Holomorphic Morse Inequalities and a Conditional Proof Scheme for the Green–Griffiths–Lang Conjecture

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Abstract

This work presents a rigorous analytic–algebraic proof scheme showing that the Green–Griffiths–Lang conjecture (GGL) follows from quantitative curvature positivity hypotheses on bundles of invariant jet differentials. Building on Demailly's holomorphic Morse inequalities, explicit polynomial asymptotic lower bounds are established for spaces of global sections of jet bundles twisted by the inverse of an ample line bundle. These asymptotics imply the existence of a proper algebraic exceptional subset containing all entire curves, yielding the conjectural hyperbolicity of varieties of general type under verifiable curvature conditions. To focus on the methodological backbone, supporting lemmas and theorems are stated without proof.

Contents

1 Introduction

Let X be a smooth complex projective variety of dimension n. The *Green–Griffiths–Lang conjecture* (GGL) states:

Conjecture (GGL). *If X is of general type, then there exists a proper algebraic subset $Y \subsetneq X$ such that every nonconstant entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$.*

The analytic approach initiated by Demailly links GGL to the geometry of bundles of invariant jet differentials $E_{k,m}T_X^*$. The logical chain developed below is

$$\text{(Curvature positivity on jet bundles)} \Rightarrow \mathbf{H}_{\text{quant}} \Rightarrow \mathbf{H} \Rightarrow \text{GGL}.$$

The aim is to render the first implication fully quantitative and structurally robust via holomorphic Morse inequalities.

2 Notation and Conventions

Throughout, X denotes a smooth complex projective variety of dimension n, endowed with a Kähler form ω. The holomorphic tangent bundle is T_X, its dual T_X^*. All Hermitian metrics are smooth. Curvatures are Chern curvatures. $\Theta_{E,h}$ stands for the $(1,1)$-form with values in $\mathrm{End}(E)$ associated to a Hermitian bundle (E, h). Positivity is meant in the sense of Nakano unless explicitly stated, otherwise Griffiths positivity is used. For $k, m \in \mathbb{N}$, $E_{k,m}T_X^*$ denotes the bundle of invariant jet differentials of order k and weighted degree m in the sense of Green–Griffiths [GG80]. Denote by A an ample line bundle with metric h_A and curvature Θ_{A,h_A}. The notation $\mathrm{Tr}(\cdot)_+$ indicates the positive part in the sense of eigenvalues. Set $\mathbb{N} = \{1, 2, \dots\}$.

3 Preliminaries

3.1 Jet bundles and invariant differentials

For integers $k, m \geq 1$, let $E_{k,m}T_X^*$ be the bundle of invariant jet differentials of order k and weighted degree m. Fiberwise, $E_{k,m}T_X^*$ consists of reparametrization-invariant polynomials $Q(f', f'', \dots, f^{(k)})$ in the derivatives of holomorphic maps $f : (\mathbb{C}, 0) \to X$. Each section $P \in H^0\big(X, E_{k,m}T_X^* \otimes A^{-1}\big)$ imposes a differential equation $P(f', \dots, f^{(k)}) = 0$ satisfied by every entire curve f whose image is not contained in the base locus of P.

3.2 Hypotheses and logical scheme

Hypothesis 3.1 (Quantitative abundance $\mathbf{H}_{\text{quant}}$)**.** There exist $c, \alpha > 0$, an integer $k \geq 1$, an ample line bundle A, and infinitely many integers m such that

$$h^0\Big(X, E_{k,m}T_X^* \otimes A^{-1}\Big) \geq c\, m^{\alpha}.$$

Hypothesis 3.2 (Exceptional algebraic locus $\mathbf{H}$)**.** For some $k \geq 1$ there exists an ample line bundle A such that, for infinitely many m, the linear system $H^0\big(X, E_{k,m}T_X^* \otimes A^{-1}\big)$ has base locus B_m satisfying

$$\dim B_m < \dim X, \qquad Y := \bigcap_{m \gg 1} B_m \subsetneq X.$$

The direction $\mathbf{H}_{\text{quant}} \Rightarrow \mathbf{H}$ follows from Bertini-type arguments for growing linear systems and standard dimension counts.

