $S_n = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$
- $S_n = \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$

Operations on Finite-Length Sequences

Classification of Sequences (Energy and Power Signals)

- Energy Signals:
  - The total energy of a sequence $x[n]$ is defined by
  - $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad (2.34)$
  - An infinite-length sequence with finite sample values may or may not have finite energy.
  - If the energy of a signal $\mathcal{E}_x$ is finite, then its average power $\mathcal{P}_x$ is zero.
  - A finite energy signal with zero average power is called an energy signal.
    - For instance, a finite length sequence is an energy signal because it has finite energy and zero average power.

Operations on Finite-Length Sequences

Classification of Sequences (Energy and Power Signals)

- Power Signals:
  - The average power of an aperiodic sequence $x[n]$ is defined by
  - $\mathcal{P}_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2 \qquad (2.37)$
  - The average power of a sequence can be related to its energy by defining its energy over a finite interval $-K \leq n \leq K$ as
  - $\mathcal{E}_{x,K} = \sum_{n=-K}^{K} |x[n]|^2 \qquad (2.38)$
  - Then,
  - $\mathcal{P}_x = \lim_{K\to\infty} \frac{1}{2K+1} \mathcal{E}_{x,K} \qquad (2.39)$
  - Eq. (2.36) suggest that if the energy of a signal is finite then its average power is zero.
  - The average power of a periodic sequence $\tilde{x}[n]$ with a period N is given by
  - $\mathcal{P}_x = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2 \qquad (2.40)$

Operations on Finite-Length Sequences

Classification of Sequences (Energy and Power Signals)

- Power Signals:
  - A infinite energy signal with finite average power is called a power signal.
  - For example, a periodic signal has finite average power but infinite energy.
- Example. 2.9: Find whether the given infinite sequence is an energy signal or power signal.
- $x_1[n] = \begin{cases} \frac{1}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$
- Solution:
- The energy of above signal is obtained as:
- $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{0} |x[n]|^2 + \sum_{n=1}^{\infty} |x[n]|^2 =$
- $\mathcal{E}_x = \sum_{n=-\infty}^{0} |0|^2 + \sum_{n=1}^{\infty} (\frac{1}{n})^2 = \sum_{n=1}^{\infty} (\frac{1}{n})^2 = \frac{\pi^2}{6}$

Since, the energy is finite, so average power is zero. Hence given signal is an energy signal.

Operations on Finite-Length Sequences
Classification of Sequences (Energy and Power Signals)

- Example. 2.9: Find whether the given infinite sequence is an energy signal or power signal.
- $x_2[n] = \begin{cases} \frac{1}{\sqrt{n}}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$
- Solution:
- The energy of above signal is obtained as:
- $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{0} |x[n]|^2 + \sum_{n=1}^{\infty} |x[n]|^2 =$
- $\mathcal{E}_x = \sum_{n=-\infty}^{0} |0|^2 + \sum_{n=1}^{\infty} (\frac{1}{\sqrt{n}})^2 = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$

This is a harmonic series

Operations on Finite-Length Sequences

Classification of Sequences (Energy and Power Signals)

This is neither an energy signal nor a power signal.

- Solution:
- The energy of above signal is obtained as:
- $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{0} |x[n]|^2 + \sum_{n=1}^{\infty} |x[n]|^2 =$
- $\mathcal{E}_x = \sum_{n=-\infty}^{0} |0|^2 + \sum_{n=1}^{\infty} (\frac{1}{\sqrt{n}})^2 = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \frac{1}{1-1} = \infty$
- Since, the energy is infinite, we will calculate its power
- $\mathcal{P}_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=1}^{K} |x[n]|^2 = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=1}^{K} \frac{1}{n} = \infty$

Operations on Finite-Length Sequences

Classification of Sequences (Energy and Power Signals)

- Example. 2.10: Find whether the given infinite sequence is an energy signal or power signal.

Since, the energy is infinite, so average power is not zero. Hence given signal is a power signal

- $x[n] = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$
- Solution:

This minus sign is removed due to even power (also there is modulus)

- The energy of above signal is obtained as:
- $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{-1} |x[n]|^2 + \sum_{n=0}^{\infty} |x[n]|^2 =$
- $\mathcal{E}_x = \sum_{n=-\infty}^{-1} |0|^2 + \sum_{n=0}^{\infty} |3(-1)^n|^2 = 9 \sum_{n=0}^{\infty} (1)^{2n} = \frac{1}{1-1} = \infty$

Operations on Finite-Length Sequences

Classification of Sequences (Energy and Power Signals)

- Example. 2.10: Find whether the given infinite sequence is an energy signal or power signal.

Since, the energy is infinite, so average power is not zero. Hence given signal is a power signal

- $x[n] = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$
- Solution:
- Since, the energy is infinite, we will calculate its power

Since, there are (K+1) ones, so there some will be K+1

- $\mathcal{P}_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=0}^{K} |x[n]|^2 = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=0}^{K} |3(-1)^n|^2 =$
- $= \lim_{K\to\infty} \frac{9}{2K+1} \sum_{n=0}^{K} (1)^{2n} = \lim_{K\to\infty} \frac{9}{2K+1} (K+1) = \lim_{K\to\infty} \frac{9(1+\frac{1}{K})}{2\left(1+\frac{1}{K}\right)} = 4.5$

Operations on Finite-Length Sequences

Classification of Sequences (Other types of Classifications)

- **Bounded Sequence:**
  - A sequence $x[n]$ is called bounded if each of its sample is of magnitude less than or equal to a finite positive number B_x: that is,
  - $|x[n]| \leq B_x \leq \infty \qquad (2.41)$
- **Absolutely Summable:**
  - A sequence $x[n]$ is said to be absolutely summable if
  - $\sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad (2.42)$
- **Square-Summable Sequence:**
  - A sequence $x[n]$ is said to be square-summable if
  - $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$
  - A square-summable sequence has finite energy, and it would be called energy signal if its average power is zero.

Operations on Finite-Length Sequences

Classification of Sequences (Other types of Classifications)

- Following sequence is square-summable but not absolutely summable
- $x_a[n] = \begin{cases} \dfrac{sin w_c n}{\pi n} & n < 0, n > 0 \\ \dfrac{w_c n}{\pi} & n = 0 \end{cases}$
- Following sequences are neither absolutely summable nor square-summable
- $x_b[n] = sin w_c n, \quad -\infty < n < \infty$ and
- $x_c[n] = K, \quad -\infty < n < \infty$ (where, K is a constant)

Typical Sequences and Sequences Representation

Some Basic Sequences (Unit Sample (Impulse) Sequence)

- Unit Sample (Impulse) Sequence:
  - Also called the discrete time impulse or unit impulse.
  - It is defined by
  - $\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \qquad (2.45)$
  - The unit sample sequence shifted by k samples is thus given by
  - $\delta[n-k] = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases} \qquad (2.45)$
- Uses:
  - Any arbitrary sequence can be represented as a sum of weighted time-shifted unit sample sequence.
  - A certain class of discrete-time systems is completely characterized in time domain by its output to a unit impulse input, called impulse response.
  - Knowing impulse response of the system, we can compute its response to any arbitrary input sequence.

Typical Sequences and Sequences Representation

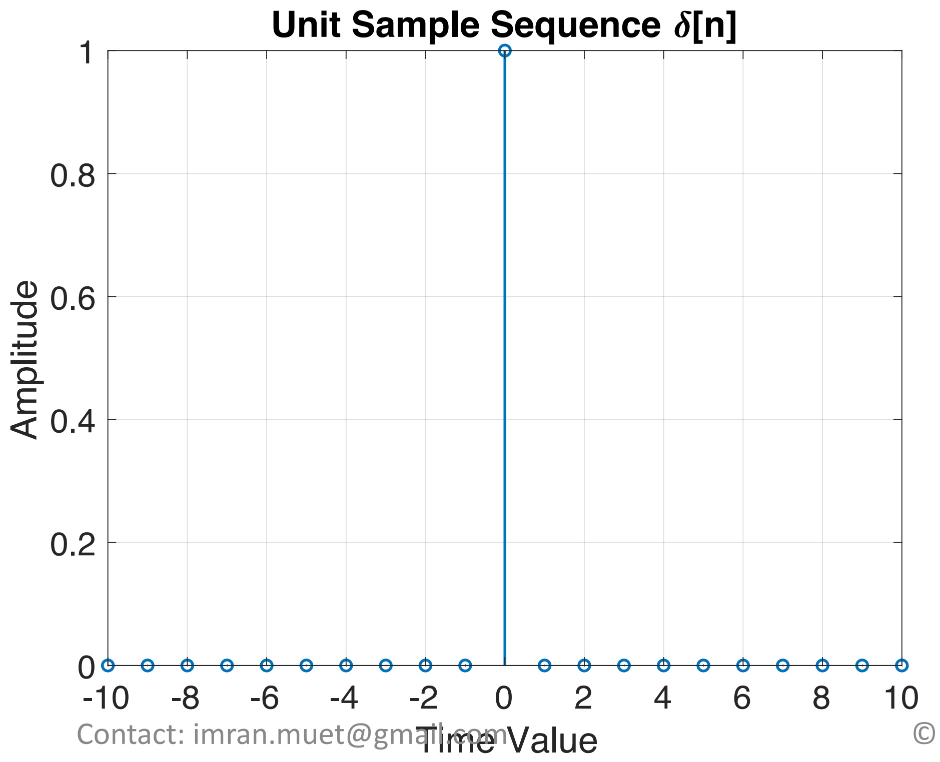
Some Basic Sequences (Unit Sample (Impulse) Sequence)

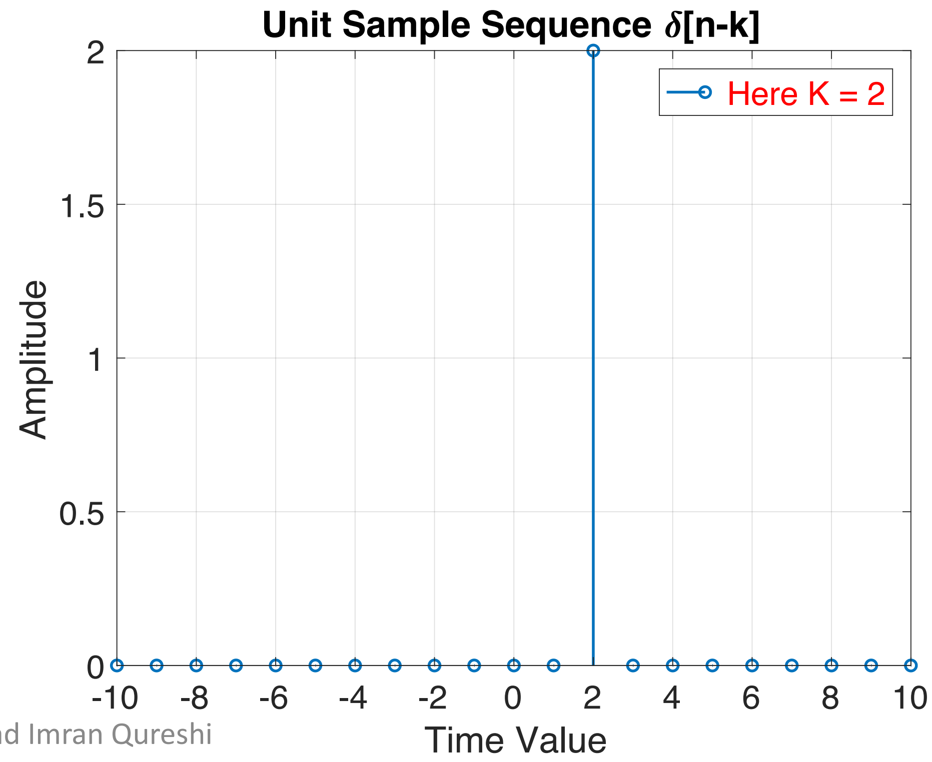
Unit Sample (Impulse) Sequence:

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \qquad (2.45)$$

The unit sample sequence shifted by k samples is thus given by

$$\delta[n-k] = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases} \qquad (2.45)$$





Typical Sequences and Sequences Representation

Some Basic Sequences (Unit Step Sequence)

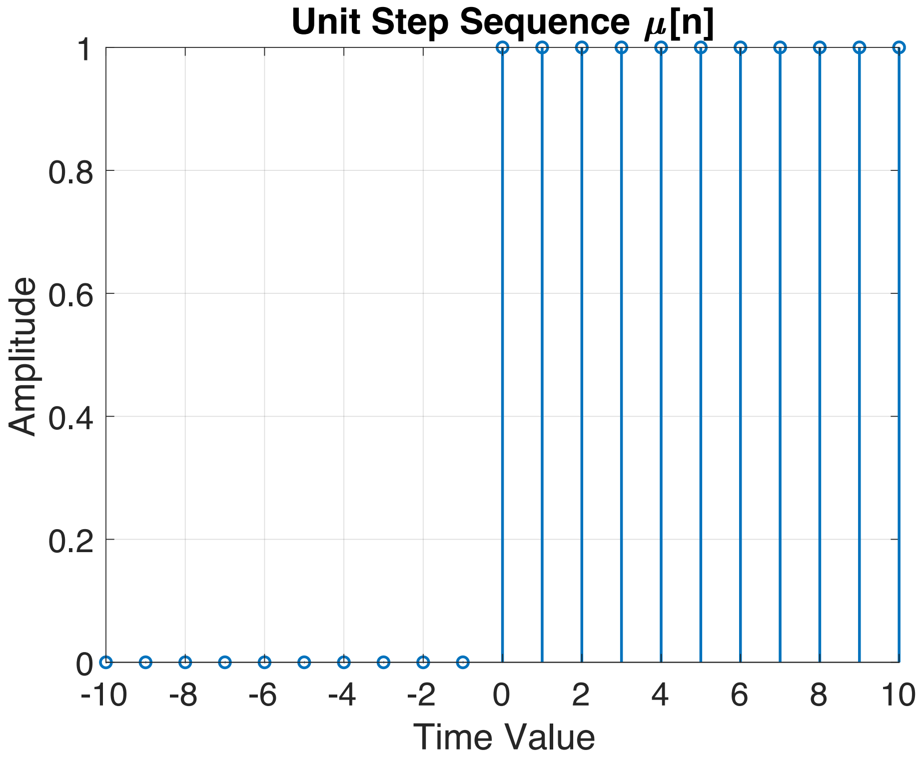
- Unit Step Sequence:
  - It is denoted by $\mu[n]$ and is defined by
  - $\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \qquad (2.46)$
  - The unit step sequence shifted by k samples is thus given by
  - $\mu[n-k] = \begin{cases} 1, & n \geq k, \\ 0, & n < k. \end{cases}$
  - The unit sample and the unit step sequences are related as follows:
    - $\mu[n] = \sum_{m=-\infty}^{\infty} \delta[n-m]$
    - $\delta[n] = \mu[n] - \mu[n-1]$

Typical Sequences and Sequences Representation

Some Basic Sequences (Unit Step Sequence)

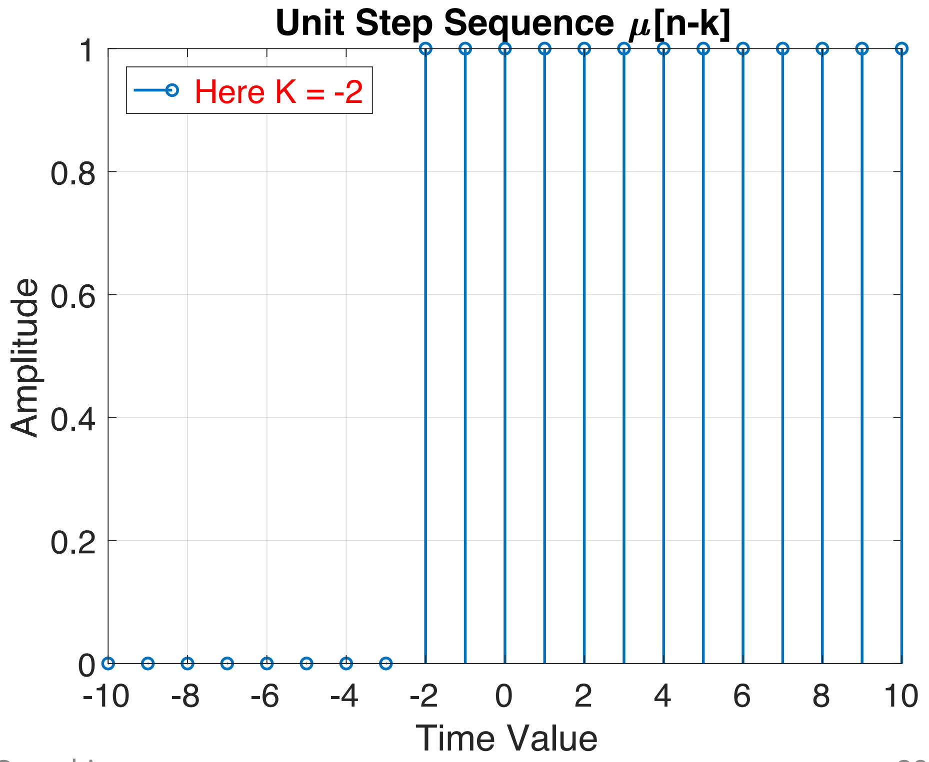
Unit Step Sequence is defined by

$$\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$



The unit step sequence shifted by k samples is thus given by

$$\mu[n-k] = \begin{cases} 1, & n \geq k, \\ 0, & n < k. \end{cases}$$



Typical Sequences and Sequences Representation

Some Basic Sequences (Sinusoidal and Exponential Sequences)

- Sinusoidal Sequence:
- A real sinusoidal sequence with constant amplitude (A) has the form
- $x[n] = A\cos(w_o n + \phi) \quad -\infty \leq n \leq \infty \qquad (2.48)$
- Where and are real numbers.
- The parameter A is called the amplitude.
- The parameter w_o is called the normalized angular frequency.
- The parameter ϕ is called the phase.
- The real sinusoidal sequence can be written as

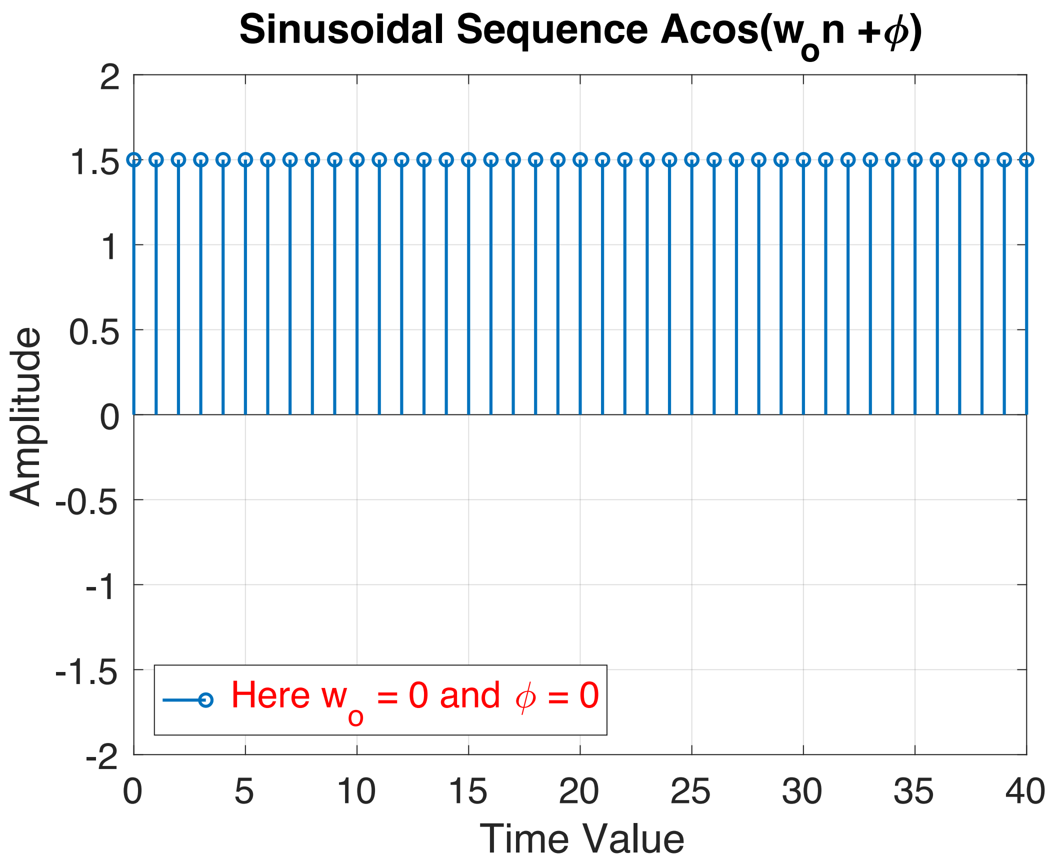
Typical Sequences and Sequences Representation

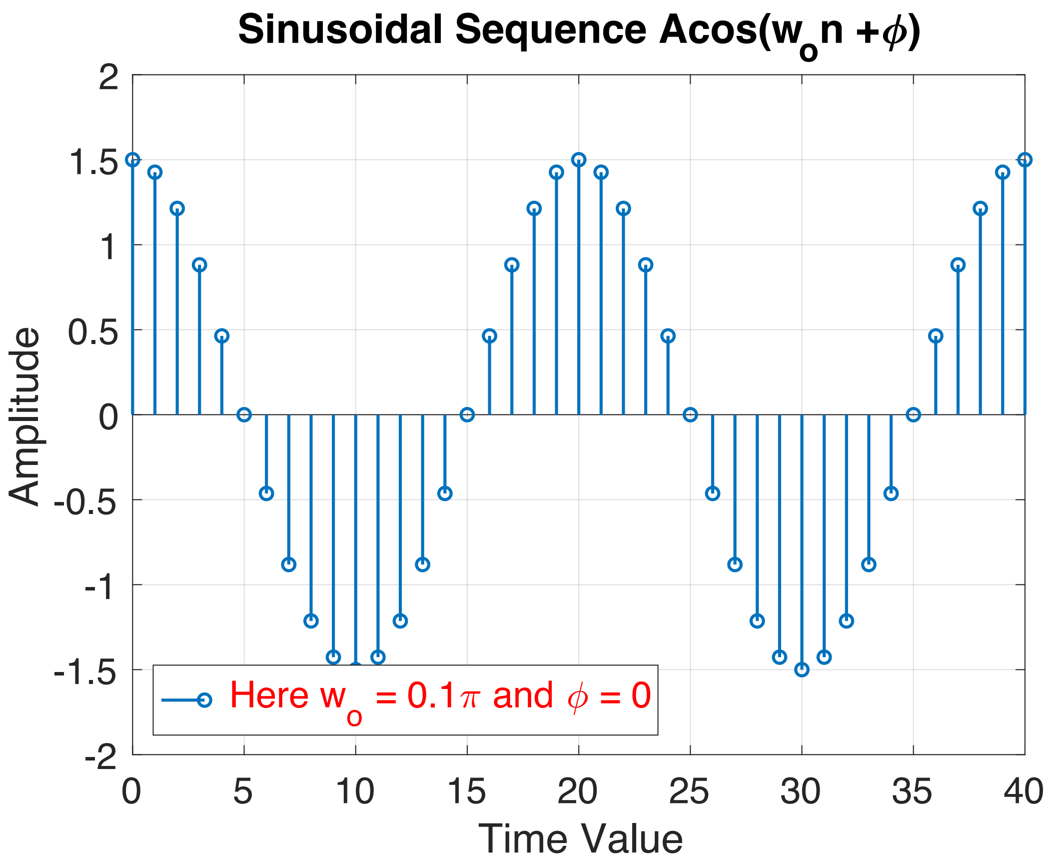
Some Basic Sequences (Sinusoidal and Exponential Sequences)

- **Sinusoidal Sequence:**
- The real sinusoidal sequence can be written as
- $x[n] = x_i[n] + x_q[n]$ (2.49)
- Where, $x_i[n] = Acos(\phi)\cos(w_o n)$ is called, the in-phase component with amplitude $Acos(\phi)$, and $x_q[n] = -Asin(\phi)\sin(w_o n)$ is quadrature component with amplitude $Asin(\phi)$.

