Infinite Sequences:

Def: (Informal) A sequence is a succession of #'s that are listed according to a given prescription or rule.

Specifically, if n is a +ve integer, the sequence whose nth term is the # a_n can be written as: $a_1, a_2, \ldots, a_n, \ldots$ or more simply $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

a_n is called the general term of the sequence.

Note: The number a_1 is called the 1st term.
The number a_2 is called the 2nd term.
$\vdots$

Example: Write the 1st four terms of the following sequences $\{a_n\}$:

(a) $a_n = \dfrac{3n+1}{n+2}$

Solution:

$a_1 =$ $\qquad\qquad$ $a_2 =$

$a_3 =$ $\qquad\qquad$ $a_4 =$

$$\therefore \{a_n\} = \left\{\frac{3n+1}{n+2}\right\} = \{$$

(b) $a_n = \cos\dfrac{n\pi}{6}$

Solution:

$a_1 =$ $\qquad\qquad$ $a_2 =$

$a_3 =$ $\qquad\qquad$ $a_4 =$

(c) $a_n = \dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{n!}$

Solution:

$a_1 =$

$a_2 =$

$a_3 =$

$a_4 =$

Note: Even though we write $a_n = \dfrac{3n+1}{n+2}$, remember that this is a function. That is, $f: \mathbb{N} \to \mathbb{R}$ defined by:

$$f(n) = \frac{3n+1}{n+2}$$

So, $a_1 = f(1) = \dfrac{3(1)+1}{1+2} = \dfrac{4}{3}$, $a_2 = f(2) = \dfrac{3(2)+1}{2+2} = \dfrac{7}{4}$

$\vdots$ etc. $\qquad$ $\vdots$ etc.

Example: Find a formula for the nth term of the sequence

$$\frac{1}{1}, -\frac{1}{8}, \frac{1}{27}, \ldots$$

Solution:

denominators are 3rd power of the +ve integers starting with n=1. Also the signs of the terms a alternating. Thus,

$a_1 =$ $\qquad$ $a_2 =$

$a_3 =$

$\vdots$

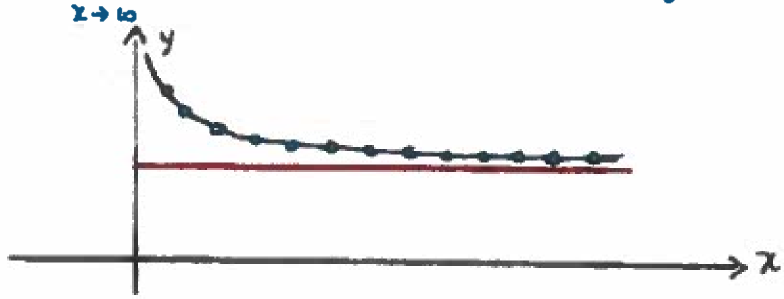
$a_n =$

<u>Def:</u> A sequence of real #'s is a function on the set $\mathbb{N}$ of natural #'s whose range is contained in $\mathbb{R}$.

<u>Def:</u> A sequence $\{a_n\}$ has the limit L, written as $\lim_{n \to \infty} a_n = L$ or $a_n \to L$ as $n \to \infty$, if the terms of the sequence $\{a_n\}$ can be made as close to L as may be desired by taking n sufficiently large.

If $\lim_{n \to \infty} a_n$ exists, we say the sequence ____________ or is convergent. Otherwise, we say the sequence is ____________

<u>Note:</u> The difference between the definitions $\lim_{n \to \infty} a_n = L$ and $\lim_{x \to \infty} f(x) = L$ is that _____ is required to be an integer.



<u>Theorem:</u> If $\lim_{x \to \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \to \infty} a_n = L$.

Note: The converse is not True!!
Consider the sequence $\left\{ \cos 2\pi n \right\}_{n=1}^{\infty}$

Now $\lim_{n\to\infty} \cos 2\pi n =$ since $\cos(2\pi n) =$ for all $n \in \mathbb{Z}$

but $\lim_{x\to\infty} \cos(2\pi x)$ ________ _______ _________, because

it _____________________ between _____ and _____.