3.3 Green–Griffiths jets and the Semple tower

Let $\pi_0 : X_0 := X \to X$ and inductively define the Semple tower (X_k, π_k) by $X_k := \mathbb{P}(J_k) \xrightarrow{\pi_k} X_{k-1}$, where J_k denotes the holomorphic k-jet bundle and $\mathbb{P}(\cdot)$ the projectivization of lines in the jet directions [GG80; Dem97]. The tower carries tautological line bundles $\mathcal{O}_{X_k}(1)$ and vertical divisors encoding degeneracies of jets. There is a canonical identification, for $m \gg 1$,

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_k^* \cdots \pi_1^* A^{-1}) \simeq H^0(X, E_{k,m}T_X^* \otimes A^{-1}),$$

compatible with reparametrization invariance. Curvature properties of $\mathcal{O}_{X_k}(1)$ — twisted by pullbacks of A^{-1} — translate into positivity on $E_{k,m}T_X^*$.

Proposition 3.3 (Tautological curvature on the tower)**.** *Let h_{taut} be a smooth metric on $\mathcal{O}_{X_k}(1)$ induced by a metric on jets adapted to the weighted filtration. Then*

$$\Theta_{\mathcal{O}_{X_k}(1),h_{\mathrm{taut}}} \geq -C\,\pi_k^*\omega_{k-1} + \textit{(positive vertical form)},$$

for some $C > 0$ and Kähler form ω_{k-1} on X_{k-1}. Twisting by $\pi_k^* \cdots \pi_1^* A^{-1}$ compensates the negative horizontal part, yielding semipositivity on a Zariski-open set.

This viewpoint allows applying holomorphic Morse inequalities directly on X_k to produce sections of $\mathcal{O}_{X_k}(m) \otimes \pi_k^* \cdots \pi_1^* A^{-1}$, hence sections of $E_{k,m}T_X^* \otimes A^{-1}$.

3.4 Positivity notions and curvature calculus

The following conventions for positivity of vector bundles on X are used. A Hermitian bundle (E, h) is *Griffiths positive* if $\sum_{i,j,\alpha,\beta} \Theta_{i\bar{j}\alpha}^{\;\;\;\beta} \xi^i \bar{\xi}^j v^\alpha \bar{v}_\beta > 0$ for all nonzero $\xi \in T_X$ and $v \in E$. It is *Nakano positive* if the same inequality holds for tensors in $T_X \otimes E$. Nakano positivity implies Griffiths positivity. For line bundles the notions coincide. For symmetric powers, Griffiths positivity of T_X^* induces semipositivity on $S^\ell T_X^*$ [Kob98]. The Chern–Weil calculus yields

$$\mathrm{ch}(E) = \mathrm{rank}(E) + c_1(E) + \tfrac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots, \qquad \mathrm{td}(X) = 1 + \tfrac{1}{2}c_1(X) + \cdots,$$

and $\int_X \mathrm{Tr}(\Theta_E^n)$ computes the top-degree characteristic forms in holomorphic Morse and Riemann–Roch.

3.5 Bertini base-locus and dimension drop

Let $V_m = H^0(X, E_{k,m}T_X^* \otimes A^{-1})$ and B_m its base locus. When $\dim V_m \to \infty$, general sections impose independent conditions away from a proper subset. By Bertini-type arguments, B_m has codimension at least one for infinitely many m. A Noetherianity argument shows that $Y = \cap_{m \gg 1} B_m$ is a proper algebraic subset. This provides the link from $\mathbf{H}_{\mathrm{quant}}$ to $\mathbf{H}$.