Typical Sequences and Sequences Representation

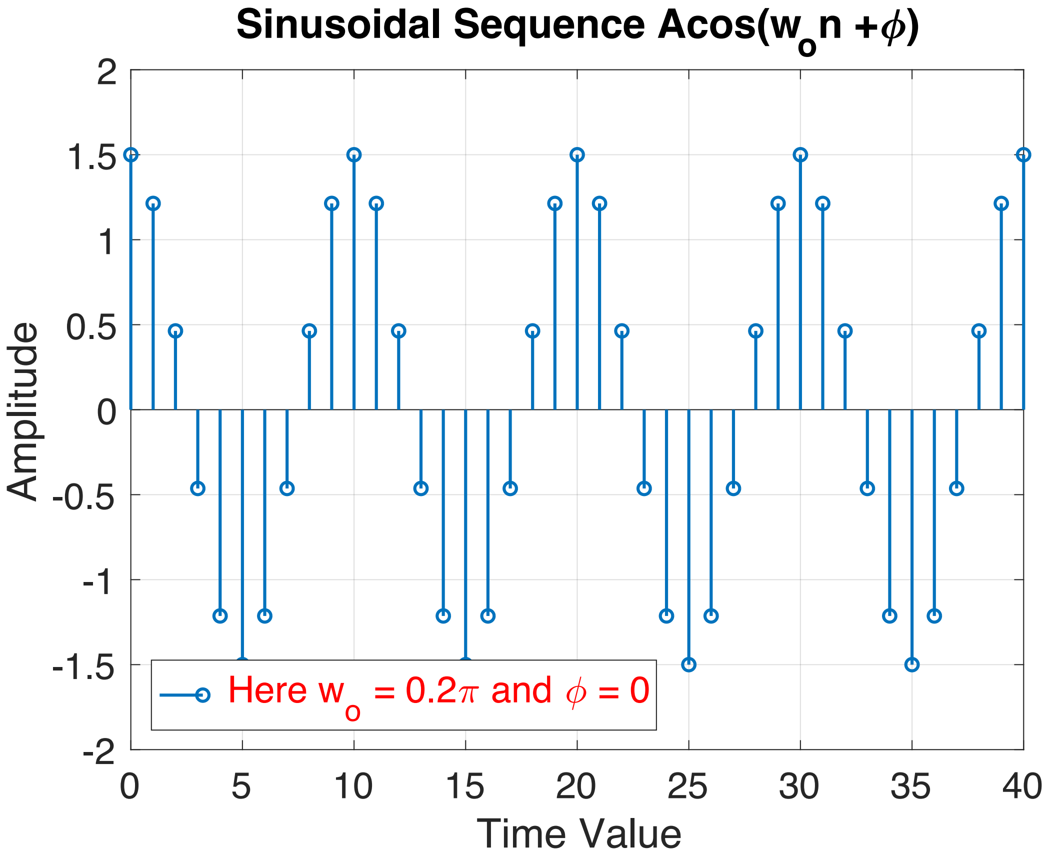
Some Basic Sequences (Sinusoidal Sequences)

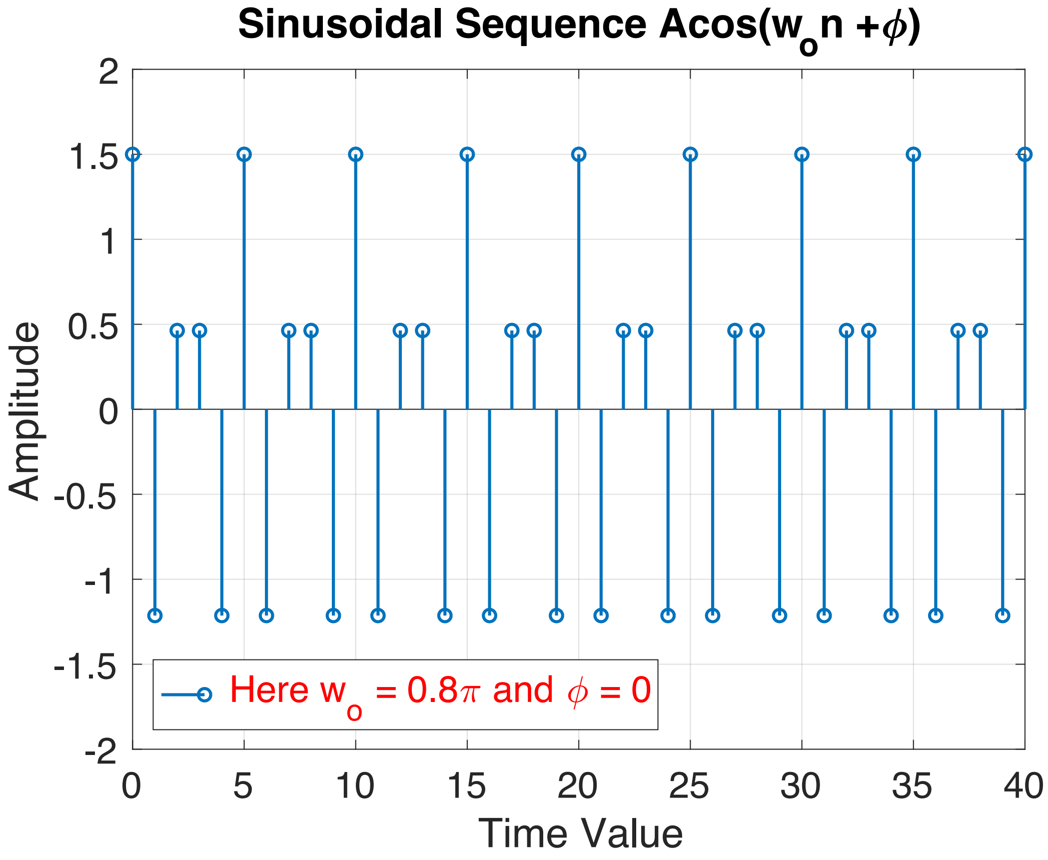




Typical Sequences and Sequences Representation

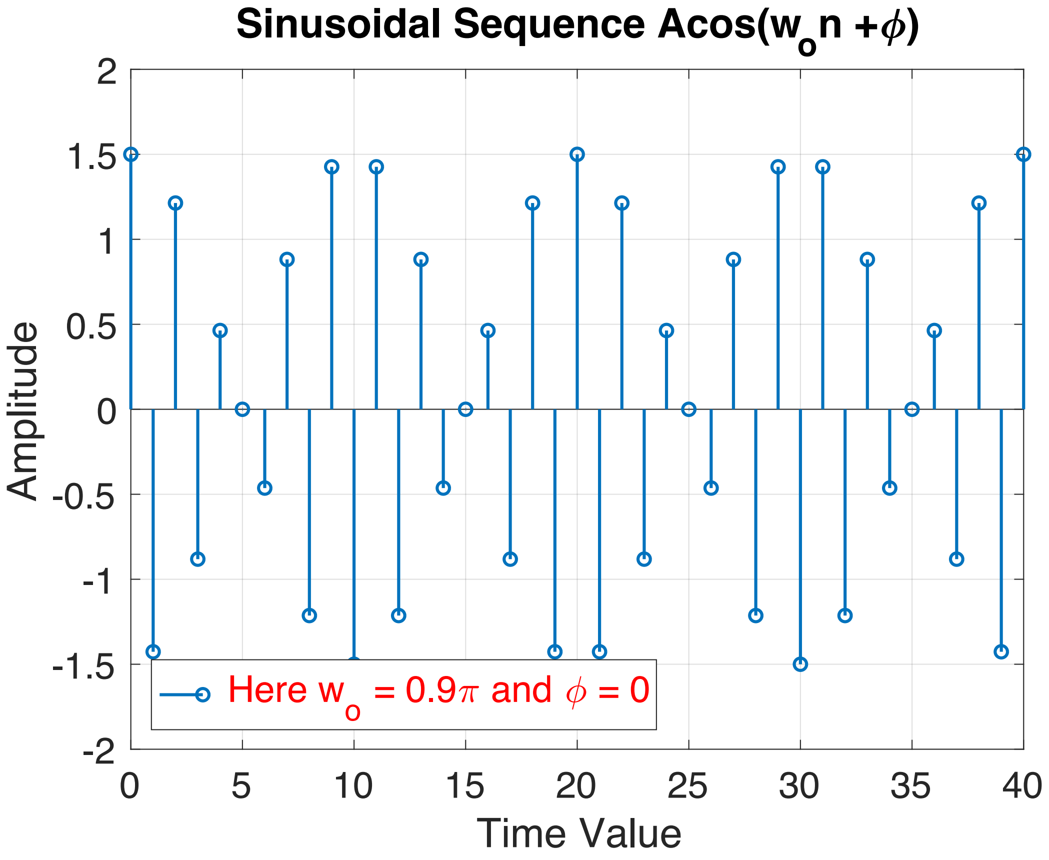
Some Basic Sequences (Sinusoidal Sequences)

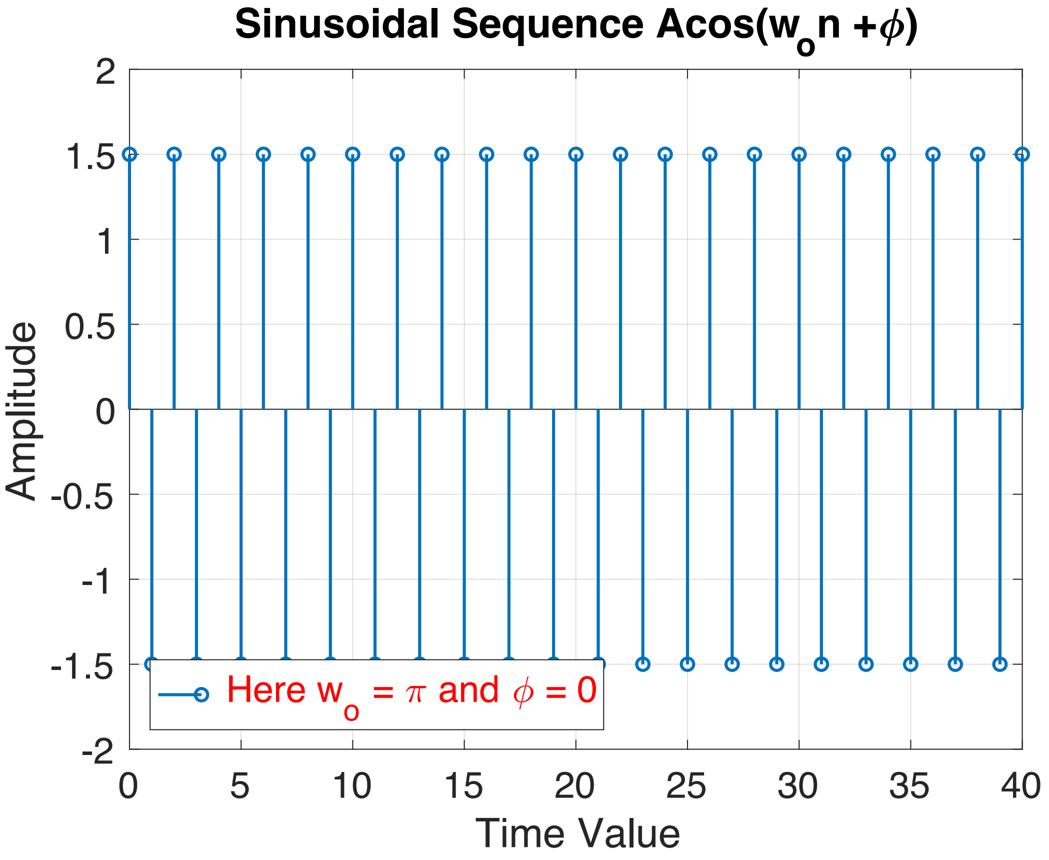




Typical Sequences and Sequences Representation

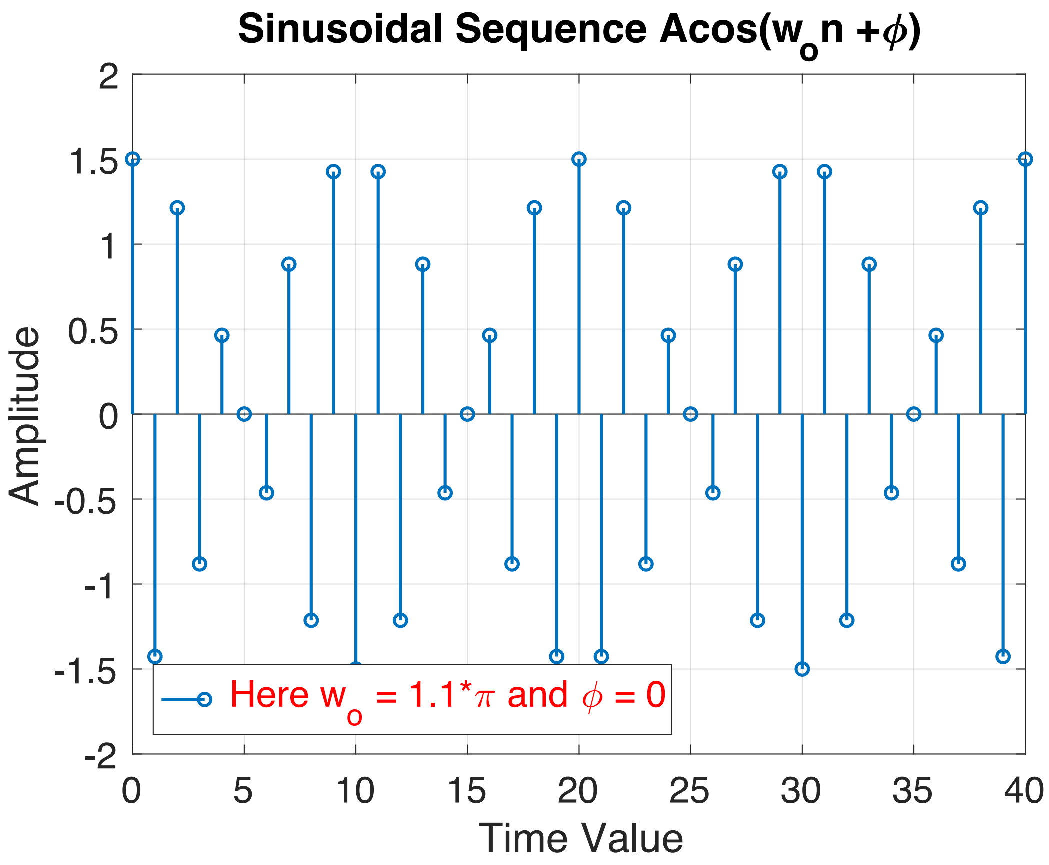
Some Basic Sequences (Sinusoidal Sequences)

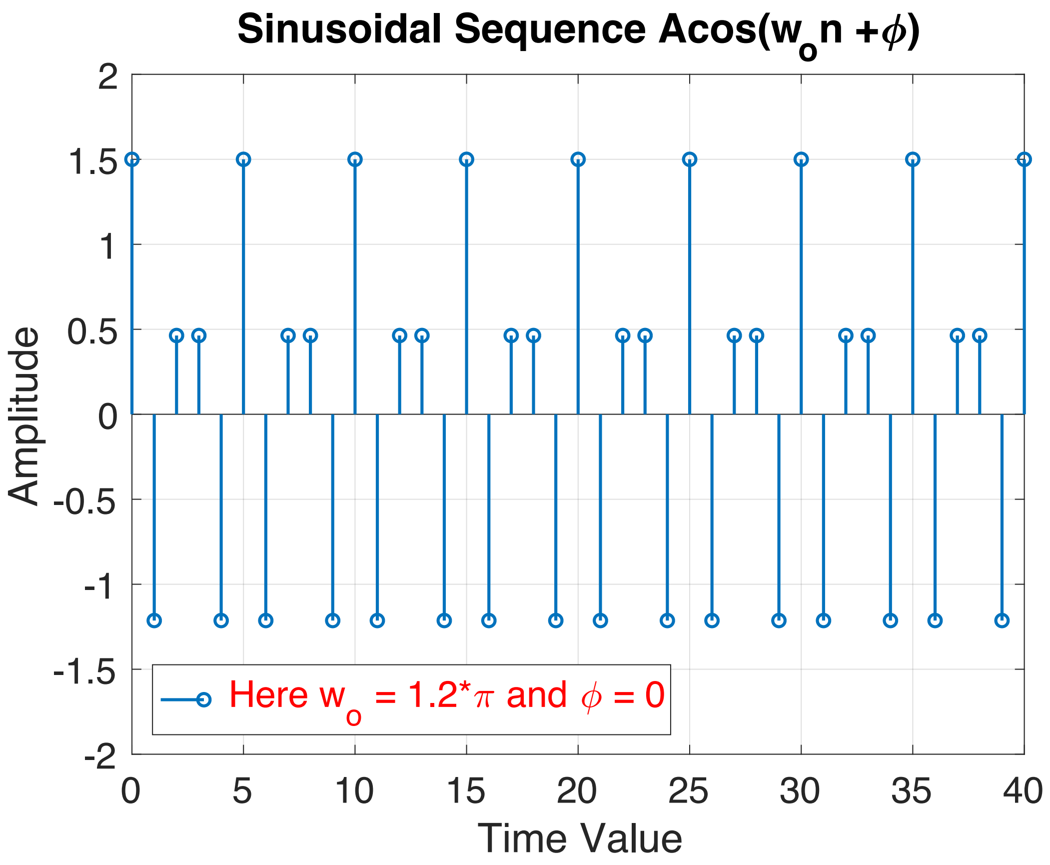




Typical Sequences and Sequences Representation

Some Basic Sequences (Sinusoidal Sequences)





Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

$Euler's\ Formula:$
$e^{j\theta} = cos\theta + jsin\theta$
$e^{-j\theta} = cos\theta - jsin\theta$

- An exponential sequence is defined by
- $x[n] = A\alpha^n \quad -\infty \leq n \leq \infty \quad (2.51)$
- Where A and α can be real or complex numbers.
- Let $\alpha = e^{(\sigma_o + jw_o)}$, and $A = |A|e^{j\phi}$, putting these values in Eq. (2.51) will result in:
- $x[n] = A\alpha^n = Ae^{(\sigma_o + jw_o)} = |A|e^{j\phi}e^{(\sigma_o + jw_o)} = |A|e^{\sigma_o n}e^{j(w_o n + \phi)} =$
- $= |A|e^{\sigma_o n}e^{j(w_o n + \phi)} = |A|e^{\sigma_o n}\{\cos(w_o n + \phi) + jsin(w_o n + \phi)\} =$
- $x[n] = |A|e^{\sigma_o n}\cos(\mathrm{w_o n} + \phi) + j|A|e^{\sigma_o n}\sin(w_o n + \phi)$,

Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

$Euler's\ Formula:$
$e^{j\theta} = cos\theta + jsin\theta$
$e^{-j\theta} = cos\theta - jsin\theta$

- Different forms of exponential sequence:
- $x[n] = A\alpha^n \quad -\infty \le n \le \infty$ (2.51)
- $x[n] = |A|e^{\sigma_o n}e^{j(w_o n+\phi)}$ (2.52a)
- $x[n] = |A|e^{\sigma_o n}\cos(\mathrm{w_o n} + \phi) + j|A|e^{\sigma_o n}\sin(w_o n + \phi)$, (2.52b)
- Now $\sigma_o, \phi, and\ w_o$ are real numbers.
- $x[n] = x_{im}[n] + x_{re}[n]$
- Where
- $x_{im}[n] = |A|e^{\sigma_o n}\cos(\mathrm{w_o n} + \phi)$ and $x_{re}[n] = |A|e^{\sigma_o n}\sin(\mathrm{w_o n} + \phi)$

Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

$Euler's\ Formula:$
$e^{j\theta} = cos\theta + jsin\theta$
$e^{-j\theta} = cos\theta - jsin\theta$

- $x[n] = x_{im}[n] + x_{re}[n]$
- Where
- $x_{im}[n] = |A|e^{\sigma_o n}\cos(\mathrm{w_o n} + \phi)$ and
- $x_{re}[n] = |A|e^{\sigma_o n}\sin(\mathrm{w_o n} + \phi)$
- Thus, the real and imaginary parts of a complex exponential sequence are real sinusoidal sequences with constant (σ_o)
  - For $\sigma_o > 0$, the exponential sequence is growing, provided $n > 0$.
  - For $\sigma_o < 0$, the exponential sequence is decaying, provided $n > 0$.

Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

$$\textit{Euler's Formula:}\quad e^{j\theta} = cos\theta + jsin\theta \qquad e^{-j\theta} = cos\theta - jsin\theta$$

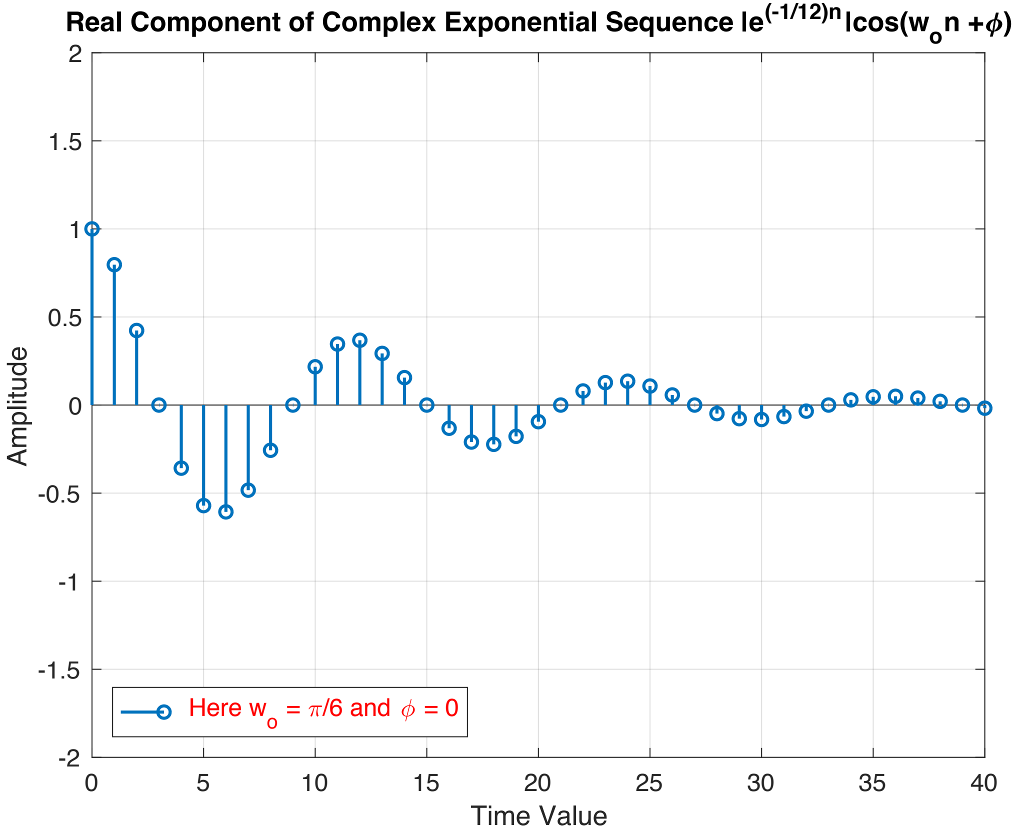
- Plot the following complex exponential sequence
- $x[n] = e^{\left(-\frac{1}{12}+\frac{j\pi}{6}\right)n} \quad 0 \le n \le 40$
- Solution:
- Step 1: The given sequence will be decomposed into real and imaginary parts
- $x[n] = e^{\left(-\frac{1}{12}+\frac{j\pi}{6}\right)n} = e^{-\frac{1}{12}n} e^{\frac{j\pi}{6}n} = e^{-\frac{1}{12}n}\left(\cos\left(\frac{\pi}{6}n\right) + jsin\left(\frac{\pi}{6}n\right)\right)$
- The real and imaginary parts are as follows:
- $x_{re}[n] = \left|e^{-\frac{1}{12}n}\right| \cos\left(\frac{\pi}{6}n\right)$ and
- $x_{im}[n] = \left|e^{-\frac{1}{12}n}\right| sin\left(\frac{\pi}{6}n\right)$

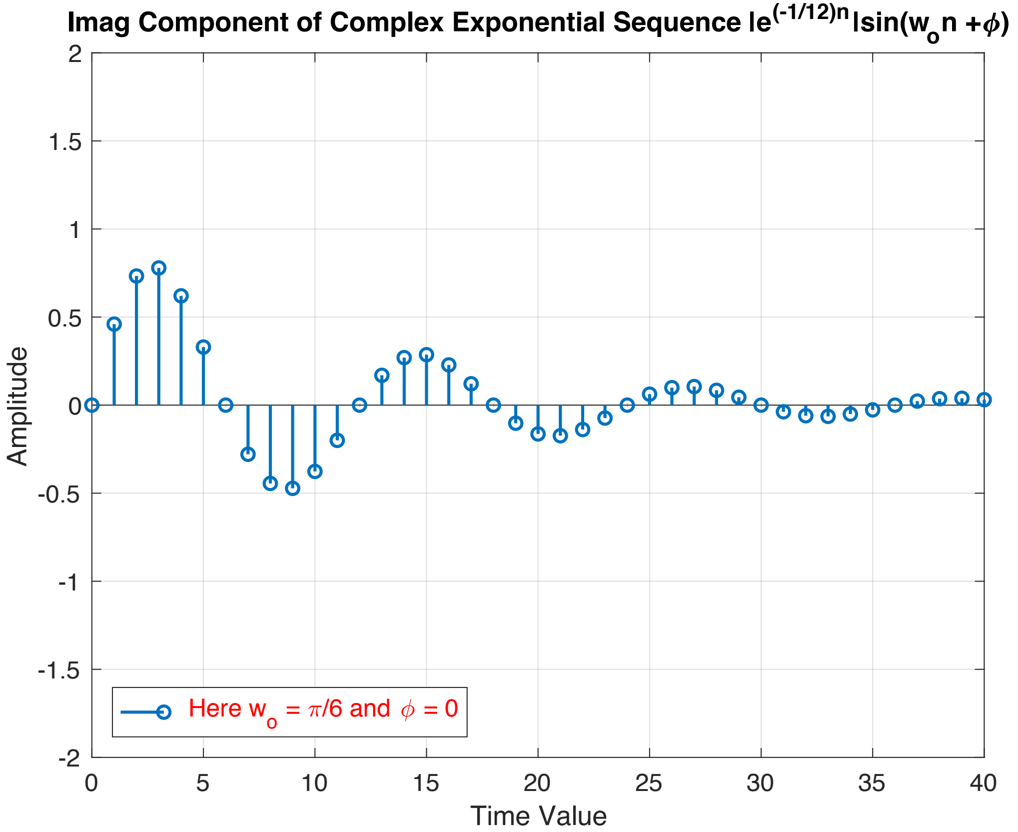
Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

$$x_{re}[n] = \left|e^{-\frac{1}{12}n}\right| \cos\left(\frac{\pi}{6}n\right)$$

$$x_{im}[n] = \left|e^{-\frac{1}{12}n}\right| sin\left(\frac{\pi}{6}n\right)$$





Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

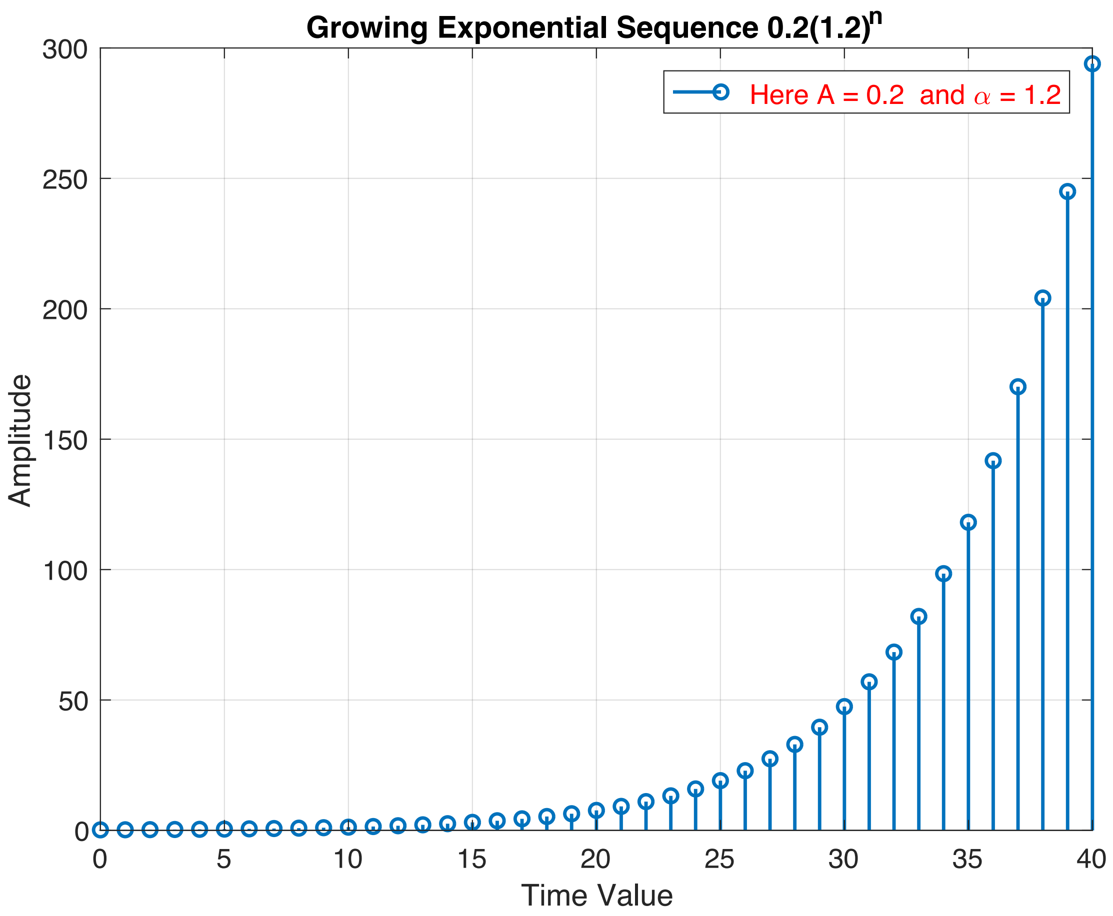
- If both $A\ and\ \alpha$ are real number then the sequence
- $x[n] = A\alpha^n \quad -\infty \leq n \leq \infty \quad (2.51)$
- Becomes a real exponental sequence
  - For $\alpha > 0$, the exponential sequence is growing, provided $n > 0$.
  - For $\alpha < 0$, the exponential sequence is decaying, provided $n > 0$.
- Plot following sequences
  1. $x[n] = 0.2(1.2)^n$ Here $\alpha = 1.2 > 0\ and\ A = 0.2$
  2. $x[n] = 20(0.9)^n$ Here $\alpha = 0.9 < 0\ and\ A = 20$

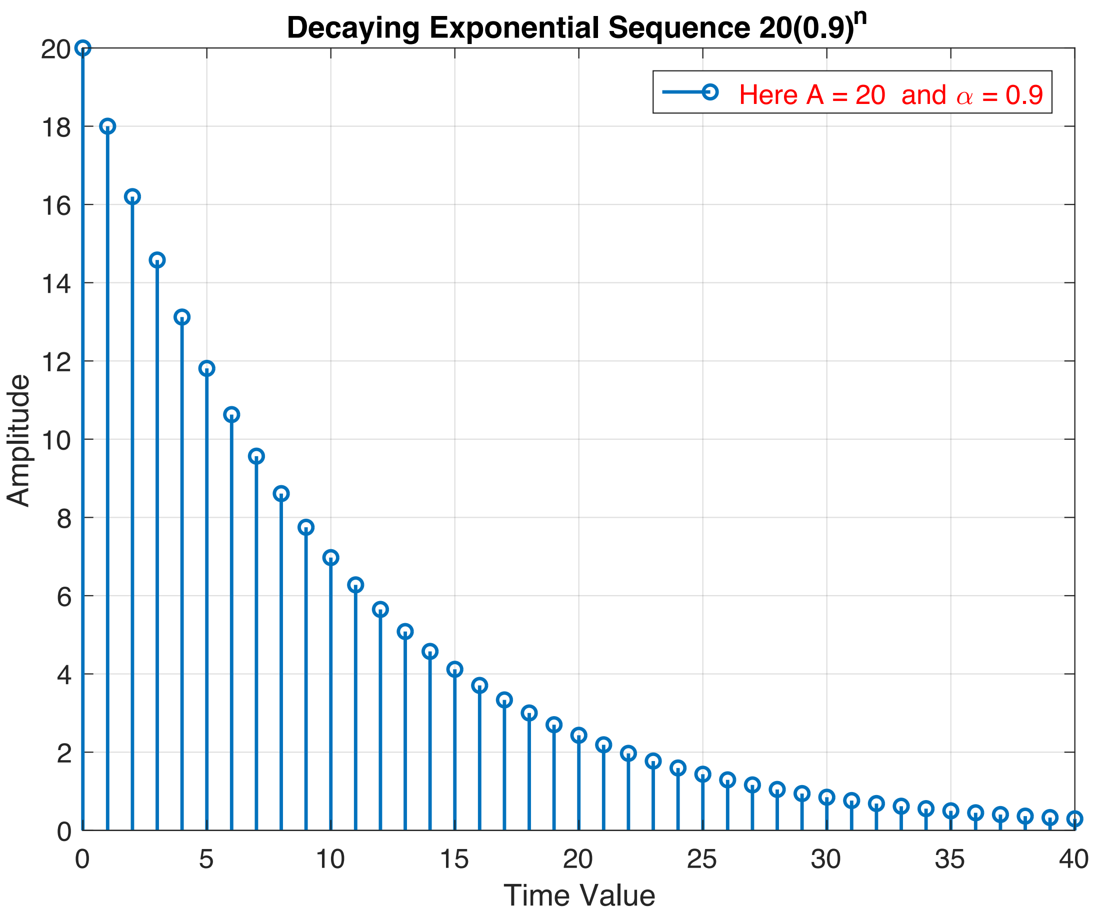
Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

$x[n] = 0.2(1.2)^n$

$x[n] = 20(0.9)^n$





Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

- The real sinusoidal sequence '$x[n] = A\cos(w_o n + \phi)$' is a periodic sequence with period N as long as $w_o N$ is an integer multiple of 2π (i.e. $w_o N = 2\pi k$, where N and k are positive integers).
- The smallest possible value of N is called fundamental period of the sequence.
- Similarly, the complex exponential (provided $\sigma_o = 0$) sequence '$x[n] = |A|e^{\sigma_o n}e^{j(w_o n+\phi)}$' is a periodic sequence with period N as long as $w_o N$ is an integer multiple of 2π (i.e. $w_o N = 2\pi k$, where N and k are positive integers).
- $\frac{2\pi}{w_o} = \frac{N}{k} \qquad (2.53b)$
- If $2\pi/w_o$ is a non-integer number, then the period will be a multiple of $2\pi/w_o$.
- If $2\pi/w_o$ is not a rational number, then the sequence is aperiodic.
  - For example, $x[n] = \cos\left(\sqrt{3n} + \phi\right)$ is an aperiodic sequence.

Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

- Final the fundamental time period (N) for following real sinusoidal sequences:
  1. $x[n] = \text{Acos}(0)$
  2. $x[n] = \text{Acos}(0.1\pi n)$
  3. $x[n] = \text{Acos}(0.2\pi n)$
  4. $x[n] = \text{Acos}(0.8\pi n)$
  5. $x[n] = \text{Acos}(0.9\pi n)$
  6. $x[n] = \text{Acos}(1.1\pi n)$
  7. $x[n] = \text{Acos}(1.2\pi n)$

$x[n] = \text{Acos}(\pi n)$

Sol:

$$\frac{2\pi}{w_o} = \frac{N}{k} \Longrightarrow \frac{2\pi}{\pi} = N \Longrightarrow \frac{\text{N}}{\text{k}} = \frac{2}{1}$$

$$\frac{\text{N}}{\text{k}} = \frac{5}{3}$$

Here $N = 2$ *and* $k = 1$

we will use the expression

$$\frac{2\pi}{w_o} = \frac{N}{k}$$

For fundamental time period, k =1

$x[n] = \text{Acos}(1.2\pi n)$

Sol:

$$\frac{2\pi}{w_o} = \frac{N}{k} \Longrightarrow \frac{2\pi}{1.2\pi} = N \Longrightarrow \frac{\text{N}}{\text{k}} = \frac{20}{12}$$

$$\frac{\text{N}}{\text{k}} = \frac{5}{3}$$

Here $N = 5$ *and* $k = 3$

Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

- $x[n] = \text{Acos}(0)$

Solution:

N can be any number.

- $x[n] = \text{Acos}(0.1\pi n)$

Solution:

$$\frac{2\pi}{w_o} = \frac{N}{k} \Longrightarrow \frac{2\pi}{0.1\pi} = \frac{N}{k} \Longrightarrow \frac{N}{\text{k}} = \frac{20}{10} = \frac{2}{1}$$

Here $N = 2\ and\ k = 1$

- $x[n] = \text{Acos}(0.2\pi n)$

Solution:

$$\frac{2\pi}{w_o} = \frac{N}{k} \Longrightarrow \frac{2\pi}{0.2\pi} = \frac{N}{k} \Longrightarrow \frac{N}{\text{k}} = \frac{20}{20} = \frac{1}{1}$$

Here $N = 1\ and\ k = 1$

- $x[n] = \text{Acos}(0.8\pi n)$

Solution:

$$\frac{2\pi}{w_o} = \frac{N}{k} \Longrightarrow \frac{2\pi}{0.8\pi} = \frac{N}{k} \Longrightarrow \frac{N}{\text{k}} = \frac{20}{8} = \frac{5}{4}$$

Here $N = 5\ and\ k = 4$

- $x[n] = \text{Acos}(0.9\pi n)$

Solution:

$$\frac{2\pi}{w_o} = \frac{N}{k} \Longrightarrow \frac{2\pi}{0.9\pi} = \frac{N}{k} \Longrightarrow \frac{N}{\text{k}} = \frac{20}{9}$$

Here $N = 20\ and\ k = 9$

- $x[n] = \text{Acos}(1.1\pi n)$

Solution:

$$\frac{2\pi}{w_o} = \frac{N}{k} \Longrightarrow \frac{2\pi}{1.1\pi} = \frac{N}{k} \Longrightarrow \frac{N}{\text{k}} = \frac{20}{11}$$

Here $N = 20\ and\ k = 11$

Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

- Final the fundamental time period (N) for following composite real sinusoidal sequences:
  - $\tilde{x}[n] = 3\cos(1.3\pi n) - 4\sin(0.5\pi n + 0.5\pi)$

  Solution:

  We will first find the fundamental period for each par

  - $\tilde{x}_a[n] = 2\cos(1.3\pi n)$

  $$\frac{2\pi}{w_o} = \frac{N_a}{k} \Longrightarrow \frac{2\pi}{1.3\pi} = \frac{N_a}{k} \Longrightarrow \frac{\mathrm{N_a}}{\mathrm{k}} = \frac{20}{13}$$

  Here $N = 20 \; and \; k = 1$

  - $\tilde{x}_b[n] = 4\cos(0.5\pi n + 0.5\pi)$

  Solution:

  $$\frac{2\pi}{w_o} = \frac{N_b}{k} \Longrightarrow \frac{2\pi}{0.5\pi} = \frac{N_b}{k} \Longrightarrow \frac{\mathrm{N_b}}{\mathrm{k}} = \frac{20}{5}$$

  Here $N = 4 \; and \; k = 1$

Now we will take LCM of both the time periods to calculate the time period of composite signal:

$$N = LCM(N_a, N_b)$$
$$N = LCM(20,4)$$
$$N = 20$$

Typical Sequences and Sequences Representation

Some Basic Sequences (Exponential Sequences)

- The parameter w_o is called normalized angular frequency.
  - When the time instant is dimensionless, the unit of normalized angular frequency w_o is radians (also the unit of phase ϕ is also radians).
  - When the unit of time instant is designed as samples , the unit of normalized angular frequency w_o is radians per sample (also the unit of phase ϕ is also radians per sample).
  - When the unit of time instant is designed as seconds , the unit of normalized angular frequency w_o is radians per seconds (also the unit of phase ϕ is also radians per seconds).
- The normalized angular frequency w_o and normalized linear frequency f are related as:
  - $w = 2\pi f \qquad (2.54)$
  - Where, f is measured in cycles per samples.

Typical Sequences and Sequences Representation

Some Basic Sequences (Two Properties of Sinusoids and Exponential)

- Consider two of the following sequences:
- $x_1[n] = e^{jw_1 n} \quad 0 \le w_1 \le 2\pi$
- $x_2[n] = e^{jw_2 n} \quad 2\pi k \le w_2 \le 2\pi(k+1)$ $(where\ k\ is\ any\ positive\ integer)$
- If
- $w_2 = w_1 + 2\pi k. \qquad (2.55)$
- Then
- $x_2[n] = e^{jw_2 n} = e^{j(w_1+2\pi k)n} = e^{jw_1 n} = x_1[n]$
- This means the two sequences $x_1[n] = e^{jw_1 n}$ and $x_1[n] = e^{jw_1 n}$ are indistinguishable.

Any exponential sequence of a normalized frequency w_2 with a value outside the frequency range $0 \le w \le 2\pi$ is equivalent to an exponential sequence of a normalized angular frequency of value w_2 modulo 2π

Typical Sequences and Sequences Representation

Some Basic Sequences (Two Properties of Sinusoids and Exponential)

- Consider two of the following sequences:
- $x_1[n] = \cos(w_1 n + \phi) \quad 0 \le w_1 \le 2\pi$
- $x_2[n] = \cos(w_2 n + \phi)\, 2\pi k \le w_2 \le 2\pi(k+1) \quad (where\ k\ is\ any\ positive\ integer)$
- If
- $w_2 = \mathrm{w}_1 + 2\pi k. \qquad (2.55)$
- Then

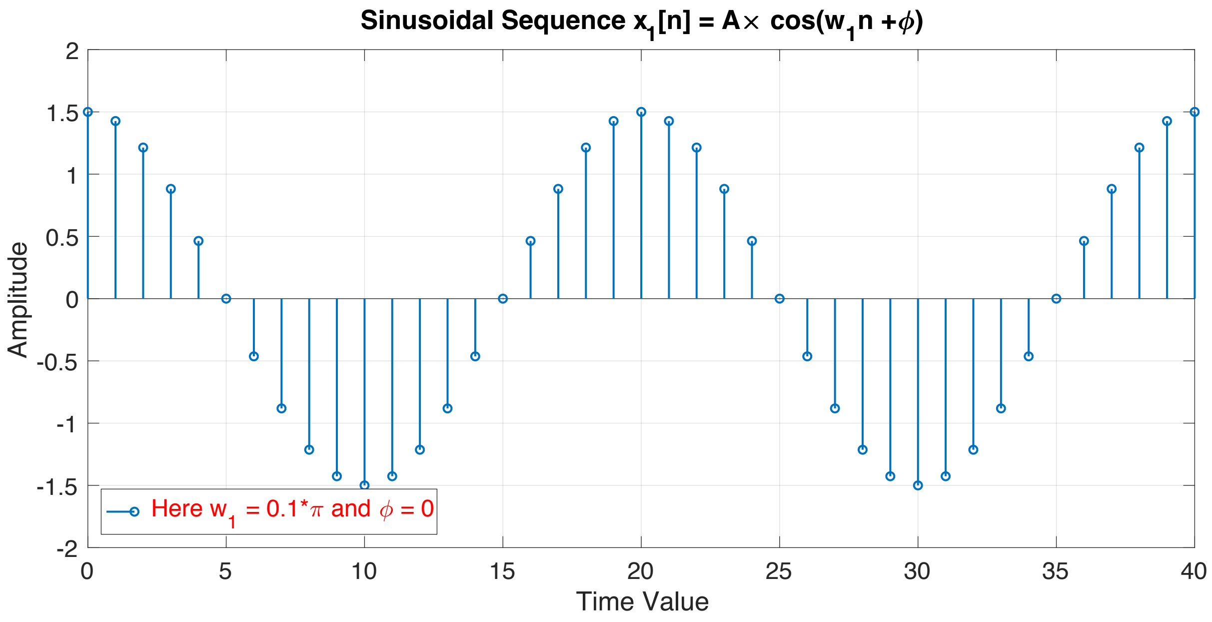
Any sinusoidal sequence of a normalized frequency w_2 with a value outside the frequency range $0 \le w \le 2\pi$ is equivalent to an sinusoidal sequence of a normalized angular frequency of value w_2 modulo 2π

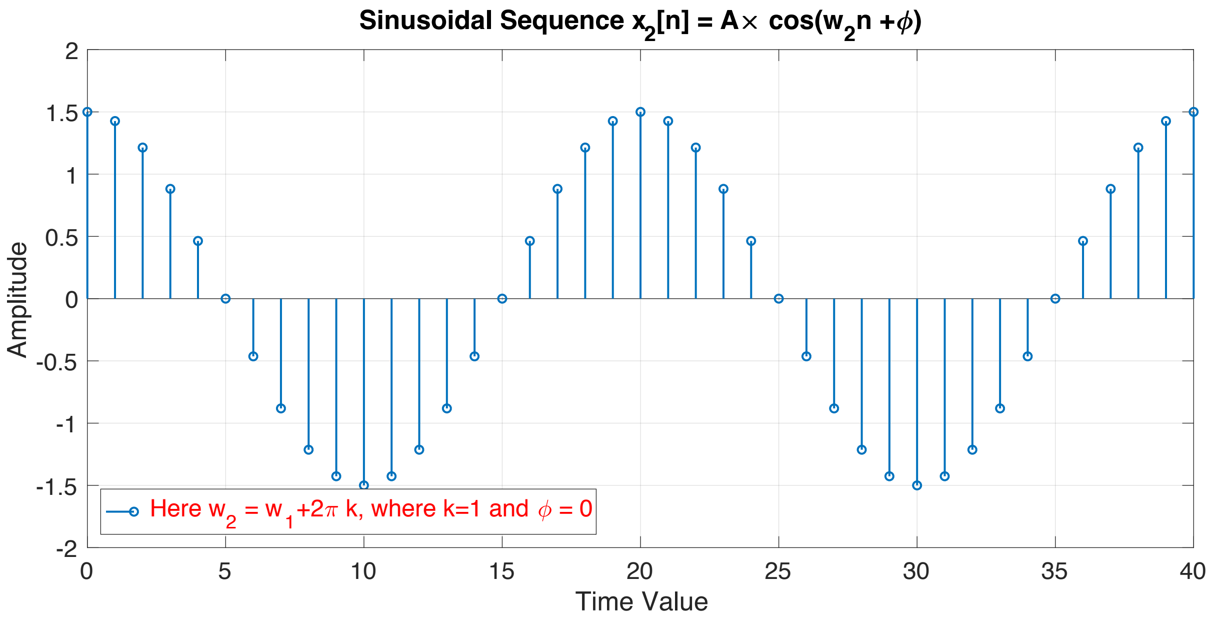
- $x_2[n] = \cos(w_2 n + \phi) = \cos\big((w_1 + 2\pi k)n + \phi\big) = \cos\big((w_1 n + \phi) + 2\pi kn\big)$
- $\cos(w_1 n + \phi)\cos(2\pi kn) - \sin(w_1 n + \phi)\sin(2\pi kn) = \cos(w_1 n + \phi) = x_1[n]$
- This means the two sequences $x_1[n] = \cos(w_1 n + \phi)$ and $x_2[n] = \cos(w_2 n + \phi)$ are indistinguishable.

Typical Sequences and Sequences Representation

Some Basic Sequences (Two Properties of Sinusoids and Exponential)

$$x_1[n] = \cos(0.1\pi n + \phi) = \cos\big((0.1\pi + 2\pi)n + \phi\big) = x_2[n], \textit{here } k = 1$$





Typical Sequences and Sequences Representation

Some Basic Sequences (Two Properties of Sinusoids and Exponential)

- Consider two sinusoidal sequences:
- $x_1[n] = \cos(w_1 n) \quad 0 \leq w_1 \leq \pi$
- $x_2[n] = \cos(w_2 n) \quad \pi \leq w_2 \leq 2\pi$ $(where\ k\ is\ any\ positive\ integer)$
- If
- $w_2 = 2\pi - \mathrm{w}_1.$
- Then
- $x_2[n] = \cos(w_2 n) = \cos\big((2\pi - w_1)n\big) = \cos(2\pi n - w_{1n})$
- $\cos(-w_1 n)\cos(2\pi n) - \sin(-w_1 n)\sin(2\pi n) = \cos(w_1 n) = x_1[n]$
- This means the two sequences $x_1[n]$ and $x_2[n]$ are indistinguishable.
- The frequency π is called folding frequency.

A sinusoidal sequence of a normalized frequency w_2 with a value outside the frequency range $\pi \leq w \leq 2\pi$ is equivalent to an sinusoidal sequence of a normalized angular frequency $w_1 = 2\pi - w_2$

Typical Sequences and Sequences Representation

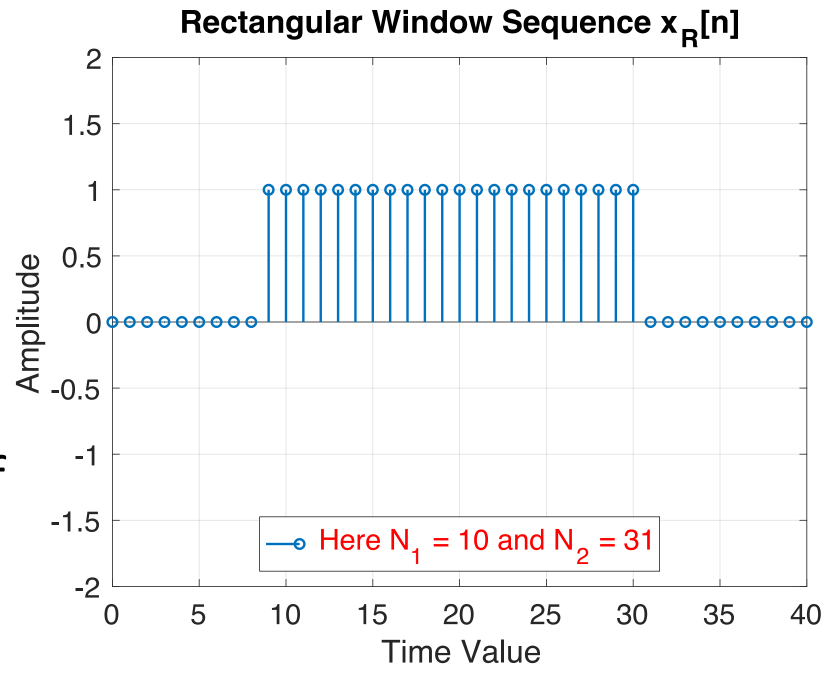
Some Basic Sequences (Two Properties of Sinusoids and Exponential)

- The frequency of oscillation of the discrete-time sinusoidal sequence $x[n] = Acos(w_o n)$ increases as w_o increases from $0\ to\ \pi$.
- The frequency of oscillation of the discrete-time sinusoidal sequence $x[n] = Acos(w_o n)$ decreases as w_o increases from $\pi\ to\ 2\pi$.
- The frequencies in the neighborhood of $w_o = 2\pi k$ $(\text{where k is any positive integer})$ are called low frequencies.
- The frequencies in the neighborhood of $w_o = \pi(2k + 1)$ $(\text{where k is any positive integer})$ are called high frequencies.
- $cos(0.1\pi n)$ and $cos(1.9\pi n)$ are examples of low-frequency signal.
- $cos(0.8\pi n)$ and $cos(1.2\pi n)$ are examples of high-frequency signal.

Typical Sequences and Sequences Representation

Some Basic Sequences (Rectangular Window Sequence)

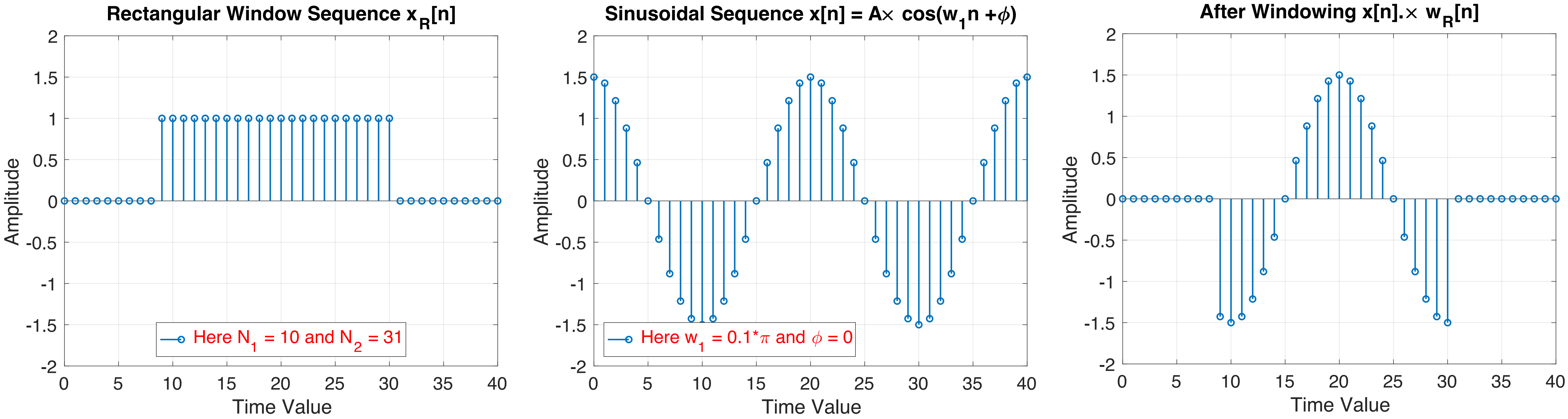
- A rectangular window sequence, also called a box-car sequence, is a sequence $x_R[n]$ with unity sample values in a finite range $N_1 \le n \le N_2$ and zero-valued samples outside this range; that is
- $w_R[n] = \begin{cases} 0, & n < N_1, \\ 1, & N_1 \le n \le N_2, \\ 0, & n > N_2. \end{cases}$
- The $x_R[n]$ is used to extract the parts of a sequence $x[n]$ in a certain range and make the sample values outside that range equal to zeros. This is called windowing, used in the design of filters.
- $x[n].\times w_R[n] = \begin{cases} 0, & n < N_1, \\ x[n], & N_1 \le n \le N_2, \\ 0, & n > N_2. \end{cases}$



Typical Sequences and Sequences Representation

Some Basic Sequences (Rectangular Window Sequence)

It can be seen multiplying rectangular window sequence with a sequence x[n], extracts the sample values in given range, and made all other values zeros.



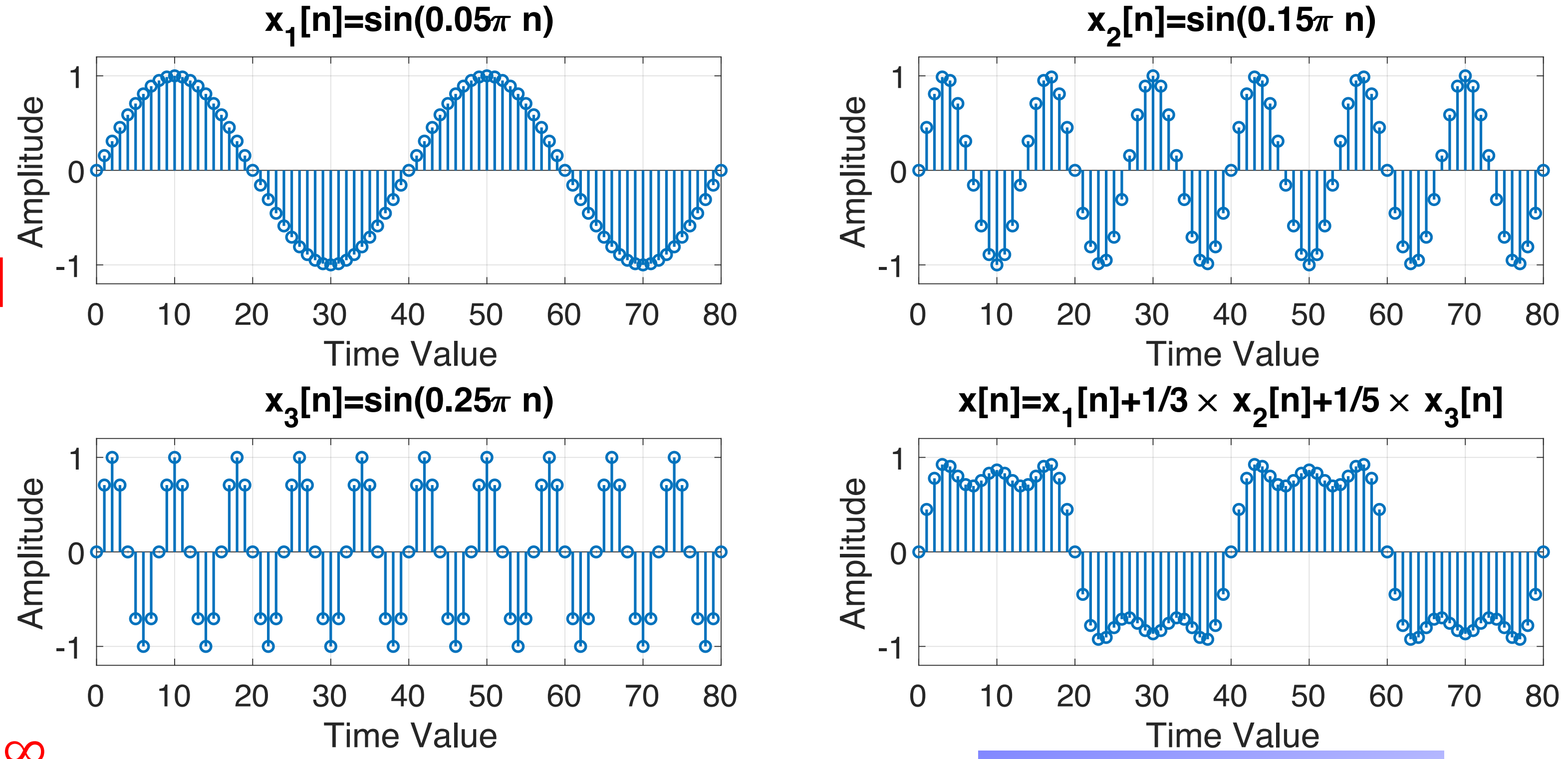
Typical Sequences and Sequences Representation

Some Basic Sequences (Generation of Sequence with Complex Waveforms)

- **Example 2.13**: Plot the following composite signals along with each component:
- $y[n] = x_1[n] + \frac{1}{3}x_2[n] + \frac{1}{3}x_3[n]$
- Where,
- $x_1[n] = \sin(0.05\pi n)\,\mu[n]$
- $x_2[n] = \sin(0.15\pi n)\,\mu[n]$
- $x_3[n] = \sin(0.25\pi n)\,\mu[n]$
- Here $\mu[n]$ indicates that each sequence is defined for $0 \le n \le \infty$

Fundamental Component with fundamental normalized angular frequency $w_1 = 0.05\pi$

Third harmonic with normalized angular frequency $w_1 = 0.15\pi$



Fifth harmonic with normalized angular frequency $w_1 = 0.25\pi$

Composite sequence with normalized angular frequency $w = LCM(0.05, 0.15, 0.25)$

Typical Sequences and Sequences Representation

Some Basic Sequences (Generation of Sequence with Complex Waveforms)

- **Example 2.13**: Plot the following composite signals along with each component:
- $y[n] = x_1[n] + \frac{1}{3}x_2[n] + \frac{1}{3}x_3[n]$
- Where,
- $x_1[n] = \sin(0.05\pi n)\,\mu[n]$
- $x_2[n] = \sin(0.15\pi n)\,\mu[n]$
- $x_3[n] = \sin(0.25\pi n)\,\mu[n]$
- Here $\mu[n]$ indicates that each sequence is defined for $0 \le n \le \infty$

- **Fundamental and Harmonic Components:**
  - The normalized angular frequency of $x_1[n]$ is $\mathrm{w}_1 = 0.05\pi$, the normalized angular frequency of $x_2[n]$ is $\mathrm{w}_2 = 0.15\pi$, and the normalized angular frequency of $x_3[n]$ is $\mathrm{w}_3 = 0.25\pi$,
  - It can be seen that w_2 is three times w_1 i.e $(w_2 = 3w_1)$.
  - $\frac{w_3}{w_1} = \frac{0.25}{0.05} = \frac{5}{1}$ $(i.e\ w_3 = 5\times w_1)$
  - The component $x_1[n]$ is called fundamental component and w_1 is called fundamental frequency (the sinusoidal sequence with lower frequency is called fundamental component and corresponding frequency is called fundamental frequency)
  - The components $x_2[n]$ and $x_3[n]$ are called 3rd and 5th harmonics, and their corresponding corresponding frequencies are 3rd and 5th harmonics (The sinusoidal sequences whose angular frequencies are integer multiple of a sinusoidal sequence with of lower angular frequency are called harmonics).

Typical Sequences and Sequences Representation

Some Basic Sequences (Generation of Sequence with Complex Waveforms)

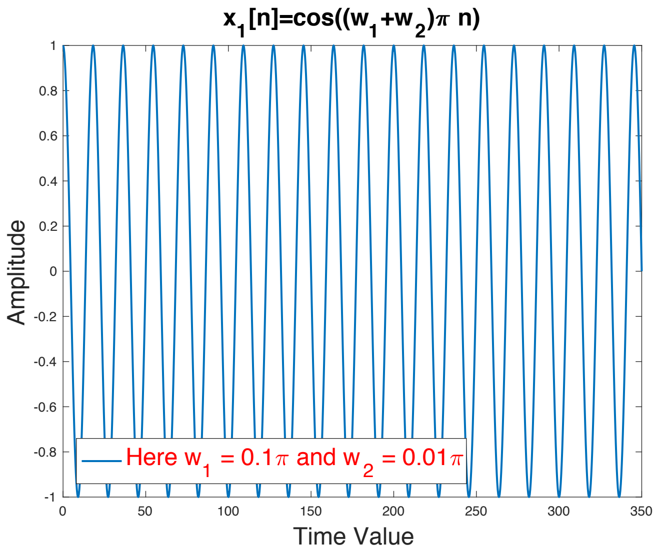
- **Fundamental and Harmonic Components:**
  - The normalized angular frequency of $x_1[n]$ is $w_1 = 0.05\pi$, the normalized angular frequency of $x_2[n]$ is $w_2 = 0.15\pi$, and the normalized angular frequency of $x_3[n]$ is $w_3 = 0.25\pi$,
  - It can be seen that w_2 is three times w_1 i.e $(w_2 = 3w_1)$.
  - $\frac{w_3}{w_1} = \frac{0.25}{0.05} = \frac{5}{1}$ $(i.e\ w_3 = 5 \times w_1)$
  - The component $x_1[n]$ is called fundamental component and w_1 is called fundamental frequency (the sinusoidal sequence with lower frequency is called fundamental component and corresponding frequency is called fundamental frequency)
  - The components $x_2[n]$ and $x_3[n]$ are called 3rd and 5th harmonics, and their corresponding corresponding frequencies are 3rd and 5th harmonics (The sinusoidal sequences whose angular frequencies are integer multiple of a sinusoidal sequence with of lower angular frequency are called harmonics).

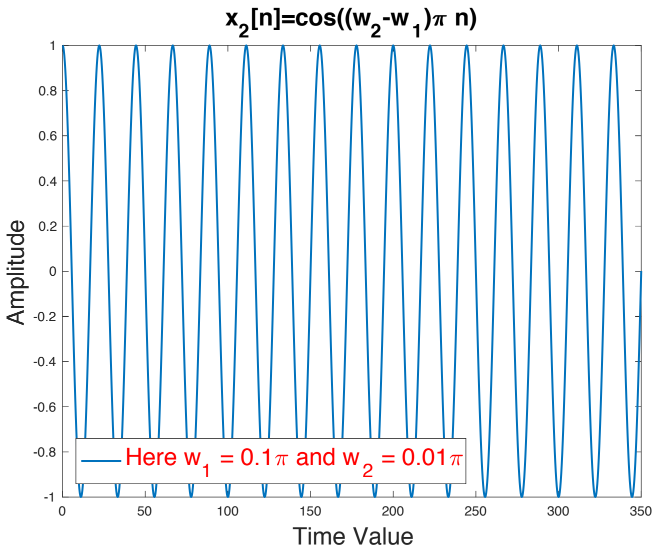
- **Fourier Series Expansion**
- Any periodic sequence can be expressed in the form of linearly weighted combinations of a fundamental and a series of harmonic components called the Fourier Series Expansion.
- The weights associated with each component in the expansion are called Fourier series coefficients.
  - In our last example, the Fourier series coefficients are. $1, \frac{1}{3}, and\ \frac{1}{5}$.

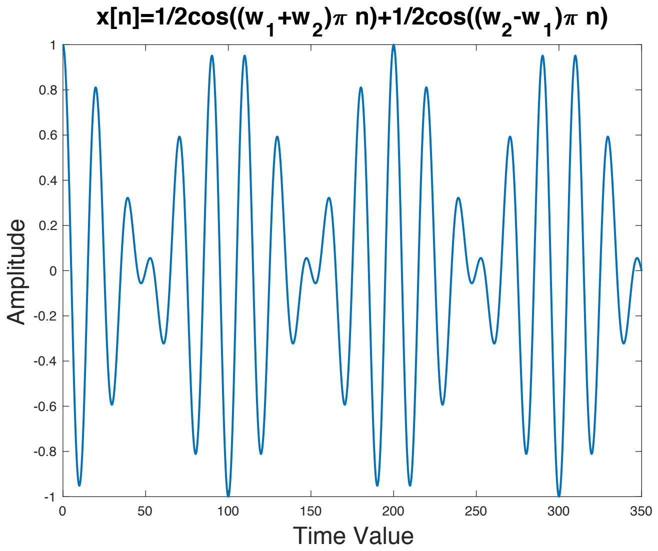
Typical Sequences and Sequences Representation

Some Basic Sequences (Generation of Sequence with Complex Waveforms)

- **Example 2.14**: Perform modulation:
- $y[n] = x_1[n].x_2[n]$
- Where,
- $x_1[n] = \cos(w_1 \pi n)$
- $x_2[n] = \cos(w_2 \pi n)$
- With $\pi > w_2 \gg w_1 > 0$
- Sol:
- $y[n] = \cos(w_1 \pi n)\cos(w_2 \pi n)$
- $= \frac{1}{2}\cos((w_1 + w_2)\pi n) + \frac{1}{2}\cos((w_2 - w_1)\pi n)$



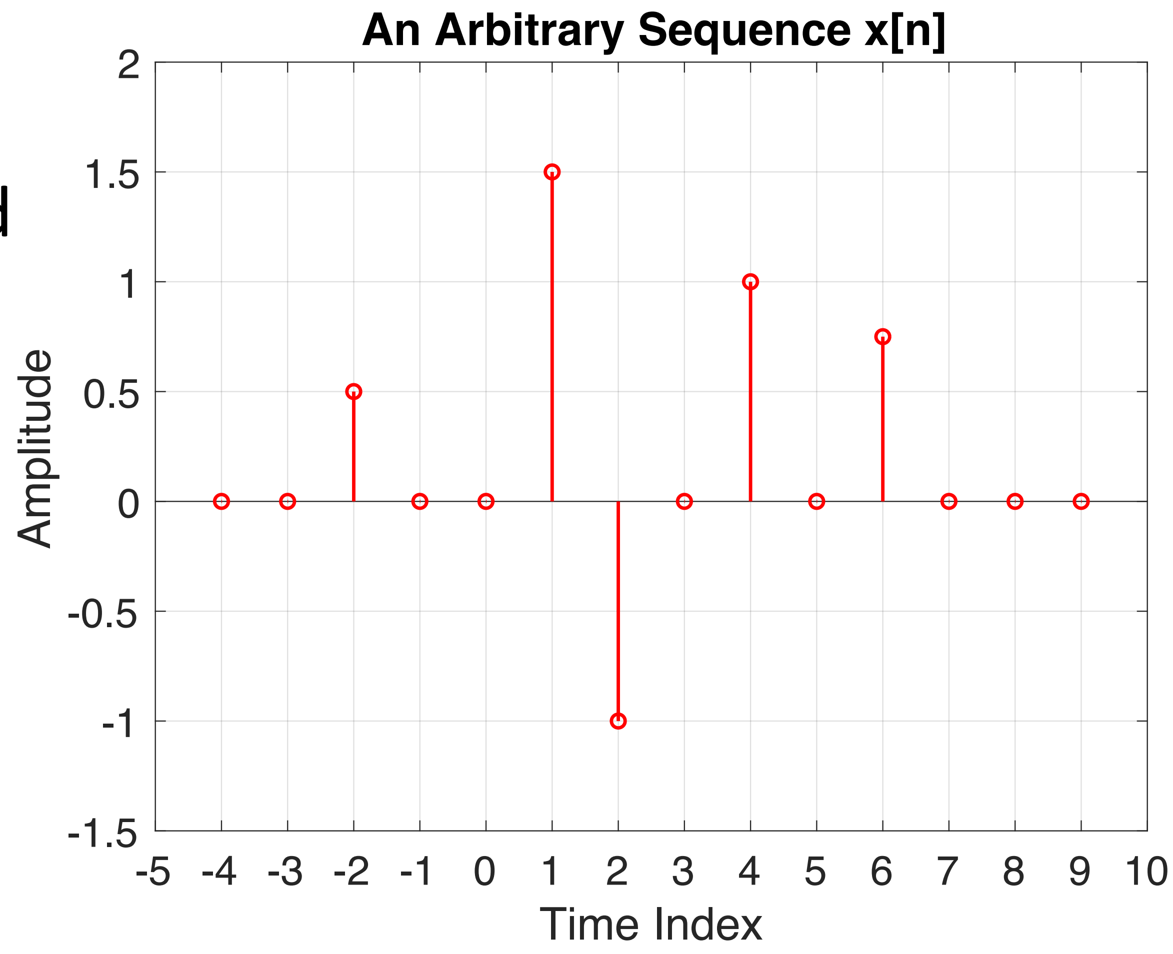




Typical Sequences and Sequences Representation

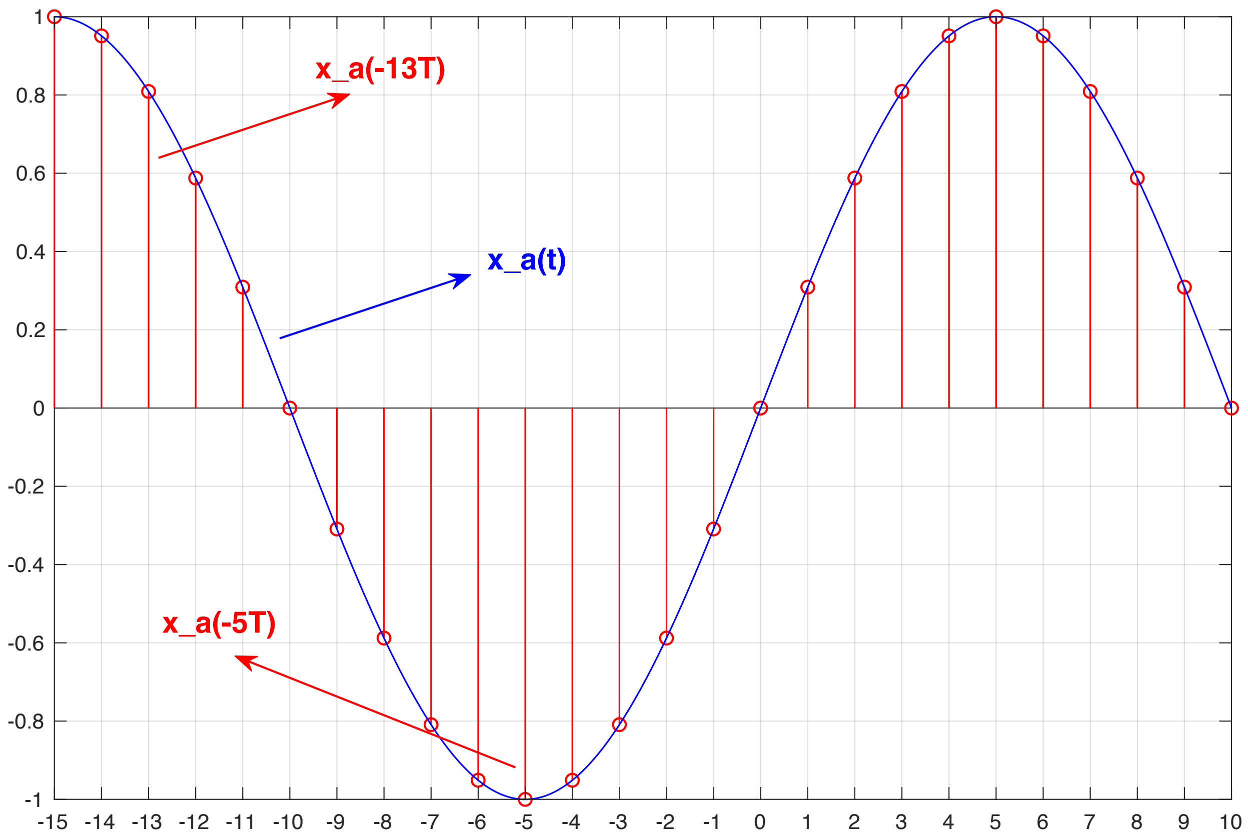
Representation of an Arbitrary Sequence

- An arbitrary sequence can be represented in the time domain as a weighted sum of the delayed and advanced versions of a basic sequence
- $x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + \delta[n-4] + 0.75\delta[n-6]$



The Sampling Process

- A discrete-time sequence $\{x[n]\}$ is generated by periodically sampling a continuous time signal $x_a(t)$ at uniform intervals:
- $x[n] = x_a(t)|_{t=nT} = x_a(nT)$ (2.2)
- Where, $n = \cdots, -2, -1, 0, 1, 2, \ldots$
- This procedure is shown in Figure.
- Where T (fundamental time period) means that after each T seconds the sample of $x_a(t)$ is taken to create discrete-time domain sequence $x[n]$.



The Sampling Process

- A discrete-time sequence $\{x[n]\}$ is generated by periodically sampling a continuous time signal $x_a(t)$ at uniform intervals:
- $x[n] = x_a(t)|_{t=nT} = x_a(nT)$ (2.2)
- Where, $n = \cdots, -2, -1, 0, 1, 2, \ldots$
- This procedure is shown in Figure.
- Where T (fundamental time period) means that after each T seconds the sample of $x_a(t)$ is taken to create discrete-time domain sequence $x[n]$.
- The time variable t of the continuous-time signal $x_a(t)$ is related to the time variable n of the discrete-time sequence $x[n]$ only at discrete-time instants t_n given by
- $t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$ (2.61)

The Sampling Process

- The time variable t of the continuous-time signal $x_a(t)$ is related to the time variable n of the discrete-time sequence $x[n]$ only at discrete-time instants t_n given by
- $t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$ (2.61)
- Where, $F_T = \frac{1}{T}$ is the sampling frequency.
- Ω_T is the sampling angular frequency.
- The sampling frequency F_T and sampling angular frequency are related as
- $\Omega_T = 2\pi F_T$

The Sampling Process

- For a given continuous-time signal
  - $x_a(t) = A cos(2\pi f_o t + \phi) = A cos(\Omega_o t + \phi)$ (2.62)
- The corresponding discrete-time signal is given by
  - $x[n] = A cos(\Omega_o nT + \phi)$
- From Eq. (2.61), we know that $T = \frac{2\pi}{\Omega_T}$ putting this value in above eq. will result
  - $x[n] = A cos\left(\frac{2\pi\Omega_o}{\Omega_T} + \phi\right) = A cos(w_o n + \phi)$ (2.63)
- Here:
  - Ω_o is the normalized angular frequency (randians per second) of continuous-time signal.
  - Ω_T is the angular sampling frequency.
  - w_o is the normalized angular frequency (radians per sample) of the discrete-time sequence.
  - f_0 is the the linear frequency (cycles per second or Hertz (Hz)) of continuous-time signal.
- The normalized angular frequency of continuous-time signal is related with the normalized angular frequency of discrete time sequence through
  - $w_o = \frac{2\pi\Omega_o}{\Omega_T} = \Omega_o T$ (2.64)

The Sampling Process

- Example: Using the sampling rate $10\ Hz$, find the discrete-time sequences for following continuous-time signals:
  1. $g_1[n] = \cos(6\pi t)$
  2. $g_2[n] = \cos(14\pi t)$
  3. $g_3[n] = \cos(26\pi t)$
- Solution:
- $F_T = 10\ Hz\ \left(or\ T = \frac{1}{F_T} = \frac{1}{10} = 0.1\ seconds\right)$

  It means after each 0.1 second (or 10 samples) the sampled-value for discrete-time sequence is to be computed.

1. $g_1[n] = \cos(6\pi t)$

Solution:

Here:

$F_T = 10\ Hz\ (\mathrm{T} = 0.1)$

$\Omega_o = 6\pi \frac{\text{radians}}{\text{second}}$

$\Omega_T = 2\pi F_T = 20\pi$

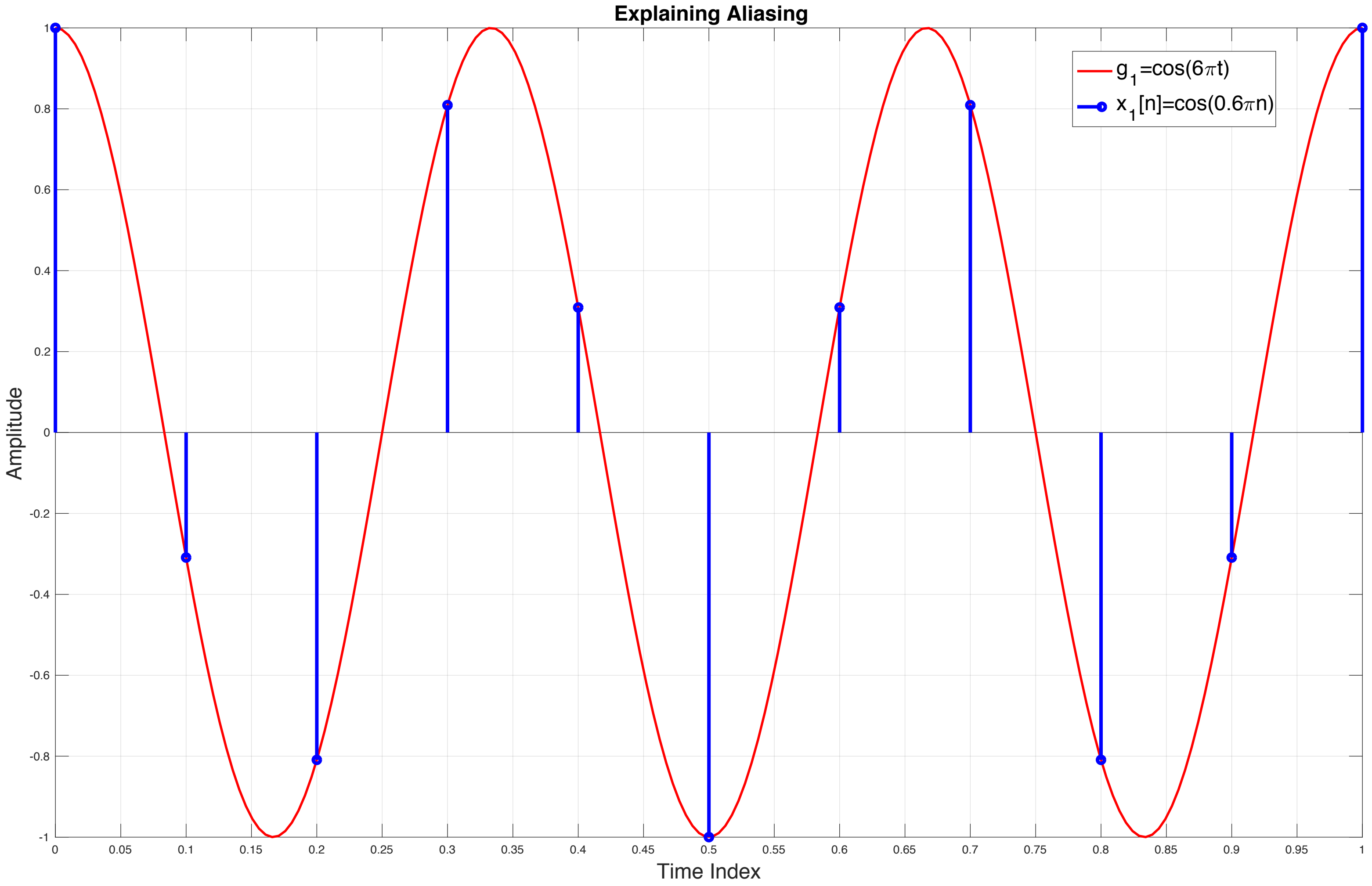
The equivalent discrete-time sequence is obtained as

$x[n] = \cos(\Omega_o nT) = \cos(6\pi n \times 0.1)$

$x[n] = \cos(0.6\pi n)$

Here the normalized angular frequency $w_o = 0.6\pi$ radians per sample.

The Sampling Process



The Sampling Process

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- $F_T = 10\ Hz \left(or\ T = \frac{1}{F_T} = \frac{1}{10} = 0.1\ seconds\right)$

  It means after each 0.1 second (or 10 samples) the sampled-value for discrete-time sequence is to be computed.

2. $g_2[n] = \cos(14\pi t)$

Solution:

Here:

$$F_T = 10\ Hz\ (\mathrm{T} = 0.1)$$

$$\Omega_o = 14\pi \frac{\mathrm{radians}}{\mathrm{second}}$$

$$\Omega_T = 2\pi F_T = 20\pi$$

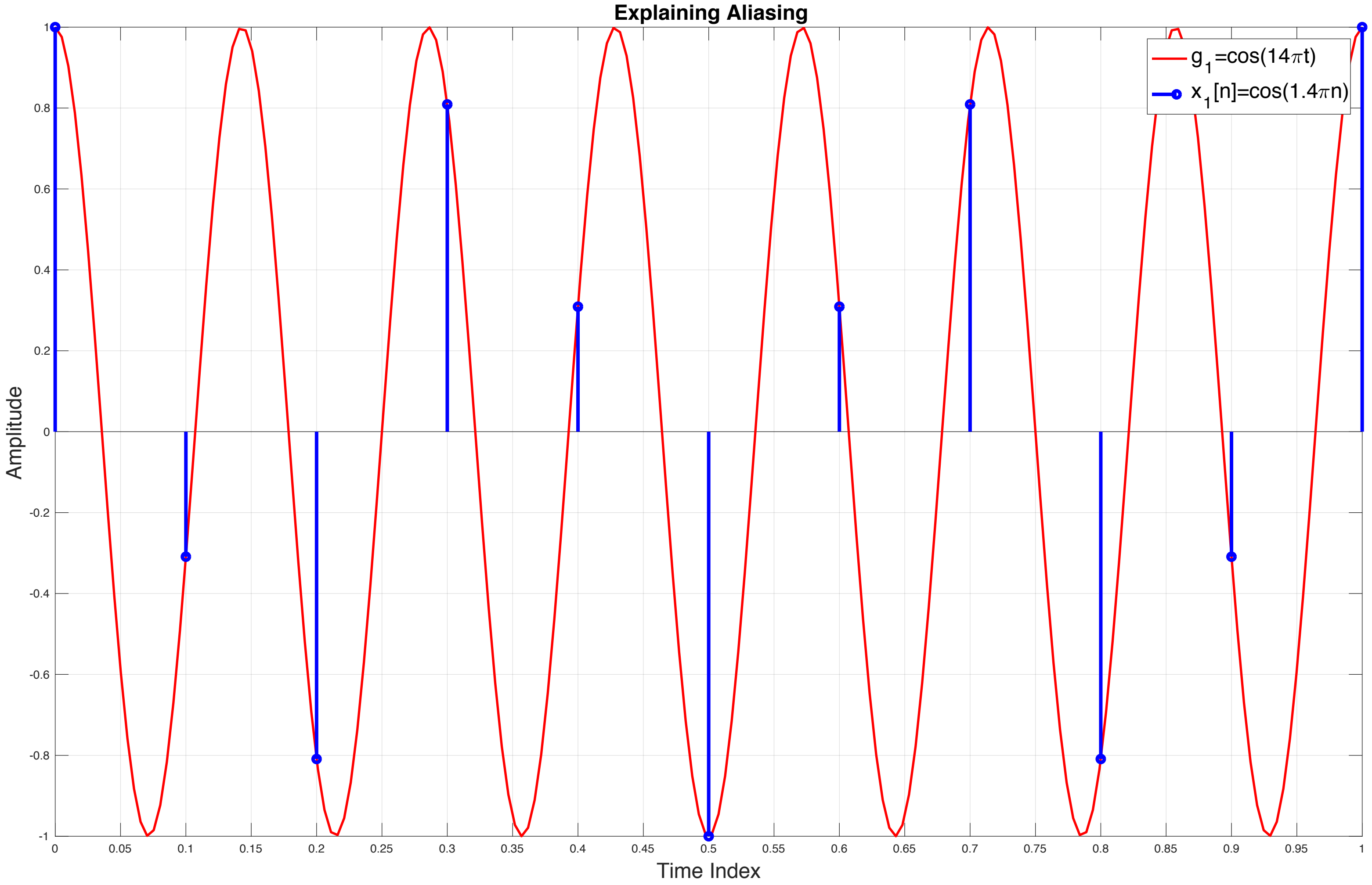
The equivalent discrete-time sequence is obtained as

$$x[n] = \cos(\Omega_o nT) = \cos(14\pi n \times 0.1)$$

$$x[n] = \cos(1.4\pi n)$$

Here the normalized angular frequency $w_o = 1.4\pi$ radians per sample.