Note: (i) If a_n becomes larger as n becomes large, we write

$$\lim_{n\to\infty} a_n = \infty$$

(ii) Since $\lim_{x\to\infty} \frac{1}{x^r} = 0$ when $r>0$, we have:

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \text{ if } r>0$$

(iii) Since $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$, we have that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Example. Compute the following limits:

(a) $\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n$

Solution:

$$\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n =$$

$$=$$

=

=

Alternate Solution:

$$\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$$

Let $\frac{2}{n} = \frac{1}{m} \Rightarrow m = \frac{n}{2}$ ($\Rightarrow$ =

As $n\to\infty$, $m\to\infty$

Hence,

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{m\to\infty}\left(\quad\right)$$

$$= \lim_{m\to\infty}\left[\left(\quad\right)\right]$$

$$=$$

We can also use L'Hospital's Rule:

$$a_n = \left(\frac{n+2}{n}\right)^n .$$

Let $f(x) = \left(\frac{x+2}{x}\right)^x$ so that $a_n = f(n)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$$ indeterminate of type 1^∞

So, $f(x) = \left(1 + \frac{2}{x}\right)^x$

$$\ln f(x) = \ln\left(1 + \frac{2}{x}\right)^x$$

$$= x \ln\left(1 + \frac{2}{x}\right)$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} x \ln\left(1 + \frac{2}{x}\right)$$

indeterminate of type $\infty \cdot 0$

$$= \lim_{x\to\infty}$$

$$\overset{H}{=} \lim_{x\to\infty}$$

$$=$$

$$=$$

$$\therefore \lim_{x\to\infty}\left(1 + \frac{2}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)}$$

$$= e^{\lim_{x\to\infty} \ln f(x)}$$

$$=$$

<u>Note:</u> The limit laws that we have seen in Ma. 1000 also hold for limits of sequences:

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and k is a constant, then the following also hold for sequences:

(i) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

(ii) $\lim\limits_{n\to\infty}(k\,a_n) = k\lim\limits_{n\to\infty} a_n$

(iii) $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

(iv) $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ if $\lim\limits_{n\to\infty} b_n \neq 0$

(v) $\lim\limits_{n\to\infty} a_n^{\,p} = \left[\lim\limits_{n\to\infty} a_n\right]^p$ if $p>0$ and $a_n>0$

<u>The Squeeze Theorem:</u> If $a_n \le b_n \le c_n$ for $n \geq n_0$ and

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$

<u>Application:</u> If $\lim\limits_{n\to\infty}|a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

<u>Proof:</u>

If $\lim\limits_{n\to\infty}|a_n| = 0$, then $\lim\limits_{n\to\infty} -|a_n| = -\lim\limits_{n\to\infty}|a_n| = 0$

Now, $\quad -|a_n| \le a_n \le |a_n|$

$$\lim_{n\to\infty}(-|a_n|) \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty}|a_n|$$

$$0 \le \lim_{n\to\infty} a_n \le 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0 \qquad \text{by the Squeeze Theorem}$$

Example: Is the sequence with n^{th} term $a_n = \dfrac{(-1)^n \sqrt{n}}{n+1}$ convergent?

Solution:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^n \sqrt{n}}{n+1} \right|$$

$$=$$

$$=$$

$$=$$

$$=$$

$\Rightarrow \lim_{n\to\infty} a_n = 0$

$\therefore \ \{a_n\}$ ________________ to _____.

Theorem: If $\lim_{n\to\infty} a_n = L$ and the function f is cont. at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. That is,

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

Example: Determine whether the sequence with nth term $a_n = \tan^{-1}\left(\dfrac{2n}{2n+1}\right)$ is convergent.

Solution:

<u>Example</u>: Determine whether each of the following sequences with given nth term converges or diverges.

(a) $a_n = \dfrac{2n^2+5n-7}{n^3}$

<u>Solution</u>:

$$\lim_{n\to\infty} \frac{2n^2+5n-7}{n^3} =$$

$$=$$

$$=$$

$$=$$

$\therefore \{a_n\}$ ______________ to ____.

(b) $a_n = \dfrac{n^5+n^3+2}{7n^4+n^2+3}$

<u>Solution</u>:

$$\lim_{n\to\infty} \frac{n^5+n^3+2}{7n^4+n^2+3} =$$

$$=$$

$$=$$

$\therefore \{a_n\}$ ______________

(c) $a_n = \dfrac{n^2}{n+4} - \dfrac{n^2}{n+9}$

<u>Solution:</u>

Note: $\lim\limits_{n\to\infty}\left(\dfrac{n^2}{n+4} - \dfrac{n^2}{n+9}\right) \neq \lim\limits_{n\to\infty}\dfrac{n^2}{n+4} - \lim\limits_{n\to\infty}\dfrac{n^2}{n+9}$

because neither limit exists. It is also incorrect to use this as a reason to say that the limit does not exist.

Thus, $\lim\limits_{n\to\infty}\left(\dfrac{n^2}{n+4} - \dfrac{n^2}{n+9}\right) =$

$=$

$=$

$=$

$=$

$\therefore \{a_n\}$ ________________ to ____.