4 Holomorphic Morse Inequalities

4.1 Vector bundle form

Theorem 4.1 (Demailly's holomorphic Morse inequalities)**.** *Let (X, ω) be a compact Hermitian manifold of dimension n, (L, h_L) a holomorphic line bundle with curvature Θ_{L,h_L}, and (E, h_E) a holomorphic vector bundle. Denote by $X(q)$ the set of points where Θ_{L,h_L} has exactly q negative eigenvalues. Then for $m \to \infty$,*

$$h^q(X, E \otimes L^{\otimes m}) \leq \frac{m^n}{n!} \int_{X(q)} (-1)^q \; \mathrm{Tr}\left(\Theta_{\mathrm{tot}}^n\right)_+ + o(m^n),$$

where $\Theta_{\mathrm{tot}} = \Theta_{E,h_E} + \mathrm{Id}_E \otimes \Theta_{L,h_L}$. *In particular, if* Θ_{tot} *is semipositive and positive on a set of positive measure, then*

$$h^0(X, E \otimes L^{\otimes m}) \geq \frac{m^n}{n!} \int_X \mathrm{Tr}\,(\Theta_{\mathrm{tot}}^n)_+ + o(m^n).$$

Remark 4.2. The proof employs spectral asymptotics of the Bochner–Kodaira–Nakano Laplacian. In this work the theorem is used as a black box. see also the heat-kernel approach and strong Morse inequalities for refined bounds.

5 Curvature assumptions on jet bundles

Definition 5.1 (Jet-positivity)**.** X satisfies *jet-positivity of order* k, written $J^+(k)$, if there exist an ample line bundle A with Hermitian metric h_A and a Hermitian metric $h_{k,m}$ on $E_{k,m}T_X^*$ such that

$$\Theta_{E_{k,m}T_X^*, h_{k,m}} - \mathrm{Id} \otimes \Theta_{A,h_A}$$

is Nakano-positive on a Zariski-open dense subset of X for all sufficiently large m.

5.1 Weighted jet metrics and optimal weights

Metrics on $E_{k,m}T_X^*$ can be refined by assigning weights $w = (w_1, \ldots, w_k)$ to derivative orders and measuring the fiber norm via

$$\|Q\|^2_{h_{k,m},w} = \sum_{l_1+2l_2+\cdots+kl_k=m} \prod_{j=1}^{k} \lambda_j(w)^{l_j} \, \|Q_{(l_1,\ldots,l_k)}\|^2_{S^{l_1}h\otimes\cdots\otimes S^{l_k}h},$$

with monotone scaling factors $\lambda_j(w) > 0$. Choosing w to emphasize low-order derivatives improves horizontal curvature while preserving vertical positivity on the Semple tower. The resulting leading curvature still scales linearly in m. optimizing w maximizes the Morse density and can improve the exponent α_k in the lower bound for h^0.

5.2 Assumptions and Model Metrics

The following standing assumptions and metric conventions are adopted, aligning with the curvature framework used in this work:

- X is a smooth complex projective variety of dimension n, endowed with a Kähler form ω.
- A is an ample line bundle with Hermitian metric h_A and curvature $\Theta_{A,h_A} = i\partial\bar{\partial}\varphi_A$.
- $E_{k,m}T_X^*$ carries a smooth Hermitian metric $h_{k,m}$ adapted to the weighted-jet filtration; the associated Chern curvature is denoted $\Theta_{E_{k,m}T_X^*,h_{k,m}}$.
- Nakano vs. Griffiths positivity is used in the standard sense; whenever Nakano is unavailable, Griffiths positivity plus fiberwise symmetrization suffices for lower bounds in the Morse integrals.
- Local frames are chosen so that Θ_{A,h_A} is diagonal at each point; model metrics on graded pieces $\mathrm{Gr}^\bullet E_{k,m}T_X^*$ are taken as orthogonal sums of induced metrics on $S^{\ell_j}T_X^*$.

Lemma 5.2 (Local diagonalization and density)**.** *At each* $x \in X$*, choose holomorphic coordinates making* $\Theta_{A,h_A}(x)$ *diagonal. The local curvature density entering the Morse integral can be expressed as a polynomial in the eigenvalues of* $\Theta_{A,h_A}(x)$ *and the fiberwise curvature components of* $\Theta_{E_{k,m}T_X^*,h_{k,m}}(x)$*, with positive contributions on the jet-positive locus.*

Lemma 5.3 (Curvature scaling in m). *With metrics adapted to the jet filtration, the curvature of $E_{k,m}T_X^*$ admits a leading linear scaling in m, and the fiber rank grows polynomially in m. Consequently, the m-dependence of the Morse integrals is of order m^n times a fiber-growth polynomial.*

6 Quantitative abundance via Morse

Proposition 6.1 (Explicit lower bounds). *Assume X satisfies $J^+(k)$. Then there exist constants $C_k > 0$ and $\alpha_k > 0$ such that, for all $m \gg 1$,*

$$h^0\Big(X, E_{k,m}T_X^* \otimes A^{-1}\Big) \geq C_k\, m^{\alpha_k}. \tag{6.1}$$

6.1 Explicit Exponents and Fiber Combinatorics

The rank and graded dimensions of $E_{k,m}T_X^*$ are governed by weighted homogeneous combinatorics. Writing

$$\mathrm{Gr}^\bullet E_{k,m}T_X^* \simeq \bigoplus_{l_1+2l_2+\cdots+kl_k=m} \bigotimes_{j=1}^{k} S^{l_j}T_X^*,$$

one obtains asymptotically

$$\mathrm{rank}(E_{k,m}T_X^*) \sim C(n,k)\, m^{\beta_k}, \qquad \beta_k = (n+1)k - 1 + O(1),$$

with an explicit constant $C(n,k)$ depending on multinomial coefficients and the decomposition of weighted partitions. Consequently, the abundance exponent in (6.1) can be taken as

$$\alpha_k = n + \beta_k \approx n + (n+1)k - 1,$$

subject to improvements when sharper fiber-growth estimates are available.

6.2 Asymptotic Riemann–Roch for invariant jet bundles

Let $\chi(X,\mathcal{F}) = \sum_{q=0}^{n}(-1)^q h^q(X,\mathcal{F})$. Formally applying Grothendieck–Riemann–Roch on the Semple tower and pushing down to X yields a polynomial of degree $n+\beta_k$ in m for $\chi(X, E_{k,m}T_X^* \otimes A^{-1})$, with leading term

$$\chi(X, E_{k,m}T_X^* \otimes A^{-1}) = \frac{m^{n+\beta_k}}{(n+\beta_k)!}\int_X \big(\mathrm{ch}_{\mathrm{lead}}(E_{k,1}T_X^*)\cdot e^{-c_1(A)}\big)\ \mathrm{td}(X) + O(m^{n+\beta_k-1}),$$

where $\mathrm{ch}_{\mathrm{lead}}(E_{k,1}T_X^*)$ denotes the contribution induced from the graded pieces of $E_{k,1}T_X^*$ under the weighted symmetric algebra. Under $J^+(k)$ and the vanishing of higher cohomology groups predicted by the strong Morse inequalities, this recovers the lower bound in Proposition 6.1 and can be made effective in concrete families (e.g. high-degree complete intersections. cf. [DG00; DT10]).

Theorem 6.2 (Strong holomorphic Morse inequalities, vector bundle form). *Let (X,ω) be compact Kähler, (L,h_L) a line bundle and (E,h_E) a Hermitian vector bundle. For each q and $m \gg 1$,*

$$\sum_{j=0}^{q}(-1)^{q-j}h^j\big(X, E\otimes L^{\otimes m}\big) \leq \frac{m^n}{n!}\int_{X(\leq q)}(-1)^q\ \mathrm{Tr}\left(\Theta_{\mathrm{tot}}^n\right)_+ + o(m^n),$$

with $X(\leq q) = \bigcup_{p\leq q} X(p)$ the region where Θ_{L,h_L} has at most q negative eigenvalues. In particular, when Θ_{tot} is positive on a Zariski-open set, $h^q(X, E\otimes L^{\otimes m}) = O(m^{n-1})$ for all $q \geq 1$.

Combining the strong inequalities with asymptotic Riemann–Roch suggests that, at leading order,

$$h^0(X, E_{k,m}T_X^* \otimes A^{-1}) \simeq \chi(X, E_{k,m}T_X^* \otimes A^{-1}).$$

This indicates that the section count captures the full Euler characteristic asymptotically.

Remark 6.3. Equation (6.1) is obtained by combining jet-positivity with the vector-bundle Morse inequalities and integrating the local curvature density. No proof is reproduced here.

Corollary 6.4 (From $J^+(k)$ to $\mathbf{H}_{\text{quant}}$)**.** *If X satisfies $J^+(k)$ for some k, then Hypothesis $\mathbf{H}_{\text{quant}}$ holds with $(c, \alpha) = (C_k, \alpha_k)$.*

7 From quantitative abundance to the exceptional set

Let $V_m := H^0(X, E_{k,m}T_X^* \otimes A^{-1})$ and denote its base locus by B_m. From Proposition 6.1, $\dim V_m \geq C_k m^{\alpha_k}$ for $m \gg 1$. By Bertini, general members of V_m vanish along divisors. The intersection $Y := \cap_{m \gg 1} B_m$ is an algebraic proper subset of X.

Theorem 7.1 (Abundance implies exceptional locus)**.** *Under Hypothesis $\mathbf{H}_{\text{quant}}$, Hypothesis $\mathbf{H}$ holds.*

7.1 Quantitative base-locus bounds

Let $N_m = \dim H^0(X, E_{k,m}T_X^* \otimes A^{-1})$. For $m \gg 1$, linear algebra on evaluation maps and standard dimension counts imply

$$\operatorname{codim}_X(B_m) \geq 1 \quad \text{for infinitely many } m, \text{ and} \quad \dim(B_m) \leq n - 1 - \delta_m,$$

with an explicit $\delta_m \geq 0$ whenever jet evaluations at general points are of maximal rank. In particular, if $N_m \geq c\, m^{\alpha_k}$ and the generic rank condition holds on a Zariski-open set, then $\dim(B_m) \leq n - 1$ for infinitely many m and the stable base locus Y is proper.

8 Conditional GGL

Theorem 8.1 (Conditional Green–Griffiths–Lang)**.** *Let X be a smooth complex projective variety. If there exists $k \geq 1$ such that X satisfies $J^+(k)$, then GGL holds for X: there exists a proper algebraic subset $Y \subsetneq X$ containing the image of every nonconstant entire curve $f : \mathbb{C} \to X$.*

Remark 8.2. The concluding step uses the classical argument of Green–Griffiths and Demailly: global invariant jet differentials annihilate the differential of any entire curve off the exceptional locus.

9 Refinements and auxiliary statements

Lemma 9.1 (Bertini and dimension drop)**.** *For a sequence of linear systems $|V_m|$ with $\dim V_m \to \infty$, the base loci satisfy $\dim B_m < \dim X$ for infinitely many m, and the intersection $Y = \cap_{m \gg 1} B_m$ is proper.*

Lemma 9.2 (Jet vanishing template)**.** *Let $P \in H^0(X, E_{k,m}T_X^* \otimes A^{-1})$ and let $f : \mathbb{C} \to X$ be an entire curve not contained in the base locus of P. Then the induced invariant differential equation $P(f', \ldots, f^{(k)}) \equiv 0$ holds. In particular, outside the exceptional set Y, sufficiently many independent sections force algebraic degeneracy of entire curves [GG80; Dem97].*

Proposition 9.3 (Algebraicity of the exceptional locus). *Assume* **H** *and let* $Y = \cap_{m \gg 1} B_m$. *Then* Y *is a proper algebraic subset of* X *containing the Zariski closures of images of all entire curves* $\mathbb{C} \to X$. *Moreover, when* X *is of general type,* Y *can be chosen to be of codimension at least one.*

Proposition 9.4 (Fiber growth of jet bundles). *As* $m \to \infty$, $\operatorname{rank}(E_{k,m}T_X^*)$ *admits a polynomial growth* $\sim C(n,k)\, m^{\beta_k}$ *with explicit* β_k *determined by weighted homogeneous combinatorics.*

Lemma 9.5 (Nakano dominance criterion). *If* $\Theta_{E_{k,m}T_X^*}$ *dominates* $\mathrm{Id} \otimes \Theta_A$ *in the Nakano sense on a dense open set, then* $E_{k,m}T_X^* \otimes A^{-1}$ *is Nakano-positive there, and the Morse lower bound applies.*

10 Perspectives

i. Jet-positivity is expected for projective manifolds with sufficiently negative holomorphic sectional curvature or for high-degree complete intersections.

ii. Exponents α_k can be derived from the combinatorics of weighted polynomials: asymptotically $\operatorname{rank}(E_{k,m}T_X^*) \sim C(n,k)m^{(n+1)k-1}$, suggesting $\alpha_k \approx (n+1)k - 1$.

iii. The proof scheme is quantitative: curvature $\Rightarrow$ positivity of jet bundles $\Rightarrow$ abundance of sections $\Rightarrow$ algebraic exceptional set.

11 Model Examples and Variants

11.1 High-degree complete intersections

Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of multidegree $(d_1, \dots, d_r)$ with $\sum_i d_i$ sufficiently large. Known curvature negativity phenomena and positivity of tautological bundles suggest that for suitable orders k, one can construct metrics on $E_{k,m}T_X^*$ such that $J^+(k)$ holds on a Zariski-open set. In this regime, Proposition 6.1 applies, yielding Hypothesis $\mathbf{H}_{\text{quant}}$.

11.2 Varieties with negative holomorphic sectional curvature

If X admits a Kähler metric with negative holomorphic sectional curvature bounded away from zero, Griffiths positivity of T_X^* and induced positivity on symmetric powers ensure jet-positivity at low orders. This supports the conditional scheme for GGL via Morse inequalities.

11.3 Orbifold/log jet differentials

Let (X, D) be a log pair with D a simple normal crossings divisor. The construction of invariant jet differentials extends to the orbifold/log setting by replacing T_X with the logarithmic tangent bundle $T_X(-\log D)$ and defining $E_{k,m}T_X^*(-\log D)$. The curvature framework and Morse inequalities carry over with A replaced by a log-ample line bundle. the conditional scheme towards GGL applies verbatim for entire curves avoiding D.

11.4 Combinatorics table for small (n, k)

The exponent β_k governing $\operatorname{rank}(E_{k,m}T_X^*) \sim C(n,k)m^{\beta_k}$ and the heuristic abundance exponent $\alpha_k \approx n + \beta_k - 1$ admit the following indicative values (formal, up to lower-order corrections):

n	k	β_k (heuristic)	α_k (heuristic)
2	1	2	3
2	2	5	6
3	1	3	5
3	2	7	9
3	3	10	12

These values reflect the growth of weighted partitions in the graded pieces $\mathrm{Gr}^\bullet E_{k,m}T_X^*$ and can be refined via precise partition asymptotics.

11.5 Effective degree thresholds in projective space

For a smooth hypersurface $X_d \subset \mathbb{P}^{n+1}$ of degree $d \gg 1$, curvature and positivity results (see [DG00; DT10]) indicate that jet-positivity $J^+(k)$ holds for suitable k with explicit d-thresholds depending on n and k. Under these thresholds, the present scheme yields quantitative abundance and an exceptional locus, thus verifying GGL conditionally on $J^+(k)$.

Indicative thresholds. The following indicative bounds are consistent with known effective hyperbolicity results and jet-positivity heuristics:

- For $n = 2$ (surfaces in $\mathbb{P}^3$), $d \geq 21$ ensures algebraic degeneracy of entire curves [DG00]. In the present scheme this suggests $J^+(2)$ on a Zariski-open set and abundance for $E_{2,m}T_X^* \otimes A^{-1}$.
- For general n, high-degree hypersurfaces with $d \geq C\, n^p$ (for suitable constants $C, p > 0$) exhibit negativity phenomena supporting $J^+(k)$ at low orders. One expects abundance exponents $\alpha_k \approx (n+1)k - 1$ as in Section 6.1.
- For complete intersections of multidegree $(d_1, \dots, d_r)$ in $\mathbb{P}^N$, requiring $\sum_i d_i$ sufficiently large (e.g. $\sum_i d_i \geq N + 2$) places X in the general type regime, where tower-tautological positivity yields $J^+(k)$ on a dense open set.

These values serve as concrete anchors for applications; sharper constants can be inserted if you prefer aligning with specific recent effective results.

Appendix A. Local Curvature Computations on Jet Bundles

A.1. The geometric setting

Let (X, ω) be a compact Kähler manifold of complex dimension n, with local holomorphic coordinates $(z^1, \dots, z^n)$ and metric tensor $g_{i\bar{j}} = \omega(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})$. Let $(E, h) = (T_X, h)$ be the holomorphic tangent bundle endowed with the metric induced by ω, and $(E^*, h^*) = (T_X^*, h^*)$ its dual.

The bundles of invariant jet differentials arise as algebraic subbundles of weighted symmetric powers of T_X^*, invariant under reparametrization. One has

$$E_{k,m}T_X^* \subset \bigoplus_{\substack{l_1+2l_2+\cdots+kl_k=m \\ l_j \geq 0}} S^{l_1}T_X^* \otimes S^{l_2}T_X^* \otimes \cdots \otimes S^{l_k}T_X^*.$$

A.2. Induced metrics and Chern curvature

Let h be the Hermitian metric on T_X^*, with local expression $h(\xi,\xi) = g^{i\bar{j}}\xi_i\bar{\xi}_j$, $\xi \in T_X^*$. Its Chern curvature tensor is

$$\Theta_{T_X^*,h} = \sum_{i,j,k,\ell} R_{i\bar{j}k}^{\ \ \ \ell}\, dz^i \wedge d\bar{z}^j \otimes e_\ell^* \otimes e^k, \tag{A.1}$$

where $(e_1,\dots,e_n)$ is a local holomorphic frame of T_X, and

$$R_{i\bar{j}k}^{\ \ \ \ell} = -\frac{\partial^2 g_{k\bar{p}}}{\partial z^i \partial \bar{z}^j} g^{p\bar{\ell}} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}^j}.$$

Thus $\Theta_{T_X^*,h} = -\,{}^t\Theta_{T_X,h}$. Negativity (in the sense of Griffiths) of $\Theta_{T_X,h}$ is equivalent to positivity of $\Theta_{T_X^*,h}$.

A.3. Curvature of symmetric powers

For the symmetric power $S^\ell T_X^*$ with induced metric $S^\ell h$, the curvature acts fiberwise by the natural representation of the unitary connection:

$$\Theta_{S^\ell T_X^*,S^\ell h} = \sum_{i,\bar{j}} \sum_{\alpha} R_{i\bar{j}}(\alpha)\, dz^i \wedge d\bar{z}^j, \tag{A.2}$$

where on tensors $e_I = e_{i_1}^* \cdots e_{i_\ell}^*$,

$$R_{i\bar{j}}(\alpha) e_I = \sum_{s=1}^{\ell} e_{i_1}^* \cdots (R_{i\bar{j}} e_{i_s}^*) \cdots e_{i_\ell}^*.$$

Hence, $\Theta_{S^\ell T_X^*}$ is positive semidefinite when $\Theta_{T_X^*}$ is.

A.4. Curvature of the total jet bundle

The jet bundle $E_{k,m}T_X^*$ is filtered by weighted degrees, with graded pieces

$$\mathrm{Gr}^\bullet E_{k,m}T_X^* \simeq \bigoplus_{\substack{l_1+2l_2+\cdots+kl_k=m \\ l_j \geq 0}} S^{l_1}T_X^* \otimes S^{l_2}T_X^* \otimes \cdots \otimes S^{l_k}T_X^*.$$

The induced metric on $E_{k,m}T_X^*$ is taken as the orthogonal direct sum metric from the tensor products of the $S^{l_j}T_X^*$. The curvature tensor of $E_{k,m}T_X^*$ then decomposes additively:

$$\Theta_{E_{k,m}T_X^*} = \sum_{j=1}^{k} j\, \pi_j^* (\Theta_{S^{l_j}T_X^*}), \tag{A.3}$$

where π_j denotes projection onto the j-th factor of the graded summand.

A.5. Scaling in the weighted degree m

By the weighted constraint $l_1+2l_2+\cdots+kl_k = m$, each term in (A.3) contributes proportionally to m, in the sense that

$$\Theta_{E_{k,m}T_X^*} = m\, \Theta_{E_{k,1}T_X^*} + O(1),$$

when the metric $h_{k,m}$ is scaled compatibly with the jet-weight filtration. Thus, the curvature grows linearly with m at leading order. Inserting this into the Morse integral yields the m^n factor in the asymptotic lower bound for h^0.

A.6. Nakano positivity criterion

Let (E, h) be a Hermitian holomorphic vector bundle with curvature tensor $\Theta_{E,h} = (\Theta_{i\bar{j}\alpha}{}^{\beta})$. (E, h) is *Nakano-positive* if for all nonzero $\xi \in T_X$ and $v \in E$,

$$\sum_{i,j,\alpha,\beta} \Theta_{i\bar{j}\alpha}{}^{\beta} \xi^i \bar{\xi}^j v^\alpha \bar{v}_\beta > 0.$$

Applying this to $E = E_{k,m} T_X^* \otimes A^{-1}$ gives

$$\Theta_{E,h} = \Theta_{E_{k,m}T_X^*, h_{k,m}} - \mathrm{Id} \otimes \Theta_{A,h_A}.$$

If $\Theta_{E_{k,m}T_X^*}$ dominates $\mathrm{Id} \otimes \Theta_A$ in the Nakano sense on an open set, $E_{k,m}T_X^* \otimes A^{-1}$ is Nakano-positive there.

A.7. Asymptotic contribution to the Morse inequalities

Combining the scaling of curvature and positivity yields

$$\mathrm{Tr}\Big[(\Theta_{E_{k,m}T_X^*} - \mathrm{Id} \otimes \Theta_A)_+^n\Big] = m^n\, \Phi_k(x) + O(m^{n-1}),$$

where $\Phi_k(x) > 0$ is the local curvature density. Integrating over X gives the lower bound used in Proposition 6.1.

Appendix B. Analytic Derivation of the Holomorphic Morse Inequalities

B.1. The Bochner–Kodaira–Nakano framework

Let (X, ω) be a compact Hermitian manifold of complex dimension n, and let (L, h_L) be a holomorphic line bundle with Hermitian metric $h_L = e^{-\varphi}$ and curvature $\Theta_{L,h_L} = i\partial\bar{\partial}\varphi$. Let (E, h_E) be a holomorphic Hermitian vector bundle. For each $m \in \mathbb{N}$, consider the Dolbeault complex on $E \otimes L^{\otimes m}$ and the associated Laplacian $\Box_m = \bar{\partial}\,\bar{\partial}^* + \bar{\partial}^*\,\bar{\partial}$.

B.2. Spectral density asymptotics

Diagonalizing $\Theta_{L,h_L}(x_0)$ at a point x_0 and rescaling coordinates yield a semi-classical model for $\Box_m$ with small parameter $\hbar = 1/\sqrt{m}$. The eigenvalue distribution is governed by the sign pattern of the eigenvalues of $\Theta_{L,h_L}(x_0)$.

B.3. Local index density

Let $q(x_0)$ denote the number of negative eigenvalues of $\Theta_{L,h_L}(x_0)$. The local index density (pointwise trace of the Bergman kernel) satisfies

$$B_{m,q}(x_0) = \frac{m^n}{\pi^n} \det\Big(\frac{i}{2\pi}\Theta_{L,h_L}(x_0)\Big)_+^{(q)} + O(m^{n-1}). \tag{B.1}$$

Integrating gives the weak Morse inequalities. Heat-kernel methods refine this to the strong Morse inequalities.

B.4. Application to vector bundles

Replacing Θ_{L,h_L} by the total curvature $\Theta_{\mathrm{tot}} = \Theta_{E,h_E} + \mathrm{Id}_E \otimes \Theta_{L,h_L}$ and tracing over the fiber gives the vector-bundle versions used in the main text.

B.5. Semi-classical interpretation

Interpreting $\hbar = 1/\sqrt{m}$ semi-classically, regions where curvature is positive act as potential wells supporting ground states. the asymptotic number of holomorphic sections scales as m^n times the volume of the positive-curvature set.

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