The Sampling Process



The Sampling Process

- Example: Using the sampling rate $10\ Hz$, find the discrete-time sequences for following continuous-time signals:
  1. $g_1[n] = \cos(6\pi t)$
  2. $g_2[n] = \cos(14\pi t)$
  3. $g_3[n] = \cos(26\pi t)$
- Solution:
- $F_T = 10\ Hz\ \left(or\ T = \frac{1}{F_T} = \frac{1}{10} = 0.1\ seconds\right)$

  It means after each 0.1 second (or 10 samples) the sampled-value for discrete-time sequence is to be computed.

3. $g_3[n] = \cos(26\pi t)$

Solution:

Here:

$F_T = 10\ Hz\ (\mathrm{T} = 0.1)$

$\Omega_o = 26\pi \frac{\mathrm{radians}}{\mathrm{second}}$

$\Omega_T = 2\pi F_T = 20\pi$

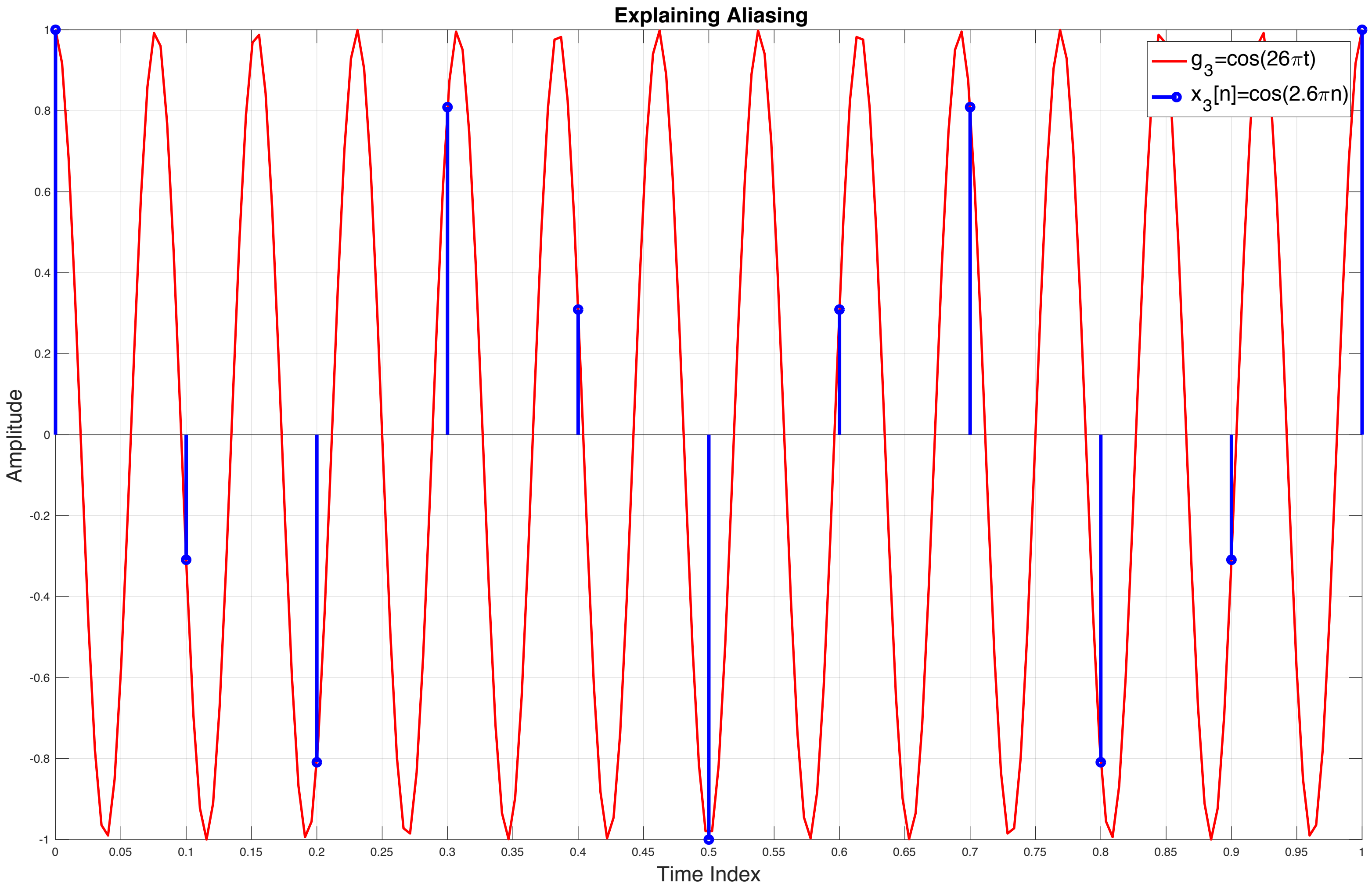
The equivalent discrete-time sequence is obtained as

$x[n] = \cos(\Omega_o nT) = \cos(26\pi n\times 0.1)$

$x[n] = \cos(2.6\pi n)$

Here the normalized angular frequency $w_o = 2.6\pi$ radians per sample.

The Sampling Process



It can be see that same discrete-time sequence is obtained from three different continuous time signal. This phenomemon is called aliasing.
This can further be explained if all plots are ploted in same figue.

Reason:

A sinusoidal sequence of a normalized frequency w_2 with a value outside the frequency range $\pi \leq w \leq 2\pi$ is equivalent to an sinusoidal sequence of a normalized angular frequency $w_1 = 2\pi - w_2$

The Sampling Process

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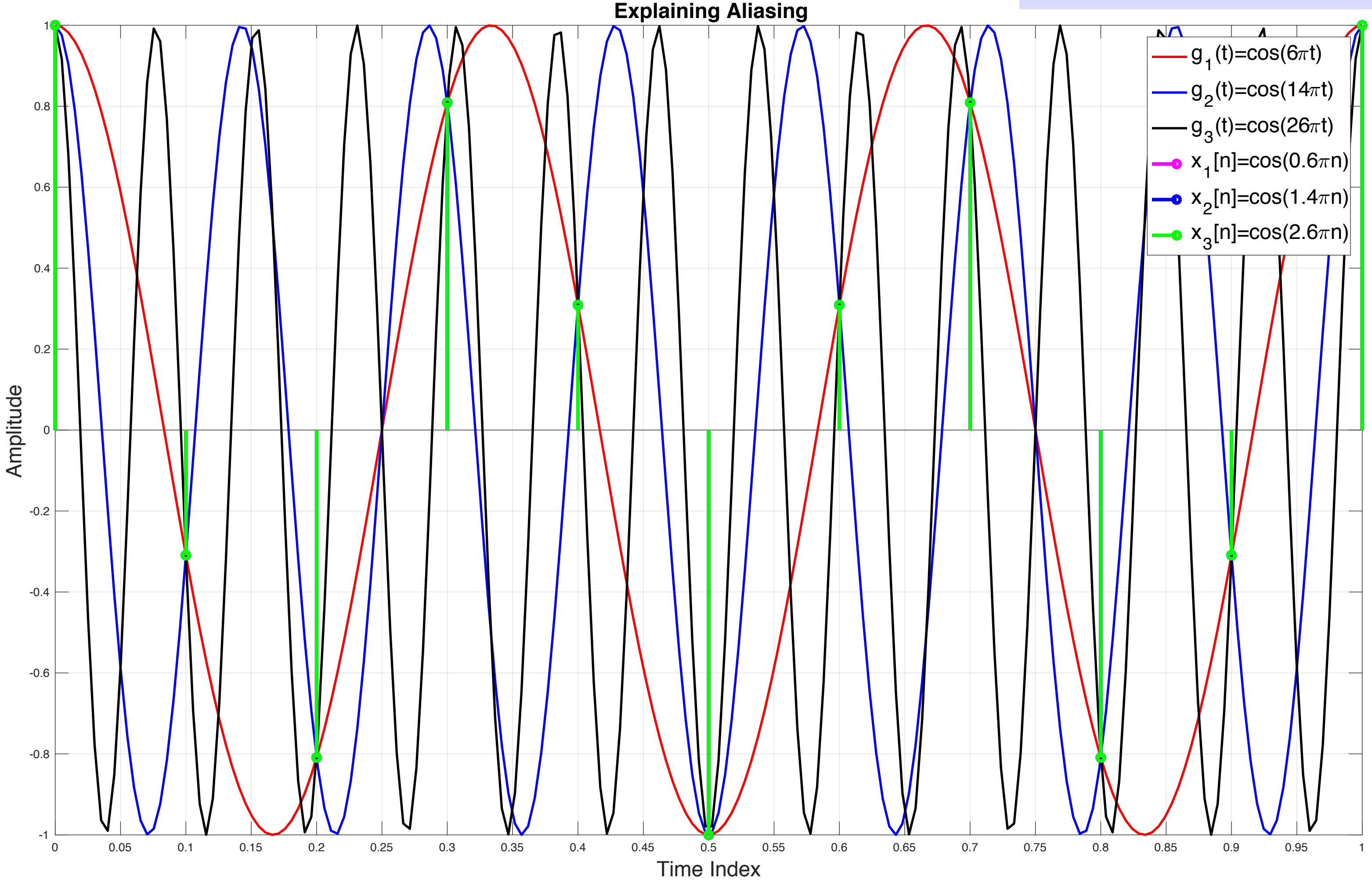

Reason:

A sinusoidal sequence of a normalized frequency w_2 with a value outside the frequency range $\pi \leq w \leq 2\pi$ is equivalent to an sinusoidal sequence of a normalized angular frequency $w_1 = 2\pi - w_2$

$$x_{2[n]} = \cos(1.4\pi n)$$
$$= \cos\big((2\pi - 0.6\pi)n\big) = x_1[n]$$

$$x_{3[n]} = \cos(2.6\pi n)$$
$$= \cos\big((2\pi + 0.6\pi)n\big) = x_1[n]$$

The Sampling Process

- Aliasing:
  - The phenomenon of a continuous-time sinusoidal signal of higher frequency acquiring the identity of a sinusoidal sequence of lower frequency after sampling is called Aliasing.
    - The Aliasing happens, whenever, $\Omega_T \leq 2\Omega_o$
  - Since there are an infinite number of continuous-time functions, additional conditions must be imposed so that each discrete-time sequence $\{x[n]\}$ obtained as a result of sampling, must uniqly represent its parent continuous time function $x_a(t)$.
  - In such case, $x_a(t)$ can be fully recovered from a knowledge of discrete-time sequence $\{x[n]\}$.

The Sampling Process

- Aliasing:
  - If $\Omega_T > 2\Omega_o$, then the normalized angular frequency w_o of obtained discrete-time sequence (obtained as a result of sampling of a continuous-time sequence $x_a(t)$) is in the range $-\pi < w < \pi$, implying no aliasing.
    - The value of w_o is given by $\frac{2\pi\Omega_o}{\Omega_T}$ $modulo\ 2\pi$. i.e
    - $w_o = \left\langle \frac{2\pi\Omega_o}{\Omega_T} \right\rangle_{2\pi}$
    - Hence, to prevent aliasing, the sampling frequency Ω_T should be greater than 2 times the frequency Ω_o of the sinusoidal signal beign samped.
    - If $x_a(t)$ is a composite signal (i.e. $x_a(t)$ is weighted sum of different sinusoids), then $x_a(t)$ can be uniquely represented by sampled version $\{x[n]\}$, if the sampling frequency Ω_T is chosen to be greater than 2 **times the higher frequency contained in** $x_a(t)$.
  - If $\Omega_T < 2\Omega_o$, then w_o is in the range $\pi < w < 2\pi$ and will fold into lower digital frequency in the range $-\pi < w < \pi$ because of aliasing.

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$
$$\sin(x \pm y) = \cos(x)\sin(y) \pm \sin(x)\cos(y)$$

The Sampling Process

- Example 2.16: Determine the discrete-time signal $v[n]$ obtained by uniformly sampling a continuous-time signal $v_a(t)$ composed of weighted sum of five sinusoidal signals of frequencies $(f_o = 30\ Hz)$, $(f_o = 150\ Hz)$, $(f_o = 170\ Hz)$, $(f_o = 250\ Hz)$, and $(f_o = 330\ Hz)$, at a sampling rate of $(F_T = 200\ Hz)$, as given by
- $v_a(t) = 6\cos(60\pi t) + 3\sin(300\pi t) + 2\cos(340\pi t) + 4\cos(500\pi t) + 10\sin(660\pi t)$
- Solution: Here we have
- Sampling Frequencey, $F_T = 200\ Hz$.
- Samplling Time Period, $T = \frac{1}{F_T} = 0.02$ **(This means after each 0.02 sec the sample-values are taken)**
- $v[n] = 6\cos(\Omega_o nT) + 3\sin(\Omega_o nT) + 2\cos(\Omega_o nT) + 4\cos(\Omega_o nT) + 10\sin(\Omega_o nT)$
- $v[n] = 6\cos(0.3\pi n) + 3\sin(1.5\pi n) + 2\cos(1.7\pi n) + 4\cos(2.5\pi n) + 10\sin(3.3\pi n)$
- This is the sampled version, but it can further be solved, as some components have angular frequencis in the range $\pi < w < 2\pi$, which wil fold in the range $0 < w < \pi$, and aliasing will occur.
- $v[n] = 6\cos(0.3\pi n) + 3\sin\big((2\pi - 0.5)\pi n\big) + 2\cos\big((2\pi - 0.3\pi)n\big) + 4\cos\big((2\pi + 5\pi)n\big) + 10\sin\big((2\pi + 3\pi)n\big)$
- $v[n] = 6\cos(0.3\pi n) - 3\sin(0.5\pi n) + 2\cos(0.3\pi n) + 4\cos(0.5\pi n) - 10\sin(0.7\pi n)$

$$\Omega_o = 2\pi f_o$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$
$$\sin(x \pm y) = \cos(x)\sin(y) \pm \sin(x)\cos(y)$$

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- $v[n] = 6\cos(0.3\pi n) + 3\sin(1.5\pi n) + 2\cos(1.7\pi n) + 4\cos(2.5\pi n) + 10\sin(3.3\pi n)$
- $v[n] = 6\cos(0.3\pi n) - 3\sin(0.5\pi n) + 2\cos(0.3\pi n) + 4\cos(0.5\pi n) - 10\sin(0.7\pi n)$
- It can be seen that the components $3\sin(1.5\pi n)$, $2\cos(1.7\pi n)$, $4\cos(2.5\pi n)$, and $10\sin(3.3\pi n)$, have been aliased into the components $-3\sin(0.5\pi n)$, $2\cos(0.3\pi n)$, $\cos(0.5\pi n)$, and $-10\sin(0.7\pi n)$ the resulting discrete-time sequence is
- $\boldsymbol{v[n] = 8\cos(0.3\pi n) + 5\cos(0.5\pi n + 0.6435) - 10\sin(0.7\pi n)}$
- The reultng discrete-time sequence has only free frequencies: $0.3\pi, 0.5\pi, and\ 0.7\pi$.

This alising effect can be avoided by choosing sampling frequency to be greater thand 2 times the higher frequency i.e. $F_T > 2\times 330$

The Sampling Process

- Example 2.16: Determine the discrete-time signal $v[n]$ obtained by uniformly sampling a continuous-time signal $v_a(t)$ composed of weighted sum of five sinusoidal signals of frequencies $(f_o = 30\ Hz)$, $(f_o = 50\ Hz)$, and $(f_o = 70\ Hz)$, at a sampling rate of $(F_T = 200\ Hz)$, as given by
- $v_a(t) = 8\cos(60\pi t) + 5\cos(100\pi t + 0.6435) - 10\sin(140\pi t)$
- Solution: After Simplification we get
- $\boldsymbol{v[n] = 8\,cos(0.3\pi n) + 5\,cos(0.5\pi n + 0.6435) - 10\,sin(0.7\pi n)}$
- The reultng discrete-time sequence has only free frequencies: $0.3\pi, 0.5\pi, and\ 0.7\pi$.

This is same discrete-time sequence which we obtained in last example for different sinusoid with different frequencies. It means same discrete-time sequence is obtained for different sinusoids due to aliasing.

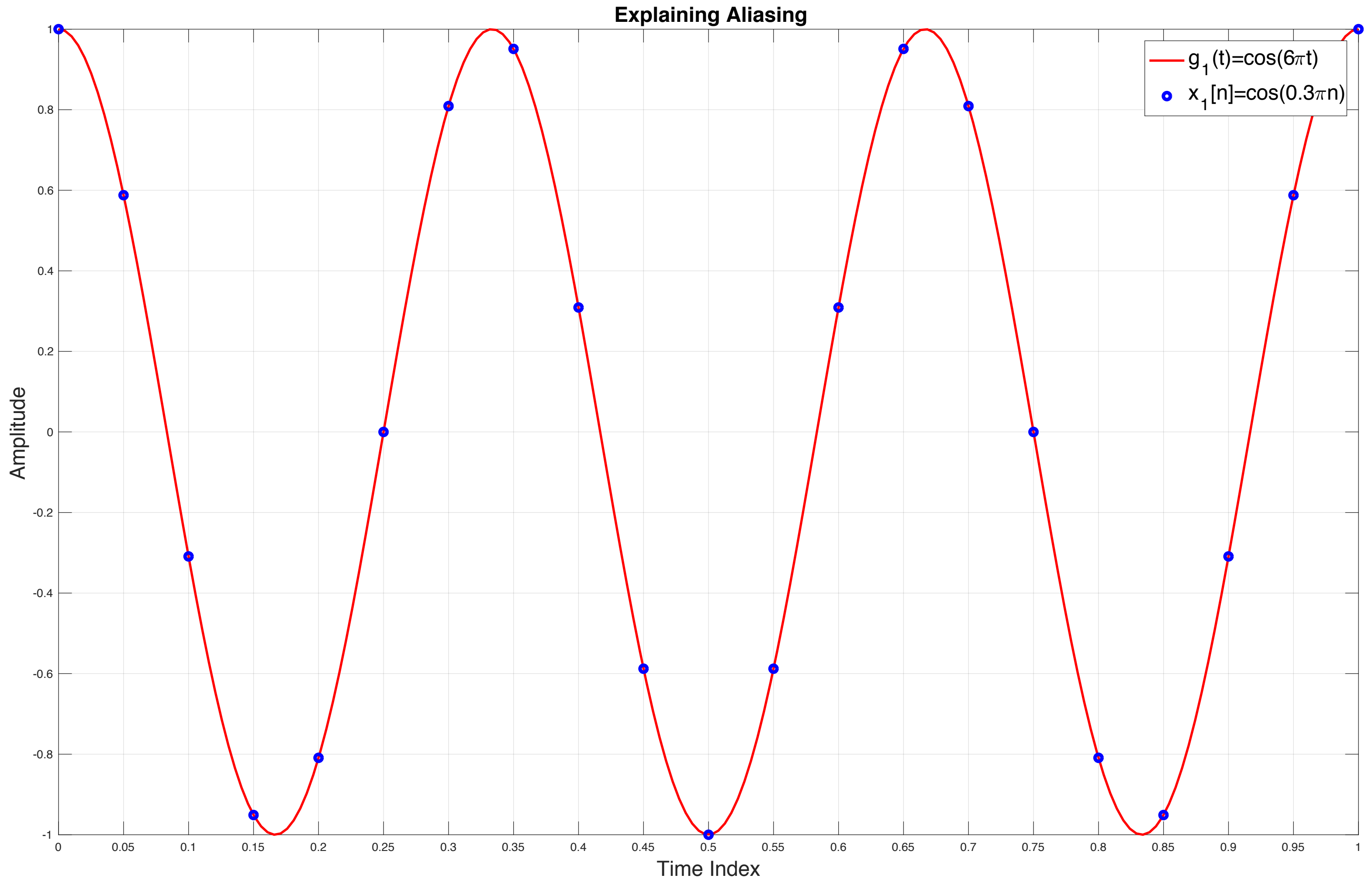
The Sampling Process

- Example 2.16: Determine the discrete-time signal $v[n]$ obtained by uniformly sampling a continuous-time signal $v_a(t)$ composed of weighted sum of five sinusoidal signals of frequencies $(f_o = 30\ Hz)$, $(f_o = 50\ Hz)$, $(f_o = 130\ Hz)$, $(f_o = 230\ Hz)$ and $(f_o = 350\ Hz)$, at a sampling rate of $(F_T = 200\ Hz)$, as given by
- $v_a(t) = 2\cos(60\pi t) + 4\cos(100\pi t) + 10\sin(260\pi t) + 6\cos(460\pi t) + 10\sin(700\pi t)$

- Solution: After Simplification we get
- $\boldsymbol{v[n] = 8\,cos(0.3\pi n) + 5\,cos(0.5\pi n + 0.6435) - 10\,sin(0.7\pi n)}$
- The reultng discrete-time sequence has only free frequencies: $0.3\pi, 0.5\pi, and\ 0.7\pi$.

This is same discrete-time sequence which we obtained in last two example for different sinusoid with different frequencies. It means same discrete-time sequence is obtained for different sinusoids due to aliasing.

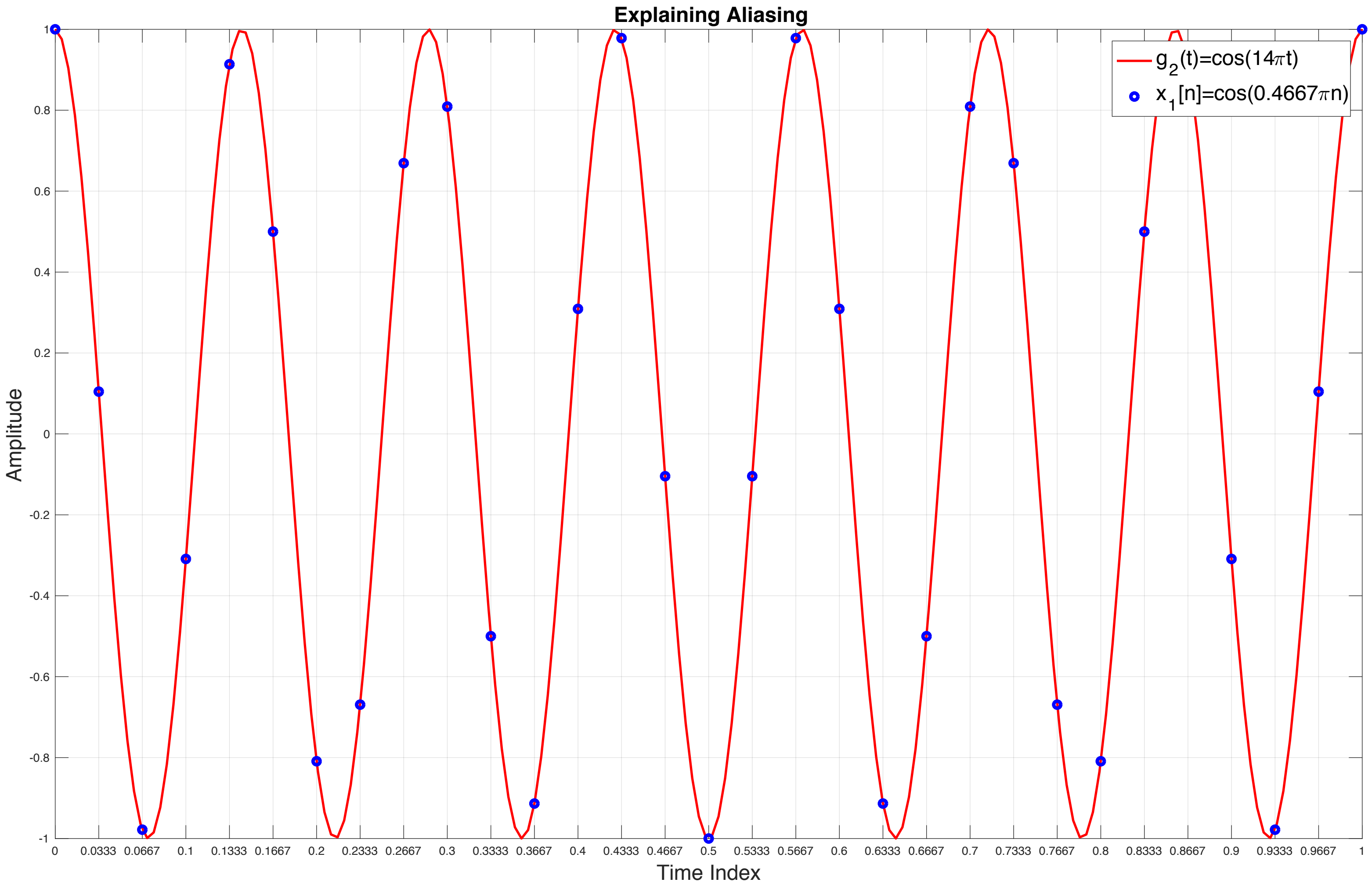
The Sampling Process



Here the sampling frequency is 6 i.e. $f_o = 6\ Hz$. So, to avoid aliasing, we will choose a sampling frequency greater than 2 times f_o (i.e. $F_T > 2f_o = 20\ Hz$.

As it can be seen from figure, that now the discrete samples obtained uniquely representing the continuous time sequence.

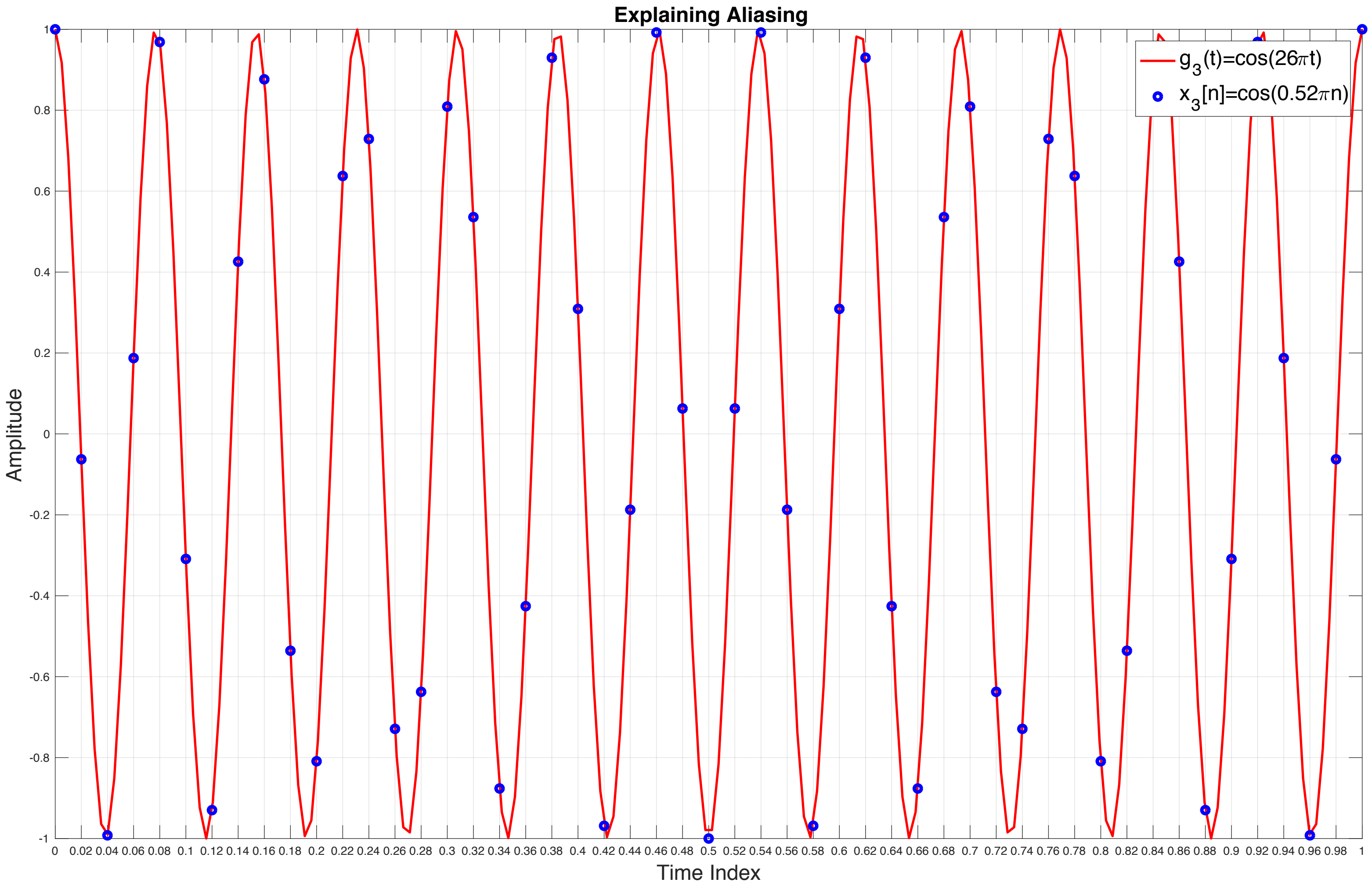
The Sampling Process



Here the sampling frequency is 6 i.e. $f_o = 14\ Hz$. So, to avoid aliasing, we will choose a sampling frequency greater than 2 times f_o (i.e. $F_T > 2f_o = 30\ Hz$.

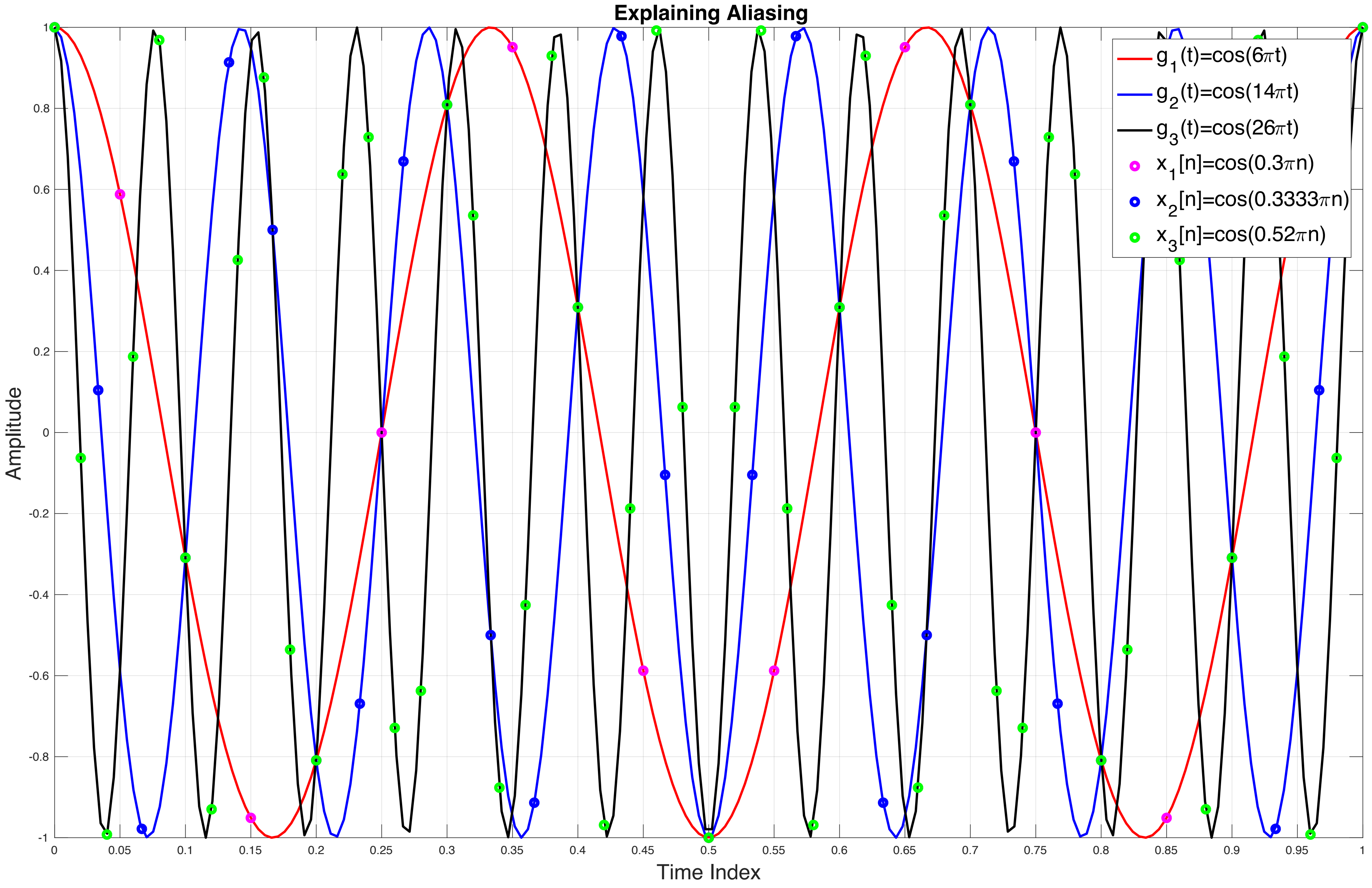
As it can be seen from figure, that now the discrete samples obtained uniquely representing the continuous time sequence.

The Sampling Process



Here the sampling frequency is 6 i.e. $f_o = 26\ Hz$. So, to avoid aliasing, we will choose a sampling frequency greater than 2 times f_o (i.e. $F_T > 2f_o = 50\ Hz$.

As it can be seen from figure, that now the discrete samples obtained uniquely representing the continuous time sequence.



Here the sampling frequency is choosen to be 2 times greater than the frequency of each signal.

As a result it can be seen from figure, that now each discrete-time sequence representing its parent continuous time sequence, uniquely.

Correlation of Signals

- Correlation is one of the way to measure to similarity between a reference signal and one or more signals.
- Examples:
  - In digital communication, a set of data symbols are represented by a set of unique discrete-time sequences. If one of these sequence is transmitted, the receiver determines which particular sequence has been received by comparing it with every member of of possible sequences from the set.
  - In sonar and radar applications, the received signal reflected from the target is the delayed version of the transmitted signal, and by measuring delay one can determine the location of the target.

Correlation of Signals

Definitions

- A measure of similarity between a pair of energy signal, $x[n]$ and $y[n]$, is given by the **cross-correlation sequence** $r_{xy}[l]$ defined by
- $r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]\,, \qquad l = 0, \pm 1, \pm 2, \dots\,, \qquad (2.67)$
- Provided the above infinite sum converges.
- The parameter l called **lag**, indicates the time-shift between the pair, $x[n]$ and $y[n]$.
- The ordering in Eq. (2.67) indicates that $x[n]$ is reference signal and is fixed with respect to time, whereas, $y[n]$ is shifted with respect to time.
- If you want to make $y[n]$ a reference signal, then the ordering of Eq. (2.67) has to changed, as shown below:
- $r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l]\,,$
- Let $n = m + l$, the above expression reduces to
- $r_{yx}[l] = \sum_{m=-\infty}^{\infty} x[m+l]y[m] = r_{xy}[-l], \qquad (2.68)$
- Thus, $r_{yx}[l]$ is obtained by time-reversing the sequence $r_{xy}[l]$.

Correlation of Signals

Definitions

- A measure of similarity of a sequence with itself, $(\text{i.e } x[n]$ and $x[n])$, is called **auto-correlation sequence** $r_{xx}[l]$ and is defined by
- $r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l], \qquad l = 0, \pm 1, \pm 2, \dots, \qquad (2.69)$
- Let $l = 0$, put it in Eq. (2.69)
- $r_{xx}[0] = \sum_{n=-\infty}^{\infty} x[n]x[n] = \sum_{n=-\infty}^{\infty} x^2[n] = \mathcal{E}_x$
- Where $\mathcal{E}_x$ is the energy of the signal $x[n]$.
- $r_{xx}[l]$ is an even sequence of $x[n]$, as $r_{xx}[l] = r_{xx}[-l]$

Correlation of Signals

Definitions (Relationship Between Correlation and Convolution)

- $r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l] = \sum_{n=-\infty}^{\infty} x[n]y[-(-n+l)]$
- $r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[-(l-n)] \qquad (A)$
- From the definition of convolution, we know that
- $x[l] * y[l] = \sum_{n=-\infty}^{\infty} x[n]y[l-n]$
- Let $\mathrm{y}[l] = y[-l]$, the above expression reduces to
- $x[l] * y[-l] = \sum_{n=-\infty}^{\infty} x[n]h[-l+n] = \sum_{n=-\infty}^{\infty} x[n]h[-(l-n)] \ (\mathrm{B})$
- Comparing (A) and (B), will result:
- $r_{xy}[l] = x[l] * y[-l]$
- Cross-correlation of the sequence $x[n]$ with $y[n]$ is equivalent to convolution of $x[n]$ with $y[-n]$.

Correlation of Signals

Properties of Autocorrelation and Cross-Correlation Sequences

- Consider two finite-energy sequences $x[n]$ and $y[n]$. The energy of the combined sequence $ax[n] + y[n-l]$ (alos finite and nonnegative) is obtained as:
- $\sum_{n=-\infty}^{\infty}(ax[n] + y[n-l])^2 = \sum_{n=-\infty}^{\infty}(a^2x^2[n] + 2ax[n]y[n-l] + y^2[n-l])$
- $\sum_{n=-\infty}^{\infty}(ax[n] + y[n-l])^2 = a^2\sum_{n=-\infty}^{\infty} x^2[n] + 2a\sum_{n=-\infty}^{\infty} x[n]y[n-l] + \sum_{n=-\infty}^{\infty} y^2[n-l]$
- $\sum_{n=-\infty}^{\infty}(ax[n] + y[n-l])^2 = a^2 r_{xx}[0] + 2a r_{xy}[l] + r_{yy}[0] \geq 0.$
- Where, $r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \mathcal{E}_x > 0$ (Energy of sequence x[n])
- $r_{yy}[0] = \sum_{n=-\infty}^{\infty} y^2[n-l] = \mathcal{E}_y > 0$ (Energy of Sequence y[n])

Correlation of Signals

Properties of Autocorrelation and Cross-Correlation Sequences

- $\sum_{n=-\infty}^{\infty}(ax[n] + y[n-l])^2 = a^2 r_{xx}[0] + 2a r_{xy}[l] + r_{yy}[0] \geq 0.$
- The above equation can be written in matrix form as follows:
- $\begin{bmatrix} a & 1 \end{bmatrix} \begin{bmatrix} r_{xx}[0] & r_{xy}[l] \\ r_{xy}[l] & r_{yy}[0] \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \geq 0$ (For any finite value of a)
- The matrix
- $\begin{bmatrix} r_{xx}[0] & r_{xy}[l] \\ r_{xy}[l] & r_{yy}[0] \end{bmatrix}$
- is positive semi-definite. This implies
- $r_{xx}[0] r_{yy}[0] - r_{xy}^2[l] \geq 0$
- Or alternatively
- $r_{xy}[l] = \sqrt{r_{xx}[0] r_{yy}[0]} = \sqrt{\mathcal{E}_x \mathcal{E}_y}$ (2.72)
- This is the upper bound for the cross-correlation sequence.

Correlation of Signals

Properties of Autocorrelation and Cross-Correlation Sequences

- $r_{xy}[l] = \sqrt{r_{xx}[0] r_{yy}[0]} = \sqrt{\mathcal{E}_x \mathcal{E}_y} \qquad (2.72)$
- If we put $x[n] = y[n]$, above equation becomes
- $|r_{xx}[l]| \leq r_{xx}[0] = \mathcal{E}_x$ (this implies that at zero lag, the autocorrelation has maximum value)
- For $y[n] = \pm b x[n - N]$
- We get
- $-b r_{xx}[0] \leq r_{xy}[l] \leq b r_{xx}[0]$

Correlation of Signals

Properties of Autocorrelation and Cross-Correlation Sequences

- Example 2.17: Find the cross correlation $r_{xy}[l]$ between two finite-length sequence:
- $x[n] = [1, 3, -2, 1, 2, -1, 4, 4, 2], 0 \leq n \leq 8$ (i.e $N_1 \leq n \leq N_2$)
- $y[n] = [2, -1, 4, 1, -2, 3].\quad 0 \leq n \leq 5$ (i.e $N_3 \leq n \leq N_4$)

Solution:

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

The limite should be from $l\ is\ from\ \ -(|N_1|+|N_4|)\ to\ (|N_2|+|N_4|)$

So, the limit would be from -5 to 13

$$r_{xy}[l] = \sum_{n=0}^{8} x[n]y[n-l]$$

This limit would be max(N2,N4)

This limit would be min(N1,N3)

Correlation of Signals

Properties of Autocorrelation and Cross-Correlation Sequences

$$r_{xy}[l] = \sum_{n=0}^{13} x[n]y[n-l]$$

For $l = -5$

$$r_{xy}[-5] = \sum_{n=0}^{8} x[n]y[n+5]$$

$= x[0]y[5] + x[1]y[6] + x[2]y[7] + x[3]y[8] + x[4]y[9] + x[5]y[10]$
$+ x[6]y[11] + x[7]y[12] + x[8]y[13]$
$= 3 - 0 - 0 + 0 - 0 + 0 + 0 + 0 = 3$

For $l = -4$

$$r_{xy}[-4] = \sum_{n=0}^{8} x[n]y[n+4]$$

$= x[0]y[4] + x[1]y[5] + x[2]y[6] + x[3]y[7] + x[4]y[8] + x[5]y[9]$
$+ x[6]y[10] + x[7]y[11] + x[8]y[12]$
$= -2 + 9 - 0 + 0 - 0 + 0 + 0 + 0 = 7$

For $l = -3$

$$r_{xy}[-3] = \sum_{n=0}^{8} x[n]y[n+3]$$

$= x[0]y[3] + x[1]y[4] + x[2]y[5] + x[3]y[6] + x[4]y[7] + x[5]y[8]$
$+ x[6]y[9] + x[7]y[10] + x[8]y[11]$
$= 1 - 6 - 6 - 0 + 0 - 0 + 0 + 0 + 0 = -11$

Similarly $r_{xy}[-2] = 14, and\ \ r_{xy}[-1] = 13$

For $l = 0$

$$r_{xy}[0] = \sum_{n=0}^{8} x[n]y[n-0]$$

$= x[0]y[0] + x[1]y[1] + x[2]y[2] + x[3]y[3] + x[4]y[5]$
$+ x[5]y[5] + x[6]y[6] + x[7]y[7] + x[8]y[8]$
$= 2 - 3 - 8 + 1 - 4 - 3 + 0 + 0 = -15$

For $l = 1$

$$r_{xy}[1] = \sum_{n=0}^{8} x[n]y[n-1]$$

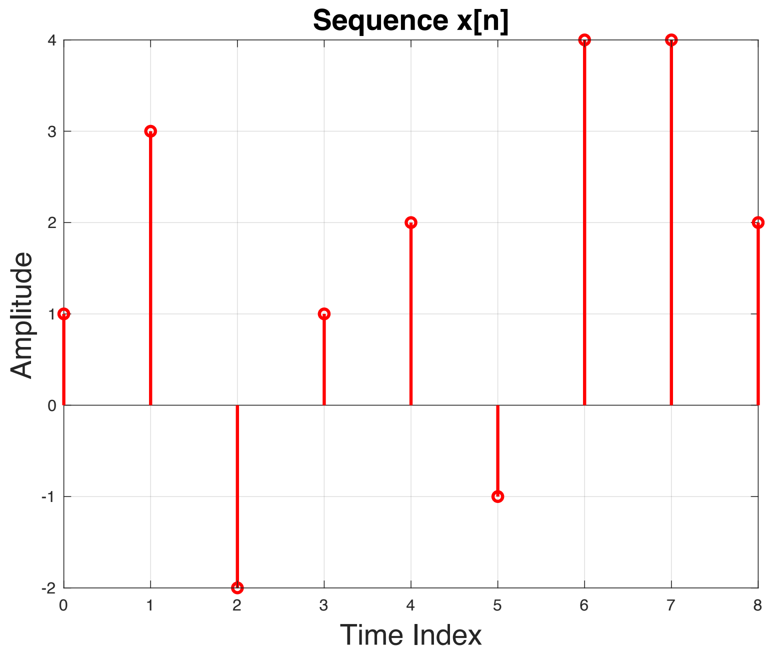
$= x[0]y[-1] + x[1]y[0] + x[2]y[1] + x[3]y[2] + x[4]y[3]$
$+ x[5]y[4] + x[6]y[5] + x[7]y[4] + x[8]y[7]$
$= 0 + 6 + 2 + 4 + 2 + 2 + 12 + 0 = 28$

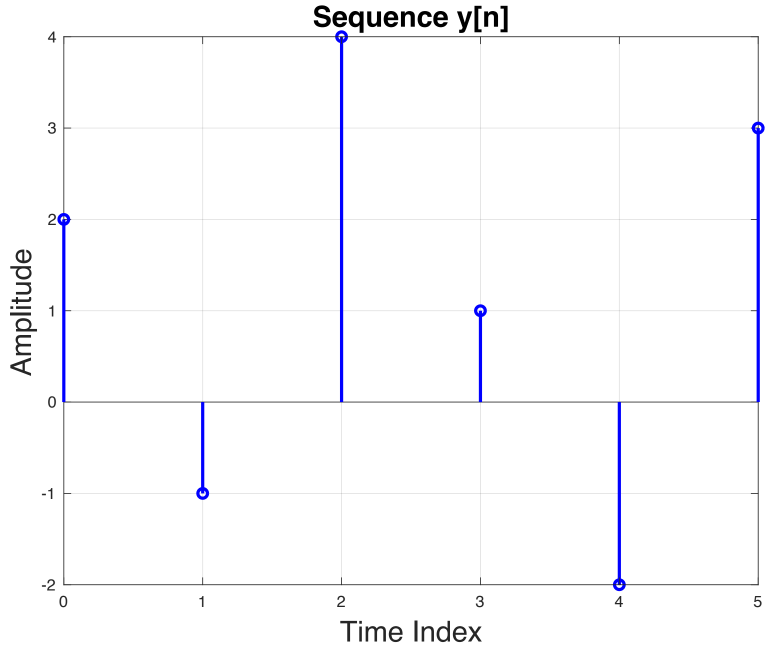
For $l = 2$

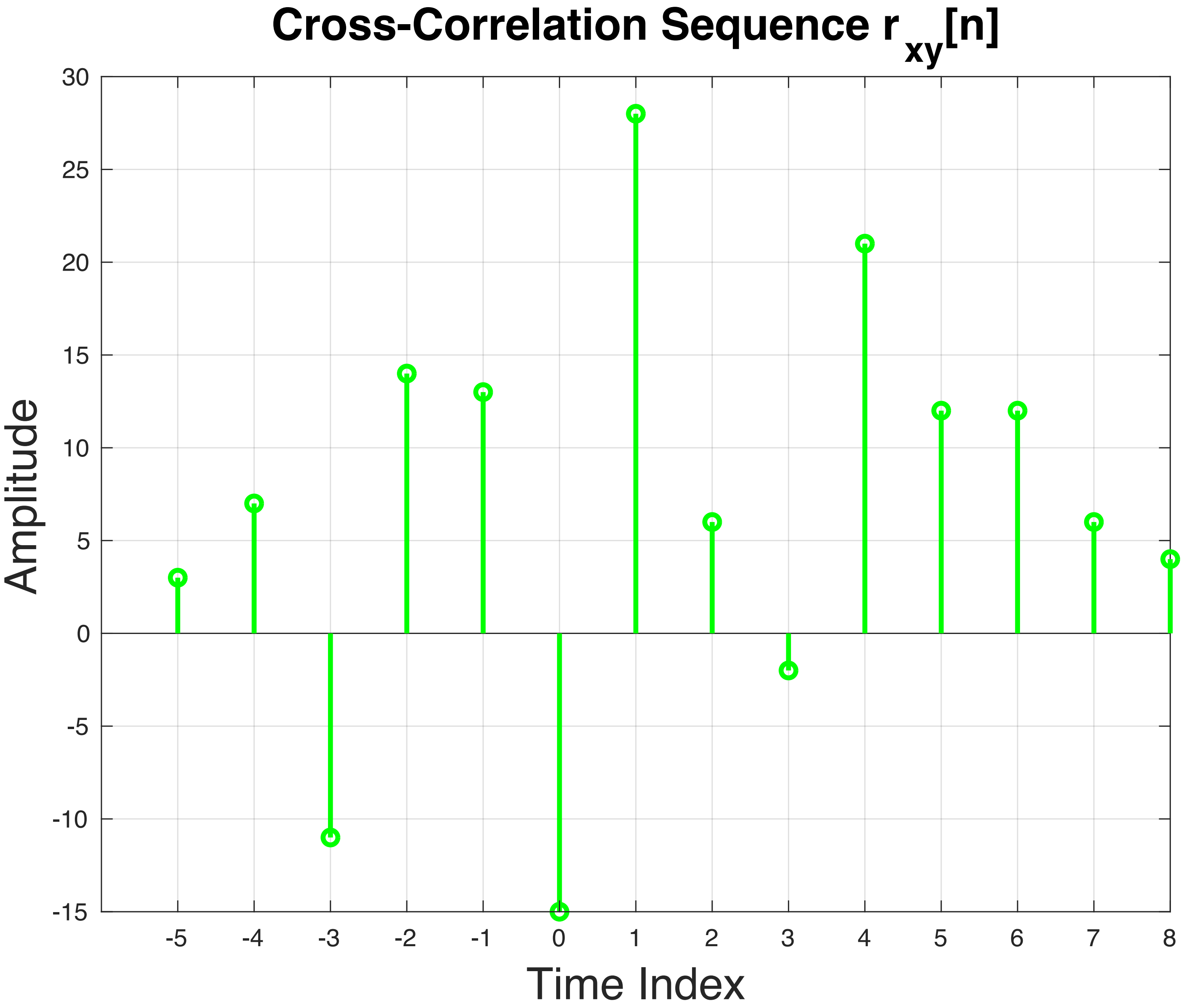
$$r_{xy}[2] = \sum_{n=0}^{8} x[n]y[n-2]$$

$= x[0]y[-2] + x[1]y[-1] + x[2]y[0] + x[3]y[1] + x[4]y[2]$
$+ x[5]y[3] + x[6]y[4] + x[7]y[5] + x[8]y[6]$
$= -4 - 1 + 8 - 1 - 8 + 12 + 0 + 0 + 0 = 6$

Similarly $r_{xy}[3] = -2, r_{xy}[4] = 21, r_{xy}[5] = 12, r_{xy}[6] = 12, r_{xy}[7] = 6, and\ \ r_{xy}[8] = 4$

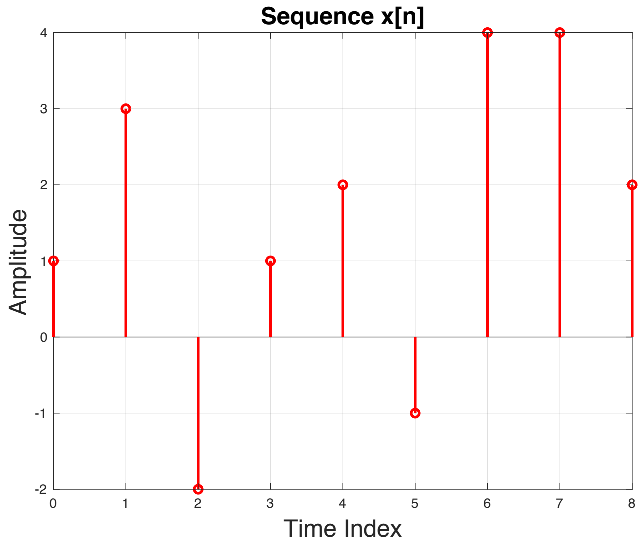

Sequence x[n]
Amplitude
Time Index

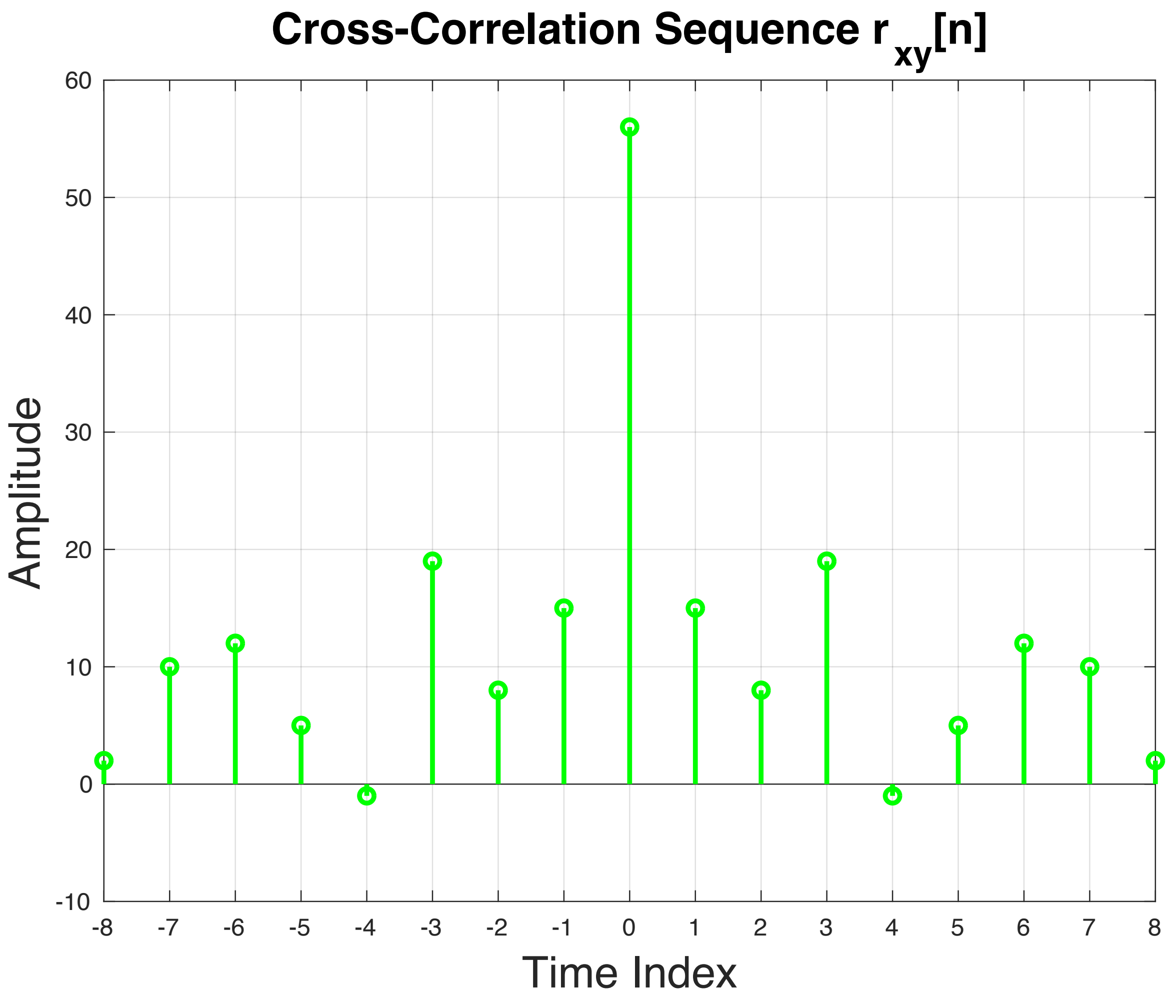

Sequence y[n]
Amplitude
Time Index


Cross-Correlation Sequence $r_{xy}[n]$
Amplitude
Time Index

Autocorrelation

- Let find the auto-correlation of sequence given below:
- $x[n] = [1, 3, -2, 1, 2, -1, 4, 4, 2]$, $0 \leq n \leq 8$ (i.e $N_1 \leq n \leq N_2$)

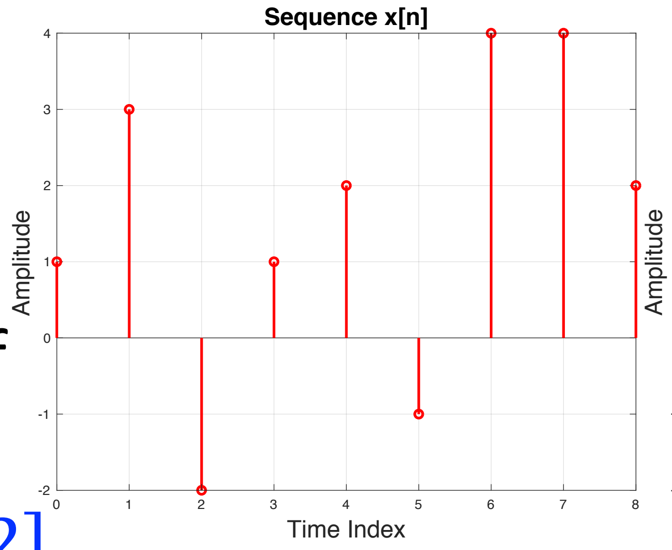


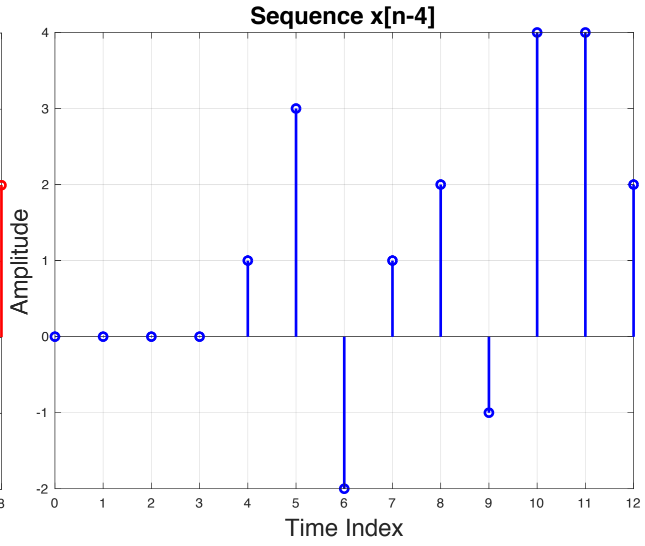


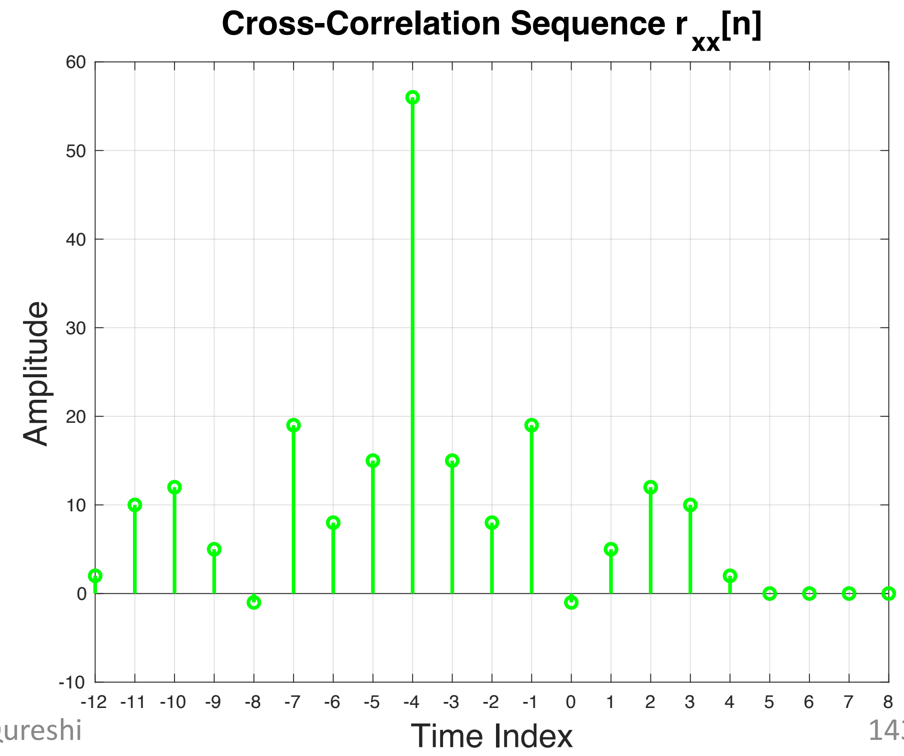
As expected at zero lag, $r_{xx}[0]$ is the maximum

Computing the Delay using Cross-Correlation

- Let find the cross-correlation of sequence given below:
- $x[n] = [1, 3, -2, 1, 2, -1, 4, 4, 2],$ $0 \le n \le 8$ (i.e $N_1 \le n \le N_2$)
- With its delayed version as shown below:
- $x[n-4] =$ $[0, 0, 0, \underset{\uparrow}{1}, 3, -2, 1, 2, -1, 4, 4, 2],$ $-3 \le n \le 4$







$r_{xx}[l-4]$ is the maximum, indicating the delay N=4

Normalized Forms of Correlation

- For convenience in comparing and displaying, normalized autocorrelation (Eq. (2.74)) and cross-correlation (Eq. (2.75)) are used, given by
- $r_{xx}[l] = \frac{r_{xx}[l]}{r_{xx}[0]} \leq 1 \qquad (2.74)$
- $r_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{r_{xx}[0] r_{yy}[0]}} \leq 1 \qquad (2.75)$
- Both autocorrelation and cross-correlation are independent of the range of values of $x[n]$ and $y[n]$.

Correlation Computation for Power and Periodic Signals

- The autocorrelation and cross-correlation for power and periodic signals are slightly different.
- For a pair of **power signals** $\boldsymbol{x}[\boldsymbol{n}]$ and $\boldsymbol{y}[\boldsymbol{n}]$, the cross-correlation is defined as
- $r_{xy}[l] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]y[n-l]$ (2.76)
- The autocorrelation of sequence $x[n]$ is defined below
- $r_{xx}[l] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]x[n-l]$ (2.77)

Correlation Computation for Power and Periodic Signals

- For a pair of **periodic signals** $\tilde{\boldsymbol{x}}[\boldsymbol{n}]$ and $\tilde{\boldsymbol{y}}[\boldsymbol{n}]$, with period N, the cross-correlation is defined as
- $r_{\tilde{x}\tilde{y}}[l] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]\tilde{y}[n-l]$ (2.78)
- The autocorrelation of sequence $\tilde{x}[n]$ is defined below
- $r_{\tilde{x}\tilde{x}}[l] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]\tilde{x}[n-l]$ (2.79)

- Both $r_{\tilde{x}\tilde{y}}[l]$ and $r_{\tilde{x}\tilde{x}}[l]$ are periodic sequences with a period N.

Random Signals

- Deterministic Signals:
  - The discrete-time signals that can be uniquely determined by well-defined processes such as a mathematical expression or a lookup table..
  - The sinusoidal or exponential sequences are just a few types of deterministic signals.
- Random Signals:
  - Signals for which each sample value is generated in a random fashion and cannot be predicted ahead of time are called random signals or stochastic signals.
  - Speech, music, or seismic are just a few examples of random signals.

References:

- Sanjit K. Mitra "Discrete Time Signal in Time Domain," *In: Digital Signal Processing, A Computer Based Approach*, 4th Ed. McGraw Hill, pp. 41—87, 2013.