(d) $a_n = \dfrac{\sin 4n}{2^n}$

(e) $a_n = (-1)^n$

Solution:

(f) $a_n = n \tan^{-1}\left(\frac{1}{n}\right)$

Solution:

(g) $a_n = 2\ln n - \ln(n^2+1)$

Solution:

More applications of the Squeeze Theorem:

Example: Determine whether the sequence converges or diverges:

(a) $\{b_n\} = \left\{ \dfrac{n!}{n^n} \right\}$

Solution:

(b) $\{a_n\} = \left\{ \dfrac{n!}{2^n} \right\}$

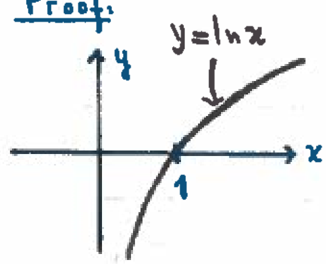
Solution:

$$a_n = \frac{n!}{2^n}$$

Note: The following theorem has frequent applications in the study of series.

Theorem: If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

Proof:



Example: Find $\lim_{n\to\infty} \dfrac{3^n + 4^n + 5^n}{5^n}$

Solution:

<u>Some Important limits:</u>

(i) $\lim_{n\to\infty} x^{\frac{1}{n}} =$ for all $x>0$

(ii) $\lim_{n\to\infty} x^n = \begin{cases} & \text{if } |x|<1 \\ & \text{if } x=1 \end{cases}$

i.e. $\{x^n\}$ is convergent if ______________ and divergent for all values of x.

(iii) $\lim_{n\to\infty} \frac{1}{n^k} =$ for all $k>0$

(iv) $\lim_{n\to\infty} \frac{x^n}{n!} =$ for all $x \in \mathbb{R}$

(v) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ (Use ______________ ______

(vi) $\lim_{n\to\infty} n^{\frac{1}{n}} =$

Proof of (vi):

Bounded and Monotonic Sequences:

Def: A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$.
i.e. $a_1 < a_2 < a_3 < \cdots < \cdots$

It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. i.e.
$a_1 > a_2 > a_3 > \cdots > \cdots$

It is called monotonic if it is either increasing or decreasing

Note: In order to show that a sequence is increasing or decreasing, we follow the following procedure:

(i) $\{a_n\}$ is $\uparrow \Leftrightarrow a_n < a_{n+1}$ for all $n \geq 1$.

$$\Leftrightarrow \frac{a_{n+1}}{a_n} > 1 \quad \text{for all } n \geq 1$$

$$\Leftrightarrow a_{n+1} > a_n \quad \text{for all } n \geq 1$$

$$\Leftrightarrow a_{n+1} - a_n > 0 \quad \text{for all } n \geq 1$$

Similarly for $\{a_n\} \downarrow$

(ii) Use Calculus to show f is $\downarrow$ or $\uparrow$ using the derivative.

Example: Show that the sequence $\left\{\frac{n}{n+1}\right\}$ is monotonic.

Solution:

Method 2:

Example: Show that the sequence $\left\{ \dfrac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots (2n)} \right\}$ is monotonic.

Solution:

Def: A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \le M$ for all $n \ge 1$.

It is bounded below if there is a number m such that $m \le a_n$ for all $n \ge 1$.

If it is bounded above and below, then $\{a_n\}$ is said to be a **bounded** sequence.

Example: (i) The sequence $\{n\}$ is bounded below since $0 < a_n$ for all n. but it is not bounded above.
$\rightarrow \{n\}$ is not a bounded sequence.

(ii) Let $a_n = \dfrac{n}{n^2+1}$. Is $\{a_n\}$ bounded?

(*) Note: (i) If a sequence is ↑ and bounded above, it is convergent.
(ii) If a sequence is ↓ and bounded below, it is convergent.

Solution:

$a_1 =$, $a_2 =$, $a_3 =$

(*) Theorem: The Bounded, Monotonic, Convergence Theorem (BMCT)
Every bounded, monotonic sequence is convergent.

Example: Use the BMCT to show that the following sequences are convergent:

(a) $\left\{\dfrac{n}{2^n}\right\}$

Solution:

(b) $\left\{\frac{\ln n}{\sqrt{n}}\right\}$

<u>Solution:</u>

Recursively defined Sequences:

Example: Let a_n be defined recursively by:

$$a_1 = 1, \quad a_{n+1} = \sqrt{6 + a_n} \quad (n = 1, 2, \ldots)$$

Find $\lim_{n\to\infty} a_n$ assuming it exists.

Solution: